Factorial Q-functions and Tokuyama identities for classical Lie groups

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Abstract

Factorial characters of each of the classical Lie groups have recently been defined algebraically as rather simple deformations of irreducible characters. Each such factorial character has been shown to satisfy a flagged Jacobi-Trudi identity, thereby allowing for its combinatorial realisation in terms of first a non-intersecting lattice path model and then a tableau model. Here we propose algebraic definitions of factorial Q-functions of the classical Lie groups and translate these definitions into combinatorial realisations in terms of non-intersecting lattice path and primed shifted tableaux models. By way of some justification of our chosen definitions, it is then shown that our factorial Q-functions satisfy Tokuyama-type identities and relate some special case of these to other identities that have appeared in the literature.

1 Introduction

In the last 15 years, Tokuyama identities [41] have been the subject of intense research activity, with a host of papers appearing from both a number theoretic and a combinatorial perspective (see for example [30, 9, 11, 1, 2, 3, 12, 5, 29]).Recent interest has focused on extending these results to the factorial domain (e.g. [4, 15], among others), and we advance this pursuit by continuing our recent work [16] on factorial characters of the classical Lie groups, defining corresponding factorial Q-functions and deriving Tokuyama identities that involve these characters.

Before introducing our definition of factorial Q-functions for the classical groups and embarking on their study, it is instructive to prepare the ground with a summary of the definitions and some properties of factorial characters of these groups.

Let $n \in \mathbb{N}$ be fixed. Let $\mathbf{x} = (x_1, x_2, \ldots, x_n)$ and $\overline{\mathbf{x}} = (\overline{x}_1, \overline{x}_2, \ldots, \overline{x}_n)$ with $\overline{x}_i = x_i^{-1}$ for $i = 1, 2, \ldots, n$, and let $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n)$ be a partition of length $\ell(\lambda) \leq n$. Then

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each of the classical groups $G = GL(n,\mathbb{C})$, $SO(2n+1,\mathbb{C})$, $Sp(2n,\mathbb{C})$, and $O(2n,\mathbb{C})$ possesses a finite dimensional irreducible representation V_G^λ of highest weight λ whose character may be denoted by $\mathrm{ch}\, V_G^\lambda(\mathbf{z})$ where $\mathbf{z}$ is a suitable parametrisation of the eigenvalues of the group elements of G, namely $\mathbf{x}$, $(\mathbf{x},\overline{\mathbf{x}},1)$, $(\mathbf{x},\overline{\mathbf{x}})$ and $(\mathbf{x},\overline{\mathbf{x}})$, respectively. Equivalently, these characters may be identified with irreducible characters $g_\lambda(\mathbf{z})$ of the corresponding Lie algebras $g = gl(n)$, $so(2n+1)$, $sp(2n)$, and $o(2n)$, where each parameter x_i in $\mathbf{z}$ is to be interpreted as a formal exponential e^{ϵ_i} of a Euclidean basis vector ϵ_i in the weight space of the Lie algebras g. Accordingly we may write

$$\begin{aligned} &\mathrm{ch}\, V^\lambda_{GL(n,\mathbb{C})}(\mathbf{x}) = gl_\lambda(\mathbf{x})\,; \qquad \mathrm{ch}\, V^\lambda_{SO(2n+1,\mathbb{C})}(\mathbf{x},\overline{\mathbf{x}},1) = so_\lambda(\mathbf{x},\overline{\mathbf{x}},1)\,; \\ &\mathrm{ch}\, V^\lambda_{Sp(2n,\mathbb{C})}(\mathbf{x},\overline{\mathbf{x}}) = sp_\lambda(\mathbf{x},\overline{\mathbf{x}})\,; \qquad \mathrm{ch}\, V^\lambda_{O(2n,\mathbb{C})}(\mathbf{x},\overline{\mathbf{x}}) = o_\lambda(\mathbf{x},\overline{\mathbf{x}})\,. \end{aligned} \tag{1.1}$$

Explicit formulae for these characters $g_\lambda(\mathbf{z})$ as ratios of determinants are well known, see for example [25, 6]. The passage to factorial characters $g_\lambda(\mathbf{z}|\mathbf{a})$ involves an infinite sequence of factorial parameters $\mathbf{a} = (a_1, a_2, \ldots)$ and is effected by replacing non-negative powers x_i^k and $\overline{x}_i^k$ by factorial powers:

$$(x_i|\mathbf{a})^k = \begin{cases} (x_i+a_1)(x_i+a_2)\cdots(x_i+a_k) & \text{if } k>0; \\ 1 & \text{if } k=0, \end{cases} \tag{1.2}$$

$$(\overline{x}_i|\mathbf{a})^k = \begin{cases} (\overline{x}_i+a_1)(\overline{x}_i+a_2)\cdots(\overline{x}_i+a_k) & \text{if } k>0; \\ 1 & \text{if } k=0, \end{cases} \tag{1.3}$$

in a manner explained for each g in [16]. If we introduce $h_m^g(\mathbf{z}|\mathbf{a}) = g_{(m)}(\mathbf{z}|\mathbf{a})$ for all $m \geq 0$ with $h_m^g(\mathbf{z}|\mathbf{a}) = 0$ for $m < 0$, then it has been shown that each factorial character satisfies a flagged Jacobi-Trudi identity of the form

$$g_\lambda(\mathbf{z}|\mathbf{a}) = \left|\, h^g_{\lambda_j - j + i}(\mathbf{z}^{(i)}|\mathbf{a}) \,\right|, \tag{1.4}$$

where the precise definition of $\mathbf{z}^{(i)}$ for each g is given in Theorem 4.

These results amount to a factorial generalisation of the flagged Jacobi-Trudi formulae for characters of the classical Lie groups first provided by Okada [31]. Inspired in part by preliminary presentations [14, 15] of the results given here, Okada [32] has also obtained by slightly different means not only the same flagged Jacobi-Trudi identities, but also flagged dual Jacobi-Trudi and unflagged Giambelli identities for all the factorial characters $g_\lambda(\mathbf{z}|\mathbf{a})$.

Generating function techniques have been used in [16] to interpret each term contributing to $h_m^g(\mathbf{x}|\mathbf{a})$ as a suitable weighted sequence of edges constituting a path in a g-dependent lattice. Each determinant of (1.4) may then be evaluated as a signed sum of contributions from n-tuples of such lattice paths that can be reduced to a sum over n-tuples of non-intersecting lattice paths by means of the Lindström-Gessel-Viennot Theorem [24, 7, 8]. Moreover, for each g, a bijective correspondence between such n-tuples and tableaux leads to a combinatoral realisation of factorial characters of the form

$$g_\lambda(\mathbf{z}|\mathbf{a}) = \sum_{T \in \mathcal{T}^g_\lambda} 2^{\zeta(T)} \prod_{(i,j)\in F^\lambda} \mathrm{wgt}(T_{ij})\,, \tag{1.5}$$

with the sum taken over a set $\mathcal{T}^g$ of tableaux T consisting of arrays of entries in a Young diagram F^λ of shape λ. The entries, taken from alphabets appropriate to each g, are subject to a variety of conditions such as the semistandardness condition that applies in the $gl(n)$ case. The sets $\mathcal{T}^g$ coincide with familiar sets of classical group tableaux [25, 21, 22, 40]. Here $\mathrm{wgt}(T_{ij})$ is the factorial weight of the entry T_{ij} at position (i,j) in F^λ and generally takes the form of $(x_k + a_\ell)$ and $(\overline{x}_k + a_\ell)$ for some k and ℓ. The factor $2^{\zeta(T)}$ is peculiar to the $o(2n)$ case.

In order to go further and arrive at appropriate Tokuyama-type identities for all such characters, analogous to those obtained already for the factorial $gl(n)$ case [4, 13], it is necessary to generalise classical Schur Q-functions [35, 38] not only from the general linear case to that of the other classical Lie groups, but also to factorial versions of these. In the case of $gl(n)$, factorial Schur Q-functions were introduced by Ivanov [18, 19] based on an algebraic definition due to Okounkov but given a combinatorial interpretation by Ivanov in terms of primed shifted tableaux. However, these factorial Q-functions are not general enough for our purposes since they do not involve enough independent parameters to encompass the non-factorial Q-functions that were introduced in [11]. These were defined in terms of primed shifted tableaux for both the $gl(n)$ and $sp(2n)$ cases, and were shown to give rise to Tokuyama type identities by purely combinatorial arguments. Here, in Section 3, we go further by defining factorial Q-functions $Q^g_\lambda(\mathbf{w};\mathbf{z}|\mathbf{a})$ for each g. This is done algebraically in terms of determinants whose elements are simple multiples of supersymmetric factorial q-functions $q^g_m(\mathbf{w};\mathbf{z}|\mathbf{a})$, which are themselves defined by means of generating functions. These algebraic definitions of $Q^g_\lambda(\mathbf{w};\mathbf{z}|\mathbf{a})$ and $q^g_m(\mathbf{w};\mathbf{z}|\mathbf{a})$ are converted into combinatorial expressions in Section 4 by following the same procedure as that used for $g_\lambda(\mathbf{z}|\mathbf{a})$ and $h_m(\mathbf{z}|\mathbf{a})$ in [16], that is proceeding by way of a lattice path model, as introduced in the non-factorial $gl(n)$ case by Okada [30], and bijections this time between $\ell(\lambda)$-tuples of non-intersecting lattice paths and sets $\mathcal{P}^g_\lambda$ of primed shifted tableaux P of shifted shape SF^λ with λ a strict partition, of length $\ell(\lambda)$ all of whose parts are distinct. This culminates in our first main result, namely Theorem 13 which takes the form:

$$Q^g_\lambda(\mathbf{w};\mathbf{z}|\mathbf{a}) = \sum_{P\in\mathcal{P}^g_\lambda} \prod_{(i,j)\in SF^\lambda} \mathrm{wgt}(P_{ij})\,. \tag{1.6}$$

The conditions on $P \in \mathcal{P}^g_\lambda$ are provided for each g, and the weight, $\mathrm{wgt}(P_{ij})$, of each entry P_{ij} is tabulated[1].

The original Tokuyama identity [41] expressed a sum over suitably weighted strict Gelfand-Tsetlin patterns, indexed by strict partitions $\lambda = \mu+\delta$ with μ a partition and $\delta = (n, n-1, \ldots, 1)$, as a product of a deformation of Weyl's denominator formula for $gl(n)$ and the Schur function $s_\mu(\mathbf{x})$. When the strict Gelfand-Tsetlin patterns are translated into the language of tableaux they coincide with those shifted primed tableaux $P \in \mathcal{P}^{gl(n)}_\lambda$ having no primed entries on the main diagonal. Allowing for

[1] We take this opportunity to point out that in [14] entries $P_{ij} = 0$, a_{j-i} and a horizontal edge were omitted from the three empty boxes in the last row of (6.2), and in Figure 6 the tableau entry 0 should have been given a weight a_2 rather than $1 - a_2$ and associated with a horizontal rather than a diagonal edge in $LP(P)$.

this, in our current notation Tokuyama's formula is none other than the identity

$$Q^{gl}_\lambda(\mathbf{x};t\mathbf{x}|\mathbf{0}) = \prod_{1\le i\le j\le n} (x_i + tx_j)\, gl_\mu(\mathbf{x}|\mathbf{0})\,, \tag{1.7}$$

where $t\mathbf{x} = (tx_1, tx_2, \ldots, tx_n)$ and $0 = (0, 0, \ldots)$.

Subsequently numerous Tokuyama-type identities have appeared in the literature, including those appropriate to other classical Lie groups. In particular, in the $sp(2n)$ case a symplectic Tokuyama identity exactly analogous to (1.7) has been derived [9, 10] within various intimately related settings, namely those of primed shifted tableaux, strict Gelfand-Tsetlin patterns, alternating sign matrices and square ice models. Bijective tableaux based proofs of both the $gl(n)$ and $sp(2n)$ Tokuyama-type identities were provided in [11] in which the parametrisation was generalised through replacing $t\mathbf{x}$ by $\mathbf{y}$. Equivalent generalisations in the $gl(n)$ case have been obtained by Brubaker *et al.* [1, 2] in a quite different manner through the application of the Yang-Baxter equation to the partition functions, $\mathcal{Z}(\mathcal{S}^\Gamma_\lambda)$ and $\mathcal{Z}(\mathcal{S}^\Delta_\lambda)$, of a six-vertex square ice model assigned different Boltzmann weights. Further extensions of this approach by Brubaker and Schultz [3] have led to the conclusion that the partition functions, $\mathcal{Z}(\mathcal{M}^\lambda)$, of six-vertex ice models, $\mathcal{M}^\lambda$, corresponding to $gl(n)$, $so(2n+1)$, $sp(2n)$ and $so(2n)$ are each divisible by a deformation, $\mathcal{Z}(\mathcal{M}^\delta)$, of the corresponding Weyl denominator, where $\lambda = \mu + \delta$. While in the cases $gl(n)$ and $sp(2n)$ these results are Tokuyama-type identities, in the sense that the quotient $\mathcal{Z}(\mathcal{M}^\lambda)/\mathcal{Z}(\mathcal{M}^\rho)$ is a trivial multiple of the irreducible character of $gl(n)$ and $sp(2n)$ of highest weight μ, the same is not the case for $so(2n+1)$ or $so(2n)$, where in the case of $so(2n+1)$ this quotient has been shown to be a certain sum of irreducible characters of $gl(n)$ [13]. On the other hand Friedberg and Zhang [5] have succeeded in deriving a Tokuyama-type identity appropriate to $so(2n+1)$ involving a quotient that is an irreducible character of $so(2n+1)$. They did so by working in a Gelfand-Tsetlin framework and using an inductive formula for the Whittaker coefficients of an appropriate Eisenstein series. In fact, using as they do the metaplectic cover of $so(2n+1)$ their result applies not only to ordinary characters of highest weight λ, with λ a partition, but also to spin characters of highest weight λ, with the parts of λ all half odd integers in our basis.

In a quite different approach, the factorial Schur Q-functions that were introduced by Ivanov [18, 19] enabled Ikeda, Mihalcea and Naruse [17] to derive rather easily a special case, namely $t = 1$, of the following factorial version of Tokuyama's identity:

$$Q^{gl}_\lambda(\mathbf{x};t\mathbf{x}|\mathbf{a}) = \prod_{1\le i\le j\le n} (x_i + tx_j)\, gl_\mu(\mathbf{x}|\mathbf{a})\,. \tag{1.8}$$

Within the context of a six-vertex ice model based on free-fermionic Boltzmann weights and the use of the Yang-Baxter equation an equivalent result involving factorial characters of $gl(n)$ was derived for its partition function by Bump, McNamara and Nakasuji [4]. Again within the context of a free-fermion model Motegi [29] has derived for its wavefunction under certain boundary conditions an analogous result involving factorial $sp(2n)$ characters, using arguments introduced by Ivanov [20] in the non-factorial case. The precise result (1.8) was derived in [13] by quite different means using a definition of $Q^{gl}_\lambda(\mathbf{x};\mathbf{y}|\mathbf{a})$ in terms of the primed shifted tableaux

$P \in \mathcal{P}^{gl}_\lambda$. We are now in a position to generalise this to the case not only of $sp(2n)$ but also of $so(2n+1)$ and $o(2n)$, treating all cases in a uniform manner.

This is done in Section 5, where it is shown that for each strict partition $\lambda = \mu + \delta$ with $\delta = (n, n-1, \ldots, 1)$ and μ a partition of length $\ell(\mu) \leq n$, our definitions of factorial characters and Q-functions are such that they satisfy a Tokuyama-type identity. The derivation is algebraic and leads to our second main result, Theorem 14, which for all g takes the form:

$$Q^g_\lambda(\mathbf{w};\mathbf{z}|\mathbf{a}) = Q^g_\delta(\mathbf{w};\mathbf{z})\ g_\mu(\mathbf{w};\mathbf{z}|\mathbf{a}) \tag{1.9}$$

where $Q^g_\delta(\mathbf{w};\mathbf{z})$ is a simple product of factors independent of the factorial parameters $\mathbf{a}$.

In view of what follows, it is convenient to point out here that, unless otherwise stated, we not only let

$$\mathbf{x} = (x_1, \ldots, x_n), \mathbf{y} = (y_1, \ldots, y_n), \overline{\mathbf{x}} = (\overline{x}_1, \ldots, \overline{x}_n), \overline{\mathbf{y}} = (\overline{y}_1, \ldots, \overline{y}_n), \tag{1.10}$$

but also let

$$\mathbf{x}^{(d)} = (x_d, \ldots, x_n), \mathbf{y}^{(d)} = (y_d, \ldots, y_n), \overline{\mathbf{x}}^{(d)} = (\overline{x}_d, \ldots, \overline{x}_n), \overline{\mathbf{y}}^{(d)} = (\overline{y}_d, \ldots, \overline{y}_n), \tag{1.11}$$

for $1 \leq d \leq n$.

Following Macdonald [26] it is also useful to introduce here the shift operator τ defined in such a way that

$$\tau^r\mathbf{a} = (a_{r+1}, a_{r+2}, \ldots) \quad \text{for any integer } r \text{ and any } \mathbf{a} = (a_1, a_2, \ldots)\,. \tag{1.12}$$

This necessarily extends the sequence $\mathbf{a} = (a_1, a_2, \ldots)$ of factorial parameters to a doubly infinite sequence $(\ldots, a_{-2}, a_{-1}, a_0, a_1, a_2, \ldots)$.

2 Ordinary and factorial characters

The first of our characters, $gl_\lambda(\mathbf{x})$, in (1.1) is none other than the Schur polynomial $s_\lambda(\mathbf{x})$ which has a well known definition as a ratio of alternants [25, 27]. Thanks to Weyl's character formula for the corresponding Lie algebras each of the characters $g_\lambda(\mathbf{x})$ can be expressed in a similar form [6]:

Definition 1 *For any partition $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_n)$ of length $\ell(\lambda) \le n$ let*

$$gl_\lambda(\mathbf{x}) = \frac{\left| x_i^{\lambda_j+n-j} \right|}{\left| x_i^{n-j} \right|}; \tag{2.1}$$

$$so_\lambda(\mathbf{x}, \overline{\mathbf{x}}, 1) = \frac{\left| x_i^{\lambda_j+n-j+1/2} - \overline{x}_i^{\lambda_j+n-j+1/2} \right|}{\left| x_i^{n-j+1/2} - \overline{x}_i^{n-j+1/2} \right|}; \tag{2.2}$$

$$sp_\lambda(\mathbf{x}, \overline{\mathbf{x}}) = \frac{\left| x_i^{\lambda_j+n-j+1} - \overline{x}_i^{\lambda_j+n-j+1} \right|}{\left| x_i^{n-j+1} - \overline{x}_i^{n-j+1} \right|}; \tag{2.3}$$

$$o_\lambda(\mathbf{x}, \overline{\mathbf{x}}) = \frac{\eta \left| x_i^{\lambda_j+n-j} + \overline{x}_i^{\lambda_j+n-j} \right|}{\frac{1}{2} \left| x_i^{n-j} + \overline{x}_i^{n-j} \right|} \quad \textit{with} \quad \eta = \begin{cases} \frac{1}{2} & \textit{if } \lambda_n = 0; \\ 1 & \lambda_n > 0. \end{cases} \tag{2.4}$$

where all the determinants, both here and hereafter, are $n \times n$, and we have specified in each case the element in the ith row and jth column.

As anticipated in the Introduction, the factorial characters $g_\lambda(\mathbf{z}|\mathbf{a})$ of the classical Lie algebras g are obtained from $g_\lambda(\mathbf{z})$ by replacing various x_i^k and $\overline{x}_i^k$ by $(x_i|\mathbf{a})^k$ and $(\overline{x}_i|\mathbf{a})^k$, respectively. This amounts to the adoption of the following definition [16]:

Definition 2 *For any partition $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_n)$ of length $\ell(\lambda) \le n$ and any $\mathbf{a} = (a_1, a_2, \dots)$ let*

$$gl_\lambda(\mathbf{x}|\mathbf{a}) = \frac{\left| (x_i|\mathbf{a})^{\lambda_j+n-j} \right|}{\left| (x_i|\mathbf{a})^{n-j} \right|}; \tag{2.5}$$

$$so_\lambda(\mathbf{x}, \overline{\mathbf{x}}, 1|\mathbf{a}) = \frac{\left| x_i^{1/2}(x_i|\mathbf{a})^{\lambda_j+n-j} - \overline{x}_i^{1/2}(\overline{x}_i|\mathbf{a})^{\lambda_j+n-j} \right|}{\left| x_i^{1/2}(x_i|\mathbf{a})^{n-j} - \overline{x}_i^{1/2}(\overline{x}_i|\mathbf{a})^{n-j} \right|}; \tag{2.6}$$

$$sp_\lambda(\mathbf{x}, \overline{\mathbf{x}}|\mathbf{a}) = \frac{\left| x_i(x_i|\mathbf{a})^{\lambda_j+n-j} - \overline{x}_i(\overline{x}_i|\mathbf{a})^{\lambda_j+n-j} \right|}{\left| x_i(x_i|\mathbf{a})^{n-j} - \overline{x}_i(\overline{x}_i|\mathbf{a})^{n-j} \right|}; \tag{2.7}$$

$$o_\lambda(\mathbf{x}, \overline{\mathbf{x}}|\mathbf{a}) = \frac{\eta \left| (x_i|\mathbf{a})^{\lambda_j+n-j} + (\overline{x}_i|\mathbf{a})^{\lambda_j+n-j} \right|}{\frac{1}{2} \left| (x_i|\mathbf{a})^{n-j} + (\overline{x}_i|\mathbf{a})^{n-j} \right|} \quad \textit{with} \quad \eta = \begin{cases} \frac{1}{2} & \textit{if } \lambda_n = 0; \\ 1 & \lambda_n > 0. \end{cases} \tag{2.8}$$

where each $(x_i|\mathbf{a})^k$ and $(\overline{x}_i|\mathbf{a})^k$ with $k \ge 0$ is defined by (1.2) and (1.3) as appropriate.

The definition (2.5) is that of Macdonald [27] for factorial Schur functions, and the others have been drawn up as rather natural generalisations of this that all have the merit of reducing to the classical non-factorial characters if one sets $\mathbf{a} = \mathbf{0} = (0, 0, \dots)$. In each case the denominators are independent of $\mathbf{a}$ and coincide with the Weyl denominators of (2.1)-(2.4).

To establish flagged Jacobi-Trudi identities for these factorial characters use was made in [16] of the special one part partition cases $h_m^g(\mathbf{z}|\mathbf{a}) = g_{(m)}(\mathbf{z}|\mathbf{a})$ whose definition by means of generating functions took the form:

Definition 3 *For any integer m let*

$$h_m^{gl}(\mathbf{x}|\mathbf{a}) = [t^m]\ \prod_{i=1}^{n} \frac{1}{1-tx_i}\ \prod_{j=1}^{n+m-1} (1+ta_j)\,; \tag{2.9}$$

$$h_m^{oo}(\mathbf{x},\overline{\mathbf{x}},1|\mathbf{a}) = [t^m]\ (1+t)\ \prod_{i=1}^{n} \frac{1}{(1-tx_i)(1-t\overline{x}_i)}\ \prod_{j=1}^{n+m-1} (1+ta_j)\,; \tag{2.10}$$

$$h_m^{sp}(\mathbf{x},\overline{\mathbf{x}}|\mathbf{a}) = [t^m]\ \prod_{i=1}^{n} \frac{1}{(1-tx_i)(1-t\overline{x}_i)}\ \prod_{j=1}^{n+m-1} (1+ta_j)\,; \tag{2.11}$$

$$h_m^{eo}(\mathbf{x},\overline{\mathbf{x}}|\mathbf{a}) = [t^m]\ \begin{cases} \left(\dfrac{1}{1-tx_1} + \dfrac{1}{1-t\overline{x}_1} - \delta_{m0}\right)\ \displaystyle\prod_{j=1}^{m}(1+ta_j) & \textit{if } n=1\,; \\ (1-t^2)\ \displaystyle\prod_{i=1}^{n} \frac{1}{(1-tx_i)(1-t\overline{x}_i)}\ \prod_{j=1}^{n+m-1} (1+ta_j) & \textit{if } n>1\,. \end{cases} \tag{2.12}$$

Then for $m=0$ we have $h_0^{gl}(\mathbf{x}|\mathbf{a}) = h_0^{oo}(\mathbf{x},\overline{\mathbf{x}},1|\mathbf{a}) = h_0^{sp}(\mathbf{x},\overline{\mathbf{x}}|\mathbf{a}) = h_0^{eo}(\mathbf{x},\overline{\mathbf{x}}|\mathbf{a}) = 1$, while for $m<0$ we have $h_m^{gl}(\mathbf{x}|\mathbf{a}) = h_m^{oo}(\mathbf{x},\overline{\mathbf{x}},1|\mathbf{a}) = h_m^{sp}(\mathbf{x},\overline{\mathbf{x}}|\mathbf{a}) = h_m^{eo}(\mathbf{x},\overline{\mathbf{x}}|\mathbf{a}) = 0$.

In terms of these we have [16]

Theorem 4 (Flagged factorial Jacobi-Trudi identities) *For any partition $\lambda = (\lambda_1,\lambda_2,\ldots,\lambda_n)$ and any $\mathbf{a} = (a_1,a_2,\ldots)$ we have*

$$gl_\lambda(\mathbf{x}|\mathbf{a}) = \left| h^{gl}_{\lambda_j-j+i}(\mathbf{x}^{(i)}|\mathbf{a}) \right|\,; \tag{2.13}$$

$$so_\lambda(\mathbf{x},\overline{\mathbf{x}},1|\mathbf{a}) = \left| h^{oo}_{\lambda_j-j+i}(\mathbf{x}^{(i)},\overline{\mathbf{x}}^{(i)},1|\mathbf{a}) \right|\,; \tag{2.14}$$

$$sp_\lambda(\mathbf{x},\overline{\mathbf{x}}|\mathbf{a}) = \left| h^{sp}_{\lambda_j-j+i}(\mathbf{x}^{(i)},\overline{\mathbf{x}}^{(i)}|\mathbf{a}) \right|\,; \tag{2.15}$$

$$o_\lambda(\mathbf{x},\overline{\mathbf{x}}|\mathbf{a}) = \left| h^{eo}_{\lambda_j-j+i}(\mathbf{x}^{(i)},\overline{\mathbf{x}}^{(i)}|\mathbf{a}) \right|\,, \tag{2.16}$$

This leaves us in a position to introduce in the next Section factorial Q-functions defined by determinants somewhat resembling those appearing in the above flagged factorial Jacobi-Trudi type identities, but dependent on two sequences of parameters $\mathbf{z}$ and $\mathbf{w}$, as well as the factorial parameters $\mathbf{a}$, and with the various $h_m^g(\mathbf{z}|\mathbf{a})$ of Definition 3 replaced by certain $q_m^g(\mathbf{w};\mathbf{z}|\mathbf{a})$ that are once again defined by means of generating functions.

3 Factorial q-functions and Q-functions

To discuss Q-functions we move into the realm of supersymmetric functions. Just as the complete homogeneous functions $h_m^{gl}(\mathbf{x})$ and their factorial counterparts $h_m^{gl}(\mathbf{x}|\mathbf{a})$ played an essential role in our discussion of characters and factorial characters, so here an essential role is played by the supersymmetric functions $q_m^{gl}(\mathbf{x};\mathbf{y})$ and their factorial generalisations $q_m^{gl}(\mathbf{x};\mathbf{y}|\mathbf{a})$. These and their counterparts for the orthogonal and symplectic groups are defined for our required range of parameters by means of generating functions in t as follows:

Definition 5 *For* $\mathbf{x}=(x_1,x_2,\ldots,x_r)$*,* $\mathbf{y}=(y_1,y_2,\ldots,y_s)$*,* $\overline{\mathbf{x}}=(\overline{x}_1,\overline{x}_2,\ldots,\overline{x}_r)$*,* $\overline{\mathbf{y}}=(\overline{y}_1,\overline{y}_2,\ldots,\overline{y}_s)$*, with* $r\geq s$*, and* $\mathbf{a}=(a_1,a_2,\ldots)$ *and each integer* m *let*

$$q_m^{gl}(\mathbf{x};\mathbf{y}|\mathbf{a}) = [t^m]\ \frac{\prod_{j=1}^{s}(1+ty_j)\prod_{k=1}^{m+r-s-1}(1+ta_k)}{\prod_{i=1}^{r}(1-tx_i)}; \tag{3.1}$$

$$q_m^{oo}(\mathbf{x},\overline{\mathbf{x}};\mathbf{y},\overline{\mathbf{y}},1|\mathbf{a}) = [t^m]\ \frac{(1+t)\prod_{j=1}^{s}(1+ty_j)(1+t\overline{y}_j)\prod_{k=1}^{m+r-s-1}(1+ta_k)}{\prod_{i=1}^{r}(1-tx_i)(1-t\overline{x}_i)}; \tag{3.2}$$

$$q_m^{sp}(\mathbf{x},\overline{\mathbf{x}};\mathbf{y},\overline{\mathbf{y}}|\mathbf{a}) = [t^m]\ \frac{\prod_{j=1}^{s}(1+ty_j)(1+t\overline{y}_j)\prod_{k=1}^{m+r-s-1}(1+ta_k)}{\prod_{i=1}^{r}(1-tx_i)(1-t\overline{x}_i)}; \tag{3.3}$$

$$q_m^{eo}(\mathbf{x},\overline{\mathbf{x}};\mathbf{y},\overline{\mathbf{y}}|\mathbf{a}) = [t^m]\frac{(1-t^2)\prod_{j=1}^{s}(1+ty_j)(1+t\overline{y}_j)\prod_{k=1}^{m+r-s-1}(1+ta_k)}{\prod_{i=1}^{r}(1-tx_i)(1-t\overline{x}_i)} + \ a_1a_2\cdots a_m\,\delta_{r,s+1}\,, \tag{3.4}$$

where $a_1a_2\cdots a_m=0$ *for* $m\leq 0$*. For* $m=0$ *we have* $q_0^{gl}(\mathbf{x};\mathbf{y}|\mathbf{a}) = q_0^{oo}(\mathbf{x},\overline{\mathbf{x}};\mathbf{y},\overline{\mathbf{y}},1|\mathbf{a}) = q_0^{sp}(\mathbf{x},\overline{\mathbf{x}};\mathbf{y},\overline{\mathbf{y}}|\mathbf{a}) = q_0^{eo}(\mathbf{x},\overline{\mathbf{x}};\mathbf{y},\overline{\mathbf{y}}|\mathbf{a}) = 1$*, while for all* $m<0$ *we have* $q_m^{gl}(\mathbf{x};\mathbf{y}|\mathbf{a}) = q_m^{oo}(\mathbf{x},\overline{\mathbf{x}};\mathbf{y},\overline{\mathbf{y}},1|\mathbf{a}) = q_m^{sp}(\mathbf{x},\overline{\mathbf{x}};\mathbf{y},\overline{\mathbf{y}}|\mathbf{a}) = q_m^{eo}(\mathbf{x},\overline{\mathbf{x}};\mathbf{y},\overline{\mathbf{y}}|\mathbf{a}) = 0$*.*

As usual each factorial q-function reduces to a corresponding non-factorial q-function through setting $a_k=0$ for all k. With these definitions the gl factorial q-function $q_m^{gl}(\mathbf{x};\mathbf{y}|\mathbf{a})$ is clearly supersymmetric [37] in the sense that it is symmetric with respect to independent permutations of $x_1,x_2,\ldots,x_r$ and $y_1,y_2,\ldots,y_s$, and it is independent of z if we set $x_i=z=-y_j$ for any $i\in\{1,2,\ldots,r\}$ and $j\in\{1,2,\ldots,s\}$. Similarly, the orthogonal and symplectic factorial q-functions are Weyl supersymmetric in the sense that they are symmetric with respect to independent permutations of $x_1,x_2,\ldots,x_r$ and $y_1,y_2,\ldots,y_s$, invariant under the interchange of any x_i with $\overline{x}_i$ or of any y_j with $\overline{y}_j$, as well as being independent of z if we set either x_i or $\overline{x}_i$ equal to z and either y_j or $\overline{y}_j$ equal to $-z$ for any $i\in\{1,2,\ldots,r\}$ and $j\in\{1,2,\ldots,s\}$.

In what follows it is the case $r=s+1$ that is of particular relevance, as will become apparent through the following definition of factorial Q-functions, each indexed by a strict partition, where a partition is said to be strict if and only if its non-zero parts are distinct.

Definition 6 *For any strict partition λ of length $\ell(\lambda)\leq n$ and $\mathbf{a}=(a_1,a_2,\ldots)$:*

$$Q_\lambda^{gl}(\mathbf{x};\mathbf{y}|\mathbf{a}) = \sum_{\mathbf{d}} \left|\, (x_{d_i}+y_{d_i})\ q^{gl}_{\lambda_j-1}(\mathbf{x}^{(d_i)};\mathbf{y}^{(d_i+1)}|\mathbf{a})\,\right|; \tag{3.5}$$

$$Q_\lambda^{oo}(\mathbf{x},\overline{\mathbf{x}};\mathbf{y},\overline{\mathbf{y}},1|\mathbf{a}) = \sum_{\mathbf{d}} \left|\, (x_{d_i}+y_{d_i}+\overline{x}_{d_i}+\overline{y}_{d_i})\ q^{oo}_{\lambda_j-1}(\mathbf{x}^{(d_i)},\overline{\mathbf{x}}^{(d_i)};\mathbf{y}^{(d_i+1)},\overline{\mathbf{y}}^{(d_i+1)},1|\mathbf{a})\,\right|; \tag{3.6}$$

$$Q_\lambda^{sp}(\mathbf{x},\overline{\mathbf{x}};\mathbf{y},\overline{\mathbf{y}}|\mathbf{a}) = \sum_{\mathbf{d}} \left|\, (x_{d_i}+y_{d_i}+\overline{x}_{d_i}+\overline{y}_{d_i})\ q^{sp}_{\lambda_j-1}(\mathbf{x}^{(d_i)},\overline{\mathbf{x}}^{(d_i)};\mathbf{y}^{(d_i+1)},\overline{\mathbf{y}}^{(d_i+1)}|\mathbf{a})\,\right|; \tag{3.7}$$

$$Q_\lambda^{eo}(\mathbf{x},\overline{\mathbf{x}};\mathbf{y},\overline{\mathbf{y}},1|\mathbf{a}) = \sum_{\mathbf{d}} \left|\, (x_{d_i}+y_{d_i}+\overline{x}_{d_i}+\overline{y}_{d_i})\ q^{eo}_{\lambda_j-1}(\mathbf{x}^{(d_i)},\overline{\mathbf{x}}^{(d_i)};\mathbf{y}^{(d_i+1)},\overline{\mathbf{y}}^{(d_i+1)}|\mathbf{a})\,\right|, \tag{3.8}$$

where each determinant is $\ell(\lambda)\times\ell(\lambda)$ and each sum is over all $\mathbf{d}=(d_1,d_2,\ldots,d_{\ell(\lambda)})$ such that $1\leq d_1<d_2<\cdots<d_{\ell(\lambda)}\leq n$.

In order to arrive at combinatorial realisations of these factorial Q-functions it is helpful to express each of them as far as possible in terms of $q_m^{gl}(\mathbf{w};\mathbf{z}|\mathbf{a})$ for some $\mathbf{w}=(w_1,w_2,\ldots,w_r)$ and $\mathbf{z}=(z_1,z_2,\ldots,z_s)$ with $r=s+1$, as in the following:

Lemma 7 *For any strict partition λ of length $\ell(\lambda)\leq n$:*

$$Q_\lambda^{gl}(\mathbf{x};\mathbf{y}|\mathbf{a}) = \sum_{\mathbf{d}} \left|\, (x_{d_i}+y_{d_i})\ q^{gl}_{\lambda_j-1}(\mathbf{x}^{(d_i)};\mathbf{y}^{(d_i+1)}|\mathbf{a})\,\right|; \tag{3.9}$$

$$\begin{aligned} Q_\lambda^{oo}(\mathbf{x},\overline{\mathbf{x}};\mathbf{y},\overline{\mathbf{y}},1|\mathbf{a}) = \sum_{\mathbf{d}} \big|\, & (x_{d_i}+y_{d_i})\ q^{gl}_{\lambda_j-1}(\mathbf{x}^{(d_i)},\overline{\mathbf{x}}^{(d_i)},0;\mathbf{y}^{(d_i+1)},\overline{\mathbf{y}}^{(d_i)},1|\mathbf{a}) \\ & + (\overline{x}_{d_i}+\overline{y}_{d_i})\ q^{gl}_{\lambda_j-1}(\mathbf{x}^{(d_i+1)},\overline{\mathbf{x}}^{(d_i)},0;\mathbf{y}^{(d_i+1)},\overline{\mathbf{y}}^{(d_i+1)},1|\mathbf{a})\,\big|; \end{aligned} \tag{3.10}$$

$$\begin{aligned} Q_\lambda^{sp}(\mathbf{x},\overline{\mathbf{x}};\mathbf{y},\overline{\mathbf{y}}|\mathbf{a}) = \sum_{\mathbf{d}} \big|\, & (x_{d_i}+y_{d_i})\ q^{gl}_{\lambda_j-1}(\mathbf{x}^{(d_i)},\overline{\mathbf{x}}^{(d_i)};\mathbf{y}^{(d_i+1)},\overline{\mathbf{y}}^{(d_i)}|\mathbf{a}) \\ & + (\overline{x}_{d_i}+\overline{y}_{d_i})\ q^{gl}_{\lambda_j-1}(\mathbf{x}^{(d_i+1)},\overline{\mathbf{x}}^{(d_i)};\mathbf{y}^{(d_i+1)},\overline{\mathbf{y}}^{(d_i+1)}|\mathbf{a})\,\big|; \end{aligned} \tag{3.11}$$

$$\begin{aligned} Q_\lambda^{eo}(\mathbf{x},\overline{\mathbf{x}};\mathbf{y},\overline{\mathbf{y}}|\mathbf{a}) = \sum_{k=0}^{\ell(\lambda)} \Bigg(& \big(\,\delta_{k0}+(-1)^{\ell(\lambda)-k}a_1a_2\cdots a_{\lambda_k-1}(1-\delta_{k0})\,\big) \\ & \times \Big(\sum_{\mathbf{d}} \big|\,(x_{d_i}+y_{d_i})\ q^{gl}_{\kappa_j^{(k)}-1}(\mathbf{x}^{(d_i)},\overline{\mathbf{x}}^{(d_i)},0,0;\mathbf{y}^{(d_i+1)},\overline{\mathbf{y}}^{(d_i)},1,-1|\,\mathbf{a}) \\ & + (\overline{x}_{d_i}+\overline{y}_{d_i})\ q^{gl}_{\kappa_j^{(k)}-1}(\mathbf{x}^{(d_i+1)},\,\overline{\mathbf{x}}^{(d_i)},0,0;\mathbf{y}^{(d_i+1)},\,\overline{\mathbf{y}}^{(d_i+1)},1,-1|\mathbf{a})\,\big|\,\Big)\Bigg), \end{aligned} \tag{3.12}$$

where $\kappa^{(0)}=\lambda$ and $\kappa^{(k)}$ for $1\leq k\leq\ell(\lambda)$ is the partition obtained from λ by deleting the part λ_k and adding a final part 1.

Proof: The first of these is just (3.5). To verify the validity of the symplectic case it suffices to note that that for $1 \leq d \leq n$

$$
\begin{aligned}
&(x_d + y_d + \overline{x}_d + \overline{y}_d)\ q_m^{sp}(\mathbf{x}^{(d)},\overline{\mathbf{x}}^{(d)};\mathbf{y}^{(d+1)},\overline{\mathbf{y}}^{(d+1)}|\mathbf{a}) \\
&= [t^m]\ ((x_d{+}y_d)(1{+}t\overline{y}_d){+}(\overline{x}_d{+}\overline{y}_d)(1{-}tx_d))\frac{\prod_{j=d+1}^n(1{+}ty_j)(1{+}t\overline{y}_j)\prod_{k=1}^m(1{+}ta_k)}{\prod_{i=d}^n(1{-}tx_i)(1{-}t\overline{x}_i)} \\
&= (x_d{+}y_d)\ q_m^{gl}(\mathbf{x}^{(d)},\overline{\mathbf{x}}^{(d)};\mathbf{y}^{(d+1)},\overline{\mathbf{y}}^{(d)}|\mathbf{a}) + (\overline{x}_d{+}\overline{y}_d)\ q_m^{gl}(\mathbf{x}^{(d+1)},\overline{\mathbf{x}}^{(d)};\mathbf{y}^{(d+1)},\overline{\mathbf{y}}^{(d+1)}|\mathbf{a})\,.
\end{aligned}
$$

In the odd orthogonal case it the follows in the same way that

$$
\begin{aligned}
&(x_d + y_d + \overline{x}_d + \overline{y}_d)\ q_m^{oo}(\mathbf{x}^{(d)},\overline{\mathbf{x}}^{(d)};\mathbf{y}^{(d+1)},\overline{\mathbf{y}}^{(d+1)},1|\mathbf{a}) \\
&= [t^m]\ ((x_d{+}y_d)(1{+}t\overline{y}_d){+}(\overline{x}_d{+}\overline{y}_d)(1{-}tx_d))\frac{(1{+}t1)\prod_{j=d+1}^n(1{+}ty_j)(1{+}t\overline{y}_j)\prod_{k=1}^m(1{+}ta_k)}{(1{-}t0)\prod_{i=d}^n(1{-}tx_i)(1{-}t\overline{x}_i)} \\
&= (x_d{+}y_d)\ q_m^{gl}(\mathbf{x}^{(d)},\overline{\mathbf{x}}^{(d)},0;\mathbf{y}^{(d+1)},\overline{\mathbf{y}}^{(d)},1|\mathbf{a}) + (\overline{x}_d{+}\overline{y}_d)\ q_m^{gl}(\mathbf{x}^{(d+1)},\overline{\mathbf{x}}^{(d)},0;\mathbf{y}^{(d+1)},\overline{\mathbf{y}}^{(d+1)},1|\mathbf{a})\,,
\end{aligned}
$$

where the factor $(1{+}t)$ in the q_m^{oo}-function has been written in the form $(1{+}t1)/(1{-}t0)$ in order to preserve the condition $r = s+1$ in the pair of q_m^{gl}-functions.

The even orthogonal case is rather different because of the term $a_1a_2\cdots a_m$ appearing in (3.4). As a result the determinants appearing in our definition (3.8) of $Q_\lambda^{eo}(\mathbf{x},\overline{\mathbf{x}};\mathbf{y},\overline{\mathbf{y}}|\mathbf{a})$ can be written in the form

$$
\begin{aligned}
&\left|\,(x_{d_i}{+}y_{d_i}{+}\overline{x}_{d_i}{+}\overline{y}_{d_i})\ q_{\lambda_j-1}^{eo}(\mathbf{x}^{(d_i)},\overline{\mathbf{x}}^{(d_i)};\mathbf{y}^{(d_i+1)},\overline{\mathbf{y}}^{(d_i+1)}|\mathbf{a})\,\right| \\
&= \prod_{i=1}^{\ell(\lambda)}(x_{d_i}{+}y_{d_i}{+}\overline{x}_{d_i}{+}\overline{y}_{d_i})\ \left|\,Q_{i,\lambda_j-1} + A_{\lambda_j-1}\,Q_{i,0}\,\right|\ ,
\end{aligned}
$$

where for typographical convenience we have set

$$
Q_{i,m} = [t^m]\,\frac{(1-t^2)\prod_{j=d_{i+1}}^n(1+ty_j)(1+t\overline{y}_j)\prod_{k=1}^m(1+ta_k)}{\prod_{i=d_i}^n(1-tx_i)(1-t\overline{x}_i)}\ ,
$$

and

$$
A_m = a_1a_2\cdots a_m \quad \text{for } m>0 \quad \text{with} \quad A_0 = 0\,.
$$

Since A_{λ_j-1} is independent of the row number i and $Q_{i,0}=1$ for all i we have

$$
\left|\,Q_{i,\lambda_j-1} + A_{\lambda_j-1}\,Q_{i,0}\,\right| = \left|\,Q_{i,\kappa_j^{(0)}-1}\,\right| + \sum_{k=1}^{\ell(\lambda)}(-1)^{\ell(\lambda)-k}A_{\lambda_k-1}\,\left|\,Q_{i,\kappa_j^{(k)}-1}\,\right|\ , \qquad (3.13)
$$

where $\kappa^{(0)}=\lambda$ and in the summation over k the kth column of the determinant with elements Q_{i,λ_k-1} has been replaced by a column whose elements are all $A_{\lambda_k-1}\,Q_{i,0}$. The common factor A_{λ_k-1} has then been extracted before permuting columns to move the column of elements $Q_{i,0}=1$ to the rightmost position. This involves $\ell(\lambda)-k$ column interchanges and has the effect of replacing labels λ_j-1 by $\kappa_j^{(k)}-1$ for all $j=1,2,\ldots,\ell(\lambda)$. Then once again using the identity $(x_d+y_d+\overline{x}_d+\overline{y}_d) =$

$(x_d+y_d)(1+t\overline{y}_d)+(\overline{x}_d+\overline{y}_d)(1-tx_d)$ and rewriting $(1-t^2)$ in the form $(1+t1)(1+t(-1))/(1-t0)(1-t0)$ it can be seen that

$$
\begin{aligned}
(x_{d_i}+y_{d_i}+\overline{x}_{d_i}+\overline{y}_{d_i})\,Q_{i,m} &= (x_{d_i}+y_{d_i})\;q_m^{gl}(\mathbf{x}^{(d_i)},\overline{\mathbf{x}}^{(d_i)},0,0;\mathbf{y}^{(d_i+1)},\overline{\mathbf{y}}^{(d_i)},1,-1|\mathbf{a})\\
&\quad+(\overline{x}_{d_i}+\overline{y}_{d_i})\;q_m^{gl}(\mathbf{x}^{(d_i+1)},\overline{\mathbf{x}}^{(d+i)},0,0;\mathbf{y}^{(d_i+1)},\overline{\mathbf{y}}^{(d_i+1)},1,-1|\mathbf{a})\,.
\end{aligned}
$$

Resinserting the factors $(x_{d_i}+y_{d_i}+\overline{x}_{d_i}+\overline{y}_{d_i})$ back into the determinants on the right hand side of (3.13) and using this last identity completes the proof of (3.8).

It might be remarked here that in the case of a partition λ with $\lambda_{\ell(\lambda)}=1$ (3.13) reduces to

$$\left|\,Q_{i,\lambda_j-1}+A_{\lambda_j-1}\,Q_{i,0}\,\right|=\left|\,Q_{i,\lambda_j-1}\,\right|\,. \tag{3.14}$$

This is because $A_{\lambda_{\ell(\lambda)}-1}=A_0=0$ and $Q_{i,\lambda_{\ell(\lambda)}-1}=Q_{i,0}=1$ so that the rightmost $\ell(\lambda)$th column of the determinant on the left is a column of 1s. Subtracting A_{λ_j-1} times this column from the jth column for each $j<\ell(\lambda)$ eliminates all the dependence on A_m leaving just the determinant on the right. This can also be seen by noting on the right of (3.13) that $\kappa^{(k)}$ is a partition of the form $(\ldots,1,1)$ for $0<k<\ell(\lambda)$. In such a situation the determinant $\left|\,Q_{i,\kappa_j^{(k)}-1}\,\right|$ has two equal columns and must vanish. The term with $k=\ell(\lambda)$ also vanishes since $A_{\lambda_{\ell(\lambda)}-1}=0$ leaving just the single term involving $\kappa^{(0)}=\lambda$, as claimed in (3.14). □

In order to exploit Lemma 7 to construct combinatorial models of our factorial Q-functions it is necessary to express each of the q_m^{gl} functions appearing in (3.9)-(3.12) in a more amenable form. This can be done by means of the following:

Lemma 8 *Let $\mathbf{x}^{(d)}=(x_d,x_{d+1},\ldots,x_n)$ and $\mathbf{y}^{(d+1)}=(y_{d+1},y_{d+2},\ldots,y_n)$ for $1\leq d\leq n$ with $\mathbf{y}^{(n+1)}=0$. Then for all $\mathbf{a}=(a_1,a_2,\ldots)$ and integer $m>0$*

$$q_m^{gl}(\mathbf{x}^{(d)};\mathbf{y}^{(d+1)}|\mathbf{a})=\sum_{1\dot{\leq}i_1\dot{\leq}i_2\dot{\leq}\cdots\dot{\leq}i_m\dot{\leq}2n-2d+1}(w_{i_1}\pm a_1)(w_{i_2}\pm a_2)\cdots(w_{i_m}\pm a_m)\,, \tag{3.15}$$

with $\mathbf{w}=(w_1,w_2,\ldots,w_{2n-2d+1})=(x_d,y_{d+1},x_{d+1},y_{d+2},x_{d+2},\ldots,y_n,x_n)$ and the notation $\dot{\leq}$ indicates a weakly increasing order that allows factors $(w_i\pm a_\ell)=(x_k+a_\ell)$ or (y_k-a_ℓ) to appear according as $w_i=x_k$ or y_k, with several factors of the form $(x_k+a_\ell)(x_k+a_{\ell+1})\cdots$ allowed, but at most one factor (y_k-a_ℓ) for each k.

Proof: For $1\leq d\leq n$ and $m>0$ it is helpful to introduce

$$f_m(\mathbf{x}^{(d)};\mathbf{y}^{(d+1)}|\mathbf{a})=\sum_{1\dot{\leq}i_1\dot{\leq}i_2\dot{\leq}\cdots\dot{\leq}i_m\dot{\leq}2n-2d+1}(w_{i_1}\pm a_1)(w_{i_2}\pm a_2)\cdots(w_{i_m}\pm a_m)\,, \tag{3.16}$$

with the same interpretation as that given to the right hand side of (3.15). In the case $m=1$ this yields

$$f_1(\mathbf{x}^{(d)};\mathbf{y}^{(d+1)}|\mathbf{a})=\sum_{k=d}^{n}(x_k+a_1)+\sum_{k=d+1}^{n}(y_k-a_1)=(x_d+a_1)+\sum_{k=d+1}^{n}(x_k+y_k)\,, \tag{3.17}$$

while for $m>1$ we have the recurrence relation

$$f_m(\mathbf{x}^{(d)};\mathbf{y}^{(d+1)}|\mathbf{a}) = (x_d{+}a_1) f_{m-1}(\mathbf{x}^{(d)};\mathbf{y}^{(d+1)}|\tau\mathbf{a})$$
$$+\sum_{k=d+1}^{n}(x_k{+}a_1)f_{m-1}(\mathbf{x}^{(k)};\mathbf{y}^{(k+1)}|\tau\mathbf{a}) + \sum_{k=d+1}^{n}(y_k{-}a_1)f_{m-1}(\mathbf{x}^{(k)};\mathbf{y}^{(k+1)}|\tau\mathbf{a}), \quad (3.18)$$

as can be seen by considering all possible initial factors $(w_{i_1}\pm a_1)$.

It only remains to show that $q^{gl}_m(\mathbf{x}^{(d)};\mathbf{y}^{(d+1)}|\mathbf{a})$, as defined through (3.1), satisfies both (3.17) and (3.18). In the case $m=1$, expanding in powers of t gives

$$q^{gl}_1(\mathbf{x}^{(d)};\mathbf{y}^{(d+1)}|\mathbf{a}) = [t]\ \frac{\prod_{j=d+1}^{n}(1+ty_j)\prod_{k=1}^{m}(1+ta_k)}{\prod_{i=d}^{n}(1-tx_i)} = a_1+x_d+\sum_{k=d+1}^{n}(x_k+y_k)\,,$$

as required. While for $m>1$ we have

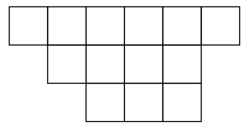
$$\begin{aligned}
q^{gl}_m(\mathbf{x}^{(d)};\mathbf{y}^{(d+1)}|\mathbf{a}) &= [t^m]\ \frac{\prod_{j=d+1}^{n}(1+ty_j)\prod_{k=1}^{m}(1+ta_k)}{\prod_{i=d}^{n}(1-tx_i)} \\
&= [t^m]\ \left(1+\frac{t(x_d+a_1)}{1-tx_d}\right)\prod_{j=d+1}^{n}\frac{1+ty_j}{1-tx_j}\prod_{k=2}^{m}(1+ta_k) \\
&= (x_d+a_1)q^{gl}_{m-1}(\mathbf{x}^{(d)};\mathbf{y}^{(d+1)}|\tau\mathbf{a}) \\
&\quad + [t^m]\ \left(1+\frac{t(x_{d+1}+y_{d+1})}{1-tx_{d+1}}\right)\prod_{j=d+2}^{n}\frac{1+ty_j}{1-tx_j}\prod_{k=2}^{m}(1+ta_k) \\
&= (x_d+a_1)\,q^{gl}_{m-1}(\mathbf{x}^{(d)};\mathbf{y}^{(d+1)}|\tau\mathbf{a}) + (x_{d+1}+y_{d+1})\,q^{gl}_{m-1}(\mathbf{x}^{(d+1)};\mathbf{y}^{(d+2)}|\tau\mathbf{a}) \\
&\quad + [t^m]\ \left(1+\frac{t(x_{d+2}+y_{d+2})}{1-tx_{d+2}}\right)\prod_{j=d+3}^{n}\frac{1+ty_j}{1-tx_j}\prod_{k=2}^{m}(1+ta_k) \\
&= \quad \cdots \\
&= (x_d+a_1)\,q^{gl}_{m-1}(\mathbf{x}^{(d)};\mathbf{y}^{(d+1)}|\tau\mathbf{a}) + \sum_{k=d+1}^{n}(x_k+y_k)\,q^{gl}_{m-1}(\mathbf{x}^{(k)};\mathbf{y}^{(k+1)}|\tau\mathbf{a})\,,
\end{aligned}$$

where advantage has been taken of the fact that $[t^m]\prod_{k=2}^{m}(1+ta_k)=0$. Then by the simple expedient of setting $(x_k+y_k)=(x_k+a_1)+(y_k-a_1)$ it can be seen that this takes the same form as our required recurrence relation (3.18), thereby completing the proof of (3.15). □

4 Primed shifted tableaux and factorial Q-functions

The passage from Schur functions to Schur Q-functions can be effected by replacing tableaux by primed shifted tableaux [42, 34]. We replicate this in the factorial setting by expressing our factorial Q-functions in terms of certain primed shifted tableaux. To this end we first define shifted Young diagrams.

Each strict partition λ of length $\ell(\lambda) \leq n$ specifies a shifted Young diagram SF^λ consisting of rows of boxes of lengths λ_i for $i = 1, 2, \ldots, \ell(\lambda)$ left adjusted to a diagonal line. This is exemplified in the case $\lambda = (6, 4, 3)$ by

$SF^{6531} \;=$ 

This allows us to define various primed shifted tableaux.

Definition 9 *[42, 34] Let $\mathcal{P}^{gl}_\lambda$ be the set of all primed shifted tableaux P of shape λ that are obtained by filling each box of SF^λ with an entry from the alphabet*

$$\{1' < 1 < 2' < 2 < \cdots < n' < n\}$$

in such a way that: **(Q1)** *entries weakly increase from left to right across rows;* **(Q2)** *entries weakly increase from top to bottom down columns;* **(Q3)** *no two identical unprimed entries appear in any column;* **(Q4)** *no two identical primed entries appear in any row.*

Definition 10 *[11] Let $\mathcal{P}^{sp}_\lambda$ be the set of all primed shifted tableaux P of shape λ that are obtained by filling each box of SF^λ with an entry from the alphabet*

$$\{1' < 1 < \overline{1}' < \overline{1} < 2' < 2 < \overline{2}' < \overline{2} < \cdots < n' < n < \overline{n}' < \overline{n}\}$$

in such a way that the conditions **(Q1)-(Q4)** *are satisfied together with:* **(Q5)** *at most one of $\{k', k, \overline{k}', \overline{k}\}$ appears on the main diagonal for each $k = 1, 2, \ldots, n$.*

Definition 11 *Let $\mathcal{P}^{oo}_\lambda$ be the set of all primed shifted tableaux P of shape λ that are obtained by filling each box of SF^λ with an entry from the alphabet*

$$\{1' < 1 < \overline{1}' < \overline{1} < 2' < 2 < \overline{2}' < \overline{2} < \cdots < n' < n < \overline{n}' < \overline{n} < 0' < 0\}$$

in such a way that the conditions **(Q1)-(Q5)** *are satisfied, together with:* **(Q6)** *the entries $0'$ and 0 do not appear on the main diagonal.*

Definition 12 *Let $\mathcal{P}^{eo}_\lambda$ be the set of all primed shifted tableaux P of shape λ that are obtained by filling each box of SF^λ with an entry from the alphabet*

$$\{1' < 1 < \overline{1}' < \overline{1} < 2' < 2 < \overline{2}' < \overline{2} < \cdots < n' < n < \overline{n}' < \overline{n} < 0' < 0 < \overline{0}' < \overline{0} < \emptyset' < \emptyset\}$$

in such a way that the conditions **(Q1)-(Q5)** *are satisfied, together with:* **(Q7)** *the entries $0'$, 0, $\overline{0}'$, $\overline{0}$, $\emptyset'$ and $\emptyset$ do not appear on the main diagonal; and* **(Q8)** *the entry $\emptyset'$ also does not appear on the neighbouring (second) diagonal.*

In the case $\lambda = (6,5,3)$ each of these types of shifted primed tableaux is illustrated as follows for $gl(4)$, $sp(8)$, $so(9)$ and $so(8)$ from left to right

$$\begin{array}{|c|c|c|c|c|c|} \hline 1' & 1 & 2' & 2 & 3' & 4 \\ \hline \multicolumn{1}{c|}{} & 2 & 3' & 3 & 3 \\ \cline{2-5} \multicolumn{2}{c|}{} & 4' & 4 & 4 \\ \cline{3-5} \end{array} \quad \begin{array}{|c|c|c|c|c|c|} \hline 1 & \overline{1} & 2' & \overline{2}' & 3' & 3 \\ \hline \multicolumn{1}{c|}{} & 2' & 2 & 3 & 4' \\ \cline{2-5} \multicolumn{2}{c|}{} & 4' & 4 & \overline{4} \\ \cline{3-5} \end{array} \quad \begin{array}{|c|c|c|c|c|c|} \hline 1 & \overline{1} & 2' & \overline{2}' & 3 & 0 \\ \hline \multicolumn{1}{c|}{} & \overline{2}' & \overline{2} & 3 & 0' \\ \cline{2-5} \multicolumn{2}{c|}{} & 4' & 0' & 0 \\ \cline{3-5} \end{array} \quad \begin{array}{|c|c|c|c|c|c|} \hline 1 & \overline{1} & 2' & 3 & \overline{0}' & \overline{0} \\ \hline \multicolumn{1}{c|}{} & 2' & 2 & 0 & \emptyset' \\ \cline{2-5} \multicolumn{2}{c|}{} & 4' & \emptyset & \emptyset \\ \cline{3-5} \end{array} \tag{4.1}$$

Our factorial Q-functions can be expressed in terms of these primed shifted tableaux by means of the following:

Theorem 13 *For* $\mathbf{a} = (a_1, a_2, \ldots)$, $a_0 = 0$, *and any strict partition* λ *of length* $\ell(\lambda) \le n$,

$$Q^g_\lambda(\mathbf{w}|\mathbf{a}) = \sum_{P \in \mathcal{P}^g_\lambda} \prod_{(i,j) \in SF^\lambda} \mathrm{wgt}(P_{ij}) \tag{4.2}$$

where P_{ij} *is the entry in the* i*th row and* j*th column of* SF^λ, *with* $Q^g_\lambda(\mathbf{w}|\mathbf{a})$ *and the weight* $\mathrm{wgt}(P_{ij})$ *of each entry given by*

g	$Q^g_\lambda(\mathbf{w}\vert\mathbf{a})$
gl	$Q^{gl}_\lambda(\mathbf{x};\mathbf{y}\vert\mathbf{a})$
oo	$Q^{oo}_\lambda(\mathbf{x},\overline{\mathbf{x}};\mathbf{y},\overline{\mathbf{y}},1\vert\mathbf{a})$
sp	$Q^{sp}_\lambda(\mathbf{x},\overline{\mathbf{x}};\mathbf{y},\overline{\mathbf{y}}\vert\mathbf{a})$
eo	$Q^{eo}_\lambda(\mathbf{x},\overline{\mathbf{x}};\mathbf{y},\overline{\mathbf{y}}\vert\mathbf{a})$

and

P_{ij}	$\mathrm{wgt}(P_{ij})$	P_{ij}	$\mathrm{wgt}(P_{ij})$
k	$x_k + a_{j-i}$	k'	$y_k - a_{j-i}$
$\overline{k}$	$\overline{x}_k + a_{j-i}$	$\overline{k}'$	$\overline{y}_k - a_{j-i}$
0	a_{j-i}	$0'$	$1 - a_{j-i}$
$\overline{0}$	a_{j-i}	$\overline{0}'$	$-1 - a_{j-i}$
$\emptyset$	a_{j-i}	$\emptyset'$	$-a_{j-i}$

(4.3)

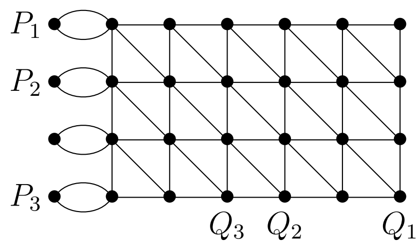
It is clear that for these factorial Q-functions the dependence on the factorial parameters $\mathbf{a}$ is simpler than it is for factorial characters as given in [16] since the factors in (4.3) are all of the form $w_k \pm a_{j-i}$ with the subscript on a completely independent of that on w.

The case $Q^{gl}_\lambda(\mathbf{x};\mathbf{y}|\mathbf{a})$ has been introduced and studied elsewhere [13]. The special case $Q^{gl}_\lambda(\mathbf{x};\mathbf{x}|-\mathbf{a})$ obtained by setting $\mathbf{y} = \mathbf{x}$ and $\mathbf{a} = -\mathbf{a}$ coincides with the generalized Q-function $Q_\lambda(\mathbf{x}|\mathbf{a})$ introduced by Ivanov [18, 19] and studied further by Ikeda, Milhalcea and Naruse [17]. If one further sets $\mathbf{a} = \mathbf{0}$ one recovers the combinatorial primed shifted tableaux formula [42, 34, 38, 30] for the original Schur Q-functions $Q_\lambda(\mathbf{x})$.

In order to establish the above combinatorial expressions for our factorial Q-functions in terms of primed shifted tableaux we follow the method of Okada [30] first to construct lattice path models based directly on the determinantal formulae for factorial Q-functions given in Lemma 7, and then to exploit the bijective correspondences between sets of $\ell(\lambda)$-tuples of non-intersecting lattice paths and the above sets of primed shifted tableaux. Thanks to Lemma 8 we are in position to embark on the proof of Theorem 13 as follows.

Proof: **Case** $gl(n)$. Here we introduce a rectangular lattice augmented by pairs of curved edges. As usual we adopt matrix coordinates (k, ℓ) for lattice points with

$k = 1, 2, \dots, n$ and $\ell = (0, 1, \dots, \lambda_1)$. For example, if $n = 4$ and $\lambda = (6, 4, 3)$ the lattice takes the form:



(4.4)

On this lattice the (i, j)th term of the determinant

$$Q^{gl}_\lambda(\mathbf{x}; \mathbf{y}|\mathbf{a}) = \sum_{\mathbf{d}} \left| \, (x_{d_i} + y_{d_i}) \; q^{gl}_{\lambda_j - 1}(\mathbf{x}^{(d_i)}; \mathbf{y}^{(d_i+1)}|\mathbf{a}) \, \right| \tag{4.5}$$

may be represented by the sum of all possible suitably signed and weighted continuous lattice paths from the starting point $P_i = (d_i, 0)$ to the end point $Q_j = (n, \lambda_j)$.

We associate two initial curved edges with the factor $(x_{d_i} + y_{d_i})$: one a concave downward edge carrying weight $x_{d_i} + a_0$ and the other a concave upward edge carrying weight $y_{d_i} - a_0$. Here the parameter a_0 is both arbitrary and redundant since the sum of both contributions is required. However, it is notationally convenient to include it in our edge weighting, while remembering that it may be ignored or set to 0 at any time.

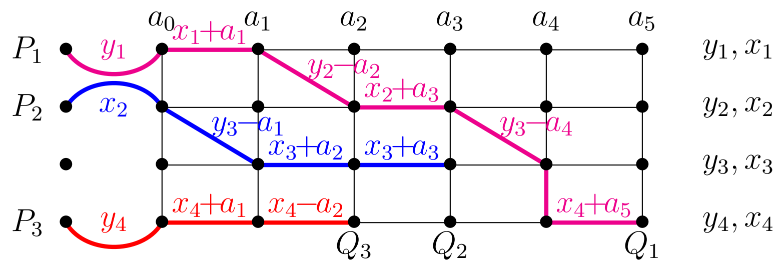
The horizontal and diagonal edges are associated with $q^{gl}_{\lambda_j - 1}(\mathbf{x}^{(d_i)}; \mathbf{y}^{(d_i+1)}|\mathbf{a})$ whose expansion is given in Lemma 8 by (3.15) with $m = \lambda_j - 1$: each horizontal edge with right hand end at (k, ℓ) carries weight $x_k + a_{\ell-1}$ and each diagonal edge with right hand end at (k, ℓ) carries weight $y_k - a_{\ell-1}$. As usual, the vertical edges necessary to make each lattice path continuous all carry weight 1. Then the product of all such horizontal, diagonal and vertical edge weights on a given path gives a summand of $q^{gl}_{\lambda_j - 1}(\mathbf{x}^{(d_i)}; \mathbf{y}^{(d_i+1)}|\mathbf{a})$.

The expansion of each determinant specified by $\mathbf{d}$ in (4.5) therefore involves a signed sum of weighted $\ell(\lambda)$-tuples of lattice paths $P_iQ_{\pi(i)}$ for $i = 1, 2, \dots, \ell(\lambda)$, with $P_i = (d_i, 0)$ and $Q_{\pi(i)} = (n, \lambda_{\pi(i)})$, and the sum is taken over all permutations $\pi \in S_{\ell(\lambda)}$. The sign factor is just $\mathrm{sgn}(\pi)$. Then thanks as usual to the Lindström-Gessel-Viennot Theorem [24, 7, 8] the only surviving contributions to the expansion of each determinant are those arising from $\ell(\lambda)$-tuples of non-intersecting lattice paths P_iQ_i for $i = 1, 2, \dots, \ell(\lambda)$ for which π is the identity element.

Furthermore, the set of all non-intersecting $\ell(\lambda)$-tuples of lattice paths from fixed starting points $P_i = (d_i, 0)$ to fixed end points $Q_i = (n, \lambda_i)$ for $i = 1, 2, \dots, \ell(\lambda)$ is in bijective correspondence with the set of all primed shifted tableaux $P \in \mathcal{P}^{gl}_\lambda$ as specified in Definition 9 but with diagonal entries $P_{ii} = d_i$ or d_i' for $i = 1, 2, \dots, \ell(\lambda)$. The required bijective correspondence is such that edges with labels $x_k + a_{\ell-1}$ or $y_k - a_{\ell-1}$ on the path P_iQ_i map to entries k or k' having weights $x_k + a_{\ell-1}$ or $y_k - a_{\ell-1}$ in the ℓth position of ith row of P.

A typical non-intersecting $\ell(\lambda)$-tuple of lattice paths LP and the corresponding primed shifted tableau P are illustrated below, along with both the lattice path edge labels and tableau entry weights, with the redundant parameter a_0, which in any case does not survive the sum over all such primed shifted tableaux, set equal to 0.

$LP \ =$



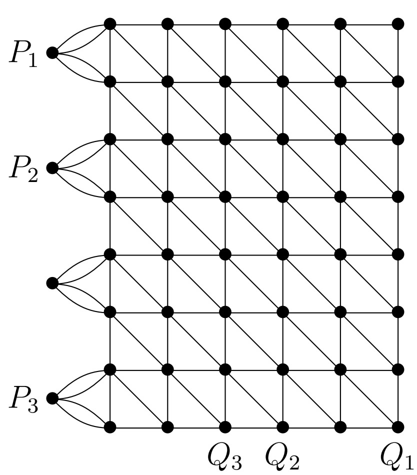
$$P \ = \ \begin{array}{|c|c|c|c|c|c|} \hline 1' & 1 & 2' & 2 & 3' & 4 \\ \hline \multicolumn{1}{c|}{} & 2 & 3' & 3 & 3 & \multicolumn{1}{c}{} \\ \cline{2-5} \multicolumn{2}{c|}{} & 4' & 4 & 4 & \multicolumn{1}{c}{} \\ \cline{3-5} \end{array} \qquad \mathrm{wgt}(P) \ = \ \begin{array}{|c|c|c|c|c|c|} \hline y_1 & x_1{+}a_1 & y_2{-}a_2 & x_2{+}a_3 & y_3{-}a_4 & x_4{+}a_5 \\ \hline \multicolumn{1}{c|}{} & x_2 & y_3{-}a_1 & x_3{+}a_2 & x_3{+}a_3 & \multicolumn{1}{c}{} \\ \cline{2-5} \multicolumn{2}{c|}{} & y_4 & x_4{+}a_1 & x_4{+}a_2 & \multicolumn{1}{c}{} \\ \cline{3-5} \end{array} \tag{4.6}$$

The nature of the map from LP to P described above implies that the ith row of P has length λ_i with entries satisfying (**Q1**) and (**Q4**). When these rows are combined to form a primed shifted tableau P of shape λ it can be seen that the non-intersecting condition implies the conditions (**Q2**) and (**Q3**). The map is clearly invertible and these conditions (**Q1**)-(**Q4**) on P are necessary and sufficient for the corresponding $\ell(\lambda)$-tuple to be non-intersecting. This completes the proof of (4.2) in the gl case with weights as given in the top line of the right hand table in (4.3).

Case $sp(2n)$. In the symplectic case we have

$$\begin{aligned} Q^{sp}_\lambda(\mathbf{x},\overline{\mathbf{x}};\mathbf{y},\overline{\mathbf{y}}|\mathbf{a}) = \sum_{\mathbf{d}} \Big| \, & (x_{d_i}{+}y_{d_i}) \; q^{gl}_{\lambda_j-1}(\mathbf{x}^{(d_i)},\overline{\mathbf{x}}^{(d_i)};\mathbf{y}^{(d_i+1)},\overline{\mathbf{y}}^{(d_i)}|\mathbf{a}) \\ & + (\overline{x}_{d_i}{+}\overline{y}_{d_i}) \; q^{gl}_{\lambda_j-1}(\mathbf{x}^{(d_i+1)},\overline{\mathbf{x}}^{(d_i)};\mathbf{y}^{(d_i+1)},\overline{\mathbf{y}}^{(d_i+1)}|\mathbf{a}) \, \Big|, \end{aligned} \tag{4.7}$$

so that our alphabet is extended to include not only x_k and y_k, but also $\overline{x}_k$ and $\overline{y}_k$ for $k = 1, 2, \ldots, n$. To accomodate this the underlying lattice takes the typical form:

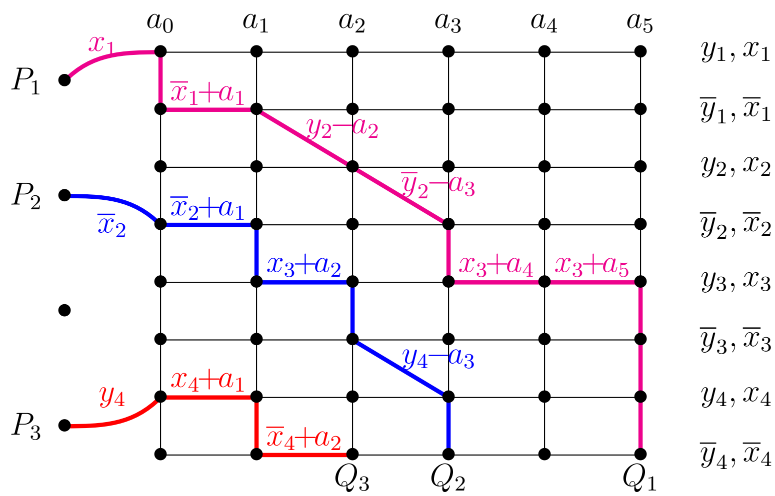


(4.8)

The four curved edges emanating from each starting point $P_i = (d_i, 0)$ correspond to the four possibe pre-factors x_{d_i}, y_{d_i}, $\overline{x}_{d_i}$ and $\overline{y}_{d_i}$ multiplying one or other of the $q^{gl}_{\lambda_j-1}$ terms.

Lattice paths are then constructed just as in the $gl(n)$ case, but taking into account the extended alphabet. A typical $\ell(\lambda)$-tuple LP of non-intersecting lattice paths is as shown below in the case $n = 4$, $\ell(\lambda) = 3$, $\mathbf{d} = (1, 2, 4)$ and $\lambda = (6, 4, 3)$, where once again we have for simplicity set $a_0 = 0$ in specifying weights.

$LP =$



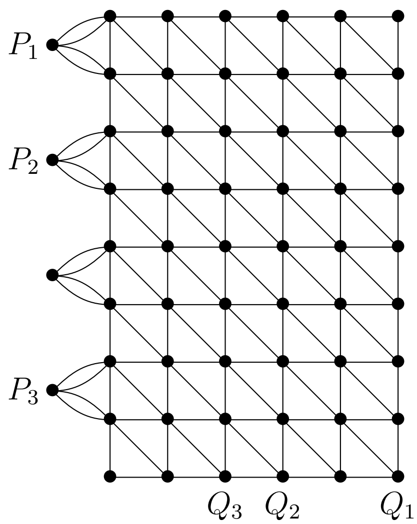
$$P \;=\; \begin{array}{|c|c|c|c|c|c|} \hline 1 & \overline{1} & 2' & \overline{2}' & 3 & 3 \\ \hline \end{array} \atop \begin{array}{c|c|c|c|c|} \multicolumn{1}{c}{} & \overline{2} & \overline{2} & 3 & 4' \\ \cline{2-5} \multicolumn{2}{c|}{} & 4' & 4 & \overline{4} \\ \cline{3-5} \end{array} \qquad \mathrm{wgt}(P) \;=\; \begin{array}{|c|c|c|c|c|c|} \hline x_1 & \overline{x}_1{+}a_1 & y_2{-}a_2 & \overline{y}_2{-}a_3 & x_3{+}a_4 & x_3{+}a_5 \\ \hline \multicolumn{1}{c|}{} & \overline{x}_2 & \overline{x}_2{+}a_1 & x_3{+}a_2 & y_4{-}a_3 & \multicolumn{1}{c}{} \\ \cline{2-5} \multicolumn{2}{c|}{} & y_4 & x_4{+}a_1 & \overline{x}_4{+}a_2 & \multicolumn{1}{c}{} \\ \cline{3-5} \end{array} \tag{4.9}$$

This figure also includes the corresponding primed shifted tableau $P \in \mathcal{P}^{sp}_\lambda$ obtained in the same way as in the $gl(n)$ case by reading off from each lattice path the consecutive entries in each row and adjusting the rows to the shifted shape λ. The bijective correspondence between $\ell(\lambda)$-tuples LP of non-intersecting lattice paths and primed shifted tableau $P \in \mathcal{P}^{sp}_\lambda$ is such that edges with labels $(x_k + a_{\ell-1})$, $(y_k - a_{\ell-1})$, $(\overline{x}_k + a_{\ell-1})$ and $(\overline{y}_k - a_{\ell-1})$ on the path P_iQ_i map to entries k, k', $\overline{k}$ and $\overline{k}'$, respectively, in the ℓth position of ith row of P. This applies, in particular to the case of the entries on the main diagonal of P which are associated with one or other of the four types of curved edge in LP, three of which have been illustrated in the above example. It is clear from this association that condition (**Q5**) applies to P, while the remaining conditions (**Q1**)-(**Q4**) are an immediate consequence of the non-intersecting nature of the paths in LP. This completes the proof of (4.2) in the sp case with weights as given in the top two lines of the right hand table in (4.3).

Case $o(2n+1)$. In the odd orthogonal case we have

$$\begin{aligned} Q^{oo}_\lambda(\mathbf{x},\overline{\mathbf{x}};\mathbf{y},\overline{\mathbf{y}},1|\mathbf{a}) = \sum_{\mathbf{d}} \Big|\, &(x_{d_i}{+}y_{d_i})\ q^{gl}_{\lambda_j-1}(\mathbf{x}^{(d_i)},\overline{\mathbf{x}}^{(d_i)},0;\mathbf{y}^{(d_i+1)},\overline{\mathbf{y}}^{(d_i)},1|\mathbf{a}) \\ &+ (\overline{x}_{d_i}{+}\overline{y}_{d_i})\ q^{gl}_{\lambda_j-1}(\mathbf{x}^{(d_i+1)},\overline{\mathbf{x}}^{(d_i)},0;\mathbf{y}^{(d_i+1)},\overline{\mathbf{y}}^{(d_i+1)},1|\mathbf{a})\,\Big|. \end{aligned} \tag{4.10}$$

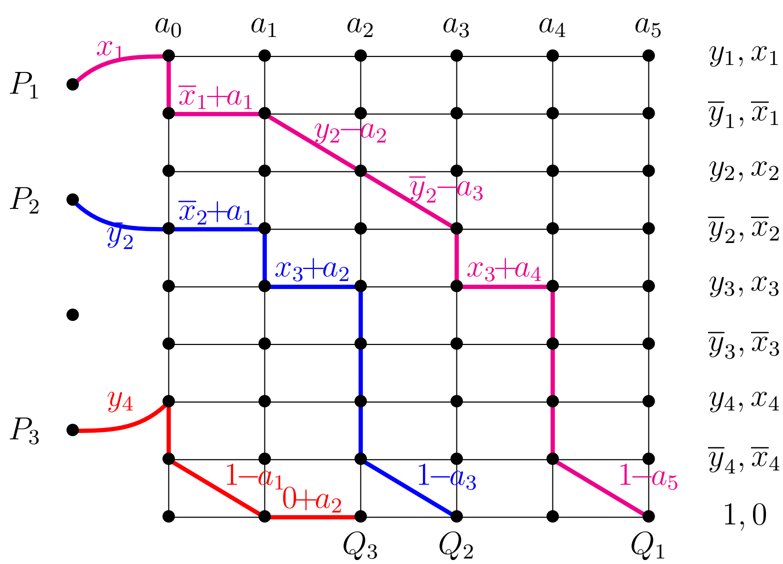
The only difference from the symplectic case is the further extension of the alphabet to include additional entries 0 and 1. To deal with this one just extends the underlying lattice through the addition of single row of lattice points as indicated below:



(4.11)

As in the $sp(2n)$ case a typical $\ell(\lambda)$-tuple LP of non-intersecting lattice paths is as shown below in the case $n=4$, $\ell(\lambda)=3$, $\mathbf{d}=(1,2,4)$ and $\lambda=(6,4,3)$, where the additional 0 and 1 in the alphabet lead to contributions to the weight of the form $0+a_{\ell-1}$ and $1-a_{\ell-1}$, respectively, but always with $\ell>1$.

LP =



$$
P \;=\; \begin{array}{|c|c|c|c|c|c|}
\hline
1 & \overline{1} & 2' & \overline{2}' & 3 & 0' \\
\hline
\multicolumn{1}{c|}{} & \overline{2}' & \overline{2} & 3 & 0' & \multicolumn{1}{|c}{} \\
\cline{2-5}
\multicolumn{1}{c}{} & \multicolumn{1}{c|}{} & 4' & 0' & 0 & \multicolumn{1}{|c}{} \\
\cline{3-5}
\end{array}
\qquad
\mathrm{wgt}(P) \;=\; \begin{array}{|c|c|c|c|c|c|}
\hline
x_1 & \overline{x}_1+a_1 & y_2-a_2 & \overline{y}_2-a_3 & x_3+a_4 & 1-a_5 \\
\hline
\multicolumn{1}{c|}{} & \overline{y}_2 & \overline{x}_2+a_1 & x_3+a_2 & 1-a_3 & \multicolumn{1}{|c}{} \\
\cline{2-5}
\multicolumn{1}{c}{} & \multicolumn{1}{c|}{} & y_4 & 1-a_1 & a_2 & \multicolumn{1}{|c}{} \\
\cline{3-5}
\end{array}
\tag{4.12}
$$

This time we have a bijective correspondence between $\ell(\lambda)$-tuples LP of non-intersecting lattice paths and primed shifted tableau $P \in \mathcal{P}^{oo}_{\lambda}$ such that edges with labels $x_k+a_{\ell-1}$,

$y_k - a_{\ell-1}$, $\overline{x}_k + a_{\ell-1}$,$\overline{y}_k - a_{\ell-1}$, $0 + a_{\ell-1}$ and $1 - a_{\ell-1}$ on the path P_iQ_i map to entries k, k', $\overline{k}$, $\overline{k}'$, 0 and $0'$, respectively, in the ℓth position of ith row of P. As in the symplectic case the conditions (**Q1**)-(**Q2**) apply to P, while the remaining condition (**Q6**) reflects the fact that the weights $0 + a_{\ell-1}$ and $1 - a_{\ell-1}$ only arise in cases for which $\ell > 1$. This is a consequence of the fact that the lattice has no curved edges attached to its bottom line, which is due in turn to the fact that the only pre-factors to the $q^{gl}_{\lambda_j-1}$ terms in (4.10) are x_{d_i}, y_{d_i}, $\overline{x}_{d_i}$ and $\overline{y}_{d_i}$. This completes the proof of (4.2) in the *oo* case with weights as given in the top three lines of the right hand table in (4.3).

Case $o(2n)$. In the even orthogonal case we have

$$
\begin{aligned}
Q^{eo}_\lambda(\mathbf{x},\overline{\mathbf{x}};\mathbf{y},\overline{\mathbf{y}}|\mathbf{a}) &= \sum_{k=0}^{\ell(\lambda)} \Bigg(\Big(\delta_{k0} + (-1)^{\ell(\lambda)-k} a_1a_2\cdots a_{\lambda_k-1}(1-\delta_{k0}) \Big) \\
&\times \Big(\sum_{\mathbf{d}} \Big| \, (x_{d_i}+y_{d_i}) \; q^{gl}_{\kappa^{(k)}_j-1}(\mathbf{x}^{(d_i)},\overline{\mathbf{x}}^{(d_i)},0,0;\mathbf{y}^{(d_i+1)},\overline{\mathbf{y}}^{(d_i)},1,-1|\,\mathbf{a}) \\
&+ (\overline{x}_{d_i}+\overline{y}_{d_i}) \; q^{gl}_{\kappa^{(k)}_j-1}(\mathbf{x}^{(d_i+1)},\,\overline{\mathbf{x}}^{(d_i)},0,0;\mathbf{y}^{(d_i+1)},\,\overline{\mathbf{y}}^{(d_i+1)},1,-1|\mathbf{a}) \,\Big|\, \Big) \Bigg).
\end{aligned}
\tag{4.13}
$$

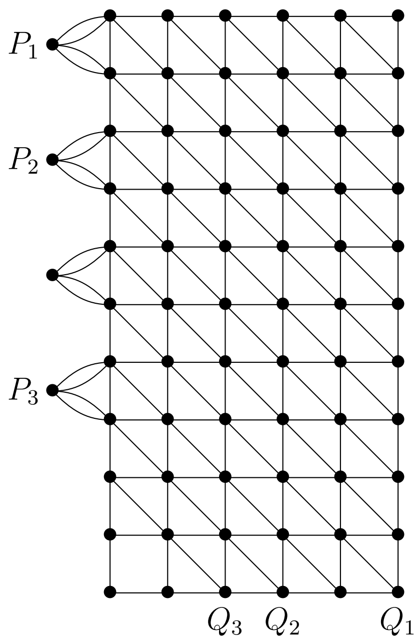
Clearly, one immediate difference from the odd orthogonal case is the further extension of the alphabet to include a second entry 0 and a new entry -1. This may be dealt with by adding an additional horizontal line of lattice points to the foot of our odd orthogonal lattice. However, the second difference arises from the necessity of coping not only with the case $k = 0$ for which $\kappa^{(0)} = \lambda$, but also with those cases $k > 0$ which involve factors $(-1)^{\ell(\lambda)-k}a_1a_2\cdots a_{\lambda_k-1}$ and partitions $\kappa^{(k)}$ obtained, it will be recalled, from λ through the deletion of its kth part and adding a final part 1.

In our usual example with $n = 4$, $\ell(\lambda) = 3$, $\mathbf{d} = (1, 2, 4)$ and $\lambda = (6, 4, 3)$, this implies that our sum over k yields contributions arising from $\kappa^{(0)} = (6, 4, 3)$, $\kappa^{(1)} = (6, 4, 1)$, $\kappa^{(2)} = (6, 3, 1)$, $\kappa^{(3)} = (4, 3, 1)$ weighted by factors 1, a_1a_2, $-a_1a_2a_3$ and $a_1a_2a_3a_4a_5$. respectively. It will be noted that these contributions can be accounted for in terms of shifted diagrams by weightings of the form:

$$
\lambda=\kappa^{(0)}: \;
\begin{array}{cccccc}
\square&\square&\square&\square&\square&\square\\
&\square&\square&\square&\square&\\
&&\square&\square&\square&
\end{array}
\qquad
\kappa^{(1)}: \;
\begin{array}{cccccc}
\square&\square&\square&\square&-a_4&a_5\\
&\square&\square&\square&-a_3&\\
&&\square&a_1&a_2&
\end{array}
$$
$$
\kappa^{(2)}: \;
\begin{array}{cccccc}
\square&\square&\square&\square&\square&\square\\
&\square&\square&\square&-a_3&\\
&&\square&a_1&a_2&
\end{array}
\qquad
\kappa^{(3)}: \;
\begin{array}{cccccc}
\square&\square&\square&\square&\square&\square\\
&\square&\square&\square&\square&\\
&&\square&a_1&a_2&
\end{array}
\tag{4.14}
$$

which involve the deletion of continuous strips of boxes from the shifted shape λ starting at the foot of the second diagonal and ending at the right hand end of each row, and weighting them by consecutive entries $a_1, a_2, \ldots$ with a factor of (-1)Closi for each row that they cover except the lowest.

In anticipation of all this we use a lattice of the form:

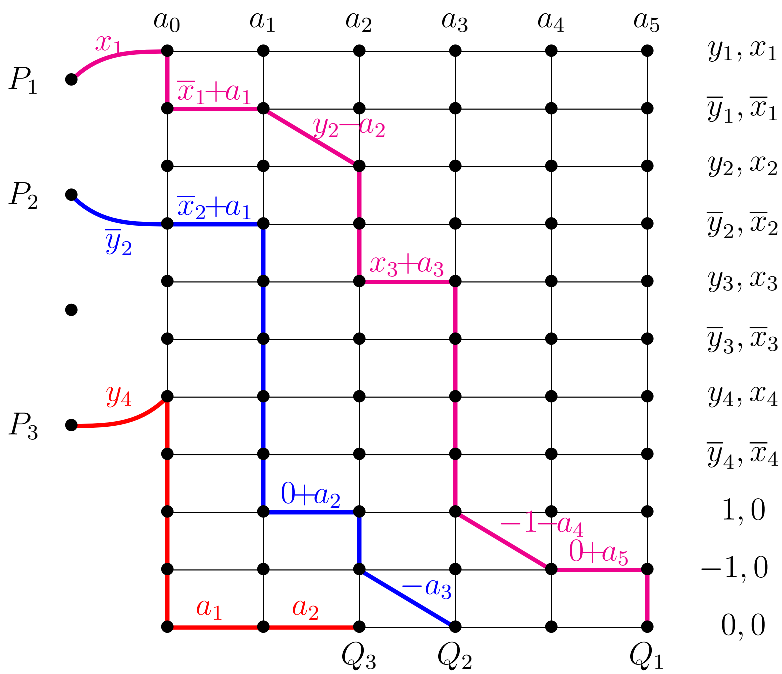


(4.15)

where it should be noted that a diagonal edge has deliberately been omitted at the lower left hand corner, and we refer to this lattice (4.15) as being punctured.

A typical $\ell(\lambda)$-tuple LP of non-intersecting lattice paths is as shown below in the case $n = 4$, $\ell(\lambda) = 3$, $\mathbf{d} = (1, 2, 4)$ and $\lambda = (6, 4, 3)$, where the addition of the pairs $0, 0$ and $1, -1$ to the alphabet leads to contributions to the weight of the form $0+a_{\ell-1}$, $0 + a_{\ell-1}$ and $1 - a_{\ell-1}$,$-1 - a_{\ell-1}$, respectively, but always with $\ell > 1$, while entirely new contributions to the weights of the form $a_{\ell-1}$ and $-a_{\ell-1}$ owe their origin to the distinction between λ and $\kappa^{(k)}$.

$LP \ =$

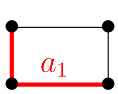


$$P \ = \ \begin{array}{|c|c|c|c|c|c|}\hline 1 & \overline{1} & 2' & 3 & \overline{0}' & \overline{0} \\ \hline \multicolumn{1}{c|}{} & \overline{2}' & \overline{2} & 0 & \emptyset' & \multicolumn{1}{|c}{} \\ \cline{2-5} \multicolumn{1}{c}{} & \multicolumn{1}{c|}{} & 4' & \emptyset & \emptyset & \multicolumn{1}{|c}{} \\ \cline{3-5} \end{array} \qquad \mathrm{wgt}(P) \ = \ \begin{array}{|c|c|c|c|c|c|}\hline x_1 & \overline{x}_1+a_1 & y_2-a_2 & x_3+a_3 & -1-a_4 & 0+a_5 \\ \hline \multicolumn{1}{c|}{} & \overline{y}_2 & \overline{x}_2+a_1 & 0+a_2 & -a_3 & \multicolumn{1}{|c}{} \\ \cline{2-5} \multicolumn{1}{c}{} & \multicolumn{1}{c|}{} & y_4 & a_1 & a_2 & \multicolumn{1}{|c}{} \\ \cline{3-5} \end{array} \tag{4.16}$$

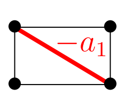
This example corresponds to a contribution to the even orthogonal character arising from a $k=2$ summand of (4.13) involving $\kappa^{(2)}=(6,3,1)$ and a multiplicative factor of $-a_1a_2a_3$. We can indeed choose to deal with all such terms by adding to our alphabet two more symbols $\emptyset$ and $\emptyset'$ carrying weights $a_{\ell-1}$ and $-a_{\ell-1}$. This almost amounts to replacing the right hand side of (4.13) by the expression

$$\begin{aligned}\Big(\sum_{\mathbf{d}} \Big|\, (x_{d_i}+y_{d_i})\ q^{gl}_{\lambda_j-1}(\mathbf{x}^{(d_i)}, \overline{\mathbf{x}}^{(d_i)}, 0, 0, \emptyset; \mathbf{y}^{(d_i+1)}, \overline{\mathbf{y}}^{(d_i)}, 1, -1, \emptyset' |\, \mathbf{a}) \\ + (\overline{x}_{d_i}+\overline{y}_{d_i})\ q^{gl}_{\lambda_j-1}(\mathbf{x}^{(d_i+1)}, \overline{\mathbf{x}}^{(d_i)}, 0, 0, \emptyset; \mathbf{y}^{(d_i+1)}, \overline{\mathbf{y}}^{(d_i+1)}, 1, -1, \emptyset' |\mathbf{a})\,\Big|\,\Big).\end{aligned} \tag{4.17}$$

The only impediment to this is our deliberate omission of the diagonal edge in the lower left hand corner of the lattice (4.15). That this omission is necessary can be seen by noting that if this were not the case then the standard interpretation of the expression (4.17) as a sum over $\ell(\lambda)$-tuples would include not only LP as above, but also a new diagram LP' differing only in the P_3Q_3 lattice path edges in the lower left hand corner with



replaced by



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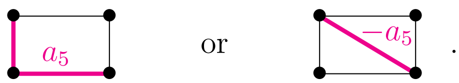
Such terms would mutual cancel by virtue of replacing the weight a_1 by $-a_1$. It is to avoid such cancellations that our lattice has been punctured, and when translated into the language of primed shifted tableaux $P \in \mathcal{P}^{eo}_\lambda$ this leads to the condition (**Q8**). As usual, the non-intersecting nature of our $\ell(\lambda)$-tuples implies and is implied by conditions (**Q1**)-(**Q4**), while (**Q5**) and (**Q7**) are a simple consequence of the four options available for the first edge of each path P_iQ_i. Thus we have a bijective correspondence between the non-intersecting path $\ell(\lambda)$-tuples on the punctured lattice (4.15) and the primed shifted tableaux $P \in \mathcal{P}^{eo}_\lambda$.

However, the use of our enlarged alphabet as in (4.17) gives rise to additional terms not encountered in the use, more properly, of (4.13). For example, the above path P_1Q_1 could reach Q_1 along alternative routes in which its final two edges are replaced either by a single diagonal edge or by a vertical edge followed by a horizontal edge, as shown below:



The corresponding contributions to the weight would be $-a_5$ and a_5, respectively, resulting in mutual cancellation. This cancellation is common to all terms other than those corresponding to some partition $\kappa^{(k)}$ and can perhaps best be seen by considering the corresponding primed shifted tableaux. In our above example this would involve replacing the rightmost entry $\overline{0}$ of P by $\emptyset'$ and $\emptyset$, respectively, as shown schematically below where all entries other than $\emptyset'$ and $\emptyset$ are suppressed.

$$
P \;=\; \begin{array}{cccccc} \square & \square & \square & \square & \square & \emptyset' \\ & \square & \square & \square & \emptyset' & \\ & & \square & \emptyset & \emptyset & \end{array}
\qquad \mathrm{wgt}(P) \;=\; \begin{array}{cccccc} \square & \square & \square & \square & \square & -a_5 \\ & \square & \square & \square & -a_3 & \\ & & \square & a_1 & a_2 & \end{array}
$$

$$
P \;=\; \begin{array}{cccccc} \square & \square & \square & \square & \square & \emptyset \\ & \square & \square & \square & \emptyset' & \\ & & \square & \emptyset & \emptyset & \end{array}
\qquad \mathrm{wgt}(P) \;=\; \begin{array}{cccccc} \square & \square & \square & \square & \square & a_5 \\ & \square & \square & \square & -a_3 & \\ & & \square & a_1 & a_2 & \end{array}
\tag{4.18}
$$

The cancellation arises if and only if the boundary strip consisting of all entries $\emptyset'$ and $\emptyset$ is not continuous or does not start at the second box of the final row. In all such cases the first box in each continuous segment may contain an entry $\emptyset'$ or $\emptyset$, of weight $-a_{\ell-1}$ or $a_{\ell-1}$ with $\ell > 1$, resulting in the required cancellation. We should stress once again that, as a consequence of our punctured lattice hypothesis and condition (**Q8**), the cancellation does not arise in the case of a continuous boundary strip starting in the second box of the final row. It might also be noted that if this box is empty, that is to say $\lambda_{\ell(\lambda)} = 1$, then any boundary strip consisting of entries $\emptyset'$ and $\emptyset$ commences with a box whose entry may be either $\emptyset'$ or $\emptyset$, leading to a cancellation of contributions. In such a case the only contributions to (4.13) are those arising from $\kappa^{(0)} = \lambda$, as pointed out earlier in the derivation of (3.14).

These observations are sufficient to establish the validity of (4.2) in the even orthogonal case with weights as given in the right hand table of (4.3). In the light of this discussion, as an alternative to condition (**Q8**) applying to $P \in \mathcal{P}^{eo}_\lambda$ one might equally well use the condition (**Q9**): any entries $\emptyset'$ and $\emptyset$ must form a continuous boundary strip starting at the second box in the final row with an entry $\emptyset$.

For the sake of summarising the situation, for each g and given n, $\ell(\lambda)$, $\mathbf{d}$ and λ, the bijective correspondence between $\ell(\lambda)$-tuples of non-intersecting lattice paths P_iQ_i for $i=1,2,\ldots,\ell(\lambda)$ and the shifted primed tableaux $P\in\mathcal{P}^g_\lambda$ we offer the following tabulation of the nature of the lattice path edges corresponding to all possible primed shifted tableaux entries P_{ij}, along with their corresponding weights $\mathrm{wgt}(P_{ij})$.

g	P_{ii}	$\mathrm{wgt}(P_{ii})$		r	P_{ii}	$\mathrm{wgt}(P_{ii})$		r
gl	k	x_k		k	k'	y_k		k
oo, sp, eo	k	x_k		$2k-1$	k'	y_k		$2k-1$
oo, sp, eo	$\overline{k}$	$\overline{x}_k$		$2k$	$\overline{k}'$	$\overline{y}_k$		$2k$
g	$P_{ij}, i<j$	$\mathrm{wgt}(P_{ij})$		r	$P_{ij}, i<j$	$\mathrm{wgt}(P_{ij})$		r
gl	k	x_k+a_{j-i}		k	k'	y_k-a_{j-i}		k
oo, sp, eo	k	x_k+a_{j-i}		$2k-1$	k'	y_k-a_{j-i}		$2k-1$
oo, sp, eo	$\overline{k}$	$\overline{x}_k+a_{j-i}$		$2k$	$\overline{k}'$	$\overline{y}_k-a_{j-i}$		$2k$
oo, eo	0	a_{j-i}		$2n+1$	$0'$	$1-a_{j-i}$		$2n+1$
eo	$\overline{0}$	a_{j-i}		$2n+2$	$\overline{0}'$	$-1-a_{j-i}$		$2n+2$
eo	$\emptyset$	a_{j-i}		$2n+3$	$\emptyset'$	$-a_{j-i}$		$2n+3$

(4.19)

The rightmost end point of each edge is at lattice point (r,c), with row r as specified in the table and column $c=j-i+1$. The start and end points of each path are specified for each of our lattices as follows

g	P_i	Q_i
gl	$(d_i,0)$	(n,λ_i)
oo	$(2d_i-\frac{1}{2},0)$	$(2n+1,\lambda_i)$
sp	$(2d_i-\frac{1}{2},0)$	$(2n,\lambda_i)$
eo	$(2d_i-\frac{1}{2},0)$	$(2n+3,\lambda_i)$

(4.20)

and each path is completed by the insertion of vertical edges of weight 1.

This precise specification of the bijective correspondence as above completes the proof of Theorem 13. □

5 Factorial Tokuyama identities

Here we restrict ourselves to the case for which $\lambda=\mu+\delta$ where μ is a partition of length $\ell(\mu)\le n$ and $\delta=(n,n-1,\ldots,1)$ so that λ is a strict partition of length

$\ell(\lambda)=n$. In such case the sums over $\mathbf{d}$ appearing in Theorem 13 reduce to a single term corresponding to the only possible case $\mathbf{d}=(1,2,\ldots,n)$. Moreover, each of the surviving determinants factorises, to yield the following factorial Tokuyama type identities.

Theorem 14 *Let $\lambda=\mu+\delta$ with $\delta=(n,n-1,\ldots,1)$ and μ a partition of length $\ell(\mu)\leq n$. Then for any $\mathbf{x}=(x_1,x_2,\ldots,x_n)$, $\mathbf{y}=(y_1,y_2,\ldots,y_n)$ and $\mathbf{a}=(a_1,a_2,\ldots)$*

$$Q^{gl}_\lambda(\mathbf{x};\mathbf{y}|\mathbf{a}) = \prod_{1\leq i\leq j\leq n}(x_i+y_j)\ s_\mu(\mathbf{x}|\mathbf{a})\,; \tag{5.1}$$

$$Q^{oo}_\lambda(\mathbf{x},\overline{\mathbf{x}};\mathbf{y},\overline{\mathbf{y}},1|\mathbf{a}) = \prod_{1\leq i\leq j\leq n}(x_i+y_j+\overline{x}_i+\overline{y}_j)\ so_\mu(\mathbf{x},\overline{\mathbf{x}},1|\mathbf{a})\,; \tag{5.2}$$

$$Q^{sp}_\lambda(\mathbf{x},\overline{\mathbf{x}};\mathbf{y},\overline{\mathbf{y}}|\mathbf{a}) = \prod_{1\leq i\leq j\leq n}(x_i+y_j+\overline{x}_i+\overline{y}_j)\ sp_\mu(\mathbf{x},\overline{\mathbf{x}}|\mathbf{a})\,; \tag{5.3}$$

$$Q^{eo}_\lambda(\mathbf{x},\overline{\mathbf{x}},\mathbf{y},\overline{\mathbf{y}}|\mathbf{a}) = \prod_{1\leq i\leq j\leq n}(x_i+y_j+\overline{x}_i+\overline{y}_j)\ \ o_\mu(\mathbf{x},\overline{\mathbf{x}}|\mathbf{a})\,. \tag{5.4}$$

Proof: For notational convenience, before embarking on the proof it is helpful to make use of the following special $r=s+1$ cases of Definition 5.

Definition 15 *For all $1\leq p\leq q\leq n$ and all integers m let*

$$f^{gl}_{m,p,q,n}(\mathbf{x};\mathbf{y}|\mathbf{a}) = q^{gl}_m(\mathbf{x}^{(p)};\mathbf{y}^{(q+1)}|\mathbf{a})\,; \tag{5.5}$$

$$f^{oo}_{m,p,q,n}(\mathbf{x},\overline{\mathbf{x}};\mathbf{y},\overline{\mathbf{y}},1|\mathbf{a}) = q^{oo}_m(\mathbf{x}^{(p)},\overline{\mathbf{x}}^{(p)};\mathbf{y}^{(q+1)},\overline{\mathbf{y}}^{(q+1)},1|\mathbf{a})\,; \tag{5.6}$$

$$f^{sp}_{m,p,q,n}(\mathbf{x},\overline{\mathbf{x}};\mathbf{y},\overline{\mathbf{y}}|\mathbf{a}) = q^{sp}_m(\mathbf{x}^{(p)},\overline{\mathbf{x}}^{(p)};\mathbf{y}^{(q+1)},\overline{\mathbf{y}}^{(q+1)}|\mathbf{a})\,; \tag{5.7}$$

$$f^{eo}_{m,p,q,n}(\mathbf{x},\overline{\mathbf{x}};\mathbf{y},\overline{\mathbf{y}}|\mathbf{a}) = q^{eo}_m(\mathbf{x}^{(p)},\overline{\mathbf{x}}^{(p)};\mathbf{y}^{(q+1)},\overline{\mathbf{y}}^{(q+1)}|\mathbf{a})\,. \tag{5.8}$$

In the special case $p=q=d$ these definitions are such that

$$f^{gl}_{m,d,d,n}(\mathbf{x};\mathbf{y}|\mathbf{a}) = q^{gl}_m(\mathbf{x}^{(d)};\mathbf{y}^{(d+1)}|\mathbf{a})\,; \tag{5.9}$$

$$f^{oo}_{m,d,d,n}(\mathbf{x},\overline{\mathbf{x}};\mathbf{y},\overline{\mathbf{y}},1|\mathbf{a}) = q^{oo}_m(\mathbf{x}^{(d)},\overline{\mathbf{x}}^{(d)};\mathbf{y}^{(d+1)},\overline{\mathbf{y}}^{(d+1)},1|\mathbf{a})\,; \tag{5.10}$$

$$f^{sp}_{m,d,d,n}(\mathbf{x},\overline{\mathbf{x}};\mathbf{y},\overline{\mathbf{y}}|\mathbf{a}) = q^{sp}_m(\mathbf{x}^{(d)},\overline{\mathbf{x}}^{(d)};\mathbf{y}^{(d+1)},\overline{\mathbf{y}}^{(d+1)}|\mathbf{a})\,; \tag{5.11}$$

$$f^{eo}_{m,d,d,n}(\mathbf{x},\overline{\mathbf{x}};\mathbf{y},\overline{\mathbf{y}}|\mathbf{a}) = q^{eo}_m(\mathbf{x}^{(d)},\overline{\mathbf{x}}^{(d)};\mathbf{y}^{(d+1)},\overline{\mathbf{y}}^{(d+1)}|\mathbf{a})\,. \tag{5.12}$$

In the case $p=d, q=n$ they reduce to

$$f^{gl}_{m,d,n,n}(\mathbf{x};\mathbf{y}|\mathbf{a}) = h^{gl}_m(\mathbf{x}^{(d)}|\mathbf{a})\,; \tag{5.13}$$

$$f^{oo}_{m,d,n,n}(\mathbf{x},\overline{\mathbf{x}};\mathbf{y},\overline{\mathbf{y}},1|\mathbf{a}) = h^{oo}_m(\mathbf{x}^{(d)},\overline{\mathbf{x}}^{(d)},1|\mathbf{a})\,; \tag{5.14}$$

$$f^{sp}_{m,d,n,n}(\mathbf{x},\overline{\mathbf{x}};\mathbf{y},\overline{\mathbf{y}}|\mathbf{a}) = h^{sp}_m(\mathbf{x}^{(d)},\overline{\mathbf{x}}^{(d)}|\mathbf{a})\,; \tag{5.15}$$

$$f^{eo}_{m,d,n,n}(\mathbf{x},\overline{\mathbf{x}};\mathbf{y},\overline{\mathbf{y}}|\mathbf{a}) = h^{eo}_m(\mathbf{x}^{(d)},\overline{\mathbf{x}}^{(d)}|\mathbf{a})\,. \tag{5.16}$$

The last case comes about because

$$f^{eo}_{m,d,n,n}(\mathbf{x},\overline{\mathbf{x}};\mathbf{y},\overline{\mathbf{y}}|\mathbf{a}) = (a_1a_2\cdots a_m)\delta_{dn} + [t^m]\frac{(1-t^2)\prod_{k=1}^{m+n-d}(1+ta_k)}{\prod_{i=d}^{n}((1-tx_i)(1-t\overline{x}_i))}$$
$$= [t^m]\left(\delta_{dn}(1-\delta_{m0}) + \frac{(1-t^2)}{\prod_{i=d}^{n}((1-tx_i)(1-t\overline{x}_i))}\right)\prod_{k=1}^{m+n-d}(1+ta_k)$$
$$= [t^m]\begin{cases}\left(\dfrac{1}{1-tx_n}+\dfrac{1}{1-t\overline{x}_n}-\delta_{m0}\right)\displaystyle\prod_{j=1}^{m}(1+ta_j) & \text{if } n=d\,;\\ (1-t^2)\displaystyle\prod_{i=d}^{n}\frac{1}{(1-tx_i)(1-t\overline{x}_i)}\prod_{j=1}^{n+m-d}(1+ta_j) & \text{if } n>d\,,\end{cases}$$
$$= h^{eo}_m(\mathbf{x}^{(d)},\overline{\mathbf{x}}^{(d)}|\mathbf{a})\,.$$

Finally, for $1\le p<q\le n$ and all m

$$f^{gl}_{m,p,q-1,n}(\mathbf{x};\mathbf{y}|\mathbf{a}) - f^{gl}_{m,p+1,q,n}(\mathbf{x};\mathbf{y}|\mathbf{a}) = (x_p+y_q)f^{gl}_{m-1,p,q,n}(\mathbf{x};\mathbf{y}|\mathbf{a})\,; \tag{5.17}$$
$$f^{oo}_{m,p,q-1,n}(\mathbf{x},\overline{\mathbf{x}};\mathbf{y},\overline{\mathbf{y}},1|\mathbf{a}) - f^{oo}_{m,p+1,q,n}(\mathbf{x},\overline{\mathbf{x}};\mathbf{y},\overline{\mathbf{y}},1|\mathbf{a})$$
$$= (x_p+y_q+\overline{x}_p+\overline{y}_q)f^{oo}_{m-1,p,q,n}(\mathbf{x},\overline{\mathbf{x}};\mathbf{y},\overline{\mathbf{y}},1|\mathbf{a})\,; \tag{5.18}$$
$$f^{sp}_{m,p,q-1,n}(\mathbf{x},\overline{\mathbf{x}};\mathbf{y},\overline{\mathbf{y}}|\mathbf{a}) - f^{sp}_{m,p+1,q,n}(\mathbf{x},\overline{\mathbf{x}};\mathbf{y},\overline{\mathbf{y}}|\mathbf{a})$$
$$= (x_p+y_q+\overline{x}_p+\overline{y}_q)f^{sp}_{m-1,p,q,n}(\mathbf{x},\overline{\mathbf{x}};\mathbf{y},\overline{\mathbf{y}}|\mathbf{a})\,; \tag{5.19}$$
$$f^{eo}_{m,p,q-1,n}(\mathbf{x},\overline{\mathbf{x}};\mathbf{y},\overline{\mathbf{y}}|\mathbf{a}) - f^{eo}_{m,p+1,q,n}(\mathbf{x},\overline{\mathbf{x}};\mathbf{y},\overline{\mathbf{y}}|\mathbf{a})$$
$$= (x_p+y_q+\overline{x}_p+\overline{y}_q)f^{eo}_{m-1,p,q,n}(\mathbf{x},\overline{\mathbf{x}};\mathbf{y},\overline{\mathbf{y}}|\mathbf{a})\,. \tag{5.20}$$

Proof of Theorem 14: If we now focus, for example, on the symplectic case (5.3) and start by using (3.7) in the case $\ell(\lambda)=n$ then, as we have said, the sum over $\mathbf{d}$ is restricted to a single term with $d_i=i$ for $i=1,2,\ldots,n$. It follows that

$$Q^{sp}_\lambda(\mathbf{x},\overline{\mathbf{x}};\mathbf{y},\overline{\mathbf{y}}|\mathbf{a}) = \prod_{i=1}^{n}(x_i+y_i+\overline{x}_i+\overline{y}_i)\left|\, f^{sp}_{\lambda_j-1;i,i,n}(\mathbf{x},\overline{\mathbf{x}};\mathbf{y},\overline{\mathbf{y}}\,|\mathbf{a})\,\right|\,, \tag{5.21}$$

where we have extracted a common factor $(x_i+y_i+\overline{x}_i+\overline{y}_i)$ from the ith row for $i=1,2,\ldots,n$, and used (5.11). Then, by the repeated subtraction of successive rows from one another and using (5.19) we have

$$\left|f^{sp}_{\lambda_j-1;i,i,n}(\mathbf{x},\overline{\mathbf{x}};\mathbf{y},\overline{\mathbf{y}}\,|\mathbf{a})\right| = \prod_{1\le i<j\le n}(x_i+y_j+\overline{x}_i+\overline{y}_j)\;\left|\, f^{sp}_{\lambda_j-1-n+i;i,n,n}(\mathbf{x},\overline{\mathbf{x}};\mathbf{y},\overline{\mathbf{y}}\,|\mathbf{a})\right|\,. \tag{5.22}$$

We are now in a position to use (5.15) which leads directly to

$$f^{sp}_{\lambda_j-1-n+i;i,n,n}(\mathbf{x},\overline{\mathbf{x}};\mathbf{y},\overline{\mathbf{y}}\,|\mathbf{a}) = \left|\, h^{sp}_{\lambda_j-(n-i+1)}(\mathbf{x}^{(i)},\overline{\mathbf{x}}^{(i)}|\mathbf{a})\,\right|$$
$$= \left|\, h^{sp}_{\mu_j-j+i}(\mathbf{x}^{(i)},\overline{\mathbf{x}}^{(i)}|\mathbf{a})\,\right| = sp_\mu(\mathbf{x},\overline{\mathbf{x}}|\mathbf{a})\,, \tag{5.23}$$

as required to complete the proof of (5.3). The final steps exploit the fact that $\lambda_j = \mu_j + n - j + 1$ for $j = 1, 2, \ldots, n$, as well as the symplectic factorial flagged Jacobi-Trudi identity of Theorem 4. The other results (5.1), (5.2) and (5.4) can be established in exactly the same way. $\square$

6 Closing remarks

Our results are based heavily on the Definitions 3 and 5 of $h_m^g(\mathbf{z}|\mathbf{a})$ and $q_m^g(\mathbf{w};\mathbf{z}|\mathbf{a})$ in terms of generating functions that are manifestly symmetric and supersymmetric, respectively, but these symmetries are not always evident in our main results. For example, in Theorem 4 we have chosen to express the result in terms of a particular choice of flag, namely one for which $\mathbf{x}^{(i)} = (x_i, x_{i+1}, \ldots, x_n)$ with $\mathbf{x}^{(1)} \supset \mathbf{x}^{(2)} \supset \cdots \supset \mathbf{x}^{(n)}$ and $\mathbf{x}^{(i)} \backslash \mathbf{x}^{(i+1)} = x_i$ for $i = 1, 2, \ldots, n-1$ and $\mathbf{x}^{(n)} = x_n$. However, the overall symmetry with respect to permutations of $\mathbf{x}$ means that the results are independent of this particular choice of flag. In particular one might equally well define $\mathbf{x}_{(i)} = (x_1, x_2, \ldots, x_i)$ and adopt a flag $\mathbf{x}_{(1)} \subset \mathbf{x}_{(2)} \subset \cdots \subset \mathbf{x}_{(n)}$ with $\mathbf{x}_{(1)} = x_1$ and $\mathbf{x}_{(i)} \backslash \mathbf{x}_{(i-1)} = x_i$ for $i = 2, 3, \ldots, n$.

This freedom of choice is particularly important when it comes to labelling rows in our non-intersecting lattice path and primed shifted tableaux models of factorial Q-functions based on the use of the supersymmetric functions $q_m^{gl}(\mathbf{x};\mathbf{y}|\mathbf{a})$. We have chosen to interweave the unprimed and primed elements associated with $\mathbf{x}$ and $\mathbf{y}$, respectively, and work with an alphabet $1' < 1 < 2' < 2 < \cdots < n' < n$ as used in this context by Macdonald [27] and Ivanov [18, 19]. But following Molev [28], for example, we might have tried to use the alphabet $1 < 2 < \cdots < n < 1' < 2' < \cdots < n'$ as he did in defining factorial supersymmetric functions. Similar variations of alphabets might be applied to all our models of factorial Q-functions, but each such variation poses a different weighting problem which we will not address here.

Our determinantal definitions of factorial Q-functions were chosen to fit alongside the flagged Jacobi-Trudi expressions for factorial characters. In the special case $\mathbf{y} = \mathbf{x}$ each of then can be expressed, perhaps more conventionally, as the $t = -1$

specializaton of a Hall-Littlewood polynomial:

$$Q^{gl}_\lambda(\mathbf{x};\mathbf{x}|\mathbf{a}) = \frac{2^{\ell(\lambda)}}{v^{gl}_\lambda} \sum_{w\in W^{gl}_n} w\left(\prod_{i=1}^n (x_i|\tau^{-1}\mathbf{a})^{\lambda_i} \prod_{1\le i<j\le n} \frac{1+x_j\overline{x}_i}{1-x_j\overline{x}_i} \right); \quad \text{with } a_0=0 \tag{6.1}$$

$$\begin{aligned} Q^{oo}_\lambda(\mathbf{x},\overline{\mathbf{x}};\mathbf{x},\overline{\mathbf{x}},1|\mathbf{a}) &= \frac{2^{\ell(\lambda)}}{v^{oo}_\lambda} \sum_{w\in W^{oo}_n} w\left(\prod_{i=1}^n (x_i|\tau^{-1}\mathbf{a})^{\lambda_i} \right. \\ &\quad \left. \times \prod_{i=1}^n \frac{1+\overline{x}_i^2}{1-\overline{x}_i^2} \prod_{1\le i<j\le n} \frac{(1+x_j\overline{x}_i)(1+\overline{x}_j\overline{x}_i)}{(1-x_j\overline{x}_i)(1-\overline{x}_j\overline{x}_i)} \right) \quad \text{with } a_0=1\,; \end{aligned} \tag{6.2}$$

$$\begin{aligned} Q^{sp}_\lambda(\mathbf{x},\overline{\mathbf{x}};\mathbf{x},\overline{\mathbf{x}}|\mathbf{a}) &= \frac{2^{\ell(\lambda)}}{v^{sp}_\lambda} \sum_{w\in W^{sp}_n} w\left(\prod_{i=1}^n (x_i|\tau^{-1}\mathbf{a})^{\lambda_i} \right. \\ &\quad \left. \times \prod_{i=1}^n \frac{1+\overline{x}_i^2}{1-\overline{x}_i^2} \prod_{1\le i<j\le n} \frac{(1+x_j\overline{x}_i)(1+\overline{x}_j\overline{x}_i)}{(1-x_j\overline{x}_i)(1-\overline{x}_j\overline{x}_i)} \right) \quad \text{with } a_0=0\,; \end{aligned} \tag{6.3}$$

$$\begin{aligned} Q^{eo}_\lambda(\mathbf{x},\overline{\mathbf{x}};\mathbf{x},\overline{\mathbf{x}}|\mathbf{a}) &= 2^n \sum_{w\in W^{oo}_n} w\left(\prod_{i=1}^n ((x_i+\overline{x}_i)(x_i|\mathbf{a})^{\lambda_i-1}\chi(\lambda_i>1) + x_i\,\chi(\lambda_i=1) + 1\,\chi(\lambda_i=0)) \right. \\ &\quad \left. \times \prod_{1\le i<j\le n} \frac{(1+x_j\overline{x}_i)(1+\overline{x}_j\overline{x}_i)}{(1-x_j\overline{x}_i)(1-\overline{x}_j\overline{x}_i)} \right) \quad \text{if } \lambda_n>0\,; \end{aligned} \tag{6.4}$$

$$\begin{aligned} Q^{eo}_\lambda(\mathbf{x},\overline{\mathbf{x}};\mathbf{x},\overline{\mathbf{x}}|\mathbf{a}) &= \frac{2^{\ell(\lambda)}}{v^{eo}_\lambda} \sum_{w\in W^{eo}_n} w\left(\prod_{i=1}^n ((x_i+\overline{x}_i)(x_i|\mathbf{a})^{\lambda_i-1}\chi(\lambda_i>1) + x_i\,\chi(\lambda_i=1) + 1\,\chi(\lambda_i=0)) \right. \\ &\quad \left. \times \prod_{1\le i<j\le n} \frac{(1+x_j\overline{x}_i)(1+\overline{x}_j\overline{x}_i)}{(1-x_j\overline{x}_i)(1-\overline{x}_j\overline{x}_i)} \right) \quad \text{if } \lambda_n=0\,. \end{aligned} \tag{6.5}$$

where $\chi(P)$ is the truth function for any proposition P, and W^g_n is the Weyl group of the Lie algebra g and

$$v^g_\lambda = \sum_{w\in W^{g,\lambda}_n} (-1)^{\ell(w)} \quad \text{where} \quad W^{g,\lambda}_n = \{\, w\in W^g_n \,:\, w(\lambda)=\lambda \,\}\,.$$

The identity (6.1) follows from [19], while (6.3) has been established in [33]. The corresponding identity (6.2) follows immediately from (6.3) by noting from (3.2) and (3.3) that if one extends the poduct over $(1+ta_k)$ to include an additional factor $(1+ta_0)$ in the latter, then one recovers the former by setting $a_0=1$ and the latter by setting $a_0=0$. In the case (6.4) with $\lambda_n>0$ we are dealing with an irreducible character of $o(2n)$ that is the sum of two irreducible characters of $so(2n)$. This is why the relevant Weyl group is that of $o(2n+1)$, allowing any number of x_i's to be replaced by $\overline{x}_i$. In this case $W^{oo,\lambda}_n$ is just the identity and $v_\lambda(t)=1$. On the other hand in the case of (6.5) with $\lambda_n=0$ the irreducible character of $o(2n)$ remains irreducible on restriction to $so(2n)$. It follows that the relevant Weyl group is that of $so(2n)$, allowing only even numbers of x_i's to be replaced by $\overline{x}_i$'s. The remaining factors $((x_i + \overline{x}_i)(x_i|\mathbf{a})^{\lambda_i - 1}\chi(\lambda_i \geq 1) + x_i\,\chi(\lambda_i = 1) + 1\,\chi(\lambda_i = 0))$ have been inserted only on the basis of computer based checks for all $n \leq 3$ and all strict partitions λ with $\lambda_1 \leq 5$.

Each of the expressions in Theorem 13 in the form of a sum over determinants may be expressed directly as Pfaffian following, for example, the prescription for dealing with non-intersecting lattice paths from a selection of fixed starting points to fixed set of end points in [39]. This has been done already in the case of the factorial Q-function of $gl(n)$ in [19, 17] and of $sp(2n)$ in [33], all by algebraic means. Such Pfaffian expressions offer another opportunity to derive Tokuyama-type identities.

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