SUSPENSION SPLITTINGS AND SELF-MAPS OF FLAG MANIFOLDS

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Abstract. If G is a compact connected Lie group and T is a maximal torus, we give a wedge decomposition of $\Sigma G/T$ by identifying families of idempotents in cohomology. This is used to give new information on the self-maps of G/T.

1. Introduction

Let G be a compact connected Lie group and let T be a maximal torus. There has been considerable interest in trying to determine the homotopy classes of the self-maps of the quotient space G/T. One method commonly adopted in [6, 9, 19, 26] is to study the image of the map

$$r\colon [G/T, G/T] \longrightarrow \mathrm{Hom}_{\mathrm{alg}}(H^*(G/T), H^*(G/T)).$$

We show that if G is simply-connected there is a bijection

$$[G/T, G/T] \cong [G/T, G] \times \mathrm{Im}(r), \tag{1}$$

where r sends a self-map to the ring homomorphism it induces in cohomology. This was earlier claimed to hold in more generality in [27], but there seems to be gaps. With (1) in hand, we consider the other factor, $[G/T, G]$, and develop an approach to understanding it.

Since G is a group it has a classifying space BG. This implies that there is a group isomorphism $[G/T, G] \cong [\Sigma G/T, BG]$. The idea is to decompose $\Sigma G/T$ into a wedge of smaller spaces which simplify the calculations. The decompositions are obtained by identifying certain idempotents in cohomology. These are p-local decompositions, where p is a prime. Two families of idempotents are considered, one coming from Adams operations on the classifying spaces of T and G, the other coming from the action of the Weyl group on G/T. These are also compatible in the sense that they can be merged to form a larger set of idempotents, giving a finer decomposition of the space.

These suspension splittings also fit into a larger framework that considers stable decompositions of homogeneous spaces. The classic example of this is Miller's stable splitting of Stiefel manifolds [16], which inspired a great many variants and refinements (e.g., [12, 18, 24, 25]). In those cases, the stable feature is prominent in the sense that multiple suspensions are usually needed to realize the decomposition, whereas in our case the decomposition occurs after a single suspension.

To demonstrate the methods we give explicit decompositions of $SU(3)/T$, $SU(4)/T$, $Sp(2)/T$ and G_2/T, and go on to calculate $[G/T, G]$ in each case (modulo 2-primary information in the $SU(4)$ and G_2 cases).

2. The cohomology of G/T

Let $W = N(T)/T$ be the Weyl group of G, which is generated by the simple reflections $s_1, \ldots, s_r$, where $r = \mathrm{rank}(T)$.

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Definition 2.1. For an element of $w \in W$, the length $l(w)$ of w is the least integer such that w can be written as a product of $l(w)$ simple reflections. So an element w of length n can be written as $w = s_{i_1} s_{i_2} \cdots s_{i_n}$ for some sequence of simple reflections. We often abbreviate this as $w = s_{i_1 i_2 \cdots i_n}$.

The following theorem proved by [8] describes $H^*(G/T;\mathbb{Z})$ as a free $\mathbb{Z}$-module.

Theorem 2.2 (Bruhat decomposition)**.** There is a cell decomposition

$$G/T = \bigcup_{w \in W} \sigma_w,$$

where σ_w are open cells with $\dim(\sigma_w) = 2l(w)$. Moreover, the closure is $\overline{\sigma_w} = \bigcup_{v \le w} \sigma_v$, where the order on W is given by the strong Bruhat order, that is, $v \le w$ iff a reduced word for w contains one of v as a sub-word. Consequently, $H^*(G/T;\mathbb{Z})$ is torsion free of rank $|W|$ and its basis is given by the Schubert classes:

$$H^*(G/T;\mathbb{Z}) \cong H^{even}(G/T;\mathbb{Z}) \simeq \mathbb{Z}\langle \sigma_w \rangle_{w \in W}. \qquad \square$$

Borel [3] gives another description of $H^*(G/T)$ as the quotient of a polynomial ring.

Theorem 2.3 (Coinvariant description)**.** Let R be a ring in which the torsion primes [4] of G are inverted. Then

$$H^*(G/T;R) = H^*(BT;R)/I$$

where I is the ideal generated by the Weyl group invariants of positive degree.

Example 2.4.

$$H^*(SU(n)/T^{n-1};\mathbb{Z}) = \mathbb{Z}[x_1, x_2, \ldots, x_n]/(e_1, e_2, \ldots, e_n),$$

where e_i is the i-th elementary symmetric function on $x_1, \ldots, x_n$. A choice of a polynomial representative of σ_w in this presentation is given by the classical *Schubert polynomial* [13].

3. A splitting of $[G/T, G/T]$

The group homomorphism $T \longrightarrow G$ classifies, giving a homotopy fibration sequence

$$T \longrightarrow G \xrightarrow{q} G/T \xrightarrow{j} BT \longrightarrow BG$$

which defines the map j. In particular, for a simply-connected space X we obtain an exact sequence

$$[X, G] \xrightarrow{q_*} [X, G/T] \xrightarrow{j_*} [X.BT],$$

where q_* is injective. Consider the map

$$r\colon [X, G/T] \longrightarrow \mathrm{Hom}_{\mathrm{alg}}(H^*(G/T;\mathbb{Z}), H^*(X;\mathbb{Z}))$$

defined by sending a map $X \longrightarrow G/T$ to the algebra homomorphism it induces in cohomology. Similarly, there is a map

$$[X, BT] \longrightarrow \mathrm{Hom}_{\mathrm{alg}}(H^*(BT;\mathbb{Z}), H^*(X;\mathbb{Z}))$$

which is an isomorphism since both sides are canonically isomorphic to $\bigoplus_{1 \le i \le r} H^2(X;\mathbb{Z})$. We obtain a commutative diagram

$$\begin{array}{ccccc} [X,G] & \hookrightarrow & [X,G/T] & \xrightarrow{j_*} & [X,BT] \\ & & \downarrow r & & \downarrow \simeq \\ & & \mathrm{Hom}_{\mathrm{alg}}(H^*(G/T;\mathbb{Z}), H^*(X;\mathbb{Z})) & \xrightarrow{J} & \mathrm{Hom}_{\mathrm{alg}}(H^*(BT;\mathbb{Z}), H^*(X;\mathbb{Z})) \end{array} \tag{2}$$

where $J(f^*) = f^* \circ j^*$. We have the holonomy action $[X, G/T] \times [X, G] \to [X, G/T]$. By the Puppe sequence [14, Lemma 1.4.7], the action is free and moreover, for $f_1, f_2 \in [X, G/T]$ we have

$j_*(f_1) = j_*(f_2)$ if and only if f_1 and f_2 are in the same orbit of the action of $[X, G]$. Therefore, we have the following non-canonical identification

$$[X, G/T] \cong [X, G] \times \mathrm{Im}(j_*). \tag{3}$$

Proposition 3.1. If X is simply-connected and $H^*(X; \mathbb{Z})$ is torsion-free then the map J is a monomorphism.

Proof. Since $H^*(G/T; \mathbb{Z}), H^*(X; \mathbb{Z})$, and $H^*(BT; \mathbb{Z})$ are torsion free, the vertical maps (rationalizations) in the following commutative diagram are injective

$$\begin{array}{ccc} \mathrm{Hom}_{\mathrm{alg}}(H^*(G/T; \mathbb{Z}), H^*(X; \mathbb{Z})) & \xrightarrow{J} & \mathrm{Hom}_{\mathrm{alg}}(H^*(BT; \mathbb{Z}), H^*(X; \mathbb{Z})) \\ \downarrow & & \downarrow \\ \mathrm{Hom}_{\mathrm{alg}}(H^*(G/T; \mathbb{Q}), H^*(X; \mathbb{Q})) & \xrightarrow{J_{(0)}} & \mathrm{Hom}_{\mathrm{alg}}(H^*(BT; \mathbb{Q}), H^*(X; \mathbb{Q})), \end{array}$$

where $J_{(0)}$ is the rationalization of J. Since $H^*(G/T; \mathbb{Q})$ is generated by the degree two elements and $H^2(G/T; \mathbb{Q}) \cong H^2(BT; \mathbb{Q})$, we see that $J_{(0)}$ is injective. The commutativity of the diagram then implies that J is also injective. □

Remark 3.2. Zhao [27, Lemma 1] claims that J is injective without the torsion-free hypothesis. However, this seems unlikely. Observe that $H^*(G/T; \mathbb{Z})$ is torsion-free and $H^*(G/T; \mathbb{Q})$ is generated by degree two elements. But $H^*(G/T; \mathbb{Z})$ is NOT generated by degree two elements in general; for example in the case of $H^*(G_2/T; \mathbb{Z})$ described in §5.3, even when an induced map f^* for $f \in [X, G/T]$ is trivial on $H^2(G_2/T; \mathbb{Z}) \simeq \mathbb{Z}[x_1, x_2, x_3]/(e_1)$, $f^*(\gamma)$ can be a non-trivial torsion element in $H^*(X; \mathbb{Z})$.

Corollary 3.3. If X is simply-connected and $H^*(X)$ is torsion-free then there is an isomorphism

$$[X, G/T] \cong [X, G] \times \mathrm{Im}(r).$$

Proof. By Proposition 3.1, the map J in (2) is a monomorphism. The square in (2) therefore implies that $\mathrm{Im}(j_*) \cong \mathrm{Im}(r)$. Now substitute this isomorphism into (3). □

By Theorem 2.2, $H^*(G/T)$ is torsion-free. So Proposition 3.1 immediately implies the following.

Corollary 3.4. Let G be a compact simply-connected Lie group. Then there is an isomorphism

$$[G/T, G/T] \cong [G/T, G] \times \mathrm{Im}(r). \qquad \square$$

Note that $G/T \simeq \hat{G}/\hat{T}$, where $\hat{G}$ is the universal cover of G and $\hat{T}$ is the maximal torus of $\hat{G}$. Hence, we can always take G to be simply-connected.

4. Idempotents for $H^*(G/T)$

Now we start to focus on $[G/T, G]$, which by Corollary 3.4 is a factor of $[G/T, G/T]$. In this section we construct two families of compatible idempotents for $H^*(G/T)$ and use them to produce wedge decompositions of $\Sigma G/T$. This begins with a general lemma (c.f. [20, §2]).

Definition 4.1. Let R be a ring. A finite collection $p_1, p_2, \ldots, p_n$ of self-maps of a connected space X is called a set of *mutually orthogonal idempotents* of $H^*(X; R)$ if:

(i) $p_i^* \circ p_i^* = p_i^*$ for $1 \le i \le n$;
(ii) $p_i^* \circ p_j^* = 0$ for all $1 \le i, j \le n$ with $i \neq j$; and
(iii) $p_1^* + \cdots + p_n^* = 1$.

Given a self-map $f\colon X \longrightarrow X$, let $\mathrm{Tel}(f)$ be the telescope of f and let $t\colon X \longrightarrow \mathrm{Tel}(f)$ be the map to the telescope. Since $t \circ f \simeq t$, the map $H^*(\mathrm{Tel}(f); R) \xrightarrow{t^*} H^*(X; R)$ induces the inclusion of $\mathrm{Im}\,(f^*)$.

Lemma 4.2. Let X be a simply-connected finite co-H-space. Let $p_1, \dots, p_n$ be a set of mutually orthogonal idempotents on $H^*(X;\mathbb{Z}/p\mathbb{Z})$. Then there is a p-local homotopy equivalence

$$X \simeq \bigvee_{i=1}^{n} \mathrm{Tel}(p_i).$$

Proof. Since $X \xrightarrow{p_i} \mathrm{Tel}(p_i)$ induces the inclusion of $\mathrm{Im}\,(p_i^*)$, the sum of the maps p_i defines a map $\psi\colon X \longrightarrow \bigvee_{i=1}^{n} \mathrm{Tel}(p_i)$ which induces an isomorphism in mod-p cohomology. Since X is simply-connected and of finite type, this implies that ψ is a p-local homotopy equivalence by [10, Chapter II, Theorem 1.14]. □

We identify two families of self maps of $\Sigma G/T$ that can be used to produce idempotents on $H^*(\Sigma G/T;\mathbb{Z}/p\mathbb{Z})$. Note that the co-$H$ structure on $\Sigma G/T$ induces a group structure on $[\Sigma G/T, \Sigma G/T]$.

4.1. **Unstable Adams operations.** We follow the argument in [25]. For $l \in \mathbb{Z}$ prime to $|W|$, there is a commutative diagram

$$\begin{array}{ccccccc} G & \longrightarrow & G/T & \longrightarrow & BT & \longrightarrow & BG \\ \downarrow{\scriptstyle \Omega\psi^l} & & \downarrow{\scriptstyle \psi^l} & & \downarrow{\scriptstyle \psi^l} & & \downarrow{\scriptstyle \psi^l} \\ G & \longrightarrow & G/T & \longrightarrow & BT & \longrightarrow & BG \end{array}$$

where $(\psi^l)^* : H^{2i}(X) \to H^{2i}(X)$ is multiplication by l^i.

For an odd prime p, choose $l \in \mathbb{Z}$ which is primitive in $\mathbb{F}_p^\times$ and define a self-map of $\Sigma G/T$ by

$$\phi_i = \Sigma\psi^l - l^i \text{ and } \varphi'_k = \prod_{1\le i\le n,\ i\not\equiv k \bmod p-1} \phi_i,$$

where $n = \max(p-1, \dim(G/T)/2)$ and $l^i : \Sigma G/T \to \Sigma G/T$ is l^i times the identity map. Note that φ'_i is trivial on $H^{2j+1}(\Sigma G/T;\mathbb{Z}/p\mathbb{Z})$ iff $j = i \bmod p-1$. So by normalizing up to unit, we obtain a set of mutually orthogonal idempotents $\varphi_1, \dots, \varphi_n$ on $H^*(\Sigma G/T;\mathbb{Z}/p\mathbb{Z})$, where $\varphi_i = u_i\varphi'_i$ for some unit $u_i \in \mathbb{Z}/p\mathbb{Z}$. Therefore, by Lemma 4.2 there is a p-local homotopy equivalence

$$\Sigma G/T \simeq \bigvee \mathrm{Tel}(\varphi_i),$$

where $\tilde{H}^{2i+1}(\mathrm{Tel}(\varphi_k);\mathbb{Z}/p\mathbb{Z}) \cong \begin{cases} \tilde{H}^{2i}(G/T;\mathbb{Z}/p\mathbb{Z}) & \text{if } i = k \mod p-1 \\ 0 & \text{if } i \ne k \mod p-1. \end{cases}$

4.2. **The Weyl group action.** The flag manifold G/T is equipped with a right Weyl group action:

$$gT \mapsto gwT$$

for $w \in W = N(T)/T$. Thus, given any $w \in W$ we obtain a self-map $w\colon G/T \longrightarrow G/T$. In particular, each simple reflection s_i induces a self-map $s_i\colon G/T \longrightarrow G/T$.

By [2], the W-action on Schubert classes is given by

$$s_i\sigma_w = \begin{cases} \sigma_w & \text{if } l(ws_i) = l(w)+1 \\ -\sigma_w - \sum_{l(ws_is_\beta)=l(w)} \dfrac{2(\beta,\alpha_i)}{(\beta,\beta)}\sigma_{ws_\beta} & \text{if } l(ws_i) = l(w)-1. \end{cases}$$

In the coinvariant description (Theorem 2.3), the W-action is simply induced by the ordinary one on $H^*(BT;R)$.

Using the co-H-structure on $\Sigma G/T$ to add maps, to each element v in the group ring $\mathbb{Z}[W]$ there associated a self-map $v\colon \Sigma G/T \longrightarrow \Sigma G/T$. Thus if we find a set of mutually orthogonal idempotents in the group ring we can find an induced set of mutually orthogonal idempotents in $H^*(\Sigma G/T;\mathbb{Z})$. The same argument works if we replace $\mathbb{Z}$-coefficients with $\mathbb{Z}/p\mathbb{Z}$ or $\mathbb{Q}$-coefficients and consider the corresponding localization of the space.

It is well-known that $H^*(G/T;\mathbb{Q})$ is the regular representation of W and decomposes into irreducible representations. However, constructing the corresponding set of mutually orthogonal idempotents even in $\mathbb{Q}[W]$ is non-trivial [1]. For our purpose, we aim to construct mutually orthogonal idempotents in $\mathbb{Z}/p\mathbb{Z}[W]$, and in the examples in Section 5, the identification of the mutually orthogonal idempotents is ad hoc.

Nevertheless, given a set of mutually orthogonal idempotents $\{c_1,\dots,c_n\}$ on $H^*(\Sigma G/T;\mathbb{Z}/p\mathbb{Z})$, by Lemma 4.2 we obtain a p-local homotopy equivalence

$$\Sigma G/T \simeq \bigvee_{i=1}^{n} \mathrm{Tel}(c_i).$$

4.3. **Putting the two decompositions together.** In general, if $p_1,\dots,p_n$ and $q_1,\dots,q_m$ are two sets of mutually orthogonal idempotents on $H^*(X;\mathbb{Z}/p\mathbb{Z})$ that commute, where X is a simply-connected finite co-H-space, then the collection $\{p_i\circ q_j \mid 1\le i\le n, 1\le j\le m\}$ is another set of mutually orthogonal idempotents on $H^*(X;\mathbb{Z}/p\mathbb{Z})$. In our case, the idempotents φ_i^* from the unstable Adams operations and the idempotents c_j^* from the action of the Weyl group commute since φ_i^* is just a projection on to the subspaces consisting of elements of specific degrees while c_j^* preserves the degrees. Thus the maps $\{\varphi_i\circ c_j \mid 1\le i\le n, 1\le j\le m\}$ form a set of mutually orthogonal idempotents on $H^*(\Sigma G/T;\mathbb{Z}/p\mathbb{Z})$, and produce a finer decomposition of $\Sigma G/T$. We think of the decomposition based on the unstable Adams operation as splitting $H^*(\Sigma G/T;\mathbb{Z}/p\mathbb{Z})$ "horizontally" while the one based on Weyl group action splits "vertically."

Note that as any space rationally splits into a wedge of spheres after suspension, we are primarily interested in p-local decompositions of $\Sigma G/T$ for a small prime p.

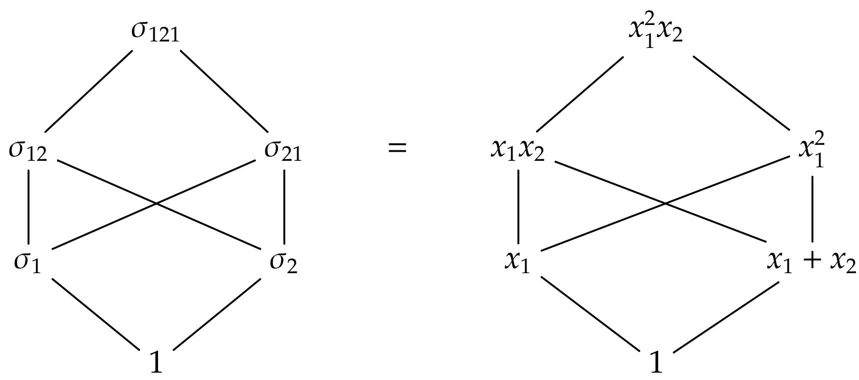
5. Examples

In identifying homotopy types of telescopes, we freely use the fact that the element $\eta\in\pi_{n+1}(S^n)$ is detected by the Steenrod operation Sq^2 and at odd primes the element $\alpha_1\in\pi_{n+2p-3}(S^n)$ is detected by the Steenrod operation $\mathcal{P}^1$. This is equivalent to saying that if $\tilde{H}^*(\mathrm{Tel}(c_i);\mathbb{Z}/2\mathbb{Z})\cong\mathbb{Z}/2\mathbb{Z}\{x,Sq^2(x)\}$ for $|x|=d\ge 3$ then there is a 2-local homotopy equivalence $\mathrm{Tel}(c_i)\simeq\Sigma^{d-2}\mathbb{C}P^2$, and if p is odd and $\tilde{H}^*(\mathrm{Tel}(c_i);\mathbb{Z}/p\mathbb{Z})\cong\mathbb{Z}/p\mathbb{Z}\{x,\mathcal{P}^1(x)\}$ for $|x|=d\ge 3$ then there is a p-local homotopy equivalence $\mathrm{Tel}(c_i)\simeq A(d,d+2p-2)$ where $A(d,d+2p-2)$ is the homotopy cofiber of $S^{d+2p-3}\xrightarrow{\alpha_1}S^d$.

In what follows, for a fixed prime p, we generically use the notation $c_1,\dots,c_n$ for a set of mutually orthogonal idempotents in the group ring $\mathbb{Z}/p\mathbb{Z}[W]$ and let $V_1,\dots,V_n$ be their images in $H^*(G/T;\mathbb{Z}/p\mathbb{Z})$. By abuse of notation, we use the same symbol c_i to denote the corresponding idempotents on $H^*(\Sigma G/T;\mathbb{Z}/p\mathbb{Z})$. The action of the Steenrod operations Sq^2 or $\mathcal{P}^1$ are determined by, for example, [7]. Computation on cohomology is carried out with the aid of a computer code described in [11]. Idempotents in the group ring are obtained by solving quadratic equations in prime fields.

5.1. **Type A_n case.** For the type A_n-case, non-modular irreducible representations V_λ are obtained by considering the Young symmetrizers for all the standard tableaux of shape λ. However, they are not always mutually orthogonal [21]. We will look at some low rank cases in an ad hoc way.

Example 5.1. Since $SU(2)/T=S^2$, the simplest non-trivial case is when $G=SU(3)$. By Theorems 2.2 and 2.3, $H^*(SU(3)/T^2;\mathbb{Z})\simeq\mathbb{Z}[x_1,x_2,x_3]/(e_1,e_2,e_3)\simeq\mathbb{Z}\langle 1,\sigma_1,\sigma_2,\sigma_{12},\sigma_{21},\sigma_{121}\rangle$. It is well-known that the type-A_r Weyl group is the symmetric group S_{r+1}. In particular, $W=\langle s_1,s_2\rangle$, where the simple reflection s_i swaps x_i and x_{i+1}. The Schubert cell decomposition looks like



where $\sigma_1^3 = \sigma_2^3 = 0$. The horizontal level indicates the degree of the cells, and the lines indicate possible non-trivial attaching maps.

We first find mutually orthogonal idempotents in the group ring (see §4.2). For $p = 2$, there is a set of mutually orthogonal idempotents in $\mathbb{Z}/2\mathbb{Z}[W]$

$$\begin{aligned} c_1 &= 1 + s_{12} + s_{21} \\ c_2 &= 1 + s_2 + s_{21} + s_{121} \\ c_3 &= 1 + s_2 + s_{12} + s_{121} \end{aligned}$$

satisfying

$$\begin{aligned} V_1 &= \langle 1, \sigma_{121} \rangle \\ V_2 &= \langle \sigma_1 + \sigma_2, \sigma_{12} + \sigma_{21} \rangle \\ V_3 &= \langle \sigma_1, \sigma_{21} \rangle \end{aligned}$$

where $\mathrm{Sq}^2(\sigma_1) = \sigma_{21}, \mathrm{Sq}^2(\sigma_2) = \sigma_{12}$. Therefore there is a 2-local homotopy equivalence

$$\Sigma SU(3)/T^2 \simeq_2 S^7 \vee \Sigma\mathbb{C}P^2 \vee \Sigma\mathbb{C}P^2.$$

For $p = 3$, there is a set of mutually orthogonal idempotents in $\mathbb{Z}/3\mathbb{Z}[W]$

$$\begin{aligned} c_1 &= 2 + s_{121} \\ c_2 &= 1 - c_1 \end{aligned}$$

satisfying

$$\begin{aligned} V_1 &= \langle 1, \sigma_1 + 2\sigma_2, \sigma_{12} + \sigma_{21} \rangle \\ V_2 &= \langle \sigma_1 + \sigma_2, \sigma_{12} + 2\sigma_{21}, \sigma_{121} \rangle \end{aligned}$$

where there is no non-trivial $\mathcal{P}^1$. Therefore each telescope is 3-locally homotopy equivalent to a wedge of spheres and we obtain a 3-local homotopy equivalence

$$\Sigma SU(3)/T^2 \simeq_3 S^3 \vee S^3 \vee S^5 \vee S^5 \vee S^7.$$

If $p > 3$, for degree reasons the unstable Adams operations (§4.1) imply that there is a p-local homotopy equivalence

$$\Sigma SU(3)/T^2 \simeq_p S^3 \vee S^3 \vee S^5 \vee S^5 \vee S^7.$$

The next example is $SU(4)/T^3$. It will be helpful to have some splitting information that comes from geometry as well as Adams operations and the action of the Weyl group.

Lemma 5.2. The stable normal bundle of any flag manifold G/T is trivial. In particular, the top cell of G/T stably splits off.

Proof. Let $\mathfrak{g}$ and $\mathfrak{t}$ be the Lie algebras of G and T respectively. Take a regular element $X \in \mathfrak{g}$ and consider the adjoint embedding $G/T \to \mathfrak{g}$ induced by $g \mapsto Ad_g(X)$. The normal bundle ν is $G \times_T \mathfrak{t}$, which is trivial since the adjoint action of T on $\mathfrak{t}$ is trivial. By the Pontrjagin-Thom construction combined with the pinching map, we obtain the splitting

$$S^{\dim(G)} \to (G/T)^{\nu} \simeq \Sigma^{\dim(T)}(G/T)_+ \to S^{\dim(G)},$$

where $(G/T)^{\nu}$ is the Thom complex of the normal bundle ν. □

Example 5.3. The rank of $H^*(SU(4)/T^3;\mathbb{Z}) \simeq \mathbb{Z}[x_1,x_2,x_3,x_4]/(e_1,e_2,e_3,e_4)$ is $\{1,3,5,6,5,3,1\}$. The Weyl group W is the symmetric group $\langle s_1,s_2,s_3\rangle$ with $|W| = 24$.

For $p = 2$, there do not appear to be so many idempotents. For example, $c_1 = s_{23} + s_{32}$ and $c_2 = 1 - c_1$ form a mutually orthogonal set of idempotents in $\mathbb{Z}/2\mathbb{Z}[W]$ but they do not help much in terms of producing splittings with identifiable wedge summands.

For $p = 3$, there is a set of mutually orthogonal idempotents in $\mathbb{Z}/3\mathbb{Z}[W]$

$$\begin{aligned}
c_1 &= (1 + s_1 + s_2 + s_{12} + s_{21} + s_{121})(1 - s_{12321})\\
c_2 &= (1 + s_{12321})(1 - s_1 - s_2 + s_{12} + s_{21} - s_{121})\\
c_3 &= 2 + 2s_1 + 2s_2 + 2s_3 + 2s_{21} + 2s_{12} + 2s_{32} + 2s_{23} + 2s_{121} + s_{321} + 2s_{232} + s_{123} + s_{1321} + s_{2321} + s_{1213}\\
&\qquad + s_{1232} + s_{21321} + s_{12321} + s_{12132} + s_{121321}\\
c_4 &= 2 + s_2 + 2s_{13} + s_{213} + s_{132} + 2s_{2132} + s_{12321} + 2s_{121321}\\
c_5 &= 2 + 2s_2 + 2s_{13} + 2s_{213} + 2s_{132} + 2s_{2132} + 2_{12321} + 2s_{121321}\\
c_6 &= 2 + s_1 + s_2 + 2s_3 + 2s_{21} + s_{13} + 2s_{12} + s_{32} + s_{23} + s_{121} + s_{321} + 2s_{213} + 2s_{132} + 2s_{232} + 2s_{123} + 2s_{1321}\\
&\qquad + 2s_{2321} + s_{1213} + s_{2132} + s_{1232} + s_{21321} + s_{12321} + 2s_{12132} + 2s_{121321}\\
c_7 &= 2 + s_{23} + s_{32} + s_3 + s_{13} + s_{213} + s_{132} + s_{232} + s_{2132} + 2s_{1213} + 2s_{12132} + 2s_{123} + 2s_{1232} + s_{1321} + s_{321}\\
&\qquad + s_{21321} + 2s_{121321} + s_{2321} + 2s_{12321} + 2s_{121} + 2s_{21} + 2s_{12} + 2s_2 + 2s_1\\
c_8 &= 1 - \sum_{i=1}^{7} c_i
\end{aligned}$$

satisfying

$$\begin{aligned}
V_1 &= \langle\sigma_3, \sigma_{23}, \sigma_{123}\rangle\\
V_2 &= \langle\sigma_{123} + \sigma_{121} + \sigma_{321} + 2\sigma_{213} + 2\sigma_{132} + \sigma_{232}, \sigma_{1213} + \sigma_{1232} + 2\sigma_{1321} + 2\sigma_{2321}, \sigma_{21321} + \sigma_{12321} + \sigma_{12132}\rangle\\
V_3 &= \langle\sigma_{321} + 2\sigma_{213} + \sigma_{123}, \sigma_{2321} + 2\sigma_{1213}, \sigma_{12321}\rangle\\
V_4 &= \langle 2\sigma_{12} + \sigma_{13} + \sigma_{21} + 2\sigma_{32} + \sigma_{23}, 2\sigma_{2321} + \sigma_{2132} + 2\sigma_{1213}, \sigma_{121321}\rangle\\
V_5 &= \langle 1, \sigma_{12} + \sigma_{13} + \sigma_{23}, \sigma_{2321} + \sigma_{1213}\rangle\\
V_6 &= \langle\sigma_{321} + 2\sigma_{213} + 2\sigma_{132} + 2\sigma_{232} + \sigma_{123}, \sigma_{2321} + \sigma_{1232} + 2\sigma_{2132} + 2\sigma_{1213}, \sigma_{12321} + 2\sigma_{12132}\rangle\\
V_7 &= \langle\sigma_1 + 2\sigma_3, \sigma_{21} + \sigma_{23}, \sigma_{321} + 2\sigma_{123}\rangle\\
V_8 &= \langle\sigma_1 + \sigma_2 + \sigma_3, \sigma_{21} + 2\sigma_{12} + \sigma_{32} + 2\sigma_{23}, \sigma_{321} + 2\sigma_{132} + \sigma_{213}\rangle.
\end{aligned}$$

The non-trivial actions of $\mathcal{P}^1$ are

$$\begin{aligned}
V_1, V_7, V_8 &: H^2 \to H^6\\
V_4, V_5 &: H^4 \to H^8\\
V_2, V_3, V_6 &: H^6 \to H^{10}.
\end{aligned}$$

For degree reasons in cohomology, the unstable Adams operations split $\mathrm{Tel}(c_i)$ for $i \in \{1,7,8\}$ into wedge summands: one inheriting the degree 3 and 7 generators in cohomology and the other inheriting the degree 5 generator. Similarly, the unstable Adams operations split $\mathrm{Tel}(c_i)$ for $i \in \{2,3,6\}$ into wedge summands: one inheriting the degree 7 and 11 generators in cohomology and the other inheriting the degree 9 generator. All together we obtain 3-local homotopy equivalences

$$\mathrm{Tel}(c_1) \simeq \mathrm{Tel}(c_7) \simeq \mathrm{Tel}(c_8) \simeq A(3,7) \vee S^5$$
$$\mathrm{Tel}(c_2) \simeq \mathrm{Tel}(c_3) \simeq \mathrm{Tel}(c_6) \simeq A(7,11) \vee S^9.$$

Note that $\mathrm{Tel}(c_5) \simeq A(5,9)$ and $\mathrm{Tel}(c_4)$ is a three-cell complex whose 9-skeleton is $A(5,9)$. We claim that $\mathrm{Tel}(c_4) \simeq A(5,9) \vee S^{13}$. To see this we show that the attaching map $g\colon S^{12} \longrightarrow A(5,9)$ for the top cell is null homotopic. Since there is no Steenrod operation connecting the 9 and 13-cells in $\mathrm{Tel}(c_4)$, the composition $S^{12} \xrightarrow{g} A(5,9) \xrightarrow{q} S^9$ must be null homotopic, where q is the pinch map to the top cell. Therefore g lifts to the homotopy fibre of q, which by a Serre spectral sequence argument, is homotopy equivalent to S^5 in dimensions ≤ 12. Thus g is homotopic to a composite $S^{12} \xrightarrow{t\cdot\alpha_2} S^5 \hookrightarrow A(5.9)$, where α_2 generates $\pi_{12}(S^5) \cong \mathbb{Z}/3\mathbb{Z}$ and $t \in \mathbb{Z}/3\mathbb{Z}$. Each map in this composite is stable, so g is stable. But by Lemma 5.2, the top cell of $\mathrm{Tel}(c_4)$ splits off stably. Thus g is stably trivial, and so g itself must be trivial. Hence, we have $\mathrm{Tel}(c_4) \simeq A(5,9) \vee S^{13}$. To sum up, there is a 3-local homotopy equivalence

$$\Sigma SU(4)/T^3 \simeq_3 3A(3,7) \vee 3S^5 \vee 3A(7,11) \vee 3S^9 \vee 2A(5,9) \vee S^{13}.$$

If $p = 5$ then for degree reasons the unstable Adams operations decompose $\Sigma SU(4)/T^3$ as a wedge $X_1 \vee \cdots \vee X_4$ where $H^*(X;\mathbb{Z}/5\mathbb{Z})$ consists of the degree 3 and 11 elements in $H^*(SU(4)/T^3;\mathbb{Z}/5\mathbb{Z})$, $H^*(X_2;\mathbb{Z}/5\mathbb{Z})$ consists of the degree 5 and 13 elements, $H^*(X_3;\mathbb{Z}/5\mathbb{Z})$ consists of the degree 7 elements, and $H^*(X_4;\mathbb{Z}/5\mathbb{Z})$ consists of the degree 9 elements. On the other hand, since $\sigma_w^5 = 0$ for all $l(w) = 1$, all $\mathcal{P}^1$ are trivial. Thus there is a 5-local homotopy equivalence

$$\Sigma SU(4)/T^3 \simeq_5 3S^3 \vee 5S^5 \vee 6S^7 \vee 5S^9 \vee 3S^{11} \vee S^{13}.$$

If $p > 5$ then for degree reasons the unstable Adams operations decompose $SU(4)/T^3$ as a wedge of spheres (the same wedge as in the $p = 5$ case.)

5.2. **Type $B_2 = C_2$.** The Weyl group W is the hyper-octahedral group $\langle s_1, s_2\rangle$ with $|W| = 8$. We have $H^*(Sp(2)/T) = \dfrac{\mathbb{Z}[x_1,x_2]}{(x_1^2+x_2^2, x_1^2x_2^2)}$ and its Betti numbers are $\{1,2,2,2,2,1\}$.

For $p = 2$, there is no non-trivial idempotents.

For $p > 2$ (the non-modular case), there is a set of mutually orthogonal idempotents in $\mathbb{Z}/p\mathbb{Z}[W]$

$$c_1 = \frac{1}{8}(1 + s_1 - s_2 - s_{12} - s_{21} - s_{121} + s_{212} + s_{1212})$$
$$c_2 = \frac{1}{8}(1 - s_1 + s_2 - s_{12} - s_{21} + s_{121} - s_{212} + s_{1212})$$
$$c_3 = \frac{1}{8}(1 - s_1 - s_2 + s_{12} + s_{21} - s_{121} - s_{212} + s_{1212})$$
$$c_4 = \frac{1}{8}(1 - s_1 + s_{212} - s_{1212})$$
$$c_5 = \frac{1}{8}(1 + s_1 - s_{212} - s_{1212})$$
$$c_6 = \frac{1}{8}\sum_{w\in W} w$$

satisfying

$$\begin{aligned} V_1 &= \langle \sigma_{12} \rangle \\ V_2 &= \langle \sigma_{21} \rangle \\ V_3 &= \langle \sigma_{1212} \rangle \\ V_4 &= \langle \sigma_1 - \sigma_2, \sigma_{121} + \sigma_{212} \rangle \\ V_5 &= \langle \sigma_2, \sigma_{212} \rangle \\ V_6 &= \langle 1 \rangle \end{aligned}$$

where $\sigma_1^3 = \sigma_{121}$ and $\sigma_2^3 = 2\sigma_{212}$. In particular, we obtain $\mathrm{Tel}(c_1) \simeq \mathrm{Tel}(c_2) \simeq S^5$, $\mathrm{Tel}(c_3) \simeq S^9$ and $\mathrm{Tel}(c_6) \simeq *$.

If $p = 3$ then the equations $\sigma_1^3 = \sigma_{121}$ and $\sigma_2^3 = 2\sigma_{212}$ imply that $\mathcal{P}^1$ is non-trivial on σ_1 and σ_2. Therefore $\mathrm{Tel}(c_4) \simeq \mathrm{Tel}(c_5) \simeq A(3,7)$. Hence there is a 3-local homotopy equivalence

$$\Sigma Sp(2)/T \simeq_3 2A(3,7) \vee 2S^5 \vee S^9.$$

If $p \geq 5$ then all Steenrod operations $\mathcal{P}^1$ are trivial so $\mathrm{Tel}(c_4) \simeq \mathrm{Tel}(c_5) \simeq S^3 \vee S^7$. Hence there is a p-local homotopy equivalence

$$\Sigma Sp(2)/T \simeq_p 2S^3 \vee 2S^5 \vee 2S^7 \vee S^9.$$

5.3. **Type G_2.** The Weyl group W is the dihedral group $D_6 = \langle s_1, s_2 \rangle$ with $|W| = 12$. The cohomology of G_2/T is computed by [5, 23] as

$$H^*(G_2/T) = \frac{\mathbb{Z}[x_1, x_2, x_3, \gamma]}{(e_1, e_2, e_3 - 2\gamma, \gamma^2)}.$$

For $p = 2$, there is a set of mutually orthogonal idempotents in $\mathbb{Z}/2\mathbb{Z}[W]$

$$\begin{aligned} c_1 &= 1 + s_{1212} + s_{2121} \\ c_2 &= 1 + s_{212} + s_{1212} + s_{12121} \\ c_3 &= 1 + s_{212} + s_{2121} + s_{12121} \end{aligned}$$

satisfying

$$\begin{aligned} V_1 &= \langle 1, \sigma_{121}, \sigma_{212}, \sigma_{121212} \rangle, \\ V_2 &= \langle \sigma_1 + \sigma_2, \sigma_{12} + \sigma_{21}, \sigma_{1212} + \sigma_{2121}, \sigma_{12121} + \sigma_{21212} \rangle, \\ V_3 &= \langle \sigma_1, \sigma_{21}, \sigma_{2121}, \sigma_{12121} \rangle, \end{aligned}$$

where $\mathrm{Sq}^2(\sigma_1) = \sigma_{21}, \mathrm{Sq}^2(\sigma_2) = \sigma_{12}, \mathrm{Sq}^2(\sigma_{1212}) = \sigma_{21212}, \mathrm{Sq}^2(\sigma_{2121}) = \sigma_{12121}$. The multiple generators of different degrees in the modules V_i imply that the telescopes of the maps c_i are not readily identifiable. So we say nothing more than there is a 2-local homotopy equivalence

$$\Sigma G_2/T \simeq_2 \mathrm{Tel}(c_1) \vee \mathrm{Tel}(c_2) \vee \mathrm{Tel}(c_3).$$

For $p = 3$, there is a set of mutually orthogonal idempotents in $\mathbb{Z}/3\mathbb{Z}[W]$

$$\begin{aligned} c_1 &= 1 + s_1 + s_{21212} + s_{121212} \\ c_2 &= 1 + s_1 - s_{21212} - s_{121212} \\ c_3 &= 1 - s_1 - s_{21212} + s_{121212} \\ c_4 &= 1 - s_1 + s_{21212} - s_{121212} \end{aligned}$$

satisfying

$$\begin{aligned} V_1 &= \langle 1, \sigma_{12}, \sigma_{1212} \rangle \\ V_2 &= \langle \sigma_2, \sigma_{212}, \sigma_{21212} \rangle \\ V_3 &= \langle \sigma_{21} + \sigma_{12}, \sigma_{2121} - \sigma_{1212}, \sigma_{121212} \rangle \\ V_4 &= \langle \sigma_1 + \sigma_2, \sigma_{121}, \sigma_{12121} - \sigma_{21212} \rangle \end{aligned}$$

The non-trivial actions of $\mathcal{P}^1$ are

$$\begin{aligned} V_1, V_3 :& H^4 \to H^8 \\ V_2 :& H^6 \to H^{10} \\ V_4 :& H^2 \to H^6. \end{aligned}$$

In particular, $\mathrm{Tel}(c_1) \simeq A(5,9)$, but the other telescopes are not as readily identifiable. So we say nothing more right now other than there is a 3-local homotopy equivalence

$$\Sigma G_2/T \simeq_3 A(5,9) \vee \mathrm{Tel}(c_2) \vee \mathrm{Tel}(c_3) \vee \mathrm{Tel}(c_4).$$

For $p > 3$ (the non-modular case), the regular representation decomposes into four 1-dimensional and four 2-dimensional irreducible representations. The maps inducing the cohomology idempotents corresponding to the 1-dimensional irreducible representations are given by

$$c_1 = \frac{1}{12}\sum_{w\in W} w \qquad c_2 = \frac{1}{12}\sum_{w\in W} (-1)^{l(w)} w$$
$$c_3 = \frac{1}{12}\sum_{w\in W} (-1)^{\#_w 1} w \qquad c_4 = \frac{1}{12}\sum_{w\in W} (-1)^{\#_w 2} w$$

where $\#_w i$ is the number of s_i's in w, and these satisfy

$$V_1 = \langle 1 \rangle \qquad V_2 = \langle \sigma_{121212} \rangle$$
$$V_3 = \langle \sigma_{121} \rangle \qquad V_4 = \langle \sigma_{212} \rangle.$$

In particular, we obtain $\mathrm{Tel}(c_1) \simeq *$, $\mathrm{Tel}(c_2) \simeq S^{13}$ and $\mathrm{Tel}(c_3) \simeq \mathrm{Tel}(c_4) \simeq S^7$.

The maps inducing the cohomology idempotents corresponding to the 2-dimensional irreducible representations are given by

$$\begin{aligned} c_5 &= \frac{1}{12}(2 + s_1 - 2s_2 - s_{12} - s_{21} + s_{121} + s_{212} - s_{1212} - s_{2121} - 2s_{12121} + s_{21212} + 2s_{121212}) \\ c_6 &= \frac{1}{12}(2 - s_1 + 2s_2 - s_{12} - s_{21} - s_{121} - s_{212} - s_{1212} - s_{2121} + 2s_{12121} - s_{21212} + 2s_{121212}) \\ c_7 &= \frac{1}{6}(1 - s_1 - s_2 + s_{21} - s_{1212} + s_{12121} + s_{21212} - s_{121212}) \\ c_8 &= \frac{1}{6}(1 + s_1 + s_2 + s_{12} - s_{2121} - s_{12121} - s_{21212} - s_{121212}) \end{aligned}$$

and these satisfy

$$\begin{aligned} V_5 &= \langle 2\sigma_{12} - \sigma_{21}, 2\sigma_{1212} + \sigma_{2121} \rangle \\ V_6 &= \langle \sigma_{21}, \sigma_{2121} \rangle \\ V_7 &= \langle 3\sigma_1 - 2\sigma_2, 3\sigma_{12121} + 2\sigma_{21212} \rangle \\ V_8 &= \langle \sigma_2, \sigma_{21212} \rangle. \end{aligned}$$

If $p = 5$ then the Steenrod operation $\mathcal{P}^1$ is trivial on V_5 and V_6 for degree reasons. So $\mathrm{Tel}(c_5) \simeq \mathrm{Tel}(c_6) \simeq S^5 \vee S^9$. On the other hand, $\mathcal{P}^1(\sigma_2) = 3\sigma_{21212}$ and $\mathcal{P}^1(3\sigma_1 - 2\sigma_2) = 2(3\sigma_{12121} + 2\sigma_{21212})$ so $\mathrm{Tel}(c_7) \simeq \mathrm{Tel}(c_8) \simeq A(3,11)$. Therefore there is a 5-local homotopy equivalence

$$\Sigma G_2/T \simeq_5 2S^5 \vee 2S^7 \vee 2S^9 \vee S^{13} \vee 2A(3,11).$$

If $p > 5$ then the Steenrod operation $\mathcal{P}^1$ is trivial on each of V_5, V_6, V_7 and V_8. Therefore there is a p-local homotopy equivalence

$$\Sigma G_2/T \simeq_p 2S^3 \vee 2S^5 \vee 2S^7 \vee 2S^9 \vee 2S^{11} \vee S^{13}.$$

Remark 5.4. It is interesting to note that factors are "Poincaré dual" to each other. For example, (V_6, V_7) in $SU(4)/T$ with $p = 3$ are dual to each other in the sense that the generators in the complimentary degrees multiply to the top degree element (e.g., $(\sigma_{2321} + \sigma_{1232} + 2\sigma_{2132} + 2\sigma_{1213})(\sigma_{21} + \sigma_{23}) = \sigma_{121321} \mod 3$). For $SU(4)/T$ with $p = 3$, $(V_1, V_2), (V_3, V_8)$, and (V_4, V_5) are all dual pairs as well. For $Sp(2)/T$ with $p > 2$, $(V_1, V_2), (V_3, V_6)$, and (V_4, V_5) are dual pairs. For G_2/T with $p > 3$, $(V_1, V_2), (V_3, V_4), (V_5, V_6)$, and (V_7, V_8) are dual pairs. It might be interesting to find a geometric explanation for this.

6. Self-maps of flag manifolds

The decomposition of $\Sigma G/T$ allows for a calculation of the factor $[G/T, G]$ of $[G/T, G/T]$. Since G is a topological group, it has a classifying space BG, and $G \simeq \Omega BG$. Therefore there is an adjunction giving an isomorphism of groups $[G/T, G] \cong [G/T, \Omega BG] \cong [\Sigma G/T, BG]$. Suppose that there is a homotopy decomposition $\Sigma G/T \simeq \bigvee_{i=1}^{k} A_i$. As this is a decomposition of spaces rather than co-H-spaces, there is a set isomorphism $[\Sigma G/T, BG] \cong [\bigvee_{i=1}^{k} A_i, BG] \cong \prod_{i=1}^{k}[A_i, BG]$. Combining these isomorphisms gives the following.

Lemma 6.1. If $\Sigma G/T \simeq \bigvee_{i=1}^{k} A_i$ then there is an isomorphism of sets $[G/T, G] \cong \prod_{i=1}^{k}[A_i, BG]$. □

Another useful general lemma is the following.

Lemma 6.2. If G is simply-connected then, rationally, $[G/T, G] \cong 0$. Consequently, $[G/T, G]$ is the product of its p-components for all primes p.

Proof. In all cases, there is a rational homotopy equivalence $G \simeq \prod_{i=1}^{m} K(\mathbb{Q}, 2n_i - 1)$ for some sequence $\{i_1, \ldots, i_m\}$. Thus $[G/T, G] \cong \prod_{i=1}^{m} H^{2n_i-1}(G/T; \mathbb{Q})$. But $H^{\mathrm{odd}}(G/T; \mathbb{Q}) \cong 0$, so we obtain a rational isomorphism $[G/T, G] \cong 0$. □

We now explicitly calculate $[G/T, G]$ when G is one of $SU(3)$, $SU(4)$, $Sp(2)$ or G_2.

Proposition 6.3. There is a group isomorphism $[SU(3)/T^2, SU(3)] \cong \mathbb{Z}/6\mathbb{Z}$ and the generator corresponds to the self-map

$$SU(3)/T^2 \xrightarrow{q} S^6 \xrightarrow{f} SU(3) \longrightarrow SU(3)/T$$

where q is the pinch to the top cell and f represents the generator of $\pi_6(SU(3)) \cong \mathbb{Z}/6\mathbb{Z}$.

Proof. By Lemma 6.2, to calculate $[SU(3)/T^2, SU(3)]$ it suffices to localize and work prime by prime.

Case 1: $p = 2$. By Example 5.1, there is a 2-local homotopy equivalence

$$\Sigma SU(3)/T^2 \simeq S^7 \vee \Sigma\mathbb{C}P^2 \vee \Sigma\mathbb{C}P^2.$$

Therefore

$$\begin{aligned}[SU(3)/T^2, SU(3)] &\cong [\Sigma SU(3)/T^2, BSU(3)]\\ &\cong [S^7, BSU(3)] \times [\Sigma\mathbb{C}P^2, BSU(3)] \times [\Sigma\mathbb{C}P^2, BSU(3)].\end{aligned}$$

By [16], the 2-component of $\pi_6(SU(3))$ is $\mathbb{Z}/2\mathbb{Z}$. Since the homotopy fibre of $BSU(3) \longrightarrow BSU(\infty)$ is 6-connected and $\Sigma\mathbb{C}P^2$ is 5-dimensional we have

$$[\Sigma\mathbb{C}P^2, BSU(3)] \cong [\Sigma\mathbb{C}P^2, BSU(\infty)] \cong \widetilde{K}(\Sigma\mathbb{C}P^2) \cong 0$$

where $\widetilde{K}$ is reduced complex K-theory. Therefore

$$[SU(3)/T^2, SU(3)] \cong \mathbb{Z}/2\mathbb{Z}$$

and this corresponds to a nontrivial 2-local self-map of $SU(3)/T^2$ given by the composite

$$SU(3)/T^2 \xrightarrow{q} S^6 \xrightarrow{f} SU(3) \longrightarrow SU(3)/T^2$$

where q is the pinch to the top cell and f represents the generator of $\pi_6(SU(3)) \cong \mathbb{Z}/2\mathbb{Z}$.

Case 2: $p > 2$. By Example 5.1, localized at a prime $p > 2$ there is a homotopy equivalence $\Sigma SU(3)/T^2 \simeq 2S^3 \vee 2S^5 \vee S^7$. Therefore, by Lemma 6.1,

$$\begin{aligned}[SU(3)/T^2, SU(3)] &\cong 2[S^3, BSU(3)] \times 2[S^5, BSU(3)] \times [S^7, BSU(3)] \\ &\cong 2\pi_2(SU(3)) \times 2\pi_4(SU(3)) \times \pi_6(SU(3)).\end{aligned}$$

By [17], $\pi_2(SU(3)) \cong 0$, $\pi_4(SU(3)) \cong 0$ and after inverting 2, $\pi_6(SU(3)) \cong \mathbb{Z}/3\mathbb{Z}$. Therefore, localized at $p > 3$ we have $[SU(3)/T^2, SU(3)] \cong 0$ and localized at 3 we have

$$[SU(3)/T^2, SU(3)] \cong \mathbb{Z}/3\mathbb{Z}.$$

In the latter case, the generator corresponds to a nontrivial 3-local self-map of $SU(3)/T^2$ given by the composite

$$SU(3)/T^2 \xrightarrow{q} S^6 \xrightarrow{f} SU(3) \longrightarrow SU(3)/T^2$$

where q is the pinch map to the top cell and f represents a generator of $\pi_6(SU(3)) \cong \mathbb{Z}/3\mathbb{Z}$.

Combining both cases, we obtain a set isomorphism $[SU(3)/T^2, SU(3)] \cong \mathbb{Z}/6\mathbb{Z}$. To upgrade this to an isomorphism of groups, it suffices to show that the group $[SU(3)/T^2, SU(3)]$ has an element of order 6. But observe that the generators of the 2 and 3-components are obtained from the same map

$$SU(3)/T^2 \xrightarrow{q} S^6 \xrightarrow{f} SU(3) \longrightarrow SU(3)/T^2$$

where f represents a generator of $\pi_6(S^3) \cong \mathbb{Z}/6\mathbb{Z}$. Thus $f \circ q$ has order 6 in $[SU(3)/T^2, SU(3)]$ and we are done. □

Proposition 6.4. Localized away from 2 there is a set isomorphism $[SU(4)/T^3, SU(4)] \cong \mathbb{Z}/15\mathbb{Z} \times 3(\mathbb{Z}/5\mathbb{Z})$. The group $[SU(4)/T^3, SU(4)]$ has a subgroup of order 15 corresponding to the self-map

$$SU(4)/T^3 \xrightarrow{q} S^{12} \xrightarrow{f} SU(4) \longrightarrow SU(4)/T^3$$

where q is the pinch map to the top cell and f represents the 3 and 5-components of $\pi_{12}(SU(4)) \cong \mathbb{Z}/60\mathbb{Z}$. The group $[SU(4)/T^3, SU(4)]$ has three subgroups of order 5 corresponding to 5-local self-maps

$$SU(4)/T^3 \xrightarrow{q'} 3S^{10} \vee S^{12} \xrightarrow{p_i} S^{10} \xrightarrow{f_i} SU(4) \longrightarrow SU(4)/T^3$$

where q' collapses the 9-skeleton of $SU(4)/T^3$ to a point, and for $1 \leq i \leq 3$ the map p_i pinches to the i^{th}-copy of S^{10} while f_i represents the generator of $\pi_{10}(SU(4)) \cong \mathbb{Z}/5\mathbb{Z}$.

Proof. By Lemma 6.2, to calculate $[SU(4)/T^3, SU(4)]$ it suffices to localize and work prime by prime.

Case 1: $p = 3$. By Example 5.3, there is a 3-local homotopy equivalence

$$\Sigma SU(4)/T^3 \simeq 3A(3,7) \vee 3S^5 \vee 3A(7,11) \vee 3S^9 \vee 2A(5,9) \vee S^{13}. \tag{4}$$

We calculate $[\Sigma SU(4)/T^3, BSU(4)]$ by using Lemma 6.1.

By [17], $\pi_m(BSU(4)) \cong 0$ for $m \in \{3,5,7,9\}$, so $[A(3,7), BSU(4)] \cong [S^5, BSU(4)] \cong [S^9, BSU(4)] \cong 0$, and $\pi_{13}(BSU(4)) \cong \mathbb{Z}/60\mathbb{Z}$. It remains to consider $[A(7,11), BSU(4)]$ and $[A(5,9), BSU(4)]$.

For $A(7,11)$, the cofibration sequence $S^7 \longrightarrow A(7,11) \longrightarrow S^{11} \xrightarrow{\alpha_1} S^8$ induces an exact sequence

$$[S^8, BSU(4)] \xrightarrow{(\alpha_1)^*} [S^{11}, BSU(4)] \longrightarrow [A(7,11), BSU(4)] \longrightarrow [S^7, BSU(4)].$$

On the one hand, $[S^7, BSU(4)] = \pi_6(SU(4)) \cong 0$. On the other hand, by [17], $[S^8, BSU(4)] \cong \mathbb{Z}$, $[S^{11}, BSU(4)] \cong \mathbb{Z}/3\mathbb{Z}$, and $(\alpha_1)^*$ is an epimorphism. Thus $[A(7,11), BSU(4)] \cong 0$.

For $A(5,9)$, the cofibration sequence $S^5 \longrightarrow A(5,9) \longrightarrow S^9 \xrightarrow{\alpha_1} S^6$ induces an exact sequence

$$[S^6, BSU(4)] \xrightarrow{(\alpha_1)^*} [S^9, BSU(4)] \longrightarrow [A(5,9), BSU(4)] \longrightarrow [S^5, BSU(4)].$$

On the one hand, $[S^5, BSU(4)] = \pi_4(SU(4)) \cong 0$. On the other hand, by [17], $[S^6, BSU(4)] \cong \mathbb{Z}$, $[S^9, BSU(4)] \cong \mathbb{Z}/3\mathbb{Z}$, and $(\alpha_1)^*$ is an epimorphism. Thus $[A(5,9), BSU(4)] \cong 0$.

Therefore, from (4) we obtain a set isomorphism

$$[SU(4)/T^3, SU(4)] \cong \mathbb{Z}/3\mathbb{Z}.$$

This is in fact a group isomorphism. It suffices to find an element of $[SU(4)/T^3, SU(4)]$ whose order when localized at 3 is 3. But this is given by the map $q \circ f$ in the composite

$$SU(4)/T^3 \xrightarrow{q} S^{12} \xrightarrow{f} SU(4) \longrightarrow SU(4)/T^3$$

where q is the pinch map to the top cell and f represents the 3-component of $\pi_{12}(SU(4)) \cong \mathbb{Z}/60\mathbb{Z}$. The map $f \circ q$ has order 3 in $[SU(4)/T^3, SU(4)]$ so

Case 2: $p \geq 5$. By Example 5.3, localized at a prime $p \geq 5$ there is a homotopy equivalence

$$\Sigma SU(4)/T^3 \simeq 3S^3 \vee 5S^5 \vee 6S^7 \vee 5S^9 \vee 3S^{11} \vee S^{13}.$$

Therefore, by Lemma 6.1,

$$[SU(4)/T^3, SU(4)] \cong 3[S^3, BSU(4)] \times 5[S^5, BSU(4)] \times 6[S^7, BSU(4)] \times$$
$$5[S^9, BSU(4)] \times 3[S^{11}, BSU(4)] \times [S^{13}, BSU(4)].$$

By [17], $\pi_2(SU(4)) \cong 0$, $\pi_4(SU(4)) \cong 0$, $\pi_6(SU(4)) \cong 0$, and after inverting 2 and 3, $\pi_8(SU(4)) \cong 0$, $\pi_{10}(SU(4)) \cong \mathbb{Z}/5\mathbb{Z}$ and $\pi_{12}(SU(4)) \cong \mathbb{Z}/5\mathbb{Z}$. Thus

$$[SU(4)/T^3, SU(4)] \cong 4(\mathbb{Z}/5\mathbb{Z})$$

and the generators correspond to two types of nontrivial 5-local self-maps. First,

$$SU(4)/T^3 \xrightarrow{q'} 3S^{10} \vee S^{12} \xrightarrow{p_i} S^{10} \xrightarrow{f_i} SU(4) \longrightarrow SU(4)/T^3$$

where q' is the map that collapses out the 9-skeleton of $SU(4)/T^3$, and for $1 \leq i \leq 3$ the map p_i pinches to the i^{th}-copy of S^{10} while f_i represents the generator of $\pi_{10}(SU(4)) \cong \mathbb{Z}/5\mathbb{Z}$. Second,

$$SU(4)/T^3 \xrightarrow{q} S^{12} \xrightarrow{f} SU(4) \longrightarrow SU(4)/T^3$$

where q is the pinch map to the top cell and f represents the generator of $\pi_{12}(SU(4)) \cong \mathbb{Z}/5\mathbb{Z}$.

Finally, notice that the same map $SU(4)/T^3 \xrightarrow{q} S^{12} \xrightarrow{f} SU(4)$ appears in the $p = 3$ and $p = 5$ cases, so $f \circ q$ has order 15 and generates a subgroup of order 15 in $[SU(4)/T^3, SU(4)]$. □

Proposition 6.5. There is a group isomorphism $[G_2/T, G_2] \cong 0$.

Proof. By Lemma 6.2, to calculate $[G_2/T, G_2]$ it suffices to localize and work prime by prime.

Case 1: $p = 2$. As in Section 5.3, there is a 2-local homotopy equivalence

$$\Sigma G_2/T \simeq \mathrm{Tel}(c_1) \vee \mathrm{Tel}(c_2) \vee \mathrm{Tel}(c_3).$$

By Lemma 6.1, to calculate the 2-component of $[G_2/T, G_2]$ it is equivalent to calculate $[\mathrm{Tel}(c_i), BG_2]$ for $1 \leq i \leq 3$. The space $\mathrm{Tel}(c_1)$ has cells in dimensions 7 and 13. By [16], $\pi_7(BG_2) \cong \pi_{13}(BG_2) \cong 0$, so $[\mathrm{Tel}(c_1), BG_2] \cong 0$. The spaces $\mathrm{Tel}(c_2)$ and $\mathrm{Tel}(c_3)$ both have cells in dimensions $3, 5, 9, 11$, and the Steenrod operation Sq^2 connects the 3 and 5 cells, and the 9 and 11 cells. Therefore, for $2 \leq i \leq 3$ there is a homotopy cofibration $\Sigma\mathbb{C}P^2 \longrightarrow \mathrm{Tel}(c_i) \longrightarrow \Sigma^7\mathbb{C}P^2$. Let $g\colon \mathrm{Tel}(c_i) \longrightarrow BG_2$ be any map. By [16], $\pi_3(BG_2) \cong \pi_5(BG_2) \cong 0$, so the restriction of g to $\Sigma\mathbb{C}P^2$ is null homotopic, implying that g factors as a composite $\mathrm{Tel}(c_i) \longrightarrow \Sigma^7\mathbb{C}P^2 \xrightarrow{h} BG_2$ for some map h. By [16], $\pi_9(BG_2) \cong \mathbb{Z}/2\mathbb{Z}$. We claim that the restriction of h to S^9 is trivial. If not, then it represents the generator γ of $\pi_9(BG_2)$. By [16], this generator has the property that the composite $S^{10} \xrightarrow{\eta} S^9 \xrightarrow{\gamma} BG_2$ is also nontrivial. But this implies that there can be no extension of γ to a map $\Sigma^7\mathbb{C}P^2 \longrightarrow BG_2$. That is, the restriction of h to S^9 cannot extend to h, a contradiction. Therefore the restriction of h to S^9 is trivial, implying that h factors as a composite $\Sigma^7\mathbb{C}P^2 \longrightarrow S^{11} \xrightarrow{k} BG_2$ for some map k. By [16], $\pi_{11}(BG_2) \cong 0$. Hence k, and therefore h, and therefore g are all trivial. Consequently, $[\mathrm{Tel}(c_i), BG_2] \cong 0$ for $2 \leq i \leq 3$. Collectively, we obtain a 2-local isomorphism $[G_2/T, G] \cong 0$.

Case 2: $p = 3$. As in Section 5.3, there is a 3-local homotopy equivalence

$$\Sigma G_2/T \simeq A(5,9) \vee \mathrm{Tel}(c_2) \vee \mathrm{Tel}(c_3) \vee \mathrm{Tel}(c_4).$$

By Lemma 6.1, to calculate the 3-component of $[G_2/T, G_2]$ it is equivalent to calculate $[A(5,9), BG_2]$ and $[\mathrm{Tel}(c_i), BG_2]$ for $2 \leq i \leq 4$. By [16], at 3 we have $\pi_m(BG_2) = 0$ for $m \in \{3,5,9,11,13\}$ while $\pi_7(BG_2) \cong \mathbb{Z}/3\mathbb{Z}$. In particular, $[A(5,9), BG_2] \cong 0$ since $A(5,9)$ has cells in dimensions 5 and 9, and $[\mathrm{Tel}(c_3), BG_2] \cong 0$ as $\mathrm{Tel}(c_3)$ has cells in dimensions 5, 9 and 13.

The space $\mathrm{Tel}(c_2)$ has cells in dimensions 3, 7 and 11 with the 7 and 11 cells connected by the Steenrod operation $\mathcal{P}^1$. The triviality of $\pi_3(BG_2)$ implies that any map $\mathrm{Tel}(c_3) \longrightarrow BG_2$ factors through $\mathrm{Tel}(c_3)/S^3$. The nontrivial Steenrod operation in cohomology implies that there is a homotopy cofibration

$$S^{10} \xrightarrow{\alpha} S^7 \longrightarrow \mathrm{Tel}(c_3)/S^3 \longrightarrow S^{11}.$$

This induces an exact sequence

$$[S^{11}, BG_2] \longrightarrow [\mathrm{Tel}(c_3)/S^3, BG_2] \longrightarrow [S^7, BG_2] \xrightarrow{\alpha^*} [S^{10}, BG_2].$$

On the one hand, $[S^{11}, BG_2] = \pi_{11}(BG_2) \cong 0$. On the other hand, by [16], $[S^7, BG_2] \cong \mathbb{Z}/3\mathbb{Z}$, $[S^{10}, BG_2] \cong \mathbb{Z}/3\mathbb{Z}$, and α^* is an isomorphism. Thus $[\mathrm{Tel}(c_3)/S^3, BG_2] \cong 0$ and hence $[\mathrm{Tel}(c_3), BG_2] \cong 0$.

The space $\mathrm{Tel}(c_4)$ also has cells in dimensions 3, 7 and 11, but this time there is no Steenrod operation connecting the 7 and 11 cells. Thus $\mathrm{Tel}(c_4)/S^3 \simeq S^7 \vee S^{11}$. As in the $\mathrm{Tel}(c_2)$ case, any map $\mathrm{Tel}(c_4) \longrightarrow BG_2$ factors through $\mathrm{Tel}(c_4)/S^3 \simeq S^7 \vee S^{11}$. The homotopy cofibration

$$S^3 \longrightarrow \mathrm{Tel}(c_4) \longrightarrow S^7 \vee S^{11} \xrightarrow{\gamma} S^4$$

induces an exact sequence

$$[S^4, BG_2] \xrightarrow{\gamma^*} [\mathrm{Tel}(c_4), BG_2] \longrightarrow [S^7 \vee S^{11}, BG_2] \longrightarrow [S^3, BG_2]. \tag{5}$$

Since the 3 and 7-cells of $\mathrm{Tel}(c_4)$ are connected by the Steenrod operation $\mathcal{P}^1$, the restriction of γ to S^7 is α. It is not clear what the restriction of γ to S^{11} is but this is not relevant since $\pi_{11}(BG_2) = 0$, so γ^* factors as $[S^4, BG_2] \xrightarrow{\alpha^*} [S^7, BG_2] \longrightarrow [S^7 \vee S^{11}, BG_2]$. By [16], $[S^4, BG_2] \cong \mathbb{Z}$, $[S^7, BG_2] \cong \mathbb{Z}/3\mathbb{Z}$, and α^* is reduction mod-3. Therefore γ^* is onto. On the other hand, $[S^3, BG^2] \cong 0$, so exactness in (5) implies that $[\mathrm{Tel}(c_4), BG_2] \cong 0$. Collectively, we obtain a 3-local isomorphism $[G_2/T, G_2] \cong 0$.

Case 3: $p = 5$. As in Section 5.3, there is a 5-local homotopy equivalence

$$\Sigma G_2/T \simeq 2S^5 \vee 2S^7 \vee 2S^9 \vee S^{13} \vee 2A(3,11).$$

Therefore by Lemma 6.1

$$[G_2/T, G_2] \cong 2\pi_4(G_2) \times 2\pi_6(G_2) \times 2\pi_8(G_2) \times \pi_{12}(G_2) \times 2[A(3,11), BG_2].$$

By [16], $\pi_4(G_2) \cong 0$ and $\pi_{12}(G_2) \cong 0$, and localized at 5, $\pi_6(G_2) \cong 0$, $\pi_8(G_2) \cong 0$. As well, $\pi_3(BG_2) \cong 0$ and $\pi_{11}(BG_2) \cong 0$ so $[A, BG_2] \cong 0$. Thus, at 5, $[G_2/T, G_2] \cong 0$.

Case 4: $p > 5$. As in Section 5.3, there is a p-local homotopy equivalence

$$\Sigma G_2/T \simeq 2S^3 \vee 2S^5 \vee 2S^7 \vee 2S^9 \vee 2S^{11} \vee S^{13}.$$

By [16], the p-component of $\pi_m(G_2)$ is 0 for $m \in \{2,4,6,8,10,12\}$. Thus, at $p > 5$, $[G/T, G] \cong 0$. □

Proposition 6.6. Localized away from 2 there is a group isomorphism $[Sp(2)/T, Sp(2)] \cong 0$.

Proof. The cells of $Sp(2)/T$ occur in dimensions $2, 4, 6, 8$, and by [17], after inverting 2 we have $\pi_m(Sp(2)) \cong 0$ for $m \in \{2, 4, 6, 8\}$. Thus $[Sp(2)/T, Sp(2)] \cong 0$. □

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