RELATIVE HOMOTOPY ABELIAN H-SPACES

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Abstract. We introduce the notion of a relatively homotopy associative and homotopy commutative H-space, construct one for any path-connected space X, and describe several useful properties, including exponent properties.

1. Introduction

If X is a pointed, path-connected space having the homotopy type of a CW-complex, Y is a homotopy associative H-space, and $f\colon X \longrightarrow Y$ is a continuous map, then the James construction shows that f extends to an H-map $\overline{f}\colon \Omega\Sigma X \longrightarrow Y$. In particular, $\Omega\Sigma X$ is constructed functorially from X and is itself homotopy associative.

An H-space is *homotopy abelian* if it is homotopy associative and homotopy commutative. If X is as above, Z is homotopy abelian, and $f\colon X \longrightarrow Z$ is a continuous map, it is natural to ask if there is an analogue of the James construction that extends f to a homotopy abelian H-space constructed functorially from X. However, past experience shows that this is unlikely: some non-functorial p-local constructions have been given for specific cases of X in [Gra1, Grb, T2] while the analysis in [Gra3, Section 2] suggests such homotopy abelian spaces do not exist in abundance. In other words, the wrong question is being asked. In this paper we change the question, and show that there is a functorial construction that produces a space that is "as close as possible" to a homotopy abelian H-space for X.

Fix X as above. Let $\mathcal{R}(X)$ be the category whose objects are pairs (f, Z) where Z is a homotopy associative H-space, $f\colon X \longrightarrow Z$ is a map, and the Samelson product $\langle f, f\rangle$ is null homotopic. A morphism between objects (f, Z) and (f', Z') in this category is a homotopy commutative diagram

$$\begin{array}{ccc} X & \xrightarrow{f} & Z \\ \Vert & & \downarrow{\scriptstyle g} \\ X & \xrightarrow{f'} & Z' \end{array}$$

where g is an H-map. A pair $(f, Z) \in \mathcal{R}(X)$ is said to be *homotopy associative and homotopy commutative relative to* X; when the map f is understood it will more loosely said that Z is homotopy

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associative and homotopy commutative relative to X. Note that Z itself need not be homotopy commutative, it is the relative feature that is the key. Note also that if Z is homotopy commutative, then any map $X \xrightarrow{f} Z$ determines an object $(f, Z) \in \mathcal{R}(X)$.

Let $J_2(\Sigma X)$ be the second stage of the James construction for ΣX. There is a homotopy cofibration

$$\Sigma X \wedge X \xrightarrow{[1,1]} \Sigma X \xrightarrow{j} J_2(\Sigma X)$$

where $[1, 1]$ is the Whitehead product of the identity map on ΣX. Let ι be the composite

$$\iota\colon X \xrightarrow{E} \Omega\Sigma X \xrightarrow{\Omega j} \Omega J_2(\Sigma X)$$

where E is the suspension map, and note that ι is the adjoint of j.

Theorem 1.1. *Let X be a pointed, path-connected space having the homotopy type of a CW-complex. The following hold:*

(a) $(\iota, \Omega J_2(\Sigma X)) \in \mathcal{R}(X)$*;*

(b) *if $(f, Z) \in \mathcal{R}(X)$ then there exists a homotopy commutative diagram*

$$\begin{array}{ccc} X & \xrightarrow{f} & Z \\ \downarrow{\scriptstyle \iota} & \nearrow{\scriptstyle \widetilde{f}} & \\ \Omega J_2(\Sigma X) & & \end{array}$$

for some map $\widetilde{f}$;

(c) *the map $\widetilde{f}$ in part (b) may be chosen so that the composite $\Omega\Sigma X \xrightarrow{\Omega j} \Omega J_2(\Sigma X) \xrightarrow{\widetilde{f}} Z$ is an H-map.*

Theorem 1.1 says that $\Omega J_2(\Sigma X)$ is homotopy associative and homotopy commutative relative to X and it has a partial universal property with respect to other spaces that are homotopy associative and homotopy commutative relative to X. The partial in the universal property is due to the map $\widetilde{f}$ perhaps not being an H-map and perhaps not satisfy a uniqueness property. However, part (c) implies that the restriction of $\widetilde{f}$ to $\Omega\Sigma X$ is an H-map, and a property of the James construction implies that this H-map is the unique one that extends f.

We use Theorem 1.1 to examine properties in the case of homotopy fibration sequences of the form $\Omega\Sigma X \xrightarrow{\delta} T \xrightarrow{*} R \xrightarrow{\varphi} \Sigma X$ where the map $*$ is a null homotopy. Note that the null homotopy implies that T is a retract of $\Omega\Sigma X$ and the fibration sequence has a homotopy action $\Omega\Sigma X \times T \longrightarrow T$ whose restrictions to $\Omega\Sigma X$ and T are δ and the identity map respectively. Gray [Gra2, Appendix] showed that the existence of such a fibration sequence is equivalent to there being a map $\Omega\Sigma X \longrightarrow T$ having a right homotopy inverse and a homotopy action. The statement of Theorem 1.2 is phrased in terms of the fibration sequence, but could equally well be phrased in its alternative equivalent form.

Fibration sequences $\Omega\Sigma X \xrightarrow{\delta} T \xrightarrow{*} R \xrightarrow{\varphi} \Sigma X$ have been studied in many contexts: to construct finite H-spaces in [CN], to produce functorial retracts of $\Omega\Sigma X$ in [SW], to establish a universal property for particular H-spaces in [Gra1, Grb, T2], and to analyze the "bottom" indecomposable factor of $\Omega\Sigma X$ in [Gra2] in the case when ΣX is indecomposable.

Let $ev\colon \Sigma\Omega X \longrightarrow X$ be the canonical evaluation map. The *universal Whitehead product* on X is the Whitehead product $\Sigma\Omega X \wedge \Omega X \xrightarrow{[ev,ev]} X$. It is universal because the Whitehead product of any two maps $\Sigma A \longrightarrow X$ and $\Sigma B \longrightarrow X$ factors through $[ev, ev]$.

Theorem 1.2. *Fix a pointed, path-connected space X having the homotopy type of a CW-complex. Suppose that there is a homotopy fibration sequence $\Omega\Sigma X \xrightarrow{\delta} T \xrightarrow{*} R \xrightarrow{\varphi} \Sigma X$ where $*$ is null homotopic. Let t be the composite $t\colon X \xrightarrow{E} \Omega\Sigma X \xrightarrow{\delta} T$. The following hold:*

(a) *if δ is an H-map and the Whitehead product $\Sigma X \wedge X \xrightarrow{[1,1]} \Sigma X$ lifts through φ, then T is homotopy associative and homotopy commutative, and there is a homotopy commutative diagram*

$$\begin{array}{ccc} \Omega\Sigma X & \xrightarrow{\delta} & T \\ \big\downarrow{\scriptstyle \Omega j} & \nearrow{\scriptstyle \widetilde{\delta}} & \\ \Omega J_2(\Sigma X) & & \end{array}$$

where $\widetilde{\delta}$ has a right homotopy inverse;

(b) *if φ factors through the universal Whitehead product on ΣX, then for any $(f, Z) \in \mathcal{R}(X)$ there is a homotopy commutative diagram*

$$\begin{array}{ccc} X & \xrightarrow{f} & Z \\ \big\downarrow{\scriptstyle t} & \nearrow{\scriptstyle \widetilde{f}} & \\ T & & \end{array}$$

where $\widetilde{f}$ is an H-map;

(c) *if the hypotheses of parts (a) and (b) both hold then $\widetilde{f}$ is the unique H-map, up to homotopy, such that $\widetilde{f} \circ t \simeq f$.*

Part (a) of Theorem 1.2 gives a criterion for determining when T is homotopy associative and homotopy commutative. This is shown to be equivalent to three other criteria in Section 4, one of which first appeared in [T1] and has been used to establish homotopy associativity and homotopy commutativity in specific cases [Gra1, Grb, T2]. Part (b) of Theorem 1.2 ideally holds together with part (a), as it does in the cases in [Gra1, Grb, T2], but it may hold independently. An example is when ΣX is the odd primary Moore space $P^{2n+1}(p^r)$; the space T was constructed in [CMN2] and shown to be neither homotopy associative nor homotopy commutative, but as φ factors through Whitehead products it will satisfy part (b).

Gray [Gra3] proved statements analogous to those in Theorem 1.2 (b) and (c) in the absolute case when Z is a homotopy associative and homotopy commutative H-space. Theorem 1.2 is therefore a generalization to the relative case. By [GTW], looped co-H-spaces have many of the same properties as looped suspensions. It would be interesting to see if the results in Theorems 1.1 and 1.2 generalize in some manner to co-H-spaces.

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2. Some properties of homotopy associative H-spaces

This section establishes some preliminary results regarding the James construction and projective planes in the context of homotopy associative H-spaces. From this point forward we make the global hypotheses that all spaces are pointed, path-connected, and have the homotopy type of CW-complexes.

2.1. **The James construction.** For $k \geq 1$, let $X^{\times k}$ be the k-fold product of X with itself and let $X^{\wedge k}$ be the k-fold smash product of X with itself. Define $J_k(X)$ as the quotient space $X^{\times k}/\sim$ where the basepoint is allowed to move freely. That is, $(x_1, \ldots, x_{i-1}, *, x_{i+1}, \ldots, x_k) \sim (x_1, \ldots, *, x_{i-1}, x_{i+1}, \ldots, x_k)$. Note that $J_1(X) = X$. Let

$$q_k \colon X^{\times k} \longrightarrow J_k(X)$$

be the quotient map. There is a map $J_k(X) \longrightarrow J_{k+1}(X)$ given by sending $(x_1, \ldots, x_k)$ to $(x_1, \ldots, x_k, *)$. Let $J(X) = \mathrm{colim} J_k(X)$ and observe that $J(X)$ has an associative multiplication defined by concatenation of sequences.

Theorem 2.1 (James [J])**.** *There are homotopy equivalences $J(X) \simeq \Omega\Sigma X$ and*

$$\Sigma J(X) \simeq \bigvee_{k=1}^{\infty} \Sigma X^{\wedge k}. \qquad \square$$

In Theorem 2.4 we will show that the James construction has a universal property with respect to homotopy associative H-spaces. To prepare, note that James used a particular choice of a homotopy equivalence $\Sigma J(X) \simeq \bigvee_{k=1}^{\infty} \Sigma X^{\wedge k}$. For our purposes, it will be more convenient to use a different choice. Let $j_k \colon J_k(X) \longrightarrow J(X)$ be the inclusion and let e_k be the composite

$$e_k \colon X^{\times k} \xrightarrow{q_k} J_k(X) \xrightarrow{j_k} J(X).$$

Since the quotient map $X^{\times k} \longrightarrow X^{\wedge k}$ has a right homotopy inverse after suspending, we obtain a composite

$$\psi_k \colon \Sigma X^{\wedge k} \longrightarrow \Sigma X^{\times k} \xrightarrow{\Sigma e_k} \Sigma J(X).$$

Taking the wedge sum of the maps ψ_k for $k \geq 1$ gives a map

$$\psi \colon \bigvee_{k=1}^{\infty} \Sigma X^{\wedge k} \longrightarrow \Sigma J(X).$$

Theorem 2.2. *The map ψ is a homotopy equivalence.*

Proof. Take homology with field coeffiicients. By the homotopy equivalence $J(X) \simeq \Omega\Sigma X$ of Theorem 2.1 and the Bott-Samelson Theorem, there is an algebra isomorphism $H_*(J(X)) \cong T(\widetilde{H}_*(X))$ where $T(\quad)$ is the free tensor algebra functor. Observe that $(\psi_k)_*$ has image isomorphic to the suspension of the submodule of length k tensors in $T(\widetilde{H}_*(X))$. Thus ψ_* is an isomorphism in homology. This holds for homology with coefficients in any field so ψ induces an isomorphism in integral homology, and hence is a homotopy equivalence by Whitehead's Theorem. $\square$

We now prove a criterion for when two maps out of $J(X)$ are homotopic.

Lemma 2.3. *Let Y be an H-space and suppose that there are maps $f, g\colon J(X) \longrightarrow Y$. If the composites $X^{\times k} \xrightarrow{e_k} J(X) \xrightarrow{f} Y$ and $X^{\times k} \xrightarrow{e_k} J(X) \xrightarrow{g} Y$ are homotopic for each $k \geq 1$, then f is homotopic to g.*

Proof. Since Y is an H-space, by [Su] it retracts off $\Omega\Sigma Y$, so to show that $f \simeq g$ it suffices to show that $\Sigma f \simeq \Sigma g$. By hypothesis, the composites $X^{\times k} \xrightarrow{e_k} J(X) \xrightarrow{f} Y$ and $X^{\times k} \xrightarrow{e_k} J(X) \xrightarrow{g} Y$ are homotopic for all $k \geq 1$. Therefore, by the definition of ψ_k, the composites $\Sigma X^{\wedge k} \xrightarrow{\psi_k} \Sigma J(X) \xrightarrow{\Sigma f} \Sigma Y$ and $\Sigma X^{\wedge k} \xrightarrow{\psi_k} \Sigma J(X) \xrightarrow{\Sigma g} \Sigma Y$ are homotopic for all $k \geq 1$. Taking the wedge sum of the maps ψ_k then implies that the composites $\bigvee_{k=1}^{\infty} \Sigma X^{\wedge k} \xrightarrow{\psi} \Sigma J(X) \xrightarrow{\Sigma f} \Sigma Y$ and $\bigvee_{k=1}^{\infty} \Sigma X^{\wedge k} \xrightarrow{\psi} \Sigma J(X) \xrightarrow{\Sigma g} \Sigma Y$ are homotopic. But ψ is a homotopy equivalence by Theorem 2.2, implying that $\Sigma f \simeq \Sigma g$. $\square$

Theorem 2.4 is something of a folk theorem: it is widely accepted as true but as far as the authors are aware there is no proof in the literature.

Theorem 2.4. *Let Z be a homotopy associative H-space. Suppose that there is a map $f\colon X \longrightarrow Z$. Then there is an extension*

$$\begin{array}{ccc} X & \xrightarrow{f} & Z \\ \downarrow{\scriptstyle j_1} & \nearrow{\scriptstyle \widetilde{f}} & \\ J(X) & & \end{array}$$

where $\widetilde{f}$ is an H-map, and it is the unique H-map, up to homotopy, with the property that $\widetilde{f} \circ j_1 \simeq f$.

Proof. The existence of an extension which is an H-map was asserted by Stasheff [St], and a proof can be found in [N, Lemma 1.4]. It remains to show uniqueness. Suppose that $\widetilde{f}, \widetilde{g}\colon J(X) \longrightarrow Z$ are two H-maps satisfying $\widetilde{f} \circ j_1 \simeq f \simeq \widetilde{g} \circ j_1$. Fix $k \geq 1$ and consider the diagram

$$\begin{array}{ccccc} X^{\times k} & \xrightarrow{j_1^{\times k}} & J(X)^{\times k} & \xrightarrow{\widetilde{f}^{\times k}} & Z^{\times k} \\ \downarrow{\scriptstyle q_k} & & \downarrow{\scriptstyle \mu_k} & & \downarrow{\scriptstyle m_k} \\ J_k(X) & \xrightarrow{j_k} & J(X) & \xrightarrow{\widetilde{f}} & Z \end{array}$$

where μ_k is the k-fold multiplication of $J(X)$ with itself and m_k is the k-fold iterated multiplication on Z. The left square strictly commutes by the concatenation multiplication on $J(X)$. The right square homotopy commutes since $\widetilde{f}$ is an H-map. Observe that the top row is homotopic to $f^{\times k}$. The homotopy commutativity of the diagram implies that $\widetilde{f} \circ j_k \circ q_k \simeq m_k \circ f^{\times k}$. By definition, $e_k = j_k \circ q_k$. Thus $\widetilde{f} \circ e_k \simeq m_k \circ f^{\times k}$. Similarly, we obtain $\widetilde{g} \circ e_k \simeq m \circ f^{\times k}$. Thus $\widetilde{f} \circ e_k \simeq \widetilde{g} \circ e_k$. As this is true for all $k \geq 1$, Lemma 2.3 implies that $\widetilde{f} \simeq \widetilde{g}$. □

Finally, it is useful to reformulate Theorem 2.4 in terms of $\Omega\Sigma X$ instead of $J(X)$. Let $E\colon X \longrightarrow \Omega\Sigma X$ be the suspension map, which is adjoint to the identity map on ΣX.

Theorem 2.5. *Let Z be a homotopy associative H-space. Suppose that there is a map $f\colon X \longrightarrow Z$. Then there is an extension*

$$\begin{array}{ccc} X & \xrightarrow{f} & Z \\ \downarrow{\scriptstyle E} & \nearrow{\scriptstyle \overline{f}} & \\ \Omega\Sigma X & & \end{array}$$

where $\overline{f}$ is an H-map, and it is the unique H-map, up to homotopy, with the property that $\overline{f} \circ E \simeq f$.

Proof. Applying Theorem 2.4 to the suspension $X \xrightarrow{E} \Omega\Sigma X$ gives an H-map $\overline{E}\colon J(X) \longrightarrow \Omega\Sigma X$ such that $\overline{E} \circ j_1 \simeq E$. James [J, Sections 3 and 4] showed that $\overline{E}$ is a homotopy equivalence. If g is the inverse homotopy equivalence for $\overline{E}$, then g is an H-map and $j_1 \simeq g \circ E$. Let $\overline{f}$ be the composite $\overline{f}\colon \Omega\Sigma X \xrightarrow{g} J(X) \xrightarrow{\widetilde{f}} Z$ where $\widetilde{f}$ is the map from Theorem 2.4. Then $\overline{f}$ is an H-map since it is a composite of H-maps, we have $\overline{f} \circ E = \widetilde{f} \circ g \circ E \simeq \widetilde{f} \circ j_1 \simeq f$, and the uniqueness property for $\overline{f}$ follows from that of $\widetilde{f}$. □

Applying Theorem 2.5 to the identity map on Z recovers a result of Stasheff [St].

Corollary 2.6 (Stasheff)**.** *If Z is a homotopy associative H-space then there is an H-map $\partial\colon \Omega\Sigma Z \longrightarrow Z$ with the property that $\partial \circ E$ is homotopic to the identity map on Z.* □

Note for comparison purposes that Sugawara's result mentioned in the proof of Lemma 2.3 shows that if Y is an H-space then there is a map $r\colon \Omega\Sigma Y \longrightarrow Y$ with a right homotopy inverse. However, r need not be an H-map. Corollary 2.6 implies that if Y is homotopy associative then there is a choice of r which is an H-map.

2.2. **The projective plane.** For spaces A and B, the *join* is defined as the quotient space $A * B = (A \times I \times B)/\sim$, where $I = [0,1]$ is the unit interval, $(a,0,b) \sim (a',0,b)$ for all $a, a' \in A$, and $(a,1,b) \sim (a,1,b')$ for all $b, b' \in B$. It is well known that there is a natural homotopy equivalence $A * B \simeq \Sigma A \wedge B$.

If Z is an H-space then there are different "Hopf constructions" $Z * Z \longrightarrow \Sigma Z$. If Z is also homotopy associative then a choice of Hopf construction can be made so that the H-map $\Omega\Sigma Z \xrightarrow{\partial} Z$ in Corollary 2.6 appears as the fibration connecting map.

Lemma 2.7. *Let Z be a homotopy associative H-space. Then there is a choice of a map $Z * Z \xrightarrow{m^*} \Sigma Z$ such that the H-map $\Omega\Sigma Z \longrightarrow Z$ in Corollary 2.6 fits in a homotopy fibration sequence*

$$\Omega\Sigma Z \xrightarrow{\partial} Z \longrightarrow Z * Z \xrightarrow{m^*} \Sigma Z.$$

Proof. Gray [Gra2, Proposition A1] showed that there is a one-to-one correspondence between homotopy classes of maps

$$\theta\colon A \times F \longrightarrow F$$

with $\theta(*, x) = x$ and fibre homotopy classes of fibrations

$$F \longrightarrow E' \longrightarrow \Sigma A.$$

Given such a fibration there is a canonical homotopy action $a\colon \Omega\Sigma A \times F \longrightarrow F$ where the restriction of a to $\Omega\Sigma A$ is the connecting map for the fibration and the restriction of a to F is the identity map. The map θ is obtained as the composite $A \times F \xrightarrow{E\times 1} \Omega\Sigma A \times F \xrightarrow{a} F$. Further, Gray [Gra2, Proposition A.2] showed that if the restriction of θ to A is a homotopy equivalence, then $E' \simeq F * F$. Gray [Gra2, Proposition A.3] also showed that if $\Omega\Sigma A$ is replaced by the James construction $J(A)$, then $J(A) \times F \xrightarrow{a} F$ is described by the formula $a((a_1, \ldots, a_k), x) = \theta(a_1, \theta(a_2, \ldots, \theta(a_k, x) \ldots)$.

In particular, if Z is an H-space with multiplication m, take $\theta\colon Z \times Z \longrightarrow Z$ to be m. Then the restriction of m to either factor of Z is homotopic to the identity map so we obtain a homotopy fibration

$$Z \longrightarrow Z * Z \xrightarrow{m^*} \Sigma Z \tag{1}$$

which can be termed a "Hopf construction". Moreover, if $a\colon \Omega\Sigma Z \times Z \longrightarrow Z$ is the canonical homotopy action associated to this fibration then the composite $Z \times Z \xrightarrow{E\times 1} \Omega\Sigma Z \times Z \xrightarrow{a} Z$ is θ. Replacing $\Omega\Sigma Z$ by $J(Z)$, the map a satisfies the formula $a((z_1, \ldots, z_k), x) = \theta(z_1, \theta(z_2, \ldots, \theta(z_k, x) \ldots)$. That is, as $\theta = m$, $a((z_1, \ldots, z_k), x) = m(z_1, m(z_2, \ldots, m(z_k, x) \ldots)$.

Now suppose that the multiplication m on Z is homotopy associative. Then the order of the multiplication in the formula for a is irrelevant so we may write $a((z_1, \ldots, z_k), x) = z_1 z_2 \cdots z_k x$. The restriction of a to $\Omega\Sigma Z$ is the connecting map $\partial'\colon \Omega\Sigma Z \longrightarrow Z$ for the homotopy fibration (1). The formula for a therefore implies that, regarding $\Omega\Sigma Z$ as $J(Z)$, we have $\partial'(z_1, \ldots, z_k) = z_1 \cdots z_k$. But if we regard the H-map ∂ as a map $J(Z) \longrightarrow Z$ then we also obtain $\partial(z_1, \ldots, z_k) = z_1 \cdots z_k$. Thus $\partial' \simeq \partial$. That is, the connecting map for (1) is homotopic to ∂. □

Define the projective plane $P_2(Z)$ and the map i by the homotopy cofibration

$$Z * Z \xrightarrow{m^*} \Sigma Z \xrightarrow{i} P_2(Z).$$

We will show that $\Omega\partial$ factors through Ωi.

Lemma 2.8. *Let Z be a homotopy associative H-space. Then there is a homotopy commutative diagram*

$$\begin{array}{ccc} \Omega\Sigma Z & \xrightarrow{\partial} & Z \\ \downarrow{\scriptstyle \Omega i} & & \| \\ \Omega P_2(Z) & \xrightarrow{\partial_2} & Z \end{array}$$

where ∂ is the H-map in Theorem 2.6 and ∂_2 is some map.

Proof. This uses the Dold-Lashof construction [DL]. In general, let $A \xrightarrow{f} B \longrightarrow C$ be a cofibration and suppose $F \longrightarrow E \xrightarrow{g} B$ is a quasifibration. Let Q be the homotopy pullback of f and g. If there is a trivialization $Q \simeq A \times F$ then there is a quasifibration $F \longrightarrow E' \longrightarrow C$ and a homotopy pullback

$$\begin{array}{ccc} E & \longrightarrow & B \\ \downarrow & & \downarrow \\ E' & \longrightarrow & C \end{array}$$

where E' is the homotopy pushout of $A \times F \longrightarrow E$ and the projection $A \times F \longrightarrow F$.

In our case, let Q be the homotopy pullback of m^* and itself. Stasheff [St] showed that the homotopy associativity of Z implies that there is a trivialization $Q \simeq (Z * Z) \times Z$. Thus the Dold-Lashof construction gives a homotopy pullback

$$\begin{array}{ccc} Z * Z & \xrightarrow{m^*} & \Sigma Z \\ \downarrow & & \downarrow{\scriptstyle i} \\ E'' & \longrightarrow & P_2(Z) \end{array}$$

where E'' is the homotopy pushout of the map $(Z*Z)\times Z \longrightarrow Z*Z$ and the projection $(Z*Z)\times Z \longrightarrow Z$. This pullback induces a homotopy fibration diagram

$$\begin{array}{ccccccc} \Omega\Sigma Z & \xrightarrow{\partial} & Z & \longrightarrow & Z * Z & \xrightarrow{m^*} & \Sigma Z \\ \downarrow{\scriptstyle \Omega i} & & \| & & \downarrow & & \downarrow{\scriptstyle i} \\ \Omega P_2(Z) & \xrightarrow{\partial_2} & Z & \longrightarrow & E'' & \longrightarrow & P_2(Z) \end{array}$$

for some map ∂_2, where Lemma 2.7 has been used to identify the map ∂ in the top row. The lefthand square is the one asserted by the lemma. □

2.3. **The projective plane and the second stage of the James construction.** For a space X, let

$$[1,1]\colon \Sigma X \wedge X \longrightarrow \Sigma X$$

be the Whitehead product of the identity map on ΣX with itself. Then there is a homotopy cofibration

$$\Sigma X \wedge X \xrightarrow{[1,1]} \Sigma X \xrightarrow{j} J_2(\Sigma X) \qquad (2)$$

where we write j for the inclusion of of $\Sigma X = J_1(\Sigma X)$ into $J_2(\Sigma X)$.

Lemma 2.9. *Let Z be a homotopy associative H-space. If $f\colon X \longrightarrow Z$ is a map with the property that the Samelson product $\langle f, f\rangle$ is null homotopic then there is a homotopy commutative diagram*

$$\begin{array}{ccc} \Sigma X \wedge X & \xrightarrow{\lambda} & Z * Z \\ \big\downarrow{\scriptstyle [1,1]} & & \big\downarrow{\scriptstyle m^*} \\ \Sigma X & \xrightarrow{\Sigma f} & \Sigma Z. \end{array}$$

for some map λ.

Proof. Taking the adjoint of $\Sigma f \circ [1,1]$ and composing with the H-map $\Omega\Sigma Z \xrightarrow{\partial} Z$ in Theorem 2.6 gives the composite

$$\gamma\colon X \wedge X \xrightarrow{E} \Omega(\Sigma X \wedge X) \xrightarrow{\Omega[1,1]} \Omega\Sigma X \xrightarrow{\Omega\Sigma f} \Omega\Sigma Z \xrightarrow{\partial} Z.$$

If γ is null homotopic then from the homotopy fibration in Lemma 2.7 we otain a lift

$$\begin{array}{ccccc} X \wedge X & & \overset{\lambda'}{\dashrightarrow} & & \Omega(Z * Z) \\ \big\downarrow{\scriptstyle E} & & & & \big\downarrow{\scriptstyle \Omega m^*} \\ \Omega(\Sigma X \wedge X) & \xrightarrow{\Omega[1,1]} & \Omega\Sigma X & \xrightarrow{\Omega\Sigma f} & \Omega\Sigma Z \end{array}$$

for some map λ'. Taking adjoints then gives the asserted homotopy commutative diagram.

It remains to show that γ is null homotopic. Observe that $\Omega[1,1] \circ E$ is the adjoint of the Whitehead product $[1,1]$, which is the Samelson product $\langle E, E\rangle$. Consider the string of homotopies

$$\partial \circ \Omega\Sigma f \circ \langle E, E\rangle \simeq \langle \partial \circ \Omega\Sigma f \circ E, \partial \circ \Omega\Sigma f \circ E\rangle \simeq \langle \partial \circ E \circ f, \partial \circ E \circ f\rangle \simeq \langle f, f\rangle.$$

From left to right, the first homotopy holds since the Samelson product is natural with respect to composition with H-maps on the left, and both $\Omega\Sigma f$ and ∂ are H-maps. The second holds by the naturality of E, and the third holds since, by Theorem 2.6, $\partial \circ E$ is homotopic to the identity map on Z. Thus $\gamma \simeq \langle f, f\rangle$, but by hypothesis, $\langle f, f\rangle$ is null homotopic. □

Corollary 2.10. *Let Z be a homotopy associative H-space. If $f\colon X \longrightarrow Z$ is a map with the property that the Samelson product $\langle f, f\rangle$ is null homotopic then:*

(a) *there is an extension*

$$\begin{array}{ccccc} \Sigma X & \xrightarrow{\Sigma f} & \Sigma Z & \xrightarrow{i} & P_2(Z) \\ \big\downarrow{\scriptstyle j} & & & \nearrow{\scriptstyle g} & \\ J_2(\Sigma X) & & & & \end{array}$$

for some map g;

(b) *there is a homotopy commutative square*

$$\begin{array}{ccc} \Omega\Sigma X & \xrightarrow{\Omega\Sigma f} & \Omega\Sigma Z \\ \downarrow{\scriptstyle \Omega j} & & \downarrow{\scriptstyle \partial} \\ \Omega J_2(\Sigma X) & \xrightarrow{\partial_2\circ\Omega g} & Z. \end{array}$$

Proof. Consider the diagram

$$\begin{array}{ccccc} \Sigma X\wedge X & \xrightarrow{\lambda} & Z * Z & & \\ \downarrow{\scriptstyle [1,1]} & & \downarrow{\scriptstyle m^*} & \searrow{\scriptstyle *} & \\ \Sigma X & \xrightarrow{\Sigma f} & \Sigma Z & \xrightarrow{i} & P_2(Z). \end{array}$$

The left square homotopy commutes by Lemma 2.9 and the right triangle homotopy commutes since i and m^* are consecutive maps in a homotopy cofibration. The diagram as a whole implies that the composite $i\circ\Sigma f\circ[1,1]$ is null homotopic. The existence of the extension asserted in part (a) now follows immediately.

For part (b), loop the square in part (a) and compose with ∂_2 to obtain $\partial_2\circ\Omega i\circ\Omega\Sigma f\simeq\partial_2\circ\Omega g\circ\Omega j$. By Lemma 2.8, $\partial\simeq\partial_2\circ\Omega i$, so we obtain $\partial\circ\Omega\Sigma f\simeq\partial_2\circ\Omega g\circ\Omega j$ as asserted. □

3. The proof of Theorem 1.1

Let X be a space and let $ev\colon\Sigma\Omega X\longrightarrow X$ be the evaluation map. Recall from the Introduction that the *universal Whitehead product* is the Whitehead product $\Sigma\Omega X\wedge\Omega X\xrightarrow{[ev,ev]}X$. It will be helpful to write this as a composite.

In general, let X and Y be spaces. Let ev_1 and ev_2 be the composites

$$ev_1\colon\Sigma\Omega X\xrightarrow{ev}X\xrightarrow{i_1}X\vee Y$$

$$ev_2\colon\Sigma\Omega Y\xrightarrow{ev}Y\xrightarrow{i_2}X\vee Y,$$

where i_1 and i_2 are the inclusions of the left and right wedge summands respectively. By [Ga], there is a homotopy fibration

$$\Sigma\Omega X\wedge\Omega Y\xrightarrow{[ev_1,ev_2]}X\vee Y\longrightarrow X\times Y$$

where the right map is the inclusion of the wedge into the product. When $X=Y$ there is a fold map $\nabla\colon X\vee X\longrightarrow X$. Let Ψ be the composite

$$\Psi\colon\Sigma\Omega X\wedge\Omega X\xrightarrow{[ev_1,ev_2]}X\vee X\xrightarrow{\nabla}X.$$

Notice that Ψ is homotopic to the universal Whitehead product $[ev,ev]$.

The key property of the universal Whitehead product that will be needed is the following.

Lemma 3.1. *Let X be a simply-connected space. Then the composite $\Sigma\Omega X\wedge\Omega X\xrightarrow{\Psi}X\xrightarrow{j}J_2(X)$ is null homotopic.*

Proof. By its definition in the James construction, $J_2(X)$ is the quotient space $(X \times X)/\sim$, where $(x, *) \sim (*, x)$. In particular, this implies that the composite $X \vee X \longrightarrow X \times X \longrightarrow J_2(X)$ is homotopic to the composite $X \vee X \xrightarrow{\nabla} X \xrightarrow{j} J_2(X)$. We obtain a homotopy commutative diagram

$$\begin{array}{ccccc} \Sigma\Omega X \wedge \Omega X & \xrightarrow{[ev_1, ev_2]} & X \vee X & \longrightarrow & X \times X \\ & & \downarrow \nabla & & \downarrow \\ & & X & \xrightarrow{j} & J_2(X) \end{array}$$

where the top row is a homotopy fibration. As $\Psi \simeq \nabla \circ [ev_1, ec_2]$, the lower direction around the diagram is $j \circ \Psi$, and the upper direction around the diagram is null homotopic since the top row is a homotopy fibration. Thus $j \circ \Psi$ is null homotopic. $\square$

Let

$$c\colon \Omega\Sigma X \wedge \Omega\Sigma X \longrightarrow \Omega\Sigma X$$

be the Samelson product of the identity map on $\Omega\Sigma X$ with itself. Observe that the adjoint of c is the universal Whitehead product on ΣX. That is, c is homotopic to the composition $\Omega\Sigma X \wedge \Omega\Sigma X \xrightarrow{E} \Omega(\Sigma\Omega\Sigma X \wedge \Omega\Sigma X) \xrightarrow{\Omega\Psi} Z$. Taking adjoints in Lemma 3.1 immediately implies the following.

Lemma 3.2. *For any path-connected space X, the composite $\Omega\Sigma X \wedge \Omega\Sigma X \xrightarrow{c} \Omega\Sigma X \xrightarrow{\Omega j} \Omega J_2(\Sigma X)$ is null homotopic.* $\square$

Fix a space X. Recall from the Introduction that $\mathcal{R}(X)$ is the category whose objects are pairs (f, Z) where Z is a homotopy associative H-space, $f\colon X \longrightarrow Z$ is a map, and the Samelson product $\langle f, f\rangle$ is null homotopic. Also, an object in $\mathcal{R}(X)$ is called a homotopy associative and homotopy commutative H-space relative to X.

Proof of Theorem 1.1. For part (a) we need to show that $(\iota, \Omega J_2(\Sigma X)) \in \mathcal{R}(X)$, where ι is the composite $\iota\colon X \xrightarrow{E} \Omega\Sigma X \xrightarrow{\Omega j} \Omega J_2(\Sigma X)$. Clearly $\Omega J_2(\Sigma X)$ is homotopy associative. The naturality of the Samelson product implies that $\langle \iota, \iota\rangle \simeq \Omega j \circ \langle E, E\rangle$. As $\langle E, E\rangle$ factors through the Samelson product c, Lemma 3.2 implies that $\langle \iota, \iota\rangle$ is null homotopic. Hence $(\iota, \Omega J_2(\Sigma X) \in \mathcal{R}(X)$.

Next, consider the diagram

$$\begin{array}{ccc} X & \xrightarrow{f} & Z \\ \downarrow E & & \downarrow E \\ \Omega\Sigma X & \xrightarrow{\Omega\Sigma f} & \Omega\Sigma Z \\ \downarrow \Omega j & & \downarrow \partial \\ \Omega J_2(\Sigma X) & \xrightarrow{\partial_2 \circ \Omega g} & Z. \end{array}$$

The upper square homotopy commutes by the naturality of E while the lower square homotopy commutes by Corollary 2.10 (b). By Corollary 2.6, $\partial \circ E$ is homotopic to the identity on Z so the upper direction around the diagram is homotopic to f. The left column is the definition of ι. So setting $\widetilde{f} = \partial_2 \circ \Omega g$, the homotopy commutativity of the diagram implies that $f \simeq \widetilde{f} \circ \iota$, proving part (b).

Finally, the homotopy commutativity of the bottom square in the previous diagram implies that $\widetilde{f} \circ \Omega j = \partial_2 \circ \Omega g \circ \Omega j$ is homotopic to $\partial \circ \Omega\Sigma f$. The latter is an H-map since $\Omega\Sigma f$ is and, by Corollary 2.6, so is ∂. That is, $\widetilde{f} \circ \Omega j$ is an H-map, proving part (c). $\square$

As a closing remark, an interesting special case is when Z is a homotopy associative, homotopy-commutative H-space. Take $X = Z$ and f to be the identity map. Then $\langle f, f\rangle = \langle 1, 1\rangle$, which is null homotopic since Z is homotopy commutative. Therefore Corollary 2.10 (a) and Theorem 1.1 (b) imply the following.

Corollary 3.3. *Let Z be a homotopy associative, homotopy commutative H-space. Then there are homotopy commutative diagrams*

$$\begin{array}{ccc} \Sigma Z & \xrightarrow{i} & P_2(Z) \\ \downarrow{\scriptstyle j} & \nearrow & \\ J_2(\Sigma Z) & & \end{array} \qquad\qquad \begin{array}{ccc} Z & = & Z \\ \downarrow{\scriptstyle j} & \nearrow & \\ \Omega J_2(\Sigma Z). & & \end{array}$$

$\square$

The left triangle in Corollary 3.3 recovers a result of Stasheff [St, Theorem 1.9], while the retraction of Z off of $\Omega J_2(\Sigma Z)$ in the right triangle recovers a result of Williams [Wi].

4. Criteria for homotopy associativity and homotopy commutativity

The purpose of this section is to prove Theorem 4.2, which gives four equivalent criteria for certain spaces to be homotopy associative and homotopy commutative. We begin with an initial result.

Suppose that T retracts off ΩX, so there are maps $s\colon T \longrightarrow \Omega X$ and $\delta\colon \Omega X \xrightarrow{\delta} T$ such that $\delta \circ s$ is homotopic to the identity map on T. Define a multiplication m on T by the composite

$$m\colon T \times T \xrightarrow{s\times s} \Omega X \times \Omega X \xrightarrow{\mu} \Omega X \xrightarrow{\delta} T$$

where μ is the standard loop multiplication.

Lemma 4.1. *If δ is an H-map then the multiplication m on T is homotopy associative.*

Proof. Since δ is an H-map with s as a right homotopy inverse, there is a homotopy commutative diagram

$$\begin{array}{ccccccc} T\times T\times T & \xrightarrow{s\times s\times s} & \Omega X\times\Omega X\times\Omega X & \xrightarrow{\mu\times 1} & \Omega X\times\Omega X & \xrightarrow{\mu} & \Omega X \\ & \searrow\!\!\searrow & \downarrow{\scriptstyle \delta\times\delta\times\delta} & & \downarrow{\scriptstyle \delta\times\delta} & & \downarrow{\scriptstyle \delta} \\ & & T\times T\times T & \xrightarrow{m\times 1} & T\times T & \xrightarrow{m} & T. \end{array}$$

Thus $m \circ (m \times 1) \simeq \partial \circ \mu \circ (\mu \times 1) \circ (s \times s \times s)$. Similarly, $m \circ (1 \times m) \simeq \partial \circ \mu \circ (1 \times \mu) \circ (s \times s \times s)$. But as $\mu \circ (\mu \times 1) \simeq \mu \circ (1 \times \mu)$ we obtain $m \circ (m \times 1) \simeq m \circ (1 \times m)$. $\square$

Now we turn to equivalent criteria for homotopy associativity and homotopy commutativity.

Theorem 4.2. *Let $\Omega\Sigma X \xrightarrow{\delta} T \xrightarrow{*} R \xrightarrow{\varphi} \Sigma X$ be a homotopy fibration sequence where $*$ is null homotopic. Then the following are equivalent:*

(a) *there is a lift of the universal Whitehead product*

$$\begin{array}{ccc} & & \Sigma\Omega\Sigma X \wedge \Omega\Sigma X \\ & \swarrow & \downarrow \Psi \\ R & \xrightarrow{\varphi} & \Sigma X; \end{array}$$

(b) *the space T is homotopy associative and homotopy commutative, and the map δ is an H-map;*

(c) *the map $\Omega\Sigma X \xrightarrow{\delta} T$ is an H-map and there is a lift*

$$\begin{array}{ccc} & & \Sigma X \wedge X \\ & \swarrow & \downarrow [1,1] \\ R & \xrightarrow{\varphi} & \Sigma X. \end{array}$$

(d) *there is an extension*

$$\begin{array}{ccc} \Omega\Sigma X & \xrightarrow{\delta} & T \\ \downarrow & \nearrow & \\ \Omega J_2(\Sigma X). & & \end{array}$$

Proof. We show that (a) implies (b), (b) implies (c), (c) implies (d) and (d) implies (a).

Part (a) implies part (b). This was first proved in [T1]; a slick proof can be found in [Gra4, Proposition 2.9].

Part (b) implies part (c). Part (b) assumes that δ is an H-map so it remains only to show that the map $\Sigma X \wedge X \xrightarrow{[1,1]} \Sigma X$ lifts through $R \xrightarrow{\varphi} \Sigma X$. Taking adjoints, it is equivalent to show that the Samelson product $\langle E, E\rangle$ lifts through $\Omega R \xrightarrow{\Omega\varphi} \Omega\Sigma X$, which in turn is equivalent to showing that the composite $\delta \circ \langle E, E\rangle$ is null homotopic. By hypothesis, T is homotopy associative and δ is an H-map so by the naturality of the Samelson product we obtain $\delta \circ \langle E, E\rangle \simeq \langle \delta \circ E, \delta \circ E\rangle$. By hypothesis, T is also homotopy commutative, so the Samelson product $\langle \delta \circ E, \delta \circ E\rangle$ is null homotopic. That is, $\delta \circ \langle E, E\rangle$ is null homotopic, as required.

Part (c) implies part (d). Since δ is an H-map, by Lemma 4.1, T is homotopy associative. Let f be the composite $f\colon X \xrightarrow{E} \Omega\Sigma X \xrightarrow{\delta} T$. The Samelson product is natural with respect to composition with H-maps on the left, so $\langle f, f\rangle$ factors as the composite $X \wedge X \xrightarrow{\langle E,E\rangle} \Omega\Sigma X \xrightarrow{\delta} T$. Since $\langle E, E\rangle$ is the adjoint of the Whitehead product $[1, 1]$, by hypothesis it lifts through $\Omega\varphi$ to ΩR. Therefore $\langle f, f\rangle$ is null homotopic. Applying Theorem 1.1 (b) with $Z = T$ and $\overline{f} = \delta$, we obtain a factorization of δ through $\Omega J_2(\Sigma X)$, as asserted.

Part (d) implies part (a). By Lemma 3.1, the composite $\Sigma\Omega\Sigma X\wedge\Omega\Sigma X\xrightarrow{\Psi}\Sigma X\xrightarrow{j}J_2(\Sigma X)$ is null homotopic. Taking adjoints, this implies that the composite $\Omega\Sigma X\wedge\Omega\Sigma X\xrightarrow{\overline{\Psi}}\Omega\Sigma X\xrightarrow{\Omega j}\Omega J_2(\Sigma X)$ is null homotopic, where $\overline{\Psi}$ is the adjoint of Ψ. By hypothesis, the map $\Omega\Sigma X\xrightarrow{\delta}T$ factors through Ωj, so the composite $\Omega\Sigma X\wedge\Omega\Sigma X\xrightarrow{\overline{\Psi}}\Omega\Sigma X\xrightarrow{\delta}T$ is null homotopic. Hence there is a lift

$$\begin{array}{ccc} & & \Omega R \\ & \overset{\lambda}{\nearrow} & \big\downarrow{\scriptstyle \Omega\varphi} \\ \Omega\Sigma X\wedge\Omega\Sigma X & \xrightarrow{\overline{\Psi}} & \Omega\Sigma X \end{array}$$

for some map λ. Suspending and using the naturality of the evaluation map, we obtain a homotopy commutative diagram

$$\begin{array}{ccccc} & & \Sigma\Omega R & \xrightarrow{ev} & R \\ & \overset{\Sigma\lambda}{\nearrow} & \big\downarrow{\scriptstyle \Sigma\Omega\varphi} & & \big\downarrow{\scriptstyle \varphi} \\ \Sigma\Omega\Sigma X\wedge\Omega\Sigma X & \xrightarrow{\Sigma\overline{\Psi}} & \Sigma\Omega\Sigma X & \xrightarrow{ev} & \Sigma X. \end{array}$$

Observe that the bottom row is the adjoint of $\overline{\Psi}$, which is Ψ. Thus $ev\circ\Sigma\lambda$ is a lift of Ψ through φ, as required. □

In [Gra4, Grb, T2] certain spaces were shown to be homotopy associative and homotopy commutative by using the (a) implies (b) part of Theorem 4.2. This involved detailed work. It would be interesting to know if the equivalent statements in Theorem 4.2 could be used to give simpler proofs.

Example 4.3. Suppose that $T=\Omega T'$ and there is a "classifying map" $\Sigma X\xrightarrow{\epsilon}T'$, so that $\delta\simeq\Omega\epsilon$. Thus δ is an H-map so the equivalent statements in Theorem 4.2 imply that the loop space T is also homotopy commutative provided that the map $\Sigma X\wedge X\xrightarrow{[1,1]}\Sigma X$ lifts through $R\xrightarrow{\varphi}\Sigma X$. That is, the obstruction to T being homotopy commutative depends only on whether the Whitehead product $\Sigma X\wedge X\xrightarrow{[1,1]}\Sigma X$ lifts through φ, or equivalently, whether the Samelson product $X\wedge X\xrightarrow{\langle E,E\rangle}\Omega\Sigma X$ composes trivially with δ.

For instance, if $X=\Sigma\mathbb{C}P^{n-1}$ then there is a canonical map $X\longrightarrow SU(n)$ inducing the inclusion of the generators in homology. Taking adjoints we obtain a map $\Sigma X\xrightarrow{\epsilon}BSU(n)$ which induces a homotopy fibration sequence $\Omega\Sigma X\xrightarrow{\Omega\epsilon}SU(n)\longrightarrow R\xrightarrow{\varphi}\Sigma X\xrightarrow{\epsilon}BSU(n)$. Localized at an odd prime p, if $n\leq(p-1)^2+1$ then $\Omega\epsilon$ has a right homotopy inverse [T3]. Therefore, for these values of n, $SU(n)$ is homotopy commutative at p if and only if the composite $X\wedge X\xrightarrow{\langle E,E\rangle}\Omega\Sigma X\xrightarrow{\Omega\epsilon}SU(n)$ is null homotopic. This could be used to simplify the argument used by McGibbon [M] to classify those n for which $SU(n)$ is homotopy commutative at p.

Theorem 4.2 can also be used to show that retracts of $\Omega\Sigma X$ fail to have certain properties.

Example 4.4. The following statement is known but is usefully recast in light of Theorem 4.2. For odd primes p, Cohen, Moore and Neisendorfer [CMN2] constructed a homotopy fibration sequence

$$\Omega P^{2n+1}(p^r)\xrightarrow{\delta}T\xrightarrow{*}R\xrightarrow{\varphi}P^{2n+1}(p^r) \tag{3}$$

where T is the indecomposable retract of $\Omega P^{2n+1}(p^r)$ that contains the bottom Moore space. The space R is a wedge of mod-p^r Moore spaces and φ factors through the universal Whitehead product on $P^{2n+1}(p^r)$. However, the universal Whitehead product does not factor through φ. For if it did, then Theorem 4.2 would imply that T is homotopy associative and homotopy commutative, but by [CMN2] it cannot have these properties since $H_*(T)$ is neither associative nor commutative.

5. Extension results

In this section we prove Theorem 1.2. Given a homotopy fibration sequence $\Omega\Sigma X \xrightarrow{\delta} T \xrightarrow{*} R \xrightarrow{\varphi} \Sigma X$, let t be the composite

$$t\colon X \xrightarrow{E} \Omega\Sigma X \xrightarrow{\delta} T.$$

Proposition 5.1. *Let $\Omega\Sigma X \xrightarrow{\delta} T \xrightarrow{*} R \xrightarrow{\varphi} \Sigma X$ be a homotopy fibration sequence where φ factors through the universal Whitehead product on ΣX and let Z be a homotopy associative H-space. If $f\colon X \longrightarrow Z$ is a map with the property that the Samelson product $\langle f, f\rangle$ is null homotopic then there is an extension*

$$\begin{array}{ccc} X & \xrightarrow{f} & Z \\ \downarrow{\scriptstyle t} & \nearrow{\scriptstyle \widetilde{f}} & \\ T & & \end{array}$$

where $\widetilde{f}$ is an H-map. Further, if δ is an H-map then $\widetilde{f}$ is the unique H-map, up to homotopy, such that $\widetilde{f} \circ t \simeq f$.

Proof. The proof proceeds in stages.

Step 1: Setting up. Since Z is homotopy associative and the Samelson product $\langle f, f\rangle$ is null homotopic, Theorem 1.1 implies that f extends across $X \xrightarrow{\iota} \Omega J_2(\Sigma X)$ to a map

$$f'\colon \Omega J_2(\Sigma X) \longrightarrow Z$$

with the property that the composite

$$\overline{f}\colon \Omega\Sigma X \xrightarrow{\Omega j} \Omega J_2(\Sigma X) \xrightarrow{f'} Z$$

is an H-map. Observe that $\overline{f} \circ \Omega\varphi$ is null homotopic since, by hypothesis, φ factors through the universal Whitehead product Ψ and by Lemma 3.1 the composite $j \circ \Psi$ is null homotopic.

Step 2: $\overline{f}$ factors through δ. By hypothesis, the map $\Omega\Sigma X \xrightarrow{\delta} T$ has a right homotopy inverse $s\colon T \longrightarrow \Omega\Sigma X$. Let $\widetilde{f}$ be the composite

$$\widetilde{f}\colon T \xrightarrow{s} \Omega\Sigma X \xrightarrow{\overline{f}} Z.$$

Consider the diagram

$$\begin{array}{ccccc}
T \times \Omega R & \xrightarrow{s \times \Omega\varphi} & \Omega\Sigma X \times \Omega\Sigma X & \xrightarrow{\mu} & \Omega\Sigma X \\
\downarrow{\scriptstyle \pi_1} & & \downarrow{\scriptstyle \overline{f} \times \overline{f}} & & \downarrow{\scriptstyle \overline{f}} \\
T & \xrightarrow{i_1 \circ \widetilde{f}} & Z \times Z & \xrightarrow{m} & Z
\end{array}$$

where μ is the loop space multiplication, m is the given homotopy associative multiplication on Z, π_1 is the projection onto the first factor, and $i_1\colon Z \longrightarrow Z \times Z$ is the inclusion of the first factor. The left square homotopy commutes since $\overline{f} \circ \Omega\varphi$ is null homotopic and the right square homotopy commutes since $\overline{f}$ is an H-map. Let $e\colon T \times \Omega R \longrightarrow \Omega\Sigma X$ be the composite along the top row. Observe that e is a homotopy equivalence and the composite $m \circ i_1 \circ \widetilde{f}$ along the bottom row is homotopic to $\widetilde{f}$. The diagram as a whole therefore shows that $\overline{f} \circ e \simeq \widetilde{f} \circ \pi_1$. Precomposing with e^{-1}, we obtain $\overline{f} \simeq \widetilde{f} \circ \pi_1 \circ e^{-1}$. Therefore, if $\pi_1 \circ e^{-1} \simeq \delta$ then $\overline{f} \simeq \widetilde{f} \circ \delta$, completing Step 2.

It remains to show that $\pi_1 \circ e^{-1} \simeq \delta$. Since the map δ is a connecting map in a homotopy fibration, there is a homotopy action $\theta\colon \Omega\Sigma X \times T \longrightarrow T$ such that the restriction of θ to $\Omega\Sigma X$ is δ and the restriction to T is the identity map. Now consider the diagram

$$\begin{array}{ccccc}
T \times \Omega R & \xrightarrow{s \times \Omega\varphi} & \Omega\Sigma X \times \Omega\Sigma X & \xrightarrow{\mu} & \Omega\Sigma X \\
\downarrow{\scriptstyle \pi_1} & & \downarrow{\scriptstyle 1 \times \delta} & & \downarrow{\scriptstyle \delta} \\
T & \xrightarrow{i_1 \circ s} & \Omega\Sigma X \times T & \xrightarrow{\theta} & T.
\end{array}$$

The left square homotopy commutes since δ and $\Omega\varphi$ are consecutive maps in a homotopy fibration and so their composite is null homotopic. The right square homotopy commutes since any homotopy action induced by a homotopy fibration connecting map has this property. Observe that the upper row in the diagram is the definition of e and the lower row is the identity map on T. The diagram as a whole therefore implies that $\delta \circ e \simeq \pi_1$. Thus $\delta \simeq \pi_1 \circ e^{-1}$, as required.

Step 3: $\widetilde{f}$ extends f. By the construction of $\widetilde{f}$ in Step 1 we have $\overline{f} \circ E \simeq f$. By Step 2, $\overline{f} \simeq \widetilde{f} \circ \delta$. By definition, $t = \delta \circ E$. Thus $\widetilde{f} \circ t = \widetilde{f} \circ \delta \circ E \simeq \overline{f} \circ E \simeq f$.

Step 4: $\widetilde{f}$ is an H-map. Consider the diagram

$$\begin{array}{ccccccc}
T \times T & \xrightarrow{s \times s} & \Omega\Sigma X \times \Omega\Sigma X & \xrightarrow{\mu} & \Omega\Sigma X & \xrightarrow{\delta} & T \\
& \searrow{\scriptstyle \widetilde{f} \times \widetilde{f}} & \downarrow{\scriptstyle \overline{f} \times \overline{f}} & & \downarrow{\scriptstyle \overline{f}} & & \downarrow{\scriptstyle \widetilde{f}} \\
& & Z \times Z & \xrightarrow{m} & Z & = & Z.
\end{array}$$

The left triangle homotopy commutes by definition of $\widetilde{f}$, the middle square homotopy commutes since $\overline{f}$ is an H-map, and the right square homotopy commutes by Step 2. Since the upper row in the diagram is the multiplication on T, the homotopy commutativity of the diagram implies that $\widetilde{f}$ is an H-map.

Step 5: Uniqueness. Now we assume the extra hypothesis that δ is an H-map. Suppose that $g, h\colon T \longrightarrow Z$ are H-maps such that $g \circ t \simeq h \circ t \simeq f$. Consider the composition $\Omega\Sigma X \xrightarrow{\delta} T \xrightarrow{g} Z$. It is an H-map since it is the composition of two H-maps. Further, by definition, $t = \delta \circ E$ so $g \circ \delta \circ E = g \circ t \simeq f$. Similarly, $h \circ \delta$ is an H-map and $h \circ \delta \circ E \simeq f$. The uniqueness property in Theorem 2.5 then implies that $h \circ \delta \simeq g \circ \delta$. Since s is a right homotopy inverse for δ, we obtain $h \simeq h \circ \delta \circ s \simeq g \circ \delta \circ s \simeq g$. □

Proof of Theorem 1.2. For part (a), since δ is an H-map and the Whitehead product $\Sigma X \wedge X \xrightarrow{[1,1]} \Sigma X$ factors through φ, Theorem 4.2 implies that T is homotopy associative and homotopy commutative. The same theorem implies that δ extends through $\Omega J_2(\Sigma X)$ and the retraction of T off $\Omega J_2(\Sigma X)$ follows from the fact that δ has a right homotopy inverse.

Part (b) is simply a rephrasing of the first assertion of Proposition 5.1 and part (c) is the second assertion of Proposition 5.1. □

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