

ODD PRIMARY HOMOTOPY TYPES OF THE GAUGE GROUPS OF EXCEPTIONAL LIE GROUPS

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ABSTRACT. The p -local homotopy types of gauge groups of principal G -bundles over S^4 are classified when G is a compact connected exceptional Lie group without p -torsion in homology except for $(G, p) = (E_7, 5)$.

1. INTRODUCTION

Let G be a topological group and P be a principal G -bundle over a base X . The *gauge group* $\mathcal{G}(P)$ is the topological group of G -equivariant self-maps of P covering the identity map of X . As P ranges over all principal G -bundles over X , we get a collection of gauge groups $\mathcal{G}(P)$. In [14], Kono classified the homotopy types in the collection of $\mathcal{G}(P)$ when $G = \mathrm{SU}(2)$ and $X = S^4$. Since then, the homotopy theory of gauge groups has been deeply developed in connection with mapping spaces and fiberwise homotopy theory incorporating higher homotopy structures. The classification of the homotopy types has been its motivation and considerable progress has been made towards it.

Suppose that G is a compact connected simple Lie group. Then there is a one-to-one correspondence between principal G -bundles over S^4 and $\pi_3(G) \cong \mathbb{Z}$. Let \mathcal{G}_k denote the gauge group of the principal G -bundle over S^4 corresponding to $k \in \mathbb{Z}$. Most of the classification of the homotopy types of gauge groups has been done in this setting, generalizing the situation studied by Kono [3, 5, 8, 9, 13, 20, 21]. As we will see, the homotopy type of \mathcal{G}_k is closely related with a certain Samelson product in G . It is shown in [12] that if p is a prime large with respect to the rank of G , then the classification of the p -local homotopy types of \mathcal{G}_k reduces to determining the order of this Samelson product. So in [11, 22], the p -local classification is obtained by calculating this Samelson product when G is a classical group.

However, when G is exceptional, the classification, up to one factor of 2, for $G = G_2$ [13] is the only one result obtained so far. The aim of this paper is to classify the p -local homotopy types of \mathcal{G}_k when G is exceptional and has no p -torsion in $H_*(G; \mathbb{Z})$. Recall that exceptional Lie groups consist of five types G_2, F_4, E_6, E_7, E_8 and they have no p -torsion in $H_*(G; \mathbb{Z})$ if and

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only if

$$(1.1) \quad G_2, p \geq 3; \quad F_4, E_6, E_7, p \geq 5; \quad E_8, p \geq 7.$$

All but one case is included in our results. We define an integer $\gamma(G)$ by:

$$\begin{aligned} \gamma(G_2) &= 3 \cdot 7 & \gamma(F_4) &= 5^2 \cdot 13 & \gamma(E_6) &= 5^2 \cdot 7 \cdot 13 \\ \gamma(E_7) &= 7 \cdot 11 \cdot 19 & \gamma(E_8) &= 7^2 \cdot 11^2 \cdot 13 \cdot 19 \cdot 31 \end{aligned}$$

For integers m and n , let (m, n) denote the greatest common divisor of m and n , and for an integer $k = p^r q$ with $(p, q) = 1$, let $\nu_p(k) = p^r$ denote the p -component of k . Now we state our main result.

Theorem 1.1. *Let G be a compact connected exceptional simple Lie group without p -torsion in $H_*(G; \mathbb{Z})$ except for $(G, p) = (E_7, 5)$. Then \mathcal{G}_k and \mathcal{G}_l are p -locally homotopy equivalent if and only if $\nu_p((k, \gamma(G))) = \nu_p((l, \gamma(G)))$.*

2. GAUGE GROUPS AND SAMELSON PRODUCTS

We first recall a relation between gauge groups and Samelson products. Let G be a compact connected simple Lie group and let ϵ be a generator of $\pi_3(G) \cong \mathbb{Z}$. Then by definition \mathcal{G}_k is the gauge group of a principal G -bundle over S^4 corresponding to $k\epsilon$. Let $\text{map}(X, Y; f)$ denote the connected component of the mapping space $\text{map}(X, Y)$ containing the map $f: X \rightarrow Y$. It is proved by Gottlieb [4] (cf. [1]) that there is a homotopy equivalence

$$B\mathcal{G}_k \simeq \text{map}(X, BG; k\bar{\epsilon}),$$

where $\bar{\epsilon}: S^4 \rightarrow BG$ is the adjoint of $\epsilon: S^3 \rightarrow G$. Then by evaluating at the basepoint of S^4 , we get a homotopy fibration sequence

$$\mathcal{G}_k \rightarrow G \xrightarrow{\partial_k} \Omega_0^3 G \rightarrow B\mathcal{G}_k \rightarrow BG,$$

so \mathcal{G}_k is homotopy equivalent to the homotopy fiber of the map ∂_k . Thus we must identify the map ∂_k , which was done by Lang [15].

Lemma 2.1. *The adjoint $S^3 \wedge G \rightarrow G$ of the map ∂_k is the Samelson product $\langle k\epsilon, 1_G \rangle$.*

By the linearity of Samelson products, we have:

Corollary 2.2. $\partial_k \simeq k \circ \partial_1$.

Hereafter we localize spaces and maps at a prime p and suppose that G has no p -torsion. We next recall the result of Kono, Tsutaya and the second author [12] on the p -local homotopy types of \mathcal{G}_k . By a classical result of Hopf, any connected finite H-space X is rationally homotopy equivalent to a product of odd dimensional spheres $S^{2n_1-1} \times \cdots \times S^{2n_l-1}$ for $n_1 \leq \cdots \leq n_l$. The sequence (n_1, \dots, n_l) is called the type of X . An H-space X is called *retractile* if the following conditions are satisfied:

- (1) there is a space A such that $H_*(X) = \Lambda(\tilde{H}_*(A))$;
- (2) there is a map $\lambda: A \rightarrow X$ which is the inclusion of the generating set in homology.
- (3) $\Sigma\lambda$ has a left homotopy inverse.

The space A is often called a homology generating subspace. Recall from [16] that since we are localizing at p , G decomposes as

$$G \simeq B_1 \times \cdots \times B_{p-1}$$

where the type of B_i consists of the integers n in the type of G with $2n - 1$ congruent to $2i + 1$ modulo $2(p - 1)$. This is called the mod p decomposition of G . The existence of the space A is well known for the classical groups and is proved in [19] for the exceptional groups by using the mod p decomposition of G . In particular, one has $A_i \subset B_i$ such that $A = A_1 \vee \cdots \vee A_{p-1}$, where A_i consists of rank B_i cells whose dimensions are congruent to $2i + 1$. Since the homology of each factor B_i is primitively generated, if the rank of B_i is less than $p - 1$, then we can apply the method of [2] - as argued in [19] in a smaller range - to retract ΣB_i onto ΣA_i so that the map $A \rightarrow G$ has a left homotopy inverse after a suspension. Then a compact connected simple Lie group G is retractile at the prime p if (G, p) is in the following table.

$SU(n)$	$(p - 1)^2 + 1 \geq n$	G_2	$p \geq 3$
$Sp(n), Spin(2n + 1)$	$(p - 1)^2 + 1 \geq 2n$	F_4, E_6	$p \geq 5$
$Spin(2n)$	$(p - 1)^2 + 1 \geq 2(n - 1)$	E_7, E_8	$p \geq 7$

Note that by (1.1), there is no p -torsion in $H_*(G; \mathbb{Z})$ if and only if G is retractile at the prime p , except for $(G, p) = (E_7, 5)$. Let $\gamma(G, p)$ be the order of the Samelson product $\langle \epsilon, \lambda \rangle$. In [12] the following reduction of the classification of the p -local homotopy types of \mathcal{G}_k is proved.

Theorem 2.3. *Let G be a compact connected simple Lie group which is retractile at the prime p except for $(G, p) = (G_2, 3), (E_7, 5)$. Then \mathcal{G}_k and \mathcal{G}_l are p -locally homotopy equivalent if and only if $(k, \gamma(G, p)) = (l, \gamma(G, p))$.*

Remark 2.4. The range of p in [12] is smaller than the above statement because the range in [12] was taken so that G is retractile and the mod p decomposition satisfies a certain universality. However, we can easily see that only the retractile property was used in [12], so we can improve the range of p as above.

Remark 2.5. As a CW-complex, $A_i = S^{2n+1} \cup e^{2n+1+2(p-1)} \cup \cdots \cup e^{2n+1+2(\text{rank } B_i - 1)(p-1)}$ for some integer n congruent to i modulo $p - 1$ in any case except $(G, p) = (G_2, 3)$, though $A_1 = S^3 \cup e^{11}$ for $(G_2, 3)$. This causes the case $(G, p) = (G_2, 3)$ to be exceptional in Theorem 2.3. However, as mentioned in Section 1, we do not have to consider G_2 in this paper owing to [13].

Hereafter, we assume that G is of type (n_1, \dots, n_l) , without p -torsion, and retractile at p . We next decompose the Samelson product $\langle \epsilon, \lambda \rangle$ in G into small pieces. As mentioned above, $A = A_1 \vee \dots \vee A_{p-1}$ for $A_i \subset B_i$. Then it follows that the Samelson product $\langle \epsilon, \lambda \rangle$ is the direct sum of $\langle \epsilon, \lambda_1 \rangle, \dots, \langle \epsilon, \lambda_{p-1} \rangle$, where $\lambda_i: A_i \rightarrow G$ is the restriction of λ to A_i . Let $\gamma_i(G, p)$ be the order of $\langle \epsilon, \lambda_i \rangle$. Then we have:

Proposition 2.6. *If G is retractile at the prime p , then*

$$\gamma(G, p) = \max\{\gamma_1(G, p), \dots, \gamma_{p-1}(G, p)\}.$$

Thus if we calculate $\gamma_i(G, p)$ for $1 \leq i \leq p-1$ for a given pair (G, p) then by Proposition 2.6 we know $\gamma(G, p)$ and by Theorem 2.3 we can describe the p -local homotopy types of the corresponding gauge groups. These calculations are carried out in the subsequent sections.

The following table lists explicit mod p decompositions of exceptional Lie groups without p -torsion in homology except for G_2 , where $B(2m_1 - 1, \dots, 2m_k - 1)$ is an H-space of type (m_1, \dots, m_k) . One can easily deduce the types of exceptional Lie groups from this table.

F_4	$p = 5$	$B(3, 11) \times B(15, 23)$
	$p = 7$	$B(3, 15) \times B(11, 23)$
	$p = 11$	$B(3, 23) \times S^{11} \times S^{15}$
	$p \geq 13$	$S^3 \times S^{11} \times S^{15} \times S^{23}$
E_6	$p = 5$	$F_4 \times B(9, 17)$
	$p \geq 7$	$F_4 \times S^9 \times S^{17}$
E_7	$p = 5$	$B(3, 11, 19, 27, 35) \times B(15, 23)$
	$p = 7$	$B(3, 15, 27) \times B(11, 23, 35) \times S^{19}$
	$p = 11$	$B(3, 23) \times B(15, 35) \times S^{11} \times S^{19} \times S^{27}$
	$p = 13$	$B(3, 27) \times B(11, 35) \times S^{15} \times S^{19} \times S^{23}$
	$p = 17$	$B(3, 35) \times S^{11} \times S^{15} \times S^{19} \times S^{23} \times S^{27}$
	$p \geq 19$	$S^3 \times S^{11} \times S^{15} \times S^{19} \times S^{23} \times S^{27} \times S^{35}$
E_8	$p = 7$	$B(3, 15, 27, 39) \times B(23, 35, 47, 59)$
	$p = 11$	$B(3, 23) \times B(15, 35) \times B(27, 47) \times B(39, 59)$
	$p = 13$	$B(3, 27) \times B(15, 39) \times B(23, 47) \times B(35, 59)$
	$p = 17$	$B(3, 35) \times B(15, 47) \times B(27, 59) \times S^{23} \times S^{39}$
	$p = 19$	$B(3, 39) \times B(23, 59) \times S^{15} \times S^{27} \times S^{35} \times S^{47}$
	$p = 23$	$B(3, 47) \times B(15, 59) \times S^{23} \times S^{27} \times S^{35} \times S^{39}$
	$p = 29$	$B(3, 59) \times S^{15} \times S^{23} \times S^{27} \times S^{35} \times S^{39} \times S^{47}$
	$p \geq 31$	$S^3 \times S^{15} \times S^{23} \times S^{27} \times S^{35} \times S^{39} \times S^{47} \times S^{59}$

If each B_i is a sphere, then we call G *p-regular*. If each B_i is a sphere or a rank 2 H-space $B(2m-1, 2m+2p-3)$, then we call G *quasi-p-regular*. In the following we calculate

$\gamma(G, p)$ for different cases. Note that, **in the case of any factor from the above table**, $A_i = S^{2n+1} \cup e^{2n+1+2(p-1)} \cup \dots \cup e^{2n+1+2(\text{rank } B_i - 1)(p-1)}$ for some integer n congruent to i modulo $p-1$.

3. THE p -REGULAR CASE

Throughout this section, we let (G, p) be

$$(3.1) \quad F_4, E_6, p \geq 13; \quad E_7, p \geq 19; \quad E_8, p \geq 31.$$

By the above table, this is equivalent to the case when G is p -locally homotopy equivalent to a product of spheres, and such a G is called p -regular. This is also equivalent to $p \geq n_l + 1$.

The homotopy groups of p -localized spheres are known for low dimension.

Theorem 3.1 (Toda [23]). *Localized at p , we have*

$$\pi_{2n-1+k}(S^{2n-1}) \cong \begin{cases} \mathbb{Z}/p\mathbb{Z} & \text{for } k = 2i(p-1) - 1, 1 \leq i \leq p-1 \\ \mathbb{Z}/p\mathbb{Z} & \text{for } k = 2i(p-1) - 2, n \leq i \leq p-1 \\ 0 & \text{other cases for } 1 \leq k \leq 2p(p-1) - 3, \end{cases}$$

By Theorem 3.1 we have $p \cdot \pi_{2k}(S^{2n-1}) = 0$ for $0 \leq k \leq n + p(p-1) - 2$, so $p \cdot \pi_{2k}(G) = 0$ for $0 \leq k \leq p(p-1)$. Then since the Samelson product $\langle \epsilon, \lambda_i \rangle$ is a map from a $2n$ -dimensional sphere into G for $n \leq n_l + 1 \leq p$, we get:

Proposition 3.2. *If G is p -regular, then $\gamma_i(G, p) \leq p$ for any i .*

On the other hand, the non-triviality of the Samelson product $\langle \lambda_i, \lambda_j \rangle$ is completely determined in [6], where λ_i is the inclusion $S^{2n_i-1} \rightarrow G$ of the sphere factor in the mod p decomposition of G .

Theorem 3.3. *The Samelson product $\langle \lambda_i, \lambda_j \rangle$ is non-trivial if and only if $n_i + n_j = n_k + p - 1$ for some n_k in the type of G .*

Then we immediately get:

Corollary 3.4. (1) *The Samelson product $\langle \epsilon, \lambda_l \rangle$ is non-trivial for $p = n_l + 1$;*
 (2) *The Samelson product $\langle \epsilon, \lambda_i \rangle$ is trivial for any i for $p > n_l + 1$.*

Thus by combining Propositions 2.6 and 3.2 and Corollary 3.4, we obtain the following.

Corollary 3.5. *We have*

$$\gamma(G, p) = \begin{cases} p & p = n_l + 1 \\ 1 & p > n_l + 1. \end{cases}$$

4. THE QUASI- p -REGULAR CASE

Throughout this section, we assume that (G, p) is

$$(4.1) \quad F_4, E_6, p = 5, 7, 11; \quad E_7, p = 11, 13, 17; \quad E_8, p = 11, 13, 17, 19, 23, 29$$

unless otherwise stated. This is equivalent to G being a product of spheres and rank 2 H-spaces, and such a G is quasi- p -regular.

We first show the triviality of $\langle \epsilon, \lambda_i \rangle$ from the homotopy groups of G . By Theorem 3.1, for $n \geq 1$ and $i \leq n + (p - 1)^2$, we have $\pi_{2i}(S^{2n+1}) = 0$ unless $i \equiv n \pmod{p-1}$ and $p \cdot \pi_{2i}(S^{2n+1}) = 0$.

Moreover, the homotopy groups of the H-spaces $B(2n - 1, 2n - 1 + 2(p - 1))$ in the mod p decomposition of G are calculated.

Theorem 4.1 (Mimura and Toda [17], Kishimoto [10]). *Localized at p , we have*

$$\pi_{2n-1+k}(B(3, 2p+1)) \cong \begin{cases} \mathbb{Z}/p\mathbb{Z} & \text{for } k = 2i(p-1) - 1, 2 \leq i \leq p-1 \\ \mathbb{Z} & \text{for } k = 2p-2 \\ 0 & \text{other cases for } 1 \leq k \leq 2p(p-1) - 3, \end{cases}$$

and

$$\pi_{2n-1+k}(B(2n-1, 2n+2p-3)) \cong \begin{cases} \mathbb{Z}/p^2\mathbb{Z} & \text{for } k = 2i(p-1) - 1, 2 \leq i \leq p-1 \\ \mathbb{Z}/p\mathbb{Z} & \text{for } k = 2i(p-1) - 2, n \leq i \leq p-1 \\ \mathbb{Z} & \text{for } k = 2p-2 \\ 0 & \text{other cases for } 1 \leq k \leq 2p(p-1) - 3. \end{cases}$$

For $n \geq 1$ and $i \leq n + (p-1)^2$, $\pi_{2i}(B(2n+1, 2n+1+2(p-1))) = 0$ unless $i \equiv n \pmod{p-1}$, and $p \cdot \pi_{2i}(B(3, 2p+1)) = 0$ for $i \leq 1 + (p-1)^2$. By Theorems 3.1 and 4.1, we obtain the following table for all possibly non-trivial Samelson products $\langle \epsilon, \lambda_i \rangle$ in G :

	$p = 5$	$p = 7$	$p = 11$	$p = 13$	$p = 19$
F_4	$i = 3$	$i = 5$			
E_6	$i = 3$	$i = 2, 5$			
E_7			$i = 3, 7$	$i = 5$	
E_8			$i = 9$	$i = 5, 11$	$i = 5, 11$

We will determine the order of these Samelson products by using the technique developed in [7] which we briefly recall here. The technique in [7] is for computing the Samelson products in the p -localized exceptional Lie group G for (G, p) in (1.1) except for $(G, p) = (G_2, 3), (E_7, 5)$. We only consider the case that $G = E_6, E_7, E_8$ since the F_4 case can be deduced from the E_6 case. Consider the 27-, 56- and 248-dimensional irreducible representations of $G = E_6, E_7, E_8$,

respectively. Let $\rho: G \rightarrow \mathrm{SU}(\infty)$ be the composite of these representations and the inclusions into $\mathrm{SU}(\infty)$. Then we have a homotopy fibration sequence

$$\Omega\mathrm{SU}(\infty) \xrightarrow{\delta} F \rightarrow G \xrightarrow{\rho} \mathrm{SU}(\infty)$$

in which all maps are loop maps, where F is the homotopy fiber of the map ρ . For any finite CW complex Z , we obtain an exact sequence of groups

$$\tilde{K}^0(Z) \cong \tilde{K}^{-2}(Z) \xrightarrow{\delta_*} [Z, F] \rightarrow [Z, G] \rightarrow \tilde{K}^{-1}(Z).$$

In particular, if Z has only even dimensional cells, then $\tilde{K}^{-1}(Z) = 0$ and we get

$$(4.2) \quad [Z, G] \cong [Z, F]/\mathrm{Im} \delta_*.$$

In [7], the right hand side of (4.2) is identified by using the cohomology of Z under some conditions and we recall it in the special case $Z = S^3 \wedge A_i$. Let $\mathrm{SU}(\infty) \simeq C_1 \times \cdots \times C_{p-1}$ be the mod p decomposition of $\mathrm{SU}(\infty)$ where C_i is of type $(i+1, i+p, \dots, i+1+k(p-1), \dots)$.

Theorem 4.2. *Let (G, p) be as in (1.1) except for $G = G_2$ and $(G, p) = (E_7, 5)$. For given $1 \leq i \leq p-1$, we suppose the following conditions:*

(1) *for $1 \leq j \leq p-1$ with $j \equiv i+2 \pmod{p-1}$,*

$$B_j = B(2j+1, 2j+1+2(p-1), \dots, 2j+1+2r(p-1));$$

(2) *the composite $B_j \xrightarrow{\mathrm{incl}} G \xrightarrow{\rho} \mathrm{SU}(\infty) \xrightarrow{\mathrm{proj}} C_j$ is surjective in cohomology;*

(3) $\dim A_i \leq 2j-3+2(r+2)(p-1)$.

Then for $n = j + (r+1)(p-1)$, there is an injective homomorphism

$$\Phi: [S^3 \wedge A_i, F] \rightarrow H^{2n}(S^3 \wedge A_i; \mathbb{Z}_{(p)}) \oplus H^{2n+2(p-1)}(S^3 \wedge A_i; \mathbb{Z}_{(p)})$$

such that

$$\Phi(\delta_*(\xi)) = (n! \mathrm{ch}_n(\xi), (n+p-1)! \mathrm{ch}_{n+p-1}(\xi))$$

for $\xi \in \tilde{K}(S^3 \wedge A_i)$, where ch_k is the $2k$ -dimensional part of the Chern character.

Proof. To better understand the statement of the theorem, we sketch the proof in [7]. Let ρ_j be the composition $B_j \xrightarrow{\mathrm{incl}} G \xrightarrow{\rho} \mathrm{SU}(\infty) \xrightarrow{\mathrm{proj}} C_j$ and let F_j be its homotopy fiber. By hypothesis, $\tilde{H}^i(F_j; \mathbb{Z}_{(p)})$ is zero for $i \leq 2j+2(r+2)(p-1)$, except for $\tilde{H}^{2n}(F_j; \mathbb{Z}_{(p)})$ and $\tilde{H}^{2n+2(p-1)}(F_j; \mathbb{Z}_{(p)})$ which are $\mathbb{Z}_{(p)}$. Moreover, the generators a_{2n} and $a_{2n+2(p-1)}$ of $\tilde{H}^{2n}(F_j; \mathbb{Z}_{(p)})$ and $\tilde{H}^{2n+2(p-1)}(F_j; \mathbb{Z}_{(p)})$ transgress to the suspensions of Chern class c_{n+1} and c_{n+p-1} modulo some decomposables respectively. Define

$$\Phi: [S^3 \wedge A_i, F_j] \rightarrow H^{2n}(S^3 \wedge A_i; \mathbb{Z}_{(p)}) \oplus H^{2n+2(p-1)}(S^3 \wedge A_i; \mathbb{Z}_{(p)})$$

by

$$\Phi(\xi) = \xi^*(a_{2n}) \oplus \xi^*(a_{2n+2(p-1)}).$$

Then calculating as in [5] implies the asserted statement. \square

This theorem can be applied to calculate Samelson products in G . Let Z_1 and Z_2 be spaces and let $\theta_1 : Z_1 \rightarrow G$ and $\theta_2 : Z_2 \rightarrow G$ be maps. Consider the Samelson product $\langle \theta_1, \theta_2 \rangle$ and put $Z = Z_1 \wedge Z_2$. Since $\mathrm{SU}(\infty)$ is homotopy commutative, $\langle \theta_1, \theta_2 \rangle$ lifts to a map $\tilde{\theta} : Z \rightarrow F$. Then if Z is a CW-complex consisting only of even dimensional cells, the order of $\tilde{\theta}$ in the coset $[Z, F]/\mathrm{Im} \delta_*$ is equal to the order of $\langle \theta_1, \theta_2 \rangle$. In [7], a convenient lift $\tilde{\theta}$ is constructed such that, if we have the injection Φ of Theorem 4.2, then $\Phi(\tilde{\theta})$ is calculated from the Chern classes of ρ .

We now apply Theorem 4.2 to our case.

Proposition 4.3. *For $(G, p, i) = (E_6, 5, 3), (E_8, 11, 9)$, there is an injection*

$$\Phi : [S^3 \wedge A, F] \rightarrow H^{2i+4+2(p-1)}(S^3 \wedge A; \mathbb{Z}_{(p)}) \oplus H^{2i+4+4(p-1)}(S^3 \wedge A; \mathbb{Z}_{(p)}) \cong (\mathbb{Z}_{(p)})^2$$

satisfying the following, where $A = A(2i+1+2(p-1), 2i+1+4(p-1))$:

- (1) *for $(G, p, i) = (E_6, 5, 3)$, $\mathrm{Im} \Phi \circ \delta_*$ is generated by $(3 \cdot 5, 2^{-3} \cdot 5 \cdot 13)$ and $(0, 5^2)$, and a lift $\tilde{\theta}$ of $\langle \epsilon, \lambda_3 \rangle$ can be chosen so that $\Phi(\tilde{\theta}) = (2^{-3} \cdot 3^2, -2^{-6} \cdot 3^{-1} \cdot 71)$;*
- (2) *for $(G, p, i) = (E_8, 11, 9)$, is generated by $(3 \cdot 11 \cdot 17 \cdot 41, 5^4 \cdot 11 \cdot 13 \cdot 31 \cdot 61)$ and $(0, 11^2)$, and a lift $\tilde{\theta}$ of $\langle \epsilon, \lambda_3 \rangle$ can be chosen so that $\Phi(\tilde{\theta}) = (3 \cdot 7 \cdot 17 \cdot 89, 2^{-2} \cdot 3 \cdot 5 \cdot 2207 \cdot 17977)$.*

Proof. The Chern classes of ρ are calculated in [7], so we can easily check that the conditions of Theorem 4.2 are satisfied in our case. We consider the case $(G, p, i) = (E_6, 5, 3)$. Let $\bar{\rho} : \Sigma A \rightarrow B\mathrm{SU}(\infty)$ be the adjoint of $\rho \circ \lambda_i$ and put $\xi = \beta \wedge \bar{\rho} \in \tilde{K}(S^3 \wedge A)$, where β is a generator of $\tilde{K}(S^2)$. Let $\eta \in \tilde{K}(S^3 \wedge A)$ be the composite of the pinch map $S^3 \wedge A \rightarrow S^{26}$ and a generator of $\pi_{26}(BU(\infty)) \cong \mathbb{Z}_{(5)}$. Then by [7], we have

$$\mathrm{ch}(\xi) = \frac{5}{8!}u_{18} + \frac{5}{32 \cdot 11!}u_{26}, \quad \mathrm{ch}(\eta) = u_{26}.$$

Thus we get $\mathrm{Im} \Phi \circ \delta_*$ as desired. As in [7], we can calculate $\Phi(\tilde{\theta})$ by using the Chern classes of ρ to get the desired result. The case of $(G, p, i) = (E_8, 11, 9)$ is proved similarly. \square

Corollary 4.4. *We have $\gamma_3(E_6, 5) = 5^2$ and $\gamma_9(E_8, 11) = 11^2$.*

The following are proved similarly to Proposition 4.3.

Proposition 4.5. *For $(G, p, i) = (E_6, 7, 5), (E_7, 11, 7), (E_7, 13, 5), (E_8, 13, 11), (E_8, 19, 11)$, there is an injection*

$$\Phi : [S^3 \wedge A, F] \rightarrow H^{2i+2p+2}(S^3 \wedge A; \mathbb{Z}_{(p)}) \cong \mathbb{Z}_{(p)}$$

such that $\mathrm{Im} \Phi \circ \delta_$ is generated by $1 \in \mathbb{Z}_{(p)}$, where $A = A(2i+1, 2i+1+2(p-1))$. Then we have $[S^3 \wedge A, F] = 0$.*

Corollary 4.6. $\gamma_5(E_6, 7) = \gamma_7(E_7, 11) = \gamma_5(E_7, 13) = \gamma_{11}(E_8, 13) = \gamma_{11}(E_8, 19) = 1$.

Proposition 4.7. *For $(G, p, i) = (E_8, 13, 5)$, there is an injection*

$$\Phi: [S^3 \wedge A(35, 59), F] \rightarrow H^{62}(S^3 \wedge A(35, 59); \mathbb{Z}_{(13)}) \cong \mathbb{Z}_{(13)}$$

such that $\text{Im } \Phi \circ \delta_$ is generated by $13 \in \mathbb{Z}_{(13)}$ and a lift $\tilde{\theta}$ of $\langle \epsilon, \lambda_5 \rangle$ can be chosen such that $\Phi(\tilde{\theta}) = 2^{-2} \cdot 3 \cdot 5 \cdot 2207 \cdot 17977 \in \mathbb{Z}_{(13)}$.*

Corollary 4.8. $\gamma_5(E_8, 13) = 13$.

Proposition 4.9. *For $(G, p, i) = (E_8, 19, 5)$, there is an injection*

$$\Phi: \pi_{50}(F) \rightarrow H^{50}(S^{50}; \mathbb{Z}_{(19)}) \cong \mathbb{Z}_{(19)}$$

such that $\text{Im } \Phi \circ \delta_$ is generated by $19 \in \mathbb{Z}_{(19)}$ and a lift $\tilde{\theta}$ of $\langle \epsilon, \lambda_i \rangle$ can be chosen such that $\Phi(\tilde{\theta}) = -2 \cdot 3^2 \cdot 5 \cdot 11 \cdot 2861$.*

Corollary 4.10. $\gamma_5(E_8, 19) = 19$.

Proposition 4.11. $\gamma_2(E_6, 7) = 7$ and $\gamma_3(E_7, 11) = 11$.

Proof. We can prove this proposition in the same manner as above, but it needs more data from the Chern characters than that obtained in [7]. Therefore we employ another method also used in [7]. Consider the case $(G, p) = (E_6, 7)$. Recall that the mod 7 cohomology of BE_6 is given by $\mathbb{Z}/7[x_4, x_{10}, x_{12}, x_{16}, x_{18}, x_{24}]$ for $|x_i| = i$. Let $\bar{\epsilon}: S^4 \rightarrow BE_6$ and $\bar{\lambda}_2: S^{18} \rightarrow BE_6$ be the adjoints of $\epsilon: S^3 \rightarrow E_6$ and $\lambda_2: S^{17} \rightarrow E_6$, respectively. Since $\bar{\epsilon}$ is the inclusion of the bottom cell and λ_2 is the inclusion of the direct product factor, we have $\bar{\epsilon}^*(x_4) = u_4$ and $\bar{\lambda}_2^*(x_{18}) = u_{18}$ for a generator u_i of $H^i(S^i; \mathbb{Z}/7)$. By [7], we have that $\mathcal{P}^1 x_{10}$ has no linear term and includes $x_4 x_{18}$. Suppose that the Samelson product $\langle \epsilon, \lambda_2 \rangle$ is trivial. Then, taking adjoints, the Whitehead product $[\bar{\epsilon}, \bar{\lambda}_2]$ is trivial too, so there is a homotopy commutative diagram

$$\begin{array}{ccc} S^4 \vee S^{18} & \xrightarrow{\bar{\epsilon} \vee \bar{\lambda}_2} & BE_6 \\ \downarrow \text{incl} & & \parallel \\ S^4 \times S^{18} & \xrightarrow{\mu} & BE_6. \end{array}$$

Let u_i be a generator of $H^i(S^i; \mathbb{Z}/7)$. It follows that $\mu^*(x_4) = u_4 \otimes 1$ and $\mu^*(x_{18}) = u_{18}$, so

$$0 = \mathcal{P}^1 \mu^*(x_{10}) = \mu^*(\mathcal{P}^1 x_{10}) = \mu^*(x_4 x_{18}) = u_4 \otimes u_{18} \neq 0,$$

a contradiction. Thus the Samelson product $\langle \epsilon, \lambda_2 \rangle$ is non-trivial. On the other hand, by Theorems 3.1 and 4.1, $\langle \epsilon, \lambda_2 \rangle$ factors through $S^9 \subset E_6$ and $7 \cdot \pi_{20}(S^9)_{(11)} = 0$. Therefore we obtain that $\gamma_2(E_6, 7) = 7$ as desired. The case of $\gamma_3(E_7, 11)$ is similarly verified using the fact from [7] that $\mathcal{P}^1 x_{12}$ has no linear term and includes $-3x_4 x_{28}$, and the fact that the mod 11 cohomology of BE_7 is $\mathbb{Z}/11[x_4, x_{12}, x_{16}, x_{20}, x_{24}, x_{28}, x_{36}]$ for $|x_i| = i$. \square

Since F_4 is a retract of E_6 , we can deduce the following from Propositions 4.3 and 4.5.

Corollary 4.12. *Possible non-trivial $\langle \epsilon, \lambda_i \rangle$ in F_4 at the prime p are the cases $(p, i) = (5, 3), (7, 5)$ and $\gamma_3(F_4, 5) = 5^2, \gamma_5(F_4, 7) = 1$.*

5. THE CASE $(G, p) = (E_7, 7), (E_8, 7)$

We first consider the case $(G, p) = (E_7, 7)$, where $E_7 \simeq B(3, 15, 27) \times B(11, 23, 35) \times S^{19}$. As in [16], the H-spaces $B(3, 15, 27)$ and $B(11, 23, 35)$ are direct product factors of the mod p decomposition of $SU(18)$. So we can determine their low dimensional homotopy groups from those of $SU(18)$. Therefore Theorems 3.1 and 4.1 imply that $\langle \epsilon, \lambda_3 \rangle$ is null homotopic. Thus we get:

Proposition 5.1. $\gamma_3(E_7, 7) = 1$.

Let $A(11, 23, 35)$ be a homology generating subspace of $B(11, 23, 35)$.

Proposition 5.2. *For $(G, p) = (E_7, 7)$, there is an injection*

$$\Phi: [S^3 \wedge A(11, 23, 35), F] \rightarrow H^{38}(S^3 \wedge A(11, 23, 35); \mathbb{Z}_{(7)}) \cong \mathbb{Z}_{(7)}$$

such that $\text{Im } \Phi \circ \delta_$ is generated by 1, implying that $[S^3 \wedge A(11, 23, 35), F] = 0$.*

Proof. The proof is similar to that for Proposition 4.3 once we calculate $\text{Im } \Phi \circ \delta_*$. Let $\bar{\rho}: \Sigma A(11, 23, 35) \rightarrow BSU(\infty)$ be the adjoint of $\rho \circ \lambda_5: A(11, 23, 35) \rightarrow SU(\infty)$ and put $\xi = \beta \wedge \bar{\rho} \in \tilde{K}(S^3 \wedge A(11, 23, 35))$ for a generator $\beta \in \tilde{K}(S^2)$. Then by (4.2) we have

$$\text{ch}(\xi) = -\frac{1}{5!}u_{14} - \frac{9}{2 \cdot 11!}u_{26} - \frac{1229}{60 \cdot 17!}u_{38}$$

where u_i is a generator of $H^i(S^3 \wedge A(11, 23, 35); \mathbb{Z}_{(7)}) \cong \mathbb{Z}_{(7)}$ for $i = 14, 26, 38$. Thus we get $\text{Im } \Phi \circ \delta_*$ as desired. \square

Corollary 5.3. $\gamma_5(E_7, 7) = 1$.

Proposition 5.4. $\gamma_1(E_7, 7) = 7$.

Proof. By Theorems 3.1 and 4.1, the Samelson product $\langle \epsilon, \lambda_1 \rangle$ in E_7 at $p = 7$ factors through $S^{19} \subset E_7$, so it is the composite of the pinch map onto the top cell $S^3 \wedge A(3, 15, 27) \rightarrow S^{30}$ and an element of $\pi_{30}(S^{19})_{(7)}$. As $7 \cdot \pi_{30}(S^{19})_{(7)} = 0$, we get $\gamma_1(E_7, 7) \leq 7$.

Next, we show that $\langle \epsilon, \lambda_1 \rangle$ is non-trivial. Recall that the mod 7 cohomology of BE_7 is

$$H^*(BE_7; \mathbb{Z}/7) = \mathbb{Z}/7[x_4, x_{12}, x_{16}, x_{20}, x_{24}, x_{28}, x_{36}], \quad |x_i| = i.$$

If we show that $\mathcal{P}^1 x_{20}$ includes $\alpha x_4 x_{28}$ for $\alpha \neq 0$ and does not include the linear term then we may proceed as in the proof of Proposition 4.11. Let $j: \text{Spin}(11) \rightarrow E_7$ be the natural inclusion. In [7], it is shown that the generators x_i ($i = 4, 12, 20, 28$) are chosen as

$$j^*(x_4) = p_1, \quad j^*(x_{12}) = p_3 + p_2 p_1, \quad j^*(x_{20}) = p_5, \quad j^*(x_{28}) = -2p_5 p_2 + \text{other terms},$$

where $H^*(B\text{Spin}(11); \mathbb{Z}/7) = \mathbb{Z}/7[p_1, p_2, p_3, p_4, p_5]$ and p_i is the i -th Pontrjagin class. By the mod p Wu formula in [18], we can see that

$$\mathcal{P}^1 p_5 = -p_5 p_3 + p_5 p_2 p_1 + \text{other terms},$$

so for degree reasons, we get that $\mathcal{P}^1 x_{20}$ includes $-x_4 x_{28}$ and does not include the linear term. \square

We next consider the case $(G, p) = (E_8, 7)$. As in the case of $(G, p) = (E_7, 7)$, we see that $\langle \epsilon, \lambda_1 \rangle$ is trivial. Let A be a homology generating subspace of $B(23, 35, 47, 59) \subset E_8$.

Proposition 5.5. *For $(G, p) = (E_8, 7)$, there is an injection*

$$\Phi: [S^3 \wedge A, F] \rightarrow H^{50}(S^3 \wedge A; \mathbb{Z}_{(7)}) \oplus H^{62}(S^3 \wedge A; \mathbb{Z}_{(7)}) \cong (\mathbb{Z}_{(7)})^2$$

such that $\text{Im } \Phi \circ \delta_*$ is generated by $(-2^2 \cdot 3 \cdot 7 \cdot 19 \cdot 199, 5^3 \cdot 7 \cdot 13 \cdot 31 \cdot 61)$ and $(0, 7^2)$, and a lift $\tilde{\theta}$ of $\langle \epsilon, \lambda_5 \rangle$ can be chosen such that $\Phi(\tilde{\theta}) = (-2 \cdot 3^2 \cdot 5 \cdot 11 \cdot 2861, 2^{-2} \cdot 3 \cdot 5 \cdot 2207 \cdot 17977)$.

Proof. Let $\xi_1 = \beta \wedge \bar{\rho} \in \tilde{K}(S^3 \wedge A)$ be as in the proof of Proposition 5.2. Let

$$\xi_2 = \frac{1}{2^{13} \cdot 3^2}(\phi^2(\xi_1) - 2^{13}\xi_1), \quad \xi_3 = \frac{1}{2^{19} \cdot 3^2 \cdot 13}(\phi^2(\xi_2) - 2^{19}\xi_2)$$

and let ξ_4 be the composite of the pinch map $S^3 \wedge A \rightarrow S^{62}$ and a generator of $\pi_{62}(BU(\infty))$. Put $a_1 = 2 \cdot 3^2 \cdot 7, a_2 = 2^2 \cdot 5 \cdot 7 \cdot 13, a_3 = 2 \cdot 3 \cdot 7 \cdot 11 \cdot 19 \cdot 199, a_4 = 2 \cdot 5^3 \cdot 7 \cdot 11 \cdot 13 \cdot 61$. Then by [7], we have:

$$\begin{aligned} \text{ch}(\xi_1) &= \frac{a_1}{11!}u_{26} + \frac{a_2}{17!}u_{38} - \frac{a_3}{5 \cdot 23!}u_{50} + \frac{a_4}{29!}u_{62} \\ \text{ch}(\xi_2) &= \frac{7a_2}{17!}u_{38} - \frac{7 \cdot 13a_3}{23!}u_{50} + \frac{3 \cdot 7 \cdot 19 \cdot 73a_4}{29!}u_{62} \\ \text{ch}(\xi_3) &= -\frac{7^2 a_3}{23!}u_{50} + \frac{3 \cdot 5 \cdot 7^2 \cdot 19 \cdot 73a_4}{29!}u_{62} \\ \text{ch}(\xi_4) &= u_{62} \end{aligned}$$

where u_i is a generator of $H^i(S^3 \wedge A; \mathbb{Z}_{(7)}) \cong \mathbb{Z}_{(7)}$ for $i = 26, 38, 50, 62$. Then $\text{Im } \Phi \circ \delta_*$ can be calculated as asserted, and $\Phi(\tilde{\theta})$ can be calculated as in [7]. \square

Corollary 5.6. *We have $\gamma_1(E_8, 7) = 1$ and $\gamma_5(E_8, 7) = 7^2$.*

Proof of Theorem 1.1. By Propositions 2.6, 4.11, 5.1, 5.4 and Corollaries 3.5, 4.4, 4.6, 4.8, 4.10, 4.12, 5.3, 5.6, we obtain the following table:

p	5	7	11	13	17	19	23	29	31	≥ 37
$\gamma(F_4, p)$	5^2	1	1	13	1	1	1	1	1	1
$\gamma(E_6, p)$	5^2	7	1	13	1	1	1	1	1	1
$\gamma(E_7, p)$		7	11	1	1	19	1	1	1	1
$\gamma(E_8, p)$		7^2	11^2	13	1	19	1	1	31	1

Therefore $\gamma(G)$ is the product of $\gamma(G, p)$ for $p \geq 5$ ($G = F_4, E_6$) and $p \geq 7$ ($G = E_7, E_8$), so the assertion of Theorem 1.1 follows from Theorem 2.3. The G_2 case was proved in [13]. \square

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