NEW PERSPECTIVES ON THE CLASSIFYING SPACE OF THE FIBRE OF THE DOUBLE SUSPENSION

PAUL SELICK AND STEPHEN THERIAULT

Abstract. We give a new construction of a classifying space for the fibre of the double suspension, and an elementary proof of Gray's conjecture at odd primes.

1. Introduction

The double suspension $E^2\colon S^{2n-1} \longrightarrow \Omega^2 S^{2n+1}$ plays a key role in the calculation of the homotopy groups of spheres, so it is important to determine its properties. One property of significance is that, when localized at any prime p, the homotopy fibre W_n of E^2 has a classifying space that fits in a homotopy fibration $S^{2n-1} \xrightarrow{E^2} \Omega^2 S^{2n+1} \longrightarrow BW_n$. Such a classifying space and homotopy fibration was first proved by Gray [G1, G2] (the construction in [G1] did not, in fact, require localization). A different construction of what may be a homotopically distinct classifying space was given by Moore and Neisendorfer [MN]. In both cases BW_n was constructed as the homotopy fibre of a map $\Omega^2 S^{2np+1} \xrightarrow{\phi} S^{2np-1}$ and had the property that the composite $\Omega^2 S^{2n+1} \longrightarrow BW_n \longrightarrow \Omega^2 S^{2np+1}$ is homotopic to ΩH, where H is the p^{th}-James-Hopf invariant. The first goal of this paper is to give a new construction of BW_n and factorization of ΩH. The construction is elementary and makes use of properties of the $(p-1)^{st}$ projective space of $\Omega^2 S^{2n+1}$.

Gray conjectured that his construction of BW_n satisfied the additional property that the composite $\Omega^2 S^{2np+1} \xrightarrow{\phi} S^{2np-1} \xrightarrow{E^2} \Omega^2 S^{2np+1}$ is homotopic to the p^{th}-power map. Harper [H] showed that at odd primes, $\Omega E^2 \circ \Omega\phi$ is homotopic to the p^{th}-power map on $\Omega^3 S^{2np+1}$. Richter [R1] then proved a similar result at the prime 2. The second author claimed in [T1] that $E^2 \circ \phi \simeq p$ at odd primes, although a gap was later found, and recently Richter [R2] gave a proof that $E^2 \circ \phi \simeq p$ at all primes. The proof in [R2] uses delicate combinatorial arguments. The second goal of this paper is to give a simpler, more conceptual proof of Gray's conjecture at odd primes.

Finally we abstract and consider a Gray map: a map $f\colon \Omega^2 S^{2np+1} \longrightarrow S^{2np-1}$ which is degree p on the bottom cell and composes trivially with ΩH. The homotopy fibre B_f of f is shown to have an H-structure which makes it H-equivalent to BW_n, and consequences are drawn.

2010 *Mathematics Subject Classification.* Primary 55Q40, Secondary 55P35.

Key words and phrases. double suspension, classifying space, James-Hopf invariant.

2. A new construction of BW_n

Let X be a simply-connected, pointed space. For $k \geq 2$, let X^{*k} and $X^{\wedge k}$ respectively be the iterated join and iterated smash product of k copies of X with itself. Observe that $X^{*k} \simeq \Sigma^{k-1} X^{\wedge k}$.

Following Milnor [M] (see also Ganea [Ga] or Stasheff [St]), for $k \geq 1$ there is a k^{th}-projective space $P_k(\Omega X)$ that fits in a homotopy fibration

$$(\Omega X)^{*(k+1)} \longrightarrow P_k(\Omega X) \xrightarrow{ev_k} X \tag{1}$$

and the projective space $P_{k+1}(\Omega X)$ is defined by the homotopy cofibration

$$(\Omega X)^{*(k+1)} \longrightarrow P_k(\Omega X) \longrightarrow P_{k+1}(\Omega X).$$

This iteration starts with $P_1(\Omega X) = \Sigma\Omega X$ and ev_1 the canonical evaluation, and the map ev_{k+1} is an extension of ev_k. Since Ωev_1 has a right homotopy inverse and each ev_k extends ev_1, the map Ωev_k also has a right homotopy inverse. Consequently, the homotopy fibration connecting map

$$\delta_k \colon \Omega X \longrightarrow (\Omega X)^{*(k+1)}$$

for (1) is null homotopic.

Fix a prime p and consider the special case when $X = \Omega S^{2n+1}$ and $k = p-1$. There is a homotopy fibration sequence

$$\Omega^2 S^{2n+1} \xrightarrow{\ *\ } (\Omega^2 S^{2n+1})^{*p} \longrightarrow P_{p-1}(\Omega^2 S^{2n+1}) \xrightarrow{ev_{p-1}} \Omega S^{2n+1}.$$

Observe that $(\Omega^2 S^{2n+1})^{*p} \simeq \Sigma^{p-1}(\Omega^2 S^{2n+1})^{\wedge p}$ is $(2np-2)$-connected. Thus, ev_{p-1} is a homotopy equivalence through dimension $2np-2$. The $(2np-2)$-skeleton of ΩS^{2n-1} is homotopy equivalent to $J_{p-1}(S^{2n})$, the $(p-1)^{st}$-stage of the James construction on S^{2n}. Thus the map $J_{p-1}(S^{2n}) \longrightarrow \Omega S^{2n+1}$ lifts to a map $J_{p-1}(S^{2n}) \longrightarrow P_{p-1}(\Omega^2 S^{2n+1})$.

Now localize all spaces and maps at p and take homology with mod-p coefficients. There is a homotopy fibration sequence

$$\Omega^2 S^{2np+1} \longrightarrow J_{p-1}(S^{2n}) \longrightarrow \Omega S^{2n+1} \xrightarrow{H} \Omega S^{2np+1}$$

where H is the p^{th}-James-Hopf invariant. Combining this with previous paragraph, we obtain a homotopy fibration diagram

$$\begin{array}{ccccccc}
\Omega^2 S^{2n+1} & \xrightarrow{\Omega H} & \Omega^2 S^{2np+1} & \longrightarrow & J_{p-1}(S^{2n}) & \longrightarrow & \Omega S^{2n+1} \\
\| & & \downarrow{\scriptstyle\gamma} & & \downarrow & & \| \\
\Omega^2 S^{2n+1} & \xrightarrow{\ *\ } & (\Omega^2 S^{2n+1})^{*p} & \longrightarrow & P_{p-1}(\Omega^2 S^{2n+1}) & \xrightarrow{ev_{p-1}} & \Omega S^{2n+1}
\end{array} \tag{2}$$

for some map γ. Evaluating twice gives a map $\Sigma^2\Omega^2 S^{2n+1} \longrightarrow S^{2n+1}$ which is a left homotopy inverse of E^2. Iterating this, we obtain a map $(\Omega^2 S^{2n+1})^{*p} \longrightarrow S^{2np-1}$ which has a right homotopy inverse. Let ϕ' be the composite

$$\phi' \colon \Omega^2 S^{2np+1} \xrightarrow{\gamma} (\Omega^2 S^{2n+1})^{*p} \longrightarrow S^{2np-1}.$$

It is useful to record the homology of $\Omega^2 S^{2n+1}$. There is an isomorphism of Hopf algebras

$$H_*(\Omega^2 S^{2n+1}) \cong \Big(\otimes_{i=0}^{\infty} \Lambda(a_{2np^i-1}) \Big) \otimes \Big(\otimes_{j=1}^{\infty} \mathbb{Z}/p\mathbb{Z}[b_{2np^j-2}] \Big) \tag{3}$$

and there are Bocksteins $\beta(a_{2np^i-1}) = b_{2np^i-1}$ for $i \geq 1$.

Lemma 2.1. *The composite $S^{2np-1} \xrightarrow{E^2} \Omega^2 S^{2np+1} \xrightarrow{\phi'} S^{2np-1}$ is of degree $\pm p$, up to multiplication by a unit in $\mathbb{Z}_{(p)}$.*

Proof. First we determine the $2np$-skeleton of $P_{p-1}(\Omega^2 S^{2n+1})$. It has already been mentioned that the $(2np-2)$-skeleton of $P_{p-1}(\Omega^2 S^{2n+1})$ is homotopy equivalent to $J_{p-1}(S^{2n})$. To go further, let L be the $(2np-1)$-skeleton of $\Omega^2 S^{2n+1}$; by (3), there is a vector space isomorphism $H_*(L) = \mathbb{Z}/p\mathbb{Z}\{a_{2n-1}, b_{2np-2}, a_{2np-1}\}$ and $\beta(a_{2np-1}) = b_{2np-2}$. For $k \geq 1$, let ℓ_k be the composite

$$\ell_k \colon \Sigma L \longrightarrow \Sigma\Omega^2 S^{2n+1} = P_1(\Omega^2 S^{2n+1}) \longrightarrow P_k(\Omega^2 S^{2n+1}).$$

Note that ℓ_1 is an isomorphism on $H_{2np-1}(\ )$ and $H_{2np}(\ )$. For $2 \leq k \leq p-1$, $(\Omega^2 S^{2n+1})^{*k}$ is homotopy equivalent to S^{2nk-1} in dimensions $\leq 2np+1$. Thus, for dimensional reasons, in each of the cofibrations $(\Omega^2 S^{2n+1})^{*k} \longrightarrow P_{k-1}(\Omega^2 S^{2n+1}) \longrightarrow P_k(\Omega^2 S^{2n+1})$, the right map is an isomorphism on $H_{2np-1}(\ )$ and $H_{2np}(\ )$. Consequently, ℓ_k is an isomorphism on $H_{2np-1}(\ )$ and $H_{2np}(\ )$. Moreover, there is a Bockstein connecting these two generators since the same is true in $H_*(\Sigma L)$. Combining the ℓ_{p-1} case with the information on the $(2np-2)$-skeleton of $P_{p-1}(\Omega^2 S^{2n+1})$, in dimensions $\leq 2np$ we obtain $H_*(P_{p-1}(\Omega^2 S^{2n+1})) = H_*(J_{p-1}(S^{2n})) \oplus H_*(P^{2np}(p))$. Hence if X is the $2np$-skeleton of $P_{p-1}(\Omega^2 S^{2n+1})$ then there is a homotopy cofibration $P^{2np-1}(p) \longrightarrow J_{p-1}(S^{2n}) \longrightarrow X$.

Define the space M by the homotopy fibration $M \longrightarrow J_{p-1}(S^{2n}) \longrightarrow P_{p-1}(\Omega^2 S^{2n+1})$. The description of X implies, by the Serre exact sequence, that the $(2np-1)$-skeleton of M is $P^{2np-1}(p)$. By (2), there is also a homotopy fibration $M \longrightarrow \Omega^2 S^{2np+1} \xrightarrow{\gamma} (\Omega^2 S^{2n+1})^{*p}$. Taking $(2np-1)$-skeletons we obtain a sequence $P^{2np-1}(p) \longrightarrow S^{2np-1} \xrightarrow{\gamma_{2np}} S^{2np-1}$, which is a homotopy cofibration by the Serre exact sequence. The only possibility for γ_{2np} that produces a long exact sequence in homology is $\pm p$, up to multiplication by a unit in $\mathbb{Z}_{(p)}$. Finally, notice that γ_{2np} is homotopic to $\phi' \circ E^2$. □

Let B_n be the homotopy fibre of ϕ'. The left square in (2) and the definition of ϕ' imply that the composite $\Omega^2 S^{2n+1} \xrightarrow{\Omega H} \Omega^2 S^{2np+1} \xrightarrow{\phi'} S^{2np-1}$ is null homotopic. Thus ΩH lifts to a map $\nu_B \colon \Omega^2 S^{2n+1} \longrightarrow B_n$.

Theorem 2.2. *There is a homotopy fibration $S^{2n-1} \xrightarrow{E^2} \Omega^2 S^{2n+1} \xrightarrow{\nu_B} B_n$.*

Proof. Since ϕ' has degree p on the bottom cell and ΩH factors as $\Omega^2 S^{2n+1} \xrightarrow{\nu_B} B_n \longrightarrow \Omega^2 S^{2np+1}$, [R2, Theorem 4.5] applies to show that the homotopy fibre of ν_B is S^{2n-1}. □

Corollary 2.3. *There is a homotopy equivalence $W_n \simeq \Omega B_n$.* □

Corollary 2.3 says that B_n is a classifying space for W_n. It would be interesting to know whether B_n is homotopy equivalent to Gray's classifying space BW_n.

3. A proof of Gray's conjecture at odd primes

In general, suppose that Y is an H-space with multiplication m. Let m^* be the composite

$$m^*\colon \Sigma Y \wedge Y \xrightarrow{s} \Sigma(Y \times Y) \xrightarrow{\Sigma m} \Sigma Y$$

where s is a natural choice of a right homotopy inverse of the suspension of the quotient map $Y \times Y \longrightarrow Y \wedge Y$. The map m^* is natural for H-maps $Y \xrightarrow{f} Z$. Further, by [St], the homotopy fibre of m^* is Y, and if $Y = \Omega X$ then the homotopy fibration (1) for $k = 1$ is $\Omega X * \Omega X \xrightarrow{m^*} \Sigma\Omega X \xrightarrow{ev_1} X$.

In our case, let $\Omega^2 S^{2np+1}\{p\}$ be the homotopy fibre of the p^{th}-power map on $\Omega^2 S^{2np+1}$. In [G2] it was shown that the map $BW_n \xrightarrow{\nu} \Omega^2 S^{2np+1}$ lifts to a map $\nu'\colon BW_n \longrightarrow \Omega^2 S^{2np+1}\{p\}$. If $p \geq 3$ then BW_n is an H-space [G2] and ν' may be chosen to be an H-map [T2]. To take advantage of this, from now on we assume that all spaces and maps are localized at an odd prime.

Let $i\colon S^{2np-1} \longrightarrow \Sigma BW_n$ be the inclusion of the bottom cell and let Q be the homotopy pullback of i and $BW_n * BW_n \xrightarrow{m^*} \Sigma BW_n$. Consider the diagram

$$\begin{array}{ccccc}
 & & & & \Omega^2 S^{2np+1} \\
 & & \nearrow^{\lambda} & & \downarrow{\scriptstyle p} \\
Q & \xrightarrow{q} & S^{2np-1} & \xrightarrow{E^2} & \Omega^2 S^{2np+1} \\
\downarrow & & \downarrow{\scriptstyle i} & & \downarrow \\
BW_n * BW_n & \xrightarrow{m^*} & \Sigma BW_n & & \\
\downarrow{\scriptstyle \nu' * \nu'} & & \downarrow{\scriptstyle \Sigma\nu'} & & \\
\Omega^2 S^{2np+1}\{p\} * \Omega^2 S^{2np+1}\{p\} & \xrightarrow{m^*} & \Sigma\Omega^2 S^{2np+1}\{p\} & \xrightarrow{ev_1} & \Omega S^{2np+1}\{p\}
\end{array} \tag{4}$$

where the map λ will be defined momentarily. The lower left square homotopy commutes since ν' is an H-map, and the square above it is the homotopy pullback defining Q and q. The right rectangle homotopy commutes since both ways round the diagram are degree one in homology. Thus all rectangular parts of the diagram homotopy commute. Observe that the composite along the bottom row is null homotopic since it consists of two consecutive maps in a homotopy fibration. Thus the homotopy commutativity of the rectanglular part of the diagram implies that $E^2 \circ q$ composes trivially to $\Omega S^{2np+1}\{p\}$, resulting in a lift λ that makes the upper triangle homotopy commute.

Since Q is the homotopy pullback of i and m^*, the homotopy fibre of q is BW_n. Thus there is a homotopy fibration sequence $\Omega S^{2np-1} \xrightarrow{\delta} BW_n \xrightarrow{a} Q \xrightarrow{q} S^{2np-1}$ where δ is the fibration connecting map.

Lemma 3.1. *The map λ is a homotopy equivalence.*

Proof. First consider the homotopy pullback defining Q in (4). Since BW_n is $(2np-3)$-connected, $BW_n * BW_n$ is $(4np-4)$-connected. Therefore in dimensions less than $4np-4$ there is a homotopy fibration $Q \xrightarrow{q} S^{2np-1} \xrightarrow{i} \Sigma BW_n$. The $2np$-skeleton of BW_n is the Moore space $P^{2np}(p)$, and i is the inclusion of the bottom cell. Therefore the Serre exact sequence implies that the $(2np-1)$-skeleton of Q is S^{2np-1} and q is of degree p. The homotopy commutativity of the top triangle in (4) therefore implies that λ_* is an isomorphism on $H_{2np-1}(\ )$.

Next, we calculate the homology of Q. By [GT], there is a coalgebra isomorphism over the dual Steenrod algebra

$$H_*(BW_n) \cong \mathbb{Z}/p\mathbb{Z}[b_{2np-2}] \otimes H_*(\Omega^2 S^{2np+1}) \tag{5}$$

with the extra relation that $\beta(a_{2np-1}) = b_{2np-2}$, where a_{2np-1} is the generated of $H_{2np-1}(\Omega^2 S^{2np+1})$. Consider the mod-$p$ homology Serre spectral sequence for the fibration $\Omega S^{2np-1} \xrightarrow{\delta} BW_n \longrightarrow Q$. Since q_* is degree p in $H_{2np-1}(\ )$, the map δ_* must be degree one in $H_{2np-2}(\ )$. Thus if we write $H_*(\Omega S^{2np-1}) \cong \mathbb{Z}/p\mathbb{Z}[c_{2np-2}]$ then $\delta_*(c_{2np-2}) = b_{2np-2}$. Since δ is a homotopy fibration connecting map, it satisfies a homotopy action, implying that δ_* sends c^k_{2np-2} to b^k_{2np-2} for all $k \geq 1$. Hence δ_* is an injection. The homotopy fibration $\Omega S^{2n+1} \xrightarrow{\delta} BW_n \xrightarrow{a} Q$ is therefore totally nonhomologous to zero, meaning that the mod-p homology Serre spectral sequence collapses at the E^2-term. Hence the description of $H_*(BW_n)$ in (5) implies that there is a coalgebra isomorphism $H_*(Q) \cong H_*(\Omega^2 S^{2np+1})$. Moreover, this is an isomorphism as modules over the dual Steenrod algebra since a_* is now seen to be a projection.

Finally, we claim that $H_*(Q) \xrightarrow{\lambda_*} H_*(\Omega^2 S^{2np+1})$ is an injection. If not, then λ_* has a nontrivial kernel with an element x of least degree. As in the atomicity arguments in [Se] or [T3], such an x would have to be primitive, annihilated by the dual Steenrod algebra (modulo $\mathcal{P}^0$), and in the image of the Hurewicz homomorphism. By [Se] there is only one such element in $H_*(Q) \cong H_*(\Omega^2 S^{2n+1})$, that being a_{2np-1}. But we have already seen that $\lambda_*(a_{2np-1}) \neq 0$. Hence the kernel of λ_* is zero, implying that λ_* is an injection. As λ_* is a map between vector spaces of the same type, it must therefore be an isomorphism. Hence λ is a homotopy equivalence. $\square$

Define $\overline{\phi}$ by the composite

$$\overline{\phi}\colon \Omega^2 S^{2np+1} \xrightarrow{\lambda^{-1}} Q \xrightarrow{q} S^{2np-1}$$

and $\overline{\nu}$ by the composite

$$\overline{\nu}\colon BW_n \xrightarrow{a} Q \xrightarrow{\lambda} \Omega^2 S^{2np+1}.$$

We now prove Gray's conjecture at odd primes, using the modified maps $\overline{\phi}$ and $\overline{\nu}$.

Theorem 3.2. *There is a homotopy fibration* $BW_n \xrightarrow{\overline{\nu}} \Omega^2 S^{2np+1} \xrightarrow{\overline{\phi}} S^{2np-1}$ *and a homotopy commutative diagram*

$$\begin{array}{ccc} \Omega^2 S^{2np+1} & \xrightarrow{p} & \Omega^2 S^{2np+1} \\ \downarrow{\scriptstyle \overline{\phi}} & & \| \\ S^{2np-1} & \xrightarrow{E^2} & \Omega^2 S^{2np+1}. \end{array}$$

Proof. The first assertion follows from the existence of the homotopy fibration $BW_n \xrightarrow{a} Q \xrightarrow{q} S^{2np-1}$ and the definitions of $\overline{\nu}$ and $\overline{\phi}$. By (4), as self-maps of $\Omega^2 S^{2np+1}$ we have $p \simeq E^2 \circ q \circ \lambda^{-1}$. So by definition of $\overline{\phi}$ we obtain $p \simeq E^2 \circ \overline{\phi}$. □

Remark 3.3. Ideally, the composite $\Omega^2 S^{2n+1} \longrightarrow BW_n \xrightarrow{\overline{\nu}} \Omega^2 S^{2np+1}$ would be homotopic to ΩH. However, it is not clear if this is the case. It would be interesting to know if this does hold, or if it holds after a modification of the construction of $\overline{\phi}$.

4. Gray Maps

Definition 4.1. A *Gray Map* is a map $f\colon \Omega^2 S^{2np+1} \longrightarrow S^{2np-1}$ that is degree p on the bottom cell and has the property that the composition $\Omega^2 S^{2n+1} \xrightarrow{\Omega H} \Omega^2 S^{2np+1} \longrightarrow S^{2np-1}$ is trivial.

For a Gray map $f\colon \Omega^2 S^{2np+1} \longrightarrow S^{2np-1}$, we write B_f for the homotopy fibre of f and $j_f\colon B_f \longrightarrow \Omega^2 S^{2np+1}$ for the inclusion of the homotopy fibre into the total space. Let G be the Gray map constructed by Gray. Thus $B_G = BW_n$.

Let f be a Gray map. Since $f \circ \Omega H$ is null, there is lift $\nu_f\colon \Omega^2 S^{2n+1} \longrightarrow B_f$. As in [G2], a homology calculation shows that there is homotopy fibration $S^{2n-1} \xrightarrow{E^2} \Omega^2 S^{2n+1} \xrightarrow{\nu_f} B_f$. Also as in [G2], this implies that $\Sigma^2\Omega^2 S^{2n+1} \simeq \Sigma^2(S^{2n-1} \times B_f)$ and thus $\Sigma^2\nu_f\colon \Sigma^2\Omega^2 S^{2n+1} \longrightarrow \Sigma^2 B_f$ has a right homotopy inverse. Further, the homotopy equivalence $\Sigma^2 S^{2n+1} \simeq \Sigma^2(S^{2n-1} \times B_f)$ holds for any Gray map f, so $H_*(B_f) \cong H_*(B_G)$ as modules over the dual Steenrod algebra. It is this structure that is used in [GT] to show that $B_G = BW_n$ is atomic. Therefore, B_f is also atomic, and to show that any map $B_f \longrightarrow B_g$ is a homotopy equivalence it suffices to use an atomicity argument.

Henceforth, assume that either $p > 2$ or that $n = 1$ or 2. That is, assume that S^{2np-1} is an H-space. Recall the following lemma from [T2].

Lemma 4.2. *Let* $f : X \longrightarrow Y$ *be a map such that* $\Sigma^2 f$ *has a right homotopy inverse and let* Z *be an* H*-space. Let* $g, h : Y \longrightarrow Z$ *be maps such that* $g \circ f \simeq h \circ f$*. Then* $g \simeq h$*.*

Theorem 4.3. *Let* f, g *be Gray maps. Then there is a homotopy commutative diagram*

$$\begin{array}{ccc} & & B_g \\ & \nearrow^{\lambda} & \downarrow{\scriptstyle j_g} \\ B_f & \xrightarrow{j_f} & \Omega^2 S^{2np+1} \end{array}$$

where λ is a homotopy equivalence.

Proof. Consider the composition $B_f \xrightarrow{j_f} \Omega^2 S^{2np+1} \xrightarrow{g} S^{2np-1}$. Since $g \circ j_f \circ \nu_f \simeq g \circ \Omega H \simeq *$ and $\Sigma^2 \nu_f$ has a right homotopy inverse, it follows from Lemma 4.2 that $g \circ j_f \simeq *$. Thus there is a lift $\lambda : B_f \to B_g$. A homology calculation (an atomicity argument) shows that λ is a homotopy equivalence. $\square$

In particular, $B_f \simeq B_G$. It follows that for any Gray map f, B_f has an H-space structure such that j_f is an H-map, since B_G has this property. Here is a direct proof (or, if preferred, here is an alternate proof that B_G has the property).

Proposition 4.4. *If $f\colon A \longrightarrow B$ is a map for which $\Sigma^2 f$ has a right homotopy inverse then $\Sigma^2 J(f)\colon \Sigma^2 J(A) \longrightarrow \Sigma^2 J(B)$ has a right homotopy inverse.*

Proof. The existence of a right homotopy inverse for $\Sigma^2 f$ implies the existence of a right homotopy inverse for $\Sigma^2 f^{(k)}\colon \Sigma^2 A^{(k)} \longrightarrow \Sigma^2 B^{(k)}$ for any $k \geq 1$. The statement now follows from the naturality of the James decomposition $\Sigma J(X) \simeq \bigvee_{k=1}^{\infty} \Sigma X^{(k)}$ $\square$

For an H-space Z write $m_Z : J(Z) \longrightarrow Z$ for the map $m_Z(z_1, \ldots, z_k) := (\cdots((z_1 z_2)z_3)\cdots z_k)$.

Proposition 4.5. *Let $F \xrightarrow{j} E \xrightarrow{\pi} Y$ be a homotopy fibration and let $h\colon X \longrightarrow F$ be a map such that $\Sigma^2 h$ has a right homotopy inverse. Suppose that X, E, Y are H-spaces and that $j \circ h\colon X \longrightarrow E$ is an H map (π need not be an H-map.) Then there is a "multiplication" map $m_F\colon J(F) \longrightarrow F$ such that $j \circ m_F \simeq m_E \circ J(j)\colon J(F) \longrightarrow E$.*

Proof. Since $j \circ h$ is an H-map, we have $m_E \circ J(j \circ h) \simeq j \circ h \circ m_X\colon J(X) \longrightarrow E$. Thus $\pi \circ m_E \circ J(j \circ h) \simeq \pi \circ j \circ h \circ m_X \simeq *\colon J(X) \longrightarrow Y$. Since $\Sigma^2 J(h)$ has a right homotopy inverse by Proposition 4.4, Lemma 4.2 gives $\pi \circ m_E \circ J(j) \simeq *\colon J(F) \longrightarrow Y$. Thus there is a lift $m_F\colon J(F) \longrightarrow F$ of $m_E \circ J(j)$ and by construction $j \circ m_F \simeq m_E \circ J(j)\colon J(F) \longrightarrow E$. $\square$

Theorem 4.6. *Let $f\colon \Omega^2 S^{2np+1} \longrightarrow S^{2np-1}$ be a Gray map. Then B_f has an H-space structure such that $j_f\colon B_f \longrightarrow \Omega^2 S^{2np+1}$ is an H-map.*

Proof. Consider the fibration $B_f \xrightarrow{j_f} \Omega^2 S^{2np+1} \xrightarrow{f} S^{2np-1}$ and map $\Omega^2 S^{2n+1} \xrightarrow{\nu_f} B_f$. Since $\Sigma^2 \nu_f$ has a right homotopy inverse and $j_f \circ \nu_f$ is the H-map ΩH, we can apply Proposition 4.5 to get a multiplication m_{B_f} on B_f which commutes with $j_f\colon B_f \longrightarrow \Omega^2 S^{2np+1}$. It remains to show that m_{B_f} produces an H-space structure on B_f. That is, we need to know that the composition $e\colon B_f \xrightarrow{i_{B_f}} J(B_f) \xrightarrow{m_{B_f}} B_f$ is a homotopy equivalence,where $i_A\colon A \longrightarrow J(A)$ denotes the inclusion into the James construction. By naturality of i_A we have $j_f \circ e \simeq j_f \circ m_{B_f} \circ i_{B_f} \simeq m_{\Omega^2 S^{2np+1}} \circ J(j_f) \circ i_{B_f} \simeq m_{\Omega^2 S^{2np+1}} \circ i_{\Omega^2 S^{2np+1}} \circ j_f \simeq 1 \circ j_f = j_f$. Since j_f is nonzero on $H_{2np-1}(\ \ )$, the equation $j_f \circ e \simeq j_f$ implies that e is nonzero on $H_{2np-1}(\ \ )$. The Bockstein connecting the bottom two cells of B_f then implies that e is nonzero on $H_{2np-2}(\quad)$, the least nonvanishing degree in homology. Since B_f is atomic, e is therefore a homotopy equivalence. □

In general, two H-spaces X and Y are *H-equivalent* if there is an H-map $X \longrightarrow Y$ which is also a homotopy equivalence.

If $p \geq 3$ then, by [T2], B_G has a unique H-space structure $m_G \colon B_G \times B_G \longrightarrow B_G$ for which $\Omega^2 S^{2n+1} \xrightarrow{\nu_G} B_G$ is an H-map. By Theorem 4.3, there is a homotopy equivalence $\lambda \colon B_f \longrightarrow B_G$ such that $j_G \circ \lambda \simeq j_f$. Define an H-space structure on B_f by the composite

$$m_f \colon B_f \times B_f \xrightarrow{\lambda \times \lambda} B_G \times B_G \xrightarrow{m_G} B_G \xrightarrow{\lambda^{-1}} B_f.$$

Then, by construction, λ is an H-equivalence. Now consider the composite

$$\Omega^2 S^{2n+1} \xrightarrow{\nu_G} B_G \xrightarrow{\lambda^{-1}} B_f.$$

On the one hand, since ν_G and λ^{-1} are both H-maps, so is their composite. On the other hand, this composite is a valid choice for ν_f since $j_f \circ \lambda^{-1} \circ \nu_G \simeq j_G \circ \nu_G \simeq \Omega H$. Thus, taking $\nu_f = \lambda^{-1} \circ \nu_G$, we have ν_f an H-map. Moreover, m_f is the unique H-structure for which ν_f is an H-map because of the corresponding statement for m_G and ν_G, the definitions of m_f and ν_f, and the fact that λ is an H-equivalence.

From now on, assume that B_f has been given the H-space structure m_f. In particular, for any two Gray maps f and g, we obtain an H-equivalence $B_f \simeq B_g$. Further, for any Gray map f, there is an H-lift $j_f' \colon B_f \longrightarrow \Omega^2 S^{2np+1}\{p\}$ of j_f since B_G has this property.

If f is a Gray map, write f' for the map constructed from f by applying the method of Section 3 to j_f'. The following proposition shows that this method is a kind of normalizing procedure.

Proposition 4.7. *Let f, g be Gray maps. Then $f' \simeq g'$.*

Proof. Let $e \colon B_f \longrightarrow B_g$ be an H-equivalence. Then there is a homotopy commutative square

$$\begin{array}{ccc} B_f * B_f & \xrightarrow{m_f^*} & \Sigma B_f \\ \downarrow{\scriptstyle e*e} & & \downarrow{\scriptstyle \Sigma e} \\ B_g * B_g & \xrightarrow{m_g^*} & \Sigma B_g \end{array}$$

so Q, the homotopy pullback of m_f^* and $S^{2np-1} \longrightarrow \Sigma B_f$, is also the homotopy pullback of m_g^* and $S^{2np-1} \longrightarrow \Sigma B_g$. □

References

[Ga] T. Ganea, Lusternik-Schirelmann category and strong category, *Illinois. J. Math.* **11** (1967), 417-427.

[G1] B. Gray, On the double suspension, *Algebraic Topology*, pp. 150-162, Springer Lecture Notes in Math. **1370**, Springer, New York-Berlin, 1989.

[G2] B. Gray, On the iterated suspension, *Topology* **27** (1988), 301-310.

[GT] B.Gray and S. Theriault, On the double suspension and the mod-p Moore space, *Contemp. Math.* **399** (2006), 101-121.

[H] J.R. Harper, A proof of Gray's conjecture, *Contemp. Math. 96* (1989), 189-195.

[M] J. Milnor, Construction of universal bundles, II, *Ann. of Math.* **63** (1956), 430-436.

[MN] J.C. Moore and J.A. Neisendorfer, Equivalence of Toda-Hopf invariants, *Israel J. Math.* **66** (1989), 300-318.

[R1] W. Richter, The H-space squaring map on $\Omega^3 S^{4n+1}$ factors through the double suspension, *Proc. Amer. Math. Soc.* **123** (1995), 3889-3900.

[R2] W. Richter, A conjecture of Gray and the p^{th}-power map on $\Omega^2 S^{2np+1}$, *Proc. Amer. Math. Soc.* **142** (2014), 2151-2160.

[Se] P.S. Selick, A reformulation of the Arf invariant one mod-p problem and applications to atomic spaces, *Pacific J. Math.* **108** (1983), 431-450.

[St] J.D. Stasheff, Homotopy associativity of H-spaces I, II, *Trans. Amer. Math. Soc.* **108** (1963), 275-292, 293-312.

[T1] S. Theriault, Proofs of two conjectures of Gray involving the double suspension, *Proc. Amer. Math. Soc.* **131** (2003), 2953-2962.

[T2] S. Theriault, The 3-primary classifying space of the fiber of the double suspension, *Proc. Amer. Math. Soc.* **136** (2008), 1489-1499.

[T3] S. Theriault, Atomicity for Anick's spaces, *J. Pure Appl. Algebra* **219** (2015), 2346-2358.

Department of Mathematics, University of Toronto, Toronto ON, M5S 2E4, Canada

E-mail address: selick@math.toronto.edu

Mathematical Sciences, University of Southampton, Southampton SO17 1BJ, United Kingdom

E-mail address: S.D.Theriault@soton.ac.uk