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Holographic Renormalisation Group Flows and Supergravity

by

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ABSTRACT

FACULTY OF PHYSICAL SCIENCES AND ENGINEERING

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This thesis can be divided into two related parts. In the first part the idea of the holographic beta function is reviewed and a new method is developed that allows to compute the scalar potential of one-scalar truncations of the five-dimensional gauged supergravity theory, provided that the beta function of the field theory is classical. A class of deformations that is likely to have a classical beta function are the $\mathcal{N} = 1$ preserving operators in short multiplets of the $\mathcal{N} = 4$. We classify all single-trace operators with such properties, and give extra emphasis to F-terms and D-terms. By writing the deformations in the most general way in terms of $\mathcal{N} = 1$ superfields we find interesting relations to pairs of Kaluza-Klein towers that originate from the same ten-dimensional field in the gravity dual. The ideas of the holographic beta function can be generalized to vacuum expectation values, we record some basic observations, and give an outlook for future work.

In the second part a full uplift of the GPPZ flow to ten dimensions is constructed using the exceptional field theory formalism. We obtain the metric, the axion-dilaton matrix, and a full set of RR potentials and fluxes, which are checked to satisfy the IIB equations of motion. The uplift contains an extended version of the GPPZ solution where the mass term m and the gaugino condensate σ are complex, and a U(1) gauge field A_μ is included for consistency. We argue that the phases of the complex scalars are related to the $\mathrm{U}(1)_{\mathrm{R}}$ and the bonus U(1) symmetries of the field theory. We complete a thorough analysis of the asymptotics of the uplift close to the conformal boundary and close to the singularity. While the near-boundary asymptotics are found to agree with the zero-temperature limit of the Freedman-Minahan analysis, we could not fully match with the Polchinski-Strassler solution. The near-singularity limits confirm and extend the results of Pilch-Warner. We show that there are conformal frames in which the singularity in the Ricci scalar is improved, but never completely eliminated. In order to relate the singularity to the presence of D-branes a search for D-brane sources is initiated and the first preliminary results are positive. In anticipation of a future Kaluza-Klein analysis of the solution we start a systematic derivation of corresponding spherical harmonic functions.

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Declaration of Authorship

I, Stanislav Schmidt, declare that this thesis entitled *Holographic Renormalisation Group Flows and Supergravity* and the work presented in it are my own and have been generated by me as the result of my own original research.

I confirm that:

1. This work was done wholly or mainly while in candidature for a research degree at this University;
2. Where any part of this thesis has previously been submitted for a degree or any other qualification at this University or any other institution, this has been clearly stated;
3. Where I have consulted the published work of others, this is always clearly attributed;
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Date:

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Chapter 1

Introduction

A thesis on holography necessarily has to start with a reference to Maldacena's original article [1] in which it was argued that a four-dimensional conformal field theory on a stack of D3 branes, and the gravitational physics on the $AdS_5 \times S^5$ geometry which arises close to these branes are dual to each other. It took a combination of remarkable knowledge, intuition, and insight to put different pieces together, take the right limits, and realise that various approximations can be trusted in order to lay foundations to what is today known as the AdS/CFT correspondence. However, one must not forget that this result was a product of an enormous collective effort of dozens of researchers in the high energy community that paved the way to this important breakthrough. The AdS/CFT correspondence is in fact one concrete realisation of the so-called "holographic principle", according to which the degrees of freedom in certain physical systems are encoded on lower-dimensional surfaces. The ideas that ultimately converged in this principle and in the creation of the duality had their origins in the study of black holes and their intriguing properties, some of which remain a mystery to this day. In the seventies Hawking and Bekenstein studied thermodynamic properties of black holes [2–7], and discovered that many of these properties are related to the two-dimensional black hole surface, rather than its volume. Much of this work was refined and improved in the years that followed, and new discoveries were made. Based on the ideas of 't Hooft [8] the holographic principle found its way into string theory in the nineties, among others through work done by Thorne and Susskind [9, 10]. From there it took several years and an enormous amount of research to arrive at the discovery of the AdS/CFT duality. A lot of effort was put in the study of black holes and black branes in string theory, AdS and other curved spaces and their symmetry properties, truncations and decoupling limits, and many other related topics were studied by a large number or remarkable scientists such as Gibbons, Townsend, Skenderis, Horowitz, Strominger, Seiberg, Sfetsos, Klebanov, and many many others. The list of citations would break all reasonable bounds. After the duality was conjectured by Maldacena, the flood gate was opened, and many researchers committed to studying the implications of the proposal and understanding it in a quantitative way. The articles by Witten, Gubser, Klebanov, and Polyakov [11, 12] were among the first important contributions and filled in many details. From then the power of the implications of the duality was recognized by a wider community and research in various directions ensued. Big reviews were written [13, 14], and year after year we gained a more refined and generalised idea of the inner workings of the holographic principle. Today, twenty years after the discovery of the AdS/CFT duality, we have a much better understanding of much of the details, however the end of the possibilities is not in sight, and holography is still one of the dominating research areas within the high energy physics community.

Maldacena's conjecture relates a superconformal field theory to gravitational physics, and since the early days of this conjecture possibilities were explored whether or not non-conformal field theories can have a holographic dual. A natural way of approaching the non-conformal case is by deforming the conformal field theory in such a fashion that the conformal symmetry is restored in some appropriate limit. At the same time it was also realised that the radial direction in the bulk gravity theory can be related to the energy scale of the field theory so that the conformal boundary of the AdS space is dual to the limit of infinite energies on the field theory side, and movement away from the conformal boundary corresponds to a field theory renormalisation group flow from the UV towards the IR [15–29]. Thus given a conformal field theory, a general deformation will break the conformal symmetry and therefore trigger a renormalisation group flow. The dual bulk description of this system can be thought of as foliated along the radial direction so that different points along the radial direction correspond to different energy scales on the field theory side. If the deformed field theory exhibits conformal fixed points, then on the gravity side the AdS geometry is restored at some points of the radial direction, away from the field theory conformal points the bulk geometry will deviate from the AdS. The standard way of describing a field theory renormalisation group flow is through the so-called beta-function which is assigned to all running coupling constants and describes the rate of change of each constant as the energy scale is varied. Over the years a precise formulation of the dual description of beta-functions was constructed [26, 30–34], which relates them to the scalar potential of the gravity theory. Since general beta-functions are complicated objects this opened up the possibility to circumvent the quantum calculations by studying the gravity dual. In the first part of this thesis we reverse this logic and study what knowledge about the gravitational theory can be inferred provided that the beta-function is known. We restrict our attention to the cases of deformations by one operator, which leads to the running of one coupling, and therefore to a domain-wall profile of one dual scalar mode in the bulk theory. As we will show in the main text, given an exact classical beta-function one can directly integrate it to the so-called “fake” superpotential. This superpotential describes the self-interactions of the dual bulk mode and can be used to determine the corresponding scalar potential. The scalar potentials so obtained match those computed purely from the gravitational perspective, which can be found in literature, and we list all possible one-scalar potentials for dimensions $d \in \{3, 4, 6\}$ that correspond to relevant deformations.

Given the success of the holographic computation of the bulk scalar potential we perform a systematic analysis of $\mathcal{N} = 1$ preserving deformations of the $\mathcal{N} = 4$ super Yang-Mills theory in four dimensions by $\frac{1}{2}$BPS chiral operators. These chiral operators fall into short representations of the PSU$(2, 2|4)$ superconformal group, the structure of which and the relation to bulk modes has been analysed before [35–41]. Our analysis provides a general prescription how the $\mathcal{N} = 1$ branching of any operator in any short multiplet can be identified with $\mathcal{N} = 1$ superfields, and determines their quantum numbers. Given this prescription it is straightforward to list all possible F-terms and D-terms that can be used to construct $\mathcal{N} = 1$ symmetric deformations of the $\mathcal{N} = 4$ super Yang-Mills theory. It was noted long before the discovery of the holographic principle that the compactification of the ten-dimensional supergravity theory on a five-sphere gives rise to pairs of Kaluza-Klein towers that originate from the same ten-dimensional field [42]. Using our analysis we show that if the operators corresponding to the modes of one of these towers are top components of an $\mathcal{N} = 1$ chiral superfield, then the operators in the twinned tower are necessarily top components of a related real superfield. Thus for scalar superfields we obtain F-terms and D-terms that are related to each other. By writing a general F-term and D-term deformation we see that the former is parametrised by a holomorphic function, the $\mathcal{N} = 1$ superpotential, while the latter is gives rise to a harmonic function. This is in agreement with what was found for supersymmetric deformations from the bulk point of view [43].

Some supersymmetric flows are accompanied by a vacuum expectation value of some operators, such as the gaugino condensate in the GPPZ flow [21–23]. Unlike coupling constants the condensates do not have to be added to the action, and therefore do not need to be top components of superfields. In the GPPZ case, for example, one can show that the gaugino bilinear operator that develops a vacuum expectation value is in fact in the bottom component. Nevertheless there are indications that the holographic beta-function can be generalised to condensates in the sense that the holographic description of their energy dependence as domain-wall profiles of some supergravity modes is the same. Nevertheless, there are still some open issues. To obtain a classical beta-function one needs to resort to particular $\mathcal{N} = 1$ non-renormalisation theorems, so that in order to apply the same formalism for condensates one needs to first show that they are similarly protected. We hope to address this issue in future.

In the second part of the thesis we construct a full uplift of the GPPZ solution of the five-dimensional supergravity theory to the ten-dimensional type IIB theory. The GPPZ solution, constructed by Girardello, Petrini, Porrati, and Zaffaroni [21–23] was one of the first studies of a deformed CFT and its holographic dual. The four-dimensional $\mathcal{N} = 4$ super

Yang-Mills theory is deformed by addition of mass terms for chiral superfields. The mass terms break the $\mathcal{N} = 4$ symmetry down to $\mathcal{N} = 1$, which is why this model is sometimes referred to as $\mathcal{N} = 1^*$, in order to distinguish it from the pure $\mathcal{N} = 1$ super Yang-Mills case. In the UV the mass terms, being a relevant deformation, become negligible and one recovers the $\mathcal{N} = 4$ super Yang-Mills. Using this fact as a boundary condition the authors constructed a domain wall solution of the five-dimensional gauged supergravity theory in AdS_5 [44, 45]. The $\mathcal{N} = 1^*$ theory is very rich in itself, exhibiting a variety of different vacua [46–48], which can be of Higgs, confining, or Coulomb type, and are classified by the solutions of the F-term equation and the gauge symmetry breaking pattern. The potential ability to better understand these vacua through the holographic dual is in itself an exciting prospect.

The original GPPZ solution of the five-dimensional gauged supergravity theory involves two real boundary conditions, m_0 and σ_0 corresponding to the field theory UV mass term and the gaugino condensate. Using these boundary conditions one finds a non-normalisable mode $m(r)$ and normalisable mode $\sigma(r)$ in the bulk that have a domain-wall profile with respect to the radial coordinate r and are interpreted as describing the energy dependence of the field theory mass and condensate respectively. Because the scalars m and σ couple to the geometry one also obtains a domain-wall profile of the metric which asymptotes to the AdS space at infinity. One notable feature of this GPPZ solution is a singularity that appears in the IR at some finite value of the radial coordinate. Whether the singularity in m or in σ appears at a larger value of r is controlled by a parameter λ, so that for $\lambda < 1$ one finds the singularity in m first, and for $\lambda > 1$ in σ. For $\lambda = 1$ both singularities coincide. It was argued by GPPZ [23] that on physical grounds one should expect $\lambda \leq 1$, which is the case we restricted to in this thesis.

The appearance of the singularity introduces difficulties in interpreting the field theory dual. Even though the GPPZ solution exhibits some qualitative features that favour the interpretation of one of the confining vacua, it is not possible to extract a definitive answer from the five-dimensional singular solution a priori. It is known that sometimes uplifting a five-dimensional solution to ten dimensions resolves singularities that appear in the lower dimension [29, 48–50]. However, even though the GPPZ solution and partial uplifts have been studied in literature [25, 51–56] before, a full ten-dimensional uplift was never constructed. We construct such an uplift using the approach developed by Baguet, Hohm, and Samtleben [57], which is based on exceptional field theory [58, 59]. We obtain the full ten-dimensional metric, which agrees with that of the partial uplift by Pilch and Warner [54], the axion-dilaton, and a full set of form-potentials and fluxes. To verify that the uplift is indeed a solution of the IIB theory we checked explicitly the Einstein equation, as well as the equations of motion for all fields.

The uplift that we obtain is an extension of the GPPZ solution in the sense that we promote the real fields m and σ to complex fields, and include an additional U(1) gauge field A_μ which is necessary for the truncation to be consistent. The two complex phase rotations of m and σ are accounted for by two U(1) symmetries, one of which can be identified as the R-symmetry of the field theory and acts as coordinate diffeomorphism on in ten-dimensions, while the other corresponds to the so-called bonus U(1) symmetry [60, 61] in field theory and is part of the $\mathrm{SL}(2)_{\mathrm{IIB}}$ on the gravity side.

Given the full uplift of the GPPZ solution we perform some important checks on the asymptotics of the fields. The leading order terms in the expansion close to the conformal boundary can be seen as diagnostic of the consistency of the uplift. As expected we find the $AdS_5 \times S^5$ background and linear 2-form perturbations that correspond to the field theory deformation. After expanding the axion-dilaton, the metric and the 3-form flux we find agreement with the zero-temperature limit of Freedman and Minahan [62], however some of the subleading terms cannot be matched with the analysis by Polchinski and Strassler [48]. By taking the near singularity limit we see that the singular behaviour in the radial coordinate can be improved, however, some singularities in the compact coordinates, called the "ring singularity" by Pilch and Warner [54] persist. The exact structure depends on the parameter λ that we discussed above, but the singularity can never be completely removed. This suggests that one might need to take into account stringy effects to obtain a full description.

As advocated by Polchinski and Strassler [48] and Pilch and Warner [54] it is possible that the resolution of the singularity proceeds via the Myers' effect [50] by which the D3 branes are polarised to higher-dimensional branes by the presence of form fluxes. In order to show that such branes are indeed present we make first attempts of their detection by integrating the fluxes against a test function. The idea is that only in the case where there are non-zero delta-function sources the integral will give a non-zero result, and should be a reliable detection mechanism if applied correctly. The advantage of the integration is that it avoids the manual search of the sources in the equations of motion, but it also introduces some caveats. For example one needs to make sure that all boundaries are taken into account on which the integral gives a non-zero contribution; these could be located at the singularity or at the conformal infinity, and depend on the topology of the space-time. Another difficulty is the analytic evaluation of the integral, and if resorting to numerical methods one needs to ensure that the errors are under control. These are issues that we leave for investigation in further projects.

This thesis includes some additional detailed calculations that might not appear in published articles. In the holographic beta-function part we include the derivation of the GPPZ potential using the gauged supergravity approach. This helps to understand the associated group theory and the symmetry breaking on the one hand, but also shows its complexity compared to the holographic computation on the other hand. Also in the context of the holographic beta-function we review the $\mathcal{N} = 2$ harmonic superspace approach to $\mathcal{N} = 2$ supersymmetric actions, and show how an $\mathcal{N} = 2$ mass term can be added. Furthermore we consider the $\mathcal{N} = 2$ decomposition of $\mathcal{N} = 4$ short multiplets in a fashion similar to that of the $\mathcal{N} = 1$ decomposition. The GPPZ uplift part contains a derivation of some formulas for the Kaluza-Klein curvature tensors for the cases where the space-time manifold has a compact part and where Kaluza-Klein gauge fields are turned on. In another section we explain in detail how spherical scalar, vector, and tensor harmonics can be derived subject to a symmetry constraint. We demonstrate two different approaches, a group theoretical one, and one based on the defining equation. We find and list some solutions, however a complete set of all harmonics requires further work. The appendix contains a discussion of the $\mathfrak{so}(6)$ sigma matrices, and an extensive list their identities. These sigma matrices appear in the $\mathcal{N} = 4$ super Yang-Mills action in four dimensions and in calculations related to it. Finally, in another appendix we record some techniques for holographic calculations of $\mathcal{N} = 4$ four-point function using the Witten-diagram approach. These techniques could be useful in future in order to test predictions for the scalar potential provided by the holographic beta-function approach.

Part I

Field Theory Basics

Chapter 2

The $\mathcal{N} = 4$ Super Yang-Mills Theory

2.1 The Superconformal Algebra

The AdS/CFT correspondence in its original form relates a theory of gravity in five dimensions to a conformal field theory in four dimensions. Apart from the conformal symmetry the field theory is also maximally supersymmetric, and these two symmetries combine into a larger symmetry group, known as the superconformal group. Superconformal algebras only exist in dimensions $d \leq 6$, however in this section we will specialise to $d = 4$, in order the review the superconformal algebra in the dimension in which our field theory computations will be performed.

To study the structure of a superconformal algebra it is worth breaking it down to simpler subalgebras. Start with the Lorentz algebra, the generators of which we will denote by M. It can be extended to the Poincaré algebra by adding translations P. The Poincaré algebra can be further extended in two ways, one can either add dilatations D and special conformal transformations K to promote it to a conformal algebra, or one can add Poincaré supercharges Q and $\bar{Q}$, and R-symmetry generators R to get the usual supersymmetry algebra. One may wonder if these two extensions are compatible. Indeed, it turns out that this is the case under some conditions, and both are combined into a bigger algebra, the superconformal algebra, which contains both the conformal algebra and the supersymmetry algebra as subalgebras. To make the superconformal algebra close, one needs to add the same number of additional fermionic generators as there are Poincaré supercharges. These new supercharges are called conformal supercharges S.

The superconformal algebra is a superalgebra, and following [14] we can group the generators in four blocks, two bosonic, which form the maximal bosonic commuting subalgebras, and two fermionic as follows:

$$\left(\begin{array}{c|c} M_{\mu\nu}, P_\mu, K_\mu, D & Q^A_\alpha, \bar{S}^A_{\dot\alpha} \\ \hline \bar{Q}_{\dot\alpha A}, S_{\alpha A} & R^i \end{array}\right) \tag{2.1}$$

The statements made so far are true for any superconformal algebra, so let us now specialise to the dimension $d = 4$. In this case the algebra is denoted by $\mathfrak{psu}(2,2|4)$, the corresponding group is $\mathrm{PSU}(2,2|4)$, and constitutes the global symmetry group of the $\mathcal{N}=4$ super Yang-Mills theory in four-dimensional Lorentzian space-time. It contains 16 Poincaré supercharges Q, $\bar{Q}$, 16 conformal supercharges S, $\bar{S}$, 15 generators of the conformal subgroup $\mathrm{SU}(2,2)$ and 15 generators of the R-symmetry group $\mathrm{SU}(4)$ [13, 14, 63–66].

The block structure of the superconformal algebra in (2.1) applied to the case $d = 4$ leads to the following maximal bosonic subgroup, which makes the commuting bosonic subgroups manifest:

$$\begin{aligned} \mathrm{PSU}(2,2|4) &\supset \mathrm{SU}(2,2) \times \mathrm{SU}(4)_{\mathrm{R}} \\ &\cong \mathrm{SO}(4,2) \times \mathrm{SO}(6)_{\mathrm{R}} \end{aligned} \tag{2.2}$$

We define the algebra by listing all possible commutators. First consider the commutators of the R-symmetry generators with the rest:

$$\begin{aligned} &[R^i, R^j] = i f^{ijk} R^k && [R^i, M_{\mu\nu}] = [R^i, P_\mu] = [R^i, K_\mu] = [R^i, D] = 0 \\ &[R^i, Q^A_\alpha] = (R^i)^A{}_B Q^B_\alpha && [R^i, \bar{Q}_{\dot\alpha A}] = (\bar{R}^i)_A{}^B \bar{Q}_{\dot\alpha B} \\ &[R^i, \bar{S}^A_{\dot\alpha}] = (R^i)^A{}_B S^B_{\dot\alpha} && [R^i, S_{\alpha A}] = (\bar{R}^i)_A{}^B S_{\alpha B} \end{aligned}$$

The symbols f^{ijk} denote the structure constants of the $\mathrm{SU}(4)$, and because this $\mathrm{SU}(4)$ is one of the commuting bosonic factors, its generators R^i commute with all other bosonic generators. The matrices $(R^i)^A{}_B$ and $(\bar{R}^i)_A{}^B$ are the $\mathbf{4}$ and $\bar{\mathbf{4}}$ representations of the R-symmetry generators R^i [63], and the commutators involving supercharges reflect the fact that they transform in the fundamental or anti-fundamental representation of the R-symmetry. This implies that the R-symmetry is the automorphism group of the supersymmetry algebra.

The commutators of the conformal subalgebra are standard and can be found in many places in literature:

$$\begin{aligned} [M^{\mu\nu}, M^{\rho\sigma}] &= -i\eta^{\mu\rho} M^{\nu\sigma} + i\eta^{\nu\rho} M^{\mu\sigma} - i\eta^{\nu\sigma} M^{\mu\rho} + i\eta^{\mu\sigma} M^{\nu\rho} \\ [M^{\mu\nu}, P^\rho] &= -i\eta^{\mu\rho} P^\nu + i\eta^{\nu\rho} P^\mu \\ [M^{\mu\nu}, K^\rho] &= -i\eta^{\mu\rho} K^\nu + i\eta^{\nu\rho} K^\mu \\ [P^\mu, P^\nu] &= [K^\mu, K^\nu] = [D, D] = [D, M^{\mu\nu}] = 0 \\ [D, P^\mu] &= -iP^\mu \end{aligned}$$

$$[D, K^\mu] = iK^\mu$$
$$[P^\mu, K^\nu] = 2i(M^{\mu\nu} - \eta^{\mu\nu} D).$$

The conventions for the space-time metric can be found in Appendix A. Next consider the commutators of the supercharges with the generators of the conformal subalgebra. One can see that the commutator of the special conformal generator K with a Poincaré supercharge Q produces a conformal supercharge $\bar{S}$, which shows that without the conformal supercharges the algebra would not close.

$$[D, Q^A_\alpha] = -\tfrac{i}{2} Q^A_\alpha \qquad [P^\mu, Q^A_\alpha] = 0$$
$$[D, S_{\alpha A}] = \tfrac{i}{2} S_{\alpha A} \qquad [P^\mu, S_{\alpha A}] = i\sigma^\mu_{\alpha\dot\beta} \bar{Q}^{\dot\beta}_A$$
$$[M^{\mu\nu}, Q^A_\alpha] = \sigma^{\mu\nu}{}_\alpha{}^\beta Q^A_\beta \qquad [K^\mu, Q^A_\alpha] = i\sigma^\mu_{\alpha\dot\beta} \bar{S}^{\dot\beta A}$$
$$[M^{\mu\nu}, S_{\alpha A}] = \sigma^{\mu\nu}{}_\alpha{}^\beta S_{\beta A} \qquad [K^\mu, S_{\alpha A}] = 0.$$

The definitions of the sigma matrices σ^μ and $\sigma^{\mu\nu}$ used in the commutators can be found in Appendix B.1. The remaining commutators are those involving two supercharges:

$$\{Q^A_\alpha, \bar{Q}_{\dot\beta B}\} = 2\sigma^\mu_{\alpha\dot\beta} P_\mu \delta^A_B \qquad \{Q^A_\alpha, Q^B_\beta\} = \{S_{\alpha A}, S_{\beta B}\} = \{Q^A_\alpha, \bar{S}^B_{\dot\beta}\} = 0$$
$$\{S_{\alpha A}, \bar{S}^B_{\dot\beta}\} = 2\sigma^\mu_{\alpha\dot\beta} K_\mu \delta^A_B \qquad \{Q^A_\alpha, S_{\beta B}\} = -i\sigma^{\mu\nu}{}_{\alpha\beta} M_{\mu\nu} \delta^A_B + \epsilon_{\alpha\beta}(\delta^A_B D + R^B_A),$$

where we have defined $R^B_A = \sum_i R^i((\bar{R}^i)_A{}^B - (R^i)^A{}_B)$. Note that as above R^i are the abstract generators of the R-symmetry algebra $\mathfrak{su}(4)$, while the matrices $(R^i)^A{}_B$ and $(\bar{R}^i)_A{}^B$ are the representations $\mathbf{4}$ and $\bar{\mathbf{4}}$ of R^i. The reason why this peculiar combination of R-symmetry generators in form of R^B_A appears on the right-hand side of the anti-commutator $\{Q^A_\alpha, S_{\beta B}\}$ is the consistency with the commutators of the R-symmetry with the supercharges. Using the Jacobi identity $[R, \{Q, S\}] = \{Q, [R, S]\} + \{S, [R, Q]\}$ one finds that the matrices $(R^i)^A{}_B$ and $(\bar{R}^i)_A{}^B$ appear through the commutators $[R, Q]$ and $[R, S]$ and form a combination as indicated. It is also useful to note that for unitary representations in terms of hermitian matrices a representation and its conjugate are related by $\bar{R} = -R^* = -R^t$ so that it is true that $(\bar{R}^i)_A{}^B = -(R^i)^B{}_A$. This allows us to rewrite R^B_A as $R^B_A = -2\sum_i R^i (R^i)^B{}_A = 2\sum_i R^i (\bar{R}^i)_A{}^B$.

2.2 The $\mathcal{N} = 4$ SU(N) Super Yang-Mills Theory in Four Dimensions

The $\mathcal{N} = 4$ super Yang-Mills theory will be the basis for many computations and results in what follows, and therefore in this section we will write down the formulation of this theory in the conventions, notation and terminology which are in accord with the rest of the thesis.

This theory was originally derived from the $\mathcal{N}=1$ super Yang-Mills theory in ten dimensions [67, 68] by dimensional reduction on a six-torus. It is an instructive exercise to perform this reduction, however, since this computation can be found at many places in literature and is not very useful in the current context, we shall not repeat it here.

2.2.1 Action and Fields

We will take the SU(4) covariant formulation available in literature [69–72] as a basis and adopt it to our needs. In particular, we take the fermions λ_A to transform in the anti-fundamental representation rather than fundamental. This convention is more common in literature related to holography [13, 14, 21], and in practical terms this change amounts to swapping the left-chiral and right-chiral Weyl fermions in the transformation rules. Other differences are the use of the definitions for sigma matrices that we think are more standard and intuitive, which are listed in Appendix B, and the mostly negative metric. All this leads to some sign differences in the action and the supersymmetry transformation as compared to some of the literature. Thus the action we will use in this text is given by

$$\begin{aligned} S_4 = \operatorname{tr}\int \mathrm{d}^4x \Big\{ &-\frac{1}{2}F_{\mu\nu}F^{\mu\nu} - \frac{1}{2}D_\mu\bar{\phi}_{AB}D^\mu\phi^{AB} + 2i\bar{\lambda}^A\bar{\sigma}^\mu D_\mu\lambda_A \\ &+ \sqrt{2}g\left(\phi^{AB}[\lambda_A,\lambda_B] - \bar{\phi}_{AB}[\bar{\lambda}^A,\bar{\lambda}^B]\right) + \frac{g^2}{8}[\phi^{AB},\phi^{CD}][\bar{\phi}_{AB},\bar{\phi}_{CD}]\Big\} \end{aligned} \tag{2.3}$$

Note that it is also possible to include the topological term $\delta\mathcal{L} = \operatorname{tr}\left(\frac{\theta}{16\pi^2}F_{\mu\nu}\tilde{F}^{\mu\nu}\right)$ in the action. This term does not have any effect on the local properties of the theory, and in particular no symmetries of the theory are affected. Since we will not make reference to this term in this text we will choose not to include it in the action. It is also possible to rescale all fields by the coupling constant $\{A_\mu,\lambda_A,\phi^{AB}\} \to \frac{1}{g}\{A_\mu,\lambda_A,\phi^{AB}\}$, which removes the coupling constant from the field strength, covariant derivatives, and interaction terms, so that it appears only as an overall factor $\frac{1}{g^2}$ in front of the action. The topological term and the rescaling will be briefly addressed in Section 2.4.

Sometimes it is more convenient to work with explicit colour indices. To this end we can evaluate the trace in (2.3) using

$$\operatorname{tr}(T^aT^b) = \frac{1}{2}\delta^{ab} \qquad \operatorname{tr}(T^a[T^b,T^c]) = \frac{i}{2}f^{abc} \qquad \operatorname{tr}([T^a,T^b][T^c,T^d]) = -\frac{1}{2}f^{abe}f^{cde} \tag{2.4}$$

to get the action in component form with respect to the SU(N) gauge group:

$$\begin{aligned} S_4 = \int \mathrm{d}^4x \Big\{ &-\frac{1}{4}F^a_{\mu\nu}F^{a\mu\nu} - \frac{1}{4}D_\mu\bar{\phi}^a_{AB}D^\mu\phi^{aAB} + i\bar{\lambda}^{aA}\bar{\sigma}^\mu D_\mu\lambda^a_A \\ &+ \frac{ig}{\sqrt{2}}f^{abc}\left(\phi^{aAB}\lambda^b_A\lambda^c_B - \bar{\phi}^a_{AB}\bar{\lambda}^{bA}\bar{\lambda}^{cB}\right) - \frac{g^2}{16}f^{abe}f^{cde}\phi^{a\,AB}\phi^{b\,CD}\bar{\phi}^c_{AB}\bar{\phi}^d_{CD}\Big\}. \end{aligned} \tag{2.5}$$

From the action S_4 we can read off that the field content of the $\mathcal{N}=4$ SYM theory consists of six real scalars, four chiral Weyl fermions and one gauge vector field, which together constitute the $\mathcal{N}=4$ gauge multiplet. The $\mathrm{SU}(4)_\mathrm{R}$ and the Lorentz quantum numbers of these basic fields are the given by

$$\phi^{AB}=\mathbf{6}_{(00)} \qquad \lambda_A=\bar{\mathbf{4}}_{(\frac{1}{2}0)} \qquad A_\mu=\mathbf{1}_{(\frac{1}{2}\frac{1}{2})} \tag{2.6}$$

If one thinks of the $\mathcal{N}=4$ theory as the dimensional reduction of the $\mathcal{N}=1$ super Yang-Mills theory in ten dimensions, then the six scalars are the six real components of the ten-dimensional gauge potential A_M which are aligned along the compact directions. In the notation ϕ^{AB} the $\mathbf{6}$ is represented by an anti-symmetric pair of SU(4) indices, and it is not manifest that this gives six real scalar fields rather than six complex ones. A manifestly real representation can be obtained by applying the Lie-algebra isomorphism $\mathfrak{su}(4)\cong\mathfrak{so}(6)$, in which case the scalars are in the fundamental $\mathbf{6}$ of the SO(6) and can be written as

$$\phi^I(x)\in\mathbb{R}^6, \qquad I=1,\dots,6. \tag{2.7}$$

The linear transformation that relates ϕ^I and ϕ^{AB} is given by the six-dimensional sigma matrices $\Sigma^{I\,AB}$ which are described in Appendix B.2, and reads

$$\phi^{AB}=\frac{1}{\sqrt{2}}\Sigma^{I\,AB}\phi^I. \tag{2.8}$$

A representation is said to be real if it is equivalent to its conjugate representation. In other words, the conjugate representation can be obtained from the original one by a linear transformation. In our case the scalar fields ϕ^{AB} are in the real representation $\mathbf{6}$ of the SU(4) and the reality condition reads

$$\bar{\phi}_{AB}=\left(\phi^{AB}\right)^*=\frac{1}{2}\epsilon_{ABCD}\phi^{CD}. \tag{2.9}$$

This identity follows directly from the analogous one that holds for the six-dimensional sigma matrices described in Appendix B.2. A way of understanding the reality condition is that it means that among the six complex scalars ϕ^{AB} three are in fact complex conjugates of the other three, and thus ϕ^{AB} contains only six real degrees of freedom. One can also see this by spelling out the definition of $\Sigma^{I\,AB}$ and the resulting components of ϕ^{AB}:

$$\phi^{AB}=\frac{1}{\sqrt{2}}\begin{pmatrix} 0 & \phi^3+i\phi^6 & -(\phi^2+i\phi^5) & \phi^1-i\phi^4 \\ -(\phi^3+i\phi^6) & 0 & \phi^1+i\phi^4 & \phi^2-i\phi^5 \\ \phi^2+i\phi^5 & -(\phi^1+i\phi^4) & 0 & \phi^3-i\phi^6 \\ -(\phi^1-i\phi^4) & -(\phi^2-i\phi^5) & -(\phi^3-i\phi^6) & 0 \end{pmatrix}. \tag{2.10}$$

One may wonder about the normalisation of the kinetic term for the scalar fields in (2.3), and it is instructive to transform to the manifestly real SO(6) basis. Using the identities from Appendix B.2 we see that $\frac{1}{4}D_\mu\phi^a_{AB}D^\mu\bar{\phi}^{aAB}=\frac{1}{2}D_\mu\phi^{aI}D^\mu\bar{\phi}^{aI}$, and so because the scalar fields ϕ^{aI} are real, the normalisation of the kinetic term is indeed canonical.

2.2.2 Gauge Transformations

In this subsection we would like to discuss our conventions related to the SU(N) gauge group and gauge transformations which we used in writing down the actions (2.3) and (2.5).

As customary in physics we will use the conventions for Lie algebras where the imaginary unit appears explicitly in the commutator, while the structure constants f^{abc} are real:

$$[T^a,T^b]=if^{abc}T^c. \tag{2.11}$$

Gauge transformations are associated with a propagating gauge field A^a_μ and the corresponding field strength $F^a_{\mu\nu}$, both of which transform in the adjoint representation of the gauge group and can therefore be thought of as elements of the algebra by writing $A_\mu=A^a_\mu T^a$ and $F_{\mu\nu}=F^a_{\mu\nu}T^a$. The field strength is computed from the gauge field as follows

$$F_{\mu\nu}=\partial_\mu A_\nu-\partial_\nu A_\mu-ig[A_\mu,A_\nu], \tag{2.12}$$
$$F^a_{\mu\nu}=\partial_\mu A^a_\nu-\partial_\nu A^a_\mu+gf^{abc}A^b_\mu A^c_\nu. \tag{2.13}$$

This definition is such that the gauge coupling g appears explicitly in the field strength and in the covariant derivatives. It is also common to rescale the gauge field according to $A_\mu\to\frac{1}{g}A_\mu$ and eliminate the explicit gauge coupling g from both the field strength and the covariant derivative. This possibility will be further commented on in Section 2.4.

Gauge transformations act on ordinary fields $\vec{\phi}(x)$ as well as on the gauge fields A_μ and the field strength $F_{\mu\nu}$. Gauge transformations that are close to the identity can be described as exponentials of vectors on the tangent space of the gauge group, which is the gauge algebra, and we can write an element of the group as $g(x)=e^{\alpha(x)}$. For a given choice for a basis $\{T^a\}$ for the algebra, the vector $\alpha(x)\in\mathfrak{su}(N)$ can be decomposed into it as $\alpha(x)=\alpha(x)^aT^a$. With these definitions gauge transformation act on fields as follows

$$\vec{\phi}\to e^{i\alpha}\vec{\phi} \qquad \delta\vec{\phi}=i\alpha\vec{\phi} \tag{2.14}$$
$$F_{\mu\nu}\to e^{i\alpha}F_{\mu\nu}e^{-i\alpha} \qquad \delta F_{\mu\nu}=i[\alpha,F_{\mu\nu}] \tag{2.15}$$
$$A_\mu\to e^{i\alpha}\left(A_\mu+\frac{i}{g}\,\partial_\mu\right)e^{-i\alpha} \qquad \delta A_\mu=i[\alpha,A_\mu]+\frac{1}{g}\partial_\mu\alpha. \tag{2.16}$$

Note that the field $\vec{\phi}$ can be in any representation “R” of the gauge group, and the group and algebra elements that act on it are represented by the corresponding matrices g_R and

T_R^a. The covariant derivative of $\vec{\phi}$ is given by

$$D_\mu \vec{\phi} = (\partial_\mu - igA_\mu)\vec{\phi}. \tag{2.17}$$

The gauge field A_μ entering this definition is implicitly assumed to be in the correct representation "R", so that it can also be written as $A_\mu = A_\mu^a T_R^a$. The covariant derivative is defined such that it transforms covariantly under gauge transformations, and one can check that these transformations are given by

$$(D_\mu \vec{\phi}) \to e^{i\alpha} D_\mu \vec{\phi} \qquad\qquad \delta(D_\mu \vec{\phi}) = i\alpha D_\mu \vec{\phi}. \tag{2.18}$$

Fields that transform in the adjoint representation make no exception and have the same gauge transformation rules and covariant derivatives as for fields in any other representation as in (2.14) and (2.17), with the algebra generators T^a_{adj} in the adjoint representation. However, the fact that the generators in the adjoint representation are given by the structure constants as

$$(T^b_{\text{adj}})^{ac} = if^{abc} \tag{2.19}$$

allows us to write the transformations in (2.14) and (2.17) in an alternative way. Given some field $B^a(x)$ that transforms in the adjoint representation one can think of it as an element of the algebra by writing $B(x) = B(x)^a T^a$. Then the gauge transformations and the covariant derivative can be written as

$$\delta B = i[\alpha, B] \tag{2.20}$$

$$D_\mu B = \partial_\mu B - ig[A_\mu, B]. \tag{2.21}$$

2.2.3 Supersymmetry

Superconformal multiplets are built by applying supercharges to primary operators. Since such multiplets will be the subject of our study later in this text, it makes sense to spell out the basic supersymmetry transformations in detail. This will show that the action in (2.3) is indeed supersymmetric and that our definitions are consistent. Moreover it will also provide elementary building blocks for supersymmetry transformations of composite operators, which can be build from supersymmetry transformations of basic fields. Some of the steps in the variations involve transformations that might not be obvious on the first sight, and carrying them out will serve us as a honing steel for our supersymmetry transformation techniques. The action in (2.3) is invariant under the following $\mathcal{N} = 4$ supersymmetry transformations

$$\delta_\xi \phi^{AB} = \quad 2\sqrt{2}i \left(\bar{\lambda}^{[A} \bar{\xi}^{B]} - \frac{1}{2} \epsilon^{ABCD} \lambda_C \xi_D \right) \tag{2.22a}$$

$$\delta_\xi \bar\phi_{AB} = -2\sqrt{2}i\left(\lambda_{[A}\xi_{B]} - \frac{1}{2}\epsilon_{ABCD}\bar\lambda^C\bar\xi^D\right) \tag{2.22b}$$

$$\delta_\xi \lambda_{\alpha A} = -iF_{\mu\nu}\sigma^{\mu\nu}{}_\alpha{}^\beta\xi_{\beta A} + \sqrt{2}(D_\mu\bar\phi_{AB})\sigma^\mu_{\alpha\dot\beta}\bar\xi^{\dot\beta B} + ig[\bar\phi_{AB},\phi^{BC}]\xi_{\alpha C} \tag{2.22c}$$

$$\delta_\xi \bar\lambda^{\dot\alpha A} = -iF_{\mu\nu}\bar\sigma^{\mu\nu\,\dot\alpha}{}_{\dot\beta}\bar\xi^{\dot\beta A} - \sqrt{2}(D_\mu\phi^{AB})\bar\sigma^{\mu\,\dot\alpha\beta}\xi_{\beta B} + ig[\phi^{AB},\bar\phi_{BC}]\bar\xi^{\dot\alpha C} \tag{2.22d}$$

$$\delta_\xi A^\mu = i\bar\lambda^A\bar\sigma^\mu\xi_A + i\lambda_A\sigma^\mu\bar\xi^A\,. \tag{2.22e}$$

These infinitesimal transformations are related to the supercharges by

$$\delta_\xi\mathcal{O} = [\xi_A Q^A + \bar\xi^A\bar Q_A, \mathcal{O}] \equiv (\xi_A Q^A + \bar\xi^A\bar Q_A)\mathcal{O} \tag{2.23}$$

which allows us to read off the action of Q and $\bar Q$ on the fields from the infinitesimal transformations by setting either ξ or $\bar\xi$ to zero. The result is

$$Q^{\alpha A}\phi^{BC} = \sqrt{2}i\,\epsilon^{ABCD}\lambda^\alpha_D \qquad \bar Q_{\dot\alpha A}\phi^{BC} = -2\sqrt{2}i\,\delta^{[B}_A\bar\lambda^{C]}_{\dot\alpha} \tag{2.24a}$$

$$Q^{\alpha A}\bar\phi_{BC} = 2\sqrt{2}i\,\delta^A_{[B}\lambda^\alpha_{C]} \qquad \bar Q_{\dot\alpha A}\bar\phi_{BC} = -\sqrt{2}i\,\epsilon_{ABCD}\bar\lambda^D_{\dot\alpha} \tag{2.24b}$$

$$Q^{\alpha A}\lambda_{\beta B} = iF_{\mu\nu}\sigma^{\mu\nu}{}_\beta{}^\alpha\delta^A_B + ig[\phi^{AC},\bar\phi_{CB}]\delta^\alpha_\beta \qquad \bar Q_{\dot\alpha A}\lambda_{\beta B} = \sqrt{2}D_\mu\bar\phi_{AB}\sigma^\mu_{\beta\dot\alpha} \tag{2.24c}$$

$$Q^{\alpha A}\bar\lambda^{\dot\beta B} = -\sqrt{2}D_\mu\phi^{AB}\bar\sigma^{\mu\,\dot\beta\alpha} \qquad \bar Q_{\dot\alpha A}\bar\lambda^{\dot\beta B} = iF_{\mu\nu}\bar\sigma^{\mu\nu\,\dot\beta}{}_{\dot\alpha}\delta^B_A + ig[\bar\phi_{AC},\phi^{CB}]\delta^{\dot\beta}_{\dot\alpha} \tag{2.24d}$$

$$Q^{\alpha A}A^\mu = i\bar\lambda^A_{\dot\alpha}\bar\sigma^{\mu\,\dot\alpha\alpha} \qquad \bar Q_{\dot\alpha A}A^\mu = i\lambda^\alpha_A\sigma^\mu_{\alpha\dot\alpha} \tag{2.24e}$$

Let us show that the action (2.3) is indeed invariant under the supersymmetry transformations in (2.22). Schematically the terms in the action transform as follows

$$\delta F^2_{\mu\nu} \sim F_{\mu\nu}\,\xi\sigma^{[\mu}D^{\nu]}\bar\lambda + \text{c.c.} \tag{2.25a}$$

$$\delta(D_\mu\phi)^2 \sim D^\mu\phi\,\xi D_\mu\lambda + gD_\mu\phi[\xi\sigma^\mu\bar\lambda,\phi] + \text{c.c.} \tag{2.25b}$$

$$\delta\bar\lambda\sigma^\mu D_\mu\lambda \sim F_{\mu\nu}\,\xi\sigma^{\mu\nu}\sigma^\rho D_\rho\bar\lambda + D_\mu\phi\,\xi D^\mu\lambda \tag{2.25c}$$

$$+\, g[F_{\mu\nu},\phi]\xi\sigma^{\mu\nu}\lambda + g\lambda\sigma^\mu[\xi\sigma_\mu\bar\lambda,\lambda] + g[\phi,\phi]\xi\sigma^\mu D_\mu\lambda + \text{c.c.} \tag{2.25d}$$

$$\delta(g\lambda[\lambda,\phi] + \text{c.c.}) \sim g\xi F^+[\phi,\lambda] + gD_\mu\phi\,\xi\sigma^\mu[\phi,\bar\lambda] + g\lambda[\xi\lambda,\lambda] + g^2[\phi,\phi]\xi[\phi,\lambda] + \text{c.c.} \tag{2.25e}$$

$$\delta(g^2[\phi,\phi][\phi,\phi]) \sim g^2[\phi,\phi]\xi[\phi,\lambda] + \text{c.c.} \tag{2.25f}$$

One can see that that terms of different orders in g match structurally, which leads to their cancellation in the variation of the action. Let us now compute the variations exactly. Note that throughout these computations one often needs to use various identities for six-dimensional sigma matrices listed in Appendix B.2. We will not include the symbol for the trace over the colour indices in our computations to avoid the clutter, but still occasionally cyclically permute fields as if the trace was there.

Consider first the variations that give terms proportional to g^2. We see from the schematic variations (2.25) that there are two such terms and both come from variations of interaction terms. The variation of the quartic scalar interaction terms is easily evaluated:

$$\delta\frac{g^2}{8}[\phi^{AB},\phi^{CD}][\bar{\phi}_{AB},\bar{\phi}_{CD}] = \frac{g^2}{2}[\delta\phi^{AB},\phi^{CD}][\bar{\phi}_{AB},\bar{\phi}_{CD}] \\ = \sqrt{2}ig^2\Big([\xi_A\lambda_B,\bar{\phi}_{CD}][\phi^{AB},\phi^{CD}] - [\bar{\xi}^A\bar{\lambda}^B,\phi^{CD}][\bar{\phi}_{AB},\bar{\phi}_{CD}]\Big). \quad (2.26)$$

The only other term proportional to g^2 comes from the cubic interaction term involving fermions:

$$\delta\sqrt{2}g\left(\lambda^{\alpha}_{A}[\lambda_{\alpha B},\phi^{AB}] - \bar{\lambda}^{A}_{\dot{\alpha}}[\bar{\lambda}^{\dot{\alpha}B},\bar{\phi}_{AB}]\right)\Big|_{g^2} = 2\sqrt{2}g\left(\delta\lambda^{\alpha}_{A}[\lambda_{\alpha B},\phi^{AB}] - \delta\bar{\lambda}^{A}_{\dot{\alpha}}[\bar{\lambda}^{\dot{\alpha}B},\bar{\phi}_{AB}]\right) \\ = 2\sqrt{2}ig^2\Big([\bar{\phi}_{AC},\phi^{CD}][\xi_D\lambda_B,\phi^{AB}] - [\phi^{AC},\bar{\phi}_{CD}][\bar{\xi}^D\bar{\lambda}^B,\bar{\phi}_{AB}]\Big). \quad (2.27)$$

To make progress note that using the definition $\phi^{AB} = \frac{1}{\sqrt{2}}\Sigma^{I\,AB}\phi^I$ we can obtain a sigma matrix from each of the scalar fields, so that we end up with a product of three sigma matrices from each term. Moreover, the commutator of the scalar fields makes sure that two of the six-dimensional indices of the three sigma matrices are anti-symmetrise. Using identities for products of three sigma matrices from Appendix B.2 we obtain

$$\Sigma^I\overline{\Sigma}^{[J}\Sigma^{K]} = -2\delta^{I[J}\Sigma^{K]} + \Sigma^{[I}\overline{\Sigma}^{J}\Sigma^{K]}. \quad (2.28)$$

Let us show that the last term in this equation inserted into the variation which we were computing gives a vanishing result. If one evaluates the colour trace in the variation one obtains two structure constants with one index contracted of the form $f^{abc}f^{ade}$. The anti-symmetry of the term $\Sigma^{[I}\overline{\Sigma}^{J}\Sigma^{K]}$ makes it cyclic in the three indices, and because it couples to the scalar fields the cyclicity can be transferred onto three of the four free colour indices in $f^{abc}f^{ade}$, but this vanishes by the Jacobi identity for structure constants, concluding the proof. Thus we only need to keep the term $-2\delta^{I[J}\Sigma^{K]}$, and after inserting it into the variation it is straightforward to see that the result is

$$\delta\sqrt{2}g\left(\lambda^{\alpha}_{A}[\lambda_{\alpha B},\phi^{AB}] - \bar{\lambda}^{A}_{\dot{\alpha}}[\bar{\lambda}^{\dot{\alpha}B},\bar{\phi}_{AB}]\right)\Big|_{g^2} \\ = -\sqrt{2}ig^2\Big([\xi_A\lambda_B,\bar{\phi}_{CD}][\phi^{AB},\phi^{CD}] - [\bar{\xi}^A\bar{\lambda}^B,\phi^{CD}][\bar{\phi}_{AB},\bar{\phi}_{CD}]\Big). \quad (2.29)$$

This is exactly the negative of the variation of the quartic scalar interaction term, and we can conclude that the action is supersymmetry-invariant at order g^2.

Next consider the variation terms of the order g^0. The variation of the gauge kinetic term is straightforward to compute and gives

$$\delta\left(-\frac{1}{2}F_{\mu\nu}F^{\mu\nu}\right) = -2iF^{\mu\nu}\left(\xi_A\sigma_\mu D_\nu\bar{\lambda}^A + \bar{\xi}^A\bar{\sigma}_\mu D_\nu\lambda_A\right). \quad (2.30)$$

To compute the variation of the scalar and fermion kinetic terms note first that the covariant derivative contains the gauge field, which also needs to be varied. Thus for any field ψ in the adjoint representation the variation is given by

$$\begin{aligned}\delta D_\mu\psi &= (\delta D_\mu)\psi + D_\mu\delta\psi \\ &= -g[\xi_A\sigma_\mu\bar{\lambda}^A + \bar{\xi}^A\bar{\sigma}_\mu\lambda_A, \psi] + D_\mu\delta\psi.\end{aligned} \tag{2.31}$$

Since the terms that arise from the variation of the covariant derivative are of order g we can neglect them for order g^0 calculations, and will come back to them later. With this in mind the variation of the kinetic term for the scalar is given by

$$\delta\left(-\frac{1}{2}D_\mu\bar{\phi}_{AB}D^\mu\phi^{AB}\right)\Big|_{g^0} = 2\sqrt{2}i\left(D_\mu\bar{\phi}_{AB}\bar{\xi}^A D^\mu\bar{\lambda}^B - D_\mu\phi^{AB}\xi_A D^\mu\lambda_B\right). \tag{2.32}$$

Next we will show that the order g^0 terms coming from the variation of the gauge and scalar kinetic terms are cancelled by the variation of the fermion kinetic term. The application of the variation leads to the following intermediate result

$$\begin{aligned}\delta\left(2i\bar{\lambda}^A\bar{\sigma}^\mu D_\mu\lambda_A\right)\Big|_{g^0} &= -2i(D_\mu\bar{\lambda}^A\bar{\sigma}^\mu\delta\lambda_A + D_\mu\lambda_A\sigma^\mu\delta\bar{\lambda}^A) \\ &= -2\left(D_\rho\bar{\lambda}^A F_{\mu\nu}\bar{\sigma}^\rho\sigma^{\mu\nu}\xi_A + D_\rho\lambda_A F_{\mu\nu}\sigma^\rho\bar{\sigma}^{\mu\nu}\bar{\xi}^A\right) \\ &\quad - 2\sqrt{2}i\left(D_\rho\bar{\lambda}^A D_\mu\bar{\phi}_{AB}\bar{\sigma}^\rho\sigma^\mu\bar{\xi}^B - D_\rho\lambda_A D_\mu\phi^{AB}\sigma^\rho\bar{\sigma}^\mu\xi_B\right).\end{aligned} \tag{2.33}$$

Consider the first term after the last equality. According to the definitions in Appendix B.1 the sigma matrices with two indices are given by $\sigma^{\mu\nu} = \frac{i}{2}\sigma^{[\mu}\bar{\sigma}^{\nu]}$ and $\bar{\sigma}^{\mu\nu} = \frac{i}{2}\bar{\sigma}^{[\mu}\sigma^{\nu]}$. In the same appendix we find that $\bar{\sigma}^\rho\sigma^{[\mu}\bar{\sigma}^{\nu]} = -2\eta^{\rho[\mu}\bar{\sigma}^{\nu]} + i\epsilon^{\rho\mu\nu\sigma}\bar{\sigma}_\sigma$ and $\sigma^\rho\bar{\sigma}^{[\mu}\sigma^{\nu]} = -2\eta^{\rho[\mu}\sigma^{\nu]} - i\epsilon^{\rho\mu\nu\sigma}\sigma_\sigma$. So in sum we can say that $\bar{\sigma}^\rho\sigma^{\mu\nu} = -i\eta^{\rho[\mu}\bar{\sigma}^{\nu]} - \frac{1}{2}\epsilon^{\rho\mu\nu\sigma}\bar{\sigma}_\sigma$ and $\sigma^\rho\bar{\sigma}^{\mu\nu} = -i\eta^{\rho[\mu}\sigma^{\nu]} + \frac{1}{2}\epsilon^{\rho\mu\nu\sigma}\sigma_\sigma$. In the second term we can integrate by parts and split the resulting pair of covariant derivatives $D_\mu D_\rho$ into its symmetric and anti-symmetric parts. The anti-symmetric part is proportional to the field strength, and a careful calculation shows that for any field ψ in the adjoint representation it holds that

$$([D_\mu, D_\nu]\psi)^a = gf^{abc}F^b_{\mu\nu}\psi^c. \tag{2.34}$$

Since this term is proportional to g it can be discarded at the g^0 order. The symmetric combination $D_{(\mu}D_{\rho)}$, in contrast, symmetrises the two sigma matrices, and we should use the formula $\bar{\sigma}^{(\rho}\sigma^{\mu)} = \sigma^{(\rho}\bar{\sigma}^{\mu)} = -\eta^{\rho\mu}$. After these intermediate transformations we obtain

$$\begin{aligned}\delta\left(2i\bar{\lambda}^A\bar{\sigma}^\mu D_\mu\lambda_A\right)\Big|_{g^0} = 2i&\left(D^\mu\bar{\lambda}^A F_{\mu\nu}\bar{\sigma}^\nu\xi_A + D^\mu\lambda_A F_{\mu\nu}\sigma^\nu\bar{\xi}^A\right) \\ &+ \epsilon^{\rho\mu\nu\sigma}\left(D_\rho\bar{\lambda}^A F_{\mu\nu}\bar{\sigma}_\sigma\xi_A - D_\rho\lambda_A F_{\mu\nu}\sigma_\sigma\bar{\xi}^A\right) \\ &+ 2\sqrt{2}i\left(D^\mu\bar{\lambda}^A\xi^B D_\mu\bar{\phi}_{AB} - D^\mu\lambda_A\xi_B D_\mu\phi^{AB}\right).\end{aligned} \tag{2.35}$$

To simplify further recall the Bianchi identity for the field strength, $\epsilon^{\rho\mu\nu\sigma} D_\rho F_{\mu\nu} = 0$. To prove it one uses the relation in equation (2.34) to show that $[D_\rho, [D_\mu, D_\nu]]\psi^c = g f^{abc} (D_\rho F_{\mu\nu})^b \psi^c$ for any adjoint field ψ. We then contract with the Levi-Civita tensor to get $\epsilon^{\rho\mu\nu\sigma}[D_\rho, [D_\mu, D_\nu]]\psi = -ig[\epsilon^{\rho\mu\nu\sigma} D_\rho F_{\mu\nu}, \psi]$. The left-hand side vanishes by the Jacobi identity, and since ψ was arbitrary the proof is concluded. This eliminates the second term from the variation and after re-ordering some terms we obtain

$$\begin{aligned} \delta \left(2i\bar{\lambda}^A \bar{\sigma}^\mu D_\mu \lambda_A \right)\Big|_{g^0} = 2iF^{\mu\nu} \left(\xi_A \sigma_\mu D_\nu \bar{\lambda}^A + \bar{\xi}^A \bar{\sigma}_\mu D_\nu \lambda_A \right) \\ - 2\sqrt{2}i \left(D_\mu \bar{\phi}_{AB} \bar{\xi}^A D^\mu \bar{\lambda}^B - D_\mu \phi^{AB} \xi_A D^\mu \lambda_B \right). \end{aligned} \tag{2.36}$$

This cancels exactly the other order g^0 variations we obtained earlier, so that supersymmetry of the action (2.3) is established at this order.

Finally let us consider the order g. Such terms will appear in the variation of the scalar and fermion kinetic terms, as well as in the variation of the cubic interaction term. The order g terms in the variation of the scalar kinetic term arise from the variation of the covariant derivative. Using the result in (2.31) derived earlier we obtain

$$\delta \left(-\frac{1}{2} D_\mu \bar{\phi}_{AB} D^\mu \phi^{AB} \right)\Big|_g = -g \left(\lambda_D \sigma^\mu \bar{\xi}^D [\bar{\phi}_{AB}, D_\mu \phi^{AB}] + \bar{\lambda}^D \bar{\sigma}^\mu \xi_D [\phi^{AB}, D_\mu \bar{\phi}_{AB}] \right). \tag{2.37}$$

The variation of the fermion kinetic term contains several order g contributions, namely one from the variation of the covariant derivative and two from the variation of the fermions themselves. After substituting the variations one obtains

$$\begin{aligned} \delta \left(2i\bar{\lambda}^A \bar{\sigma}^\mu D_\mu \lambda_A \right)\Big|_g = -2ig\bar{\lambda}^A \bar{\sigma}^\mu [\xi_B \sigma_\mu \bar{\lambda}^B + \bar{\xi}^B \sigma_\mu \lambda^B, \lambda_A] \\ + 2g \left(D_\mu \lambda_A \sigma^\mu \bar{\xi}^B [\phi^{AC}, \bar{\phi}_{CB}] + D_\mu \bar{\lambda}^A \bar{\sigma}^\mu \xi_B [\bar{\phi}_{AC}, \phi^{CB}] \right) \\ - \sqrt{2}i \left(\lambda_A \sigma^\rho \bar{\sigma}^\mu \xi_B [D_\rho, D_\mu] \phi^{AB} - \bar{\lambda}^A \bar{\sigma}^\rho \sigma^\mu \bar{\xi}^B [D_\rho, D_\mu] \bar{\phi}_{AB} \right). \end{aligned} \tag{2.38}$$

Remember that the last term was obtained from the splitting of a pair of covariant derivatives into its symmetric and anti-symmetric parts. While the symmetric combination contributed at order g^0, the anti-symmetric one is proportional to g, which is why we must include it here.

Finally we need the variation of the cubic interaction term at order g. The fermions contribute two terms, and the scalar contributes one, so that we get

$$\delta\sqrt{2}g\left(\lambda^{\alpha}_{A}[\lambda_{\alpha B},\phi^{AB}]-\bar{\lambda}^{A}_{\dot{\alpha}}[\bar{\lambda}^{\dot{\alpha}B},\bar{\phi}_{AB}]\right)\Big|_{g}=$$
$$2\sqrt{2}igF_{\mu\nu}\left([\lambda_{A}\sigma^{\mu\nu}\xi_{B},\phi^{AB}]-[\bar{\lambda}^{A}\bar{\sigma}^{\mu\nu}\bar{\xi}^{B},\bar{\phi}_{AB}]\right)$$
$$+4g\left(\lambda_{D}\sigma^{\mu}\bar{\xi}^{B}[\phi^{AD},D_{\mu}\bar{\phi}_{AB}]+\bar{\lambda}^{D}\bar{\sigma}^{\mu}\xi_{B}[\bar{\phi}_{AD},D_{\mu}\phi^{AB}]\right)$$
$$-4ig\left(\lambda_{A}[\lambda_{B},\bar{\xi}^{[A}\bar{\lambda}^{B]}-\frac{1}{2}\epsilon^{ABCD}\xi_{C}\lambda_{D}]+\bar{\lambda}^{A}[\bar{\lambda}^{B},\xi_{[A}\lambda_{B]}-\frac{1}{2}\epsilon_{ABCD}\bar{\xi}^{C}\bar{\lambda}^{D}]\right). \tag{2.39}$$

One can see that the order g variations are lengthy, and it makes sense to split the problem of matching the terms. We can single out the terms that will cancel by the fields that they contain, and one finds by inspection that there are three types of such terms, namely $(F\lambda\phi)$, (λ^3), and $(D\lambda\phi^2)$. Let us study these terms one by one.

First consider the terms of the form $(F\lambda\phi)$, there are two of them, one from the interaction term, and one from the fermion kinetic term. In the fermion kinetic term we have to evaluate the anti-commutator of two covariant derivatives first. We have already derived the result in (2.34), which can be conveniently written as

$$[D_\rho,D_\mu]\psi=-ig[F_{\rho\mu},\psi] \tag{2.40}$$

for any adjoint field ψ. One can now substitute this result into the corresponding term in the variation of the fermion kinetic term, use the cyclicity of the colour trace, and the fact that $\sigma^{[\rho}\bar{\sigma}^{\mu]}=-2i\sigma^{\mu\nu}$ and $\bar{\sigma}^{[\rho}\sigma^{\mu]}=-2i\bar{\sigma}^{\mu\nu}$ to obtain exactly the negative of the analogous term in the variation of the cubic interaction. Thus we see that the terms of the form $(F\lambda\phi)$ cancel exactly.

Next study the terms of the form (λ^3). Again, the cubic interaction term and the fermion kinetic term each contribute one term that matches this pattern. First note that in the variation of the interaction term there are terms with three fermions of the same chirality. After evaluating the colour trace these terms are of the form

$$f^{abc}\epsilon^{ABCD}(\lambda^{a}_{A}\lambda^{b}_{B})(\lambda^{c}_{C}\xi_{D})\qquad\text{and}\qquad f^{abc}\epsilon_{ABCD}(\bar{\lambda}^{aA}\bar{\lambda}^{bB})(\bar{\lambda}^{cC}\bar{\xi}_{D}). \tag{2.41}$$

Since such terms do not appear anywhere else we expect that they should be identically zero. This can indeed be shown using an appropriate Fierz identity, which states that a product of two fermion bilinears vanishes if three of the four fermions are cyclically symmetrised [73]. This is just another Jacobi-type identity, and can be written as follows. For any fermions ψ_i it is true that

$$(\psi_1\psi_2)(\psi_3\psi_4)+(\psi_2\psi_3)(\psi_1\psi_4)+(\psi_3\psi_1)(\psi_2\psi_4)=0. \tag{2.42}$$

Thanks to the contraction with the structure constants and the Levi-Civita tensor the fermions λ and $\bar{\lambda}$ in (2.41) can be symmetrised in exactly such a cyclic way, and therefore vanish. To see that the remaining terms cancel one should now apply the identity $\sigma^{\mu}_{\alpha\dot{\alpha}}\bar{\sigma}^{\dot{\beta}\beta}_{\mu} = -2\delta^{\beta}_{\alpha}\delta^{\dot{\beta}}_{\dot{\alpha}}$ to the fermion kinetic term. After some careful reshuffling of the terms one obtains

$$\delta(\bar{\lambda}D\lambda) = 4ig\left([\lambda_A, \lambda_B]\bar{\xi}^A\bar{\lambda}^B + [\bar{\lambda}^A, \bar{\lambda}^B]\xi_A\lambda_B\right) + \dots \tag{2.43}$$

$$\delta(g\lambda\lambda\phi) = -4ig\left([\lambda_A, \lambda_B]\bar{\xi}^A\bar{\lambda}^B + [\bar{\lambda}^A, \bar{\lambda}^B]\xi_A\lambda_B\right) + \dots \tag{2.44}$$

It is manifest that both terms cancel.

Finally we would like to show the cancellation of the terms which are of the form $(D\lambda\phi^2)$. Each of the three variations under consideration contributes one such term, which all have to cancel simultaneously. To bring these terms to the same form we integrate the variation of the fermion kinetic term by parts, rename some indices, and regroup some terms. This gives the following intermediate result

$$\delta(D\phi D\phi) = -g\left(\lambda_D\sigma^{\mu}\bar{\xi}^B\delta^D_B[\bar{\phi}_{AC}, D_{\mu}\phi^{AC}] - \bar{\lambda}^D\bar{\sigma}^{\mu}\xi_B\delta^B_D[\phi^{AC}, D^{\mu}\bar{\phi}_{AC}]\right) + \dots \tag{2.45}$$

$$\begin{aligned}\delta(\bar{\lambda}D\lambda) = 2g\Big(&\lambda_D\sigma^{\mu}\bar{\xi}^B\left\{[\phi^{DC}, D_{\mu}\bar{\phi}_{CB}] - [\bar{\phi}_{CB}, D_{\mu}\phi^{DC}]\right\}\\ &+ \bar{\lambda}^D\bar{\sigma}^{\mu}\xi_B\left\{[\bar{\phi}_{DC}, D_{\mu}\phi^{CB}] - [\phi^{CB}, D_{\mu}\bar{\phi}_{DC}]\right\} + \dots\end{aligned} \tag{2.46}$$

$$\delta(g\lambda\lambda\phi) = 4g\left(\lambda_D\sigma^{\mu}\bar{\xi}^B[\phi^{AD}, D_{\mu}\bar{\phi}_{AB}] + \bar{\lambda}^D\bar{\sigma}^{\mu}\xi_B[\bar{\phi}_{AD}, D_{\mu}\phi^{AB}]\right) + \dots \tag{2.47}$$

In the next step one adds these three terms together, and separates the six-dimensional sigma-matrices off the scalar fields. The result of this operation is

$$\begin{aligned}&\delta(D\phi D\phi + \bar{\lambda}D\lambda + g\lambda\lambda\phi) =\\ &\qquad g\lambda_D\sigma^{\mu}\bar{\xi}^B[\phi^I, D_{\mu}\phi^J]\left(-\delta^D_B\overline{\Sigma}^I_{AC}\Sigma^{J\,AC} + 4\Sigma^{[I\,DC}\overline{\Sigma}^{J]}_{CB} + 4\Sigma^{I\,AD}\overline{\Sigma}^J_{AB}\right)\\ &\qquad + g\bar{\lambda}^D\bar{\sigma}^{\mu}\xi_B[\phi^I, D_{\mu}\phi^J]\left(-\delta^B_D\Sigma^{I\,AC}\overline{\Sigma}^J_{AC} + 4\overline{\Sigma}^I_{DC}\Sigma^{J\,CB} + 4\overline{\Sigma}^I_{AD}\Sigma^{J\,AB}\right) + \dots\end{aligned} \tag{2.48}$$

The cancellation is now all due to the sigma-matrix identities, which can be found in Appendix B.2. The identities that we need here are the following

$$\Sigma^{I\,AC}\overline{\Sigma}^J_{AC} = 4\delta^{IJ} \tag{2.49}$$

$$\Sigma^{[I\,DC}\overline{\Sigma}^{J]}_{CB} \equiv -2i(\Sigma^{IJ})^D{}_B \tag{2.50}$$

$$\Sigma^{I\,AD}\overline{\Sigma}^J_{AB} = \delta^{IJ}\delta^C_B + 2i(\Sigma^{IJ})^D{}_B. \tag{2.51}$$

The cancellation is now indeed obvious. At this point we have applied all supersymmetry variations in (2.22) to the action in (2.3) and have shown that they leave that action invariant. To conclude the section we would like to list the complete variations of all terms in the action, which can be used as a reference.

The gauge kinetic term

$$\delta\left(-\frac{1}{2}F_{\mu\nu}F^{\mu\nu}\right) = -\,2iF^{\mu\nu}\left(\xi_A\sigma_\mu D_\nu\bar{\lambda}^A + \bar{\xi}^A\bar{\sigma}_\mu D_\nu\lambda_A\right). \tag{2.52}$$

The scalar kinetic term

$$\begin{aligned}\delta\left(-\frac{1}{2}D_\mu\bar{\phi}_{AB}D^\mu\phi^{AB}\right) = &\quad 2\sqrt{2}i\left(D_\mu\bar{\phi}_{AB}\bar{\xi}^A D^\mu\bar{\lambda}^B - D_\mu\phi^{AB}\xi_A D^\mu\lambda_B\right)\\ &- g\left(\lambda_D\sigma^\mu\bar{\xi}^B\delta^D_B[\bar{\phi}_{AC}, D_\mu\phi^{AC}] - \bar{\lambda}^D\bar{\sigma}^\mu\xi_B\delta^B_D[\phi^{AC}, D^\mu\bar{\phi}_{AC}]\right).\end{aligned} \tag{2.53}$$

The fermion kinetic term

$$\begin{aligned}\delta\left(2i\bar{\lambda}^A\bar{\sigma}^\mu D_\mu\lambda_A\right) = &\quad 2iF^{\mu\nu}\left(\xi_A\sigma_\mu D_\nu\bar{\lambda}^A + \bar{\xi}^A\bar{\sigma}_\mu D_\nu\lambda_A\right)\\ &- 2\sqrt{2}i\left(D_\mu\bar{\phi}_{AB}\bar{\xi}^A D^\mu\bar{\lambda}^B - D_\mu\phi^{AB}\xi_A D^\mu\lambda_B\right)\\ &+ 4ig\left([\lambda_A,\lambda_B]\bar{\xi}^A\bar{\lambda}^B + [\bar{\lambda}^A,\bar{\lambda}^B]\xi_A\lambda_B]\right)\\ &+ 2g\left(D_\mu\lambda_A\sigma^\mu\bar{\xi}^B[\phi^{AC},\bar{\phi}_{CB}] + D_\mu\bar{\lambda}^A\bar{\sigma}^\mu\xi_B[\bar{\phi}_{AC},\phi^{CB}]\right)\\ &+ 2\sqrt{2}ig\left(\lambda_A\sigma^{\rho\mu}\xi_B[F_{\rho\mu},\phi^{AB}] - \bar{\lambda}^A\bar{\sigma}^{\rho\mu}\bar{\xi}^B[F_{\rho\mu},\bar{\phi}_{AB}]\right).\end{aligned} \tag{2.54}$$

The cubic interaction

$$\begin{aligned}&\delta(\sqrt{2}g(\lambda^\alpha_A[\lambda_{\alpha B},\phi^{AB}] - \bar{\lambda}^A_{\dot{\alpha}}[\bar{\lambda}^{\dot{\alpha}B},\bar{\phi}_{AB}])) =\\ &\qquad = -\,4ig\left([\lambda_A,\lambda_B]\bar{\xi}^A\bar{\lambda}^B + [\bar{\lambda}^A,\bar{\lambda}^B]\xi_A\lambda_B\right)\\ &\qquad\quad + 4g\left(\lambda_D\sigma^\mu\bar{\xi}^B[\phi^{AD}, D_\mu\bar{\phi}_{AB}] + \bar{\lambda}^D\bar{\sigma}^\mu\xi_B[\bar{\phi}_{AD}, D_\mu\phi^{AB}]\right)\\ &\qquad\quad + 2\sqrt{2}igF_{\mu\nu}\left([\lambda_A\sigma^{\mu\nu}\xi_B,\phi^{AB}] - [\bar{\lambda}^A\bar{\sigma}^{\mu\nu}\bar{\xi}^B,\bar{\phi}_{AB}]\right)\\ &\qquad\quad - \sqrt{2}ig^2\Big([\xi_A\lambda_B,\bar{\phi}_{CD}][\phi^{AB},\phi^{CD}] - [\bar{\xi}^A\bar{\lambda}^B,\phi^{CD}][\bar{\phi}_{AB},\bar{\phi}_{CD}]\Big).\end{aligned} \tag{2.55}$$

The quartic interaction

$$\delta\left(\frac{g^2}{8}[\phi^{AB},\phi^{CD}][\bar{\phi}_{AB},\bar{\phi}_{CD}]\right) = \quad \sqrt{2}ig^2\Big([\xi_A\lambda_B,\bar{\phi}_{CD}][\phi^{AB},\phi^{CD}] - [\bar{\xi}^A\bar{\lambda}^B,\phi^{CD}][\bar{\phi}_{AB},\bar{\phi}_{CD}]\Big). \tag{2.56}$$

2.3 The $\mathcal{N}=4$ Theory in $\mathcal{N}=1$ Language

The $\mathcal{N}=4$ SYM theory can also be viewed as a theory with fewer supersymmetries by only keeping the generators of some symmetry subgroup and disregarding the rest. One might want to do this for example to simplify the multiplet structure or to break the symmetry down to subgroups that one might want to consider. In our case we will make use of specific non-renormalisation theorems that are known to hold in a theory with $\mathcal{N}=1$ supersymmetry. Thus it makes sense to reformulate the $\mathcal{N}=4$ SYM theory that we have been considering so far in the $\mathcal{N}=1$ language.

Restricting from $\mathcal{N}=4$ to $\mathcal{N}=1$ means breaking the R-symmetry according to $\mathrm{SU}(4)_{\mathrm{R}} \to \mathrm{SU}(3)\times\mathrm{U}(1)_{\mathrm{R}}$. In a generic $\mathcal{N}=1$ theory the $\mathrm{U}(1)_{\mathrm{R}}$ symmetry will be broken by interactions and is therefore not a symmetry of the theory. However, in the case of the $\mathcal{N}=4$ theory the $\mathrm{SU}(4)_{\mathrm{R}}$ *is* a symmetry of the action, and therefore after the restriction to $\mathcal{N}=1$ the $\mathrm{U}(1)_{\mathrm{R}}$ is a symmetry of the action too. The SU(3) factor, in contrast, is an ordinary global flavour symmetry for the chiral $\mathcal{N}=1$ multiplet, as we will see shortly. With respect to this decomposition the representations in which the $\mathcal{N}=4$ fields are branch as follows

$$\mathbf{6} \to \mathbf{3}_{-2} + \bar{\mathbf{3}}_{2} \tag{2.57}$$

$$\mathbf{4} \to \mathbf{3}_{1} + \mathbf{1}_{-3}. \tag{2.58}$$

The U(1) charges in the subscript correspond to the generator given by the following matrix

$$T_{\mathrm{U}(1)} = \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & -3 \end{pmatrix}. \tag{2.59}$$

Because $\mathrm{tr}(T_{\mathrm{U}(1)}T_{\mathrm{U}(1)}) = 12 \neq \frac{1}{2}$ it is not canonically normalised. Since we will use the U(1) charges merely for bookkeeping we prefer not to rescale the generator and so to keep integer charges. We see that under the branching just provided the six scalars decompose into a complex triplet of scalar fields and the fermions split into a triplet and a singlet. The gauge potential being a singlet does not decompose at all. After splitting the fundamental SU(4) index to $A \to (i,4)$ we obtain the following field decomposition

$$\phi^{AB} \to \{\phi^{ij}, \phi^{i4}\} \equiv \{\epsilon^{ijk} z_k, \bar{z}^i\} \tag{2.60}$$

$$\lambda_A \to \{\lambda_i, \lambda\}. \tag{2.61}$$

The complex scalar $\bar{z}^i$ is in the $\mathbf{3}_{-2}$ of the $\mathrm{SU}(3)\times\mathrm{U}(1)$ and z_i is in the $\bar{\mathbf{3}}_{2}$. One can see that this decomposition makes sense by writing out the components of the field ϕ^{AB}. This was already done in equation (2.10) and we just copy the same result once again:

$$\phi^{AB} = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & \phi^3 + i\phi^6 & -(\phi^2 + i\phi^5) & \phi^1 - i\phi^4 \\ -(\phi^3 + i\phi^6) & 0 & \phi^1 + i\phi^4 & \phi^2 - i\phi^5 \\ \phi^2 + i\phi^5 & -(\phi^1 + i\phi^4) & 0 & \phi^3 - i\phi^6 \\ -(\phi^1 - i\phi^4) & -(\phi^2 - i\phi^5) & -(\phi^3 - i\phi^6) & 0 \end{pmatrix}. \tag{2.62}$$

This shows that the new complex scalars are given by

$$z_i = \frac{1}{\sqrt{2}} \left(\phi^i + i\phi^{i+3} \right) \tag{2.63}$$

$$\bar{z}^i = \frac{1}{\sqrt{2}} \left(\phi^i - i\phi^{i+3} \right) \tag{2.64}$$

and are indeed complex conjugates of one another. Overall, this branching suggests that in the $\mathcal{N}=1$ language the gauge multiplet branches into three chiral multiplets and one gauge multiplet, which is indeed the case [74]. The scalars and the fermions which are in the **3** of the SU(3) are components of chiral multiplets, and the remaining fermion and the vector, which are both singlets of the SU(3) form the gauge multiplet. We can now take the $\mathcal{N}=4$ SYM action in (2.3) and decompose all fields in it as just explained. This gives the formulation of the $\mathcal{N}=4$ SYM theory in terms of $\mathcal{N}=1$ fields

$$\begin{aligned} S_4^{\mathcal{N}=1} = \operatorname{tr} \int \mathrm{d}^4 x \Big\{ & -\frac{1}{2} F_{\mu\nu} F^{\mu\nu} - 2 D_\mu \bar{z}^i D^\mu z_i + 2i \bar{\lambda}^i \bar{\sigma}^\mu D_\mu \lambda_i + 2i \bar{\lambda} \bar{\sigma}^\mu D_\mu \lambda \\ & + \sqrt{2} g \left(\epsilon^{ijk} \lambda_i^\alpha [\lambda_{\alpha j}, z_k] - \epsilon_{ijk} \bar{\lambda}^i_{\dot{\alpha}} [\bar{\lambda}^{\dot{\alpha} j}, \bar{z}^k] \right) + 2\sqrt{2} g \left(\lambda_i^\alpha [\lambda_\alpha \bar{z}^i] - \bar{\lambda}^i_{\dot{\alpha}} [\bar{\lambda}^{\dot{\alpha}}, z_i] \right) \\ & + g^2 [z_k, z_l][\bar{z}^k, \bar{z}^l] + g^2 [z_k, \bar{z}^l][\bar{z}^k, z_l] \Big\}. \end{aligned} \tag{2.65}$$

By construction this action has only the $\mathcal{N}=1$ subgroup of the full $\mathcal{N}=4$ symmetry manifestly exposed, the rest of the symmetries of the $\mathcal{N}=4$ superconformal group are not manifest and are therefore called hidden. The fact that all cubic and quartic interactions are proportional to the same coupling constant g is the only hint that the hidden symmetries are there. Note that it is possible to regroup the four scalars in the quartic interaction, and one finds different ways of writing this interaction in literature. After expanding the commutators one realises that some terms are equal and can be grouped together. One can also evaluate the colour trace, which produces two SU(N) structure constants contracted in one index. This allows one to apply the Jacobi identity to further transform this interaction term. We will just leave this term as it is.

Unlike the $\mathcal{N}=4$ theory, the $\mathcal{N}=1$ viewpoint admits a fully off-shell superfield formulation, while the former would need an infinite number of auxiliary fields. To write the $\mathcal{N}=1$ action in this way we view the $\mathcal{N}=4$ fields as components of the following $\mathcal{N}=1$ superfields:

$$Z_i = z_i + \sqrt{2}\theta\lambda_i + \theta^2 F_i \tag{2.66}$$

$$V = \theta\sigma^\mu\bar{\theta}A_\mu + \theta^2\bar{\theta}\bar{\lambda} + \bar{\theta}^2\theta\lambda + \frac{1}{2}\theta^2\bar{\theta}^2 D. \tag{2.67}$$

As we anticipated before, the whole field content of the $\mathcal{N}=4$ theory fits into a triplet of chiral superfields Z_i and a vector gauge superfield V. As usual, chiral fields have to be taken to depend on the chiral coordinate y^μ rather than x^μ, see Appendix A.5 for a summary of some standard supersymmetry definitions used here. The vector superfield has already been gauge-fixed to the Wess-Zumino gauge, and one can construct the field strength multiplet W_α from V in the standard way:

$$\begin{aligned} W_\alpha &= -\frac{1}{4}\bar{D}^2 e^{-2gV} D_\alpha e^{2gV} \\ &= 2g\left(\lambda_\alpha + \theta^\beta\left(\epsilon_{\alpha\beta}D + F^+_{\alpha\beta}\right) + \theta^2 i D_{\alpha\dot{\alpha}}\bar{\lambda}^{\dot{\alpha}}\right), \end{aligned} \tag{2.68}$$

where $F^+ = \sigma^{\mu\nu}F_{\mu\nu}$ is the self-dual part of the vector field strength, and $D_{\alpha\dot{\alpha}} = \sigma^\mu_{\alpha\dot{\alpha}}D_\mu$ the covariant derivative with respect to the SU(N) gauge group, which should not be confused with the auxiliary field D in the θ^β component of W_α. It is possible to remove the overall factor of $2g$ in the components of the field strength W_α by rescaling $V \to \frac{1}{2g}V$. This in fact allows to factor out the gauge coupling g out of the action, and such a normalisation is sometimes referred to as holomorphic. See Section 2.4 for more details on this.

After this preparation it is now straightforward to show that in terms of $\mathcal{N}=1$ superfields the $\mathcal{N}=4$ action takes the following form

$$\begin{aligned} S_4^{\mathcal{N}=1} = 2\,\mathrm{tr}\int \mathrm{d}^4x \Bigg\{ &\int \mathrm{d}^4\theta\, e^{-2gV}\bar{Z}_i e^{2gV} Z^i + \frac{1}{16g^2}\int \mathrm{d}^2\theta\, W^\alpha W_\alpha + h.c. \\ &+ ig\frac{\sqrt{2}}{3!}\left(\int \mathrm{d}^2\theta\,\epsilon^{ijk}Z_i[Z_j,Z_k] + \int \mathrm{d}^2\bar{\theta}\,\epsilon_{ijk}\bar{Z}^i[\bar{Z}^j,\bar{Z}^k]\right)\Bigg\}. \end{aligned} \tag{2.69}$$

The proof that this action indeed reproduces the action in component form (2.65) introduced earlier is elementary but tedious, therefore we will not present it here. To sum up, the $\mathcal{N}=4$ theory written in the language of $\mathcal{N}=1$ superfields contains a gauge vector multiplet and a triplet of chiral multiplets with the superpotential given by

$$\mathcal{W} = 2\,\mathrm{tr}\left(ig\frac{\sqrt{2}}{3!}\epsilon^{ijk}Z_i[Z_j,Z_k]\right). \tag{2.70}$$

2.4 The Holomorphic Gauge Coupling

In this section we would like to discuss two main normalisations for the Yang-Mills field strength called the canonical and the holomorphic normalisations, which are related by rescaling of the gauge potential, and in the supersymmetric case by the rescaling of the gauge vector superfield. These normalisations are often mentioned in literature [66, 75, 76], but here we would like to give a self-consistent overview compatible with our conventions. Note that in this section we will suppress the colour indices in expressions linear and quadratic in fields to make the notation more transparent. To restore them simply add a colour index to each field and contract the colour indices for all field bilinears. To distinguish quantities in canonical normalisation from those in holomorphic normalisation sometimes we will use the superscripts 'c' for canonical and 'h' for holomorphic

Let us start with the Yang-Mills action, and later generalise to the super Yang-Mills case. The usual way of writing the Yang-Mills action is the following:

$$S_{\mathrm{YM}}^{(\mathrm{c})} = \int \mathrm{d}^4 x \left(-\frac{1}{4} F_{\mu\nu} F^{\mu\nu} \right). \tag{2.71}$$

In this case the gauge kinetic term is said to be canonically normalised and the field strength is given by

$$F_{\mu\nu}^{(c)a} = \partial_\mu A_\nu^a - \partial_\nu A_\mu^a + g f^{abc} A_\mu^b A_\nu^c. \tag{2.72}$$

To change to the holomorphic normalisation rescale the gauge potential to $A_\mu \to \frac{1}{g} A_\mu$. This leads the rescaling of the field strength $F_{\mu\nu}^{(\mathrm{c})} \to \frac{1}{g} F_{\mu\nu}^{(\mathrm{h})}$ so that the gauge coupling g does not appear in the definition of the field strength and other quantities related to the gauge field like the covariant derivatives and the gauge transformation of the gauge field itself:

$$F_{\mu\nu}^{(\mathrm{c})a} \to F_{\mu\nu}^{(\mathrm{h})a} = \partial_\mu A_\nu^a - \partial_\nu A_\mu^a + f^{abc} A_\mu^b A_\nu^c. \tag{2.73}$$

In this case the gauge coupling appears in the action as a factor in front of the gauge kinetic term

$$S_{\mathrm{YM}}^{(\mathrm{h})} = \int \mathrm{d}^4 x \left(-\frac{1}{4g^2} F_{\mu\nu} F^{\mu\nu} \right). \tag{2.74}$$

This way of writing the Yang-Mills action seems more natural as it separates the geometric nature of the gauge field and its field strength from the field theoretic coupling. This form of the Yang-Mills action is often used in the study of instantons, while the canonically normalised one is more appropriate for perturbative calculations, where it is useful to have the coupling g appear in the action so that it can be used as the perturbation expansion parameter for perturbation theory.

The idea of the holomorphic coupling can be generalised to the super Yang-Mills case. We recall the definition of the field strength superfield W_α in (2.68), which corresponds to the canonical normalisation:

$$\begin{aligned} W_\alpha^{(\mathrm{c})} &= -\frac{1}{4} \bar{D}^2 e^{-2gV} D_\alpha e^{2gV} \\ &= 2g \left(\lambda_\alpha + \theta^\beta \left(\epsilon_{\alpha\beta} D + F_{\alpha\beta}^{+} \right) + \theta^2 i D_{\alpha\dot{\alpha}} \bar{\lambda}^{\dot{\alpha}} \right). \end{aligned} \tag{2.75}$$

In order to derive the super Yang-Mills action it is useful to recall the definition of the dual field strength $\tilde{F}_{\mu\nu} \equiv \frac{1}{2}\epsilon_{\mu\nu\rho\sigma}F^{\rho\sigma}$, and it is possible to show using the identities and definitions in Appendix B.1 that the following relations for various field strength bilinears hold:

$$\tilde{F}_{\mu\nu} \equiv \frac{1}{2}\epsilon_{\mu\nu\rho\sigma}F^{\rho\sigma} \tag{2.76a}$$

$$\tilde{F}_{\mu\nu}\tilde{F}^{\mu\nu} = -F_{\mu\nu}F^{\mu\nu} \tag{2.76b}$$

$$(F^+)^{\alpha\beta}(F^+)_{\alpha\beta} = -2(F^+)_{\mu\nu}(F^+)^{\mu\nu} \tag{2.76c}$$

$$(F^-)_{\dot{\alpha}\dot{\beta}}(F^-)^{\dot{\alpha}\dot{\beta}} = -2(F^-)_{\mu\nu}(F^-)^{\mu\nu} \tag{2.76d}$$

$$(F^+)_{\mu\nu}(F^+)^{\mu\nu} + (F^-)_{\mu\nu}(F^-)^{\mu\nu} = F_{\mu\nu}F^{\mu\nu} \tag{2.76e}$$

$$i\left[(F^+)_{\mu\nu}(F^+)^{\mu\nu} - (F^-)_{\mu\nu}(F^-)^{\mu\nu}\right] = F_{\mu\nu}\tilde{F}^{\mu\nu}. \tag{2.76f}$$

Thus we see that the F-terms of the field strength superfield bilinears are given by

$$W^\alpha W_\alpha = 4g^2\theta^2\left(2i\lambda\sigma^\mu D_\mu\bar{\lambda} + D^2 - F^+_{\mu\nu}F^{+\,\mu\nu}\right) + \ldots \tag{2.77}$$

$$\overline{W}_{\dot{\alpha}}\overline{W}^{\dot{\alpha}} = 4g^2\bar{\theta}^2\left(2i\bar{\lambda}\sigma^\mu D_\mu\lambda + D^2 - F^-_{\mu\nu}F^{-\,\mu\nu}\right) + \ldots \tag{2.78}$$

and the super Yang-Mills part of the action therefore reads

$$\begin{aligned} S^{(c)}_{\text{SYM}} &= \int \mathrm{d}^4x \int \mathrm{d}^2\theta\, \frac{1}{16g^2}W^\alpha W_\alpha + \text{c.c.} \\ &= \int \mathrm{d}^4x \left(-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + i\bar{\lambda}\bar{\sigma}^\mu D_\mu\lambda + \frac{1}{2}D^2\right). \end{aligned} \tag{2.79}$$

To change to the holomorphic normalisation we should now rescale the whole gauge vector superfield, rather than just the potential, therefore we set $V \to \frac{1}{2g}V$. This gives a definition of the field-strength superfield that is independent of the gauge coupling

$$\begin{aligned} W^{(h)}_\alpha &= -\frac{1}{4}\bar{D}^2 e^{-V}D_\alpha e^V \\ &= \lambda_\alpha + \theta^\beta\left(\epsilon_{\alpha\beta}D + F^+_{\alpha\beta}\right) + \theta^2 iD_{\alpha\dot{\alpha}}\bar{\lambda}^{\dot{\alpha}}. \end{aligned} \tag{2.80}$$

Analogously to the Yang-Mills case we now get an action in which the gauge coupling has been factored out of the action:

$$S^{(h)}_{\text{SYM}} = \int \mathrm{d}^4x \int \mathrm{d}^2\theta\, \frac{1}{4g^2}W^\alpha W_\alpha + \text{c.c.} \tag{2.81}$$

$$= \frac{1}{g^2}\int \mathrm{d}^4x \left(-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + i\bar{\lambda}\bar{\sigma}^\mu D_\mu\lambda + \frac{1}{2}D^2\right). \tag{2.82}$$

As in the Yang-Mills case such a rescaling has the advantage of emphasising the geometric nature of the action rather than the perturbative one. Apart from this it also allows one to introduce a generalised complexified gauge coupling, usually defined as follows

$$\tau = \frac{\theta}{2\pi} + \frac{4\pi i}{g^2}. \tag{2.83}$$

The θ-angle in the real part of τ is a new coupling, and is not related to the Grassmann coordinates θ^α. The key insight is now that in the holomorphic normalisation the gauge coupling does not appear in any of the field definitions nor in the covariant derivative, and enters the action in a holomorphic way as a multiplicative factor in front of the $W^\alpha W_\alpha$ F-term. This in fact allows us to replace the real gauge coupling constant g by the complex coupling τ.

Using the same identities (2.76) for the field strength as before it is straightforward to show that

$$\frac{\tau}{16\pi i} F^{+}_{\mu\nu} F^{+\,\mu\nu} + \text{c.c.} = -\frac{1}{4g^2} F_{\mu\nu}F^{\mu\nu} + \frac{\theta}{32\pi^2} F_{\mu\nu}\tilde{F}^{\mu\nu}. \tag{2.84}$$

It follows immediately that the holomorphically normalised super Yang-Mills action can be written as

$$S_{\text{SYM}} = \int \mathrm{d}^4x \int \mathrm{d}^2\theta \, \frac{\tau}{16\pi i} W_\alpha W^\alpha + \text{c.c.} \tag{2.85}$$

$$= \int \mathrm{d}^4x \left(-\frac{1}{4g^2} F_{\mu\nu}F^{\mu\nu} + \frac{\theta}{32\pi^2} F_{\mu\nu}\tilde{F}^{\mu\nu} + \frac{1}{g^2} i\bar{\lambda}\bar{\sigma}^\mu D_\mu \lambda + \frac{1}{2g^2} D^2 \right). \tag{2.86}$$

We see that the topological term $\frac{\theta}{32\pi^2} F^a_{\mu\nu}\tilde{F}^{a\,\mu\nu}$ now appears naturally in the action. Since in this text we will not touch upon topics related to this term, we will abstain from using the complexified gauge coupling and will use the canonical way of writing the super Yang-Mills term rather than the holomorphic one. In this way all kinetic terms will be automatically canonically normalised.

To conclude this section let us contrast the quantities that involve the gauge coupling g in the canonical normalisation with those in the holomorphic normalisation, where g disappears. These quantities are the field strength, the covariant derivative, and the gauge transformations of the gauge potential.

Recalling the Section 2.2.2 the canonical quantities are given by

Field strength:

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - ig[A_\mu, A_\nu], \tag{2.87}$$

$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g f^{abc} A^b_\mu A^c_\nu \tag{2.88}$$

Gauge transformations:

$$A_\mu \to e^{i\alpha}(A_\mu + \frac{i}{g}\,\partial_\mu)e^{-i\alpha} \tag{2.89}$$

$$\delta A_\mu = i[\alpha, A_\mu] + \frac{1}{g}\partial_\mu \alpha \tag{2.90}$$

Covariant derivative for any field $\vec{\phi}$:

$$D_\mu \vec{\phi} = (\partial_\mu - igA_\mu)\vec{\phi} \tag{2.91}$$

Covariant derivative for a field B in adjoint representation:

$$D_\mu B = \partial_\mu B - ig[A_\mu, B]. \tag{2.92}$$

The same quantities in holomorphic normalisation are given by the following expressions.

Field strength:

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - i[A_\mu, A_\nu], \tag{2.93}$$

$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + f^{abc} A^b_\mu A^c_\nu \tag{2.94}$$

Gauge transformations:

$$A_\mu \to e^{i\alpha}(A_\mu + i\partial_\mu)e^{-i\alpha} \tag{2.95}$$

$$\delta A_\mu = i[\alpha, A_\mu] + \partial_\mu \alpha \tag{2.96}$$

Covariant derivative for any field $\vec{\phi}$:

$$D_\mu \vec{\phi} = (\partial_\mu - iA_\mu)\vec{\phi} \tag{2.97}$$

Covariant derivative for a field B in adjoint representation:

$$D_\mu B = \partial_\mu B - i[A_\mu, B]. \tag{2.98}$$

Thus we see that in the holomorphic normalisation the coupling constant g disappears from all definitions.

Chapter 3

The Operator Spectrum of the $\mathcal{N} = 4$ Super Yang-Mills Theory

The holographic duality ensures that gauge invariant operators of the SYM theory, and their correlation functions can be related to the gravity modes, and their dynamics. Thus it seems essential to understand and classify the possible field theory operators, understand their symmetry transformations and multiplets, and explore ways of systematically constructing them.

Recall that the field content of the $\mathcal{N} = 4$ SYM theory in four dimensions consists of six scalars ϕ^I, four Weyl fermions $\lambda_{\alpha A}$, and one gauge field A_μ, all in the adjoint representation of the $\mathrm{SU}(N)$ gauge group. All of these fields can be identified as components of one vector multiplet of the $\mathcal{N} = 4$ supersymmetry. Thus to construct a gauge invariant operator one can take products of various fields and their derivatives, and trace the result with respect to the gauge group: $\mathcal{O} = \mathrm{tr}(F^{(1)} F^{(2)} \ldots)$, where $F^{(i)}$ can be any of the fields listed above. Operators of this form are called single trace operators, and correspond to single particle states of the gravity theory. These are the objects that are most interesting to us. One can construct more general operators called multi-trace operators by taking products of single trace operators. These operators can have identical quantum numbers as single trace operators, but correspond holographically to multi-particle states of the gravity theory and are generally suppressed as $1/N$. In what follows we shall restrict our attention to single-trace operators.

In a theory with conformal symmetry a common approach to classifying operator multiplets is by finding primary operators and their descendants. Since the R-symmetry group $SU(4)_\mathrm{R}$ with generators R^i and the Lorentz group $\mathrm{SO}(1,3)$ with generators $M_{\mu\nu}$ are commuting bosonic subgroups of the superconformal group $\mathrm{PSU}(2,2|4)$, representations of the latter can be labelled by the quantum numbers of these subgroups. Additionally, there is one more generator of the superconformal group that commutes with both R^i and $L_{\mu\nu}$, the dilaton D, and thus can also be diagonalised at the same time [14], so that the corresponding quantum number is the so-called conformal dimension Δ. In unitary representations the value of Δ is bounded from below [14], and this fact is the basis for the classification by primaries and descendants. Some generators can be assigned a conformal dimension Δ' in the sense that when applied to a state of conformal dimension Δ they create another eigenstate of the dilaton generator D with conformal dimension $\Delta + \Delta'$. One such generator is P_μ and its commutator with D can be written as $[iD, iP_\mu] = iP_\mu$, from where we immediately see that P_μ has the conformal dimension of unity. There are three other generators in the superconformal algebra that have a non-zero conformal dimension:

$$[P_\mu]_D = 1 \qquad [K_\mu]_D = -1 \qquad [Q]_D = \frac{1}{2} \qquad [S]_D = -\frac{1}{2} \tag{3.1}$$

where Q and S stand for all Poincaré supercharges Q^A_α, $\bar{Q}_{\dot{\alpha}A}$ and conformal supercharges $S_{\alpha A}$, $\bar{S}^A_{\dot{\alpha}}$. These pairs of conformal raising and lowering operators can now be used to construct multiplets of the superconformal group. Because the conformal dimension must be bounded from below, every multiplet has an operator of the lowest dimension $\mathcal{O}_P$, called a primary. In pure conformal theory without supersymmetry this means that $[K_\mu, \mathcal{O}_P] = 0$. In a superconformal theory the supercharges constitute a second pair of raising and lowering operators, and in theory for a superconformal primary we should demand $[K_\mu, \mathcal{O}_P] = 0$ and $[S, \mathcal{O}_P] = 0$. However, since the conformal dimension of S is $-\frac{1}{2}$ and that of K_μ is -1, $[S, \mathcal{O}_P] = 0$ is a stronger condition and already implies the other one. Thus we have found that superconformal primary operators $\mathcal{O}_P$ are those for which $[S, \mathcal{O}_P] = 0$, and the superconformal multiplet that they generate can be constructed by computing their descendants by applying Poincaré supercharges Q and Poincaré translations P_μ. Since according to the superconformal algebra $\{Q^A_\alpha, \bar{Q}_{\dot{\beta}B}\} = 2\sigma^\mu_{\alpha\dot{\beta}} P_\mu \delta^A_B$ some combinations of supercharges can be replaced by the operator P_μ, and the number of combinations of supercharges that can be applied to a primary without generating derivatives is therefore finite. In fact, in a theory with 16 supercharges like the $\mathcal{N}=4$ in four dimensions, the maximal number of such combinations is 16, and the highest dimension operator is $Q^8\bar{Q}^8\mathcal{O}_P$.

3.1 Superconformal Primaries

At this point the task of constructing superconformal multiplets is reduced to finding superconformal primaries. To find them observe that descendants are by construction commutators of Poincaré supercharges Q with other operators, thus a primary operator is one that cannot be written as such a commutator [14]. Upon inspection of the supersymmetry transformations of all fields of the $\mathcal{N}=4$ SYM theory we realise that scalar fields appear on the right hand side only as commutators. Thus a symmetric combination of scalar fields cannot be a descendant, and therefore it must be a superconformal primary.

Another refinement of the classification of multiplets is the fact that some primaries are annihilated by certain combination of supercharges and the state of maximal conformal dimension is never reached. Such primaries are called chiral, and the multiplets that are based on them are called short, otherwise the primary is non-chiral and the multiplet is long. The most typical case of a chiral primary is annihilated if more than half of all supercharges are applied to it, thus for an $\mathcal{N} = 4$ theory in $d = 4$ the highest dimension operator in such short representation would be of the form $Q^4\bar{Q}^4\mathcal{O}_P$ rather than $Q^8\bar{Q}^8\mathcal{O}_P$. Such operators are sometimes also called $\frac{1}{2}$BPS because they preserve half of the supersymmetry.

The reason we are interested in $\frac{1}{2}$BPS operators is that it was shown that they are exactly those that correspond to supergravity modes, which also turn out to be in short representations. This makes sense since supercharges have helicities $\pm\frac{1}{2}$, and in a theory with 16 Poincaré supercharges, 8 of positive and 8 of negative helicity, one would generate operators that correspond to supergravity modes of spin higher than 2, which should not be the case for a theory of gravity. Short representations based on $\frac{1}{2}$BPS primaries, in contrast, admit operators that are generated by at most 4 supercharges of the same helicity, which corresponds to supergravity modes with spin at most 2. Representations in long multiplets must thus correspond to stringy modes that were truncated by the supergravity limit and are therefore not accessible in the supergravity approximation.

As we found out above, superconformal primaries are traces over symmetrised products of scalars of the $\mathcal{N} = 4$ theory. It can be shown [13] that chiral primaries are exactly those primaries that are traceless with respect to the $\mathrm{SU}(4)_\mathrm{R}$ indices, which is commonly denoted by

$$\mathcal{O}_p = \operatorname{tr}\phi^{\{I_1}\ldots\phi^{I_p\}} \tag{3.2}$$

where the curly braces mean that the SU(4) indices are symmetrised and the trace is removed. Because the scalar fields are in the $\mathbf{6} = [0, 1, 0]$ of the SU(4), a symmetric traceless product of p such fields corresponds to the irreducible representation $[0, p, 0]$ of the SU(4). Further, it can be shown that the conformal dimension if this operator is $\Delta = p$.

A common way to denote the chiral primaries is

$$\mathcal{O}_p = C^A_{I_1\ldots I_p}\operatorname{tr}(\phi^{I_1}\ldots\phi^{I_p}), \tag{3.3}$$

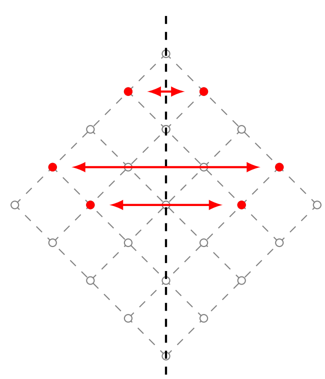
where $C^A_{I_1\ldots I_p}$ is a symmetric traceless SU(4) tensor with respect to its lower indices. The upper index A counts the multiplicity of independent symmetric traceless tensors with p indices, and therefore its range is equal to the dimension of the representation $[0, p, 0]$ in which the operator is. Thus a choice of the tensors $C^A_{I_1\ldots I_p}$ corresponds to a choice of basis for the corresponding SU(4) representation and replaces p SU(4) indices, each of which goes from 1 to 6 by one index that goes from 1 to the dimension of the representation. For example, the operator $\operatorname{tr} \phi^{\{I_1} \phi^{I_2\}}$ is in the $[0, 2, 0] = \boxplus = \mathbf{20'}$ of the SU(4), and can be written as $C^A_{I_1 I_2} \operatorname{tr}(\phi^{I_1} \phi^{I_2})$ with $A \in \{1, \ldots, 20\}$. The fact that $C^A_{I_1 I_2} \operatorname{tr}(\phi^{I_1} \phi^{I_2})$ is in the $\mathbf{20'}$ of the SU(4) manifests itself in the fact that SU(4) transformations that act on it can be represented by 20×20 matrices T^{AB} and the transformation rule becomes $C^A_{I_1 \ldots I_k} \to T^{AB} C^B_{I_1 \ldots I_k}$.

3.2 Superconformal Descendants

To construct the descendants recall again that given a superconformal primary $\mathcal{O}_P$ only a finite number of Poincaré supercharges can be applied to it before we start generating derivatives, which are conformal descendants. If we only pick out those superconformal descendants that cannot be written as derivatives, we get operators that are primaries of the conformal subalgebra, and as we have seen, there are only a finite number of such operators in a given superconformal multiplet.

The group theoretical treatment of superconformal descendants was performed in [77], and in the appendix of [39] the result is presented in a convenient graphical form that we reproduce in Figure 3.2. The down-left arrow corresponds to the action of Q, the down-right to $\bar{Q}$, and the number in the left column is the conformal dimension Δ, which increases by $\frac{1}{2}$ at each level because the Poincaré supercharges Q and $\bar{Q}$ have the conformal dimension of $\frac{1}{2}$. Because the supercharges $\bar{Q}$ and Q are complex conjugates of each other, the whole short multiplet build by acting with Q and $\bar{Q}$ on the primary is invariant under complex conjugation. The action of the complex conjugation on an operator in the short multiplet corresponds to mirroring with respect to a vertical axis, which is schematically represented in Figure 3.1. One can convince oneself this is indeed the case by comparing the representations

Figure 3.1: The complex conjugation of operators in short multiplets interchanges operators that are mirror-equivalent to each other with respect to the central vertical axis.



appearing in the multiplet in Figure 3.2 with their mirror images.

The representations that appear in square brackets in Figure 3.2 are the SU(4) highest weights, and one can see that the representations that appear in an order p multiplet have weights up to $p-4$. Because negative weights are not possible this means that representations built on a superconformal primary with $p \geq 4$ are generic, while those for $p=1$, $p=2$ and $p=3$ will be shorter. These ultra-short multiplets can be found, at least for on-shell fields by removing representations with negative weights. More technical details on these multiplets can be found in [40]. The numbers in the subscript are the Lorentz quantum numbers.

Figure 3.2: The short multiplet of the $\mathcal{N}=4$ super Yang-Mills theory in four dimensions

p	$[0,p,0]_{(0,0)}$
$p+\frac{1}{2}$	$[0,p-1,1]_{(\frac{1}{2},0)}$ $[1,p-1,0]_{(0,\frac{1}{2})}$
$p+1$	$[0,p-1,0]_{(1,0)}$ $[0,p-2,2]_{(0,0)}$ $[1,p-2,1]_{(\frac{1}{2},\frac{1}{2})}$ $[0,p-1,0]_{(0,1)}$ $[2,p-2,0]_{(0,0)}$
$p+\frac{3}{2}$	$[0,p-2,1]_{(\frac{1}{2},0)}$ $[1,p-2,0]_{(1,\frac{1}{2})}$ $[1,p-3,2]_{(0,\frac{1}{2})}$ $[0,p-2,1]_{(\frac{1}{2},1)}$ $[2,p-3,1]_{(\frac{1}{2},0)}$ $[1,p-2,0]_{(0,\frac{1}{2})}$
$p+2$	$[0,p-2,0]_{(0,0)}$ $[1,p-3,1]_{(\frac{1}{2},\frac{1}{2})}$ $[0,p-2,0]_{(1,1)}$ $[2,p-3,0]_{(1,0)}$ $[0,p-3,2]_{(0,1)}$ $[2,p-4,2]_{(0,0)}$ $[1,p-3,1]_{(\frac{1}{2},\frac{1}{2})}$ $[0,p-2,0]_{(0,0)}$
$p+\frac{5}{2}$	$[1,p-3,0]_{(0,\frac{1}{2})}$ $[0,p-3,1]_{(\frac{1}{2},1)}$ $[2,p-4,1]_{(\frac{1}{2},0)}$ $[1,p-3,0]_{(1,\frac{1}{2})}$ $[1,p-4,2]_{(0,\frac{1}{2})}$ $[0,p-3,1]_{(\frac{1}{2},0)}$
$p+3$	$[0,p-3,0]_{(0,1)}$ $[2,p-4,0]_{(0,0)}$ $[1,p-4,1]_{(\frac{1}{2},\frac{1}{2})}$ $[0,p-3,0]_{(1,0)}$ $[0,p-4,2]_{(0,0)}$
$p+\frac{7}{2}$	$[1,p-4,0]_{(0,\frac{1}{2})}$ $[0,p-4,1]_{(\frac{1}{2},0)}$
$p+4$	$[0,p-4,0]_{(0,0)}$

The first non-trivial example for which traces over the colour indices do not give zero is the $p=2$ short multiplet. It is the supermultiplet built on the superconformal primary $\mathrm{tr}\,\phi^{\{I}\phi^{J\}} = [0,2,0] = \mathbf{20'}$, and operators of this multiplet were used by GPPZ [21–23] to study field theory deformations. This is exactly the multiplet that corresponds to the massless multiplet of the graviton on the bulk side. Because of this, and since the graviton multiplet is the only one present in the gauged supergravity approach, the $p=2$ short multiplet was one of the first and basic operator multiplets to undergo tests of the holographic principle.

We can check the first few terms of the $p=2$ supermultiplet against the structure in Figure 3.2. Consider the repeated application of the Poincaré supercharged Q, we expect the following result: $[0,2,0]_{(0,0)} \overset{Q}{\to} [0,1,1]_{(\frac{1}{2},0)} \overset{Q}{\to} [0,1,0]_{(1,0)} \oplus [0,0,2]_{(0,0)}$, or, in other words

$$\mathbf{20'}_{(0,0)} \overset{Q}{\to} \mathbf{20}_{(\frac{1}{2},0)} \overset{Q}{\to} \mathbf{6}_{(1,0)} \oplus \overline{\mathbf{10}}_{(0,0)} \tag{3.4}$$

these are exactly the representations we obtain if we apply the super-transformation rules introduced earlier:

$$\mathcal{O} = \operatorname{tr}\phi^{\{I}\phi^{J\}} = \frac{1}{2}\left\{\overline{\Sigma}^{(I}_{AB}\overline{\Sigma}^{J)}_{CD} - \frac{1}{3}\delta^{IJ}\epsilon_{ABCD}\right\}\operatorname{tr}(\phi^{AB}\phi^{CD}) \tag{3.5}$$

$$Q^{\alpha A}\mathcal{O} = \frac{i}{\sqrt{2}}\left\{\Sigma^{(I\,AG}\overline{\Sigma}^{J)}_{EF} - \frac{2}{3}\delta^{IJ}\delta^{[A}_{E}\delta^{G]}_{F}\right\}\operatorname{tr}(\lambda^{\alpha}_{G}\phi^{EF}) \tag{3.6}$$

$$\begin{aligned} Q^{\beta B}Q^{\alpha A}\mathcal{O} = &-2\left\{\Sigma^{(I\,AG}\Sigma^{J)BK} - \frac{1}{3}\delta^{IJ}\epsilon^{AGBK}\right\}\operatorname{tr}(\lambda^{(\alpha}_{[G}\lambda^{\beta)}_{K]}) \\ &- \frac{1}{\sqrt{2}}\left\{\Sigma^{(I\,AB}\Sigma^{J)GK} - \frac{1}{3}\delta^{IJ}\epsilon^{ABGK}\right\}\operatorname{tr}(F^{+\,\alpha\beta}\phi_{GK}) \\ &+ \epsilon^{\alpha\beta}\Sigma^{(I\,AG}\Sigma^{J)BK}\operatorname{tr}(\lambda_{(G}\lambda_{K)}) \\ &- \epsilon^{\alpha\beta}\frac{g}{\sqrt{2}}\left\{\Sigma^{(I\,AG}\Sigma^{J)DH} - \frac{1}{3}\delta^{IJ}\epsilon^{AGDH}\right\}\operatorname{tr}(\bar{\phi}_{DH}[\phi^{BC},\bar{\phi}_{CG}]). \end{aligned} \tag{3.7}$$

In the last descendant the first two terms, which are symmetric in the spinor indices $(\alpha\beta)$, form the operator in the $\mathbf{6}_{(1,0)}$, and the other two terms, which are proportional to $\epsilon^{\alpha\beta}$, form together the operator in the $\overline{\mathbf{10}}_{(0,0)}$. These are exactly the representations we expected to find.

3.3 Matching with the Bulk Representations

The $\mathcal{N}=4$ SYM theory is holographically related to the type IIB string theory in 10 dimensions, and therefore also to the type IIB supergravity theory on AdS_5 that is obtained by dimensionally reducing the 10-dimensional theory on the S^5. By the holographic principle, everything that appears on the field theory side must be mapped to quantities on the string theory side, in particular the quantum numbers that appear on either side must be equivalent. However, it is not clear that all these quantities will still appear after taking the large N or the supergravity limit on the bulk side. In fact it must be that this is not true since by taking these approximations we simplify the theory by eliminating degrees of freedom. However, it has been established that the operators in short representations do map to bulk modes that survive the supergravity limit, and in this section we would like to study what this correspondence exactly is.

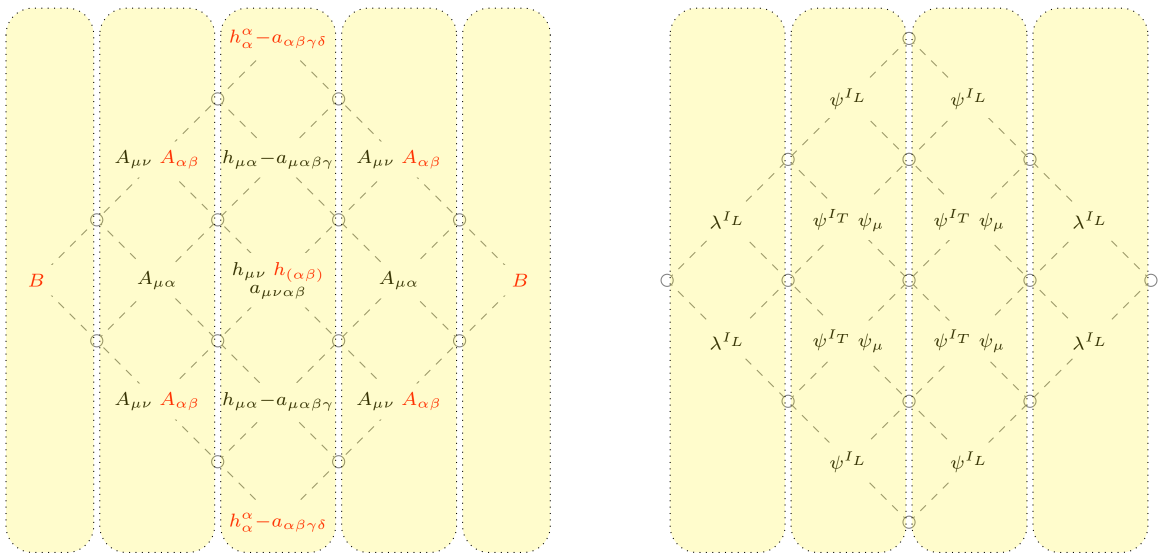
Let us start with the bulk side of the correspondence. To dimensionally reduce from ten to five dimensions one compactifies the string theory on a five-dimensional sphere. The resulting field content of the type IIB supergravity theory on $AdS_5 \times S^5$ was systematically constructed in [42] long time before the holographic correspondence was discovered. One starts by taking the so-called Freund-Rubin ansatz for the background solution. The ten-dimensional modes around this solution are then expanded into harmonics around the five-dimensional sphere, and the expansion coefficients of each ten-dimensional field become towers of five-dimensional modes on AdS_5, the so-called Kaluza-Klein modes. The isometry group of the five-sphere is SO(6), and the spherical harmonics determine the SO(6) representation of the modes that they correspond to. By analysing the linearised equations of motion one can obtain expressions for the mass of every mode. The result of this analysis is summarised in Table III of [42]. The isometry group of AdS_5 is $\mathrm{SO}(4,2)$, and so one obtains the total isometry group $\mathrm{SO}(4,2) \times \mathrm{SO}(6)$. Because the ten-dimensional theory that we started with also had an $\mathcal{N} = 2$ supersymmetry, the five-dimensional theory inherits supersymmetry transformations, and one realises that the resulting total symmetry group is $\mathrm{PSU}(2,2|4)$, the superconformal group, which is exactly the symmetry group of the $\mathcal{N} = 4$ SYM theory in $d = 4$ that we discussed in the previous sections.

The supergravity theory involves only modes of at most spin 2, so it is a proper theory of gravity, and not some higher spin theory. This is possible since all fields that are kept in the supergravity limit fit into short representations of the superconformal group. Since the early times of the holographic duality it was found [11, 12] that gauge-invariant operators of the SYM theory should correspond to supergravity modes, and their correlation functions to supergravity solutions with boundary conditions dictated by the SYM operators under consideration. This conjecture has been subjected to substantial testing ever since, and the body of literature on this topic is immense. It emerges that in terms of representation theory the contents of short supermultiplets of Figure 3.2 can be exactly matched with Kaluza-Klein towers of the supergravity theory.

If one inspects more closely which field theory operators correspond to which bulk modes some patterns start to emerge. First remember that tensor indices of ten-dimensional fields split into those along the S^5 and those that point into AdS_5 directions. This gives rise to bulk fields with various tensor structures. For example a ten-dimensional 2-form field $A_{\hat{\mu}\hat{\nu}}$, where the hatted indices refer to ten-dimensional quantities will split into a 2-form field $A_{\mu\nu}$, a 1-form field $A_{\mu\alpha}$, and a scalar $A_{\alpha\beta}$ after dimensional reduction. After studying the quantum numbers of such fields and comparing them to the quantum numbers in short multiplets one realizes that 5-dimensional fields that originate from the same 10-dimensional one fall into columns, as shown in Figure 3.3.

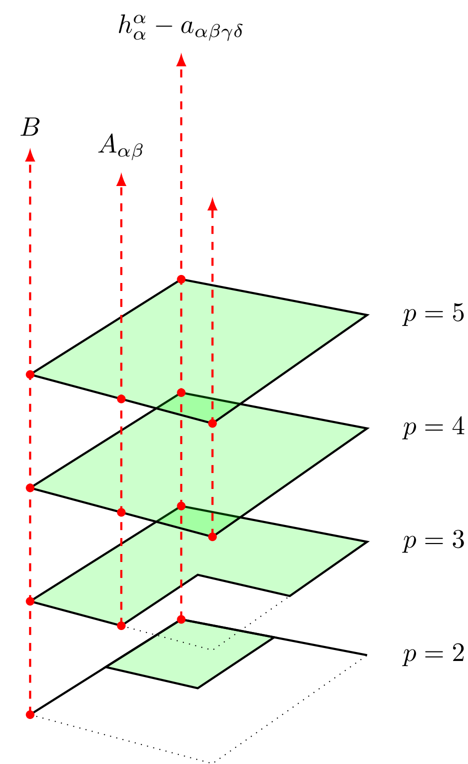
The splitting of the vector indices of fields is not the only thing that happens upon dimensional reduction—also the space-time coordinates on which the fields depend separate into coordinates on the lower-dimensional AdS spacetime and the coordinates on the internal S^5.

Figure 3.3: The mapping of five-dimensional fields that emerge after dimensional reduction on the S^5 to operators in short multiplets in the $\mathcal{N} = 4$ SYM theory. Those five-dimensional fields that correspond to the same ten-dimensional field sit in columns that sweep across short multiplets. On the left graph one can see the bosonic fields, and on the right graph the fermionic ones. The labels in red correspond to the five-dimensional scalars. The labels of the fields correspond to those used in [42].



An observer living on the AdS space does not see the internal space, instead the isometries of that space manifest themselves as symmetries of the five-dimensional fields. Instead of one ten-dimensional field, an observer in five-dimensions will see a host of five-dimensional fields distinguished by the way they transform under these internal isometries, thus measuring irreducible representations. Formally speaking what happens is that ten-dimensional fields $\phi(x, y)$ with the coordinates x on the AdS space and y on the internal space are expanded into a basis on the S^5 that provides various irreducible representations of the S^5 isometry group SO(6). These are exactly the spherical harmonics, and the expansion can be schematically written as $\phi(x, y) = \sum_k \phi(x)_k Y^k(y)$. The infinite towers of five-dimensional modes $\phi(x)_k$ is what people refer to as Kaluza-Klein fields, and it is those fields that are physical fields from the five-dimensional perspective. One may wonder how these Kaluza-Klein fields are mapped to the short multiplets on the field theory side. Remember that each superconformal primary composed of p scalar fields gives rise to a short multiplet that we call of order p. It turns out that starting with the lowest spherical harmonics in the decomposition of the ten-dimensional bulk fields each consecutive layer in the Kaluza-Klein tower corresponds to an operator of a short multiplet of a higher order. The location of this operator in the short multiplet is the same as that of all other operators corresponding to the same Kaluza-Klein tower. Thus viewing fields mapped across the short multiplet in Figure 3.3 as roots of Kaluza-Klein towers one can imagine the towers growing perpendicularly to those multiplets with each new level corresponding to an order p increased by one. Figure 3.4 demonstrates what we mean on a three-dimensional picture. The precise mapping of Kaluza-Klein towers,

Figure 3.4: Mapping of Kaluza-Klein towers on AdS_5 to short multiplets on the field theory side. The green surfaces and the black contours represent the short multiplets, while the red arrows running across them represent the Kaluza-Klein towers, where only a selection of towers is shown as an example. One can see that the towers pierce the multiplets perpendicularly so that the location of the operators corresponding to one Kaluza-Klein tower stays the same. The lowest short multiplet is that with $p = 2$, however, some towers only start from $p = 3$ or $p = 4$, which leads to ultra-short $p = 2$ and $p = 3$ multiplets.

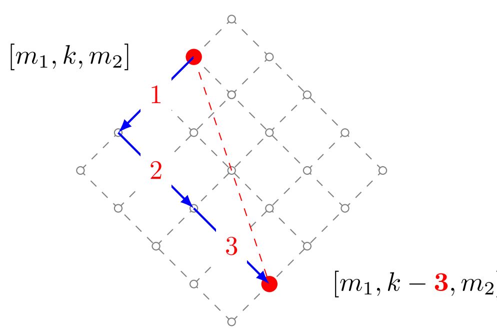


and their quantum numbers and masses to the field theory operators is summarized in Table 3.1 at the end of this section.

Before ending the section we need to explore one more detail. After looking at the mapping of the bulk fields to field theory operators in Figure 3.3 one may notice that some bulk modes appear multiple times. One reason for this is due to the complex conjugation symmetry of short multiplets explained in the previous section, so that operators related by mirroring with respect a horizontal axis as shown in Figure 3.1 in fact correspond to the same complex bulk field. A corollary of this is that fields that sit on the symmetry axis are real. The mirror symmetry is however not the whole story, and one can see that there are copies of the same bulk fields in Figure 3.3 that are obtained not by mirroring, but rather by reflecting through the centre of the diamond-shaped multiplet. In [42] it was shown that some bulk fields give rise to two Kaluza-Klein towers, instead of just one. The additional appearances of bulk fields in the short multiplets corresponds exactly to two such "twinned" towers that emerge from the same ten-dimensional field. We can formulate this more precisely: in an order p short multiplet any operator of conformal dimension $\Delta = p + x$, thus one that one obtains by applying a supercharge to the primary $2x$ times has a twin operator of dimension $\Delta = p + (4 - x)$. We see that almost all operators have twin operators, except those that map to themselves or their own complex conjugates. These correspond to $x = 2$, and are therefore operators that sit on the horizontal line exactly half way between to top and the bottom corners of the multiplet.

It is interesting that even though the representations that appear on such twinned towers are exactly the same, the masses of the corresponding modes are actually different. This asymmetry is also visible in the mapping to the short multiplets and manifests itself in the fact that in general twinned towers do not start in the same short multiplet, so that the location of their roots is skewed. To better explain what we mean by skewed we plotted the roots of all Kaluza-Klein towers and the multiplets in which they start in Figure 3.6. The way in which the twinned towers are skewed gives rise to an interesting relation between operators of *the same* short multiplet that sit in twinned Kaluza-Klein towers: if the distances of the shortest connection between these operators along the edges along which the supercharges act is $2n$ for some integer n, then one of the representations will be $[m_1, p, m_2]$, and the other one $[m_1, p - n, m_2]$ for the same m_1, m_2, and p. We have plotted this relation in Figure 3.5.

Figure 3.5: Relation between SU(4) quantum numbers of twinned operators. In the example shown there are $2n = 6$ edges that connect both operators, so that $n = 3$. Therefore, if one of the operators is in the representation $[m_1, p, m_2]$ then the other one is in the representation $[m_1, p - 3, m_2]$. This relation is true for all twinned operators.



To conclude, let us describe the matching Table 3.1. Each column in this table corresponds to one short supermultiplet, and we have written out the three lowest multiplets for $p = 2, 3, 4$ according to the multiplet structure in Figure 3.2. Each row corresponds to one of the 21 infinite Kaluza-Klein towers found in [42]. One can clearly see how the multiplets corresponding to $p = 2$ and $p = 3$ are shortened because some Kaluza-Klein towers start at a higher value of p, starting with $p = 4$ all higher short multiplets are generic and the representations can be read off the table. The relations between the masses of the Kaluza-

Klein modes and the quantum numbers of the operators are given in separate columns. After looking at the mass formula for twinned operators one discovers an interesting detail: if the mass squared of one of the twins expressed in terms of the order of the multiplet is given by $m^2(p)$ then the mass of the other twin is given by $m^2(-p)$, thus the masses of twins are related by $p \to (-p)$.

The lowest multiplet corresponding to $p = 2$ is the graviton, or massless, multiplet which constitutes the field content of the gauged supergravity theory [44, 77]. It contains the massless graviton in the $\mathbf{1}_{(1,1)}$ at $\Delta = p + 2 = 4$, and the scalars in the **20**, the $\mathbf{10_c}$, and the $\mathbf{1_c}$ with subscript 'c' standing for complex representation, and thus giving a total of 42 real scalar degrees of freedom. In particular, the $\mathbf{10_c}$ contains the scalars studied by GPPZ [21–23] which, after the symmetry breaking SU(4) $\to$ SU(3) $\times$ U(1) branches as $\mathbf{10} \to \mathbf{6} + \mathbf{3} + \mathbf{1}$, and what in GPPZ are the scalars m_{ij} and σ are the **6** and the **1** under this branching.

Figure 3.6: Skewed roots of Kaluza-Klein towers with respect to short multiplets. The horizontal lines represent short multiplets of different orders. The horizontal separation distinguishes superconformal descendants that are obtained from the primary by applying the supercharge Q a given number of times. On the bulk side this corresponds to different bulk fields. The roots of Kaluza-Klein towers are represented by big single and double circles. Red colour was used for bosons and blue colour for fermions. Double circles mean that there are multiple towers that start at the same point. Small red and blue dots that run upwards from the roots represent higher Kaluza-Klein modes. The twinned towers are those connected by dashed red and blue lines, and one can see that the twins always correspond to the same ten-dimensional field. Moreover one can see that twinned roots are all skewed in the same way so that the connecting lines are all parallel. In fact the twin relation does not only hold for the roots of the towers, but for all modes on all towers, so that the diagonal lines should be copied and transposed vertically to connected all red and blue dots. Finally note that for the cases $p = 0$ and $p = 1$ no gauge-invariant operators can be constructed, even though from the group theoretical perspective it is perfectly possible to write down the representations. On the bulk side these cases can be identified with the so-called singleton fields that can be gauged away except on the boundary of AdS, and therefore decouple from the other operators [13]. On the graph these modes correspond to the greyed out area on the bottom of the graph, and the washed out red and blue dots. Thus the parts of the towers that fall into the grey area should be removed and the corresponding towers should start at the $p = 2$ multiplet. In spite of this the $p = 0$ and $p = 1$ layers have been included to better demonstrate the symmetry of the set-up and the twin relations.

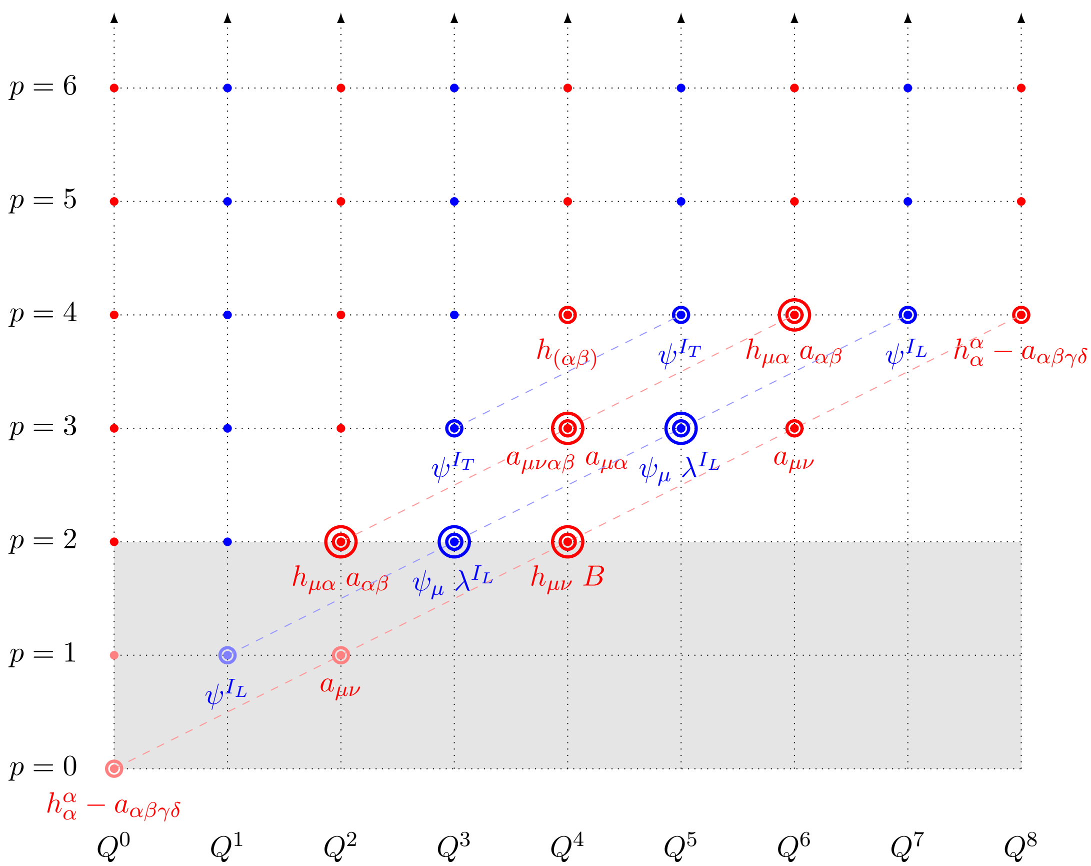


Table 3.1: Short supermultiplets vs. Kaluza-Klein towers. Each row corresponds to one of the 21 Kaluza-Klein towers, each column corresponds to a short supermultiplet. The first column corresponds to the conformal dimension Δ, the second column shows the Lorentz quantum numbers, the third and the fourth columns show the masses of the bulk modes expressed through the order p of the short multiplet, and through the conformal dimension Δ of the corresponding operator respectively. The representations in solid boxes have $\Delta < d = 4$ and therefore correspond to relevant deformations, those in dashed boxed have $\Delta = 4$ and correspond to marginal deformations, the rest are irrelevant deformations

Δ	(l_1, l_2)	m^2	m^2	p	$p=2$	$p=3$	$p=4$	$p=5$	mode
p	$(0,0)$	$p(p-4)$	$\Delta(\Delta-4)$	$[0,p,0]$	$\boxed{\mathbf{20'}}$	$\boxed{\mathbf{50}}$	$\dbox{\mathbf{105}}$	$\mathbf{196}$	$h^\alpha_\alpha - a_{\alpha\beta\gamma\delta}$
$p+\frac{1}{2}$	$(\frac{1}{2},0),(0,\frac{1}{2})$	$\frac{3}{2}-p$	$2-\Delta$	$[0,p-1,1] \oplus$ c.c.	$\boxed{\mathbf{20}}$	$\boxed{\overline{\mathbf{60}}}$	$\mathbf{140'}$	$\overline{\mathbf{280}}'$	ψ^{I_L}
$p+1$	$(1,0),(0,1)$	$(p-1)^2$	$(\Delta-2)^2$	$[0,p-1,0] \times 2$	$\boxed{\mathbf{6}}$	$\dbox{\mathbf{20'}}$	$\mathbf{50}$	$\mathbf{105}$	$a_{\mu\nu}$
	$(0,0)$	$(p+1)(p-3)$	$\Delta(\Delta-4)$	$[0,p-2,2] \oplus$ c.c.	$\boxed{\overline{\mathbf{10}}}$	$\dbox{\overline{\mathbf{45}}}$	$\overline{\mathbf{126}}$	$\overline{\mathbf{280}}$	$a_{\alpha\beta}$
	$(\frac{1}{2},\frac{1}{2})$	$p(p-2)$	$(\Delta-1)(\Delta-3)$	$[1,p-2,1]$	$\boxed{\mathbf{15}}$	$\dbox{\mathbf{64}}$	$\mathbf{175}$	$\mathbf{384}$	$h_{\mu\alpha} - a_{\mu\alpha\beta\gamma}$
$p+\frac{3}{2}$	$(\frac{1}{2},0),(0,\frac{1}{2})$	$\frac{1}{2}-p$	$2-\Delta$	$[0,p-2,1] \oplus$ c.c.	$\boxed{\bar{\mathbf{4}}}$	$\mathbf{20}$	$\overline{\mathbf{60}}$	$\mathbf{140'}$	λ^{I_L}
	$(1,\frac{1}{2}),(\frac{1}{2},1)$	$-2+p$	$\Delta-\frac{7}{2}$	$[1,p-2,0] \oplus$ c.c.	$\boxed{\mathbf{4}}$	$\overline{\mathbf{20}}$	$\mathbf{60}$	$\overline{\mathbf{140}}'$	ψ_μ
	$(0,\frac{1}{2}),(\frac{1}{2},0)$	$-\frac{3}{2}-p$	$-\Delta$	$[1,p-3,2] \oplus$ c.c.	—	$\overline{\mathbf{36}}$	$\mathbf{140}$	$\overline{\mathbf{360}}$	ψ^{I_T}
$p+2$	$(0,0)$	$(p+2)(p-2)$	$\Delta(\Delta-4)$	$[0,p-2,0] \times 2$	$\dbox{\mathbf{1}}$	$\mathbf{6}$	$\mathbf{20'}$	$\mathbf{50}$	B
	$(\frac{1}{2},\frac{1}{2})$	$(p+1)(p-1)$	$(\Delta-1)(\Delta-3)$	$[1,p-3,1] \times 2$	—	$\mathbf{15}$	$\mathbf{64}$	$\mathbf{175}$	$a_{\mu\alpha}$
	$(1,1)$	$(p+2)(p-2)$	$\Delta(\Delta-4)$	$[0,p-2,0]$	$\dbox{\mathbf{1}}$	$\mathbf{6}$	$\mathbf{20'}$	$\mathbf{50}$	$h_{\mu\nu}$
	$(1,0),(0,1)$	p^2	$(\Delta-2)^2$	$[2,p-3,0] \oplus$ c.c.	—	$\mathbf{10}$	$\mathbf{45}$	$\mathbf{126}$	$a_{\mu\nu\alpha\beta}$
	$(0,0)$	$(p+2)(p-2)$	$\Delta(\Delta-4)$	$[2,p-4,2]$	—	—	$\mathbf{84}$	$\mathbf{300}$	$h_{(\alpha\beta)}$
$p+\frac{5}{2}$	$(0,\frac{1}{2}),(\frac{1}{2},0)$	$\frac{1}{2}+p$	$\Delta-2$	$[1,p-3,0] \oplus$ c.c.	—	$\mathbf{4}$	$\overline{\mathbf{20}}$	$\mathbf{60}$	λ^{I_L}
	$(\frac{1}{2},1),(1,\frac{1}{2})$	$-2-p$	$\frac{1}{2}-\Delta$	$[0,p-3,1] \oplus$ c.c.	—	$\bar{\mathbf{4}}$	$\mathbf{20}$	$\overline{\mathbf{60}}$	ψ_μ
	$(\frac{1}{2},0),(0,\frac{1}{2})$	$-\frac{3}{2}+p$	$\Delta-4$	$[2,p-4,1] \oplus$ c.c.	—	—	$\mathbf{36}$	$\overline{\mathbf{140}}$	ψ^{I_T}
$p+3$	$(0,1),(1,0)$	$(p+1)^2$	$(\Delta-2)^2$	$[0,p-3,0] \times 2$	—	$\mathbf{1}$	$\mathbf{6}$	$\mathbf{20'}$	$a_{\mu\nu}$
	$(0,0)$	$(p-1)(p+3)$	$\Delta(\Delta-4)$	$[2,p-4,0] \oplus$ c.c.	—	—	$\mathbf{10}$	$\mathbf{45}$	$a_{\alpha\beta}$
	$(\frac{1}{2},\frac{1}{2})$	$p(p+2)$	$(\Delta-1)(\Delta-3)$	$[1,p-4,1]$	—	—	$\mathbf{15}$	$\mathbf{64}$	$h_{\mu\alpha} - a_{\mu\alpha\beta\gamma}$
$p+\frac{7}{2}$	$(0,\frac{1}{2}),(\frac{1}{2},0)$	$\frac{3}{2}+p$	$\Delta-2$	$[1,p-4,0] \oplus$ c.c.	—	—	$\mathbf{4}$	$\overline{\mathbf{20}}$	ψ^{I_L}
$p+4$	$(0,0)$	$p(p+4)$	$\Delta(\Delta-4)$	$[0,p-4,0]$	—	—	$\mathbf{1}$	$\mathbf{6}$	$h^\alpha_\alpha - a_{\alpha\beta\gamma\delta}$

Part II

Holographic Beta Function

Chapter 4

Scalar Potential in Gauged Supergravity

In the course of this part of the thesis we will establish a holographic method for the computation of the bulk scalar potential, put prior to this let us review the traditional way of obtaining it. Thus in this chapter we will demonstrate the non-holographic computation of the scalar potential in the framework of gauged supergravity. We will review some basic facts first, then specialise to the GPPZ potential [23], and indicate the steps that lead to its computation. This calculation demonstrates how fundamentally different the holographic computation is on the one hand, but also allows us to introduce and study some technical aspects of maximal gauged supergravity in five dimensions that will be useful later.

4.1 Gauged Supergravity in Five Dimensions

In this section we would like to recall some facts about gauged supergravity. The GPPZ computation is done for the gauged $\mathcal{N} = 8$ supergravity in a five-dimensional AdS_5 space-time, and therefore this is the theory we would like to focus on at the moment.

Gauged supergravity can be derived from ungauged supergravity by promoting some of its global symmetries to gauge transformations. The ungauged $\mathcal{N} = 8$ supergravity in five dimensions was first considered in [78] and has a global $E_{6(6)}$ symmetry, which is the split, real form of the complex E_6. Since E_6 has complex dimension 78, the split real form $E_{6(6)}$ is of real dimension 78. The 6 in the parenthesis in the subscript stands for the difference of the numbers of non-compact and compact generators, thus in our case the 78 generators consist of 42 non-compact, and 36 compact ones. Apart from the global $E_{6(6)}$, the theory has a composite local Sp(4) symmetry. It is local in the usual sense that it depends on the spatial coordinates, but it is not a gauge group since it is not associated to any gauge fields.

Among other fields the five-dimensional supergravity theory contains 42 scalars. These are described by a non-linear sigma model and take values in the coset manifold $E_{6(6)}/Sp(4)$. These 42 scalars correspond to the 42 non-compact directions of the $E_{6(6)}$, and the local composite symmetry group Sp(4), which is the maximal compact subgroup of $E_{6(6)}$ and therefore of dimension 36, mods out the 36 compact directions through the quotient $E_{6(6)}/Sp(4)$. In short, the 42 scalars are in fact equivalence classes in $E_{6(6)}$ and the local Sp(4) transformations correspond to the choice of a representative in each equivalence class. Fixing a "gauge" in Sp(4) corresponds to fixing the representatives for the scalars equivalence classes.

The gauging of the supergravity theory means that we choose a subgroup of the global $E_{6(6)}$ and promote it to a gauge group. This is a non-trivial procedure, as appropriate gauge fields need to be found, and gauge couplings have to be introduced. At the end of the day it can be shown that it is consistent to gauge an SU(4) subgroup. It can be embedded in the first factor of the maximal $SL(6) \times SL(2)$ subgroup of the $E_{6(6)}$, which is obvious from the algebra relation $\mathfrak{su}(4) \cong \mathfrak{so}(6) \subset \mathfrak{sl}(6)$. After the subgroup SU(4) is promoted to a gauge group, the $E_{6(6)}$ global symmetry is broken, and only its subgroup SL(2), which appears as the second factor above, survives. The composite local Sp(4) remains unaffected. In total, starting with the ungauged supergravity with the symmetry

$$E_{6(6)} \times Sp(4)^{CL}, \tag{4.1}$$

where "CL" stands for composite local, we promoted an SU(4) subgroup of the global $E_{6(6)}$ to a gauge group breaking the latter to SL(2). The gauged supergravity theory has therefore the symmetry

$$SU(4)^{gauge} \times SL(2)^{global} \times Sp(4)^{CL}. \tag{4.2}$$

More details can be found in the original literature in which the $\mathcal{N} = 8$ gauged supergravity is derived [44, 45].

4.2 The Scalar Potential

The gauged supergravity theory described in the previous section has a potential for its 42 scalars that can be written in the following way (see equation (5.4) in [45]):

$$P = -\frac{g^2}{32}\left(2W_{ab}W^{ab} - W_{abcd}W^{abcd}\right). \tag{4.3}$$

The indices $a, b, c, \ldots \in \{1, \ldots, 8\}$ refer to the fundamental representation of the Sp(4), and are raised and lowered using the symplectic metric $\Omega_{ab} = \left(\begin{smallmatrix} 0 & \mathbf{1}_4 \\ -\mathbf{1}_4 & 0 \end{smallmatrix}\right)$. Furthermore $W_{ab} = W^c{}_{acb} = \Omega^{cd} W_{dacb}$. The scalar fields in the potential P are hidden in the object

W_{abcd}, which is quadratic in elements called "vielbein" which parametrise the coset manifold $E_{6(6)}/Sp(4)$, on which the scalars live.

In order to describe the scalars as an $E_{6(6)}/Sp(4)$ coset let us first find how the relevant representations branch. The Lie algebra embedding $\mathfrak{sp}(4) \subset \mathfrak{e}_{6(6)}$ is special [79], and the representation branching can be found by standard Lie algebra techniques, or by using computer software [80]. We find that the fundamental, the anti-fundamental, and the adjoint representations of the $\mathfrak{e}_{6(6)}$ branch as follows

$$\mathbf{27} \to \mathbf{27} \tag{4.4a}$$

$$\overline{\mathbf{27}} \to \mathbf{27} \tag{4.4b}$$

$$\mathbf{78} \to \mathbf{36} + \mathbf{42}. \tag{4.4c}$$

We see that the adjoint representation **78** of the $\mathfrak{e}_{6(6)}$ branches into the adjoint representation **36** of the $\mathfrak{sp}(4)$ and an additional **42**. The quotient $E_{6(6)}/Sp(4)$ means that we have to remove those compact 36 generators of the $\mathfrak{sp}(4)$ from the 78 generators of the $\mathfrak{e}_{6(6)}$, leaving just the **42**. To construct the coset one starts with an adjoint field of the $\mathfrak{e}_{6(6)}$, which can be thought of as living on the 78-dimensional tangent space of the $E_{6(6)}$. The group Sp(4) is 36-dimensional so that the tangent space of the coset $E_{6(6)}/Sp(4)$ is 42-dimensional, and corresponds exactly to the 42 scalars. These 42 scalars can be represented by a vielbein ${V_A}^{cd}$ as follows:

$${V_A}^{cd} = \left(\quad \mathbf{42} \quad \right) \in E_{6(6)}/Sp(4)\,. \tag{4.5}$$

Recall that the scalars have to transform both under the $E_{6(6)}$ and the Sp(4), and therefore the vielbein ${V_A}^{cd}$ carries both $E_{6(6)}$ indices $A, B, \ldots \in \{1, \ldots 27\}$, and Sp(4) indices $a, b, c, \ldots \in \{1, \ldots 8\}$. Here we are following the standard index conventions where an upper index refers to a fundamental representation, and a lower index to the corresponding anti-fundamental representation. The fundamental representation of $E_{6(6)}$ is the **27**, thus the lower index A transforms in the anti-fundamental $\overline{\mathbf{27}}$ of the $E_{6(6)}$. The fundamental representation of Sp(4) is the **8**, so that the anti-symmetric and symplectic-traceless index pair cd corresponds to the **27** of the Sp(4).

In the ungauged theory the $E_{6(6)}$ is a global symmetry of the theory. The gauging is achieved by promoting an SO(6) subgroup of the $E_{6(6)}$ to a gauge group, which sits in the $E_{6(6)}$ as described by the following inclusions:

$$E_{6(6)} \supset Sp(4) \supset SL(6) \times SL(2) \supset SO(6) \times SO(2). \tag{4.6}$$

After the promotion of the SO(6) to a gauge group the global symmetry $E_{6(6)}$ is broken down to SL(2), which is the maximal subgroup of the $E_{6(6)}$, which commutes with the SO(6), as one can see above. Thus after the gauging the representations need to be branched under the algebra inclusions

$$\mathfrak{e}_{6(6)} \supset \mathfrak{sp}(4) \supset \mathfrak{so}(6) \times \mathfrak{so}(2). \tag{4.7}$$

We saw the branching of the relevant representations under $\mathfrak{sp}(4) \subset \mathfrak{e}_{6(6)}$ in (4.4), the further branching under $\mathfrak{so}(6) \times \mathfrak{so}(2) \subset \mathfrak{sp}(4)$ is given by

$$\mathbf{27} \to (\mathbf{15}, \mathbf{1}) + (\mathbf{6}, \mathbf{2}) \tag{4.8a}$$

$$\mathbf{42} \to (\mathbf{20}', \mathbf{1}) + (\mathbf{10}, \mathbf{2}) + (\mathbf{1}, \mathbf{2}). \tag{4.8b}$$

Thus given the branching of the fundamental representation we see that the $\mathrm{E}_{6(6)}$ index of the vielbein has to be transformed as follows:

$$V_A{}^{cd} \to V_{ab}{}^{cd} \to \begin{pmatrix} V^{IJcd} \\ V_{I\alpha}{}^{cd} \end{pmatrix} \tag{4.9}$$

with indices $I, J, K, \ldots \in \{1, \ldots 6\}$ and $\alpha, \beta, \gamma, \ldots \in \{1, 2\}$ transforming in the fundamental $\mathbf{6}$ of the SO(6) and the fundamental $\mathbf{2}$ of the SO(2) respectively. It is in this basis that the symbols W_{abcd} in (4.3) are expressed in terms of the vielbein [45]:

$$W_{abcd} = \epsilon^{\alpha\beta} \delta^{IJ} V_{I\alpha ab} V_{J\beta cd}\,, \tag{4.10}$$

with $\epsilon^{12} = 1$, and δ^{IJ} a six-dimensional identity matrix.

4.3 The Scalar Potential in GPPZ

In order to specialize to the GPPZ case [21–23] and apply the formula (4.3) to compute the scalar potential a number of steps have to be taken. First, the scalar fields that one turns on break the global symmetry, so that one needs to determine the symmetry breaking pattern, and parametrise the tensors in terms of the remaining symmetry groups. Then the scalars to which one truncates need to be parametrised in terms of the non-compact generators on the scalar coset manifold. The resulting algebra element X containing the scalars is then exponentiated to get the vielbein V. Once the vielbein is computed the formulas (4.10) and (4.3) yield the scalar potential. In the following subsections we will go through the details of the steps just described.

4.3.1 Symmetry Breaking and Group Theory

In the previous section the vielbein was parametrised in terms of the $\mathfrak{so}(6) \times \mathfrak{so}(2)$ subalgebra of the $\mathfrak{e}_{6(6)}$. This was convenient because precisely the first factor, the $\mathfrak{so}(6)$, is the one that gets gauged, and the second one, the $\mathfrak{so}(2)$, is part of the $\mathfrak{sl}(2)$ algebra of the residual SL(2) global symmetry that survives the gauging. Since SO(2) is the maximal compact subgroup of SL(2), it can be used to label representations of the latter.

Recall that in holography the SU(4) gauge group of the supergravity maps to the global SU(4) R-symmetry group of the field theory. In GPPZ [21–23] field theory deformations are considered that break the $\mathcal{N} = 4$ supersymmetry down to $\mathcal{N} = 1$. This breaks the R-symmetry down to $\mathrm{SU}(3) \times \mathrm{U}(1) \subset \mathrm{SU}(4)$, where U(1) is now the R-symmetry of the $\mathcal{N} = 1$ supersymmetry, and the SU(3) becomes the flavour symmetry that mixes the three chiral multiplets that are formed after the supersymmetry breaking.

Because of the symmetry breaking pattern on the field theory side it is natural to choose an algebra basis on the gravity side that reflects the corresponding subgroups, namely

$$\mathfrak{su}(4) \supset \mathfrak{su}(3) \times \mathfrak{u}(1)\,, \tag{4.11}$$

similar to the previous section, taking subgroups further branches the indices of the vielbein. The relevant representation branchings that we need are (see equations (4.8))

$$\mathbf{15} \to \mathbf{1}_0 + \mathbf{3}_4 + \overline{\mathbf{3}}_{-4} + \mathbf{8}_0 \tag{4.12}$$

$$\mathbf{6} \to \mathbf{3}_{-2} + \overline{\mathbf{3}}_2\,. \tag{4.13}$$

In total, the labelling of the representations changes according to

$$\mathfrak{so}(6) \times \mathfrak{so}(2) \cong \mathfrak{su}(4) \times \mathfrak{u}(1) \supset \mathfrak{su}(3) \times \mathfrak{u}(1) \times \mathfrak{u}(1)\,. \tag{4.14}$$

with

$$(\mathbf{15}, \mathbf{1}) \to \mathbf{15}_0 \tag{4.15}$$

$$\to \mathbf{1}_{(0,0)} + \mathbf{3}_{(4,0)} + \overline{\mathbf{3}}_{(-4,0)} + \mathbf{8}_{(0,0)} \tag{4.16}$$

$$(\mathbf{6}, \mathbf{2}) \to \mathbf{6}_2 + \mathbf{6}_{-2} \tag{4.17}$$

$$\to \mathbf{3}_{(-2,2)} + \overline{\mathbf{3}}_{(2,2)} + \mathbf{3}_{(-2,-2)} + \overline{\mathbf{3}}_{(2,-2)}\,. \tag{4.18}$$

From these branching rules we can now induce the index structure of the vielbein:

$$\begin{pmatrix} V^{IJcd} \\ V_{I\alpha}{}^{cd} \end{pmatrix} \to \left[\begin{pmatrix} V_{(1)} & V^{i}_{(2)} & V^{\bar{\imath}}_{(3)} & V^{i\bar{\jmath}}_{(4)} & V^{i}_{(5)} & V^{\bar{\imath}}_{(6)} & V^{i}_{(7)} & V^{\bar{\imath}}_{(8)} \end{pmatrix}^{cd}\right]^t \tag{4.19}$$

$$= \left(\mathbf{1}_{(0,0)}, \mathbf{3}_{(4,0)}, \overline{\mathbf{3}}_{(-4,0)}, \mathbf{8}_{(0,0)}, \mathbf{3}_{(-2,2)}, \overline{\mathbf{3}}_{(2,2)}, \mathbf{3}_{(-2,-2)}, \overline{\mathbf{3}}_{(2,-2)}\right)\,. \tag{4.20}$$

Note that the conventions for the SU(3) indices are now different: instead of denoting the fundamental representation by an upper index and the anti-fundamental by a lower index, we are now following the conventions of GPPZ to denote the fundamental representation of the SU(3), the **3**, by a lower-case Latin letter from the middle of the alphabet, $i, j, k, \ldots \in \{1,2,3\}$, and the anti-fundamental by the barred indices $\bar{\imath}, \bar{\jmath}, \bar{k}, \ldots \in \{1,2,3\}$. The upper and lower SU(3) indices, in contrast, shall now denote rows and columns, respectively, which makes it easy to write the vielbein as a matrix, as we will see in what follows.

The next step is to write the vielbein V explicitly in terms of the 42 scalars. Recall that these scalars are described by a sigma-model, and therefore take values in the target manifold $E_{6(6)}/\mathrm{Sp}(4)$. One standard way to model this is to take the scalars to be proportional to the non-compact generators of the $E_{6(6)}$ and exponentiate the resulting Lie algebra element to get the vielbein with values in the group manifold:

$$V = \exp\left(\sum_{i=1}^{42} \phi_i T_i\right) . \tag{4.21}$$

Since we are considering a symmetry breaking pattern where the $E_{6(6)}$ group is broken to $\mathrm{SU}(3) \times \mathrm{U}(1)$, we need to write the generators of the $E_{6(6)}$ in this basis. In particular, this means that also the column index *cd* in (4.19) will have to be branched to the corresponding subgroups, as it was done with the row index.

Recall that the 42 scalars of the theory are in the **42** of the $E_{6(6)}$ in the ungauged theory, and after gauging we have to consider the subalgebras according to

$$\mathfrak{e}_{6(6)} \supset \mathfrak{sp}(4) \supset \mathfrak{so}(6) \times \mathfrak{so}(2) \cong \mathfrak{su}(4) \times \mathfrak{u}(1) . \tag{4.22}$$

Under these inclusions the **42** branches as

$$\mathbf{42} \quad \to \quad \mathbf{42} \quad \to \quad \mathbf{20'}_0 + \mathbf{10}_{-2} + \overline{\mathbf{10}}_2 + \mathbf{1}_4 + \mathbf{1}_{-4}. \tag{4.23}$$

In the GPPZ case we are breaking the symmetry further to

$$\mathfrak{su}(4) \times \mathfrak{u}(1) \supset \mathfrak{su}(3) \times \mathfrak{u}(1) \times \mathfrak{u}(1) , \tag{4.24}$$

and the representations branch as

$$\mathbf{20'}_0 \to \mathbf{8}_{(0,0)} + \mathbf{6}_{(-4,0)} + \overline{\mathbf{6}}_{(4,0)} \tag{4.25}$$
$$\mathbf{10}_{-2} \to \mathbf{6}_{(2,-2)} + \mathbf{3}_{(-2,-2)} + \mathbf{1}_{(-6,-2)} \tag{4.26}$$
$$\overline{\mathbf{10}}_2 \to \overline{\mathbf{6}}_{(-2,2)} + \overline{\mathbf{3}}_{(2,2)} + \mathbf{1}_{(6,2)} \tag{4.27}$$
$$\mathbf{1}_4 \to \mathbf{1}_{(0,4)} \tag{4.28}$$
$$\mathbf{1}_{-4} \to \mathbf{1}_{(0,-4)} . \tag{4.29}$$

All in all, recalling that the row and column indices of the vielbein were in the **27** of the

Sp(4), and the scalars in the **42**, we can use all branching rules listed above to write the vielbein in the following block form:

$$
V = \begin{array}{c} \\ \mathbf{27} \end{array}\!\!
\begin{array}{c} \mathbf{27} \\ \left(\begin{array}{c} \quad \mathbf{42} \quad \end{array}\right) \end{array}
\;\rightarrow\;
\begin{array}{c} \\ \mathbf{15}_0 \\ \mathbf{6}_{-2} \\ \mathbf{6}_2 \end{array}
\begin{array}{ccc} \mathbf{15}_0 & \mathbf{6}_2 & \mathbf{6}_{-2} \end{array}
\left(\begin{array}{ccc}
\mathbf{20}'_0 & \overline{\mathbf{10}}_2 & \mathbf{10}_{-2} \\
\mathbf{10}_{-2} & \mathbf{20}'_0 & \mathbf{1}_{-4} \\
\overline{\mathbf{10}}_2 & \mathbf{1}_4 & \mathbf{20}'_0
\end{array}\right)
\;\rightarrow
$$

$$
\begin{array}{c|cccc|cc|cc}
 & \mathbf{1}_{(0,0)} & \mathbf{3}_{(4,0)} & \overline{\mathbf{3}}_{(-4,0)} & \mathbf{8}_{(0,0)} & \mathbf{3}_{(-2,2)} & \overline{\mathbf{3}}_{(2,2)} & \mathbf{3}_{(-2,-2)} & \overline{\mathbf{3}}_{(2,-2)} \\
\hline
\mathbf{1}_{(0,0)} & 0 & 0 & 0 & \mathbf{8}_{(0,0)} & 0 & \overline{\mathbf{3}}_{(2,2)} & \mathbf{3}_{(-2,-2)} & 0 \\
\overline{\mathbf{3}}_{(-4,0)} & 0 & \mathbf{8}_{(0,0)} & 0 & \mathbf{6}_{(-4,0)} & 0 & \overline{\mathbf{6}}_{(-2,2)} & \mathbf{1}_{(-6,-2)} & \mathbf{3}_{(-2,-2)} \\
\mathbf{3}_{(4,0)} & 0 & 0 & 0 & \overline{\mathbf{6}}_{(4,0)} & \overline{\mathbf{3}}_{(2,2)} & \mathbf{1}_{(6,2)} & \mathbf{6}_{(2,-2)} & 0 \\
\mathbf{8}_{(0,0)} & \mathbf{8}_{(0,0)} & \overline{\mathbf{6}}_{(4,0)} & \mathbf{6}_{(-4,0)} & \mathbf{8}_{(0,0)} & \overline{\mathbf{6}}_{(-2,2)} & \overline{\mathbf{3}}_{(2,2)} & \mathbf{3}_{(-2,-2)} & \mathbf{6}_{(2,-2)} \\
\hline
\overline{\mathbf{3}}_{(2,-2)} & 0 & 0 & \mathbf{3}_{(-2,-2)} & \mathbf{6}_{(2,-2)} & \mathbf{8}_{(0,0)} & \overline{\mathbf{6}}_{(4,0)} & \mathbf{1}_{(0,-4)} & 0 \\
\mathbf{3}_{(-2,-2)} & \mathbf{3}_{(-2,-2)} & \mathbf{6}_{(2,-2)} & \mathbf{1}_{(-6,-2)} & \mathbf{3}_{(-2,-2)} & \mathbf{6}_{(-4,0)} & \mathbf{8}_{(0,0)} & 0 & \mathbf{1}_{(0,-4)} \\
\hline
\overline{\mathbf{3}}_{(2,2)} & \overline{\mathbf{3}}_{(2,2)} & \mathbf{1}_{(6,2)} & \overline{\mathbf{6}}_{(-2,2)} & \overline{\mathbf{3}}_{(2,2)} & \mathbf{1}_{(0,4)} & 0 & \mathbf{8}_{(0,0)} & \overline{\mathbf{6}}_{(4,0)} \\
\mathbf{3}_{(-2,2)} & 0 & \overline{\mathbf{3}}_{(2,2)} & 0 & \overline{\mathbf{6}}_{(-2,2)} & 0 & \mathbf{1}_{(0,4)} & \mathbf{6}_{(-4,0)} & \mathbf{8}_{(0,0)}
\end{array}
\tag{4.30}
$$

This parametrisation of the $E_{6(6)}/Sp(4)$ coset vielbein in terms of the $\mathfrak{su}(3)\times\mathfrak{u}(1)\times\mathfrak{u}(1)$ representations leads to the following tensor structure for the vielbein and the algebra elements:

$$
\left(\begin{array}{cccc|cc|cc}
z & z_{\bar k} & z_k & z_{\bar k l} & z_{\bar k} & z_k & z_{\bar k} & z_k \\
z^i & z^i{}_{\bar k} & z^i{}_k & z^i{}_{\bar k l} & z^i{}_{\bar k} & z^i{}_k & z^i{}_{\bar k} & z^i{}_k \\
z^{\bar\imath} & z^{\bar\imath}{}_{\bar k} & z^{\bar\imath}{}_k & z^{\bar\imath}{}_{\bar k l} & z^{\bar\imath}{}_{\bar k} & z^{\bar\imath}{}_k & z^{\bar\imath}{}_{\bar k} & z^{\bar\imath}{}_k \\
z^{i\bar\jmath} & z^{i\bar\jmath}{}_{\bar k} & z^{i\bar\jmath}{}_k & z^{i\bar\jmath}{}_{\bar k l} & z^{i\bar\jmath}{}_{\bar k} & z^{i\bar\jmath}{}_k & z^{i\bar\jmath}{}_{\bar k} & z^{i\bar\jmath}{}_k \\
\hline
z^i & z^i{}_{\bar k} & z^i{}_k & z^i{}_{\bar k l} & z^i{}_{\bar k} & z^i{}_k & z^i{}_{\bar k} & z^i{}_k \\
z^{\bar\imath} & z^{\bar\imath}{}_{\bar k} & z^{\bar\imath}{}_k & z^{\bar\imath}{}_{\bar k l} & z^{\bar\imath}{}_{\bar k} & z^{\bar\imath}{}_k & z^{\bar\imath}{}_{\bar k} & z^{\bar\imath}{}_k \\
\hline
z^i & z^i{}_{\bar k} & z^i{}_k & z^i{}_{\bar k l} & z^i{}_{\bar k} & z^i{}_k & z^i{}_{\bar k} & z^i{}_k \\
z^{\bar\imath} & z^{\bar\imath}{}_{\bar k} & z^{\bar\imath}{}_k & z^{\bar\imath}{}_{\bar k l} & z^{\bar\imath}{}_{\bar k} & z^{\bar\imath}{}_k & z^{\bar\imath}{}_{\bar k} & z^{\bar\imath}{}_k
\end{array}\right) .
\tag{4.31}
$$

The next step is to pick the scalars to turn on, fit them into the coset element just described, and exponentiate it to get the vielbein.

4.3.2 Computing the Vielbein of the Scalar Coset

The scalars considered by GPPZ [21–23] are in the representations $\mathbf{6}_{(2,-2)}$ and the $\mathbf{1}_{(-6,-2)}$ that come from the branching of the $\mathbf{10}_{-2}$, and are denoted by m_{ij} and σ. On the field theory side m_{ij} corresponds to a mass term for the $\mathcal{N}=1$ chirals, and σ corresponds to the gaugino condensate. The authors showed that after an appropriate choice for the generators, which lead to canonical kinetic terms, the supergravity scalars m_{ij} and σ can be mapped into the algebra as follows:

$$X = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{m_{ij}}{\sqrt{3}} & \overline{\sigma} & 0 \\ 0 & 0 & 0 & 0 & 0 & \sigma & \frac{\overline{m}_{\bar{\imath}\bar{\jmath}}}{\sqrt{3}} & 0 \\ 0 & 0 & 0 & 0 & \epsilon_{\bar{\jmath}\bar{k}\bar{l}}\frac{m_{il}}{\sqrt{3}} & 0 & 0 & \epsilon_{jkl}\frac{\overline{m}_{\bar{\imath}\bar{l}}}{\sqrt{3}} \\ 0 & 0 & 0 & \epsilon_{jkl}\frac{\overline{m}_{\bar{\imath}\bar{l}}}{\sqrt{3}} & 0 & 0 & 0 & 0 \\ 0 & \frac{\overline{m}_{\bar{\imath}\bar{\jmath}}}{\sqrt{3}} & \overline{\sigma} & 0 & 0 & 0 & 0 & 0 \\ 0 & \sigma & \frac{m_{ij}}{\sqrt{3}} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \epsilon_{\bar{\jmath}\bar{k}\bar{l}}\frac{m_{il}}{\sqrt{3}} & 0 & 0 & 0 & 0 \end{pmatrix} \tag{4.32}$$

The unbarred indices refer to the fundamental $\mathbf{3}$ of the SU(3), the barred ones to the anti-fundamental $\overline{\mathbf{3}}$. Repeated indices are summed over, and necessarily involve one unbarred and one barred index to produces an SU(3) invariant contraction. It is necessary to choose m_{ij} to be diagonal to project out an additional chiral primary operator that would otherwise lead to a third bulk mode. Therefore the authors of GPPZ take

$$m_{ij} = m\delta_{ij} \equiv \sqrt{3}\alpha\delta_{ij}\,. \tag{4.33}$$

In the last equality we defined α as a rescaling of the mass term m. Note that in (4.32) we have reproduced the original form of X found in [23], in which the distinction between row and column indices which we used in writing down (4.31) was not used. We shall reinstate this distinction and denote rows by an upper index and columns by a lower index, so that X is given by

$$X = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \alpha\delta^i{}_k & \overline{\sigma}\delta^i{}_{\bar{k}} & 0 \\ 0 & 0 & 0 & 0 & 0 & \sigma\delta^{\bar{\imath}}{}_k & \overline{\alpha}\delta^{\bar{\imath}}{}_{\bar{k}} & 0 \\ 0 & 0 & 0 & 0 & \alpha\epsilon^{i\bar{\jmath}}{}_{\bar{k}} & 0 & 0 & \overline{\alpha}\epsilon^{i\bar{\jmath}}{}_k \\ 0 & 0 & 0 & \overline{\alpha}\epsilon^i{}_{\bar{k}l} & 0 & 0 & 0 & 0 \\ 0 & \overline{\alpha}\delta^{\bar{\imath}}{}_{\bar{k}} & \overline{\sigma}\delta^{\bar{\imath}}{}_k & 0 & 0 & 0 & 0 & 0 \\ 0 & \sigma\delta^i{}_{\bar{k}} & \alpha\delta^i{}_k & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \alpha\epsilon^{\bar{\imath}}{}_{\bar{k}l} & 0 & 0 & 0 & 0 \end{pmatrix}. \tag{4.34}$$

To get the vielbein V we need to exponentiate X, which can in fact be done by hand. Notice that X is in block form, and that each entry in the block formed by rows and columns (2, 3, 6, 7) is an identity matrix. Therefore we can separate out this block, and exponentiate it like a normal matrix:

$$\exp\begin{pmatrix} 0 & 0 & \alpha & \overline{\sigma} \\ 0 & 0 & \sigma & \overline{\alpha} \\ \overline{\alpha} & \overline{\sigma} & 0 & 0 \\ \sigma & \alpha & 0 & 0 \end{pmatrix} =$$
$$= \begin{pmatrix} \cosh|\sigma|\cosh|\alpha| & \frac{\overline{\sigma}\alpha}{|\sigma||\alpha|}\sinh|\sigma|\sinh|\alpha| & \frac{\alpha}{|\alpha|}\cosh|\sigma|\sinh|\alpha| & \frac{\overline{\sigma}}{|\sigma|}\sinh|\sigma|\cosh|\alpha| \\ \frac{\sigma\overline{\alpha}}{|\sigma||\alpha|}\sinh|\sigma|\sinh|\alpha| & \cosh|\sigma|\cosh|\alpha| & \frac{\sigma}{|\sigma|}\sinh|\sigma|\cosh|\alpha| & \frac{\overline{\alpha}}{|\alpha|}\cosh|\sigma|\sinh|\alpha| \\ \frac{\overline{\alpha}}{|\alpha|}\cosh|\sigma|\sinh|\alpha| & \frac{\overline{\sigma}}{|\sigma|}\sinh|\sigma|\cosh|\alpha| & \cosh|\sigma|\cosh|\alpha| & \frac{\overline{\sigma\alpha}}{|\sigma||\alpha|}\sinh|\sigma|\sinh|\alpha| \\ \frac{\sigma}{|\sigma|}\sinh|\sigma|\cosh|\alpha| & \frac{\alpha}{|\alpha|}\cosh|\sigma|\sinh|\alpha| & \frac{\sigma\alpha}{|\sigma||\alpha|}\sinh|\sigma|\sinh|\alpha| & \cosh|\sigma|\cosh|\alpha| \end{pmatrix} \tag{4.35}$$

The rest of X is formed by rows and columns in positions (1, 4, 5, 8) and is given by:

$$X = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & \alpha\epsilon^{i\bar{\jmath}}{}_{\bar{k}} & \overline{\alpha}\epsilon^{i\bar{\jmath}}{}_{k} \\ 0 & \overline{\alpha}\epsilon^{i}{}_{\bar{k}l} & 0 & 0 \\ 0 & \alpha\epsilon^{\bar{\imath}}{}_{\bar{k}l} & 0 & 0 \end{pmatrix} \tag{4.36}$$

To exponentiate this block we need to carefully multiply the ϵ-tensors. It is useful to notice that $X^2 = (2|\alpha|)^2 Y$, where Y has the properties $Y^2 = Y$, and $YX = XY = X$, and is given by

$$Y = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & \delta^{i\bar{\jmath}}{}_{\bar{k}l} & 0 & 0 \\ 0 & 0 & \frac{1}{2}\delta^{i}{}_{\bar{k}} & \frac{1}{2}\frac{\overline{\alpha}^2}{|\alpha|^2}\delta^{i}{}_{k} \\ 0 & 0 & \frac{1}{2}\frac{\alpha^2}{|\alpha|^2}\delta^{\bar{\imath}}{}_{\bar{k}} & \frac{1}{2}\delta^{\bar{\imath}}{}_{k} \end{pmatrix} \qquad \delta^{i\bar{\jmath}}{}_{\bar{k}l} = \delta^{[i}{}_{\bar{k}}\delta^{\bar{\jmath}]}{}_{l} = \frac{1}{2}\left(\delta^{i}{}_{k}\delta^{\bar{\jmath}}{}_{l} - \delta^{i}{}_{l}\delta^{\bar{\jmath}}{}_{\bar{k}}\right) \tag{4.37}$$

Using these properties it is now easy to exponentiate X:

$$\begin{aligned}\exp X &= \sum_{n=0}^{\infty}\frac{1}{(2n)!}\left(X^2\right)^n + \sum_{n=0}^{\infty}\frac{1}{(2n+1)!}\left(X^2\right)^n X \\ &= \sum_{n=0}^{\infty}\frac{1}{(2n)!}(2|\alpha|)^{2n}Y + (\mathbf{1}-Y) + \sum_{n=0}^{\infty}\frac{1}{(2n+1)!}(2|\alpha|)^{2n+1}\frac{1}{2|\alpha|}YX \\ &= \cosh(2|\alpha|)Y + \frac{1}{2|\alpha|}(\mathbf{1}-Y) + \frac{1}{2|\alpha|}\sinh(2|\alpha|)X \\ &= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cosh(2|\alpha|)\delta^{i\bar{j}}{}_{\bar{k}l} & \frac{\alpha}{2|\alpha|}\sinh(2|\alpha|)\epsilon^{i\bar{j}}{}_{\bar{k}} & \frac{\bar{\alpha}}{2|\alpha|}\sinh(2|\alpha|)\epsilon^{i\bar{j}}{}_{k} \\ 0 & \frac{\bar{\alpha}}{2|\alpha|}\sinh(2|\alpha|)\epsilon^{i}{}_{\bar{k}l} & \cosh(|\alpha|)^2\delta^{i}{}_{\bar{k}} & \frac{\bar{\alpha}^2}{|\alpha|^2}\sinh(|\alpha|)^2\delta^{i}{}_{k} \\ 0 & \frac{\alpha}{2|\alpha|}\sinh(2|\alpha|)\epsilon^{\bar{i}}{}_{\bar{k}l} & \frac{\alpha^2}{|\alpha|^2}\sinh(|\alpha|)^2\delta^{\bar{i}}{}_{\bar{k}} & \cosh(|\alpha|)^2\delta^{\bar{i}}{}_{k} \end{pmatrix}.\end{aligned} \tag{4.38}$$

After combining the two blocks of X in (4.35) and (4.38) we thus obtain the vielbein, in which both rows and columns are written in the basis in which the **27** of the Sp(4) is split to SU(3) × U(1) × U(1) indices. Recalling the tensor W^{abcd} is computed in terms of the vielbein in the following way

$$W^{abcd} = \epsilon^{\alpha\beta}\delta^{IJ}V_{I\alpha}{}^{ab}V_{J\beta}{}^{cd}, \tag{4.39}$$

we see that the column indices of the vielbein have to be converted back to the Sp(4) basis, in which the **27** is parametrised by an antisymmetric index pair ab. In GPPZ the authors indicated that this basis change can be performed using certain gamma matrices [23], which are contracted with the column index of the vielbein. In the basis in which we are working the following vector of gamma matrices has to be taken:

$$\vec{\Gamma} = \left(\mathbf{1}_8\,, \frac{\epsilon^{ijk}\gamma_{\bar{j}\bar{k}}}{4\sqrt{2}}\,, \frac{\epsilon^{\bar{i}\bar{j}\bar{k}}\gamma_{jk}}{4\sqrt{2}}\,, \frac{\gamma^{i\bar{j}}}{2\sqrt{2}}\,, \frac{\gamma^{i}(1-\Gamma_0)}{4}\,, \frac{\gamma^{\bar{i}}(1-\Gamma_0)}{4}\,, \frac{\gamma^{i}(1+\Gamma_0)}{4}\,, \frac{\gamma^{\bar{i}}(1+\Gamma_0)}{4}\right)^{ab}. \tag{4.40}$$

The capital gamma matrices used here are the SO(7) gamma matrices in the basis described in [25] so that

$$\Gamma_0 = i\Gamma_1\Gamma_2\Gamma_3\Gamma_4\Gamma_5\Gamma_6 = i\begin{pmatrix} 0 & -\mathbf{1}_4 \\ \mathbf{1}_4, & 0 \end{pmatrix}. \tag{4.41}$$

The small gamma matrices were defined by GPPZ [23] in terms of the SO(7) gamma matrices, and their bilinear are given by

$$\gamma_{ij} = \gamma_{[i}\gamma_{j]}, \quad \gamma_{\bar{i}\bar{j}} = \gamma_{[\bar{i}}\gamma_{\bar{j}]}, \quad \gamma_{i\bar{j}} = \gamma_{[i}\gamma_{\bar{j}]} - \frac{1}{3}\delta^{k\bar{l}}\gamma_{[k}\gamma_{\bar{l}]}. \tag{4.42}$$

After this change of basis we will denote the resulting components of the vielbein as follows:

$$\exp(X).\vec{\Gamma} = \frac{1}{4\sqrt{2}}\Big(V_{(1)}, V_{(2)}, V_{(3)}, V_{(4)}, V_{(5)}, V_{(6)}, V_{(7)}, V_{(8)}\Big)^{ab}. \tag{4.43}$$

Using the results for the vielbein in (4.35) and (4.38) we obtain the following vielbein components:

$$V^{i\bar{\jmath}}_{(1)} = \frac{1}{4\sqrt{2}}\mathbf{1}_8 \tag{4.44}$$

$$\begin{aligned} V^{i}_{(2)} &= \epsilon^{i}{}_{jk}\gamma^{\bar{\jmath}\bar{k}}\cosh|\sigma|\cosh|\alpha| + \epsilon^{i}{}_{\bar{\jmath}\bar{k}}\gamma^{jk}\frac{\overline{\sigma}\alpha}{|\sigma||\alpha|}\sinh|\sigma|\sinh|\alpha| + \\ &\sqrt{2}\delta^{i}{}_{k}\gamma^{\bar{k}}(1-\Gamma_0)\frac{\alpha}{|\alpha|}\cosh|\sigma|\sinh|\alpha| + \sqrt{2}\gamma^{i}(1+\Gamma_0)\frac{\overline{\sigma}}{|\sigma|}\sinh|\sigma|\cosh|\alpha| \end{aligned} \tag{4.45}$$

$$\begin{aligned} V^{\bar{\imath}}_{(3)} &= \epsilon^{\bar{\imath}}{}_{jk}\gamma^{\bar{\jmath}\bar{k}}\frac{\sigma\overline{\alpha}}{|\sigma||\alpha|}\sinh|\sigma|\sinh|\alpha| + \epsilon^{\bar{\imath}}{}_{\bar{\jmath}\bar{k}}\gamma^{jk}\cosh|\sigma|\cosh|\alpha| + \\ &\sqrt{2}\gamma^{\bar{\imath}}(1-\Gamma_0)\frac{\sigma}{|\sigma|}\sinh|\sigma|\cosh|\alpha| + \sqrt{2}\delta^{\bar{\imath}}{}_{\bar{k}}\gamma^{k}(1+\Gamma_0)\frac{\overline{\alpha}}{|\alpha|}\cosh|\sigma|\sinh|\alpha| \end{aligned} \tag{4.46}$$

$$\begin{aligned} V^{i\bar{\jmath}}_{(4)} &= 2\gamma^{i\bar{\jmath}}\cosh(2|\alpha|) + \\ &\sqrt{2}\epsilon^{i\bar{\jmath}}{}_{\bar{k}}\gamma^{k}(1-\Gamma_0)\frac{\alpha}{2|\alpha|}\sinh(2|\alpha|) + \sqrt{2}\epsilon^{i\bar{\jmath}}{}_{k}\gamma^{\bar{k}}(1+\Gamma_0)\frac{\overline{\alpha}}{2|\alpha|}\sinh(2|\alpha|) \end{aligned} \tag{4.47}$$

$$V^{i}_{(5)} = \epsilon^{i}{}_{\bar{k}l}\gamma^{k\bar{l}}\frac{\overline{\alpha}}{|\alpha|}\sinh(2|\alpha|) + \sqrt{2}\gamma^{i}(1-\Gamma_0)\cosh^2|\alpha| + \sqrt{2}\delta^{i}{}_{k}\gamma^{\bar{k}}(1+\Gamma_0)\frac{\overline{\alpha}^2}{|\alpha|^2}\sinh^2|\alpha| \tag{4.48}$$

$$\begin{aligned} V^{\bar{\imath}}_{(6)} &= \epsilon^{\bar{\imath}}{}_{jk}\gamma^{\bar{\jmath}\bar{k}}\frac{\overline{\alpha}}{|\alpha|}\cosh|\sigma|\sinh|\alpha| + \epsilon^{\bar{\imath}}{}_{\bar{\jmath}\bar{k}}\gamma^{jk}\frac{\overline{\sigma}}{|\sigma|}\sinh|\sigma|\cosh|\alpha| + \\ &\sqrt{2}\gamma^{\bar{\imath}}(1-\Gamma_0)\cosh|\sigma|\cosh|\alpha| + \sqrt{2}\delta^{\bar{\imath}}{}_{\bar{k}}\gamma^{k}(1+\Gamma_0)\frac{\overline{\sigma\alpha}}{|\sigma||\alpha|}\sinh|\sigma|\sinh|\alpha| \end{aligned} \tag{4.49}$$

$$\begin{aligned} V^{i}_{(7)} &= \epsilon^{i}{}_{jk}\gamma^{\bar{\jmath}\bar{k}}\frac{\sigma}{|\sigma|}\sinh|\sigma|\cosh|\alpha| + \epsilon^{i}{}_{\bar{\jmath}\bar{k}}\gamma^{jk}\frac{\alpha}{|\alpha|}\cosh|\sigma|\sinh|\alpha| + \\ &\sqrt{2}\delta^{i}{}_{k}\gamma^{\bar{k}}(1-\Gamma_0)\frac{\sigma\alpha}{|\sigma||\alpha|}\sinh|\sigma|\sinh|\alpha| + \sqrt{2}\gamma^{i}(1+\Gamma_0)\cosh|\sigma|\cosh|\alpha| \end{aligned} \tag{4.50}$$

$$V^{\bar{\imath}}_{(8)} = \epsilon^{\bar{\imath}}{}_{\bar{k}l}\gamma^{k\bar{l}}\frac{\overline{\alpha}}{|\alpha|}\sinh(2|\alpha|) + \sqrt{2}\delta^{\bar{\imath}}{}_{\bar{k}}\gamma^{k}(1-\Gamma_0)\frac{\alpha^2}{|\alpha|^2}\sinh^2|\alpha| + \sqrt{2}\gamma^{\bar{\imath}}(1+\Gamma_0)\cosh^2|\alpha|. \tag{4.51}$$

The remaining step is to compute the tensor W^{abcd}, which eventually gives the scalar potential P via the formula in equation (4.3). Before this can be done, however, there is another intermediate step that one has to take. Recalling the definition of W^{abcd} again

$$W^{abcd} = \epsilon^{\alpha\beta}\delta^{IJ}V_{I\alpha}{}^{ab}V_{J\beta}{}^{cd}, \tag{4.52}$$

we see that the contraction of the rows of the vielbein is given in the $\mathfrak{so}(6)\times\mathfrak{so}(2)$ basis, while our vielbein is written in the $\mathfrak{su}(3)\times\mathfrak{u}(1)\times\mathfrak{u}(1)$ basis. Therefore we need to investigate how the contractions in the two bases are related to one another. The following subsection is devoted to this calculation.

4.3.3 Contractions in the Sub-Algebra Basis

In this subsection we will show how the index contraction in (4.52), which is written with respect to the $\mathfrak{so}(6) \times \mathfrak{so}(2)$ basis, translates to a contraction in terms of the subalgebra $\mathfrak{su}(3) \times \mathfrak{u}(1) \times \mathfrak{u}(1) \subset \mathfrak{so}(6) \times \mathfrak{so}(2)$. Keeping only the relevant indices and considering the most general case the contraction we are looking to rewrite is of the form

$$\epsilon^{\alpha\beta}\delta^{IJ}Y^I_\alpha Z^J_\beta, \tag{4.53}$$

where Y^I_α and Z^J_β are some generic tensors in the $(\mathbf{6}, \mathbf{2})$ of the $\mathfrak{so}(6) \times \mathfrak{so}(2)$. Since the fundamental representations of both the $\mathfrak{so}(6)$ and the $\mathfrak{so}(2)$ are real there is no meaning to writing the indices upstairs or downstairs, and in (4.53) we simply chose a convenient way of writing.

First consider the $\mathfrak{so}(6)$ contraction. To take the $\mathfrak{su}(3) \times \mathfrak{u}(1) \subset \mathfrak{so}(6)$ subalgebra it is convenient to apply the isomorphism $\mathfrak{so}(6) \cong \mathfrak{su}(4)$ first. We used this isomorphism in the other direction in Chapter 2, where we saw that $Y^{AB} = \Sigma^{I\,AB}Y^I/\sqrt{2}$. One can use the sigma matrix identities in Appendix B.2 to invert this relation to get $Y^I = \overline{\Sigma}^I_{AB}Y^{AB}/(2\sqrt{2})$ so that after applying another sigma matrix identity the $\mathfrak{so}(6)$ part of the contraction in (4.53) can be written as

$$\delta^{IJ}Y^I Z^J = \frac{1}{8}\overline{\Sigma}^I_{AB}\overline{\Sigma}^I_{CD}Y^{AB}Y^{CD} = \frac{1}{4}\epsilon_{ABCD}Y^{AB}Z^{CD}. \tag{4.54}$$

Next we need to branch the $\mathbf{6}$ under the subgroup $\mathfrak{su}(3) \times \mathfrak{u}(1) \subset \mathfrak{su}(4)$. Using the embedding in which the $\mathfrak{u}(1)$ generator is given by

$$H = \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & -3 \end{pmatrix} \tag{4.55}$$

the $\mathbf{6}$ branches as

$$\mathbf{6} \to \mathbf{3}_{-2} + \bar{\mathbf{3}}_2. \tag{4.56}$$

We also saw in the field theory discussion of Chapter 2 how this translates to the branching of the tensors. After splitting the $\mathfrak{su}(4)$ index A to $(i, 4)$ the tensors in the $\mathbf{6}$ branch as follows

$$\mathbf{6} = Y^{AB} \to \{Y^{ij}, Y^{i4}\} \equiv \{\epsilon^{ijk}\bar{Y}_k, Y^i\} = \{\bar{\mathbf{3}}_2, \mathbf{3}_{-2}\}, \tag{4.57}$$

where we have introduced the $\mathfrak{su}(3)$ tensors $\bar{Y}_k$ and Y^i. To check that we have assigned Y^{ij} and Y^{i4} to the $\bar{\mathbf{3}}_2$ and $\mathbf{3}_{-2}$ correctly we can apply the $\mathfrak{u}(1)$ transformations and compute the corresponding charges. Using the U(1) generator H we defined in equation (4.55) the following result is obtained:

$$\delta_{\mathrm{U}(1)} Y^{ij} = H^{iA}Y^{Aj} + H^{jA}Y^{iA} = \delta^{iA}Y^{Aj} + \delta^{jA}Y^{iA} = 2Y^{ij} \tag{4.58}$$

$$\delta_{\mathrm{U}(1)} Y^{i4} = H^{iA}Y^{A4} + H^{4A}Y^{iA} = \delta^{iA}Y^{Aj} - 3\delta^{4A}Y^{iA} = -2Y^{i4}, \tag{4.59}$$

so the assignment is indeed correct. To work out the contraction in (4.54) one needs to decompose the $\mathfrak{su}(4)$ indices as just explained, and substitute the definitions $Y^{ij} \equiv \epsilon^{ijk}\bar{Y}_k$ and $Y^{i4} \equiv Y^i$. The result is

$$\delta^{IJ}Y^I Z^J = \frac{1}{4}\epsilon_{ABCD}Y^{AB}Z^{CD} = \bar{Y}_i Z^i + Y^i \bar{Z}_i. \tag{4.60}$$

The same can be written using the representations and their charges rather than tensors in the following way

$$\langle \mathbf{6}, \mathbf{6} \rangle \to \langle \bar{\mathbf{3}}_2, \mathbf{3}_{-2} \rangle + \langle \mathbf{3}_{-2}, \bar{\mathbf{3}}_2 \rangle. \tag{4.61}$$

Next consider the $\mathfrak{so}(2)$ contraction given by

$$\epsilon^{\alpha\beta}Y_\alpha Z_\beta. \tag{4.62}$$

The generator of counter-clockwise $\mathfrak{so}(2)$ rotations is given by

$$T_\alpha{}^\beta = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix} \qquad \exp(i\alpha T) = \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix}, \tag{4.63}$$

and has eigenvalues ± 1. Therefore carrying out the algebra isomorphism $\mathfrak{so}(2) \cong \mathfrak{u}(1)$ amounts to diagonalising $T_\alpha{}^\beta$ and branching the fundamental representation according to

$$\mathbf{2} \to \mathbf{1}_1 + \mathbf{1}_{-1}. \tag{4.64}$$

The generator $T_\alpha{}^\beta$ is diagonalised by the unitary matrix $U = \frac{1}{\sqrt{2}}\begin{pmatrix} -i & 1 \\ i & 1 \end{pmatrix}$ so that the isomorphism is given by

$$T \mapsto UTU^{-1} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{4.65}$$

$$Y_\alpha = \begin{pmatrix} Y_1 \\ Y_2 \end{pmatrix} \mapsto U \begin{pmatrix} Y_1 \\ Y_2 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} Y_2 - iY_1 \\ Y_2 + iY_1 \end{pmatrix} \equiv \begin{pmatrix} Y^+ \\ Y^- \end{pmatrix}. \tag{4.66}$$

It is manifest that Y^+ correspond to the eigenvalue $+1$, and so is in the $\mathbf{1}_1$, and similarly Y^- is in the $\mathbf{1}_{-1}$. To carry out the contraction invert the isomorphism just spelled out to get

$$Y_1 = \frac{i}{\sqrt{2}}(Y^+ - Y^-) \qquad Y_2 = \frac{1}{\sqrt{2}}(Y^+ + Y^-). \tag{4.67}$$

Thus we see that the $\mathfrak{su}(2)$ contraction transformed to the $\mathfrak{u}(1)$ basis is given by

$$\epsilon^{\alpha\beta} Y_\alpha Z_\beta = i(Y^+ Z^- - Y^- Z^+). \tag{4.68}$$

In terms of representations this can be written as

$$\langle \mathbf{2}, \mathbf{2} \rangle \to i\langle \mathbf{1}^+, \mathbf{1}^- \rangle - i\langle \mathbf{1}^-, \mathbf{1}^+ \rangle. \tag{4.69}$$

We can now combine the $\mathfrak{so}(6)$ and the $\mathfrak{so}(2)$ contractions to find the contraction of two tensors in the $(\mathbf{6}, \mathbf{2})$. Note that according to equation (4.23) the $\mathfrak{so}(2)$ charges that we need are in fact twice as large as those considered in this section. Therefore in order to apply our contraction formulas to the representation we encountered in the previous section we need to take $\mathbf{2} \to \mathbf{1}_2 + \mathbf{1}_{-2}$ under the isomorphism $\mathfrak{so}(2) \cong \mathfrak{u}(1)$. With this in mind the combined contraction in tensor form reads

$$\epsilon^{\alpha\beta} \delta^{IJ} Y^I_\alpha Z^J_\beta = i \left(\bar{Y}^+_i Z^{i-} - \bar{Y}^-_i Z^{i+} + Y^{i+} \bar{Z}^-_i - Y^{i-} \bar{Z}^+_i \right). \tag{4.70}$$

It is again convenient to write this in terms of representations and their $\mathfrak{u}(1)$ charges:

$$\langle (\mathbf{6}, \mathbf{2}), (\mathbf{6}, \mathbf{2}) \rangle \to i \Big[\langle \bar{\mathbf{3}}_{(2,2)}, \mathbf{3}_{(-2,-2)} \rangle - \langle \bar{\mathbf{3}}_{(2,-2)}, \mathbf{3}_{(-2,2)} \rangle + \langle \mathbf{3}_{(-2,2)}, \bar{\mathbf{3}}_{(2,-2)} \rangle - \langle \mathbf{3}_{(-2,-2)}, \bar{\mathbf{3}}_{(2,2)} \rangle \Big]. \tag{4.71}$$

The two $\mathfrak{u}(1)$ charges we used above are with respect to the decomposition under $\mathfrak{so}(6) \times \mathfrak{so}(2) \to \mathfrak{su}(3) \times \mathfrak{u}(1) \times \mathfrak{u}(1)$ so that the first charge in the subscript refers to the $\mathfrak{u}(1)$ contained in the $\mathfrak{su}(4)$ and the second to the $\mathfrak{u}(1)$ from the the $\mathfrak{so}(2)$.

4.3.4 Evaluation of the Scalar Potential

To apply the contraction rules just derived to equation (4.52) and compute W^{abcd} recall that in the notation (4.43) we have

$$\left(V_{(5)}, V_{(6)}, V_{(7)}, V_{(8)} \right) = (\mathbf{3}_{(-2,2)}, \bar{\mathbf{3}}_{(2,2)}, \mathbf{3}_{(-2,-2)}, \bar{\mathbf{3}}_{(2,-2)}). \tag{4.72}$$

After comparing with the contraction rules in (4.71) we see that the tensor W^{abcd} is given by

$$W^{abcd} = \epsilon^{\alpha\beta} \delta^{IJ} V_{I\alpha}{}^{ab} V_{J\beta}{}^{cd} \tag{4.73}$$

$$= i \left(V^{\bar{\imath},ab}_{(6)} V^{i,cd}_{(7)} - V^{\bar{\imath},ab}_{(8)} V^{i,cd}_{(5)} + V^{i,ab}_{(5)} V^{\bar{\imath},cd}_{(8)} - V^{i,ab}_{(7)} V^{\bar{\imath},cd}_{(6)} \right). \tag{4.74}$$

At this point everything is explicit in the $\mathfrak{su}(3)\times\mathfrak{u}(1)\times\mathfrak{u}(1)$ basis and we can insert W^{abcd} into the formula (4.75) to obtain the scalar potential. After the substitution we obtain

$$\begin{aligned} P &= -\frac{g^2}{32}\left(2W_{ab}W^{ab} - W_{abcd}W^{abcd}\right) \\ &= -\frac{g^2}{32}\left(2\Omega_{ac}\Omega_{a'c'}\Omega_{bb'}\Omega_{dd'} - \Omega_{aa'}\Omega_{cc'}\Omega_{bb'}\Omega_{dd'}\right) W^{a'b'c'd'}W^{abcd} \end{aligned} \tag{4.75}$$

$$\text{with} \quad \Omega_{ab} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \tag{4.76}$$

In GPPZ [23] the authors set $g^2 = 4$. The expression in (4.75) is best evaluated in a computer algebra software such as Mathematica [81], and to this end it is convenient to rewrite the scalar potential in the following form:

$$\begin{aligned} P = \frac{-i^2}{8}\sum_{A,B=1}^{4}\Bigg(&2\,\mathrm{tr}\left[\Gamma_0 V^i_{(A,2)}\Gamma_0 (V^j_{(B,2)})^t\Gamma_0 V^j_{(B,1)}\Gamma_0 (V^i_{(A,1)})^t\right] \\ &- \mathrm{tr}\left[\Gamma_0 V^i_{(A,2)}\Gamma_0 (V^j_{(B,2)})^t\right]\mathrm{tr}\left[\Gamma_0 V^j_{(B,1)}\Gamma_0 (V^i_{(A,1)})^t\right]\Bigg). \end{aligned} \tag{4.77}$$

where we have defined

$$V^i_{A,\alpha} = \left[\left(V^i_{(5)}, V^i_{(8)}\right), \left(V^i_{(6)}, V^i_{(7)}\right), \left(-V^i_{(7)}, V^i_{(6)}\right), \left(-V^i_{(8)}, V^i_{(5)}\right)\right], \tag{4.78}$$

and Γ_0 is the same as in equation (4.41). After inserting the expression for the vielbein found in (4.44) we do indeed reproduce the GPPZ scalar potential [23]:

$$P = -\frac{3}{8}\Big(\cosh^2(2\alpha) + 4\cosh(2\alpha)\cosh(2\sigma) - \cosh^2(2\sigma) + 4\Big). \tag{4.79}$$

To summarise we see that to compute the scalar potential for a given truncation of gauged supergravity one needs to go through some lengthy and non-trivial steps. This is why it is quite surprising that the holographic computation of the same potential is rather simple and can be performed in just a few lines, as we will see in the next chapter. Nonetheless it was useful to review the direct approach presented in this chapter for various reasons. It gave us an opportunity to study the group theory of the truncations in which we are interested, and also allows to cover multi-scalar cases in a systematic way, which is not yet well understood in the holographic computation. Even though the expression of the GPPZ potential was known, it might be useful to repeat this calculation for other truncations in order to check new analogous results obtained holographically.

Chapter 5

Holographic Beta Function

Let us first recall how holography allows us to reconstruct bulk data from field theory. Start with a field theory in d dimensions which becomes conformal in the UV. Assuming the holographic principle holds, there is a dual description of this theory in terms of a theory of gravity in one dimension more. The extra dimension can be interpreted as the energy scale of the field theory, and the UV regime is matched to the asymptotic infinity of the spacetime of the dual gravitational theory. This is why the field theory is sometimes said to live on the boundary of the bulk. From this it follows that if the field theory is conformal in the UV, the dual space-time is asymptotically AdS.

The degrees of freedom of the field theory are given a dual bulk description via the so-called operator-field correspondence. Given a scalar gauge-invariant operator $\mathcal{O}_\Delta$ of conformal dimension Δ, the dual description involves a solution to the bulk theory with a non-trivial scalar field ϕ of mass $m^2 = \Delta(\Delta - d)$. To find the correct dual interpretation of this field one needs to look at its asymptotic behaviour at the bulk infinity in more detail [82, 83]. In this near-boundary analysis one expands the field ϕ in an asymptotic series in the radial bulk coordinate r and looks at terms of the form $\sim \phi_0 e^{-(d-\Delta)r}$ and $\sim \phi_0 e^{-\Delta r}$. If the former term is non-trivial, then ϕ is interpreted as a source for a field theory deformation by the operator $\mathcal{O}_\Delta$:

$$\mathcal{L}_{\text{CFT}} \to \mathcal{L}_{\text{CFT}} + \phi_0 \mathcal{O}_\Delta. \tag{5.1}$$

If, in contrast, the latter term is non-trivial, then the field theory remains undeformed, but the operator $\mathcal{O}_\Delta$ acquires a vacuum expectation value given by

$$\langle \mathcal{O}_\Delta \rangle \sim \phi_0. \tag{5.2}$$

In either case the bulk solution is of domain-wall type and the metric can be cast in the following form

$$ds^2 = dr^2 + e^{2A(r)} \eta_{\mu\nu} dx^\mu dx^\nu, \qquad \lim_{r\to\infty} A(r) = r. \tag{5.3}$$

The limit $r \to \infty$ at spacial infinity ensures that the solution is asymptotically AdS. This domain-wall profile is precisely the description of the RG-flow that is triggered on the field theory side by the operator $\mathcal{O}_\Delta$. One of the desired properties of such RG-flows is that $\mathcal{O}_\Delta$ does not mix with other operators as one flows towards the IR. If it does, then additional non-trivial modes have to be included on the bulk side. It has proven fruitful to find such consistent truncations of the bulk theory with solutions to the field equations that involve only few non-trivial fields.

Now assume we have found a field theory deformation by an operator $\mathcal{O}_\Delta$ which does not mix with any other operators along the corresponding RG-flow. As explained above in the dual picture this corresponds to a consistent truncation of the bulk theory to a single scalar $\phi(r)$. In theory one should now solve the field equations for $\phi(r)$ and $A(r)$ to determine the radial domain-wall profile, and since this profile is related to the running coupling on the field theory side, we should be able to extract quantitative information about the beta-function. The bulk equations of motion can be derived from the action of the truncated theory which is of the following form:

$$S = \frac{1}{2\kappa^2} \int \mathrm{d}^{d+1}x \sqrt{-g} \left(-R + (\partial\phi)^2 + 2\kappa^2 V(\phi) \right). \tag{5.4}$$

The variation of this action leads to coupled second order equations of motion for the scalar field ϕ and the metric function A. For the specific case of domain-wall type solutions as in equation (5.3) it was shown [26, 32, 33] that the solutions for ϕ and A can be obtained as solutions of certain first order equations, and the second order equations of motion will be automatically satisfied. In such a set-up the scalar potential $V(\phi)$ can be written as

$$V(\phi) = \frac{1}{2\kappa^2} \left((\partial_\phi \mathcal{W})^2 - \frac{d}{d-1} \mathcal{W}^2 \right) \tag{5.5}$$

in terms of what is usually called a fake superpotential $\mathcal{W}(\phi)$ and the first order differential equations for ϕ and A are the following:

$$\partial_r \phi(r) = \partial_\phi \mathcal{W} \tag{5.6}$$

$$\partial_r A(r) = -\frac{1}{d-1} \mathcal{W}. \tag{5.7}$$

As mentioned before the domain-wall profile of the solution relates directly to the field theory beta-function in the dual description. The precise correspondence was worked out in [30, 31, 55, 84], and the beta-function can be given directly in terms of the fake superpotential in the following way:

$$\beta(r) = -(d-1) \frac{1}{\mathcal{W}} \frac{d\mathcal{W}}{d\phi}. \tag{5.8}$$

From here one could proceed as follows. Choose a CFT with a holographic dual, then find a deformation by an operator $\mathcal{O}$ which does not mix with other operators along the RG-flow that it triggers. Next find the gravity dual and determine the corresponding truncation. On this truncation find the scalar potential and the superpotential, and finally compute the beta-function for the running coupling of $\mathcal{O}$ using (5.8). Such computations have been done before, however the computation of the superpotential is rarely trivial. Therefore we would like the ask the opposite question: are there cases where the beta-function is known exactly? In this case we could reverse the procedure and learn about the superpotential of the gravity theory. Let us see how this works.

Assume we know $\beta(r)$. It follows from the first-order equation of motion (5.6) for $\phi(r)$ that

$$\frac{d\mathcal{W}(\phi(r))}{dr} = \frac{d\mathcal{W}}{d\phi}\frac{d\phi}{dr} = \left(\frac{d\mathcal{W}}{d\phi}\right)^2. \tag{5.9}$$

This can be used to turn the equation (5.8) for the holographic beta-function into a first-order differential equation with respect to the radial coordinate r, and to integrate it. The integration constant can be determined by matching the leading constant term in the large r expansion of $\mathcal{W}$ with the cosmological constant of the asymptotic AdS space. For large radii r we must get

$$\mathcal{L}_{\text{EH}} \sim \left(R - 2\kappa^2 V\right) \to \left(R + \frac{d(d-1)}{L^2}\right), \tag{5.10}$$

where L is the radius of the asymptotic AdS space. From this one can infer that the leading constant term in $\mathcal{W}$ should be $-(d-1)/L$. Furthermore we can insert the first order field equation (5.6) for ϕ in the beta-function equation (5.8) and integrate to get ϕ in terms of the beta-function, where the integration constant is fixed by the asymptotic behaviour of $\phi(r)$ for large radii. From now on we will measure distances in term of the AdS radius L, and thus effectively set $L = 1$. The procedure just outlined leads to the following results:

$$\mathcal{W}(r) = \frac{(d-1)^2}{-(d-1) - \int^r \mathrm{d}r' \beta(r')^2} \tag{5.11}$$

$$\phi(r) = -\frac{1}{d-1}\int^r \mathrm{d}r' \beta(r')\mathcal{W}(r'). \tag{5.12}$$

Using the fact that $\phi(r)$ is monotonic we can invert it to $r(\phi)$ and then insert into $\mathcal{W}(r)$ to obtain $\mathcal{W}(\phi)$, and eventually, using (5.5), the scalar potential of the bulk theory $V(\phi)$. The question to ask now is: are there interesting cases in which the beta-function is known exactly? The beta-function $\beta_\lambda(\mu)$ corresponding to a generic coupling λ receives several contributions. It has a classical part, which is in essence given by the dimension $[\lambda]$ of the coupling. Quantum loop corrections might renormalise the coupling as well and add additional terms to the beta function. Finally, the wave function renormalisation $Z_\mathcal{O}$ might be not scale-invariant itself, and contribute corrections to the beta-function known as anomalous dimension $\gamma_\mathcal{O}(\mu)$. One of the classical results is that supersymmetry is strong enough to provide cancellations of many quantum loop corrections, in fact, it is known that in $\mathcal{N}=1$ theories the superpotential does not renormalise at all [85]. Thus in supersymmetric theories where interactions for component fields arise as F-terms of a superpotential, the beta-functions for the corresponding couplings only renormalise through wave function renormalisations which contribute the anomalous dimension. On the other hand it is also known that there are classes of operators with protected dimensions, those are the operators for which the wave function renormalisation Z is scale invariant. For the couplings that multiply these operators this means that the contribution of the anomalous dimension vanishes. In fact there are theories with operators that are both $\mathcal{N}=1$ F-terms, and have a protected dimension. One such class of operators, which we will discuss in detail later, are operators in superconformal theories sitting in short multiplets. These operators are particularly interesting for holography because they are precisely those with the dual gravity modes surviving the supergravity limit. Some of these protected operators are also F-terms of some $\mathcal{N}=1$ subgroup of the full supersymmetry, and therefore the beta-function for their couplings is free of quantum corrections and anomalous dimension, and is simply given by its classical value. For an operator $\mathcal{O}_\Delta$ in d dimension the coupling has dimension $[\lambda]=d-\Delta$, and therefore the classical beta-function reads

$$\beta_\lambda(\mu) = -(d-\Delta)\lambda(\mu). \tag{5.13}$$

With this result we can now go on to evaluate equations (5.11) and (5.12) to determine the superpotential $\mathcal{W}$ and the radial profile of the gravity mode ϕ. Using the definition $\beta_\lambda(\mu)=\frac{d\lambda(\mu)}{d\log\mu}$ we can integrate (5.13) to get $\lambda(\mu)=\lambda_0\mu^{-(d-\Delta)}$. In holography we interpret ϕ as the source for the operator $\mathcal{O}_\Delta$, and the radial bulk direction as the energy scale, therefore we need to identify λ with ϕ, and μ with e^r and get

$$\beta(r) = -(d-\Delta)\phi_0 e^{-(d-\Delta)r}. \tag{5.14}$$

As expected this matches exactly the derivative of the leading term of the asymptotic expansion of $\phi(r)$ for large r. This form of the beta-function can now be directly integrated as shown in equations (5.11) and (5.12) to get

$$\mathcal{W}(r) = \frac{-(d-1)}{1-\left(\frac{\phi_0}{\alpha}\right)^2 e^{-2(d-\Delta)r}} \qquad \text{with} \quad \alpha = \sqrt{\frac{2(d-1)}{d-\Delta}} \tag{5.15}$$

$$\phi(r) = \frac{\alpha}{2}\ln\left(\frac{\alpha+\phi_0\, e^{-(d-\Delta)r}}{\alpha-\phi_0\, e^{-(d-\Delta)r}}\right). \tag{5.16}$$

One can check that $\mathcal{W}(r)$ produces the correct cosmological constant for the asymptotic AdS space and $\phi(r)$ has the correct asymptotic behaviour near the boundary. As mentioned before, we can now invert $\phi(r)$ and insert into $\mathcal{W}(r)$ to obtain the superpotential as a function ϕ. The result is:

$$\mathcal{W}(\phi) = -(d-1)\cosh^2\left(\frac{\phi}{\alpha}\right). \tag{5.17}$$

Finally, using (5.5) one can use the superpotential to construct the scalar potential:

$$V(\phi) = \frac{d-1}{4\kappa^2}\cosh^2\left(\frac{\phi}{\alpha}\right)\left[(d-2\Delta)\cosh\left(\frac{2\phi}{\alpha}\right) - (3d-2\Delta)\right]. \tag{5.18}$$

Before turning to a general discussion we can test our result on the well-known holographic flow discovered by GPPZ [21–23]. The authors discuss the deformation of the $\mathcal{N}=4$ SYM theory in dimension $d=4$ by mass terms that preserve $\mathcal{N}=1$ supersymmetry. Written in $\mathcal{N}=1$ language the $\mathcal{N}=4$ SYM theory contains a triplet Z_i of chiral superfields, and the deformation is given by the superpotential

$$\delta\mathcal{W} = m\,\delta^{ij}\operatorname{tr} Z_i Z_j \tag{5.19}$$

We will see later that the operator $\mathcal{O} = \delta^{ij}\operatorname{tr} Z_i Z_j$ is in a short multiplet of the $\mathcal{N}=4$ SYM, and therefore fits our requirements. Its F-term is a relevant operator of conformal dimension $\Delta = 3$, and m is the coupling that holographically corresponds to a supergravity mode. Plugging $d=4$ and $\Delta=3$ into our formulas, and transforming conventions via $\kappa^2 \to 2$, $\phi \to \sqrt{2}m$, $\mathcal{W} \to 2\mathcal{W}$ we recover exactly the superpotential and the scalar potential given in equations (25) and (21) in [23]:

$$\mathcal{W} = -\frac{3}{4}\left[\cosh\left(\frac{2m}{\sqrt{3}}\right) + 1\right] \tag{5.20}$$

$$V = -\frac{3}{8}\left[\cosh^2\left(\frac{2m}{\sqrt{3}}\right) + 4\cosh\left(\frac{2m}{\sqrt{3}}\right) + 3\right]. \tag{5.21}$$

Let us now generalise this construction and explore further deformations of $\mathcal{N}=4$ SYM in $d=4$ in a systematic way.

Chapter 6

$\mathcal{N}=1$ Deformations of the $\mathcal{N}=4$ Theory

In the previous chapter we noted that for deformations of the $\mathcal{N}=4$ super Yang-Mills theory that preserve an $\mathcal{N}=1$ supersymmetry non-renormalisation properties of the superpotential can be used to facilitate the computation of the beta function. Moreover, it is known that operators in short representations of the $\mathcal{N}=4$ super Yang-Mills theory that we discussed in Chapter 3 are protected and therefore have vanishing anomalous dimension [14]. In this chapter we are going to combine these two ideas and study deformations by chiral operators that preserve an $\mathcal{N}=1$ supersymmetry. We will use the formulation of the $\mathcal{N}=4$ super Yang-Mills theory in the $\mathcal{N}=1$ language which we introduced in Section 2.3, and will recall the most important facts in the next few paragraphs.

The representations of the $\mathcal{N}=4$ fields are as follows. The six Lorentz scalars ϕ^I are in the $\mathbf{6}$ of the $\mathrm{SU}(4)_{\mathrm{R}}$ symmetry, the four Lorentz fermions λ_A in the $\bar{\mathbf{4}}$ of the $\mathrm{SU}(4)_{\mathrm{R}}$, and the gauge fields A_μ are $\mathrm{SU}(4)_{\mathrm{R}}$ singlets. Restricting to $\mathcal{N}=1$ is equivalent to the decomposition under $\mathrm{U}(2,2|1)\times\mathrm{SU}(3)\times\mathrm{U}(1)\subset\mathrm{SU}(2,2|4)$ [37]. In particular, the R-symmetry decomposes as $\mathrm{SU}(4)_{\mathrm{R}}\to\mathrm{SU}(3)\times\mathrm{U}(1)_{\mathrm{R}}$. Under this decomposition the fundamental representations of the SU(4) branch as

$$
\begin{aligned}
\mathbf{4} &\to \mathbf{3}_1+\mathbf{1}_{-3}\\
\mathbf{6} &\to \mathbf{3}_{-2}+\bar{\mathbf{3}}_2\\
\bar{\mathbf{4}} &\to \bar{\mathbf{3}}_{-1}+\mathbf{1}_3\,.
\end{aligned}
\tag{6.1}
$$

with the normalisation of the U(1) charges chosen such that the $\mathrm{U}(1)\subset\mathrm{SU}(4)$ generator is given by $\mathrm{diag}(1,1,1,-3)$. Thus the basic fields of the $\mathcal{N}=4$ theory decompose as

$$
\begin{aligned}
\phi^I &\to \{\bar{z}^i, z_i\}\\
\lambda_A &\to \{\lambda_i,\lambda\},
\end{aligned}
\tag{6.2}
$$

and we get complex scalar triplets $\bar{z}^i$ and z_i in the $\mathbf{3}_{-2}$ and $\overline{\mathbf{3}}_2$ of the SU(3) × U(1), one fermion triplet λ_i in the $\overline{\mathbf{3}}_{-1}$, and a singlet fermion λ in the $\mathbf{1}_3$. The vector A_μ was an SU(4) singlet to begin with and therefore remains an SU(3) singlet with no U(1) charge. While the $\mathcal{N}=4$ SYM theory does not admit an off-shell superfield description with a finite number of auxiliary fields, its $\mathcal{N}=1$ description does, and we can pack all fields into three chiral multiplets and one vector multiplet as described in Section 2.3:

$$\begin{aligned} Z_i &= z_i + \sqrt{2}\theta\lambda_i + \theta^2 F_i \\ V &= \theta\sigma^\mu\bar{\theta}A_\mu + \bar{\theta}^2\theta\lambda + \theta^2\bar{\theta}\bar{\lambda} + \tfrac{1}{2}\theta^2\bar{\theta}^2 D\,. \end{aligned} \tag{6.3}$$

Note that since the complex scalar z_i has two units of $\mathrm{U}(1)_\mathrm{R}$ charge, and λ_i one negative unit, the fermionic coordinate θ_α must have a charge of three positive units under the $\mathrm{U}(1)_\mathrm{R}$ automorphisms. The vector multiplet V is in Wess-Zumino gauge, and can be used to construct the field strength multiplet:

$$\begin{aligned} W_\alpha &= -\tfrac{1}{4}\bar{D}^2 e^{-V} D_\alpha e^V \\ &= \lambda_\alpha + \theta^\beta\left(\epsilon_{\alpha\beta}D + F^+_{\alpha\beta}\right) + \theta^2 i D_{\alpha\dot{\alpha}}\bar{\lambda}^{\dot{\alpha}}\,, \end{aligned} \tag{6.4}$$

where $F^+ = \sigma^{\mu\nu}F_{\mu\nu}$ is the self-dual part of the vector field strength with $\sigma^{\mu\nu} = \frac{i}{2}\sigma^{[\mu}\bar{\sigma}^{\nu]}$, and $D_{\alpha\dot{\alpha}} = \sigma^\mu_{\alpha\dot{\alpha}}D_\mu$ the covariant derivative with respect to the SU(N) gauge group. The on-shell values of the auxiliary fields can be determined by writing the $\mathcal{N}=4$ action in terms of $\mathcal{N}=1$ off-shell multiplets and by deriving their equations of motion, the result is

$$F_i^a = \frac{g}{\sqrt{2}} f^{abc}\epsilon_{ijk}\bar{z}^{jb}\bar{z}^{kc} \tag{6.5a}$$

$$D^a = igf^{abc}\bar{z}^{ia}z_i^a. \tag{6.5b}$$

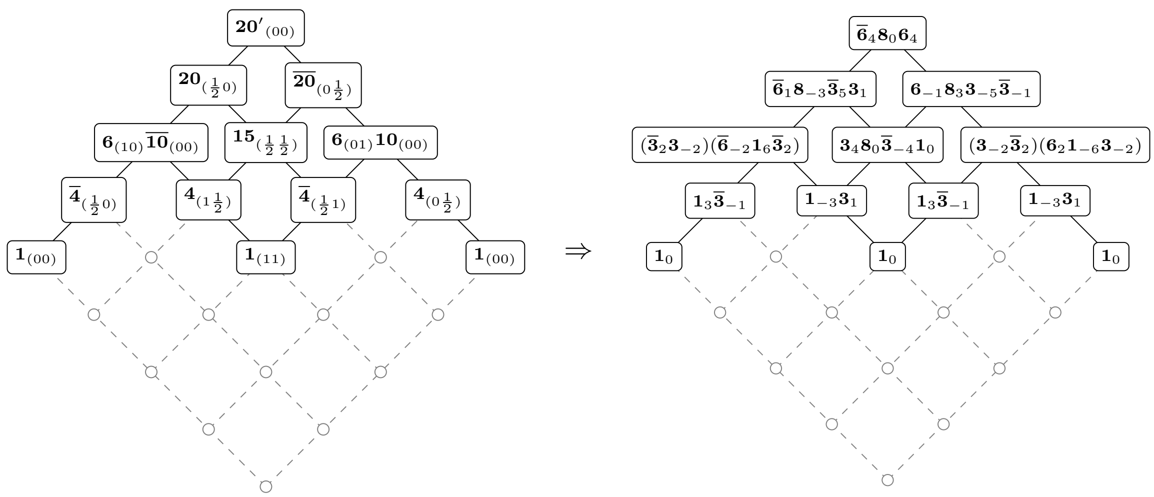
Next we would like to discuss the branching of short multiplets. In particular, we would like to describe the branched operators in terms of $\mathcal{N}=1$ superfields, find their quantum numbers, and indicate a way to obtain all operators explicitly. A way of explicitly constructing short multiplets was described in [38]. One starts with the so-called “twisted chiral superfield” $W_{[AB]}$, which was defined in [86], and contains all $\mathcal{N}=4$ fields. Then the operators that comprise the order-p short multiplet are found as components of the superfield $W^p_{[AB]}$. This is a good starting point for the decomposition under an $\mathcal{N}=1$ sub-algebra, and the method was outlined in [37]. The crucial point is that $W_{[AB]}$ decomposes to $\mathcal{N}=1$ superfields as

$$W_{[AB]} \quad\rightarrow\quad \{Z_i, \bar{Z}^i, W_\alpha, \bar{W}^{\dot{\alpha}}\}\,, \tag{6.6}$$

with the chiral superfields Z_i and W_α as above. Because the fields of the order-p short multiplet can be found as components of $W^p_{[AB]}$, after the restriction to $\mathcal{N}=1$ the branched operators can be found as components of superfields that are constructed by taking products of p superfields in $\{Z_i, W_\alpha\}$ and their conjugates. For example, for $p = 2$ some of the combinations include $Z_{(i}Z_{j)}$, $Z_i\bar{Z}^j - \frac{1}{3}\delta_i^j Z_k\bar{Z}^k$, Z_iW_α, W^2, and so forth. Note that the superfields have to be combined in such a way that they form irreducible representations of the SU(3) group. All possible combinations for $p \in \{2, 3, 4\}$ can be found in [37]. Starting with this construction we would like to understand in more detail how these $\mathcal{N} = 1$ superfields fit into the $\mathcal{N} = 4$ short multiplet.

First consider the easiest case, the order-2 multiplet. Under $\mathrm{SU}(3) \times \mathrm{U}(1)_\mathrm{R} \subset SU(4)_\mathrm{R}$ the representations of this multiplet branch as in Figure 6.1. Now we know that the branched

Figure 6.1: Branching of the order-2 short multiplet under an $\mathcal{N} = 1 \subset \mathcal{N} = 4$ subalgebra, which induces the branching $\mathfrak{su}(3) \times \mathfrak{u}(1)_\mathrm{R} \subset \mathfrak{su}(4)_\mathrm{R}$ of the R-symmetry algebra.



representations on the right diagram have to be components of some $\mathcal{N} = 1$ superfields, but how to determine which exactly? First of all we know the quantum numbers of the chiral superfield Z_i and the field strength superfield W_α that capture all fields of the $\mathcal{N} = 4$ theory in the $\mathcal{N} = 1$ language:

$$
\begin{aligned}
Z_i &= \overline{\mathbf{3}}_{2(00)} \\
W_\alpha &= \mathbf{1}_{3(\frac{1}{2}0)},
\end{aligned}
\tag{6.7}
$$

where the first number in the subscript is the $\mathrm{U}(1)_\mathrm{R}$ charge and the two numbers in parentheses are the Lorentz quantum numbers in the $\mathrm{SU}(2) \times \mathrm{SU}(2)$ notation. Additionally, as explained above, we also know that fields in order-p short multiplets are contained in superfields build of p-fold products of Z_i and W_α. This information is enough to reconstruct the decomposition in terms of $\mathcal{N} = 1$ superfields. Consider the superfield $Z^2 \equiv Z_{(i}Z_{j)}$. Here we have just introduced a short-hand notation in which the SU(3) indices are suppressed. In this case we will implicitly assume that all SU(3) indices are symmetrised and the traces are removed, so that a general product $(Z)^m(\bar{Z})^n$ corresponds to the SU(3) representation $[n, m]$. Given the representations in (6.7) we see that the resulting superfield corresponds to the representation

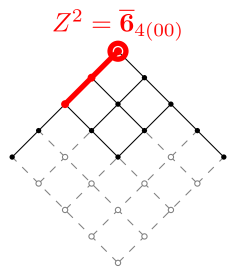
$$Z_{(i}Z_{j)} = \overline{\mathbf{6}}_{4(00)}\,. \tag{6.8}$$

Since Z_i is a chiral superfield, so is $Z_{(i}Z_{j)}$, and given that the representation of θ is

$$\theta_\alpha = \mathbf{1}_{3(\frac{1}{2}0)} \tag{6.9}$$

we observe that the θ^0, θ^1, and θ^2 components correspond to the $\overline{\mathbf{6}}_{4(00)}$, the $\overline{\mathbf{6}}_{1(\frac{1}{2},0)}$, and the $\overline{\mathbf{6}}_{-2(00)}$, respectively. The way these components fit into the decomposition can be read off from the multiplet decomposition above, which is presented in Figure 6.2.

Figure 6.2: The way the components of the $\mathcal{N}=1$ chiral superfield $Z_{(i}Z_{j)}$ fit in the $\mathcal{N}=4$ order $p=2$ short multiplet.



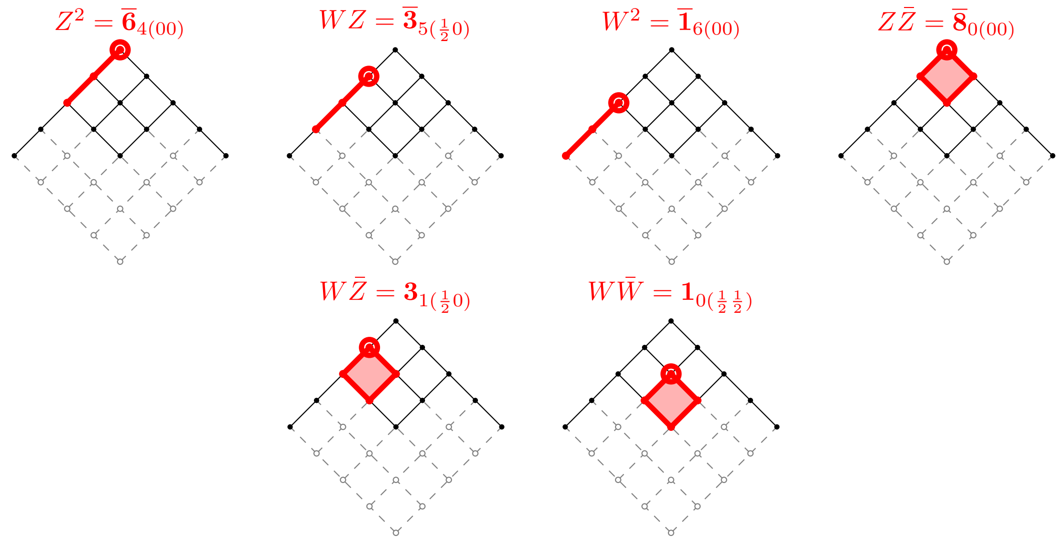
The lowest component of $Z_{(i}Z_{j)}$ is encircled and can be found at the top of the diagram. As we take superfield components with higher powers in θ we descend down the graph, θ corresponds to going down left, and $\bar{\theta}$ corresponds to going down right. The fact that $Z_{(i}Z_{j)}$ is a chiral superfield is manifest in the sense that its representation is a straight line going down left. Proceeding in the same way we can find other combinations of Z_i, $\bar{Z}^i$, W_α, and $\overline{W}_{\dot{\alpha}}$ that cover the whole order-2 multiplet. The result is presented in Figure 6.3, and can also be found in Appendix F along with higher order short multiplets up to $p=5$.

We can generalize the $\mathcal{N}=1$ decomposition to $p \geq 2$ along the same lines as the $p=2$ case. First fix the order p, and list all p-fold products of superfields Z_i and W_α and their complex conjugates, where W and $\overline{W}$ can at most occur quadratically. The representation of the resulting superfield and its components can be derived from those of Z_i, W_α, and the coordinate θ as shown for the $p=2$ case. For a general product of superfields it is given by

$$(Z)^{n_1}(\bar{Z})^{n_2}(W)^{k_1}(\overline{W})^{k_2} = [n_2,n_1]_{2(n_1-n_2)+3(k_1-k_2)(\frac{k_1 \bmod 2}{2},\frac{k_2 \bmod 2}{2})}. \tag{6.10}$$

The way the resulting superfield sits in the short multiplet can be determined without doing explicit branchings. If one denotes the sites in the short multiplet where representations sit by two-dimensional coordinates (k_1,k_2) with the origin being in the top corner and the first coordinate going down left, and the second down right, then the bottom component of the superfield $(Z)^{n_1}(\bar{Z})^{n_2}(W)^{k_1}(\overline{W})^{k_2}$ sits precisely at the site with coordinates (k_1,k_2),

Figure 6.3: Complete decomposition of the $\mathcal{N} = 4$ order-2 short multiplet under the $\mathcal{N} = 1$ subalgebra up to complex conjugation. The diagrams for conjugated superfields can be found by mirroring with respect to a vertical axis.



and has a rectangular shape. The length of the side of this rectangle that corresponds to the first coordinate is equal to the number of chiral superfields Z_i and W_α that appear in the product, the length of the other side is equal to the number of anti-chiral superfields $\bar{Z}^i$ and $\overline{W}_{\dot{\alpha}}$. However the length of either side can at most be 2, as we cannot go beyond θ^2.

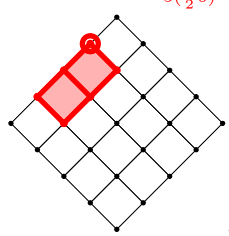
Let us take $WZ^2\bar{Z}$ as an example. It is a product of 4 basic superfields, and therefore sits in the $p = 4$ short multiplet. Because W is an SU(3) singlet, the SU(3) representation is determined by Z and $\bar{Z}$, and is equal to $[1,2] = \yng(3,2) = \overline{\mathbf{15}}$. The superfields Z and $\bar{Z}$ contribute a $\mathrm{U}(1)_\mathrm{R}$ charge equal to ± 2 and W and $\overline{W}$ a charge equal to ± 3, thus the total U(1) charge equals to 5. The superfields Z and $\bar{Z}$ are Lorentz scalars, and W_α is a left-handed Weyl fermion, thus the Lorentz quantum numbers are $(\frac{1}{2}, 0)$. Since there is one superfield W and no $\overline{W}$, the top corner that corresponds to the lowest component is located at $(1, 0)$. The total number of chiral superfields is 3, the number of anti-chiral superfields is 1, and therefore the dimensions of the rectangular region are $(2, 1)$. The result of this construction can be found in Figure 6.4.

The representations of the component fields that appear read

$$
\begin{array}{llllll}
1: & \overline{\mathbf{15}}_{5(\frac{1}{2},0)} & \theta: & \overline{\mathbf{15}}_{3(1,0)}, \overline{\mathbf{15}}_{3(0,0)} & \theta^2: & \overline{\mathbf{15}}_{-1(\frac{1}{2},0)} \\
\bar{\theta}: & \overline{\mathbf{15}}_{8(\frac{1}{2},\frac{1}{2})} & \theta\bar{\theta}: & \overline{\mathbf{15}}_{5(0,\frac{1}{2})}, \overline{\mathbf{15}}_{5(1,\frac{1}{2})} & \theta^2\bar{\theta}: & \overline{\mathbf{15}}_{2(\frac{1}{2},\frac{1}{2})}
\end{array}
$$

The corresponding fields can be derived by simply extracting the correct components from $WZ^2\bar{Z}$, and one can check that these representations really appear by starting with the $\mathcal{N} = 4$ representations in Figure 3.2 and performing the decomposition under $\mathrm{SU}(3) \times \mathrm{U}(1) \subset \mathrm{SU}(4)$.

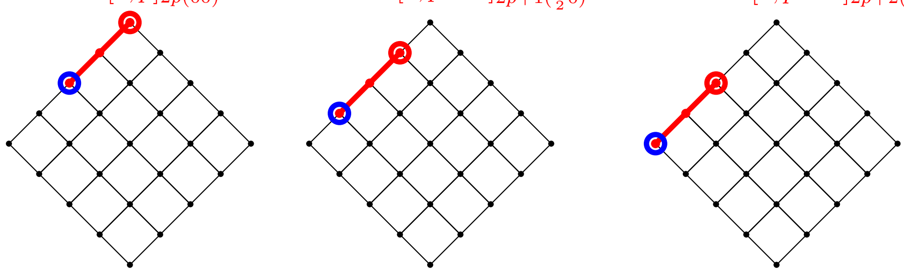
Figure 6.4: Fitting of the $\mathcal{N} = 1$ superfield $\left(W_\alpha Z_{(i} Z_{j)} \bar{Z}^k - \text{traces}\right)$ into the $\mathcal{N} = 4$ order $p = 4$ short multiplet.



6.1 Composite Chiral Superfields and F-Terms

With this complete decomposition of the $\mathcal{N} = 4$ chiral multiplets in terms of the $\mathcal{N} = 1$ subgroup it is now straightforward to extract operators which are $\mathcal{N} = 1$ F-terms. The only chiral superfields into which a short multiplet decomposes are shown in Figure 6.5. Due to our construction everything is explicit, the representations can be read off, and

Figure 6.5: Chiral $\mathcal{N} = 1$ multiplets in order-p $\mathcal{N} = 4$ short multiplet. The bottom components are encircled in red, the top components are encircled in blue. The corresponding anti-chiral superfields can be constructed by vertically mirroring the diagrams and conjugating the representations.

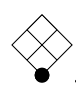


the operators can be extracted as corresponding superfield components. These are exactly the operators we set out to look for. The superfield Z^p is a generalization of the GPPZ deformation Z^2, and its F-term can be used as an $\mathcal{N} = 1$ deformation. The superfield $Z^{p-1}W_\alpha$ is chiral, but also has a free spinor index which makes it less useful for constructions of supersymmetric deformations. Lastly, the chiral superfield $Z^{p-2}W^\alpha W_\alpha$ is a Lorentz scalar, but it also contains gauge kinetic terms, so that after the addition of its top component to the action a deformation of these kinetic terms will be produced which depends on the scalars z_i.

It is interesting to note which supergravity modes the top components of these chiral superfields correspond to. This can be done in a straightforward way by looking up the corresponding representations in Table 3.1 in Section 3.3. As pointed out previously, each of the three types of chiral top components in Figure 6.5 corresponds to a Kaluza-Klein tower in the bulk, so that there are only three Kaluza-Klein towers with modes corresponding to top components of chiral superfields on the field theory side. The chiral top components in the lowest possible short multiplet $p = 2$ correspond to three modes in the bulk graviton multiplet, and top components in higher short multiplets with $p > 2$ build Kaluza-Klein towers on top of these modes. More concretely, we see from Figure 3.2 that the F-term in Z^p originates from the SU(4) representation $[0, p-2, 2]_{(00)}$ and has conformal dimension $\Delta = p + 1$. From Table 3.1 we read off that this operator corresponds to the bulk mode $A_{\alpha\beta}$ with mass $m^2 = \Delta(\Delta - 4)$. The next case, the top component of $Z^{p-1}W_\alpha$, sits within the representation $[0, p-2, 1]_{(\frac{1}{2}0)}$ and has conformal dimension $\Delta = p + \frac{3}{2}$ so that according to Table 3.1 it corresponds to the supergravity mode λ^{I_L} with mass $m^2 = 2 - \Delta$. Finally the top component of $Z^{p-2}W^2$ comes from the representation $[0, p-2, 0]_{(00)}$, has $\Delta = p + 2$, and corresponds to the supergravity mode B with mass $m^2 = \Delta(\Delta - 4)$.

6.2 Composite Real Superfields and D-Terms

As it is known, another way of generating supersymmetric actions is by taking top components of real superfields, which for scalar superfields are known as the D-terms. Let us therefore study which $\mathcal{N} = 4$ chiral operators can be written as top components of $\mathcal{N} = 1$ real superfields. Given our construction of the $\mathcal{N} = 1$ decomposition of the $\mathcal{N} = 4$ short multiplet one can immediately make one observation: since the top component is the one proportional to $\theta^2\bar{\theta}^2$ we need a product of at least two basic chiral and two basic anti-chiral superfields in $\{Z_i, \bar{Z}^i, W_\alpha, \overline{W}_{\dot{\alpha}}\}$. In the language of the pictorial diagrams that we drew this is reflected by the fact that the $\mathcal{N} = 1$ multiplet with a $\theta^2\bar{\theta}^2$ term has the following shape

 .

Thus the simplest $\mathcal{N} = 4$ chiral operators which correspond to a $\theta^2\bar{\theta}^2$ component of a $\mathcal{N} = 1$ superfield can be found in the order $p = 4$ short multiplet, while operators corresponding θ^2 components of chiral superfields can be found in all short multiplets, as we saw in the previous section.

A further observation is that there are many more operators which can be found in top components of real superfields than in top components of chiral superfields. Recall that we showed in the previous section that up to complex conjugation any order-p short multiplet has exactly three chiral composite superfields Z^p, $W_\alpha Z^{p-1}$ and W^2Z^{p-2}, which lead to three different types of θ^2 components. To obtain a $\theta^2\bar{\theta}^2$ component one does not require a chiral superfield, but rather some generic superfield plus its complex conjugate. This leads to a number of possible $\theta^2\bar{\theta}^2$ components that is increasing with the increasing order of the multiplet. For example, as we argued above, the $p = 2$ and $p = 3$ multiplets do not contain such operators, the $p=4$ multiplet contains six, namely $Z^2\bar{Z}^2$, $WZ\bar{Z}^2+\text{c.c.}$, $W^2\bar{Z}^2$, $W\overline{W}Z\bar{Z}$, $W^2\overline{W}\bar{Z}+\text{c.c.}$, $W^2\overline{W}^2$, the $p=5$ multiplet has 9 and so forth. They can all be easily constructed by starting with at least two chiral and two anti-chiral superfields and adding further superfields until one reaches the required order. The cases for $p=4$ and $p=5$ can be looked up in Appendix F.

While each of these operators may or may not be of relevance, we would like to point out one class of $\theta^2\bar{\theta}^2$ operators, that in some sense can be called twins of the F-term and other θ^2 operators we found in the previous section. Consider the chiral superfield Z^p. One way of including its θ^2 term in the Lagrangian is to add a term of the form

$$\int \mathrm{d}^2\theta\ \mathrm{d}^2\bar{\theta}\,(\bar{\theta}^2 Z^p). \tag{6.11}$$

Now we can use the fact that the Grassmann variable $\bar{\theta}_{\dot{\alpha}}$ has the same quantum numbers as the field strength superfield $\overline{W}_{\dot{\alpha}}$ to extend this to

$$\int \mathrm{d}^2\theta\ \mathrm{d}^2\bar{\theta}\,(\bar{\theta}^2+\overline{W}^2)Z^p+\text{c.c.} \tag{6.12}$$

Thus we have added the top component of the superfield $\overline{W}^2Z^p$. Because the SU(3) representation only depends on the Z^p piece, the top components in both $\overline{W}^2Z^p$ and Z^p transform in the same representation. In fact, as we will see in Section 6.4 the top components in both Z^p and in $\overline{W}^2Z^p$ holographically correspond to Kaluza-Klein modes of the same bulk field.

The remaining two types of θ^2 components discussed in the previous section also have their $\theta^2\bar{\theta}^2$ twins, which can be constructed in exactly the same way as follows

$$\int \mathrm{d}^2\theta\ \mathrm{d}^2\bar{\theta}\,(\bar{\theta}^2+\overline{W}^2)W_\alpha Z^p+\text{c.c.} \tag{6.13}$$

and

$$\int \mathrm{d}^2\theta\ \mathrm{d}^2\bar{\theta}\,(\bar{\theta}^2+\overline{W}^2)W^2 Z^p+\text{c.c.} \tag{6.14}$$

and all conclusions made about the Z^p-type top components also apply to these two types.

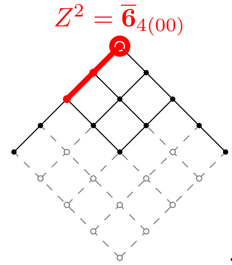
6.3 The $\mathcal{N}=1^*$ Flow and Additional Bulk Modes

After the general arguments of the previous sections now is a good time to perform a sanity check and see how the GPPZ flow fits into our general scheme. In addition we will learn how the elimination of auxiliary fields leads to additional terms which need to be analysed in detail. The GPPZ deformation breaks $\mathcal{N}=4$ to $\mathcal{N}=1$, and can be written as a deformation of the superpotential:

$$\mathcal{W} \to \mathcal{W} + \delta\mathcal{W}, \qquad \delta\mathcal{W} = m^{ij} Z_i Z_j \,. \tag{6.15}$$

In order to get a deformation by a chiral operator the coupling m^{ij} has to be taken symmetric in the SU(3) indices, and is therefore in the $\mathbf{6}$, the chiral superfield $Z_{(i}Z_{j)}$ has the quantum numbers $\bar{\mathbf{6}}_{4(00)}$. From our previous analysis it is obvious that this deformation will introduce operators from the order-2 short multiplet. The components of $Z_{(i}Z_{j)}$ sit in this multiplet as in Figure 6.6. Given that θ_α is in the $\mathbf{1}_3$ we expect the F-term of this deformation to

Figure 6.6: The $\mathcal{N} = 1$ composite superfield that was used in the GPPZ deformation is Z^2, and its components can found in the order $p = 2$ short multiplet of the $\mathcal{N} = 4$ theory.



contribute operators in the representation $\bar{\mathbf{6}}_{-2}$ and their complex conjugates. Let us see how this is realised. The deformation of the superpotential leads to the following new terms that appear in the action:

$$S \to S + \mathrm{tr} \int \mathrm{d}^4x \left(\lambda_i \lambda_j \frac{\partial^2(\delta\mathcal{W})}{\partial Z_i \partial Z_j} + \mathcal{F}_i \delta\bar{\mathcal{F}}^i + \text{c.c.} \right) + |\delta\mathcal{F}_i|^2, \tag{6.16}$$

where the F-terms are defined as usual:

$$\bar{\mathcal{F}}^i + \delta\bar{\mathcal{F}}^i = \frac{\partial\mathcal{W}}{\partial Z_i} + \frac{\partial(\delta\mathcal{W})}{\partial Z_i}\Big|_{Z_i \to z_i} . \tag{6.17}$$

The $\bar{\mathbf{6}}_{-2}$ that we were looking to obtain is the operator in parenthesis in (6.16) and is linear in $\delta\mathcal{W}$ and $\delta\mathcal{F}_i$. With $\delta\mathcal{F}_i$ integrated out it becomes of the form $m(\lambda^2 + z^3)$. This is the chiral operator of our interest, and the corresponding coupling m is interpreted as a supergravity mode living in AdS_5. However, supersymmetry forces us to include an additional term, $|\delta\mathcal{F}_i|^2$, which after integrating out the auxiliary field reads

$$\mathrm{tr} \int \mathrm{d}^4x \, (m^{ik} \bar{m}_{jk} z_i \bar{z}^j) \,. \tag{6.18}$$

The representation of the scalars is $z_i = \bar{\mathbf{3}}_2$, and therefore for generic couplings m^{ij} the representations that appear in the product are $\mathbf{3}_2 \times \bar{\mathbf{3}}_{-2} = \mathbf{8}_0 + \mathbf{1}_0$. This is just the traceless part of the tensor product, and the trace. Given that the $\mathcal{N} = 1$ decomposition of short multiplets consists of symmetric and traceless combinations of superfields Z_i and $\bar{Z}^i$, we see that the traceless part, the **8**, is indeed a chiral operator, and we therefore should include another mode in the gravity description. More precisely it is the bottom component of the superfield $(Z_i\bar{Z}^j - \text{trace})$ in the order-2 short multiplet, and corresponds to the bulk mode $h^\alpha_\alpha - a_{\alpha\beta\gamma\delta}$. For the same reasons the trace part is not a chiral operator, it is in fact the Konishi operator, and is known to correspond to a stringy mode. It is possible to project out the traceless part by a specific choice of m^{ij}, and therefore eliminate the additional gravity mode. To do so we need to choose m^{ij} such that the traceless part of $m^{ik}\bar{m}_{jk}$ vanishes, in other words that its trace is equal to the coupling itself:

$$m^{ik}\bar{m}_{jk} \stackrel{!}{=} \frac{1}{3}\delta^i_j m^{lk}\bar{m}_{lk}. \tag{6.19}$$

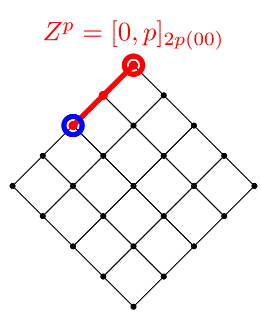
It is easy to see that a diagonal mass term $m^{ij} = m\delta^{ij}$ does indeed have this property, which is exactly the choice made by GPPZ. In the end what remains is a single chiral operator, $m(\lambda^2 + z^3)$, the relevant coupling m triggers an RG-flow, which can be described by a gravity mode.

One may wonder if this procedure can be generalised. We can start by replacing Z^2 by Z^p as deformation of superpotential:

$$\delta\mathcal{W} = m^{i_1\ldots i_p} Z_{i_1}\ldots Z_{i_p}, \tag{6.20}$$

with a coupling $m^{i_1\ldots i_p}$ completely symmetric in its indices. From the group theory perspective the situation is very similar, and the part of the $\mathcal{N}=4$ multiplet that is involved is exactly the same, as one can see in Figure 6.7. As above, the F-term operator is marked by the blue

Figure 6.7: Generalization of the GPPZ deformation from Z^2 to Z^p. In terms of the $\mathcal{N}=4$ short multiplets of order p the components of Z^p can be found at the same sites.



circle and will appear as the operator

$$\mathcal{O} = \lambda_{i_1}\lambda_{i_2}\frac{\partial^2(\delta\mathcal{W})}{\partial Z_{i_1}\partial Z_{i_2}} + \mathcal{F}_i\delta\bar{\mathcal{F}}^i \sim (\lambda\lambda + z\bar{z}\bar{z})z^{p-2} \tag{6.21}$$

and corresponds to a Kaluza-Klein mode in the same tower on top of $A_{\alpha\beta}$. Again, the additional term $|\delta\mathcal{F}_i|^2$ will make an appearance and is given by

$$\mathrm{tr}\int \mathrm{d}^4x\,|\delta\mathcal{F}_i|^2 = \mathrm{tr}\int \mathrm{d}^4x\,(m^{ki_2\dots i_p}\bar{m}_{kj_2\dots j_p} z_{i_2}\dots z_{i_p}\bar{z}^{j_2}\dots\bar{z}^{j_p}). \tag{6.22}$$

As before we have to check whether or not this operator contains any chiral parts. Group-theoretically this corresponds to the decomposition of the tensor product

$$\mathbf{3}^{p-1}\times\bar{\mathbf{3}}^{p-1} = [p-1,0]\times[0,p-1], \tag{6.23}$$

but we know that chiral operators built of scalar fields have to form symmetric and traceless combinations. Therefore an operator of this kind will be the only chiral contribution from the tensor product decomposition and is equal to the SU(3) representation $[p-1,p-1]$. To project out this operator we need to choose $m^{i_1\dots i_p}$ such that the symmetric traceless part of $m^{ki_2\dots i_p}\bar{m}_{kj_2\dots j_p}$ corresponding to the representation $[p-1,p-1]$ vanishes.

6.4 $\mathcal{N}=1$ Preserving Deformations and the $G_{(3)}$ Flux

The $\mathcal{N}=1$ preserving deformations studied so far have one thing in common: they all come from F-terms of superfields of the form Z^p, which we used as a short-hand notation for the symmetrised products $Z_{(i_1}\dots Z_{i_p)}$ of chiral SU(3) triplets. We showed that these F-terms correspond to operators in the $\mathcal{N}=4$ short multiplets which are of the same type, namely those with the quantum numbers $[0,p-2,2]_{(0,0)}$ and their complex conjugates. In holography this has as consequence that these operators correspond to Kaluza-Klein modes in the same Kaluza-Klein tower, in this particular case the tower originating from the harmonic expansion of the anti-symmetric potential $A_{\alpha\beta}$ with indices pointing in the internal directions in the bulk, where we have used the notation of [42]. The corresponding field strength is the $G_{(3)}$ form field, and therefore the deformations we are considering correspond to $G_{(3)}$ flux in the bulk. This type of deformations has previously been discussed in literature, including GPPZ [23], Distler-Zamora [20], Leigh-Strassler [87], Polchinski-Strassler [48], Polchinski-Graña [43] and others.

We can write this type of deformations in a general way. Consider the series expansion of a holomorphic function $f(Z_i)$ around the origin:

$$f(Z_i) = f(0) + Z_i\partial^i f(0) + \frac{1}{2!}Z_iZ_j\partial^i\partial^j f(0) + \frac{1}{3!}Z_iZ_jZ_k\partial^i\partial^j\partial^k f(0) + \dots \tag{6.24}$$

We can rename f evaluated at the origin and its derivatives to m in order to bring the expansion to the following form:

$$f(Z_i) = m + m^iZ_i + \frac{1}{2(2-1)}m^{ij}Z_iZ_j + \frac{1}{3(3-1)}m^{ijk}Z_iZ_jZ_k + \dots \tag{6.25}$$

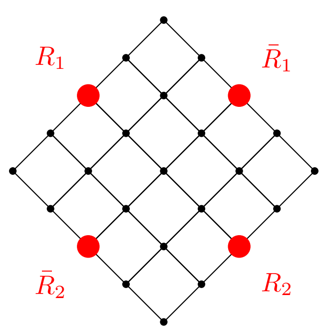
which gives $\mathfrak{su}(3)$ tensors $m^{i_1 i_2 \cdots}$ which are symmetrised in their indices. This allows us to write the most general superpotential deformation by short multiplets as follows

$$\begin{aligned}\delta\mathcal{W} &= Z_i Z_j \partial^i \partial^j f(Z) \\ &= m^{ij} Z_i Z_j + m^{ijk} Z_i Z_j Z_k + m^{ijkl} Z_i Z_j Z_k Z_l + \ldots\end{aligned} \tag{6.26}$$

We took the second derivative of f because the constant leading term in its expansion only contributes a constant shift when inserted into the action, and the linear terms will vanish after taking the colour-trace. This is reflected in the fact that the short multiplets start at $p=2$, and are therefore quadratic in $\mathcal{N}=1$ superfields. Thus one class of $\mathcal{N}=1$ preserving deformation by chiral operators using $\mathcal{N}=1$ F-terms and leading to $G_{(3)}$ flux in the bulk can be parametrised by the second derivative of a holomorphic function $f(Z_i)$.

At this point it is worth remembering that upon harmonic expansion on the sphere many of the ten-dimensional bulk fields give rise to two Kaluza-Klein towers. In terms of the SO(6) quantum numbers these two towers are identical up to a few low-lying modes that can only be found in one of the towers [42]. However, the masses of these modes are different, and in the notation of Section 3.3 are related by $m(p)^2 \to m(-p)^2$, see said Section for details. The anti-symmetric potential $A_{\alpha\beta}$ which we found to be dual to the F-terms of the type Z^p also has this property, and gives rise to two Kaluza-Klein towers. Therefore a natural question to ask is what kind of operators corresponds to this second tower and whether there are operators among them that can be used as $\mathcal{N}=1$ preserving deformations. First let us locate these operators in the $\mathcal{N}=4$ short multiplet in Figure 3.2. After comparing with the bulk modes in Table 3.1 we see that the Kaluza-Klein towers of $A_{\alpha\beta}$ correspond to operators sitting at the locations depicted in Figure 6.8. The operators in representations

Figure 6.8: The bulk field $A_{\alpha\beta}$ gives rise to two Kaluza-Klein towers. The modes on these two towers are dual to $\mathcal{N}=4$ operators in representations R_1 and R_2 and their conjugates, which can be found in short multiplets of any order p. Exceptions are the multiplets $p=2$ and $p=3$, which are ultra short and therefore do not contain R_2.

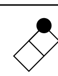
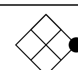
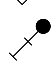
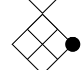
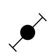
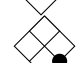
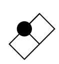
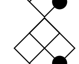
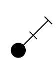
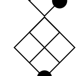
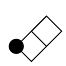
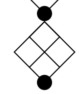


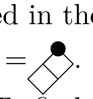
$R_1 = [0, p-2, 2]_{(0,0)}$ and their complex conjugates $\bar{R}_1$ are the ones discussed above and are associated with one of the two Kaluza-Klein towers. The other tower is covered by the operators in representations $R_2 = [0, p-4, 2]_{(0,0)}$ and its conjugate $\bar{R}_2$. We see that as expected the representations are really the same, with the only difference being that the order of the short multiplet in which R_1 and R_2 sit differ by 2, and therefore for $p \in \{2,3\}$ only R_1 is present, but not R_2. The corresponding dual bulk modes are the low-lying modes that can only be found in one of the towers. Because the representations of R_1 and R_2 are the same, they decompose in the same way under the $\mathcal{N}=1$ subgroup. In particular, the $\mathfrak{su}(3) \subset \mathfrak{su}(4)_{\mathrm{R}}$ decomposition is identical, and since W_α and $\overline{W}_{\dot\alpha}$ are $\mathfrak{su}(3)$ singlets, R_2 will be covered by $\mathcal{N}=1$ superfields that are composed of the same number of Z_i and $\bar{Z}^i$, but since the order of R_2 differs from R_1 by 2, we need to add two powers of W_α and $\overline{W}_{\dot\alpha}$. It is easy to see that only $\overline{W}^2$ works out, and so the fields in R_1 which sit at the θ^{t_1} component in the superfield $Z^{k_1}\bar{Z}^{k_2}$ will sit at the $\theta^{t_1}\bar\theta^2$ component of $\overline{W}^2 Z^{k_1}\bar{Z}^{k_2}$ in R_2. The superfield $\overline{W}_{\dot\alpha}$ and $\bar\theta_{\dot\alpha}$ have the same quantum numbers, and therefore compensate each other. To illustrate what we just said consider the case $p=3$ for R_1 and $p=3+2=5$ for R_2. For both cases the $A_{\alpha\beta}$ harmonics are in the $\overline{\mathbf{45}}$, which under $\mathfrak{su}(3) \subset \mathfrak{su}(4)_{\mathrm{R}}$ decomposes as

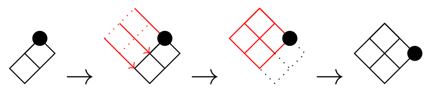
$$\overline{\mathbf{45}} \to \mathbf{3}_4 + \bar{\mathbf{3}}_8 + \bar{\mathbf{6}}_4 + \mathbf{8}_0 + \overline{\mathbf{10}}_0 + \overline{\mathbf{15}}_{-4}. \tag{6.27}$$

The operators in R_1 and R_2 can then be found in the $\mathcal{N}=1$ superfields listed in Table 6.1 below.

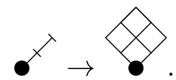
Table 6.1: The bulk field $A_{\alpha\beta}$ gives rise to two Kaluza-Klein towers, both of which contain a mode in the $\overline{\mathbf{45}}$ of the $\mathfrak{so}(6)$. The gauge theory duals of these modes are two operators in the $\overline{\mathbf{45}}$ of the $\mathfrak{su}(4)_{\mathrm{R}}$, which are in the $p=3$ and $p=5$ short multiplets. The table shows the $\mathcal{N}=1$ decomposition of these operators and the components of the $\mathcal{N}=1$ superfields in which they can be found.

R_1 and R_2	Superfield Component in R_1		Superfield Component in R_1	
$\mathbf{3}_4$	$W^2\bar{Z}$		$\overline{W}^2W^2\bar{Z}$	
$\bar{\mathbf{3}}_8$	W^2Z		$\overline{W}^2W^2Z$	
$\bar{\mathbf{6}}_4$	WZ^2		$\overline{W}^2WZ^2$	
$\mathbf{8}_0$	$WZ\bar{Z}$		$\overline{W}^2WZ\bar{Z}$	
$\overline{\mathbf{10}}_0$	Z^3		$\overline{W}^2Z^3$	
$\overline{\mathbf{15}}_{-4}$	$Z^2\bar{Z}$		$\overline{W}^2Z^2\bar{Z}$	

One can see that the effect of adding $\overline{W}^2$ is to push the representations of the $\mathcal{N}=1$ multiplet contained in the components of the superfields by 2 in the $\bar\theta$ direction. Consider, for example $W^2\bar{Z} =$ . The marked operator is the bottom component of this superfield and is in the $\mathbf{3}_4$. To find where the operators with the same quantum numbers are located in $\overline{W}^2W^2\bar{Z}$ proceed as in the following cartoon:

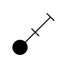


At this point we can make several observations. The first is that top components of the superfields Z^p are always mapped to top components of $\overline{W}^2 Z^p$. These are exactly the $\theta^2\bar{\theta}^2$ operators that we called twins of θ^2 operators in Section 6.2. This follows directly from the procedure just discussed, and pictorially the mapping is given by

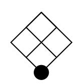


The precise operators that appear are $\lambda_{(i}\lambda_{j)}z^{p-2}$ in the θ^2 component and $(F^-)^2\lambda_{(i}\lambda_{j)}z^{p-2}$ in the $\theta^2\bar{\theta}^2$ component, plus additional terms enforced by supersymmetry which involve auxiliary fields and derivatives.

A further observation is that while for a given order p there is always only one possible θ^2 component corresponding to the mode $A_{\alpha\beta}$, namely

$$Z^p|_{\theta^2} =$$  $$\sim \lambda_{(i}\lambda_{j)}z^{p-2}, \tag{6.28}$$

which sits in the representation R_1, there are generally multiple different possible $\theta^2\bar{\theta}^2$ components contained in R_2. We have seen that there is at least one, with the same quantum numbers as $Z^p|_{\theta^2}$ and is given by

$$\overline{W}^2 Z^p\Big|_{\theta^2\bar{\theta}^2} =$$  $$, \tag{6.29}$$

but there are other superfields of the same order which correspond to the same box diagram as in equation (6.29), and are therefore also $\theta^2\bar{\theta}^2$ components. Those are superfields of the form $\overline{W}^2 Z^{k_1}\bar{Z}^{k_2}$ with $k_1+k_2=p$, and $k_1\geq 2$, and it is easy to see that there are $p-1$ such superfields.

As with the deformations of θ^2-type in R_1 we would now like to write down the most general $\theta^2\bar{\theta}^2$-type deformation in R_2. Consider the series expansion of an analytic function $g(Z_i,\bar{Z}^i)$:

$$g(Z_i,\bar{Z}^i) = g(0,0) + Z_i\partial^i g(0,0) + \bar{Z}^i\partial_i g(0,0) + Z_i\bar{Z}^j\partial^i\partial_j g(0,0) + \frac{1}{2!}Z_iZ_j\partial^i\partial^j g(0,0) + \ldots \tag{6.30}$$

As before we can rename g evaluated at the origin and its derivatives to $\tilde{m}$ in order to obtain a nicer representation:

$$g(Z_i,\bar{Z}^i) = \tilde{m} + \tilde{m}^i Z_i + \tilde{m}_i\bar{Z}^i + \tilde{m}^i_j Z_i\bar{Z}^j + \frac{1}{2(2-1)}\tilde{m}^{ij}Z_iZ_j + \ldots \tag{6.31}$$

and the components $\tilde{m}^{i\cdots}_{j\cdots}$ are automatically symmetrised in their upper and lower indices because partial derivatives commute. To represent $\overline{W}^2 Z^{k_1}\bar{Z}^{k_2}$ with $k_1\geq 2$ we again have to take the second derivative with respect to Z_i, and therefore the deformation reads

$$\delta S = \int \mathrm{d}^2\theta\,\mathrm{d}^2\bar\theta\,\left(\overline{W}^2 Z_i Z_j \partial^i\partial^j g(Z_i,\bar Z^j) + \text{c.c.}\right) \tag{6.32}$$

This is almost correct, except for the fact that in order to give irreducible representations of the $\mathfrak{su}(3)$ algebra, additionally to being symmetric, the products $Z^{k_1}\bar Z^{k_2}$ also have to be traceless. Equivalently, we should demand that the component tensors $\tilde m$ be traceless: $\tilde m^{i\cdots}_{i\cdots} = 0$. This property is enforced on $g(Z_i,\bar Z^i)$ by simply demanding that

$$\partial^i\partial_i g \equiv \partial^2 g = 0. \tag{6.33}$$

In other words, g should be a harmonic function. These results are in accord, and provide further insights into the findings of Polchinski and Graña [43]. In their paper the authors analysed perturbations to the $G_{(3)}$ flux in the bulk which are compatible with the boundary $\mathcal{N}=1$ supersymmetry. They found that the flux receives two contributions. The first one is expressed in terms of a holomorphic function $\phi(z)$, and matches with the quantum numbers of operators of the form $\lambda^i\lambda^j\partial_i\partial_j\phi$. This is exactly the superpotential deformation by operators in R_1 we found, and the holomorphic function ϕ is exactly what we called holomorphic function f. Furthermore the authors noted that after integrating a second order differential equation for the $G_{(3)}$ flux a second contribution appears as an integration "constant". This contribution appeared to be a harmonic function $\psi(z,\bar z)$, and the operators parametrised by it had the same quantum numbers as the ones parametrised by the holomorphic function ϕ, and a conformal dimension Δ' greater than the conformal dimension Δ of the first contribution by 4. Our results show that ψ, which corresponds to operators of the form $F^2\bar\lambda^i\bar\lambda^j\partial_{\bar\imath}\partial_{\bar\jmath}\psi$, is exactly the harmonic function g that we found above. The field theoretical explanation as to why ψ has to be harmonic rather than holomorphic is that the short multiplets are built on top of superconformal primaries which are symmetric and traceless products of the $\mathcal{N}=4$ scalars ϕ^I:

$$\mathcal{O}_p = \operatorname{tr}\phi^{\{I_1}\ldots\phi^{I_p\}}. \tag{6.34}$$

As one decomposes these multiplets under the $\mathcal{N}=1$ subgroup and re-expresses all operators in terms of $\mathcal{N}=1$ superfields the property of being symmetric and traceless descends down to $\mathfrak{su}(3)$ indices of the superfields Z_i and $\bar Z^i$. Therefore any deformation involving both Z_i and $\bar Z^i$ will be traceless, or equivalently vanishing under the action of ∂^2, and therefore a harmonic function of Z_i and $\bar Z^i$.

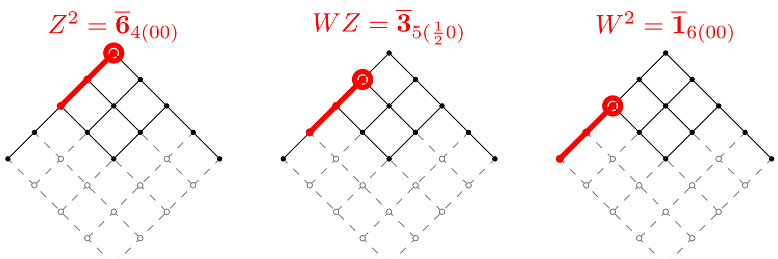
6.5 Condensates

A further feature of supersymmetric theories that is worth exploring is the condensation of gaugino bilinears. As pointed out in Chapter 5 true deformations of the gauge theory are related to non-normalisable asymptotics of bulk modes, while normalisable modes correspond to vacuum expectation values of gauge-invariant operators. In the previous sections we explored operator deformations of the $\mathcal{N}=4$ super Yang-Mills theory that break the supersymmetry to $\mathcal{N}=1$. Let us now look for potential operators that might acquire expectation values and fit them into our scheme.

Recall that in the case of GPPZ [23] it was found that after the deformation of the $\mathcal{N}=4$ theory by mass terms for $\mathcal{N}=1$ chirals the gaugino bilinear does indeed acquire an expectation value. The authors then proceeded to show how both the mass term and the gaugino condensate can be simultaneously described as a particular supergravity solution. Let us briefly review which operators are involved.

The authors considered the $\mathcal{N}=4$ descendant $Q^2_{\mathcal{N}=4}\operatorname{tr}\phi^{\{I}\phi^{J\}}$ of the order $p=2$ chiral primary $\operatorname{tr}\phi^{\{I}\phi^{J\}}$. It is in the $\overline{\mathbf{10}}$ of the $\mathrm{SU}(4)_\mathrm{R}$, and under $\mathcal{N}=1\subset\mathcal{N}=4$ decomposes as $\overline{\mathbf{10}}\to\bar{\mathbf{6}}+\bar{\mathbf{3}}+\mathbf{1}$. In terms of $\mathcal{N}=1$ operators the $\bar{\mathbf{6}}$, the $\bar{\mathbf{3}}$, and the $\mathbf{1}$ correspond to the θ^2-component of Z^2, the θ-component of $W_\alpha Z$, and the bottom component of W^2 respectively. This is reflected in the multiplet diagrams in Figure 6.9. Note that it is only

Figure 6.9: The descendant $Q^2_{\mathcal{N}=4}\operatorname{tr}\phi^{\{I}\phi^{J\}}$ is in the $\overline{\mathbf{10}}$ of the $\mathrm{SU}(4)_\mathrm{R}$, and under $\mathcal{N}=1\subset\mathcal{N}=4$ decomposes as $\overline{\mathbf{10}}\to\bar{\mathbf{6}}+\bar{\mathbf{3}}+\mathbf{1}$. In terms of $\mathcal{N}=1$ operators the $\bar{\mathbf{6}}$, the $\bar{\mathbf{3}}$, and the $\mathbf{1}$ correspond to the θ^2-component of Z^2, the θ-component of $W_\alpha Z$, and the bottom component of W^2 respectively.



true for the order-2 multiplet that the second descendant of the chiral primary decomposes into $\mathcal{N}=1$ superfields that are all chiral. Starting with the order-3 multiplet the second descendant $Q^2_{\mathcal{N}=4}\operatorname{tr}\phi^{\{I}\phi^J\phi^{K\}}$ decomposes into chiral and non-chiral $\mathcal{N}=1$ superfields, we saw the explicit decomposition as an example in Section 6.4.

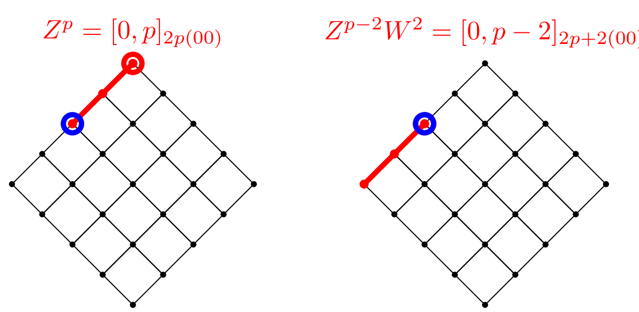
The conclusion of this analysis is that the mass terms correspond to the top component of Z^2, which is a $\bar{\mathbf{6}}$, and the gaugino bilinear which acquires an expectation value is the bottom component of the $\mathcal{N}=1$ operator W^2, which is the singlet $\mathbf{1}$. We can now try and apply our holographic beta-function formalism developed in Chapter 5 to the condensate. The conformal dimension of both m and σ is $\Delta = 3$, but the fact that m is a non-normalisable solution and goes as $e^{-(d-\Delta)r}$, and σ, in contrast, is normalisable and goes as $e^{-\Delta r}$ suggests that for σ we should take $\Delta \to d - \Delta = 1$. Changing conventions in the same way we did for m in Chapter 5 and using formulas (5.17) and (5.18) for the superpotential and the scalar potential we obtain the following result

$$\mathcal{W} = -\frac{3}{4}(\cosh(2\sigma) + 1) \tag{6.35}$$

$$V = -\frac{3}{8}\left[5 + 4\cosh(2\sigma) - \cosh^2(2\sigma)\right]. \tag{6.36}$$

This indeed reproduces correctly the results in (A.19) and (A.22) in GPPZ [23] if one sets $m = 0$. This result is very pleasant, but requires an explanation. The formulas that we just used to derive the superpotential and the scalar potential were derived in Chapter 5 from the point of view of the running coupling to which modes like m correspond to, and their description by the beta function and its relation (5.8) to the bulk superpotential eventually led to the result. The same description is not possible for condensates, as they do not appear as couplings in the action for which we could write a beta function. It seems that the way to proceed here is to study the energy dependence of the condensates, which is what the beta function describes for the couplings. For the gaugino condensate we know that it is proportional to Λ^3, where Λ is the UV scale. From this we can now compute $d\langle\sigma\rangle/d\log\Lambda = 3\Lambda^3$, which indeed does look like the classical beta function (5.13) for the coupling of an operator of dimension $\Delta = 1$. However there are still several questions to answer. First of all one needs to argue why the relation (5.8) between the beta function and the bulk superpotential is also true for the condensates. Next one may wonder whether or how this scheme generalises to higher chiral multiplets. We saw that in the case of F-terms it is rather straightforward to go from Z^2 to Z^p. Extending this to the condensates it is now tempting to go from the bottom component of W^2 to the bottom component of W^2Z^{p-2}. The superfield components in which we are looking for deformation operators and condensates are marked by blue circles in Figure 6.10. However, it is not obvious which of

Figure 6.10: The F-term operators are the top components of Z^p, the generalizations of the gaugino condensate $\langle\lambda\lambda\rangle$ to $\langle\lambda\lambda z^{p-2}\rangle$ are the bottom components of $Z^{p-2}W^2$. Both types of operators are highlighted by blue circles in the diagrams above.



the corresponding condensates $\langle\lambda\lambda z^{p-2}\rangle$ are non-zero and how to compute their values. It was shown Cachazo, Douglas, Seiberg, and Witten [88] that in $\mathcal{N}=1$ theories one can write down certain Ward-identities for chiral operators:

$$\langle \operatorname{tr} W^2 \frac{\partial f(Z,W)}{\partial Z}\rangle \sim \langle \operatorname{tr} \frac{\partial \mathcal{W}}{\partial Z} f(Z,W)\rangle, \tag{6.37}$$

where $f(Z,W)$ is a holomorphic function and $\mathcal{W}$ is the classical superpotential. We can now substitute $f(Z,W)=Z^{p-1}$ to obtain exactly the operators W^2Z^{p-2} that we need:

$$\langle \operatorname{tr} W^2 Z^{p-2}\rangle \sim \langle \operatorname{tr} Z \frac{\partial \mathcal{W}}{\partial Z} Z^{p-2}\rangle. \tag{6.38}$$

This relates the expectation values of operators W^2Z^{p-2} to expectation values of chiral operators made up of only the chiral superfields Z, and so to answer the question whether some operator W^2Z^{p-2} acquires an expectation value we may study operators on the right hand side of (6.38). The exploration of the answers to this and other related questions is part of an ongoing project.

Chapter 7

$\mathcal{N} = 2$ Deformations of the $\mathcal{N} = 4$ Theory

In this section the $\mathcal{N} = 4$ theory is written in the $\mathcal{N} = 2$ harmonic superspace language. We derive the kinematic equations and eliminate auxiliary fields to obtain the component form of the action. It is known that any potential term that is a function of the the $\mathcal{N} = 2$ hypermultiplet superfield leads to a non-trivial Kähler potential and a non-trivial metric for the resulting sigma-model. A mass-term that preserves the $\mathcal{N} = 2$ supersymmetry can be added by a Scherk-Schwarz construction. We analyse the operators that the mass deformation adds to the action by writing out the physical component fields. It is found that this mass deformation produces two scalar chiral operators, thus in order to holographically describe this deformation one needs to turn on two supergravity modes. Both modes are in the graviton multiplet.

7.1 Decomposition of the Fields

Recall that the field content of the $\mathcal{N} = 4$ super Yang-Mills theory consists of six real scalars $\phi^{AB} = \frac{1}{\sqrt{2}}\Sigma^{I\,AB}\phi^I$ in the $\mathbf{6}$ of the $\mathrm{SU}(4)_\mathrm{R}$, four fermions λ_A in the $\bar{\mathbf{4}}$, and a gauge field A_μ, which are all in the adjoint of the colour $\mathrm{SU}(N)$ gauge group. The description the $\mathcal{N} = 4$ theory in the $\mathcal{N} = 2$ language amounts to the branching of representations under the following subgroup of the R-symmetry:

$$\mathrm{SU}(2)_\mathrm{H} \times \mathrm{SU}(2)_\mathrm{R} \times \mathrm{U}(1)_\mathrm{X} \subset \mathrm{SU}(4)_\mathrm{R}\,. \tag{7.1}$$

The index 'H' stand for 'hypermultiplet' as the corresponding $\mathrm{SU}(2)_\mathrm{H}$ group will become the flavour symmetry for the hypermultiplets. The index 'R' is for R-symmetry, and 'X' has no meaning other than to distinguish the $\mathrm{U}(1)_\mathrm{X}$ form another $\mathrm{U}(1)$ that will appear later.

The $\mathrm{U}(1)_{\mathrm{X}}$ also acts as an R-symmetry, and is only allowed in scale-invariant theories [89]. The embedding of the product subgroup into the $\mathrm{SU}(4)_{\mathrm{R}}$ can be specified by providing the Cartan generators of the corresponding Lie algebra. Disregarding any normalisation of the generators we choose the following embedding:

$$H_{\mathrm{H}} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad H_{\mathrm{R}} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \quad H_{\mathrm{X}} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}. \tag{7.2}$$

Using this subalgebra embedding we can deduce the branching of the representations of the $\mathcal{N}=4$ fields. The U(1) charges will be specified with respect to the normalisation of the H_{X} generator as specified above, which we will not normalise to any conventions in order to have integer charges for the bookkeeping. The branching is as follows:

$$\mathbf{6} \to (\mathbf{2},\mathbf{2})_0 + (\mathbf{1},\mathbf{1})_2 + (\mathbf{1},\mathbf{1})_{-2} \tag{7.3}$$

$$\mathbf{4} \to (\mathbf{2},\mathbf{1})_1 + (\mathbf{1},\mathbf{2})_{-1} \tag{7.4}$$

$$\bar{\mathbf{4}} \to (\mathbf{2},\mathbf{1})_{-1} + (\mathbf{1},\mathbf{2})_1 \,. \tag{7.5}$$

To apply this branching to the fields we choose to denote the $\mathrm{SU}(2)_{\mathrm{H}}$ index by "$a, b, \ldots$" and the $\mathrm{SU}(2)_{\mathrm{R}}$ by "$i, j, \ldots$". With this choice the fundamental SU(4) index decomposes as $A \to (a, i)$, and the $\mathcal{N}=4$ fields split into the following $\mathcal{N}=2$ fields:

$$\phi^{AB} \to \begin{cases} \phi^{ab} \equiv \epsilon^{ab}\phi & = (\mathbf{1},\mathbf{1})_2 \\ \phi^{ai} \equiv f^{ai} & = (\mathbf{2},\mathbf{2})_0 \\ \phi^{ij} \equiv \epsilon^{ij}\bar{\phi} & = (\mathbf{1},\mathbf{1})_{-2} \end{cases} \quad \lambda_A \to \begin{cases} \psi_a = (\mathbf{2},\mathbf{1})_{-1} \\ \lambda_i = (\mathbf{1},\mathbf{2})_1 \end{cases} \quad \bar{\lambda}^A \to \begin{cases} \bar{\psi}^a = (\mathbf{2},\mathbf{1})_1 \\ \bar{\lambda}^i = (\mathbf{1},\mathbf{2})_{-1} \end{cases}. \tag{7.6}$$

Using the explicit components of ϕ^{AB} in equation (2.10) one can check that $\bar{\phi}$ is really the complex conjugate of ϕ, and that f^{ai} contains a pair of complex scalars and their conjugates, which makes a total of four real degrees of freedom.

A more careful analysis reveals that in terms of $\mathcal{N}=2$ supermultiplets these branched fields fall into a gauge supermultiplet and a hypermultiplet doublet [89, 90]. One way of representing these supermultiplet is by writing them into superfields. A more detailed off-shell superfield analysis will be preformed in the following sections, but at this point we can write out the physical components in a schematic way. The hypermultiplet doublet contains all fields that are doublets under $\mathrm{SU}(2)_{\mathrm{H}}$, and we can write it as an $\mathcal{N}=2$ superfield Q_a^i in which the physical components sit as follows:

$$Q_a^i = f_a^i + \theta^i \psi_a + \bar{\theta}^i \bar{\psi}_a = (\mathbf{2},\mathbf{2})_0. \tag{7.7}$$

The Grassmann variables θ used above are doublets under the $\mathrm{SU}(2)_\mathrm{R}$, and consistency of the superfield Q^i_a and its components require that the rest of the quantum numbers of the fermionic coordinates are as follows:

$$\theta^i = (\mathbf{1},\mathbf{2})_1 \qquad \bar{\theta}^i = (\mathbf{1},\mathbf{2})_{-1}\,. \tag{7.8}$$

We can see that θ^i have a non-zero $\mathrm{U}(1)_\mathrm{X}$ charge, which shows that $\mathrm{U}(1)_\mathrm{X}$ is really an R-symmetry. The remaining fields are singlets under the $\mathrm{SU}(2)_\mathrm{H}$ and make up the gauge multiplet which fits into a real vector superfield in the following way

$$V = \theta^i\sigma^\mu\bar{\theta}_i A_\mu + \theta^2\bar{\phi} + \bar{\theta}^2\phi + \bar{\theta}^2\theta^i\lambda_i + \theta^2\bar{\theta}^i\bar{\lambda}_i = (\mathbf{1},\mathbf{1})_0\,. \tag{7.9}$$

Here we have suppressed the $\mathrm{SU}(2)_\mathrm{R}$ doublet indices in θ^2 and $\bar{\theta}^2$ for simplicity. Analogously to $\mathcal{N}=1$ we can apply certain superderivatives to this superfield to get a chiral field strength [91], and its anti-chiral conjugate. It has the following form:

$$W = \phi + \theta^i\lambda_i + \theta^{i\alpha}\theta^\beta_i F^+_{\alpha\beta} = (\mathbf{1},\mathbf{1})_2 \tag{7.10a}$$

$$\overline{W} = \bar{\phi} + \bar{\theta}_i\bar{\lambda}^i + \bar{\theta}^{i\dot{\alpha}}\bar{\theta}^{\dot{\beta}}_i F^-_{\dot{\alpha}\dot{\beta}} = (\mathbf{1},\mathbf{1})_{-2}\,, \tag{7.10b}$$

where F^+ and F^- are the self-dual and anti-self-duel parts of the fields strength corresponding to A_μ. This structure can also be motivated by the fact that the field strength multiplet can be written in terms of $\mathcal{N}=1$ superfields as follows [90]:

$$W(x,\theta^i) = \Phi(x,\theta^1) + (\theta^2)^\alpha W_\alpha(x,\theta^1) + (\theta^2)^2 G(x,\theta^1)\,, \tag{7.11}$$

with G being auxiliary, and Φ and W_α the two $\mathcal{N}=1$ superfields into which the $\mathcal{N}=2$ gauge multiplet decomposes. Their physical components can be written as follows:

$$\Phi(x,\theta^1) = \phi + \theta^1\lambda \tag{7.12}$$

$$W_\alpha(x,\theta^1) = \tilde{\lambda}_\alpha + (\theta^1)^\beta F^+_{\alpha\beta}\,. \tag{7.13}$$

After inserting these component into (7.11) and omitting the auxiliary superfield G we recover exactly (7.10) upon identification $\lambda_i = (\lambda,\tilde{\lambda})$.

We can make a connection to the $\mathcal{N}=1$ representation of the $\mathcal{N}=4$. We obtained the $\mathcal{N}=1$ description of the fields by branching the representations of the fields under $\mathrm{SU}(3)\times\mathrm{U}(1)_\mathrm{Y}\subset\mathrm{SU}(4)$, where $\mathrm{U}(1)_\mathrm{Y}$ is the R-symmetry of the $\mathcal{N}=1$ theory, and we have used the subscript 'Y' instead of 'R' in order not to confuse it with the R-symmetry of the $\mathcal{N}=2$ description. To relate $\mathcal{N}=1$ and $\mathcal{N}=2$ fields one needs to branch the global SU(3) symmetry of the $\mathcal{N}=1$ theory further as follows:

$$\mathrm{SU}(2)_{\mathrm{H}} \times \mathrm{U}(1)_{\mathrm{N}} \times \mathrm{U}(1)_{\mathrm{Y}} \subset \mathrm{SU}(3) \times \mathrm{U}(1)_{\mathrm{Y}} \subset \mathrm{SU}(4)\,. \tag{7.14}$$

The Cartan generators of the $\mathcal{N}=1$ description are given by the following matrices:

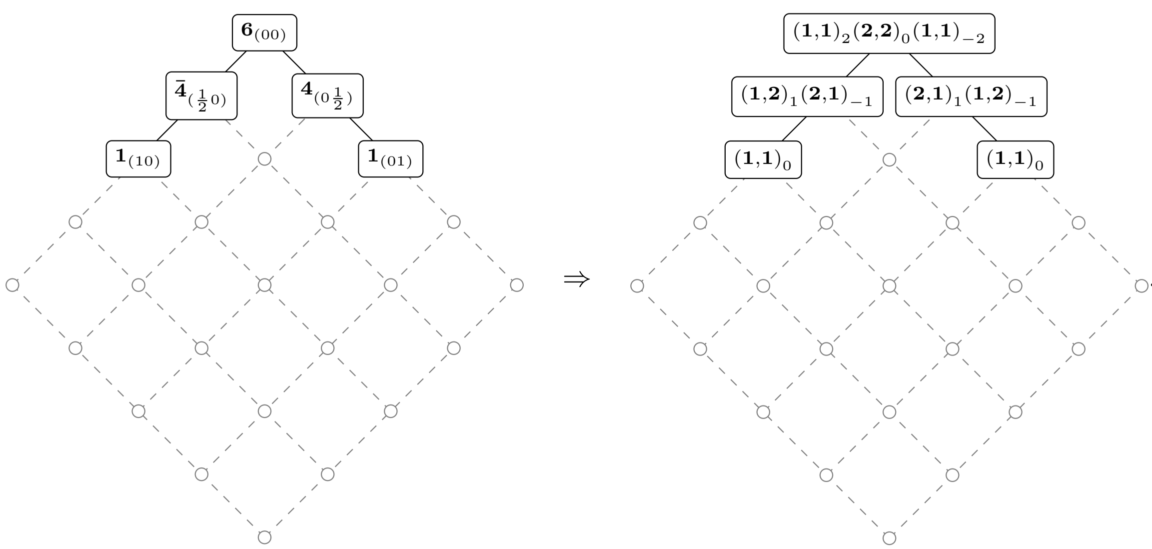
$$H_{\mathrm{N}} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \qquad H_{\mathrm{Y}} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -3 \end{pmatrix} \tag{7.15}$$

and we see that the $\mathrm{SU}(2)_{\mathrm{H}}$ factor is the same as in the $\mathcal{N}=2$ case. To match the U(1) charges one needs to relate the corresponding U(1) generators, which is:

$$\frac{2}{3}H_{\mathrm{N}} + \frac{1}{3}H_{\mathrm{Y}} = H_{\mathrm{X}} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \tag{7.16}$$

Thus the representations in the $\mathcal{N}=2$ language can be recovered up to the $\mathrm{SU}(2)_{\mathrm{R}}$ factor by branching the SU(3) global symmetry and summing the U(1) charges as above.

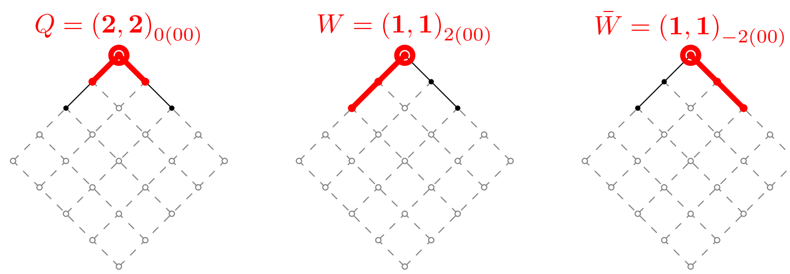
The decomposition we just did can also be understood from the perspective of short multiplets. As explained previously one usually defines short multiplets for orders $p \geq 2$. The case $p=1$ corresponds to the $\mathcal{N}=4$ fields themselves, and is usually not considered as a short multiplet because the fields by themselves are not gauge invariant. Nonetheless, from the group theory point of view there is nothing that stops one to consider the $p=1$ case and the fields can be arranged in a multiplet diagram very much like the $p \geq 2$ cases. Together with the branching to the $\mathcal{N}=2$ quantum numbers we obtain the following picture:



We saw the the **6**, the **4** and the $\bar{\mathbf{4}}$ correspond to the scalars and the fermions. The singlets are given by the self-dual and anti-self-dual curvature tensors of the connection A_μ by $\mathbf{1}_{(10)} = F^+ = F_{\mu\nu}\sigma^{\mu\nu}$ and $\mathbf{1}_{(01)} = F^- = F_{\mu\nu}\bar{\sigma}^{\mu\nu}$. The decomposition to the $\mathcal{N} = 2$ sub-symmetry performed earlier can by summarized as follows:

$$
\begin{aligned}
V &= \{\phi, \lambda^i, F^+\} &&= \{(\mathbf{1},\mathbf{1})_2, (\mathbf{1},\mathbf{2})_1, (\mathbf{1},\mathbf{1})_0\} \\
Q^a &= \{q^{ai}, \psi^a, \bar{\psi}^a\} &&= \{(\mathbf{2},\mathbf{2})_0, (\mathbf{2},\mathbf{1})_{-1}, (\mathbf{2},\mathbf{1})_1\}\,.
\end{aligned}
\tag{7.17}
$$

By comparing the content of the superfields with the representations in the multiplet diagram it is obvious how the $\mathcal{N} = 2$ superfields fit into the $\mathcal{N} = 4$ multiplet. This can be summarised by the following diagrams:



The construction goes very much like in the $\mathcal{N} = 1$ case and can be readily generalised to the short multiplets with $p \geq 2$. For any given p one needs to construct all possible products of the superfields Q, W, and $\overline{W}$ with exactly p terms. The representations of the composite superfield is determined from the quantum numbers of Q, W and $\overline{W}$ since W is a singlet with respect to both $\mathrm{SU}(2)_\mathrm{H}$ and $\mathrm{SU}(2)_\mathrm{R}$ all non-trivial SU(2) quantum numbers arise from the hypermultiplet $Q = (\mathbf{2},\mathbf{2})_0$. To obtain the representation of Q^n for some $n \geq 1$ remember that the $\mathcal{N} = 4$ primaries are given by symmetric products of the scalar fields, this property descends down to $\mathcal{N} = 2$ so that in the tensor product decomposition of Q^n only the completely symmetric representations must be kept. For SU(2) the dimension of any representation $[k]$ is given by $\mathbf{k+1}$, in particular the fundamental representation is $\square = [1] = \mathbf{2}$. The completely symmetric part of the product of n fundamentals, $[1]^n$ is given by $[n]$, from which it follows that that representation of Q^n is given by

$$
Q^n = (\mathbf{n+1}, \mathbf{n+1})_0. \tag{7.18}
$$

This completely determines the quantum numbers of any product of $\mathcal{N} = 2$ superfields. After obtaining the quantum numbers for the composite superfields the representation of the component fields can be determined using compatibility with $\theta^i = (\mathbf{1},\mathbf{2})_1$. Given the quantum numbers of the component fields the last step requires the matching of the representation to those obtained from the branching of the $\mathcal{N} = 4$ representations in the short multiplet. Unlike for the $\mathcal{N} = 1$ case there does not seem to be a straightforward rule that determines which components of the superfields enter the short multiplet. We performed the decomposition for the cases $p=2$ and $p=3$ manually by matching all components, and it turned out the superfield components can be assigned to the multiplet in a unique way just from their representations alone. The complete decomposition is presented in Appendix F.

7.2 $\mathcal{N}=4$ Theory in $\mathcal{N}=2$ Language

Our ultimate goal is to find deformations of the $\mathcal{N}=4$ super Yang-Mills theory that preserve an $\mathcal{N}=2$ supersymmetry subgroup. We cannot simply pick a deformation of our choice, for example an operator in $\mathcal{N}=2$ decomposition of a short multiplet and add it to the action, since a general deformation would break the supersymmetry completely. It is therefore necessary to find a way of adding deformations to the action in which the preservation of an $\mathcal{N}=2$ supersymmetry is ensured. Analogously to the $\mathcal{N}=1$ case there is a way to formulate action for $\mathcal{N}=2$ supersymmetric theories in terms of superfields [91]. The construction is more subtle and technical since an infinite number of auxiliary fields in form of S^2 harmonics have to be added. The $\mathcal{N}=4$ theory can be written in the $\mathcal{N}=2$ harmonic superspace language as follows:

$$\begin{aligned} S &= S_{\mathrm{SYM}} + S_{\mathrm{q}} \\ &= \frac{1}{4}\int \mathrm{d}^4x\,\mathrm{d}^4\bar{\theta}\;\mathrm{tr}(\overline{W}\overline{W}) + \frac{1}{2}\int \mathrm{d}^4x_A\,\mathrm{d}u\,\mathrm{d}^4\theta^+\;\mathrm{tr}(Q^+_a\mathcal{D}^{++}Q^{+a})\,. \end{aligned} \tag{7.19}$$

The du integral integrates over the coset $\mathrm{SU}(2)/\mathrm{U}(1)=S^2$ which is parametrised by the vielbeins $u_i^\pm$, and the coordinates $x_A^\mu = x^\mu - 2i\theta^{(i}\sigma^\mu\bar{\theta}^{j)}u_i^+u_j^-$ are used to parametrise the analytic superspace, which comprises half of the whole harmonic superspace. The $\mathrm{SU}(2)_{\mathrm{R}}$ doublet indices can be contracted with the vielbeins $u_i^\pm$, so that for examples $Q^+=Q^iu_i^+$, which converts any number of $\mathrm{SU}(2)_{\mathrm{R}}$ indices to a number of '+' and '−' superscripts. For a large number of superscripts an abbreviation is used which allows to write a number followed by a plus or minus sign, so that one could write $Q^+=Q^{(+1)}$ and $\mathcal{D}^{++}=\mathcal{D}^{(+2)}$. The θ-integration is normalised so that $\int\mathrm{d}^4\theta^+(\theta^+)^4=\int\mathrm{d}^4\bar{\theta}(\bar{\theta})^4=1$, and $\overline{W}$ is the non-abelian field strength constructed from the pre-potential V^{++}. Note that $\overline{W}$ does not depend on the harmonic coordinate u. All fields in the action above are in the adjoint representation of the gauge group $\mathrm{SU}(N)$, and therefore are elements of the corresponding algebra. In order to follow the conventions in [91] we choose the N by N matrices T^a spanning the algebra to be normalised so that $\mathrm{tr}(T^aT^b)=\delta^{ab}$.

A particularity of the harmonic superspace approach is that the introduction of the S^2 harmonics $u_i^\pm$ leads to infinite towers of auxiliary fields, which is equivalent to the standard harmonic expansion on the two-sphere. On the other hand, we are dealing with a gauge theory, and in particular the gauge parameters $\lambda(x,\theta,\bar{\theta},u)$ will also depend on the harmonic coordinates on S^2 and have an infinite harmonic expansion. One can show [91] that the gauge can be chosen in such a way that the infinite harmonic towers of $\lambda(x, \theta, \bar{\theta}, u)$ and $V^{++}(x, \theta, \bar{\theta}, u)$ cancel leaving only a finite number of fields in $V^{++}(x, \theta, u)$. This is similar to choosing the Wess-Zumino gauge in $\mathcal{N} = 1$ theories, which also reduces the number of fields in the vector superfield V. After this gauge fixing the following non-vanishing components remain in the pre-potential:

$$\begin{aligned} V^{++}_{\text{WZ}}(x, \theta, \bar{\theta}, u) = & -2i\theta^+\sigma^\mu\bar{\theta}^+ A_\mu(x) - i\sqrt{2}(\theta^+)^2\bar{\phi}(x) + i\sqrt{2}(\bar{\theta}^+)^2\phi(x) \\ & + 4(\bar{\theta}^+)^2\theta^+\lambda^i(x)u^- - 4(\theta^+)^2\bar{\theta}^+\bar{\lambda}^i(x)u^-_i \\ & + 3(\theta^+)^2(\bar{\theta}^+)^2 D^{ij}(x)u^-_i u^-_j \,. \end{aligned} \tag{7.20}$$

The construction of the field strength $\overline{W}$ from V^{++} is non-trivial, however, the details are not important at this point and can be found in [91]. What matters is the form of the super Yang-Mills part of the $\mathcal{N} = 4$ action which is obtained after the superspace integration is carried out. In the Wess-Zumino gauge mentioned above the component form of this part of the action is given by the following:

$$\begin{aligned} S_{\text{SYM}} = \operatorname{tr}\int \mathrm{d}^4x \Big\{ & -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + D_\mu\phi D^\mu\bar{\phi} - i\lambda^i \not{D}\bar{\lambda}_i \\ & -\frac{1}{2}[\phi, \bar{\phi}]^2 - \frac{1}{\sqrt{2}}\lambda^i[\bar{\phi}, \lambda_i] - \frac{1}{\sqrt{2}}\bar{\lambda}_i[\phi, \bar{\lambda}^i] + \frac{1}{4}D^{ij}D_{ij}\Big\}. \end{aligned} \tag{7.21}$$

We see the kinetic terms of all fields in the vector multiplet appearing in their canonical form, as well as the auxiliary fields D^{ij}. The superspace integration also generates some interaction terms, which take a form similar to the $\mathcal{N} = 4$ action. Next, consider the matter part of the $\mathcal{N} = 4$ action in (7.19):

$$S_{\text{q}} = \frac{1}{2}\operatorname{tr}\int \mathrm{d}^4x_A\, \mathrm{d}u\, \mathrm{d}^4\theta^+ \left(Q^+_a \mathcal{D}^{++}Q^{+a}\right) , \qquad \mathcal{D}^{++}Q^{+a} = D^{++}Q^{+a} + i[V^{++}, Q^{+a}] \tag{7.22}$$

The superfield $Q^{+a}(x, \theta^+\bar{\theta}^+, u)$ can be expanded in its Grassmann parameters to a finite sum of components, and reads:

$$\begin{aligned} Q^{+a} = F^{+a} + \theta^+\psi^a + \bar{\theta}^+\bar{\psi}^a + (\theta^+)^2 M^{-a} + (\bar{\theta}^+)^2 N^{-a} + i\theta^+\sigma^\mu\bar{\theta}^+ A^{-a}_\mu \\ + (\bar{\theta}^+)^2\theta^+\eta^{(-2)a} + (\theta^+)^2\bar{\theta}^+\bar{\chi}^{(-2)a} + (\theta^+)^2(\bar{\theta}^+)^2 P^{(-3)a} \end{aligned} \tag{7.23}$$

where all component fields F^{+a}, ψ^a, $\bar{\psi}^a$, M^{-a}, N^{-a}, A^{-a}_μ, $\eta^{(-2)a}$, $\bar{\chi}^{(-2)a}$, and $P^{(-3)a}$ still depend on both the space-time coordinate x, and the harmonic coordinate u on the sphere. Thus each of these component fields leads to an infinite harmonic expansion in the coordinate u.

The standard way to carry on from here is to first insert the expansion of Q^{+a} into the action and integrate over the Grassmann coordinates. Since everything is explicit here the computation is very straightforward. Next one varies the action with respect to the component fields. Many of the resulting equations of motion are not dynamical with respect to space-time coordinates x, and reveal that the only dynamical fields are contained in the towers of F^{+a}, ψ^a, and $\bar{\psi}^a$. Thus one uses the kinematic equations of motion to eliminate all auxiliary component fields, and all auxiliary fields in the harmonic towers of F^{+a}, ψ^a, and $\bar{\psi}^a$. This leads to the action written in terms of physical fields only. Since we will need the equations of motions just mentioned for the mass deformation, let us write them out more explicitly. After integrating over the Grassmann coordinates the matter action S_q take the following form:

$$\begin{aligned} S_{\mathrm{q}} = \int \mathrm{d}^4x\,\mathrm{d}u\, \Big\{ & P_a^{(-3)\bar{a}}\partial^{++}F^{+\bar{a}a} - \frac{1}{2}\eta_a^{(-2)\bar{a}}\partial^{++}\psi^{\bar{a}a} - \frac{1}{2}\bar{\chi}_a^{(-2)\bar{a}}\partial^{++}\bar{\psi}^{\bar{a}a} - \frac{i}{2}\psi_a^{\bar{a}}\not{D}\bar{\psi}^{\bar{a}a} \\ & + M_a^{-\bar{a}}\partial^{++}N^{-\bar{a}a} - \frac{1}{4}A_a^{-\mu\bar{a}}\partial^{++}A_\mu^{-\bar{a}a} + A_a^{-\mu\bar{a}}D_\mu F^{+\bar{a}a}\Big\} \\ + \frac{i}{2}f^{\bar{a}\bar{b}\bar{c}}\Big\{ & 3F_a^{+\bar{a}}F^{+\bar{b}a}D^{ij\bar{c}}(x)u_i^-u_j^- + 4F_a^{+\bar{a}}\bar{\lambda}^{\bar{b}i}(x)\bar{\psi}^{\bar{c}a}u_i^- - 4F_a^{+\bar{a}}\lambda^{\bar{b}i}(x)\psi^{\bar{c}a}u_i^- \\ & + i2\sqrt{2}F_a^{+\bar{a}}\phi^{\bar{b}}(x)M^{-\bar{c}a} - i2\sqrt{2}F_a^{+\bar{a}}\bar{\phi}^{\bar{b}}(x)N^{-\bar{c}a} \\ & + \frac{i}{\sqrt{2}}\psi_a^{\bar{a}}\psi^{\bar{b}a}\phi^{\bar{c}}(x) - \frac{i}{\sqrt{2}}\bar{\psi}_a^{\bar{a}}\bar{\psi}^{\bar{b}a}\bar{\phi}^{\bar{c}}(x)\Big\}. \end{aligned} \tag{7.24}$$

The barred indices $\bar{a},\bar{b} \in \{1,\ldots,N^2-1\}$ refer to the adjoint representation of the gauge group SU(N), and the corresponding covariant derivative is denoted by D_μ. The fields where the dependence on x is explicit depend on x only, while all other fields depend on both x and the harmonic coordinate u. After varying the action above we get the following equations of motion:

$$\delta/\delta P_a^{(-3)\bar{a}}: \quad \partial^{++}F^{+\bar{a}a} = 0 \tag{7.25}$$

$$\delta/\delta \eta_a^{(-2)\bar{a}}: \quad \partial^{++}\psi^{\bar{a}a} = 0 \tag{7.26}$$

$$\delta/\delta \bar{\chi}_a^{(-2)\bar{a}}: \quad \partial^{++}\bar{\psi}^{\bar{a}a} = 0 \tag{7.27}$$

$$\delta/\delta M_a^{-\bar{a}}: \quad \partial^{++}N^{-\bar{a}a} + f^{\bar{a}\bar{b}\bar{c}}\sqrt{2}F^{+\bar{b}a}\phi(x)^{\bar{c}} = 0 \tag{7.28}$$

$$\delta/\delta N_a^{-\bar{a}}: \quad \partial^{++}M^{-\bar{a}a} - f^{\bar{a}\bar{b}\bar{c}}\sqrt{2}F^{+\bar{b}a}\bar{\phi}(x)^{\bar{c}} = 0 \tag{7.29}$$

$$\delta/\delta A_a^{-\mu\bar{a}}: \quad -\frac{1}{2}\partial^{++}A_\mu^{-\bar{a}a} + D_\mu F^{+\bar{a}a} = 0 \tag{7.30}$$

Note that we should also write down the equations of motion resulting from the variation with respect to $F^{+\bar{a}a}$, $\psi^{\bar{a}a}$, and $\bar{\psi}^{\bar{a}a}$, as well as all fields that come from V^{++}, but this is not necessary at this point.

Next one notes that the action in (7.24) is linear in $P^{(-3)}$, $\eta^{(-2)}$, and $\bar{\chi}^{(-2)}$ and therefore to integrate out these fields we just remove the corresponding terms. This action is also linear in M^- and N^-, but these two fields mix in $M_a^{-\bar{a}}\partial^{++}N^{-\bar{a}a}$, and so the total contribution from all terms containing M^- and N^- will be $-M_a^{-\bar{a}}\partial^{++}N^{-\bar{a}a}$. Similarly, for A_μ^- there is one term that is quadratic, and therefore the contribution to the action from all terms with

A_μ^- is $\frac{1}{4}A_a^{-\mu\bar a}\partial^{++}A_\mu^{-\bar a a}$. After these observations the action in (7.24) reduces to

$$\begin{aligned} S_{\rm q} = \int &\mathrm{d}^4x\,\mathrm{d}u\,\Big\{-\frac{i}{2}\psi_a^{\bar a}\not{D}\bar\psi^{\bar a a} - M_a^{-\bar a}\partial^{++}N^{-\bar a a} + \frac{1}{4}A_a^{-\mu\bar a}\partial^{++}A_\mu^{-\bar a a}\Big\} \\ &+\frac{i}{2}f^{\bar a\bar b\bar c}\Big\{3F_a^{+\bar a}F^{+\bar b a}D^{ij\bar c}(x)u_i^-u_j^- + 4F_a^{+\bar a}\bar\lambda^{\bar b i}(x)\bar\psi^{\bar c a}u_i^- - 4F_a^{+\bar a}\lambda^{\bar b i}(x)\psi^{\bar c a}u_i^-, \\ &\qquad + \frac{i}{\sqrt2}\psi_a^{\bar a}\psi^{\bar b a}\phi^{\bar c}(x) - \frac{i}{\sqrt2}\bar\psi_a^{\bar a}\bar\psi^{\bar b a}\bar\phi^{\bar c}(x)\Big\} \end{aligned} \tag{7.31}$$

where we have to substitute M^-, N^-, and A_μ^- with the corresponding solutions of equations (7.28), (7.29), and (7.30). Moreover, equations (7.25), (7.26), and (7.27) determine the dependence of F^+, ψ, and $\bar\psi$ on the harmonic coordinate u and truncate the infinite harmonic towers to only one term. Thus for $P^{(-3)}$, $\eta^{(-2)}$, and $\chi^{(-2)}$ on-shell we have

$$F^{+\bar a a}(x,u) = f^{i\bar a a}(x)u_i^+\,, \qquad \psi^{\bar a a}(x,u) = \psi^{\bar a a}(x)\,, \qquad \bar\psi^{\bar a a}(x,u) = \bar\psi^{\bar a a}(x)\,. \tag{7.32}$$

At this point all the dependence on the harmonics $u_i^\pm$ is completely explicit and we can perform the u-integration using

$$\int\mathrm{d}u\;1 = 1\,,\quad \int\mathrm{d}u\;u_i^+u_j^- = \frac{1}{2}\epsilon_{ij}\,,\quad \int\mathrm{d}u\;u_{(i}^+u_{j)}^+u_{(k}^-u_{l)}^- = \frac{1}{3}\epsilon_{im}\epsilon_{jn}\delta_{kl}^{mn} = \frac{1}{6}(\epsilon_{ik}\epsilon_{jl}+\epsilon_{il}\epsilon_{jk})\,. \tag{7.33}$$

Finally we can restore the super Yang-Mills part of the action given in equation (7.21) and integrate out the auxiliary field D^{ij}. This gives us the complete $\mathcal{N}=4$ action in $\mathcal{N}=2$ component form:

$$\begin{aligned} S &= S_{\rm SYM} + S_{\rm q} \\ &= \int\mathrm{d}^4x\,\Big\{-\frac{i}{2}\psi_a^{\bar a}\not{D}\bar\psi^{\bar a a} + \frac{1}{2}D^\mu f_{ai}^{\bar a}D_\mu f^{\bar a a i}\Big\} \qquad (7.34) \\ &\qquad +\Big\{-\frac{1}{4}F_{\mu\nu}^{\bar a}F^{\bar a\mu\nu} + D_\mu\phi^{\bar a}D^\mu\bar\phi^{\bar a} - i\lambda^{\bar a i}\not{D}\bar\lambda_i^{\bar a}\Big\} \\ &\qquad +if^{\bar a\bar b\bar c}\Big\{f_a^{\bar a i}\bar\lambda_i^{\bar b}\bar\psi^{\bar c a} - f_a^{\bar a i}\lambda_i^{\bar b}\psi^{\bar c a} + \frac{i}{2\sqrt2}\psi_a^{\bar a}\psi^{\bar b a}\phi^{\bar c} - \frac{i}{2\sqrt2}\bar\psi_a^{\bar a}\bar\psi^{\bar b a}\bar\phi^{\bar c}\Big\} \\ &\qquad +if^{\bar a\bar b\bar c}\Big\{\frac{1}{\sqrt2}\bar\lambda_i^{\bar a}\bar\lambda^{\bar b i}\phi^{\bar c} - \frac{1}{\sqrt2}\lambda_i^{\bar a}\lambda^{\bar b i}\bar\phi^{\bar c}\Big\} \\ &\quad +f^{\bar a\bar b\bar c}f^{\bar a\bar b'\bar c'}\Big\{f_{ai}^{\bar b}f^{\bar b' ia}\bar\phi^{\bar c}\phi^{\bar c'} + \frac{1}{4}f_{ai}^{\bar b}f_b^{\bar b' i}f_j^{\bar c a}f^{\bar c' bj}\Big\} \\ &\quad +f^{\bar a\bar b\bar c}f^{\bar a\bar b'\bar c'}\Big\{\frac{1}{2}\phi^{\bar b}\phi^{\bar b'}\bar\phi^{\bar c}\bar\phi^{\bar c'}\Big\} \\ &= {\rm tr}\int\mathrm{d}^4x\,\Big\{-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + D_\mu\phi D^\mu\bar\phi + \frac{1}{2}D^\mu f_{ai}D_\mu f^{ai} - i\lambda^i\not{D}\bar\lambda_i - \frac{i}{2}\psi_a\not{D}\bar\psi^a\Big\} \\ &\qquad +\Big\{f_{ai}[\lambda^i,\psi^a] - f^{ai}[\bar\lambda_i,\bar\psi_a] + \frac{i}{2\sqrt2}\phi[\psi_a,\psi^a] - \frac{i}{2\sqrt2}\bar\phi[\bar\psi_a,\bar\psi^a] + \frac{1}{\sqrt2}\phi[\bar\lambda_i,\bar\lambda^i] - \frac{1}{\sqrt2}\bar\phi[\lambda_i,\lambda^i]\Big\} \\ &\qquad -\Big\{[f_{ai},\bar\phi][f^{ai},\phi] + \frac{1}{4}[f_{ai},f_j^a][f_b^i,f^{bj}] + \frac{1}{2}[\phi,\bar\phi]^2\Big\}\,. \end{aligned} \tag{7.35}$$

7.3 Deformation by $\mathcal{N}=2$ Mass Term

In the previous section we showed how the action of the $\mathcal{N}=4$ SYM can be written in the language of $\mathcal{N}=2$ harmonic superspace. In this section we would like to consider deformations of the $\mathcal{N}=4$ theory that break the supersymmetry down to $\mathcal{N}=2$. In the $\mathcal{N}=2$ language of the previous section this amounts to adding operators to the action (7.19) that preserve the already manifest $\mathcal{N}=2$ supersymmetry.

A natural way to do so is to add potential terms to the matter part of the action in (7.22):

$$S_{\mathrm{q}} \to \frac{1}{2}\,\mathrm{tr}\int \mathrm{d}^4x_A\,\mathrm{d}u\,\mathrm{d}^4\theta^+ \left(Q^+_a \mathcal{D}^{++} Q^{+a} + L^{+4}(q^+,u)\right) . \tag{7.36}$$

In the end we would like to generate a scalar potential for the physical fields, associate the emerging couplings to bulk modes, and potentially study their RG flows from a holographic perspective. However, it was shown in [91] that a deformation of the $\mathcal{N}=2$ theory like above does not generate a scalar potential, but rather a non-trivial metric for the scalar manifold.

It turns out that a way of generating a non-trivial $\mathcal{N}=2$ scalar potential is by the Scherk-Schwarz dimensional reduction method [92]. One imagines that the theory in four dimensions comes from a dimensional reduction from another theory with one dimension more, and a supersymmetry that extends to that extra dimension. One then gauges an electric U(1) subgroup of the higher-dimensional supersymmetry so that translations along the extra dimension correspond to U(1) rotations. This gauging partially breaks the higher-dimensional supersymmetry to $\mathcal{N}=2$, and the unbroken supersymmetry algebra acquires a central charge that corresponds to the gauged U(1) [93]. After dimensional reduction to four dimensions we end up with the $\mathcal{N}=2$ theory that we started with, where the central charge is realised as shifts along the extra dimension as $Z=-i\sigma_3\partial_5$, where σ_3 is the Cartan generator of the $\mathrm{SU}(2)_{\mathrm{H}}$. The only dependence of the matter fields on the extra dimension is through a U(1) phase: $Q^{+a}(x_5)=e^{imx^5}Q^{+a}$. Thus the central charge is nothing else than a mass for the superfield

$$(ZQ^+)^a = m(\sigma_3)^a{}_b Q^b\,, \tag{7.37}$$

and can also be seen as rotations by the U(1) subgroup of the $\mathrm{SU}(2)_{\mathrm{H}}$ global symmetry. In the presence of the supercharge the harmonic derivative is altered to

$$(D^{++}_{\mathrm{cc}})^a{}_b = \delta^a_b D^{++} + i(\theta^+)^2 \bar{Z}^a{}_b + i(\bar{\theta}^+)^2 Z^a{}_b \tag{7.38}$$

and the matter part of the action now reads

$$\begin{aligned} S_{\mathrm{q}}^{\mathrm{cc}} &= \frac{1}{2}\,\mathrm{tr}\int \mathrm{d}^4x_A\,\mathrm{d}u\,\mathrm{d}^4\theta^+ \left(Q_a^+[\delta_b^a\mathcal{D}^{++} - m(\sigma_3)^a{}_b(\theta^+)^2 + m(\sigma_3)^a{}_b(\bar\theta^+)^2]Q^{+b}\right) \\ &= \mathrm{tr}\int \mathrm{d}^4x_A\,\mathrm{d}u\,\mathrm{d}^4\theta^+ \left(\frac{1}{2}Q_a^+\mathcal{D}^{++}Q^{+a} + [m(\theta^+)^2 - m(\bar\theta^+)^2]Q_1^+Q_2^+\right) . \end{aligned} \tag{7.39}$$

Note that the integrand does not depend on the extra dimension, and thus the integration goes only over the four-space. Moreover, the explicit appearance of θs in the action is only possible if the supersymmetry has a central charge. The fact that the sigma matrix σ_3, a generator of the $\mathrm{SU}(2)_{\mathrm{H}}$, appears in the action explicitly breaks $\mathrm{SU}(2)_{\mathrm{H}}$ to its Cartan subgroup, which is just the U(1) that is generated by σ_3 itself, since SU(2) has rank one.

It seems possible to generalise the massive action just derived to a complex m so that the $\mathcal{N}=2$ symmetry is still preserved. In this case we would like to write

$$\begin{aligned} S_{\mathrm{q}}^{\mathrm{cc}} &= S_{\mathrm{q}} + \delta S_{\mathrm{m}} \\ &= \mathrm{tr}\int \mathrm{d}^4x_A\,\mathrm{d}u\,\mathrm{d}^4\theta^+ \left(\frac{1}{2}Q_a^+\mathcal{D}^{++}Q^{+a} + [\bar m(\theta^+)^2 - m(\bar\theta^+)^2]Q_1^+Q_2^+\right) . \end{aligned} \tag{7.40}$$

We can see the change induced by the addition of δS_{m} by performing the θ-integration, the result of which is

$$\delta S_{\mathrm{m}} = \mathrm{tr}\int \mathrm{d}^4x\,\mathrm{d}u\; m\left(F_2^+M_1^- - \frac{1}{2}\psi_2\psi_1 + M_2^-F_1^+\right) - \bar m\left(F_2^+N_1^- - \frac{1}{2}\bar\psi_2\bar\psi_1 + N_2^-F_1^+\right) . \tag{7.41}$$

We see that because the integrand in δS_{m} is proportional to θ^2 and $\bar\theta^2$, no new terms are added for $P^{(-3)}$, $\eta^{(-2)}$, and $\bar\chi^{(-2)}$. Looking at equations (7.25), (7.26), and (7.27) we realise this has for consequence that then kinematic equations of motion that eliminate the infinite harmonic towers for the physical fields remain the same and are given by (7.32). This allows us to read off the component form of the mass deformation for fermions from the equation above:

$$\begin{aligned} \delta S_{\mathrm{m}}^{\mathrm{ferm}} &= \mathrm{tr}\int \mathrm{d}^4x\,\mathrm{d}u\,\left(-\frac{m}{2}\psi_2\psi_1 + \frac{\bar m}{2}\bar\psi_2\bar\psi_1\right) \\ &= \mathrm{tr}\int \mathrm{d}^4x\,\mathrm{d}u\,\frac{1}{4}(\sigma_3)^a{}_b\left(m\psi_a\psi^b - \bar m\bar\psi_a\bar\psi^b\right) . \end{aligned} \tag{7.42}$$

The other part of $\delta_{\mathrm{m}}S$ contributes additional terms to the equations of motion (7.28) and (7.29) that were obtained by varying with respect to M^- and N^-. The new equations are given by

$$\delta/\delta M_1^{-\bar a}: \quad \partial^{++}N^{-\bar a1} + \left(\sqrt{2}f^{\bar a\bar b\bar c}f^{\bar b1i}\phi(x)^{\bar c} + mf_2^{\bar ai}\right)u_i^- = 0 \tag{7.43}$$

$$\delta/\delta M_2^{-\bar a}: \quad \partial^{++}N^{-\bar a2} + \left(\sqrt{2}f^{\bar a\bar b\bar c}f^{\bar b2i}\phi(x)^{\bar c} + mf_1^{\bar ai}\right)u_i^- = 0 \tag{7.44}$$

$$\delta/\delta N_1^{-\bar a}: \quad \partial^{++}M^{-\bar a 1} - \left(\sqrt{2}f^{\bar a\bar b\bar c}f^{\bar b 1 i}\bar\phi(x)^{\bar c} + \bar m f_2^{\bar a i}\right)u_i^- = 0 \tag{7.45}$$
$$\delta/\delta N_2^{-\bar a}: \quad \partial^{++}M^{-\bar a 2} - \left(\sqrt{2}f^{\bar a\bar b\bar c}f^{\bar b 2 i}\bar\phi(x)^{\bar c} + \bar m f_1^{\bar a i}\right)u_i^- = 0. \tag{7.46}$$

The different components of M^- and N^- can be grouped together, which introduces the Cartan generator σ^3 in the mass term:

$$\delta/\delta M_a^{-\bar a}: \quad \partial^{++}N^{-\bar a a} + \left(\sqrt{2}f^{\bar a\bar b\bar c}f^{\bar b a i}\phi(x)^{\bar c} - m(\sigma_3)^a{}_b f^{\bar a b i}\right)u_i^- = 0 \tag{7.47}$$
$$\delta/\delta N_a^{-\bar a}: \quad \partial^{++}M^{-\bar a a} - \left(\sqrt{2}f^{\bar a\bar b\bar c}f^{\bar b a i}\bar\phi(x)^{\bar c} - \bar m(\sigma_3)^a{}_b f^{\bar a b i}\right)u_i^- = 0\,. \tag{7.48}$$

As before, the whole contribution to the action comes from the term

$$\int \mathrm{d}^4x\,\mathrm{d}u\,\left(-M_a^{-\bar a}\partial^{++}N^{-\bar a a}\right). \tag{7.49}$$

After eliminating the auxiliary fields M^- and N^- by solving their equations of motion we arrive at the following mass contribution to the action

$$\delta S_{\mathrm{m}}^{\mathrm{boson}} = \mathrm{tr}\int \mathrm{d}^4x\,\left(-|m|^2 f_1^i f_{2i} + \sqrt{2}(\bar m\phi + m\bar\phi)[f_1^i, f_{2i}]\right) \tag{7.50}$$
$$= \mathrm{tr}\int \mathrm{d}^4x\,\frac{1}{2}(\sigma_3)^a{}_b\left(-|m|^2 f_{ai}f^{bi} + \sqrt{2}(\bar m\phi + m\bar\phi)[f_{ai}, f^{bi}]\right). \tag{7.51}$$

Finally, we should combine the bosonic and fermionic contribution to obtain the total change in the $\mathcal{N}=2$ action due to the mass term:

$$\begin{aligned}\delta S_{\mathrm{m}} &= \mathrm{tr}\int \mathrm{d}^4x\,\mathrm{d}u\,\left(-|m|^2 f_1^i f_{2i} + \sqrt{2}(\bar m\phi + m\bar\phi)[f_1^i, f_{2i}] - \frac{m}{2}\psi_2\psi_1 + \frac{\bar m}{2}\bar\psi_2\bar\psi_1\right)\\ &= \mathrm{tr}\int \mathrm{d}^4x\,\mathrm{d}u\,\frac{1}{2}(\sigma_3)^a{}_b\left(-|m|^2 f_{ai}f^{bi} + \sqrt{2}(\bar m\phi + m\bar\phi)[f_{ai}, f^{bi}] + \frac{m}{2}\psi_a\psi^b - \frac{\bar m}{2}\bar\psi_a\bar\psi^b\right).\end{aligned} \tag{7.52}$$

We can see several operators appearing in this deformation. Their holographic interpretation will be the subject of the following section.

7.4 Relation to the Bulk Modes

In this section we would like to analyse the operators that emerge after the mass deformation. In particular we would like to answer the question whether they correspond to chiral operators and are therefore part of a short multiplet of the $\mathcal{N}=4$ theory. If this is the case, then the corresponding modes in the bulk have to be turned on.

We have shown in the previous section that the deformation of the $\mathcal{N}=4$ theory by $\mathcal{N}=2$

mass terms leads to the addition of the following component fields to the action:

$$\delta S_m = \mathrm{tr} \int \mathrm{d}^4x \Big\{ -|m|^2 f_1^i f_{2i} + \sqrt{2}(m\phi + \bar{m}\bar{\phi})[f_1^i, f_{2i}] - \frac{\bar{m}}{2}\psi_2\psi_1 - \frac{m}{2}\bar{\psi}^2\bar{\psi}^1 \Big\}. \tag{7.53}$$

Furthermore we recall the representations of the $\mathcal{N}=2$ fields with respect to the Lie group $\mathrm{SU}(2)_\mathrm{H} \times \mathrm{SU}(2)_\mathrm{R} \times \mathrm{U}(1)_\mathrm{X} \subset \mathrm{SU}(4)_\mathrm{R}$:

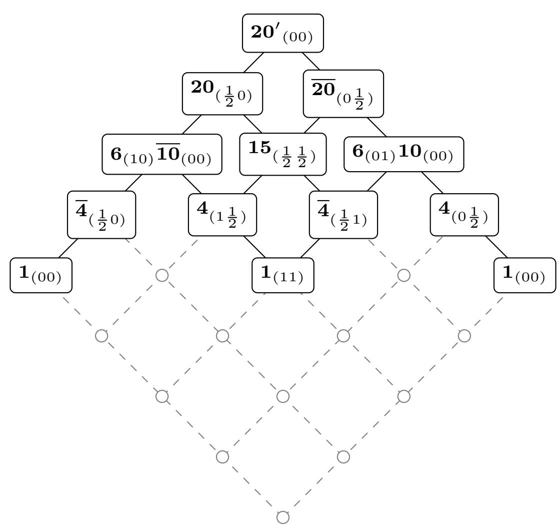
$$f_{\bar{a}}^i = (\mathbf{2},\mathbf{2})_{0(00)} \qquad \psi_{\bar{a}} = (\mathbf{2},\mathbf{1})_{-1(\frac{1}{2}0)} \qquad \phi = (\mathbf{1},\mathbf{1})_{2(00)}\,. \tag{7.54}$$

The fields $f_{\bar{a}}^i$ and $\psi_{\bar{a}}$ are in the $\mathcal{N}=2$ hypermultiplet, and ϕ is the complex scalar from the $\mathcal{N}=2$ vector multiplet. We can see explicitly that the mass term breaks the $\mathrm{SU}(2)_\mathrm{H}$ symmetry. Thus the terms in the action transform as follows under the remaining $\mathrm{SU}(2)_\mathrm{R} \times \mathrm{U}(1)_\mathrm{X}$:

$$m\,\mathrm{tr}\left(\bar{\psi}^2\bar{\psi}^1 - 2\sqrt{2}\phi[f_1^i, f_{2i}]\right) + \text{c.c.} = \mathbf{1}_{2(00)} + \mathbf{1}_{-2(00)} \tag{7.55}$$

$$|m|^2\,\mathrm{tr}\left(f_1^i f_{2i}\right) = \mathbf{1}_{0(00)} \tag{7.56}$$

It is straightforward to see that the fermion bilinears in equation (7.55) are the same as the ones that were considered by GPPZ [23], and therefore are part of the order-2 short multiplet. The $\mathcal{N}=4$ order-2 short multiplet can be represented as follows



and the operators that we are looking for sit in the $\overline{\mathbf{10}}_{(00)}$ and the $\mathbf{10}_{(00)}$, and are given by

$$\lambda_{(A}\lambda_{B)} \quad \text{and} \quad \bar{\lambda}^{(A}\bar{\lambda}^{B)} \qquad (A,B=1,\ldots,4)\,. \tag{7.57}$$

Thus we conclude that the fermion bilinears in (7.55) are chiral operators of dimension $\Delta = 2+1 = 3$, and correspond to the bulk mode $A_{\alpha\beta}$ with mass $m^2 = \Delta(\Delta - d) = -3$. Recall that under the inclusion $\mathrm{SU}(2)_\mathrm{H} \times \mathrm{SU}(2)_\mathrm{R} \times \mathrm{U}(1)_\mathrm{X} \subset \mathrm{SU}(4)_\mathrm{R}$ the $\overline{\mathbf{10}}$ decomposes as follows:

$$\overline{\mathbf{10}} \to (\mathbf{2},\mathbf{2})_0 + (\mathbf{3},\mathbf{1})_{-2} + (\mathbf{1},\mathbf{3})_2 \tag{7.58}$$

The bilinear $\psi_2\psi_1 = \mathbf{1}_{-2}$ is contained in the $(\mathbf{3},\mathbf{1})_{-2}$ since $\mathrm{SU}(2)_\mathrm{H}$ is broken and we have to omit the first factor. The operator $\bar\psi^2\bar\psi^1$ is contained in the $\mathbf{10}$ in the same way. Exactly as in the GPPZ case, there are also cubic scalar terms that appear in the $\mathbf{10}$ and $\overline{\mathbf{10}}$, these are exactly the scalars in (7.55), and therefore have to be considered together with the fermion bilinears.

Next consider the operator in (7.56) given by

$$f_1^i f_{i2} = f_1^1 f_2^2 - f_1^2 f_2^1 \tag{7.59}$$

It is a quadratic operator made of scalars only, thus we have to look in the top component of the order-2 short multiplet, which is the $\mathbf{20'}$. In terms of the 6 $\mathcal{N}=4$ scalars ϕ^I, which are in the $\mathbf{6}$ of the SU(4) the $\mathbf{20'}$ is a symmetric traceless product of two such fields and can be written as $\mathbf{20'} = C^i_{IJ}\phi^I\phi^J$ for some symmetric traceless constants C^i_{IJ}. As we saw in Section 7.1, we can write $f_1^i f_{2i}$ in terms of it's real components ϕ^I as

$$f_1^i f_{2i} = -\frac{1}{2}\left((\phi^1)^2 + (\phi^2)^2 + (\phi^4)^2 + (\phi^5)^2\right) = -\frac{1}{2}C_{IJ}\phi^I\phi^J\,, \tag{7.60}$$

$$\text{with}\quad C_{11} = C_{22} = C_{44} = C_{55} = 1\,. \tag{7.61}$$

we see that C_{IJ} is not equal to its trace so that the traceless part of it is non-zero, and therefore corresponds to a component of the $\mathbf{20'}$. In terms of the branching of the $\mathbf{20'}$ we can say that this component corresponds to the following term:

$$\mathbf{20'} \to (\mathbf{1},\mathbf{1})_0 + \dots \tag{7.62}$$

Thus we have shown that the operator in (7.56) has a chiral component with dimension $\Delta = 2$, and therefore corresponds to the bulk mode $h^\alpha_\alpha - a_{\alpha\beta\gamma\delta}$ with mass $m^2 = \Delta(\Delta-4) = -4$. The remaining part of this operator, the trace, is not chiral, and the corresponding bulk mode is not visible in supergravity.

In sum we have found that the $N=2$ mass deformation leads to two chiral operators, and therefore two different bulk modes need to be turned on. Note that in the $\mathcal{N}=1$ mass deformation of GPPZ a similar situation arises: additionally to the $\mathbf{10}$ and $\overline{\mathbf{10}}$ the following operator is produced:

$$m^{ik}\overline{m}_{jk}\phi_i\overline{\phi}^j = \mathbf{3}\times\bar{\mathbf{3}} = \mathbf{8}+\mathbf{1}\,. \tag{7.63}$$

The $\mathbf{1}$ is not chiral and therefore is not visible in supergravity. The $\mathbf{8}$, however, is, and would lead to an additional supergravity mode. In GPPZ the $\mathbf{8}$ is eliminated by choosing $m_{ij} = m\delta_{ij}$, which also eliminates the additional bulk mode. In the $\mathcal{N}=2$ case, in contrast, the additional term (7.56) is proportional to $|m|^2$, and therefore there is no freedom of choosing a non-vanishing component in m to eliminate it, and so both bulk modes have to be considered.

Part III

GPPZ Uplift

Chapter 8

IIB Uplift of $D = 5$ Maximal Supergravity

This chapter summarises the main uplift formulas which are used to construct the GPPZ uplift in the next chapter. In the first section the general formalism is presented, while in the second section a particular parametrisation of the scalar matrix tailored for the four scalar truncation is worked out.

8.1 Main Uplift Formulas

$D = 5$ maximal SO(6) gauged supergravity [44, 45, 94] is a consistent truncation of IIB supergravity around $AdS_5 \times S^5$. Its field content comprises the $D = 5$ metric $g_{\mu\nu}$ together with 42 scalar fields parametrising a 27×27 symmetric $\mathrm{E}_{6(6)}$ matrix which we parametrise in an $\mathrm{SL}(6) \times \mathrm{SL}(2)$ basis as

$$M_{MN} = \begin{pmatrix} M_{ab,cd} & M_{ab}{}^{c\beta} \\ M^{a\alpha}{}_{cd} & M^{a\alpha,c\beta} \end{pmatrix}, \quad \text{with inverse} \quad M^{MN} = \begin{pmatrix} M^{ab,cd} & M^{ab}{}_{c\beta} \\ M_{a\alpha}{}^{cd} & M_{a\alpha,c\beta} \end{pmatrix}, \tag{8.1}$$

according to the decomposition of the fundamental representation of $\mathrm{E}_{6(6)}$ as

$$\mathbf{27} \longrightarrow (\mathbf{15}, \mathbf{1}) \oplus (\mathbf{6}, \mathbf{2}) \,, \tag{8.2}$$

under $\mathrm{SL}(6) \times \mathrm{SL}(2)$. Indices $a, b, c, d = 1, \ldots, 6$, and $\alpha, \beta = 1, 2$, label the fundamental representations of SL(6) and SL(2), respectively. Index pairs ab and cd in (8.1) are antisymmetric. The remaining bosonic field content in five dimensions is given by 15 non-abelian vectors fields A_μ^{ab} and 12 topologically massive 2-forms $B_{\mu\nu,a\alpha}$. The truncation we are eventually interested in and which which carries the GPPZ solution [23] contains four scalar fields, a single vector and no 2-forms.

In this section we collect the relevant IIB uplift formulae of $D = 5$ supergravity from [57] (see also [52, 53, 95–97]), in the next section we explicitly evaluate these formulas for the four-scalar truncation. The IIB fields are expressed in terms of the $D = 5$ fields introduced above while their dependence on the five internal coordinates y^m is carried by the fundamental S^5 sphere harmonics, $\mathcal{Y}^a$, $(a = 1, \dots, 6)$, with $\mathcal{Y}^a\mathcal{Y}^a = 1$, and the S^5 Killing vectors

$$\mathcal{K}_{[ab]\,m} = -\sqrt{2}\,\mathcal{Y}^{[a}\partial_m\mathcal{Y}^{b]} , \qquad m = 1, \dots, 5 . \tag{8.3}$$

By $\mathring{G}_{mn}$ we denote the round metric on S^5 which can be expressed as

$$\mathring{G}_{mn} = \mathcal{K}_{[ab]\,m}\mathcal{K}_{[ab]\,n} , \tag{8.4}$$

in terms of the Killing vectors (8.3). We also define its volume form

$$\mathring{\omega}_{klmnp} \;\equiv\; \sqrt{\det \mathring{G}}\,\varepsilon_{klmnp} \;\equiv\; 5\,\partial_{[k}\mathring{C}_{lmnp]} , \tag{8.5}$$

in terms of a 4-form potential $\mathring{C}_{klmn}$. We will also need the tensors

$$\begin{aligned}\mathcal{K}_{[ab]\,mn} &\equiv \partial_m\mathcal{K}_{[ab]\,n} - \partial_n\mathcal{K}_{[ab]\,m} ,\\ \mathcal{K}_{[ab]\,klm} &\equiv \frac{1}{2}\,\mathring{\omega}_{klmnp}\,\mathcal{K}_{[ab]}{}^{np} ,\end{aligned} \tag{8.6}$$

where indices n, p on the right-hand side are raised with the background metric (8.4). In terms of these objects, the IIB metric takes the following form

$$\begin{aligned} ds^2 &= \Delta^{-2/3}(x,y)\,g_{\mu\nu}(x)\,dx^\mu dx^\nu \\ &\quad + G_{mn}(x,y)\left(dy^m + \mathcal{K}_{[ab]}{}^m(y)A^{ab}_\mu(x)dx^\mu\right)\left(dy^n + \mathcal{K}_{[cd]}{}^n(y)A^{cd}_\nu(x)dx^\nu\right) , \end{aligned} \tag{8.7}$$

with the internal block $G_{mn}(x,y)$ given by inverting the matrix

$$G^{mn}(x,y) = \;\Delta(x,y)^{2/3}\,\mathcal{K}_{[ab]}{}^m(y)\mathcal{K}_{[cd]}{}^n(y)\,M^{ab,cd}(x) , \tag{8.8}$$

in terms of the submatrix $M^{ab,cd}(x)$ from (8.1). We use indices μ, ν and m, n for the external five and internal five coordinates, respectively. The warp factor $\Delta(x,y)$ is defined as

$$\Delta(x,y) = \sqrt{\frac{\det G_{mn}(x,y)}{\det \mathring{G}_{mn}(y)}} . \tag{8.9}$$

The IIB dilaton and axion combine into a symmetric SL(2) matrix $m_{\alpha\beta}$ whose inverse is given by

$$m^{\alpha\beta}(x,y) = \Delta(x,y)^{4/3}\,\mathcal{Y}^a(y)\mathcal{Y}^b(y)\,M^{a\alpha,b\beta}(x) . \tag{8.10}$$

The relevant components of the IIB 2-form doublet and 4-form gauge potentials are given by

$$\begin{aligned}
C_{mn}{}^{\alpha} &= -\frac{1}{2}\,\varepsilon^{\alpha\beta}\Delta^{4/3}\,m_{\beta\gamma}\,\mathcal{Y}^c\,\mathcal{K}_{[ab]\,mn}\,M_{ab}{}^{c\gamma}\ , \\
C_{\mu kmn} &= -\frac{1}{8}\,\mathring{\omega}_{kmnpq}\mathring{\nabla}^p\mathcal{K}_{[ab]}{}^q\,A_\mu{}^{ab} - \mathcal{K}_{[ab]}{}^p A_\mu{}^{[ab]}\,\mathring{C}_{pkmn}\ , \\
C_{m\,\mu\nu\rho} &= -\frac{1}{32}\,\mathcal{K}_{[ab]\,m}\left(2\sqrt{|g|}\,\varepsilon_{\mu\nu\rho\sigma\tau}\,M_{ab,cd}F^{\sigma\tau\,cd} + 3\sqrt{2}\,\varepsilon_{abcdef}\,\partial_{[\mu}A_\nu{}^{cd}A_{\rho]}{}^{ef}\right)\ , \\
C_{klmn} &= \mathring{C}_{klmn} - \frac{1}{6}\,\mathring{\omega}_{klmnp}\,\mathring{G}^{pq}\,\Delta^{-1}\partial_q\Delta\ , \\
C_{\mu\nu\rho\sigma} &= -\frac{1}{16}\,\mathcal{Y}^a\mathcal{Y}^b\,\sqrt{|g|}\,\varepsilon_{\mu\nu\rho\sigma\tau}D^\tau M_{bc,N}M^{N\,ca} + \Lambda_{\mu\nu\rho\sigma}\ ,
\end{aligned} \tag{8.11}$$

with $F_{\mu\nu}{}^{ab}$ being the five-dimensional field strength of $A_\mu{}^{ab}$. The function $\Lambda_{\mu\nu\rho\sigma}(x)$ in the last line is defined by integrating

$$\begin{aligned}
D_{[\mu}\Lambda_{\nu\rho\sigma\tau]} &= \frac{1}{600}\,\sqrt{|g|}\,\varepsilon_{\mu\nu\rho\sigma\tau}\left(10\,\delta^d_h\delta^a_e + 2\,M^{fd,ga}M_{gh,fe} - M_{e\alpha}{}^{ga}M_{gh}{}^{d\alpha}\right)M^{bh,ec}\,\delta_{cd}\delta_{ab} \\
&\quad - \frac{1}{480}\,\sqrt{|g|}\,\varepsilon_{\mu\nu\rho\sigma\tau}D_\lambda\left(M^{N\,ac}\,D^\lambda M_{ac,N}\right) \\
&\quad + \frac{1}{240}\,\sqrt{|g|}\,\varepsilon_{\mu\nu\rho\sigma\tau}\,M_{ab,cd}\,F_{\kappa\lambda}{}^{ab}\,F^{\kappa\lambda\,cd} + \frac{1}{32}\,\sqrt{2}\,\varepsilon_{abcdef}\,F_{[\mu\nu}{}^{ab}F_{\rho\sigma}{}^{cd}A_{\tau]}{}^{ef}\ .
\end{aligned} \tag{8.12}$$

The p-forms (8.11) are given in the standard Kaluza-Klein basis

$$Dy^m = dy^m + \mathcal{K}_{[ab]}{}^m(y)A^{ab}_\mu(x)dx^\mu\ , \tag{8.13}$$

see equation (8.7). As compared to the full uplift formulas [57] we have suppressed in the p-forms (8.11) all terms anti-symmetric in more than one vector field since these will not survive in the truncation to a single vector field which we will impose next.

8.2 Parametrisation of the Scalar Matrix

In this section we spell out the explicit parametrisation of the scalar $E_{6(6)}$ matrix M_{MN} (8.1) in the 4-scalar truncation of $D = 5$ maximal supergravity. To this end, we change to a complex basis, in which the SL(6) vector decomposes according to

$$\{X_a\} \longrightarrow \{X_i, X_{\bar{i}} = X_i^*\}\ , \qquad i,\bar{i} = 1,2,3\ . \tag{8.14}$$

In this decomposition, the $E_{6(6)}$ matrix (8.1) decomposes as

$$M_{MN} = \begin{pmatrix} M_{ij,kl} & M_{ij,k\bar{l}} & M_{ij,\overline{kl}} & M_{ij}{}^{k\beta} & M_{ij}{}^{\bar{k}\beta} \\ M_{i\bar{j},kl} & M_{i\bar{j},k\bar{l}} & M_{i\bar{j},\overline{kl}} & M_{i\bar{j}}{}^{k\beta} & M_{i\bar{j}}{}^{\bar{k}\beta} \\ M_{\overline{ij},kl} & M_{\overline{ij},k\bar{l}} & M_{\overline{ij},\overline{kl}} & M_{\overline{ij}}{}^{k\beta} & M_{\overline{ij}}{}^{\bar{k}\beta} \\ M^{i\alpha}{}_{kl} & M^{i\alpha}{}_{k\bar{l}} & M^{i\alpha}{}_{\overline{kl}} & M^{i\alpha,k\beta} & M^{i\alpha,\bar{k}\beta} \\ M^{\bar{i}\alpha}{}_{kl} & M^{\bar{i}\alpha}{}_{k\bar{l}} & M^{\bar{i}\alpha}{}_{\overline{kl}} & M^{\bar{i}\alpha,k\beta} & M^{\bar{i}\alpha,\bar{k}\beta} \end{pmatrix}, \tag{8.15}$$

with its non-vanishing entries given in terms of the SL(2) vector $v^\alpha \equiv (1, -i)^\alpha$ by

$$M_{ij,kl} = -\delta_{ij}{}^{kl}\, e^{-i\,(\varphi+\omega)} \sinh\left(\frac{2m}{\sqrt{3}}\right) \sinh(2\sigma)\ , \tag{8.16}$$

$$M_{ij,\overline{kl}} = \delta_{ij}{}^{\overline{kl}} \cosh\left(\frac{2m}{\sqrt{3}}\right) \cosh(2\sigma)\ , \tag{8.17}$$

$$M_{ij}{}^{k\alpha} = \frac{1}{2}\,\varepsilon^{ijk}\, e^{-i\varphi} \sinh\left(\frac{2m}{\sqrt{3}}\right) \cosh(2\sigma)\ v^\alpha\ , \tag{8.18}$$

$$M_{\overline{ij}}{}^{k\alpha} = -\frac{1}{2}\,\varepsilon^{ijk}\, e^{i\omega} \cosh\left(\frac{2m}{\sqrt{3}}\right) \sinh(2\sigma)\ v^\alpha\ , \tag{8.19}$$

$$M_{i\bar{j}}{}^{\bar{k}\alpha} = \frac{1}{4}\,\varepsilon^{ijk}\, e^{-i\varphi} \sinh\left(\frac{4m}{\sqrt{3}}\right)\ v^\alpha\ , \tag{8.20}$$

$$M^{i\alpha,j\beta} = \frac{1}{2}\, e^{2i\varphi} \sinh^2\left(\frac{2m}{\sqrt{3}}\right) (v^\alpha v^\beta)^* - \frac{1}{2}\, e^{i(\omega-\varphi)} \sinh\left(\frac{2m}{\sqrt{3}}\right) \sinh(2\sigma)\, v^\alpha v^\beta\ , \tag{8.21}$$

$$M^{i\alpha,\bar{j}\beta} = \frac{1}{2} \cosh^2\left(\frac{2m}{\sqrt{3}}\right) (v^\alpha)^*\, v^\beta + \frac{1}{2} \cosh\left(\frac{2m}{\sqrt{3}}\right) \cosh(2\sigma)\, v^\alpha\, (v^\beta)^*\ , \tag{8.22}$$

together with those components related by complex conjugation. Plugging this explicit form of the scalar matrix into the uplift formulas of Section 8.1 yields the IIB uplift of the 4-scalar truncation of the $D = 5$ theory which we will describe in Section 9.2.

Chapter 9

Uplift of GPPZ Solution

9.1 The GPPZ Flow

The GPPZ flow [23] is a solution of the $\mathcal{N} = 8$ SO(6) gauged supergravity in five dimensions [44, 94]. The field content of the theory can be organised in representations of the $\mathrm{SO}(6) \times \mathrm{SL}(2)$ subgroup of $\mathrm{E}_{6(6)}$. It consists of 15 massless vectors fields in the adjoint of the SO(6), 12 topologically massive 2-forms transforming in the $(\mathbf{6}, \mathbf{2})$ of $\mathrm{SO}(6) \times \mathrm{SL}(2)$ and 42 scalars transforming as

$$\mathbf{42} = \mathbf{20'}_{(0)} + \mathbf{10}_{(-2)} + \overline{\mathbf{10}}_{(2)} + \mathbf{1}_{(4)} + \mathbf{1}_{(-4)} \, , \tag{9.1}$$

where the subscripts are the charges under the SO(2) subgroup of the SL(2). The masses of these scalars are $m^2 = -4$ for the $\mathbf{20'}$, $m^2 = -3$ for the $\mathbf{10}$ and $\overline{\mathbf{10}}$ and $m^2 = 0$ for the two $\mathbf{1}$s.

According to the AdS/CFT dictionary, the 42 scalars are dual to relevant and marginal operators[1] of the $\mathcal{N} = 4$ SYM. The $\mathcal{N} = 4$ SYM contains six scalars ϕ^I and four fermions λ_a transforming in the $\mathbf{6}$ and $\mathbf{4}$ of the SU(4), respectively. Then the scalars in the $\mathbf{20'}$ correspond to scalar bilinears ($\mathrm{tr}\, \phi^{(I} \phi^{J)} -$ traces) of conformal dimension $\Delta = 2$, the scalars in the $\mathbf{10}$ are massive deformations ($\Delta = 3$), schematically

$$Q^2 \, \mathrm{tr}\, \phi^{(I} \phi^{J)} \sim \mathrm{tr}(\lambda_a \lambda_b + \phi^3) \tag{9.2}$$

and the scalars in the $\mathbf{1}$ are the gauge-coupling deformations.

We are interested in massive deformations of four-dimensional $\mathcal{N} = 4$ SU(N) SYM that break supersymmetry to $\mathcal{N} = 1$. In $\mathcal{N} = 1$ notation the six scalar fields ϕ^I are arranged

[1] Recall that the mass of an AdS scalar is mapped to the conformal dimension of the dual field theory operator by $\Delta = 2 + \sqrt{4 + m^2}$.

in three complex scalars, which together with three of the four fermions form three chiral superfields Z_i, $i = 1, 2, 3$, while the remaining fermion sits in the vector multiplet. Of the full SO(6) R-symmetry only a U(1) × SU(3) subgroup is manifest, under which the vector superfield is neutral and the three chiral superfields transform in the $\mathbf{3}_{2/3}$.

To identify the relevant $5d$ scalars we need to describe the mass deformation in more detail. The mass deformation in $\mathcal{N} = 1$ language is a change in the superpotential of the theory,

$$\delta\mathcal{W} = m^{ij}\,\mathrm{tr}(Z_i Z_j) \tag{9.3}$$

and we need to decompose the SU(4) ∼ SO(6) representations into SU(3) × U(1) and single out the gaugino. The fundamental of the SU(4) splits as

$$\mathbf{4} \to \mathbf{3} + \mathbf{1} \tag{9.4}$$

and thus the λ_a, which is in the $\mathbf{4}$ of the SU(4), splits into a $\mathbf{3}$ corresponding to the three fermions in the chiral multiplets Z_i and a $\mathbf{1}$ which is the gaugino λ. The fermionic mass term in the $\mathbf{10}$ then decomposes as

$$\mathbf{10} \to \mathbf{1} + \mathbf{3} + \mathbf{6}\,. \tag{9.5}$$

and we identify the $\mathbf{6}$ with the mass deformation, while the scalar in the $\mathbf{1}$ corresponds to a gaugino condensate. Integrating out the auxiliary fields we find that $\mathcal{N} = 4$ SYM is deformed not only by the mass term but also by a part of the $\mathbf{20}'$ (and also by the Konishi operator, which however decouples in the supergravity limit), unless the three fermion masses are taken to be equal. If the masses are equal, the part corresponding to the $\mathbf{20}'$ does not appear, and there is a residual SO(3) symmetry that allows us to keep only two holomorphic scalars, $\underline{\sigma} \in \mathbf{1}$ and $\underline{m} \in \mathbf{6}$, while setting all the remaining fields consistently to zero. These two fields are dual to the operators[2]

$$\mathcal{O}_3 = \sum_{i=1}^{3} \mathrm{tr}(\lambda_i \lambda_i), \qquad \mathcal{O}_4 = \mathrm{tr}(\lambda_4 \lambda_4). \tag{9.6}$$

Similarly, we get two anti-holomorphic scalars, $\underline{\bar{\sigma}}, \underline{\bar{m}}$ (the complex conjugates of $\underline{\sigma}$ and $\underline{m}$) from the $\overline{\mathbf{10}}$. This is as expected, since $\underline{m}$ and $\underline{\sigma}$ are dual to chiral operators.

In AdS/CFT the QFT generating functional of correlation functions becomes the on-shell value of the bulk action. Since $\underline{m}$ and $\underline{\sigma}$ couple to complex operators, the generating functional will only contain the modulus of $\underline{m}$ and $\underline{\sigma}$. Indeed, in $\mathcal{N} = 4$ SYM, $\langle \mathcal{O}_3 \mathcal{O}_3 \rangle = \langle \overline{\mathcal{O}}_3 \overline{\mathcal{O}}_3 \rangle = 0$ but $\langle \mathcal{O}_3 \overline{\mathcal{O}}_3 \rangle \neq 0$ and the same is true for $\mathcal{O}_4$, which means the generating function will

[2]These operators are obtained from the $\mathbf{20}'$ by acting with two supercharges and they contain also a part proportional to ϕ^3 that we suppress here.

depend on $|\underline{m}|^2$ and $|\underline{\sigma}|^2$ but not on $\underline{m}^2, \bar{\underline{m}}^2, \underline{\sigma}^2, \bar{\underline{\sigma}}^2$, and similarly for the contributions coming from higher point functions. Indeed, we will see in the next section that there is consistent truncation of the bulk supergravity to the modulus m and σ of $\underline{m}$ and $\underline{\sigma}$,

$$\underline{m} = m\, e^{i\varphi} \,, \qquad \underline{\sigma} = \sigma\, e^{i\omega} \,. \tag{9.7}$$

We are thus lead to looking for $5d$ solutions of the form

$$ds^2 = dy^2 + e^{2\phi(y)} dx^\mu dx_\mu \tag{9.8}$$

with $\mu = 0, \ldots, 3$ and non-trivial profile for the real fields $m(y)$ and $\sigma(y)$. The radial coordinate y ranges from $-\infty$, which corresponds to the IR, to $+\infty$, which corresponds to the UV. With this truncation the Lagrangian reduces to

$$\begin{aligned} L = \sqrt{-g} \bigg\{ &-\frac{1}{4}R + \frac{1}{2}(\partial m)^2 + \frac{1}{2}(\partial\sigma)^2 \\ &- \frac{3}{8}\left[(\cosh\frac{2m}{\sqrt{3}})^2 + 4\cosh\frac{2m}{\sqrt{3}}\cosh 2\sigma - (\cosh 2\sigma)^2 + 4 \right] \bigg\} \end{aligned} \tag{9.9}$$

Because of fake supersymmetry, the fields ϕ, m and σ satisfy the first order equations

$$\dot{\phi} = \frac{1}{2}\left[\cosh\frac{2m}{\sqrt{3}} + \cosh 2\sigma \right] , \tag{9.10}$$

$$\dot{m} = -\frac{\sqrt{3}}{2}\sinh\frac{2m}{\sqrt{3}} \,, \tag{9.11}$$

$$\dot{\sigma} = -\frac{3}{2}\sinh 2\sigma \,, \tag{9.12}$$

descending form the superpotential

$$\mathcal{W} = -\frac{3}{4}\left[\cosh\frac{2m}{\sqrt{3}} + \cosh 2\sigma \right] . \tag{9.13}$$

The solution, which is often denoted as the GPPZ flow [23], is given by

$$m(y) = \frac{\sqrt{3}}{2}\log\left[\frac{1 + e^{-(y-C_1)}}{1 - e^{-(y-C_1)}} \right] = \sqrt{3}\,\mathrm{arctanh}\, e^{-(y-C_1)} \,, \tag{9.14}$$

$$\sigma(y) = \frac{1}{2}\log\left[\frac{1 + e^{-3(y-C_2)}}{1 - e^{-3(y-C_2)}} \right] = \mathrm{arctanh}\, e^{-3(y-C_2)} \,, \tag{9.15}$$

$$\begin{aligned} \phi(y) &= y + \frac{1}{2}\log\left[1 - e^{-2(y-C_1)} \right] + \frac{1}{6}\log\left[1 - e^{-6(y-C_2)} \right] \\ &= y - \log\cosh\frac{m(y)}{\sqrt{3}} - \frac{1}{3}\log\cosh\sigma(y) \,. \end{aligned} \tag{9.16}$$

where C_1 and C_2 are two arbitrary integration constants[3].

Generically, solutions of the type (9.14) can represent both deformations of the dual field theory by an operator $\mathcal{O}$ and/or different vacua of the same theory characterised by a vacuum expectation value $\langle\mathcal{O}\rangle$. The behaviour of the solution in the asymptotic AdS region, $y \to +\infty$, discriminates between the two options. For $y \to +\infty$, the asymptotic behaviour of some field φ consists of a non-normalisable part and a normalisable one

$$\varphi \underset{y\to+\infty}{\sim} e^{(\Delta-4)y}(A+\cdots) + e^{-\Delta y}(B+\cdots)\,, \tag{9.17}$$

where Δ is the conformal dimension of the dual operator and the dots in the leading non-normalisable part are local functions of A while the dots in the normalisable part are functions of both A and B. The coefficient A of the non-normalisable solution is interpreted as a deformation of the Lagrangian while the coefficient B of the normalisable solution is related to the vacuum expectation value of the dual operator, $B = 1/(2\Delta-4)\langle\mathcal{O}\rangle$, where $\mathcal{O}$ is the operator dual to φ [82, 98].

For $y \to +\infty$, the GPPZ solution behaves as

$$\phi(y) \underset{y\to+\infty}{\sim} y \tag{9.18}$$

$$m(y) \underset{y\to+\infty}{\sim} m_0 e^{-y}\,, \qquad m_0 = \sqrt{3}e^{C_1} \tag{9.19}$$

$$\sigma(y) \underset{y\to+\infty}{\sim} \frac{1}{2}\sigma_0 e^{-3y}\,, \qquad \sigma_0 = 2e^{3C_2}\,. \tag{9.20}$$

From these asymptotics we see that, since $\Delta = 3$, m_0 corresponds to a mass deformation and $\sigma_0 = \Re\langle\lambda\lambda\rangle$ is the real part of the gaugino condensate. It is then natural to interpret the solution as a flow from the mass deformed $\mathcal{N}=4$ to $\mathcal{N}=1^*$ in the IR.

The metric has a naked singularity for $y \to C_1$ (with $y \geq C_1$),

$$ds^2 = dy^2 + a(y-C_1)dx^\mu dx_\mu + \ldots\,, \tag{9.21}$$

where $a = 2e^{C_1+C_2}(2\sinh(3(C_1-C_2)))^{1/3}$. The Ricci scalar is singular

$$R = -(y-C_1)^{-2} + \ldots \tag{9.22}$$

and there is no change of frame in which the singularity disappears or is milder. Notice that

[3]The integration constants C_i used here are identical to those used by Pilch & Warner [54], and are related to those used by GPPZ [23] by $C_1^{(\mathrm{GPPZ})} = C_1$, $C_2^{(\mathrm{GPPZ})} = 3C_2$. Also the definition of $\phi(y)$ differs between Pilch & Warner and GPPZ. Here we are using the Pilch & Warner definition, which is related to GPPZ by $\phi^{(\mathrm{GPPZ})} = \phi - \frac{C_1+C_2}{2}$. See Appendix A for more details.

also the solution for m diverges at $y = C_1$

$$m(y) = -\frac{\sqrt{3}}{2}\log(y - C_1) + \dots \tag{9.23}$$

while the behaviour of σ depends on the relation between C_1 and C_2. If $C_2 \leq C_1$ then σ is regular.

Singularities of this kind are common in most $5d$ solutions and criteria have been proposed to establish whether the solutions are physically acceptable or not. In particular in [99] it was proposed that a singular solution is physically acceptable if it can be obtained as the zero temperature limit of a regular black-hole. The conditions for the existence of the black hole solution constrain the parameter of the singular solution. In this case the criterion gives $C_2 \leq C_1$. By looking at the behaviour of Wilson loops it was shown in [23] that the solutions with $C_2 \leq C_1$ confines. Such solutions should then be dual to the confining vacua of $\mathcal{N} = 1^*$.

9.2 Uplift of the GPPZ Solution

The general uplift of the $\mathcal{N} = 8$ SO(6) gauged supergravity to type IIB was constructed in [57], and we recalled the main formulas in Chapter 8. In this section we first review the 4-scalar truncation of $D = 5$ maximal supergravity in which the GPPZ solution lives and then apply the uplift formulas to obtain the full IIB uplift of the GPPZ solution. Finally, we explicitly verify that the entire set of IIB field equations is satisfied by the ten-dimensional solution.

9.2.1 Four-Scalar Truncation of $D = 5$ Supergravity

As discussed above, an important ingredient in the construction of the GPPZ solution is the invariance under an SO(3) subgroup of the gauge group SO(6) that allows to truncate the full theory to a pair of complex scalars [23]. One can actually embed the flow in a larger theory that is obtained by truncating the $\mathcal{N} = 8$ supergravity to the full set of SO(3) invariant fields [54]. This gives an $\mathcal{N} = 2$ supergravity coupled to two hyper-multiplets. Of the 42 scalars of the $\mathcal{N} = 8$ in (9.1) we only keep the 8 singlets under the

$$\mathrm{SO}(3)_{\mathrm{diag}} \subset \mathrm{SO}(3) \times \mathrm{SO}(3) \subset \mathrm{SO}(6)\ , \tag{9.24}$$

subgroup of the gauge group SO(6) $\sim$ SU(4). These form the coset space $\mathrm{G}_{2(2)}/\mathrm{SO}(4)$ and are dual to the operators

$$\mathcal{O}_1 = \mathrm{tr}\sum_{i=1}^{3}\left(\phi_i\phi_i - \phi_{i+3}\phi_{i+3}\right), \qquad \mathcal{O}_2 = \mathrm{tr}\sum_{i=1}^{3}(\phi_i\phi_{i+3}), \tag{9.25}$$

$$\mathcal{O}_3 = \mathrm{tr}\sum_{i=1}^{3}(\lambda_i\lambda_i) \qquad \mathcal{O}_4 = \mathrm{tr}(\lambda_4\lambda_4) \tag{9.26}$$

$$\mathcal{O}_5 = \mathrm{tr}(F^+F^+), \qquad \mathcal{O}_6 = \mathrm{tr}(F^-F^-), \tag{9.27}$$

where $F^{\pm}$ are the self-dual and anti-self-dual field strengths. $\mathcal{O}_1$ and $\mathcal{O}_2$ are the $\mathrm{SO}(3)_{\mathrm{diag}}$ singlets contained in the $\mathbf{20'}$, the complex operators $\mathcal{O}_3$ and $\mathcal{O}_4$ are the $\mathrm{SO}(3)_{\mathrm{diag}}$ singlets in the $\mathbf{10}$ and the $\overline{\mathbf{10}}$, and $\mathcal{O}_5$ and $\mathcal{O}_6$ correspond to the two $\mathbf{1}$s. Among the $\mathrm{SU}(4)_{\mathrm{R}}$ gauge fields, the truncation to singlets under (9.24) only keeps a single U(1) gauge field, dual to the $\mathrm{U}(1)_{\mathrm{R}}$ subgroup of $\mathrm{SU}(4)_{\mathrm{R}} \to \mathrm{SU}(3) \times \mathrm{U}(1)_{\mathrm{R}}$.[4]

The further truncation to the 2 complex scalars dual to $\mathcal{O}_3$ and $\mathcal{O}_4$ can also been shown to be consistent as it corresponds to the truncation to singlets under an additional discrete subgroup within $\mathrm{U}(1)_{\mathrm{R}} \times \mathrm{SL}(2)$, the latter being the global symmetry of the $\mathcal{N} = 8$ gauged supergravity, see [54] for details. We parametrise these scalars as

$$\underline{m} = m\, e^{i\varphi}\,, \qquad \underline{\sigma} = \sigma\, e^{i\omega}\,. \tag{9.28}$$

The five-dimensional theory [45] then reduces to

$$\frac{1}{\sqrt{g}}\,\mathcal{L} = -\frac{1}{4}\,R - \frac{1}{12}\,F_{\mu\nu}\,F^{\mu\nu} - \frac{1}{54}\,\epsilon^{\mu\nu\rho\sigma\tau}\,A_\mu F_{\nu\rho}F_{\sigma\tau} + \frac{1}{2}\,\partial_\mu m\,\partial^\mu m + \frac{1}{2}\,\partial_\mu\sigma\,\partial^\mu\sigma + \frac{3}{8}\,\sinh^2\left(\frac{2m}{\sqrt{3}}\right) D_\mu\varphi D^\mu\varphi + \frac{1}{8}\,\sinh^2(2\sigma)\,D_\mu\omega D^\mu\omega - V_{\mathrm{pot}}\,, \tag{9.29}$$

with the Maxwell and Chern-Simons terms of minimal supergravity, covariant derivatives

$$D_\mu\omega = \partial_\mu\omega + 2\,A_\mu\,, \qquad D_\mu\varphi = \partial_\mu\varphi + \frac{2}{3}\,A_\mu\,, \tag{9.30}$$

and the scalar potential

$$V_{\mathrm{pot}} = -\frac{3}{8}\left(4\cosh\left(\frac{2m}{\sqrt{3}}\right)\cosh(2\sigma) + \cosh^2\left(\frac{2m}{\sqrt{3}}\right) - \cosh^2(2\sigma) + 4\right)\,, \tag{9.31}$$

only depending on the absolute values of the complex scalars. The scalar kinetic term is an $(\mathrm{SU}(1,1)/\mathrm{U}(1))^2$ coset space, and the covariant derivatives (9.30) correspond to the gauging of $\mathrm{U}(1)_{\mathrm{R}}$ which is realised as a linear combination of the two U(1)s.

[4] We normalise the U(1) so that the charges are those of the QFT, see the discussion around (6.88) in [83].

Note that the angles φ, ω source the Maxwell equation

$$\begin{aligned}\nabla_\nu F^{\mu\nu} + \frac{1}{6}\,\epsilon^{\mu\nu\rho\sigma\tau}\,F_{\nu\rho}F_{\sigma\tau} &= \frac{3}{2}\,\sinh^2\left(\frac{2m}{\sqrt{3}}\right) D_\mu\varphi + \frac{3}{2}\,\sinh^2(2\sigma)\,D_\mu\omega \\ &\equiv \frac{3}{2}\,\mathcal{J}^\mu\ .\end{aligned} \tag{9.32}$$

Setting the vector field to zero thus implies the constraint $\mathcal{J}_\mu \equiv 0$ among the derivatives of the angles ω and φ.

The GPPZ solution (9.14) lives in the further (consistent) truncation of (9.29) to two real scalars, i.e. setting angles and the vector field to zero. For the uplift formulas we will also employ the variables [54]

$$\mu \equiv e^\sigma\ , \qquad \nu = e^{m/\sqrt{3}}\ , \tag{9.33}$$

in terms of which the flow equations (9.10) take the form

$$\dot{\mu} = \frac{3}{4\mu}\,(1-\mu^4)\ , \tag{9.34}$$

$$\dot{\nu} = \frac{1}{4\nu}\,(1-\nu^4)\ , \tag{9.35}$$

$$\dot{\phi} = \frac{1}{4\mu^2}\,(1+\mu^4) + \frac{1}{4\nu^2}\,(1+\nu^4)\ . \tag{9.36}$$

Employing the $\mathrm{U}(1)_\mathrm{R}$ gauge symmetry, we can also give a version of this solution with non-vanishing angles by setting

$$\omega = -\lambda\ , \qquad \varphi = -\frac{1}{3}\,\lambda\ , \qquad A_\mu = \frac{1}{2}\,\partial_\mu\lambda\ , \tag{9.37}$$

for some function λ.

9.2.2 Uplift of the Four-Scalar Truncation: Metric and Axion-Dilaton

In order to apply the explicit uplift formulas for the uplift given in Chapter 8, we first evaluate the matrix (8.1) for the four-scalar truncation $(\mu, \nu, \varphi, \omega)$ by exponentiating the associated generators within the group $\mathrm{E}_{6(6)}$. Since all scalars are SO(3) singlets according to (9.24) it proves useful to decompose the S^5 sphere harmonics $\mathcal{Y}^a$ into

$$\mathcal{Y}^a \longrightarrow \{u^i, v^i\}\ , \tag{9.38}$$

with $u^i u^i + v^i v^i = 1$. Moreover, for the compactness of notation, it is useful to define the rotated functions

$$U^i = \cos\left(\frac{1}{4}(\varphi+\omega)\right)\,u^i + \sin\left(\frac{1}{4}(\varphi+\omega)\right)\,v^i\ , \tag{9.39}$$

$$V^i = -\sin\left(\frac{1}{4}(\varphi+\omega)\right)\, u^i + \cos\left(\frac{1}{4}(\varphi+\omega)\right)\, v^i\ , \tag{9.40}$$

where φ and ω are the x-dependent phases of the scalars m and σ of the $D = 5$ theory defined in equation (9.28). This transformation is a local $\mathrm{U}(1)_\mathrm{R}$ transformation from the perspective of the dual QFT. Similarly, we define the rotated 1-forms

$$\Theta^i = \cos\left(\frac{1}{4}(\varphi+\omega)\right)\, Du^i + \sin\left(\frac{1}{4}(\varphi+\omega)\right)\, Dv^i\ , \tag{9.41}$$

$$\Lambda^i = -\sin\left(\frac{1}{4}(\varphi+\omega)\right)\, Du^i + \cos\left(\frac{1}{4}(\varphi+\omega)\right)\, Dv^i\ , \tag{9.42}$$

where the covariant derivatives

$$Du^i \equiv du^i - \frac{1}{3}\, v^i A_\mu\, dx^\mu\ , \tag{9.43}$$

$$Dv^i \equiv dv^i + \frac{1}{3}\, u^i A_\mu\, dx^\mu\ , \tag{9.44}$$

correspond to the Kaluza-Klein basis (8.13). Let us also note that the proper identification of the U(1) vector field A_μ among the 15 SO(6) fields ${A_\mu}^{ab}$ gives rise to the relations

$${F_{\mu\nu}}^{ab}\, M_{ab,cd}\, F^{\mu\nu\, cd} = \frac{4}{3}\, F_{\mu\nu}F^{\mu\nu}\ , \tag{9.45}$$

$$\varepsilon_{abcdef}\, F^{ab}_{\mu\nu}F^{cd}_{\rho\sigma}A^{ef}_{\tau} = -\frac{32\sqrt{2}}{9}\, F_{\mu\nu}F_{\rho\sigma}A_\tau\ . \tag{9.46}$$

We can now give the fields of the uplifted solution. The IIB metric (8.7) takes the explicit form

$$ds^2_{\mathrm{IIB}} = \Delta^{-2/3}\left(g_{\mu\nu}(x)\, dx^\mu dx^\nu + \Delta^{8/3}\, d\hat{s}^2_5\right) \tag{9.47}$$

with the warp factor Δ and the internal metric $d\hat{s}^2_5$ given by

$$\begin{aligned}\Delta^{-8/3} = {}& \frac{(1+\mu^2\nu^2)^3\,(\mu^2+\nu^6)}{16\,\mu^4\nu^6} + \frac{U^2\,V^2}{16\,\mu^4\nu^8}\left(1-\nu^4\right)^2\left(\mu^2-\nu^2\right)^2\left(1+\mu^2\nu^2\right)^2 \\ & - \frac{(U\cdot V)^2}{16\,\mu^4\nu^8}\left(1-\nu^4\right)^2(1-\mu^2\nu^2)^2\left(\mu^2+\nu^2\right)^2\ ,\end{aligned} \tag{9.48}$$

$$\begin{aligned} d\hat{s}^2_5 = {}& \frac{(1+\nu^4)\,(\mu^2+\nu^2)\,(1+\mu^2\nu^2)}{8\,\mu^2\nu^4}\left(\Theta^i\Theta^i + \Lambda^i\Lambda^i\right) \\ & - \frac{\left(1-\nu^4\right)^2}{8\,\nu^4}\left((U^2-V^2)\,(\Theta^i\Theta^i - \Lambda^i\Lambda^i) + 4\,(U\cdot V)\,\Theta^i\Lambda^i\right) \\ & + \frac{(1-\mu^4)\,(1-\nu^4)}{8\,\mu^2\nu^2}\left((U^2-V^2)\,(\Theta^i\Theta^i - \Lambda^i\Lambda^i) - 4\,(U\cdot V)\,\Theta^i\Lambda^i\right) \\ & + \frac{(1-\mu^4\nu^4)\,(1-\mu^2\nu^2)\,(\mu^2+\nu^6)}{16\,\mu^4\nu^6}\left((V^i\Theta^i)(V^j\Theta^j) + (U^i\Lambda^i)(U^j\Lambda^j)\right) \\ & + \frac{(1-\mu^4\nu^4)\,(1+\mu^2\nu^2)\,(\mu^2-\nu^6)}{8\,\mu^4\nu^6}\,(V^i\Theta^i)(U^j\Lambda^j) \end{aligned}$$

$$- \frac{(\mu^4 - \nu^4)(1-\nu^8)}{4\,\mu^2\nu^6}\,(U^i\Theta^i)(V^j\Lambda^j)\;, \tag{9.49}$$

respectively. For vanishing angles (i.e. $U^i = u^i$, $V^i = v^i$, $\Theta^i = du^i$, $\Lambda^i = dv^i$) we recover the result from [54][5]. It is important to note that the only singularities of the IIB metric can be located at $\mu, \nu = 0$ or $\mu, \nu = \infty$. Indeed, the warp factor (9.48) can be estimated using that $U^2\,V^2 \geq (U\cdot V)^2$, and $U^2\,V^2 = U^2\,(1-U^2) \leq \frac{1}{4}$ to be

$$\Delta^{-8/3} \geq \frac{\left(1+\mu^2\nu^2\right)^3\left(\mu^2+\nu^6\right)}{16\,\mu^4\nu^6} - \frac{U^2\,V^2}{4\,\mu^2\nu^6}\left(1-\mu^4\right)\left(1-\nu^4\right)^3 \tag{9.50}$$

$$\geq \frac{\left(1+\mu^2\nu^2\right)^3\left(\mu^2+\nu^6\right)}{16\,\mu^4\nu^6} - \frac{1}{16\,\mu^2\nu^6}\left(1-\mu^4\right)\left(1-\nu^4\right)^3 \tag{9.51}$$

$$= \frac{\left(\mu^2+\nu^2\right)^3\left(1+\mu^2\nu^6\right)}{16\,\mu^4\nu^6} \;>\; 0\;. \tag{9.52}$$

We will take a closer look at the possible singularities in Section 13.2. For the symmetric SL(2) dilaton/axion matrix $m_{\alpha\beta}$,

$$m_{\alpha\beta} = \frac{1}{\Im\tau}\begin{pmatrix} |\tau|^2 & -\Re\tau \\ -\Re\tau & 1 \end{pmatrix}\;, \qquad \tau = C_0 + ie^{-\Phi}\;, \tag{9.53}$$

the uplift formula (8.10) yields

$$m_{\alpha\beta} = \Delta^{4/3}\,\mathcal{S}_\alpha{}^a\mathcal{S}_\beta{}^b\,m_{ab}\;, \tag{9.54}$$

where $\mathcal{S}$ is an SO(2) rotation matrix parametrised by

$$\mathcal{S} = \begin{pmatrix} \cos\left(\frac{3}{4}\,\varphi - \frac{1}{4}\,\omega\right) & \sin\left(\frac{3}{4}\,\varphi - \frac{1}{4}\,\omega\right) \\ -\sin\left(\frac{3}{4}\,\varphi - \frac{1}{4}\,\omega\right) & \cos\left(\frac{3}{4}\,\varphi - \frac{1}{4}\,\omega\right) \end{pmatrix}\;, \tag{9.55}$$

and m_{ab} is a GL(2) matrix with entries

$$m_{11} = \frac{1+\mu^2\nu^2}{8\,\mu^2\nu^4}\Big(\left(1+\nu^4\right)\left(\mu^2+\nu^2\right) + \left(1-\nu^4\right)\left(\mu^2-\nu^2\right)\left(U^2-V^2\right)\Big)\;, \tag{9.56}$$

$$m_{12} = \frac{(1-\mu^2\nu^2)\,(1-\nu^4)\,(\mu^2+\nu^2)}{4\,\mu^2\nu^4}\,(U\cdot V)\;, \tag{9.57}$$

$$m_{22} = \frac{1+\mu^2\nu^2}{8\,\mu^2\nu^4}\Big(\left(1+\nu^4\right)\left(\mu^2+\nu^2\right) - \left(1-\nu^4\right)\left(\mu^2-\nu^2\right)\left(U^2-V^2\right)\Big)\;. \tag{9.58}$$

It is straightforward to check that the determinant of m_{ab} is given by $\Delta^{-8/3}$ (9.48) as required in order to have $m_{\alpha\beta} \in \mathrm{SL}(2)$. Again, for vanishing angles we recover the result from [54].

[5] We corrected a typo in [54] in the form of index contractions of the pen-ultimate term in (9.49).

It is remarkable that the dependence of the IIB uplift on the a priori x-dependent $5D$ angles (φ, ω) is entirely captured by a rotation of the internal coordinates (9.40) and an $\mathrm{SO}(2) \subset \mathrm{SL}(2)_{\mathrm{IIB}}$ rotation (9.55). We will see in the following that this feature persists for the full IIB uplift. In particular the 2π-periodicity of the $5D$ theory implies that the IIB uplift is invariant under the combination of an exchange $U^i \leftrightarrow V^i$ with a constant SL(2) rotation (9.55) with $\mathcal{S} = -i\sigma_2$, as is easily verified for (9.49) and (9.58).

9.2.3 Uplift of the Four-Scalar Truncation: *p*-Forms

We now evaluate the uplift formulas in (8.11) on the four scalar truncation in order to derive the IIB p-forms. For the 2-form doublet, we find

$$C_{mn\,\alpha} = \Delta^{8/3}\,\mathcal{S}_\alpha{}^a\,C_{mn\,a}\ , \tag{9.59}$$

where $\mathcal{S}$ is the SO(2) rotation matrix in (9.55), and the 2-form C_a are

$$\begin{aligned} C_1 = b_1\,\varepsilon^{ijk}\,\Big(&\Big(1-\mu^2\nu^2\Big)\Big(\mu^2+\nu^6\Big)\,V^i\,\Theta^j\wedge\Theta^k+\Big(1+\mu^2\nu^2\Big)\Big(\mu^2-\nu^6\Big)\,V^i\,\Lambda^j\wedge\Lambda^k \\ &+2\,\nu^2\Big(1-\mu^4\nu^4\Big)\,U^i\,\Theta^j\wedge\Lambda^k\Big) \\ +\,b_2\,\varepsilon^{ijk}\,\Big(&\Big(1+\mu^2\nu^2\Big)\Big(\mu^2-\nu^6\Big)\,U^i\,\Theta^j\wedge\Theta^k+\Big(1-\mu^2\nu^2\Big)\Big(\mu^2+\nu^6\Big)\,U^i\,\Lambda^j\wedge\Lambda^k \\ &+2\,\nu^2\Big(1-\mu^4\nu^4\Big)\,V^i\,\Theta^j\wedge\Lambda^k\Big)\ , \end{aligned} \tag{9.60}$$

$$C_2 = -C_1\Big|_{U^i\leftrightarrow V^i,\,\Theta^i\leftrightarrow\Lambda^i}\ , \tag{9.61}$$

with the functions

$$b_1 = -\frac{1+\mu^2\nu^2}{64\,\mu^4\nu^8}\,\Big(\Big(1+\nu^4\Big)\Big(\mu^2+\nu^2\Big)+\Big(1-\nu^4\Big)\Big(\mu^2-\nu^2\Big)\Big(U^2-V^2\Big)\Big)\ , \tag{9.62}$$

$$b_2 = \frac{1-\mu^2\nu^2}{32\,\mu^4\nu^8}\,\Big(\Big(1-\nu^4\Big)\Big(\mu^2+\nu^2\Big)\,(U\cdot V)\Big)\ . \tag{9.63}$$

Again, the dependence on the $5D$ angles (φ, ω) is entirely captured by the SO(2) rotation (9.55) and the rotated basis (9.40). The internal component of the 4-form potential takes the form

$$\begin{aligned} C = \mathring{C} &+ \frac{1}{4!}\,\Delta^{8/3}\,\varepsilon_{ijm}\varepsilon_{kln}\,\Big(f_1\,(U^mU^n-V^mV^n)+2\,f_2\,U^{(m}V^{n)}\Big)\,\Theta^i\wedge\Theta^j\wedge\Lambda^k\wedge\Lambda^l \\ &+\frac{1}{4!}\,\Delta^{8/3}\,f_3\,\Theta^i\wedge\Theta^j\wedge\Lambda^i\wedge\Lambda^j\ , \end{aligned} \tag{9.64}$$

where the background field $\mathring{C}$ is given in (8.5) and

$$f_1 = -\frac{3\,(U\cdot V)}{32\,\mu^4\nu^8}\,\Big(1-\nu^4\Big)^2\Big(\mu^2+\nu^2\Big)^2\Big(1-\mu^2\nu^2\Big)^2\ , \tag{9.65}$$

$$f_2 = \frac{3\,(U^2 - V^2)}{64\,\mu^4\nu^8}\left(1-\nu^4\right)^2\left(\mu^2-\nu^2\right)^2\left(1+\mu^2\nu^2\right)^2\ , \tag{9.66}$$

$$f_3 = \frac{3\,(U\cdot V)\,(U^2 - V^2)}{8\,\mu^2\nu^6}\left(1-\nu^4\right)^3\left(1-\mu^4\right)\ . \tag{9.67}$$

Finally, the external component $C_{\mu\nu\rho\sigma}$ is determined by integrating the y-independent function[6]

$$5\,\partial_{[\mu}\,C_{\nu\rho\sigma\tau]} = -\frac{1}{3}\,\omega_{\mu\nu\rho\sigma\tau}\left(V_{\rm pot} - \frac{1}{6}\,F_{\kappa\lambda}\,F^{\kappa\lambda}\right) - \frac{20\sqrt{2}}{9}\,F_{[\mu\nu}F_{\rho\sigma}A_{\tau]}\ , \tag{9.68}$$

with the scalar potential $V_{\rm pot}$ from equation (9.31).

9.2.4 Five-Form Field Strength and Self-Duality Equations

As a first consistency check, we compute the IIB 5-form field strength

$$F_{\hat\mu_1\ldots\hat\mu_5} \equiv 5\,\partial_{[\hat\mu_1}C_{\hat\mu_2\ldots\hat\mu_5]} - \frac{15}{4}\,\varepsilon_{\alpha\beta}\,C_{[\hat\mu_1\hat\mu_2}{}^{\alpha}\partial_{\hat\mu_3}C_{\hat\mu_4\hat\mu_5]}{}^{\beta}\ , \tag{9.69}$$

and verify that it satisfies the first order self-duality equations

$$F = \star F\ . \tag{9.70}$$

Here and in the following, indices $\hat\mu$ refer to the ten-dimensional coordinates, split as $\{x^{\hat\mu}\} = \{x^\mu, y^m\}$. After some computation we find that the internal components $F_{m_1\ldots m_5}$ calculated from the above expressions for $C_{mn}{}^{\alpha}$ and C_{klmn}, take the compact form

$$F_{m_1\ldots m_5} = -\frac{1}{3}\,\Delta^{8/3}\,\mathring{\omega}_{m_1\ldots m_5}\,V_{\rm pot}\ , \tag{9.71}$$

with $V_{\rm pot}$ the scalar potential in (9.31). The external component of the 5-form field strength $F_{\mu\nu\rho\sigma\tau}$ is computed from (9.68) and (9.69) as

$$F_{\mu\nu\rho\sigma\tau} = 5\,\partial_{[\mu}C_{\nu\rho\sigma\tau]} - 10\,F_{[\mu\nu}{}^{m}C_{\rho\sigma\tau]m} \tag{9.72}$$

$$= -\frac{1}{3}\,\omega_{\mu\nu\rho\sigma\tau}\left(V_{\rm pot} - \frac{1}{12}\,F_{\kappa\lambda}\,F^{\kappa\lambda}\right) - \frac{2}{9}\,F_{[\mu\nu}F_{\rho\sigma}A_{\tau]}$$
$$+\ \frac{5}{16}\,F_{[\mu\nu}{}^{ga}\mathcal{Y}^g\mathcal{Y}^b\left(4\,\omega_{\rho\sigma\tau]\kappa\lambda}\,M_{ab,cd}F^{\kappa\lambda\,cd} + 3\sqrt{2}\,\varepsilon_{abcdef}\,F_{\rho\sigma}{}^{cd}A_{\tau]}{}^{ef}\right) \tag{9.73}$$

$$= -\frac{1}{3}\,\omega_{\mu\nu\rho\sigma\tau}\,V_{\rm pot}\ , \tag{9.74}$$

[6] Here, and in the following, we use the notation $\omega_{\mu\nu\rho\sigma\tau} = \sqrt{|g|}\,\varepsilon_{\mu\nu\rho\sigma\tau}$ for the 5D volume form.

upon using (9.46). Comparing to (9.71), we find that the 5-form we have is indeed self-dual. Similar calculations lead to the other components of the 5-form

$$F_{\mu\, m_1\dots m_4}\, Dy^{m_1}\wedge Dy^{m_2}\wedge Dy^{m_3}\wedge Dy^{m_4} = 3\,\Delta\, \star_5\Big(U^i\Lambda^i - V^i\Theta^i\Big)\; \mathcal{J}_\mu\;, \tag{9.75}$$

where the current $\mathcal{J}_\mu$ is defined in (9.32) and the 5D Hodge dual $\star_5$ gives explicitly

$$\begin{aligned} 2\,\Delta^{-5/3}\,\star_5\Big(U^i\Lambda^i - V^i\Theta^i\Big) &= \Theta^i\wedge\Theta^j\wedge\Lambda^i\wedge\Lambda^j \\ &\quad + \frac{(1-\nu^4)^2}{\nu^4}\,\Big(U^iU^j + V^iV^j\Big)\,\Theta^i\wedge\Theta^k\wedge\Lambda^j\wedge\Lambda^k\;. \end{aligned} \tag{9.76}$$

Comparing to the result for

$$F_{\mu\nu\rho\sigma m}Dy^m = \frac{1}{12}\,\omega_{\mu\nu\rho\sigma\lambda}\,\Big(D_\kappa F^{\lambda\kappa} + \frac{1}{6}\,\omega^{\lambda\nu_1\dots\nu_4}\,F_{\nu_1\nu_2}F_{\nu_3\nu_4}\Big)\,\Big(U^i\Lambda^i - V^i\Theta^i\Big)\;, \tag{9.77}$$

shows explicitly that the IIB self-duality equations (9.70) reduce to the Maxwell equations of the $D=5$ theory (9.32). For the remaining components, we obtain after some calculation

$$F_{\mu\nu\rho mn} = -\frac{1}{12}\,\omega_{\mu\nu\rho\sigma\tau}\,F^{\sigma\tau}\,(\Theta^i\wedge\Lambda^i)_{mn}\;, \tag{9.78}$$

$$F_{\mu\nu kmn} = -\frac{\Delta^{1/3}}{12}\,F_{\mu\nu}\,\star_5\Big(\Theta^i\wedge\Lambda^i\Big)_{kmn}\;, \tag{9.79}$$

again in accordance with the self-duality of the IIB field strength.

9.2.5 Dual Six-Forms

For an explicit check of the remaining field equations, we further truncate down to two real scalar fields, i.e. we assume constant angles and set the vector field to zero, so that in particular the IIB metric is block diagonal. This is precisely compatible with the GPPZ solution (9.14). The ten-dimensional IIB 3-form field equations take the form

$$\nabla_{\hat\rho}\Big(m^{\alpha\beta}F^{\hat\mu\hat\nu\hat\rho}{}_\beta\Big) = -\frac{2}{3}\,\varepsilon^{\alpha\beta}F^{\hat\mu\hat\nu\hat\kappa\hat\lambda\hat\rho}F_{\hat\kappa\hat\lambda\hat\rho\,\beta}\;, \tag{9.80}$$

and we have explicitly checked that they are verified if the five-dimensional scalar fields satisfy five-dimensional equations of motion induced by (9.29). Rather than going through the details of this calculation, let us give an equivalent consistency check by extracting the dual 6-forms in ten dimensions. The field equations (9.80) may be rewritten as the Bianchi identities

$$\partial_{[\hat\rho_1}F_{\hat\rho_2\dots\hat\rho_8]}{}^\alpha = 28\,\varepsilon^{\alpha\beta}\,F_{[\hat\rho_1\dots\hat\rho_5}F_{\hat\rho_6\hat\rho_7\hat\rho_8]\,\beta} \tag{9.81}$$

for the dual 7-form field strength $F_{\hat{\rho}_1\ldots\hat{\rho}_7}{}^{\alpha}$ defined by

$$F_{\hat{\rho}_1\ldots\hat{\rho}_7}{}^{\alpha} \equiv \frac{1}{6}\sqrt{|G|}\,\varepsilon_{\hat{\rho}_1\ldots\hat{\rho}_7\hat{\mu}\hat{\nu}\hat{\rho}}\, m^{\alpha\beta}\, F^{\hat{\mu}\hat{\nu}\hat{\rho}}{}_{\beta}\ . \tag{9.82}$$

The Bianchi identities (9.81) may then be integrated to

$$\begin{aligned} F_{\hat{\rho}_1\ldots\hat{\rho}_7}{}^{\alpha} = 7\,\partial_{[\hat{\rho}_1} C_{\hat{\rho}_2\ldots\hat{\rho}_7]}{}^{\alpha} - 84\,\varepsilon^{\alpha\beta}\, C_{[\hat{\rho}_1\hat{\rho}_2\,\beta} F_{\hat{\rho}_3\ldots\hat{\rho}_7]} \\ - 70\,\varepsilon^{\alpha\beta}\,\varepsilon^{\gamma\delta}\, C_{[\hat{\rho}_1\hat{\rho}_2\,\beta} C_{\hat{\rho}_3\hat{\rho}_4\,\gamma} F_{\hat{\rho}_5\hat{\rho}_6\hat{\rho}_7]\,\delta}\ , \end{aligned} \tag{9.83}$$

in terms of the dual 6-form gauge potential $C_{\hat{\rho}_1\ldots\hat{\rho}_6}{}^{\alpha}$. With the above explicit expressions for the IIB gauge potentials (9.61) and field strengths (9.71), (9.74) of our ten-dimensional solution, we find that equations (9.82) can be explicitly integrated to give the following non-vanishing components of the 6-form

$$C_{\mu\nu\rho\sigma\tau\, m}{}^{\alpha} = \omega_{\mu\nu\rho\sigma\tau}\, \Xi_m{}^{\alpha} \tag{9.84}$$

$$C_{\mu\nu\rho\sigma,\, mn}{}^{\alpha} = \omega_{\mu\nu\rho\sigma\tau}\, g^{\tau\lambda}\, \Xi_{\lambda\, mn}{}^{\alpha}\ ,\ , \tag{9.85}$$

which are given in terms of the following 1-forms and 2-forms

$$\Xi^1 = -\frac{(1-\mu^2\nu^2)\left((\mu^4-\nu^4)(1-\mu^2\nu^2) + 2\,\mu^2\nu^2\,(1+\mu^2\nu^2)\right)}{8\,\mu^4\,\nu^4}\,\varepsilon_{ijk}\, U^i\, V^j\, \Lambda^k\ , \tag{9.86}$$

$$\Xi^2 = \Xi^1\Big|_{U\leftrightarrow V, \Theta\leftrightarrow\Lambda}\ , \tag{9.87}$$

$$\Xi_\lambda{}^1 = \varepsilon_{ijk}\left(\frac{\partial_\lambda \nu}{\nu} - \frac{\partial_\lambda \mu}{3\mu}\right) U^i\, \Theta^j \wedge \Theta^k + \varepsilon_{ijk}\left(\frac{\partial_\lambda \mu}{\mu} + \frac{\partial_\lambda \nu}{\nu}\right) U^i\, \Lambda^j \wedge \Lambda^k\ , \tag{9.88}$$

$$\Xi_\lambda{}^2 = \Xi_\lambda{}^1\Big|_{U\leftrightarrow V, \Theta\leftrightarrow\Lambda}\ . \tag{9.89}$$

9.2.6 Einstein Equations

It remains to check the dilaton/axion equations and the Einstein equations. In our IIB conventions, these read

$$\nabla_{\hat{\mu}}\left(m^{\beta\gamma}\partial^{\hat{\mu}} m_{\alpha\gamma}\right) = -\frac{1}{6}\, m^{\beta\gamma} F_{\hat{\mu}_1\hat{\mu}_2\hat{\mu}_3\,\gamma} F^{\hat{\mu}_1\hat{\mu}_2\hat{\mu}_3}{}_{\alpha} + \frac{1}{12}\,\delta_\alpha{}^\beta\, m^{\gamma\delta} F_{\hat{\mu}_1\hat{\mu}_2\hat{\mu}_3\,\gamma} F^{\hat{\mu}_1\hat{\mu}_2\hat{\mu}_3}{}_{\delta}\ , \tag{9.90}$$

and

$$\begin{aligned} R_{\hat{\mu}\hat{\nu}} - \frac{1}{2}\, G_{\hat{\mu}\hat{\nu}}\, R = \frac{1}{6}\, F_{\hat{\mu}\hat{\kappa}\hat{\lambda}\hat{\sigma}\hat{\tau}} F_{\hat{\nu}}{}^{\hat{\kappa}\hat{\lambda}\hat{\sigma}\hat{\tau}} + \frac{1}{4}\, F_{\hat{\mu}\hat{\sigma}\hat{\tau}}{}^{\alpha} F_{\hat{\nu}}{}^{\hat{\sigma}\hat{\tau}\,\beta} m_{\alpha\beta} - \frac{1}{24}\, G_{\hat{\mu}\hat{\nu}}\, F_{\hat{\rho}\hat{\sigma}\hat{\tau}}{}^{\alpha} F^{\hat{\rho}\hat{\sigma}\hat{\tau}\,\beta} m_{\alpha\beta} \\ - \frac{1}{4}\,\partial_{\hat{\mu}} m_{\alpha\beta}\partial_{\hat{\nu}} m^{\alpha\beta} + \frac{1}{8}\, G_{\hat{\mu}\hat{\nu}}\,\partial_{\hat{\rho}} m_{\alpha\beta}\partial^{\hat{\rho}} m^{\alpha\beta}\ . \end{aligned} \tag{9.91}$$

It is a tedious computation to check that with the above expressions for the metric (9.47), the dilaton-axion matrix (9.54), gauge potentials (9.61) and field strengths (9.71), (9.74), these field equations are indeed satisfied. We have explicitly verified all components of these equations using Mathematica [81]. Let us just note that the contribution from the 5-form field strength to the energy-momentum tensor on the right-hand side of (9.91) is simply given by

$$F_\mu{}^{\rho_1\rho_2\rho_3\rho_4}\, F_{\nu\rho_1\rho_2\rho_3\rho_4} = -\frac{8}{3}\,\Delta^{10/3}\, V_{\text{pot}}^2\, G_{\mu\nu}\ , \tag{9.92}$$

$$F_m{}^{k_1k_2k_3k_4}\, F_{nk_1k_2k_3k_4} = \frac{8}{3}\,\Delta^{10/3}\, V_{\text{pot}}^2\, G_{mn}\ . \tag{9.93}$$

In contrast, the remaining terms on the right-hand side of (9.91) produce very lengthy expressions in the scalars μ, ν, their derivatives, and the internal coordinates U^i, V^i, which we do not report in detail. They combine however precisely into the Einstein tensor computed from the metric (9.47) upon using the first order flow equations (9.36). All the ten-dimensional equations are thus satisfied.

Chapter 10

Curvatures for Kaluza-Klein Metrics

The process of dimensional reduction via compactification of some of the space-time dimensions and Kaluza-Klein expansion of the fields leads to space-time metrics of a particular form. It is clear that also the inverse process of uplifting lower-dimensional theories to higher-dimensional theories with compact dimensions and Kaluza-Klein towers leads to the same type of metrics. We saw an example of such an uplift in equations (8.7) and (9.47), and the metric that appears in these equations is of the following prototypical form:

$$\hat{ds}^2 = \Omega(x,y)^2 \Big\{ g_{\mu\nu}(x) dx^\mu dx^\nu + \tilde{g}_{mn}(x,y) \Big[dy^m - K^{im}(y) A^i_\mu(x) dx^\mu \Big] \Big[dy^n - K^{jn}(y) A^j_\nu(x) dx^\nu \Big] \Big\}. \quad (10.1)$$

A general Kaluza-Klein metric as in equation (10.1) contains an overall warp-factor $\Omega^2(x,y)$ that depends on both external and internal coordinates, an external metric $g_{\mu\nu}(x)$ which is independent of the internal coordinates and serves as the metric of the lower-dimensional theory, an internal metric $\tilde{g}_{mn}(x,y)$ which in general depends on both the external and internal coordinates, and Kaluza-Klein gauge fields $A^i_\mu(x)$, which by assumption can only depend on the external coordinates, contracted with a set of Killing vectors $K^{im}(y)$ of the internal manifold. The index conventions used in writing the metric in (10.1) are the following. The curved coordinates will be denoted by letters from the middle of the alphabet, and the flat indices by letters from the beginning of the alphabet. As in the other chapters we will use capital Latin indices for the total manifold, small Latin indices for the internal space and small Greek indices for the external manifold. Moreover the coordinates on the external manifold will be called x^μ, the coordinates on the internal manifold y^m, and the coordinates on the total manifold $Z^M = (x^\mu, y^m)$.

Kaluza-Klein theories often admit various truncations in which many fields can consistently be set to zero, leaving only a small number of non-vanishing fields. The back-reaction of these non-zero fields to the geometry will produce space-time metrics of different forms, in particular there might be a simple enough truncation that removes the warp factor, or in which the gauge fields are all zero. In such a set-up one might be interested in first computing the curvatures corresponding to $g_{\mu\nu}$ and $\tilde{g}_{mn}$, then use these curvatures to deduce the curvature of $ds^2 = g_{\mu\nu}(x)dx^\mu dx^\nu + \tilde{g}_{mn}(x,y)dy^m dy^n$, and finally turn on a non-trivial warp factor and gauge fields to obtain curvatures of the metric as in equation (10.1). Therefore in this chapter we will carry out a study of how such a computation can be done.

Let us first consider a metric as in (10.1) and assume that the warp factor is trivial. A similar example was considered by Duff, Nilsson, and Pope [100], in which it was assumed that the internal metric only depends on the internal coordinates. However, we would like to maintain a general internal metric $\tilde{g}_{mn}(x,y)$, which can depend on both the external and internal coordinates, and thus generalise the formulas found in [100]. Thus our goal is to compute the curvature for a metric in block form deformed by off-diagonal elements containing the gauge field which is of the following form

$$d\hat{s}^2 = \hat{e}^\alpha \otimes \hat{e}^\beta \eta_{\alpha\beta} + \hat{e}^a \otimes \hat{e}^b \delta_{ab}. \tag{10.2}$$

As in [100] it is useful to work in the tetrad formalism and we define the hatted vielbeins as follows:

$$\hat{e}^\alpha = e^\alpha(x) \tag{10.3}$$

$$\hat{e}^a = e^a(x,y) - K^{ia}(x,y)A^i(x). \tag{10.4}$$

The unhatted vielbeins are those of the external and internal metric in the case where the gauge field is set to zero, so that $\eta_{\alpha\beta}e^\alpha e^\beta = g_{\mu\nu}dx^\mu dx^\nu$ and $\delta_{ab}e^a e^b = \tilde{g}_{mn}dy^m dy^n$, and we have extended [100] by promoting the internal tetrad $e^a(y)$ to $e^a(x,y)$. The vectors $K^{ia}(x,y) = e^a_m(x,y)K^{im}(y)$ are components of the Killing vectors of the internal manifold. As such they satisfy the Killing equation $\nabla_{(m}K^i_{n)} = 0$. Moreover these vectors form an algebra with the commutator $[K^i, K^j] = f^{ijk}K^k$, where the vectors K^i are understood as the elements $K^i = K^{im}\frac{\partial}{\partial y^m}$ of the tangent space, which act by differentiation. Therefore the components of the Killing vectors satisfy

$$K^{im}\partial_m K^{jn} - K^{jm}\partial_m K^{in} = f^{ijk}K^{kn}. \tag{10.5}$$

As we will be working with flat indices, it is useful to note that the following identity holds

$$K^{ia}\nabla_a K^j_b - K^{ja}\nabla_a K^i_b = f^{ijk}K^k_b. \tag{10.6}$$

This identity can be proven by converting to curved indices with the tetrad and noting that $\nabla_a e^{mb} = 0$, or by spelling out the covariant derivatives and using the fact that a certain anti-symmetrisation of the spin connection can be expressed in terms of the vielbeins as follows: $\omega_{[c,b]a} = -e^m_{[c}\partial_{b]}\, e_{ma}$.

In the tetrad set-up just described the computation of the curvatures will be performed in two steps. We will start by using the first Cartan equation to determine the spin connection. Once this is done one can use the second Cartan equation which relates the Riemann tensor to the spin connection to compute the curvatures.

10.1 The Spin Connection

To compute the spin connection we will use the first Cartan structure equation. Since we are looking for a solution which corresponds to a Levi-Civita connection we need to assume that the torsion vanishes. The first Cartan equation with zero torsion is the following:

$$d\hat{e}^A + \hat{\omega}^A{}_B \wedge \hat{e}^B = 0. \tag{10.7}$$

After splitting the space-time index into its external and internal parts we see that there are six types of components of the spin connection that need to be determined:

$$\hat{\omega}_{AB} = \begin{cases} \hat{\omega}_{\alpha\beta} = \hat{\omega}_{\gamma,\alpha\beta}\hat{e}^\gamma + \hat{\omega}_{c,\alpha\beta}\hat{e}^c \\ \hat{\omega}_{ab} = \hat{\omega}_{\gamma,ab}\hat{e}^\gamma + \hat{\omega}_{c,ab}\hat{e}^c \\ \hat{\omega}_{\alpha b} = \hat{\omega}_{\gamma,\alpha b}\hat{e}^\gamma + \hat{\omega}_{c,\alpha b}\hat{e}^c \end{cases} \tag{10.8}$$

The first Cartan equation involves the computation of the differential of the tetrad, $d\hat{e}^A$. In order to compute it one needs to express the hatted tetrads in terms of the unhatted ones first, compute the differential while substituting the differentials of the unhatted tetrads by the corresponding spin connections using the unhatted version of the first Cartan equation, and finally express the unhatted tetrads of the result back in terms of the hatted ones. The external tetrads $\hat{e}^\alpha = e^\alpha$ do not receive any contributions from the gauge field, and therefore the corresponding Cartan equation is easily evaluated:

$$\begin{aligned} 0 &= d\hat{e}_\alpha + \hat{\omega}_{\alpha\beta} \wedge \hat{e}^\beta + \hat{\omega}_{\alpha b} \wedge \hat{e}^b \\ &= -\omega_{\alpha\beta} \wedge e^\beta + \hat{\omega}_{\alpha\beta} \wedge \hat{e}^\beta + \hat{\omega}_{\alpha b} \wedge \hat{e}^b \\ &= (\hat{\omega}_{\gamma,\alpha\beta} - \omega_{\gamma,\alpha\beta})\hat{e}^\gamma \wedge \hat{e}^\beta + (\hat{\omega}_{c,\alpha\beta} - \hat{\omega}_{\beta,\alpha c})\hat{e}^c \wedge \hat{e}^\beta + \hat{\omega}_{c,\alpha b}\hat{e}^c \wedge \hat{e}^b. \end{aligned} \tag{10.9}$$

Each of the three 2-forms above has to vanish separately, which allows us to obtain the following equations for the spin connection:

$$\hat{\omega}_{\gamma,\alpha\beta} - \hat{\omega}_{\beta,\alpha\gamma} = \omega_{\gamma,\alpha\beta} - \omega_{\beta,\alpha\gamma} \tag{10.10a}$$

$$\hat{\omega}_{c,\alpha\beta} - \hat{\omega}_{\beta,\alpha c} = 0 \tag{10.10b}$$

$$\hat{\omega}_{c,\alpha b} - \hat{\omega}_{b,\alpha c} = 0. \tag{10.10c}$$

Next consider the Cartan equation for $\hat{e}^a$. It is useful to compute the differential $d\hat{e}^a$ first. After some lengthy algebra we find the following result:

$$d\hat{e}_a = -\omega_{c,ab}\hat{e}^c \wedge \hat{e}^b - \frac{1}{2}K^i_a F^i_{\alpha\beta}\hat{e}^\alpha \wedge \hat{e}^\beta + (\nabla_b K^i_a A^i_\alpha + e^m_b \partial_\alpha e_{am} - \omega_{c,ab}K^{ic}A^i_\alpha)\hat{e}^\alpha \wedge \hat{e}^b. \tag{10.11}$$

After substituting $d\hat{e}_a$ just found into the Cartan equation the rest of the algebra is trivial, and we find a second set of equations for the spin connection:

$$\hat{\omega}_{c,ab} - \hat{\omega}_{b,ac} = \omega_{c,ab} - \omega_{b,ac} \tag{10.12a}$$

$$\hat{\omega}_{\alpha,a\beta} - \hat{\omega}_{\beta,a\alpha} = K^i_a F^i_{\alpha\beta} \tag{10.12b}$$

$$\hat{\omega}_{\alpha,ab} - \hat{\omega}_{b,a\alpha} = \nabla_a K^i_b A^i_\alpha - e^m_b \partial_\alpha e_{am} + \omega_{c,ab}K^{ic}A^i_\alpha \tag{10.12c}$$

It is a general fact that the six equations (10.10) and (10.12) do not determine a unique spin connection, and further conditions have to be specified to single out a particular solution. In our case the condition that we impose on the spin connection is that it be of Levi-Civita type, and it is known that the restriction to Levi-Civita solutions is sufficient to find a unique solution. By definition a Levi-Civita connection is a connection which is compatible with the metric and for which the torsion is zero. The zero torsion condition was already implemented by writing the first Cartan equation with vanishing right hand side. Metric compatibility in the tetrad basis reads $\hat{\nabla}_C \eta_{AB} = 0$, which directly translates to $\hat{\omega}_{C,(AB)} = 0$. With this condition we can now use equations (10.10) and (10.12) to find the anti-symmetric parts $\hat{\omega}_{C,[AB]}$, which automatically give the full spin connection. Let us consider one example that is demonstrative for this procedure. We pick a component type of the spin connection with a certain distribution of external and internal indices, for example $\hat{\omega}_{\alpha,ab}$. There are three cyclic permutations of the indices, for which we write down the corresponding equations obtained in (10.10) and (10.12):

$$\hat{\omega}_{\alpha,ab} - \hat{\omega}_{b,a\alpha} = \nabla_a K^i_b A^i_\alpha - e^m_b \partial_\alpha e_{am} + \omega_{c,ab}K^{ic}A^i_\alpha \tag{10.13a}$$

$$\hat{\omega}_{a,b\alpha} - \hat{\omega}_{\alpha,ba} = \nabla_a K^i_b A^i_\alpha + e^m_a \partial_\alpha e_{bm} + \omega_{c,ab}K^{ic}A^i_\alpha \tag{10.13b}$$

$$\hat{\omega}_{b,\alpha a} - \hat{\omega}_{a,\alpha b} = 0. \tag{10.13c}$$

To obtain the anti-symmetric parts of the spin-connection add two of these three equations together and subtract the remaining one. After using the fact that $\nabla_{(m}K^i_{n)} = 0$ we obtain the following results:

$$\hat{\omega}_{\alpha,[ab]} = \nabla_{[a}K^i_{b]} + \frac{1}{2}(e^m_a\partial_\alpha e_{bm} - e^m_b\partial_\alpha e_{am}) + \omega_{c,ab}K^{ic}A^i_\alpha \tag{10.14}$$

$$\hat{\omega}_{a,[b\alpha]} = \frac{1}{2}(e^m_a\partial_\alpha e_{bm} + e^m_b\partial_\alpha e_{am}) \tag{10.15}$$

$$\hat{\omega}_{b,[a\alpha]} = \frac{1}{2}(e^m_a\partial_\alpha e_{bm} + e^m_b\partial_\alpha e_{am}). \tag{10.16}$$

Thus we have found the solutions for two different types of spin connection components since one of the three solutions above is redundant due to the anti-symmetry. Proceeding in an analogous fashion we obtain the whole set of components of the spin connection

$$\hat{\omega}_{\alpha,\beta\gamma} = \omega_{\alpha,\beta\gamma} \tag{10.17}$$

$$\hat{\omega}_{c,\beta\gamma} = \frac{1}{2}K^i_cF^i_{\alpha\beta} \tag{10.18}$$

$$\hat{\omega}_{\gamma,\alpha b} = \frac{1}{2}K^i_bF^i_{\alpha\gamma} \tag{10.19}$$

$$\hat{\omega}_{c,\alpha b} = -\frac{1}{2}(e^m_c\partial_\alpha e_{bm} + e^m_b\partial_\alpha e_{cm}) \tag{10.20}$$

$$\hat{\omega}_{\gamma,ab} = \nabla_{[a}K^i_{b]} + \frac{1}{2}(e^m_a\partial_\gamma e_{bm} - e^m_b\partial_\gamma e_{am}) + \omega_{c,ab}K^{ic}A^i_\gamma \tag{10.21}$$

$$\hat{\omega}_{c,ab} = \omega_{c,ab}. \tag{10.22}$$

These components can be assembled back to 1-forms by contracting with the hatted vielbeins giving us the final solution:

$$\hat{\omega}_{\alpha\beta} = \omega_{\alpha\beta} + \frac{1}{2}F^i_{\alpha\beta}K^i_c\hat{e}^c \tag{10.23a}$$

$$\hat{\omega}_{\alpha b} = \frac{1}{2}K^i_bF^i_{\alpha\gamma}\hat{e}^\gamma - \frac{1}{2}(e^m_c\partial_\alpha e_{bm} + e^m_b\partial_\alpha e_{cm})\hat{e}^c \tag{10.23b}$$

$$\hat{\omega}_{ab} = \omega_{ab} + \nabla_{[a}K^i_{b]}A^i + \frac{1}{2}(e^m_a\partial_\gamma e_{bm} - e^m_b\partial_\gamma e_{am})\hat{e}^\gamma. \tag{10.23c}$$

10.2 Curvature Tensors

Once the spin connection $\hat{\omega}_{AB}$ has been computed one can use the second Cartan equation to obtain the Riemann tensor. The second Cartan equation can be written in terms of 2-forms as follows:

$$\frac{1}{2}\hat{R}_{ABCD}\hat{e}^C \wedge \hat{e}^D \equiv \Theta_{AB} = d\hat{\omega}_{AB} + \hat{\omega}_A{}^C \wedge \hat{\omega}_{CD}. \tag{10.24}$$

In what follows we would like to make a simplification by assuming that the gauge group is the U(1), which is abelian. The reason for taking this assumption is that on the one hand this case is of most relevance to the uplift described in the preceding section, where an abelian gauge potential appears in the metric, and on the other hand the restriction to $G = \mathrm{U}(1)$ leads to some simplification in the algebra. The generalisation to the non-abelian case should be straightforward, but tedious. After restricting to the U(1) gauge group there is only one gauge field $A_\mu(x)$ and one Killing vector $K^m(y)$. The abelian field strength is simply given by $F = dA = \frac{1}{2}F_{\alpha\beta}\hat{e}^\alpha \wedge \hat{e}^\beta$. The only change in the spin connection is that the gauge algebra index counting Killing vectors and gauge fields can now be omitted. After a direct application of the second Cartan equation (10.24) to the Levi-Civita spin connection in (10.23) the following components of the Riemann tensor are obtained:

$$\hat{R}_{\alpha\beta\gamma\delta} = R_{\alpha\beta\gamma\delta} - \frac{1}{4}K^a K_a(F_{\alpha\gamma}F_{\beta\delta} - F_{\alpha\delta}F_{\beta\gamma} + 2F_{\alpha\beta}F_{\gamma\delta}) \tag{10.25}$$

$$\hat{R}_{\alpha\beta\gamma d} = \frac{1}{2}\nabla_\gamma F_{\alpha\beta}K_d + \frac{1}{2}(F_{\alpha\beta}\partial_\gamma\tilde{g}_{mn} - F_{\gamma[\alpha}\partial_{\beta]}\tilde{g}_{mn})K^m e_d^n \tag{10.26}$$

$$\hat{R}_{\alpha\beta cd} = F_{\alpha\beta}\tilde{\nabla}_{[c}K_{d]} - \frac{1}{2}\tilde{g}^{mn}e^r_{[c}e^s_{d]}\partial_\alpha\tilde{g}_{mr}\partial_\gamma\tilde{g}_{ns} \tag{10.27}$$

$$\hat{R}_{\alpha b\gamma d} = \frac{1}{2}\tilde{\nabla}_{[b}K_{d]}F_{\alpha\gamma} + \frac{1}{4}K_bK_dF_{\alpha\beta}F_\gamma{}^\beta + \frac{1}{2}e^r_b e^s_d\left(-\nabla_\gamma\partial_\alpha\tilde{g}_{rs} + \frac{1}{2}\tilde{g}^{mn}\partial_\gamma\tilde{g}_{mr}\partial_\alpha\tilde{g}_{ns}\right) \tag{10.28}$$

$$\hat{R}_{\alpha bcd} = e^n_b e^r_{[c}e^s_{d]}\left(-\tilde{\nabla}_r\partial_\alpha\tilde{g}_{ns} + \frac{1}{2}F_{\alpha\gamma}\partial^\gamma\tilde{g}_{nr}K_s\right) \tag{10.29}$$

$$\hat{R}_{abcd} = \tilde{R}_{abcd} - \frac{1}{2}e^m_a e^n_b e^r_{[c}e^s_{d]}\partial_\alpha\tilde{g}_{mr}\partial^\alpha\tilde{g}_{ns}. \tag{10.30}$$

Note that the flat indices in the Riemann tensor above refer to the hatted vielbein, which has to be used to convert to curved indices. Thus, while $\hat{e}^\alpha = e^\alpha$, the vielbein with the internal index is $\hat{e}^a = e^a - K^a A$. After restriction to $\tilde{g}_{mn}(x,y) \to \tilde{g}_{mn}(y)$ we find a complete agreement with [100]. Additionally we provide the component $\hat{R}_{\alpha b\gamma d}$, which might have been missed by the authors. Note that the sign for the Killing vector used here is consistent with [100], but opposite to our uplift. To obtain the Ricci tensor we contract with $\eta^{\alpha\beta}$ and δ^{ab} and get

$$\hat{R}_{\alpha\gamma} = R_{\alpha\gamma} - \frac{1}{2}K_aK^aF_{\alpha\beta}F_\gamma{}^\beta \tag{10.31}$$

$$+ \frac{1}{2}\tilde{g}^{rs}\left(-\nabla_\gamma\partial_\alpha\tilde{g}_{rs} + \frac{1}{2}\tilde{g}^{mn}\partial_\gamma\tilde{g}_{mr}\partial_\alpha\tilde{g}_{ns}\right) \tag{10.32}$$

$$\hat{R}_{\alpha c} = -\frac{1}{2}\nabla^\beta F_{\alpha\beta}K_c - \frac{1}{2}F_{\alpha\beta}\partial^\beta\tilde{g}_{mn}K^m e^n_c - \frac{1}{4}F_{\alpha\beta}\partial^\beta\tilde{g}_{mn}K_c\tilde{g}^{mn} \tag{10.33}$$

$$+ \frac{1}{2}\tilde{g}^{mn}e^r_c\left(\tilde{\nabla}_n\partial_\alpha\tilde{g}_{mr} - \tilde{\nabla}_r\partial_\alpha\tilde{g}_{mn}\right) \tag{10.34}$$

$$\hat{R}_{ac} = \tilde{R}_{ac} + \frac{1}{4}K_aK_cF_{\alpha\beta}F^{\alpha\beta} \tag{10.35}$$

$$+ e^m_a e^n_c\left(-\frac{1}{2}\nabla^2\tilde{g}_{mn} - \frac{1}{4}\tilde{g}^{rs}\partial_\alpha\tilde{g}_{mn}\partial^\alpha\tilde{g}_{rs} + \frac{1}{2}\tilde{g}^{rs}\partial_\alpha\tilde{g}_{mr}\partial^\alpha\tilde{g}_{ns}\right) \tag{10.36}$$

Yet another contraction with the flat metric yields the Ricci scalar, which can be written entirely in terms of the Ricci scalars corresponding to $g_{\mu\nu}$ and $\tilde{g}_{mn}$, the Field strength $F_{\alpha\beta}$

and some covariant derivatives of the internal metric:

$$\hat{R} = R + \tilde{R} - \frac{1}{4}K_a K^a F_{\alpha\beta}F^{\alpha\beta} \tag{10.37}$$

$$- \tilde{g}^{mn}\nabla^2\tilde{g}_{mn} + \tilde{g}^{mn}\tilde{g}^{rs}\left(\frac{3}{4}\partial_\mu\tilde{g}_{mr}\partial^\mu\tilde{g}_{ns} - \frac{1}{4}\partial_\mu\tilde{g}_{mn}\partial^\mu\tilde{g}_{rs}\right). \tag{10.38}$$

10.3 Conformal Rescaling of Metric

So far we have expressed the curvatures of the total manifold in terms of the curvatures of the external and internal manifolds, the gauge field strength coupled to Killing vectors, and some derivatives of the metric. The last step in the generalisation to the metric in equation (10.1) that we started with is the restoration of the warp factor, which is achieved by a conformal rescaling as follows:

$$\hat{g}_{MN} \to \Omega^2(x,y)\hat{g}_{MN}. \tag{10.39}$$

The effect of such a rescaling of the metric on the curvatures is known [101], which have to be transformed in the following way:

$$\begin{aligned}\hat{R}_{MNRS} \to \hat{R}_{MNRS} &+ 2\hat{g}_{S[M}\hat{\nabla}_{N]}\hat{\nabla}_R \ln\Omega - 2\hat{g}_{R[M}\hat{\nabla}_{N]}\hat{\nabla}_S \ln\Omega \\ &+ 2\hat{g}_{S[N}\hat{\nabla}_{M]}\ln\Omega\hat{\nabla}_R\ln\Omega - 2\hat{g}_{R[N}\hat{\nabla}_{M]}\ln\Omega\hat{\nabla}_S\ln\Omega \\ &- 2\hat{g}_{R[M}\hat{g}_{N]S}\hat{\nabla}_L\ln\Omega\hat{\nabla}^L\ln\Omega\end{aligned} \tag{10.40}$$

$$\begin{aligned}\hat{R}_{MR} \to \hat{R}_{MR} &- (D-2)\hat{\nabla}_M\hat{\nabla}_R\ln\Omega - \hat{g}_{MR}\hat{\nabla}^2\ln\Omega \\ &+ (D-2)\hat{\nabla}_M\ln\Omega\hat{\nabla}_R\ln\Omega - (D-2)\hat{g}_{MR}\hat{g}^{SL}\hat{\nabla}_S\ln\Omega\hat{\nabla}_L\ln\Omega\end{aligned} \tag{10.41}$$

$$\hat{R} \to \frac{1}{\Omega^2}\left(\hat{R} - 2(D-1)\hat{\nabla}^2\ln\Omega - (D-2)(D-1)\hat{g}^{MR}\nabla_M\ln\Omega\nabla_R\ln\Omega\right) \tag{10.42}$$

The letter D denotes the total dimension of the space-time manifold for which $\hat{g}_{MN}$ is the metric. These rescaling formulas can be readily applied to the curvatures without the conformal factor derived in the previous subsection.

Chapter 11

Spherical Harmonics on the S^5

11.1 General Considerations

Any scalar and tensor valued analytic function on an n-sphere can be expanded into a complete set of functions that transform covariantly under the corresponding rotation algebra $\mathfrak{so}(n+1)$. These functions are the spherical harmonics. Historically one of the first examples to be considered was the reduction on 7-spheres, and the corresponding formalism was for example described in [102, 103] and applied in [104, 105]. Some details about the general expansion on n-spheres can be found in [106, 107], and the important analysis of the mass spectrum upon reduction on S^5 was done in [42]. More recent reviews and applications of this formalism include [108–110].

In this section we would like to study various scalar, vector, and tensor harmonics on the S^5 which appear upon Kaluza-Klein reductions of tensor and scalar fields on the five-sphere. For each given tensor type they constitute a complete set, and therefore any tensor can be expanded in a basis made of appropriate S^5 harmonics. The spherical harmonics we will consider in this section can be defined as solutions to the following Laplace eigenvalue equations on the five-sphere:

$$0 = \mathring{\nabla}^2 Y^{(k,m)} + k(k+4) Y^{(k,m)} \tag{11.1a}$$

$$0 = \mathring{\nabla}^2 Y^{(k,m)}_{\mathbf{5}\,n} + (k^2+6k+4) Y^{(k,m)}_{\mathbf{5}\,n} \tag{11.1b}$$

$$0 = \mathring{\nabla}^2 Y^{(k,m)}_{\mathbf{10}\,[np]} + (k^2+6k+3) Y^{(k,m)}_{\mathbf{10}\,[np]} \tag{11.1c}$$

$$0 = \mathring{\nabla}^2 Y^{(k,m)}_{\mathbf{14}\,\{np\}} + (k^2+8k+10) Y^{(k,m)}_{\mathbf{14}\,\{np\}}. \tag{11.1d}$$

$$(k = 0, 1, 2, \dots)$$

The symbol $\mathring{\nabla}$ denotes the covariant derivative on the S^5, and upper and lower Latin indices $n, p, \ldots$ refer to the S^5 tangent and cotangent spaces. The bold numerals written as subscripts serve as a distinction between harmonics of various tensor types and the value of the numeral gives the number of independent tensor components. We will omit the subscript **1** for the scalar harmonics in order to simplify the notation. For each given eigenvalue k one generally finds multiple solutions, which are distinguished using the superscript $m = m(k)$. Sometimes m can be viewed as an eigenvalue of some differential operator acting on the harmonic, however we will use it exclusively for bookkeeping to count the degeneracy for a given k.

Additionally to being solutions of their defining equations, all but the scalar harmonics have to be divergence-free, so that the defining harmonic eigenvalue problems above have to be supplemented by the following constraint equations:

$$0 = \mathring{\nabla}^n Y^{(k,m)}_{\mathbf{5}\,n} = \mathring{\nabla}^n Y^{(k,m)}_{\mathbf{10}\,[np]} = \mathring{\nabla}^n Y^{(k,m)}_{\mathbf{14}\,\{np\}}, \tag{11.2}$$

It is useful to orthonormalise the harmonics so that the harmonic expansion can be performed by projecting out the corresponding coefficients via integration on the S^5. We will choose the following normalisation for all harmonics:

$$\begin{aligned}\int_{S^5} Y^{(k,m)} Y^{(k',m')} &= \int_{S^5} Y^{(k,m)}_{\mathbf{5}\,n} Y^{(k',m')\,n}_{\mathbf{5}} = \int_{S^5} Y^{(k,m)}_{\mathbf{10}\,[np]} Y^{(k',m')\,[np]}_{\mathbf{10}} \\ &= \int_{S^5} Y^{(k,m)}_{\mathbf{14}\,\{np\}} Y^{(k',m')\,\{np\}}_{\mathbf{14}} = \frac{\pi^3}{2^{k-1}(k+1)(k+2)} \delta^{kk'} \delta^{mm'}.\end{aligned} \tag{11.3}$$

While this particular normalisation is standard for scalar harmonics, it is rather deliberate for vector and tensor harmonics, and was chosen purely for convenience reasons.

As mentioned at the beginning, all S^n spherical harmonics transform covariantly under the isometry group of the sphere, which is the $\mathrm{SO}(n+1)$ rotation group, in other words, they form irreducible representations of the $\mathfrak{so}(n+1)$ algebra. The isometry group of the S^5 is the $\mathrm{SO}(6)$, and the harmonics transform in the following $\mathrm{SO}(6)$ representations:

$$Y^{(k,m)} = [k,0,0] = \frac{1}{12}(k+1)(k+2)^2(k+3) = \mathbf{1}, \mathbf{6}, \mathbf{20'}, \ldots \tag{11.4a}$$

$$Y^{(k,m)}_{\mathbf{5}\,n} = [k,1,1] = \frac{1}{3}(k+1)(k+3)^2(k+5) = \mathbf{15}, \mathbf{64}, \mathbf{175}, \ldots \tag{11.4b}$$

$$Y^{(k,m)}_{\mathbf{10}\,[np]} = [k,2,0] \oplus [k,0,2] = \frac{1}{4}(k+1)(k+2)(k+4)(k+5) = \mathbf{10_c}, \mathbf{45_c}, \mathbf{126_c}, \ldots \tag{11.4c}$$

$$Y^{(k,m)}_{\mathbf{14}\,\{np\}} = [k,2,2] = \frac{3}{4}(k+1)(k+4)^2(k+7) = \mathbf{84}, \mathbf{300}, \mathbf{729}, \ldots. \tag{11.4d}$$

One can see that for a given harmonic type the index k fixes the representation, and the index m enumerates the states in that representation. For example if we set $k = 1$ in the vector harmonic then $Y_{\mathbf{5}\,n}^{(1,m)}$ will transform as the **64** of the $\mathfrak{so}(6)$ and therefore the index m will assume 64 different values which can be taken to be $m \in \{1, \dots, 64\}$.

The supergravity solution that we are studying in this text was obtained after a truncation of the full theory to a certain invariant sub-sector. This sub-sector corresponds to those solutions that are invariant under the $\mathrm{SO}(3)_{\mathrm{diag}}$ subgroup which is embedded in the SO(6) isometry group as

$$\mathrm{SO}(3)_{\mathrm{diag}} \subset \mathrm{SO}(3) \times \mathrm{SO}(3) \subset \mathrm{SO}(6). \tag{11.5}$$

This means that all solutions for all fields that we obtain are invariant under this $\mathrm{SO}(3)_{\mathrm{diag}}$, which in particular implies that all harmonics into which these fields can be expanded have to be invariant too. This can be either enforced at the level of their defining equations (11.1), or in terms of their representations in (11.4). For the latter this means that if a set of harmonics is known, then we must only keep those that are singlets under the $\mathrm{SO}(3)_{\mathrm{diag}}$ subgroup. As an example consider again the vector $Y_{\mathbf{5}\,n}^{(1,m)} = \mathbf{64}$. We need to determine how many singlets there are under the embedding (11.5). We have listed a number of relevant branchings in Appendix E, and we find that

$$\mathbf{64} \to 2(\mathbf{1}) + 6(\mathbf{3}) + 6(\mathbf{5}) + 2(\mathbf{7}). \tag{11.6}$$

Thus we expect that among the 64 vector harmonics there are only two that are invariant under the $\mathrm{SO}(3)_{\mathrm{diag}}$ and will show up in the expansion of vector fields in our solution. If all of the 64 harmonics are known then the singlets can be explicitly constructed by decomposing the $\mathfrak{so}(6)$ indices into $\mathfrak{so}(3)$ indices and forming the singlets in the standard way. If this is not the case then one is forced to work with the defining equations (11.1), and to find a way to enforce the invariance on the level of differential equations. In what follows we will take both routes and use the differential equation approach to find solutions for general values of k. In situations where the differential equation approach is to cumbersome the group theory method can be used to treat individual cases, which turns out to be tractable for the lowest values of k.

11.2 Scalar Spherical Harmonics with SO(3) × SO(3) Symmetry

This section can be seen as a warm up exercise for the determination of the $\mathrm{SO}(3)_{\mathrm{diag}}$ spherical harmonics. The solution strategy for the SO(3) × SO(3) invariant harmonics turns out to be simpler, and by performing it several concepts can be explained and studied that will be relevant for the $\mathrm{SO}(3)_{\mathrm{diag}}$ harmonics which we will study in the next section.

Scalar spherical harmonics corresponding to a d-dimensional sphere S^d can be identified with symmetric and traceless representations of the $\mathfrak{so}(d+1)$ algebra, which in terms of Dynkin labels can be written as $[k, 0, \ldots, 0]$. This can be seen by thinking of the S^d as embedded into $\mathbb{R}^{d+1}$, in which case the harmonics can be written as

$$Y^{(k,m)}(y^I/r) = \frac{1}{r^k} C^{(k,m)}_{I_1 \ldots I_k} y^{I_1} \ldots y^{I_k}. \tag{11.7}$$

The notation here is such that the capital indices $I, J, \ldots \in \{1, \ldots, 6\}$ refer to the embedding coordinates of the S^5 in $\mathbb{R}^6$, and can also be viewed as the fundamental index of the $\mathfrak{so}(6)$. The radius r is given by $r^2 = (y^1)^2 + \cdots + (y^{d+1})^2$ and $C^{(k,m)}_{I_1 \ldots I_k}$ is a set of linearly independent symmetric and traceless k-tensors. A straightforward counting shows that for a given k there are $\binom{d+k}{d} - \binom{d+k-2}{d}$ such tensors, and this is exactly equal to the dimension of the representation $[k, 0, \ldots, 0]$ of the $\mathfrak{so}(d+1)$, as it should be. So we see that the easiest way to find all scalar spherical harmonics for any given d is to choose the tensors $C^{(k,m)}_{I_1 \ldots I_k}$ and to express the coordinates y^I/r in terms of the angles on the sphere. One can then use the Gram-Schmidt procedure to find a basis in which the $C^{(k,m)}_{I_1 \ldots I_k}$ are normalised and orthogonal.

Yet another defining property of the spherical harmonics is that they are annihilated by the $\mathbb{R}^{d+1}$ Laplacian, which is where they derive their name "harmonics" from:

$$\nabla^2 \left(r^k Y^{(k,m)} \right) = 0. \tag{11.8}$$

This follows directly from the fact that $C^{(k,m)}_{I_1 \ldots I_k}$ are traceless. We can change to spherical coordinates on $\mathbb{R}^{d+1}$ and re-write the Laplacian to get [111]

$$0 = \nabla^2 (r^k Y^{(k,m)}) = \left(\frac{1}{r^d} \frac{\partial}{\partial r} r^d \frac{\partial}{\partial r} + \frac{1}{r^2} \mathring{\nabla}^2 \right) r^k Y^{(k,m)} = r^{k-2} \left(k(k+d-1) + \mathring{\nabla}^2 \right) Y^{(k,m)} \tag{11.9}$$

thus obtaining

$$\mathring{\nabla}^2 Y^{(k,m)} = -k(k+d-1) Y^{(k,m)}, \qquad k = 0, 1, 2, \ldots \tag{11.10}$$

For $d = 5$ this reduces to the defining eigenvalue equation in (11.1). This method was for example used in [109] to find a set of harmonics with a particular symmetry, which is different from the one considered here. The advantage of this method is that we do not have to refer to an embedding in $\mathbb{R}^{d+1}$ and can solve the Laplace equation in terms of coordinates on the sphere that are appropriate for the problem at hand.

We are interested in scalar spherical harmonics on S^5, thus we have to set $d = 5$ and the spherical harmonics will be organised in terms of representations of the $\mathfrak{so}(6)$. As we saw before the dimensions of the lowest representations are given by

$$[k, 0, 0] = \binom{5+k}{5} - \binom{5+k-2}{5} = \frac{1}{12}(k+1)(k+2)^2(k+3) = \mathbf{1}, \mathbf{6}, \mathbf{20'}, \mathbf{50}, \mathbf{105}, \mathbf{196}, \ldots \tag{11.11}$$

Further, among these harmonics we would like to pick out those that are invariant under the subgroup $\mathrm{SO}(3)\times\mathrm{SO}(3) \subset \mathrm{SO}(6)$, in other words those that correspond to the $\mathrm{SO}(3)\times\mathrm{SO}(3)$ singlets under the branching of the representations $[k,0,0]$ of the SO(6). To find these harmonics we will take the path of solving the Laplace eigenvalue problem. First we need to choose coordinates in which the action of the $\mathrm{SO}(3)\times\mathrm{SO}(3)$ is manifest. This is done by imagining $\mathbb{R}^6$ as $\mathbb{R}^3\times\mathbb{R}^3$ so that each SO(3) acts on the corresponding $\mathbb{R}^3$ factor. Next we can write each of these $\mathbb{R}^3$ factors in spherical coordinates so that the complete metric on $\mathbb{R}^6$ is given by

$$ds^2 = dr^2 + r^2 d\theta^2 + r^2\cos^2\theta d\Omega^2_{(1)} + r^2\sin^2\theta d\Omega^2_{(2)}. \tag{11.12}$$

We can now derive the form of the Laplacian $\mathring{\nabla}^2$ on the S^5 written in these coordinates. Using $\mathring{\nabla}^2_{(1)}$ and $\mathring{\nabla}^2_{(2)}$ to denote the Laplacians on the spheres corresponding to $d\Omega^2_{(1)}$ and $d\Omega^2_{(2)}$ we get

$$\mathring{\nabla}^2 = \frac{1}{\cos^2\theta\sin^2\theta}\frac{\partial}{\partial\theta}\cos^2\theta\sin^2\theta\frac{\partial}{\partial\theta} + \frac{1}{\cos^2\theta}\mathring{\nabla}^2_{(1)} + \frac{1}{\sin^2\theta}\mathring{\nabla}^2_{(2)}. \tag{11.13}$$

For $\mathrm{SO}(3)\times\mathrm{SO}(3)$-invariant solutions we must impose $\mathring{\nabla}^2_{(1)} = \mathring{\nabla}^2_{(2)} = 0$, thus the harmonics we are looking for must be solutions to the following equation:

$$\frac{1}{\cos^2\theta\sin^2\theta}\frac{\partial}{\partial\theta}\left(\cos^2\theta\sin^2\theta\frac{\partial}{\partial\theta}Y^{(k)}\right) = -k(k+4)Y^{(k)}. \tag{11.14}$$

One can show that a general solution to this equation is of the form

$$Y^{(k)} = \frac{C_1\cos(k+2)\theta + C_2\sin(k+2)\theta}{\sin 2\theta}. \tag{11.15}$$

Not all of these solutions are good spherical harmonics. Note that in the coordinates on $\mathbb{R}^6$ that we chose it is true that $r^2\cos^2\theta = (x^1)^2+(x^2)^2+(x^3)^2$ and $r^2\sin^2\theta = (x^4)^2+(x^5)^2+(x^6)^2$. To be compatible with the definition of $Y^{(k)}$ in (11.7) the solutions $Y^{(k)}$ thus have to be homogeneous polynomials in $\cos\theta$ and $\sin\theta$ of degree k. However, one can show that this is only possible when both $C_1 = 0$ and k is even. The latter assertion makes sense since otherwise one would have odd powers of $\cos\theta$ and $\sin\theta$ leading to square roots, which is not acceptable. All in all our solutions are given by

$$\begin{aligned} Y^{(2l)} &= \frac{1}{2^l\sqrt{1+3l+2l^2}}\frac{\sin 2(l+1)\theta}{\sin 2\theta} \qquad (l = 0,1,2,\dots) \\ &= \frac{1}{2^l\sqrt{1+3l+2l^2}}\sum_{n=0}^{l}\cos(2n\theta)\cos(2\theta)^{l-n} \\ &= \frac{1}{2^l\sqrt{1+3l+2l^2}}\sum_{n=0}^{l}\frac{(-1)^n}{2}\binom{2l+2}{2n+1}(\cos^2\theta)^{l-n}(\sin^2\theta)^n. \end{aligned} \tag{11.16}$$

Note that the form of $Y^{(2l)}$ in the last line is particularly useful for rewriting the harmonic to a form as in equation (11.7), one simply needs to substitute the expression of $\cos^2\theta$ and $\sin^2\theta$ in terms of the coordinates y^I that was provided above to obtain

$$Y^{(2l)} = \frac{1}{2^l\sqrt{1+3l+2l^2}} \sum_{n=0}^{l} \frac{(-1)^n}{2r^{2l}} \binom{2l+2}{2n+1} [(y^1)^2+(y^2)^2+(y^3)^2]^{l-n}[(y^4)^2+(y^5)^2+(y^6)^2]^n. \tag{11.17}$$

The solutions $Y^{(2l)}$ have been normalised to conform with the standard convention set in equation (11.3), which is also used in [111] and many other places in literature:

$$\int_{S^5} Y^k Y^{k'} = \delta^{kk'} \frac{\pi^3}{2^{k-1}(k+1)(k+2)}. \tag{11.18}$$

The reason for this particular choice of normalisation is that it implies that

$$\sum_{I_1\ldots I_k} C^k_{I_1\ldots I_k} C^{k'}_{I_1\ldots I_k} = \delta^{kk'}. \tag{11.19}$$

We can list the lowest lying harmonics corresponding to $l \in \{0,1,2,3\}$. The angles θ and ϕ can be transformed to coordinates $\vec{u}$ and $\vec{v}$, which were used in Chapter 9 and are summarised in Appendix A.4, using $\cos^2\theta = u^2$, $\sin^2\theta = v^2$, and $\sin\theta\cos\theta\cos\phi = (u\cdot v)$:

$$Y^{(0)} = 1 \tag{11.20}$$

$$Y^{(2)} = \frac{1}{\sqrt{6}}\cos 2\theta = \frac{1}{\sqrt{6}}(u^2 - v^2) \tag{11.21}$$

$$Y^{(4)} = \frac{1}{4\sqrt{15}}(2\cos 4\theta + 1) = \frac{1}{4\sqrt{15}}(3u^4 - 10u^2v^2 + 3v^4) \tag{11.22}$$

$$Y^{(6)} = \frac{1}{8\sqrt{7}}(\cos 2\theta + \cos 6\theta) = \frac{1}{4\sqrt{7}}(u^6 - 7u^4v^2 + 7u^2v^4 - v^6) \tag{11.23}$$

To sum up, we find that among all SO(6) scalar spherical harmonics there is always one for each even k that is invariant under $\mathrm{SO}(3)\times\mathrm{SO}(3) \subset \mathrm{SO}(6)$, and there are no such harmonics for odd k. In terms of group theory this means that for even k the SO(6) representation $[k,0,0]$ has exactly one singlet after the branching under $\mathrm{SO}(3)\times\mathrm{SO}(3)$, while for odd k there are no singlets. We can verify this for the lowest representations by comparing with the branching rules in Appendix E:

$$[0,0,0] = \mathbf{1} \to (\mathbf{1},\mathbf{1}) \tag{11.24}$$

$$[1,0,0] = \mathbf{6} \to (\mathbf{3},\mathbf{1}) + (\mathbf{1},\mathbf{3}) \tag{11.25}$$

$$[2,0,0] = \mathbf{20'} \to (\mathbf{1},\mathbf{1}) + (\mathbf{3},\mathbf{3}) + (\mathbf{5},\mathbf{1}) + (\mathbf{1},\mathbf{5}) \tag{11.26}$$

$$[3,0,0] = \mathbf{50} \to (\mathbf{3},\mathbf{1}) + (\mathbf{1},\mathbf{3}) + (\mathbf{5},\mathbf{3}) + (\mathbf{3},\mathbf{5}) + (\mathbf{7},\mathbf{1}) + (\mathbf{1},\mathbf{7}) \tag{11.27}$$

etc.

The fact that there are no singlets for odd k makes sense: to get a singlet all k indices have to be contracted into traces, which is only possible for even k. For the same reason it is clear that for even k there will be at least one singlet. It is not obvious to us why there is exactly one, but one can view the explicit construction of the harmonics above as a proof that this is indeed the case.

11.3 Scalar Spherical Harmonics with $\mathrm{SO}(3)_{\mathrm{diag}}$ Symmetry

In this section we would like to describe a subset of the S^5 scalar spherical harmonics that are invariant under the $\mathrm{SO}(3)_{\mathrm{diag}} \subset \mathrm{SO}(3) \times \mathrm{SO}(3) \subset \mathrm{SO}(6)$ symmetry. We will proceed as described in the previous section and solve the Laplace eigenvalue problem

$$\mathring{\nabla}^2 Y^{(k)} = -k(k+4) Y^{(k)}, \qquad k = 0, 1, 2, \dots \tag{11.28}$$

First of all we need to fix a coordinate system on $\mathbb{R}^6$ in which the $\mathrm{SO}(3)_{\mathrm{diag}}$ symmetry is manifest. A good choice are the coordinates introduced in [54], which we reviewed in Appendix A.4. The metric on the $\mathbb{R}^6$ written in these coordinates reads

$$\begin{aligned} ds^2 &= d(ru^i)d(ru^i) + d(rv^i)d(rv^i) \\ &= dr^2 + r^2 \left[d\theta^2 + \cos^2\theta\, (\sigma_1^2 + \sigma_2^2) + \sin^2\theta\, (d\phi + \sigma_1)^2 + \sin^2\theta\, (\cos\phi\, \sigma_2 - \sin\phi\, \sigma_3)^2 \right]. \end{aligned} \tag{11.29}$$

The terms proportional to r^2 form the round metric on the S^5. The left-invariant 1-forms σ_i were constructed in Appendix D, and are parametrised through three Euler angles. We are looking for solutions $Y^{(k)}(\theta, \phi)$ that do not depend on these Euler angles, and are therefore invariant under $\mathrm{SO}(3)_{\mathrm{diag}}$ rotations. In these coordinates and under the assumption that $Y^{(k)}$ do not depend on the $\mathrm{SO}(3)_{\mathrm{diag}}$ angles the Laplace equation (11.28) becomes

$$\frac{1}{\sin^2\theta\cos^2\theta}\frac{\partial}{\partial\theta}\left(\sin^2\theta\cos^2\theta\frac{\partial}{\partial\theta}Y^{(k)}\right) + \frac{1}{\sin^2\theta\cos^2\theta\sin\phi}\frac{\partial}{\partial\phi}\left(\sin\phi\frac{\partial}{\partial\phi}Y^{(k)}\right) + k(k+4)Y^{(k)} = 0. \tag{11.30}$$

This partial differential equation is separable. After separating the angles and equating the two parts to $\pm m(m+1)$ one obtains the following solutions:

$$Y(\theta, \phi) = A(\theta) B(\phi) \tag{11.31}$$

$$\begin{aligned} A(\theta) = {} & c_1 (\sin\theta\cos\theta)^m {}_2F_1(-\frac{k}{2} + m, \frac{k}{2} + m + 2; m + \frac{3}{2}; \cos^2\theta) \\ & + c_2 (\sin\theta\cos\theta)^{-(m+1)} {}_2F_1(-\frac{k}{2} - m - 1, \frac{k}{2} - m + 1; \frac{1}{2} - m; \cos^2\theta) \end{aligned} \tag{11.32}$$

$$B(\phi) = c_3 P_m(\cos\phi) + c_4 Q_m(\cos\phi). \tag{11.33}$$

The four constants c_i are integration constants, ${}_2F_1(a,b;c;x)$ is the hypergeometric function, and $P_m(x)$ and $Q_m(x)$ are the Legendre polynomials of the first and second kind. Our separation ansatz was invariant under $m \to -(m+1)$, and so are the solutions $A(\theta)$ and $B(\phi)$ up to redefinition of c_4. This is because $P_{-(m+1)} = P_m$ but $Q_{-(m+1)} = Q_m - \frac{\pi}{\tan(\pi m)} P_m$ [112].

To find which of these solutions are good spherical harmonics note first that Q_m is never a finite polynomial in its argument, so we should discard it by setting $c_4 = 0$. Because the terms proportional to c_1 and c_2 in $A(\theta)$ are interchanged upon $m \to -(m+1)$ but P_m is invariant, we can set $c_2 = 0$ without loss of generality. Furthermore, P_m is only a finite polynomial for integer values of m, and ${}_2F_1(a,b;c;x)$ is a finite polynomial only if a or b is a negative integer or zero. Therefore k must be an even integer and we should set $k = 2l$, and assume $m \leq l$. The series expansion of ${}_2F_1$ is in non-negative powers of $\cos^2\theta$ and therefore for a negative m the denominator in $(\sin\theta\cos\theta)^m$ cannot be cancelled, so we need to take $m \geq 0$. Thus we obtain the following solutions for the scalar spherical harmonics:

$$Y^{(2l,m)}(\theta,\phi) = c_{l,m}\,(\sin\theta\cos\theta)^m\,{}_2F_1(-l+m, l+m+2; m+\frac{3}{2}; \cos^2\theta) P_m(\cos\phi) \quad (11.34)$$

$$c_{l,m} = (-1)^{l+m}\sqrt{\frac{2^{1-2l+4m}}{2l+1}\frac{m!(m+1)!(l+m+1)!}{(2m)!(2m+2)!(l-m)!}}, \quad (11.35)$$
$$l = 0,1,2,\dots \quad \text{and} \quad 0 \leq m \leq l.$$

The constants $c_{l,m}$ are chosen such that the harmonics are properly normalised according to the normalisation in equation (11.3). In determining $c_{l,m}$ we made an arbitrary sign choice by adding a factor of $(-1)^{l+m}$. This sign choice makes the harmonics look nice by removing overall minus signs and additionally with this convention the $m = 0$ case matches exactly the $\mathrm{SO}(3)\times\mathrm{SO}(3)$ invariant solutions in (11.16).

Under our assumptions on the range of the variables l and m the hypergeometric function and the Legendre polynomials can be expanded into finite-order polynomials as follows:

$$\begin{aligned}{}_2F_1(-l+m, l+m+2; m+\frac{3}{2}; \cos^2\theta) = \\ = \sum_{n=0}^{l-m}\frac{(l+m+n+1)!}{(l+m+1)!}\frac{(l-m)!}{(l-m-n)!}\frac{(m+n+1)!}{(m+1)!}\frac{(2m+2)!}{(2m+2n+2)!}\frac{(-4\cos^2\theta)^n}{n!}\end{aligned} \quad (11.36)$$

$$P_m(\cos\phi) = \sum_{n=0}^{m}\binom{m}{n}\binom{m+n}{n}\frac{(\cos\phi-1)^n}{2^n}. \quad (11.37)$$

This determines the harmonics uniquely up to an overall sign. The result for the harmonics $Y^{(2l,m)}$ in (11.34) and the allowed quantum numbers are again supported by the explicit construction of representations and the appearance of $\mathrm{SO}(3)_{\mathrm{diag}}$ singlets. Under $\mathrm{SO}(3)_{\mathrm{diag}} \subset \mathrm{SO}(3)\times\mathrm{SO}(3)$ the representations $(\mathbf{r}_1, \mathbf{r}_2)$ collapse to $\mathbf{r}_1 \otimes \mathbf{r}_2$, and we see the following picture

$$[0,0,0] = \mathbf{1} \quad \to \mathbf{1} \tag{11.38}$$

$$[1,0,0] = \mathbf{6} \quad \to \mathbf{3} + \mathbf{3} \tag{11.39}$$

$$[2,0,0] = \mathbf{20'} \to (\mathbf{1}+\mathbf{1}) + \mathbf{3} + (\mathbf{5}+\mathbf{5}+\mathbf{5}) \tag{11.40}$$

$$[3,0,0] = \mathbf{50} \to (\mathbf{3}+\mathbf{3}+\mathbf{3}+\mathbf{3}) + (\mathbf{5}+\mathbf{5}) + (\mathbf{7}+\mathbf{7}+\mathbf{7}+\mathbf{7}) \tag{11.41}$$

etc.

We see the explicit appearance of one singlet in the case $l = 0$ and two singlets in the case $l = 1$. We see also that for odd k there are again no singlets at all.

Here is a list of the first few harmonics corresponding to $l \in \{0, 1, 2, 3\}$. They are properly normalised and the angles θ and ϕ were transformed to coordinates $\vec{u}$ and $\vec{v}$ using $\cos^2\theta = u^2$, $\sin^2\theta = v^2$, and $\sin\theta\cos\theta\cos\phi = (u \cdot v)$.

$$Y^{(0,0)} = 1 \tag{11.42}$$

$$Y^{(2,0)} = \frac{1}{\sqrt{6}}\cos 2\theta = \frac{1}{\sqrt{6}}(u^2 - v^2) \tag{11.43}$$

$$Y^{(2,1)} = \frac{1}{\sqrt{6}}\cos\phi\sin 2\theta = \sqrt{\frac{2}{3}}(u \cdot v) \tag{11.44}$$

$$Y^{(4,0)} = \frac{1}{4\sqrt{15}}(2\cos 4\theta + 1) = \frac{1}{4\sqrt{15}}(3u^4 - 10u^2v^2 + 3v^4) \tag{11.45}$$

$$Y^{(4,1)} = \frac{1}{2\sqrt{10}}\cos\phi\sin 4\theta = \sqrt{\frac{2}{5}}(u \cdot v)(u^2 - v^2) \tag{11.46}$$

$$Y^{(4,2)} = \frac{1}{\sqrt{30}}\sin^2\theta\cos^2\theta\ (3\cos 2\phi + 1) = \sqrt{\frac{2}{15}}[3(u \cdot v)^2 - u^2v^2] \tag{11.47}$$

$$Y^{(6,0)} = \frac{1}{8\sqrt{7}}(\cos 2\theta + \cos 6\theta) = \frac{1}{4\sqrt{7}}(u^2 - v^2)(u^4 - 6u^2v^2 + v^4) \tag{11.48}$$

$$Y^{(6,1)} = \frac{1}{8\sqrt{35}}\cos\phi\ (\sin 2\theta + 3\sin 6\theta) = \frac{1}{2\sqrt{35}}(u \cdot v)(5u^4 - 14u^2v^2 + 5v^4) \tag{11.49}$$

$$Y^{(6,2)} = \frac{1}{2\sqrt{7}}\sin^2\theta\cos^2\theta\cos 2\theta\ (3\cos 2\phi + 1) = \frac{1}{\sqrt{7}}(u^2 - v^2)[3(u \cdot v)^2 - u^2v^2] \tag{11.50}$$

$$Y^{(6,3)} = \frac{1}{\sqrt{35}}\sin^3\theta\cos^3\theta\cos\phi\ (5\cos 2\phi - 1) = \frac{2}{\sqrt{35}}(u \cdot v)[5(u \cdot v)^2 - 3u^2v^2] \tag{11.51}$$

We see that the harmonics with $m = 0$ correspond exactly to the $\mathrm{SO}(3) \times \mathrm{SO}(3)$ invariant ones found in the previous section. Note also that while $\vec{u}^2$ and $\vec{v}^2$ are invariant under both $\mathrm{SO}(3) \times \mathrm{SO}(3)$ and $\mathrm{SO}(3)_{\mathrm{diag}}$, the combination $\vec{u} \cdot \vec{v}$ is only invariant under $\mathrm{SO}(3)_{\mathrm{diag}}$. This is why in all harmonics with $m = 0$ only powers of $\vec{u}^2$ and $\vec{v}^2$ show up, while for $m \neq 0$ terms proportional to $\vec{u} \cdot \vec{v}$ appear and break $\mathrm{SO}(3) \times \mathrm{SO}(3)$ down to $\mathrm{SO}(3)_{\mathrm{diag}}$.

The scalar harmonics can also be converted to the $\{w_1, w_2\}$ basis, which is introduced in

Appendix A.4, using $w_1 = 2u^2 - 1 = 1 - 2v^2 = u^2 - v^2$ and $w_2 = 2(u \cdot v)$. Here is the result:

$$Y^{(0,0)} = 1 \tag{11.52}$$
$$Y^{(2,0)} = \frac{w_1}{\sqrt{6}} \tag{11.53}$$
$$Y^{(2,1)} = \frac{w_2}{\sqrt{6}} \tag{11.54}$$
$$Y^{(4,0)} = \frac{4w_1^2 - 1}{4\sqrt{15}} \tag{11.55}$$
$$Y^{(4,1)} = \frac{w_1 w_2}{\sqrt{10}} \tag{11.56}$$
$$Y^{(4,2)} = \frac{w_1^2 + 3w_2^2 - 1}{2\sqrt{30}} \tag{11.57}$$
$$Y^{(6,0)} = \frac{w_1(2w_1^2 - 1)}{4\sqrt{7}} \tag{11.58}$$
$$Y^{(6,1)} = \frac{w_2(6w_1^2 - 1)}{4\sqrt{35}} \tag{11.59}$$
$$Y^{(6,2)} = \frac{w_1(w_1^2 + 3w_2^2 - 1)}{4\sqrt{7}} \tag{11.60}$$
$$Y^{(6,3)} = \frac{w_2(3w_1 + 5w_2^2 - 3)}{4\sqrt{35}}. \tag{11.61}$$

11.4 Vector Spherical Harmonics with $\mathrm{SO}(3)_{\mathrm{diag}}$ Symmetry

11.4.1 Lowest Vector Harmonics from Group Theory

The lowest vector harmonic $Y_{\mathbf{5}}^{(0,m)}$ with $k = 0$ transforms in the $[0,1,1] = \mathbf{15}$ of the $\mathfrak{so}(6)$. This is the symmetric and traceless part of the product of two $\mathfrak{so}(6)$ spinors of different chiralities, and it can also be viewed as the antisymmetric part of the product of two $\mathfrak{so}(6)$ vectors. Both can be mapped to each other by some appropriately defined Clebsch-Gordan coefficients $\Sigma^{[IJ]}{}_A{}^B$, which are also discussed in Appendix B. However, it is also true that the lowest vector harmonics corresponding to the case $k = 0$ are given by the S^5 Killing vectors $K_n^{[IJ]}$, which are defined as [57]

$$Y_{\mathbf{5}\,n}^{(0,m)} \sim K_n^{[IJ]} = \partial_n y^{[I} y^{J]}. \tag{11.62}$$

The coordinates y^I are the embeddings of the S^5 into $\mathbb{R}^6$, and the anti-symmetrised index pair $[IJ]$ corresponds to the $\mathbf{15}$ of the $\mathfrak{so}(6)$ in which the lowest vector harmonics transform. This also implies that the index m will assume 15 distinct values, as it has to be mapped to the index pair $[IJ]$. These Killing vectors satisfy the Killing equation

$$\mathring{\nabla}_n K_p^{[IJ]} + \mathring{\nabla}_p K_n^{[IJ]} = 0, \tag{11.63}$$

and the overall normalisation in (11.62) was chosen such that viewed as differential operators $K^{IJ} = K^{[IJ]\,n}\partial_n$ the SO(6) commutator is canonical:

$$[K^{IJ}, K^{KL}] = \delta^{IK}K^{JL} + \delta^{JL}K^{IK} - \delta^{IL}K^{JK} - \delta^{JK}K^{IL}. \tag{11.64}$$

The decomposition of the **15** under $\mathrm{SO}(3)_{\mathrm{diag}} \subset \mathrm{SO}(3)\times\mathrm{SO}(3) \subset \mathrm{SO}(6)$ is given by

$$\mathbf{15} \to (\mathbf{3},\mathbf{1}) + (\mathbf{1},\mathbf{3}) + (\mathbf{3},\mathbf{3}) \to \mathbf{3} + \mathbf{3} + (\mathbf{1} + \mathbf{3} + \mathbf{5}). \tag{11.65}$$

We see that one singlet appears. If one decomposes the 6-index $I \to (i, \bar{\imath})$, then the first step in the decomposition above can we written as

$$K^{[IJ]} \to \{K^{[ij]}, K^{[\bar{\imath}\bar{\jmath}]}, K^{[i\bar{\jmath}]}\} \equiv \{\epsilon^{ijk}K^{k}_{(\mathbf{3},\mathbf{1})}, \epsilon^{\bar{\imath}\bar{\jmath}\bar{k}}K^{\bar{k}}_{(\mathbf{1},\mathbf{3})}, K^{i\bar{\jmath}}_{(\mathbf{3},\mathbf{3})}\}, \tag{11.66}$$

and the second step is the identification $i \leftrightarrow \bar{\imath}$. The singlet that we saw appearing above comes from $(\mathbf{3},\mathbf{3}) \to \mathbf{3}\otimes\mathbf{3} = \mathbf{1}+\mathbf{3}+\mathbf{5}$ and corresponds to the trace

$$Y_{\mathbf{5}}^{(0,0)} \sim \sum_{i=\bar{\imath}} K^{i\bar{\imath}}_{(\mathbf{3},\mathbf{3})} = K^{[14]} + K^{[25]} + K^{[36]} = y^{[1}dy^{4]} + y^{[2}dy^{5]} + y^{[3}dy^{6]}, \tag{11.67}$$

where we are viewing the components of the Killing vectors as components of some 1-forms on the sphere. We can now parametrise the embedding of the S^5 into $\mathbb{R}^6$ as in [54] in terms of two three-vectors $\vec{u}$ and $\vec{v}$ and write $y^I = (\vec{u}, \vec{v})$. The $\mathrm{SO}(3)\times\mathrm{SO}(3)$ rotations act on the components of $\vec{u}$ and $\vec{v}$ in the fundamental representation. We then decompose the differentials dy^I into left-invariant 1-forms. This gives the following expression for the vector harmonics 1-form:

$$Y_{\mathbf{5}}^{(0,0)} \sim 2\cos\theta\sin\theta\sin\phi\,\sigma^1 - \cos\phi\, d\theta + \cos\theta\sin\theta\sin\phi\, d\phi \tag{11.68}$$

$$= \sqrt{1-w_1^2-w_2^2}\,\sigma^1 + \frac{1}{2}\frac{w_2}{w_1+1}\,dw_1 - \frac{1}{2}\,dw_2. \tag{11.69}$$

We can now proceed similarly and use group theory to find the $k=1$ vector harmonics that are $\mathrm{SO}(3)_{\mathrm{diag}}$ singlets. The $k=1$ vector harmonics on the S^5 transform in the $[1,1,1] = \mathbf{64}$ of the SO(6), and the harmonics with the $\mathrm{SO}(3)_{\mathrm{diag}}$ symmetry should appear as singles of the decomposition of this representation under $\mathrm{SO}(3)_{\mathrm{diag}} \subset \mathrm{SO}(6)$. It is not straightforward to find this decomposition by appropriately symmetrising the indices since both vector and spinor indices appear in the $[1,1,1]$. We can make progress by noting that the $[1,1,1]$ appears in the following tensor product

$$\begin{aligned}\mathbf{6}\otimes\mathbf{15} &= [1,0,0]\otimes[0,1,1]\\ &= [1,0,0]+[0,2,0]+[0,0,2]+[1,1,1]\\ &= \mathbf{6}+\mathbf{10}+\overline{\mathbf{10}}+\mathbf{64}.\end{aligned} \tag{11.70}$$

All other representations in this decomposition can be written with vector indices only, and can therefore be decomposed by hand. After writing $I \to (i, \bar{\imath})$ as before we get the following decomposition under SO(3) × SO(3) ⊂ SO(6):

$$\mathbf{6} = T^I \to \{T^i, T^{\bar{\imath}}\} = (\mathbf{3}, \mathbf{1}) + (\mathbf{1}, \mathbf{3}) \tag{11.71}$$

$$\begin{aligned} \mathbf{15} = [0,1,1] = T^{[IJ]} &\to \{T^{[ij]}, T^{[i\bar{\jmath}]}, T^{[\bar{\imath}\bar{\jmath}]}\} \\ &= \{\epsilon^{ijk}T^{[jk]}, T^{[i\bar{\jmath}]}, \epsilon^{\bar{\imath}\bar{\jmath}\bar{k}}T^{[\bar{\jmath}\bar{k}]}\} \\ &= (\mathbf{3}, \mathbf{1}) + (\mathbf{3}, \mathbf{3}) + (\mathbf{1}, \mathbf{3}) \end{aligned} \tag{11.72}$$

$$\begin{aligned} (\mathbf{10} + \overline{\mathbf{10}}) = T^{[IJK]} &\to \{T^{[ijk]}, T^{[\bar{\imath}jk]}, T^{[\bar{\imath}\bar{\jmath}k]}, T^{[\bar{\imath}\bar{\jmath}\bar{k}]}\} \\ &= \{\epsilon^{ijk}T^{[123]}, \epsilon^{ijk}T^{[\bar{\imath}jk]}, \epsilon^{\bar{\imath}\bar{\jmath}\bar{k}}T^{[i\bar{\jmath}\bar{k}]}, \epsilon^{\bar{\imath}\bar{\jmath}\bar{k}}T^{[\bar{1}\bar{2}\bar{3}]}\} \\ &= (\mathbf{1}, \mathbf{1}) + (\mathbf{3}, \mathbf{3}) + (\mathbf{3}, \mathbf{3}) + (\mathbf{1}, \mathbf{1}) \end{aligned} \tag{11.73}$$

As a side remark note that the tensor $T^{[IJK]}$ is reducible because it can be decomposed into the self-dual and anti-self-dual parts using the 6-dimensional epsilon tensor ϵ^{IJKLMN}, which correspond exactly to the **10** and the $\overline{\mathbf{10}}$. This has no effect on the decomposition. We can now compute the tensor product on the left-hand side of (11.70) using the decomposed representations and compare them to the representations that appear on the right-hand side to find what the **64** decomposes into. The result is

$$\mathbf{64} \to (\mathbf{5}, \mathbf{3}) + (\mathbf{3}, \mathbf{5}) + (\mathbf{5}, \mathbf{1}) + (\mathbf{1}, \mathbf{5}) + (\mathbf{3}, \mathbf{1}) + (\mathbf{1}, \mathbf{3}) + (\mathbf{3}, \mathbf{3}) + (\mathbf{3}, \mathbf{3}). \tag{11.74}$$

To compute the decomposition under $\mathrm{SO}(3)_{\mathrm{diag}} \subset \mathrm{SO}(3) \times \mathrm{SO}(3)$ we recall that the representations $(\mathbf{r}_1, \mathbf{r}_2)$ collapse to $\mathbf{r}_1 \otimes \mathbf{r}_2$, and we get

$$\mathbf{64} \to (\mathbf{7} + \mathbf{5} + \mathbf{3}) + (\mathbf{7} + \mathbf{5} + \mathbf{3}) + \mathbf{5} + \mathbf{5} + \mathbf{3} + \mathbf{3} + (\mathbf{5} + \mathbf{3} + \mathbf{1}) + (\mathbf{5} + \mathbf{3} + \mathbf{1}). \tag{11.75}$$

We see that the **64** contains two $\mathrm{SO}(3)_{\mathrm{diag}}$ singlets, therefore we expect to find two $k = 1$ vector harmonics that are invariant under the $\mathrm{SO}(3)_{\mathrm{diag}}$.

To construct explicit expressions for the harmonics recall that the scalar harmonics were constructed by taking symmetric and traceless products of k embedding coordinates y^I, see equation (11.7). In terms of group theory, since each y^I is in the $[1,0,0]$, this corresponds to

$$[1,0,0]^k\Big|_{[k,0,0]} . \tag{11.76}$$

We can generalise this construction to vector harmonics by taking products of k coordinates $y^I = [1,0,0]$ with one Killing vector $K^{[IJ]} = [0,1,1]$ and project out the representations that we need, in other words we need to take

$$\left([1,0,0]^k \otimes [0,1,1]\right)\Big|_{[k,1,1]} . \tag{11.77}$$

For $k=1$ we get the case that we studied above:

$$y^I K^{[JK]}\Big|_{[1,1,1]} = \left([1,0,0]\otimes[0,1,1]\right)\big|_{[1,1,1]}\,. \tag{11.78}$$

As before, its not straightforward to see what the projector to the $[1,1,1]$ should look like, but we know that the singlets are contained in the $(\mathbf{3},\mathbf{3})$ parts of the $\mathbf{64}$. Two possibilities to construct a $(\mathbf{3},\mathbf{3})$ out of $x^I K^{[JK]}$ are the following:

$$\epsilon^{ijk}x^j K^{[k\bar{\imath}]} \quad \text{and} \quad \epsilon^{\bar{\imath}\bar{\jmath}\bar{k}}x^{\bar{\jmath}}K^{[\bar{k}i]}. \tag{11.79}$$

There are two other inequivalent ways to construct a $(\mathbf{3},\mathbf{3})$, they, however, correspond to those in the $(\mathbf{10}+\overline{\mathbf{10}})$, therefore we can assume that the representations we constructed in (11.79) are indeed those contained in the $\mathbf{64}$. The $\mathrm{SO}(3)_{\mathrm{diag}}$ singlets in the $(\mathbf{3},\mathbf{3})$ are the traces over the free indices after identifying i with $\bar{\imath}$. This gives us the following expressions for the harmonics

$$Y_{\mathbf{5}}^{(1,0)} \sim \sum_{i=\bar{\imath}} \epsilon^{ijk}x^j K^{[k\bar{\imath}]} = -\sin\theta\cos^2\theta\sin\phi\,\sigma^2 \tag{11.80}$$

$$Y_{\mathbf{5}}^{(1,1)} \sim \sum_{i=\bar{\imath}} \epsilon^{\bar{\imath}\bar{\jmath}\bar{k}}x^{\bar{\jmath}}K^{[\bar{k}i]} = \sin^2\theta\cos\theta\sin\phi\cos\phi\,\sigma^2 - \sin^2\theta\cos\theta\sin^2\phi\,\sigma^3. \tag{11.81}$$

Finally, we can normalise the vector harmonics as fixed in equation (11.3). With this normalisation and an arbitrary choice for the overall sign the $\mathrm{SO}(3)_{\mathrm{diag}}$ invariant $k=0$ and $k=1$ vector harmonics read

$$Y_{\mathbf{5}}^{(0,0)} = \cos\theta\sin\theta\sin\phi\,(2\sigma^1 + d\phi) - \cos\phi\, d\theta \qquad = u^i dv^i - v^i du^i \tag{11.82}$$

$$Y_{\mathbf{5}}^{(1,0)} = \frac{2}{\sqrt{3}}\sin\theta\cos^2\theta\sin\phi\,\sigma^2 \qquad = \frac{2}{\sqrt{3}}(\epsilon^{ijk}v^i u^j du^k) \tag{11.83}$$

$$Y_{\mathbf{5}}^{(1,1)} = \frac{2}{\sqrt{3}}\sin^2\theta\cos\theta\sin\phi\,(\cos\phi\,\sigma^2 - \sin\phi\,\sigma^3) \qquad = \frac{2}{\sqrt{3}}(\epsilon^{ijk}v^i u^j dv^k). \tag{11.84}$$

We can change the (θ,ϕ) coordinates to (w_1,w_2) and re-write the harmonics as follows

$$Y_{\mathbf{5}}^{(0,0)} = \sqrt{\zeta}\,\sigma^1 + \frac{1}{4}\left(\frac{1+w_1}{2}\right)^{-1} w_2\, dw_1 - \frac{1}{2}\, dw_2 \tag{11.85}$$

$$Y_{\mathbf{5}}^{(1,0)} = \sqrt{\frac{1}{3}}\left(\frac{1+w_1}{2}\right)^{1/2}\sqrt{\zeta}\,\sigma^2 \tag{11.86}$$

$$Y_{\mathbf{5}}^{(1,1)} = \sqrt{\frac{1}{12}}\left(\frac{1+w_1}{2}\right)^{-1/2}(w_2\sqrt{\zeta}\sigma^2 - \zeta\sigma^3). \tag{11.87}$$

We verified explicitly that these harmonics indeed solve the defining equation (11.1) for $k=0$ and $k=1$, and the divergence constraint (11.2). Also the orthogonality and normalisation holds as in (11.3).

In general, it is true that after branching the representation $[k,1,1]$ of the SO(6) to representations of the $\mathrm{SO}(3)_{\mathrm{diag}}$ only odd-dimensional representations with $d = 1,3,5,\ldots,2k+5$ appear. This is because the sum of the weights in $[k,1,1]$ that correspond to spin representations, $1+1$, is even, and this property is preserved by the branching. Remember that for SO(3) a representation with weight w has dimension $d = w+1$, and so even SO(3) weights give rise to odd-dimensional representations. The meaning of this is that bosons branch to bosons. The multiplicity with which an $\mathrm{SO}(3)_{\mathrm{diag}}$ representation of dimension d appears in $[k,1,1]$ seems to be given by the formula $k+1+2x(2+k-x)$, where x is defined by $d \equiv 2x+1$ and has the range $x \in \{1,\ldots,k+2\}$. For the singlets, that is $x=0$, the multiplicity becomes $k+1$ and is the upper bound on the number of $\mathrm{SO}(3)_{\mathrm{diag}}$ invariant vector harmonics that we are expecting to find. Even though this is exactly the number of singlets we found for the cases $k=0$ and $k=1$, for some singlets the harmonics might actually turn out to vanish, so this upper bound does not have to be satisfied.

11.4.2 Solving the Laplace Equation

The lowest vector harmonics found from group theory arguments allow us to make some important observations. First of all it is clear that the solutions have to be of the form

$$Y_{\mathbf{5}} = v_1(\theta,\phi)\sigma^1 + v_2(\theta,\phi)\sigma^2 + v_3(\theta,\phi)\sigma^3 + v_4(\theta,\phi)d\theta + v_5(\theta,\phi)d\phi. \tag{11.88}$$

Because we are looking for $\mathrm{SO}(3)_{\mathrm{diag}}$ invariant solutions the only 1-forms that can be considered as a basis are the invariant forms $d\theta$ and $d\phi$, and the left-invariant 1-forms σ^i. Moreover the coefficient functions $v_i(\theta,\phi)$ must not depend on the internal angles α_i. Let us recall that we are solving the following differential equation that we already gave in equation (11.1):

$$\mathring{\nabla}^2 Y^{(k)}_{\mathbf{5}\,n} = -(k^2+6k+4)Y^{(k)}_{\mathbf{5}\,n}, \qquad k = 0,1,2,\ldots \tag{11.89}$$

Note that this equation is written in some coordinate basis such as $\{d\alpha_i, d\theta, d\phi\}$, where α_i are the internal Euler angles, which is why after applying it to our ansatz in (11.88) the internal angles reappear and it is hopeless to solve the resulting coupled partial differential equations directly. It is therefore useful to rewrite the defining equation in terms of the invariant 1-form basis. The framework in which this is done is the well-known Cartan formalism. We view the invariant 1-forms as a set of local sections of the cotangent bundle, and define

$$e^a \equiv \{\sigma^1,\sigma^2,\sigma^3,d\theta,d\phi\} = e^a_m dy^m. \tag{11.90}$$

One could now define the metric $g_{ab} = e^m_a g_{mn} e^n_b$ and the Christoffel symbols $\Gamma^b_{ma} = e^n_a \Gamma^r_{mn} e^b_r - e^n_a \partial_m e^b_n$ in the new basis and start computing the curvatures. However, for our problem it is sufficient to contract the defining equation (11.89) with the transformation matrices e_a^m. Note that the divergence constraint (11.2) is invariant under the change of basis and therefore does not need to be rewritten. After carrying out the transformations just described we arrive at the following differential equations for the components $v_i(\theta, \phi)$:

$$0 = (8+6k+k^2)v_1 + \frac{1}{\sin^2\theta}\partial_\theta(\sin^2\theta\,\partial_\theta v_1) + \frac{1}{\sin^2\theta\cos^2\theta}\left(\frac{1}{\sin\phi}\partial_\phi(\sin\phi\,\partial_\phi v_1) - \frac{1}{\sin^2\phi}v_1\right) + \frac{2}{\sin\theta\cos\theta}(\partial_\phi v_4 - \partial_\theta v_5) \tag{11.91}$$

$$0 = (8+6k+k^2)v_2 + \frac{1}{\sin^2\theta}\partial_\theta(\sin^2\theta\,\partial_\theta v_2) + \frac{1}{\sin^2\theta\cos^2\theta}\left(\frac{1}{\sin\phi}\partial_\phi(\sin\phi\,\partial_\phi v_2) - \frac{1}{\sin^2\phi}v_2\right) + \frac{2}{\sin\theta\cos\theta}\left(\frac{1}{\tan\phi}\partial_\theta v_3 - \frac{1}{\sin\theta\cos\theta}\partial_\phi v_3\right) \tag{11.92}$$

$$0 = (8+6k+k^2)v_3 + \frac{1}{\cos^2\theta}\partial_\theta(\cos^2\theta\,\partial_\theta v_3) + \frac{1}{\sin^2\theta\cos^2\theta}\sin\phi\,\partial_\phi(\frac{1}{\sin\phi}\partial_\phi v_3) \tag{11.93}$$

$$0 = (8+6k+k^2)v_4 + \frac{1}{\sin^2\theta\cos^2\theta}\left(\partial_\theta(\sin^2\theta\cos^2\theta\,\partial_\theta v_4) + \frac{1}{\sin\phi}\partial_\phi(\sin\phi\,\partial_\phi v_4) - 2v_4\right) - \frac{2}{\sin\theta\cos\theta}\left(\tan^2\theta\frac{1}{\sin\phi}\partial_\phi(\sin\phi\, v_1) + (\frac{1}{\sin^2\theta} - \frac{1}{\cos^2\theta})\frac{1}{\sin\phi}\partial_\phi(\sin\phi\, v_5)\right) \tag{11.94}$$

$$0 = (8+6k+k^2)v_5 + \frac{1}{\cos^2\theta}\partial_\theta(\cos^2\theta\,\partial_\theta v_5) + \frac{1}{\sin^2\theta\cos^2\theta}\left(\frac{1}{\sin\phi}\partial_\phi(\sin\phi\,\partial_\phi v_5) - \frac{1}{\sin^2\phi}v_5\right) + \frac{2}{\sin\theta\cos\theta}\cos^2\theta\partial_\phi v_4. \tag{11.95}$$

One might notice that the original eigenvalue that we used in the Laplace equation is $(4+6k+k^2)$, while here these terms have become $(8+6k+k^2)$. This is because the Laplacian contains a part that is proportional to the Ricci tensor, which in case of the sphere S^5 is simply given by $R_{mn} = 4g_{mn}$, and therefore contributes an additional 4 to the equation.

The divergence constraint takes the following form

$$0 = \sin^2\theta\frac{1}{\sin\phi}\partial_\phi(\sin\phi\, v_1) - \partial_\theta(\sin^2\theta\cos^2\theta\, v_4) - \frac{1}{\sin\phi}\partial_\phi(\sin\phi\, v_5). \tag{11.96}$$

One can split the set of partial differential equations for v_i into their homogeneous parts and the inhomogeneities. The differential equations can then be written as

$$0 = \left[(8+6k+k^2)\delta_i^j + \mathcal{H}_i^j + \frac{2}{\sin\theta\cos\theta}\mathcal{I}_i^j\right]v_j. \tag{11.97}$$

The homogeneous differential operator $\mathcal{H}_i^j$, which by definition is diagonal, is given by

$$\mathcal{H}_1^1 = \frac{1}{\sin^2\theta}\partial_\theta \sin^2\theta\,\partial_\theta + \frac{1}{\sin^2\theta\cos^2\theta}\left(\frac{1}{\sin\phi}\partial_\phi \sin\phi\,\partial_\phi - \frac{1}{\sin^2\phi}\right) \tag{11.98}$$

$$\mathcal{H}_2^2 = \frac{1}{\sin^2\theta}\partial_\theta \sin^2\theta\,\partial_\theta + \frac{1}{\sin^2\theta\cos^2\theta}\left(\frac{1}{\sin\phi}\partial_\phi \sin\phi\,\partial_\phi - \frac{1}{\sin^2\phi}\right) \tag{11.99}$$

$$\mathcal{H}_3^3 = \frac{1}{\cos^2\theta}\partial_\theta \cos^2\theta\,\partial_\theta + \frac{1}{\sin^2\theta\cos^2\theta}\sin\phi\,\partial_\phi\frac{1}{\sin\phi}\partial_\phi \tag{11.100}$$

$$\mathcal{H}_4^4 = \frac{1}{\sin^2\theta\cos^2\theta}\left(\partial_\theta \sin^2\theta\cos^2\ \partial_\theta + \frac{1}{\sin\phi}\partial_\phi \sin\phi\partial_\phi - 2\right) \tag{11.101}$$

$$\mathcal{H}_5^5 = \frac{1}{\cos^2\theta}\partial_\theta \cos^2\theta\,\partial_\theta + \frac{1}{\sin^2\theta\cos^2\theta}\left(\frac{1}{\sin\phi}\partial_\phi \sin\phi\,\partial_\phi - \frac{1}{\sin^2\phi}\right) \tag{11.102}$$

The inhomogeneous part is given by

$$\mathcal{I}_i^j = \begin{pmatrix} 0 & 0 & 0 & \partial_\phi & -\partial_\theta \\ 0 & 0 & \frac{1}{\tan\phi}\partial_\theta - \frac{1}{\sin\theta\cos\theta}\partial_\phi & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ -\tan^2\theta\frac{1}{\sin\phi}\partial_\phi\sin\phi & 0 & 0 & 0 & -(\frac{1}{\sin^2\theta} - \frac{1}{\cos^2\theta})\frac{1}{\sin\phi}\partial_\phi\sin\phi \\ 0 & 0 & 0 & \cos^2\theta\partial_\phi & 0 \end{pmatrix}. \tag{11.103}$$

Note that the divergence constraint (11.96) is similar in structure to the inhomogeneities in the v_4 differential equation. It turns out that one can indeed use the divergence constraint equation to eliminate v_1 from that equation in favour of an additional homogeneous v_4 term. After adding $(-\tan\theta)$ times the constraint equation to the v_4 differential equation one obtains

$$\begin{aligned} 0 = (8+6k+k^2)v_4 + \frac{1}{\sin^2\theta\cos^2\theta}\Bigg(&\partial_\theta(\sin^2\theta\cos^2\theta\,\partial_\theta v_4) - 2\tan\theta\,\partial_\theta(\sin^2\theta\cos^2\theta\,v_4) \\ &+ \frac{1}{\sin\phi}\partial_\phi(\sin\phi\,\partial_\phi v_4) - 2v_4\Bigg) - \frac{2}{\sin^3\theta\cos\theta}\frac{1}{\sin\phi}\partial_\phi(\sin\phi\,v_5). \end{aligned} \tag{11.104}$$

q Consider first the system for v_2 and v_3. The homogeneous parts of each of these differential equations can be solved by separation of variables, and we need to introduce degeneracy parameters m_2 and m_3. After applying the separation rule and equating the resulting parts to $\pm m_i(m_i+1)$ we obtain the following homogeneous solution for v_2

$$v_2(\theta,\phi) = A_2(\theta)B_2(\phi) \tag{11.105}$$

$$\begin{aligned} A_2(\theta) &= c_1(\tan\theta)^{m_2}{}_2F_1(-\frac{k}{2}-1,\frac{k}{2}+2;\frac{1}{2}-m_2;\cos^2\theta) \\ &\quad + c_2(\tan\theta)^{-(m_2+1)}{}_2F_1(-\frac{k}{2}-1,\frac{k}{2}+2;\frac{3}{2}+m_2;\cos^2\theta) \end{aligned} \tag{11.106}$$

$$
\begin{aligned}
B_2(\phi) &= c_3 \sin\phi\, {}_2F_1(\frac{m_2}{2}+1, \frac{1}{2}-\frac{m_2}{2}; \frac{1}{2}; \cos^2\phi) \\
&\quad + c_4 \sin\phi \cos\phi\, {}_2F_1(\frac{m_2}{2}+\frac{3}{2}, 1-\frac{m_2}{2}; \frac{3}{2}; \cos^2\phi)
\end{aligned}
\tag{11.107}
$$

Next observe that the partial differential equation for v_3 does not have any inhomogeneities, so that the solution we just found is the full solution for v_3. Moreover, since $v_i = 0$ are also solutions to the homogeneous equations, the homogeneous solution for v_2 together with $v_1 = v_3 = v_4 = v_5 = 0$ provides a full solution to the vector harmonics Laplace equation. This solution produces good vector harmonics only for the range of parameters $k = 2l + 1$, $l = 0, 1, 2, \ldots,$ and $m_2 = 1, 2, \ldots, (l + 1)$. This can be derived as follows. Consider the solution $B_2(\phi)$. The symmetry $m_2 \to -(m_2 + 1)$ leaves all terms invariant, and has the fixed point $m_0 = -\frac{1}{2}$. Therefore we can assume $m_2 \geq m_0 = -1/2$. For this range of m_2, the c_3 term produces finite polynomials only for $m_2 = 1, 3, 5, \ldots,$ and the c_4 term only for $m_2 = 2, 4, 6, \ldots,$ so that in total we get a good $B_2(\phi)$ solution for $m_2 = 1, 2, 3, \ldots$ Next consider $A_2(\theta)$. The invariance of the eigenvalue $(k^2 + 6k + 4)$ under $k \to -(k + 6)$ is reflected in the solution and leaves both terms invariant. This symmetry has the fixed point $k_0 = -3$, and we can restrict the values of k to $k \geq k_0 = -3, -2, -1, \ldots.$ The form of the solution $A_2(\theta)$ suggests to consider the cases of even and odd k separately. For even $k = 2l$ it is obvious that the hypergeometric function turns into a finite series. However, this series starts with a constant term and both the c_1 and the c_2 parts contain overall powers of $\tan\theta$. Because we already found that $m_2 > 0$ these terms will produce negative powers of $\sin\theta$ and $\cos\theta$ which render the solution unacceptable. Therefore the even case $k = 2l$ is ruled out. For odd $k = 2l + 1$ we need to first apply the Euler transformation formula ${}_2F_1(a, b; c; z) = (1 - z)^{c-a-b}{}_2F_1(c - a, c - b; c; z)$ and obtain

$$
\begin{aligned}
A_2(\theta) \to\ & c_1(\sin\theta)^{-(m_2+1)}(\cos\theta)^{-m_2}{}_2F_1(\frac{k+3}{2} - m_2, -\frac{k+3}{2} - m_2; \frac{1}{2} - m_2; \cos^2\theta) \\
&+ c_2(\sin\theta)^{m_2}(\cos\theta)^{m_2+1}{}_2F_1(\frac{k+5}{2} + m_2, -\frac{k+1}{2} + m_2; \frac{3}{2} + m_2; \cos^2\theta).
\end{aligned}
\tag{11.108}
$$

We see again that the negative powers of the trigonometric functions in the c_1 terms spoil the solution so that we are forced to set $c_1 = 0$. The c_2 part yields a good solution whenever $-\frac{k+1}{2} + m_2 \leq 0$, which implies $m_2 \leq \frac{k+1}{2} = l + 1$. Moreover, because $m_2 \geq 1$ we need to take $k \geq 1$. Using the notation $\tilde{Y}_{\mathbf{5}}^{(k,m_2)}$ to denote the particular solution corresponding to $v_1 = v_3 = v_4 = v_5 = 0$ we get

$$
\begin{aligned}
\tilde{Y}_{\mathbf{5}}^{(2l+1,m_2)}(\theta,\phi) = c_{l,m_2}\sigma^2(\sin\theta)^{m_2}(\cos\theta)^{m_2+1}{}_2F_1(l+3+m_2, -l-1+m_2; \frac{3}{2}+m_2; \cos^2\theta)\times \\
\times \begin{cases} {}_2F_1(-\frac{m_2}{2}-\frac{1}{2}, \frac{m_2}{2}; \frac{1}{2}; \cos^2\phi) & \text{for odd } m_2 \\ {}_2F_1(-\frac{m_2}{2}, \frac{m_2}{2}+\frac{1}{2}; \frac{3}{2}; \cos^2\phi)\cos\phi & \text{for even } m_2 \end{cases}
\end{aligned}
\tag{11.109}
$$

$$l \geq 0, \qquad m_2 = 1, 2, \ldots, (l+1).$$

The constants c_{l,m_2} are an overall normalisation. Here are the few first such solutions so obtained, for which we have chosen an arbitrary overall sign, and which we have also normalised as in Subsection 11.4.1:

$$\tilde{Y}_{\mathbf{5}}^{(1,1)} = \frac{2}{\sqrt{3}} \sin\theta \cos^2\theta \sin\phi \, \sigma^2 \tag{11.110}$$

$$\tilde{Y}_{\mathbf{5}}^{(3,1)} = \sqrt{\frac{3}{5}} \sin\theta \cos^2\theta \cos(2\theta) \sin\phi \, \sigma^2 \tag{11.111}$$

$$\tilde{Y}_{\mathbf{5}}^{(3,2)} = \sqrt{\frac{3}{5}} \sin^2\theta \cos^3\theta \sin(2\phi) \, \sigma^2 \tag{11.112}$$

$$\tilde{Y}_{\mathbf{5}}^{(5,1)} = \sqrt{\frac{1}{105}} \sin\theta \cos^2(\theta)(3\cos(4\theta)+2) \sin\phi \, \sigma^2 \tag{11.113}$$

$$\tilde{Y}_{\mathbf{5}}^{(5,2)} = \frac{2}{\sqrt{7}} \sin^2\theta \cos^3\theta \cos(2\theta) \sin(2\phi) \, \sigma^2 \tag{11.114}$$

$$\tilde{Y}_{\mathbf{5}}^{(5,3)} = \sqrt{\frac{2}{35}} \sin^3\theta \cos^4\theta \sin\phi(5\cos(2\phi)+3) \, \sigma^2 \tag{11.115}$$

$$\tilde{Y}_{\mathbf{5}}^{(7,1)} = \frac{1}{12\sqrt{2}} \sin\theta \cos^2\theta(3\cos(2\theta)+2\cos(6\theta)) \sin\phi \, \sigma^2 \tag{11.116}$$

$$\tilde{Y}_{\mathbf{5}}^{(7,2)} = \frac{1}{4}\sqrt{\frac{5}{42}} \sin^2\theta \cos^3\theta(4\cos(4\theta)+3) \sin(2\phi) \, \sigma^2 \tag{11.117}$$

$$\tilde{Y}_{\mathbf{5}}^{(7,3)} = \frac{1}{2\sqrt{3}} \sin^3\theta \cos^4\theta \cos(2\theta) \sin\phi(5\cos(2\phi)+3) \, \sigma^2 \tag{11.118}$$

$$\tilde{Y}_{\mathbf{5}}^{(7,4)} = \frac{1}{6}\sqrt{\frac{5}{7}} \sin^4\theta \cos^5\theta \sin(2\phi)(7\cos(2\phi)+1) \, \sigma^2 \tag{11.119}$$

Note that the solution $\tilde{Y}_{\mathbf{5}}^{(1,1)}$ is exactly one of the solutions found by group theory arguments. As before we can change the basis from the angles $\{\theta, \phi\}$ to the variables $\{w_1, w_2\}$ and obtain

$$\tilde{Y}_{\mathbf{5}}^{(1,1)} = \sqrt{\frac{1}{3}} \left(\frac{1+w_1}{2}\right)^{1/2} \sqrt{\zeta} \, \sigma^2 \tag{11.120}$$

$$\tilde{Y}_{\mathbf{5}}^{(3,1)} = \sqrt{\frac{3}{5}} \left(\frac{1+w_1}{2}\right)^{1/2} \frac{w_1}{2} \sqrt{\zeta} \, \sigma^2 \tag{11.121}$$

$$\tilde{Y}_{\mathbf{5}}^{(3,2)} = \sqrt{\frac{3}{5}} \left(\frac{1+w_1}{2}\right)^{1/2} \frac{w_2}{2} \sqrt{\zeta} \, \sigma^2 \tag{11.122}$$

$$\tilde{Y}_{\mathbf{5}}^{(5,1)} = \sqrt{\frac{1}{105}} \left(\frac{1+w_1}{2}\right)^{1/2} \frac{6w_1^2-1}{2} \sqrt{\zeta} \, \sigma^2 \tag{11.123}$$

$$\tilde{Y}_{\mathbf{5}}^{(5,2)} = \sqrt{\frac{1}{7}} \left(\frac{1+w_1}{2}\right)^{1/2} w_1 w_2 \sqrt{\zeta} \, \sigma^2 \tag{11.124}$$

$$\tilde{Y}_{\mathbf{5}}^{(5,3)} = \sqrt{\frac{1}{70}} \left(\frac{1+w_1}{2}\right)^{1/2} \frac{w_1^2+5w_2^2-1}{2} \sqrt{\zeta} \, \sigma^2 \tag{11.125}$$

$$\tilde{Y}_{\mathbf{5}}^{(7,1)} = \frac{\sqrt{2}}{24} \left(\frac{1+w_1}{2}\right)^{1/2} \frac{(8w_1^2-3)w_1}{2} \sqrt{\zeta} \, \sigma^2 \tag{11.126}$$

$$\tilde{Y}_{\mathbf{5}}^{(7,2)} = \frac{1}{4}\sqrt{\frac{5}{42}}\left(\frac{1+w_1}{2}\right)^{1/2}\frac{(8w_1^2-1)w_2}{2}\sqrt{\zeta} \tag{11.127}$$

$$\tilde{Y}_{\mathbf{5}}^{(7,3)} = \frac{1}{4\sqrt{3}}\left(\frac{1+w_1}{2}\right)^{1/2}\frac{(w_1^2+5w_2^2-1)w_1}{2}\sqrt{\zeta}\,\sigma^2 \tag{11.128}$$

$$\tilde{Y}_{\mathbf{5}}^{(7,4)} = \frac{1}{12}\sqrt{\frac{5}{7}}\left(\frac{1+w_1}{2}\right)^{1/2}\frac{(3w_1^2+7w_2^2-3)w_2}{2}\sqrt{\zeta}\,\sigma^2 \tag{11.129}$$

The system of equations for v_2 and v_3 admit a second set of solutions, namely the one with $v_1 = v_4 = v_5 = 0$ and $v_3 \neq 0$. To find these solutions we need to first solve the homogeneous differential equation for v_3, then plug the solution into the differential equation for v_2 and solve the resulting inhomogeneous differential equation to obtain the full solution with both $v_2 \neq 0$ and $v_3 \neq 0$. Observe that there are no inhomogeneities in the equation for v_3, so that it can be solved directly by separation of variables. It is also separable and therefore splits into two ordinary differential equations that we have to set equal to a constant that we call $\pm m(m+1)$. This leads to the following solution:

$$v_3(\theta,\phi) = A_3(\theta)B_3(\phi) \tag{11.130}$$

$$\begin{aligned} A_3(\theta) = {} & c_1(\tan\theta)^{m+1}{}_2F_1(-\frac{k}{2}-1,\frac{k}{2}+2;\frac{1}{2}-m;\cos^2\theta) \\ & + c_2(\tan\theta)^{-m}{}_2F_1(-\frac{k}{2}-1,\frac{k}{2}+2;\frac{3}{2}+m;\cos^2\theta) \end{aligned} \tag{11.131}$$

$$B_3(\phi) = c_3\,{}_2F_1(-\frac{m}{2}-\frac{1}{2},\frac{m}{2};\frac{1}{2};\cos^2\phi) + c_4(\cos\phi){}_2F_1(-\frac{m}{2},\frac{m}{2}+\frac{1}{2};\frac{3}{2};\cos^2\phi) \tag{11.132}$$

First of all note that since the term $m(m+1)$ in the original ansatz is invariant under $m \to -(m+1)$ this symmetry must manifest itself in the solution. Indeed, we see that in $A_3(\theta)$ this simply swaps c_1 with c_2. In $B_3(\phi)$ each of the two terms is invariant under this symmetry. Moreover, the term proportional to c_3 is a finite polynomial only if m is an odd integer, while the same is true for the c_4 term for even m. In total we see that m has to be an integer, and also without any loss of generality we can set $c_1 = 0$. This gives the following solutions for v_3.

$$\begin{aligned} v_3(\theta,\phi) = {} & c_{k,m}(\tan\theta)^{-m}{}_2F_1(-\frac{k}{2}-1,\frac{k}{2}+2;\frac{3}{2}+m;\cos^2\theta)\times \\ & \begin{cases} {}_2F_1(-\frac{m}{2}-\frac{1}{2},\frac{m}{2};\frac{1}{2};\cos^2\phi) & \text{for odd } m \\ {}_2F_1(-\frac{m}{2},\frac{m}{2}+\frac{1}{2};\frac{3}{2};\cos^2\phi)\cos\phi & \text{for even } m \end{cases} \end{aligned} \tag{11.133}$$

The next step is to substitute this solution into the differential equation for v_2, and look for a solution. As the resulting differential equation is inhomogeneous one needs to first find the solution for its homogeneous part and then use methods like the variation of parameters to derive a particular solution for the inhomogeneous equation. The general solution can then be constructed as a sum of the particular solution and a linear combination of the homogeneous solutions. This computation is still work in progress, and the results will be reported elsewhere.

11.5 Tensor Spherical Harmonics with $\mathrm{SO}(3)_{\mathrm{diag}}$ Symmetry

In this section we will again resort to group theory methods to find a set of the lowest $\mathrm{SO}(3)_{\mathrm{diag}}$ invariant tensor harmonics $Y^{(k,m)}_{\mathbf{10}\,[np]}$. As indicated in equation (11.4) these tensor harmonics will be the singlets in the $\mathbf{10_c}$ of the $\mathfrak{so}(6)$ after the branching to the $\mathrm{SO}(3)_{\mathrm{diag}}$ subgroup. The $\mathbf{10_c} = \mathbf{10} \oplus \overline{\mathbf{10}}$ can be represented by an anti-symmetric 3-tensor of the SO(6):

$$\mathbf{10} \oplus \overline{\mathbf{10}} = T^{[abc]}. \tag{11.134}$$

The $\mathbf{10}$ and the $\overline{\mathbf{10}}$ can be extracted by splitting $T^{[abc]}$ into its self-dual and anti-self-dual parts. Taking an educated guess based on the construction of the vector harmonics we can attempt to construct these tensor harmonics as tensor products of S^5 Killing vectors $K_n^{[IJ]} = \partial_n y^{[I} y^{J]}$ and embedding coordinates y^I. We are looking for a combination that gives a harmonic of the form $Y^{[IJK]}_{[np]}$, and one such combination is

$$y^I K^{JL}_{[n} K^{KL}_{p]}\Big|_{[IJK]} = \frac{3}{4} y^{[I} \partial_{[n} y^J \partial_{p]} y^{K]}. \tag{11.135}$$

To find tensor harmonics that are invariant under the $\mathrm{SO}(3)_{\mathrm{diag}}$ we need to find the singlets that are contained in the $\mathbf{10_c}$. According to Appendix E the $\mathbf{10}$ contains two singlets, and therefore the $\mathbf{10_c}$ contains four of them. After splitting the SO(6) index to a pair of $\mathrm{SO}(3) \times \mathrm{SO}(3)$ indices as $I \to (i, \bar{\imath})$ and then taking the diagonal subgroup by identifying $i \equiv \bar{\imath}$ one obtains the following singlets

$$\epsilon^{ijk} T^{[ijk]} \qquad \sum_{i=\bar{\imath}} \epsilon^{ijk} T^{[\bar{\imath}jk]} \qquad \sum_{i=\bar{\imath}} \epsilon^{\bar{\imath}\bar{\jmath}\bar{k}} T^{[i\bar{\jmath}\bar{k}]} \qquad \epsilon^{\bar{\imath}\bar{\jmath}\bar{k}} T^{[\bar{\imath}\bar{\jmath}\bar{k}]}, \tag{11.136}$$

where we have to substitute $y^{[I} \partial_{[n} y^J \partial_{p]} y^{K]}$ for $T^{[IJK]}$. By our choice of coordinates we have $x^i = u^i$ and $x^{\bar{\imath}} = v^i$, and so we can write the singlets in (11.136) as

$$Y^{(0,0)}_{\mathbf{10}} = \epsilon^{ijk} (u^i du^j \wedge du^k) \tag{11.137}$$

$$Y^{(0,1)}_{\mathbf{10}} = \frac{1}{\sqrt{3}} \epsilon^{ijk} (v^i du^j \wedge du^k + u^j du^k \wedge v^i + u^k dv^i \wedge du^j) \tag{11.138}$$

$$Y^{(0,2)}_{\mathbf{10}} = \frac{1}{\sqrt{3}} \epsilon^{ijk} (u^i dv^j \wedge dv^k + v^j dv^k \wedge u^i + v^k du^i \wedge dv^j) \tag{11.139}$$

$$Y^{(0,3)}_{\mathbf{10}} = \epsilon^{ijk} (v^i dv^j \wedge dv^k). \tag{11.140}$$

We have checked that these four tensor harmonics indeed verify the defining Laplace equation (11.1) for $k = 0$, as well as the divergence condition (11.2). Moreover these harmonics were normalised according to the conventions set in (11.3). As for the vector harmonics we can change basis to the (w_1, w_2) variables and express the forms in terms of left-invariant forms σ^i. To further simplify the expressions these left-invariant forms can be traded for the vector harmonics $Y_{\mathbf{5}}^{(k,m)}$ which we found in the previous section, which gives:

$$Y_{\mathbf{10}}^{(0,0)} = \frac{\sqrt{3}}{\zeta}(1+w_1)\left(Y_{\mathbf{5}}^{(0,0)} - \frac{w_2 dw_1}{2(1+w_1)} + \frac{dw_2}{2}\right) \wedge Y_{\mathbf{5}}^{(1,0)} \tag{11.141}$$

$$Y_{\mathbf{10}}^{(0,1)} = \frac{1}{\zeta}(1+w_1)\left(Y_{\mathbf{5}}^{(0,0)} - \frac{w_2 dw_1}{2(1+w_1)} + \frac{dw_2}{2}\right) \wedge Y_{\mathbf{5}}^{(1,1)} + \frac{1}{\zeta}(2w_2 Y_{\mathbf{5}}^{(0,0)} - dw_1) \wedge Y_{\mathbf{5}}^{(1,0)} \tag{11.142}$$

$$Y_{\mathbf{10}}^{(0,2)} = \frac{1}{\zeta}(1-w_1)\left(Y_{\mathbf{5}}^{(0,0)} - \frac{w_2 dw_1}{2(1-w_1)} - \frac{dw_2}{2}\right) \wedge Y_{\mathbf{5}}^{(1,0)} + \frac{1}{\zeta}(2w_2 Y_{\mathbf{5}}^{(0,0)} - dw_1) \wedge Y_{\mathbf{5}}^{(1,1)} \tag{11.143}$$

$$Y_{\mathbf{10}}^{(0,3)} = \frac{\sqrt{3}}{\zeta}(1-w_1)\left(Y_{\mathbf{5}}^{(0,0)} - \frac{w_2 dw_1}{2(1-w_1)} - \frac{dw_2}{2}\right) \wedge Y_{\mathbf{5}}^{(1,1)}. \tag{11.144}$$

These are all $\mathrm{SO}(3)_{\mathrm{diag}}$ invariant $Y_{\mathbf{10}}$ harmonics with $k = 0$ since there are no more singlets contained in the $\mathbf{10_c}$. The construction of higher tensor harmonics is still in progress and will be reported elsewhere.

Chapter 12

UV Asymptotics of the Uplifted Solution

In order to interpret our ten-dimensional solution we can compute its asymptotic behaviour for large values of the radial coordinate and check whether the various fields have the fall-off expected from the AdS/CFT dictionary. It is also interesting to compare our results with the asymptotic behaviours of the other supergravity solutions that are supposed to describe $\mathcal{N} = 1^*$, namely the Polchinski-Strassler solution [48] and the zero-temperature limit of the Freedman-Minahan solution [62]. To this purpose we perform the change of variable $t = e^{C_1}/r$, where r is the radial coordinate used in [48, 62]. In this section we only give terms up to quadratic order in the deformation parameters m and σ and we fix the values of the angles φ and ω to zero.

Axion-Dilaton The expansion takes a particularly simple form for the field $B = \frac{1+i\tau}{1-i\tau}$ that appears in the Kaluza-Klein expansion around S^5 [42]. The first terms can be easily computed for any value of the angles φ and ω and are

$$B \sim -\left(\frac{4m_0^2}{9r^2}e^{-2i\varphi} - \frac{2m_0\sigma_0}{3\sqrt{3}r^4}e^{-i(\varphi-\omega)}\right)(u^2-v^2) - 2i\left(\frac{4m_0^2}{9r^2}e^{-2i\varphi} + \frac{2m_0\sigma_0}{3\sqrt{3}r^4}e^{-i(\varphi-\omega)}\right)(u\cdot v)\,, \tag{12.1}$$

where m_0 and σ_0 are given in Appendix A and are related to the UV mass deformation and the expectation value of gaugino condensate. From (12.1), setting the angles φ and ω to zero, we can compute the expansions of the dilaton and axion

$$e^{\Phi} \sim 1 + \frac{2}{3}\frac{m_0^2}{r^2}(u^2-v^2) + \frac{m_0\sigma_0}{\sqrt{3}r^4}(u^2-v^2) \tag{12.2}$$

$$C_0 \sim \frac{4}{3}\frac{m_0^2}{r^2}(u\cdot v) + \frac{2}{\sqrt{3}}\frac{m_0\sigma_0}{r^4}(u\cdot v). \tag{12.3}$$

The leading behaviour of the dilaton is the same as for the zero-temperature limit of the Freedman-Minahan solution [62]. As already discussed in their paper, the behaviour does not agrees with the asymptotic limits of the dilaton in the Polchinski-Strassler solution [48], where the leading behaviour of the dilaton is given by a scalar of the SO(6).

Metric The large r behaviour of the metric is

$$ds_{10}^2 = r^2\left(1 + \frac{m_0^2}{24r^2}\right) ds_4^2 + \left(1 + \frac{m_0^2}{16r^2}\right)\frac{dr^2}{r^2} + ds_5^2 \tag{12.4}$$

where ds_4^2 is the flat Minkowski metric in four dimensions and the internal metric ds_5^2 is given by

$$\begin{aligned} ds_5^2 = (du_i)^2 + (dv_i)^2 + du_i du_j \left(\frac{m_0^2}{r^2} A_{ij} + \frac{m_0 \sigma_0}{r^4} B_{ij}\right) \\ + dv_i dv_j \left(\frac{m_0^2}{r^2} C_{ij} + \frac{m_0 \sigma_0}{r^4} D_{ij}\right) + du_i dv_j \left(\frac{m_0^2}{r^2} E_{ij} + \frac{m_0 \sigma_0}{r^4} F_{ij}\right) \end{aligned} \tag{12.5}$$

with coefficients

$$A_{ij} = -\frac{1}{6}(3 + 4(u^2 - v^2))\delta_{ij} + \frac{1}{3} v_i v_j \tag{12.6}$$

$$B_{ij} = \frac{1}{\sqrt{3}}[(u^2 - v^2)\delta_{ij} + v_i v_j] \tag{12.7}$$

$$C_{ij} = -\frac{1}{6}(3 - 4(u^2 - v^2))\delta_{ij} + \frac{1}{3} u_i u_j \tag{12.8}$$

$$D_{ij} = \frac{1}{\sqrt{3}}[-(u^2 - v^2)\delta_{ij} + u_i u_j] \tag{12.9}$$

$$E_{ij} = \frac{1}{3}[-8(u \cdot v)\delta_{ij} - 8 u_i v_j + 6 v_i u_j] \tag{12.10}$$

$$F_{ij} = \frac{2}{\sqrt{3}}[-2(u \cdot v)\delta_{ij} + u_i v_j + v_i u_j] \tag{12.11}$$

Two-Form Potentials and Field Strengths The first terms in the expansion of the 2-forms potentials (9.59) are

$$C_1 = \frac{1}{2}\left(\frac{1}{\sqrt{3}}\frac{m_0}{r} + \frac{1}{2}\frac{\sigma_0}{r^3}\right)\epsilon_{ijk} v_i du_j \wedge du_k + \frac{1}{2}\left(\sqrt{3}\frac{m_0}{r} - \frac{1}{2}\frac{\sigma_0}{r^3}\right)\epsilon_{ijk} v_i dv_j \wedge dv_k \tag{12.12}$$

$$+ \left(\frac{1}{\sqrt{3}}\frac{m_0}{r} + \frac{1}{2}\frac{\sigma_0}{r^3}\right)\epsilon_{ijk} u_i du_j \wedge dv_k \tag{12.13}$$

$$C_2 = -\frac{1}{2}\left(\sqrt{3}\frac{m_0}{r} - \frac{1}{2}\frac{\sigma_0}{r^3}\right)\epsilon_{ijk} u_i du_j \wedge du_k - \frac{1}{2}\left(\frac{1}{\sqrt{3}}\frac{m_0}{r} + \frac{1}{2}\frac{\sigma_0}{r^3}\right)\epsilon_{ijk} u_i dv_j \wedge dv_k \tag{12.14}$$

$$- \left(\frac{1}{\sqrt{3}}\frac{m_0}{r} + \frac{1}{2}\frac{\sigma_0}{r^3}\right)\epsilon_{ijk} v_i du_j \wedge dv_k\,. \tag{12.15}$$

A simple derivation gives the asymptotic behaviour of the fully internal components of the field strengths $F_i = dC_i$

$$F_1 = \frac{3}{2}\left(\frac{1}{\sqrt{3}}\frac{m_0}{r} + \frac{1}{2}\frac{\sigma_0}{r^3}\right)\epsilon_{ijk}du_i \wedge du_j \wedge dv_k + \left(3\sqrt{3}\frac{m_0}{r} - \frac{3}{2}\frac{\sigma_0}{r^3}\right)dv_1 \wedge dv_2 \wedge dv_3 \quad (12.16)$$

$$F_2 = -\frac{3}{2}\left(\frac{1}{\sqrt{3}}\frac{m_0}{r} + \frac{1}{2}\frac{\sigma_0}{r^3}\right)\epsilon_{ijk}du_i \wedge dv_j \wedge dv_k - \left(3\sqrt{3}\frac{m_0}{r} - \frac{3}{2}\frac{\sigma_0}{r^3}\right)du_1 \wedge du_2 \wedge du_3 \quad (12.17)$$

The terms in $1/r$ in equation (12.12) reproduce the large r behaviour of the 2-form potentials of the Polchinski-Strassler background [48], but the leading terms in σ_0 disagree. For different values of the angles, $\varphi = \pi/2$ and a constant arbitrary ω we also recover the leading behaviour in the $T = 0$ limit of the 3-forms in the [62].

Five-form flux Using (9.71) it is easy to derive the large r behaviour of the purely internal component of the 5-form flux

$$F_5 = -\frac{1}{5!}\left(\frac{4}{r^6} - \frac{12m_0^2}{r^8}\right)\epsilon_{m_1\ldots m_6}y^{m_1}dy^{m_2} \wedge dy^{m_3} \wedge dy^{m_4} \wedge dy^{m_5} \wedge dy^{m_6}, \quad (12.18)$$

where y^m are the six coordinates of $\mathbb{R}^6$ that parametrise the internal manifold. Again this expression agrees with that given by Freedman and Minahan.

Chapter 13

Singularity

13.1 Uplift in Pilch-Warner Coordinates

In this section we present the uplift solution in the coordinates introduced in [54]. First let us recall the definition of the new radial coordinate t and other constants

$$t = e^{-(y-C_1)}, \qquad \lambda = e^{3(C_2-C_1)}, \qquad C_1 = \log\left(\frac{m_0}{\sqrt{3}}\right), \qquad C_2 = \frac{1}{3}\log\left(\frac{\sigma_0}{2}\right), \tag{13.1}$$

where C_1 and C_2 are the $5d$ integration constants, and m_0 and σ_0 are related to the leading asymptotic behaviour of the $5d$ fields m and σ (note that m_0 and σ_0 differ by constants relative to the ones in [54]). Defining μ, ν as in (9.33), the solution of the first order equations in terms of these variables takes the form

$$\mu(t) = \sqrt{\frac{1+\lambda t^3}{1-\lambda t^3}}, \qquad \nu(t) = \sqrt{\frac{1+t}{1-t}}. \tag{13.2}$$

For more definitions and conventions see Appendix A.

Warp-Factor The warp-factor Δ we used earlier and the warp factor ξ in [54] are related by

$$\xi^2 = \Delta^{-8/3}. \tag{13.3}$$

In the new coordinates we find

$$\xi^2 = \frac{1}{(1-t^2)^4\,(1-\lambda^2 t^6)^2}\times \left[\left(1+t^2\right)^2\left(1-\lambda^2 t^8\right)^2 - 4w_1^2\, t^4\left(1-\lambda t^2\right)^2\left(1+\lambda t^4\right)^2 - 4w_2^2\, t^4\left(1+\lambda t^2\right)^2\left(1-\lambda t^4\right)^2\right] \tag{13.4}$$

Metric The uplifted ten-dimensional metric was already obtained by Pilch and Warner [54], see equations (6.1)–(6.7) in their text. It takes a block form containing the AdS_5 and the S^5 parts as follows

$$ds^2_{10} = \xi^{1/2} ds^2_{1,4} + \xi^{-3/2} ds^2_5 \tag{13.5}$$

$$ds^2_{1,4} = e^{2\phi(y)} \eta_{\mu\nu} dx^\mu dx^\nu + dy^2 \tag{13.6}$$

$$\begin{aligned} ds^2_5 &= a_1 du^i du^i + 2a_2 du^i dv^i + a_3 dv^i dv^i \\ &\quad + a_4 (u^i dv^i + v^i du^i)^2 + 2a_5 (u^i dv^i)(v^j du^j) + 2a_6 (u^i du^i)(v^j dv^j). \end{aligned} \tag{13.7}$$

The coefficients a_i of the internal metric can be found in equation (6.3) in the Pilch and Warner text [54]. We can expand the fields $\mu(t)$ and $\nu(t)$ in terms of the radial coordinate t to get the following expressions for the coefficients a_i:

$$a_1 = \frac{(1+\lambda t^4)\left(1+t^2(1-2w_1)\left(1-\lambda t^2\right)-\lambda t^6\right)}{\left(1-t^2\right)^2\left(1-\lambda^2 t^6\right)} \tag{13.8}$$

$$a_2 = \frac{-2w_2 t^2\left(1+\lambda t^2\right)\left(1-\lambda t^4\right)}{\left(1-t^2\right)^2\left(1-\lambda^2 t^6\right)} \tag{13.9}$$

$$a_3 = \frac{(1+\lambda t^4)\left(1+t^2(1+2w_1)\left(1-\lambda t^2\right)-\lambda t^6\right)}{\left(1-t^2\right)^2\left(1-\lambda^2 t^6\right)} \tag{13.10}$$

$$a_4 = \frac{t^2\left(1+\lambda t^2\right)^2\left(1+\lambda t^4\right)\left(1+3t^2\left(1-\lambda t^2\right)-\lambda t^6\right)}{\left(1-t^2\right)^3\left(1-\lambda^2 t^6\right)^2} \tag{13.11}$$

$$a_5 = \frac{2t^2\left(1-\lambda^2 t^4\left(1+t^4\left(1-\lambda^2 t^4\right)\right)\right)}{\left(1-t^2\right)^2\left(1-\lambda^2 t^6\right)^2} \tag{13.12}$$

$$a_6 = \frac{-4t^2\left(1+t^2\right)\left(1-\lambda t^2\left(1+t^2\left(1-\lambda t^2\right)\right)\right)}{\left(1-t^2\right)^3\left(1-\lambda^2 t^6\right)}. \tag{13.13}$$

Axion/Dilaton The axion/dilaton matrix $m_{\alpha\beta}$ is given by

$$m_{\alpha\beta} = \frac{1}{\xi} m_{ab} = \frac{1}{\xi}\begin{pmatrix} m_{11} & m_{12} \\ m_{12} & m_{12} \end{pmatrix} \tag{13.14}$$

with the components

$$m_{11} = \frac{(1+\lambda t^4)}{\left(1-t^2\right)^2\left(1-\lambda^2 t^6\right)}\left[(1-\lambda t^6)+t^2(2w_1+1)-\lambda t^4(2w_2+1)\right] \tag{13.15}$$

$$m_{22} = \frac{(1+\lambda t^4)}{\left(1-t^2\right)^2\left(1-\lambda^2 t^6\right)}\left[(1-\lambda t^6)+t^2(2w_1-1)-\lambda t^4(2w_2-1)\right] \tag{13.16}$$

$$m_{12} = \frac{2w_2 t^2\left(1-\lambda t^4\right)\left(1+\lambda t^2\right)}{\left(1-t^2\right)^2\left(1-\lambda^2 t^6\right)} \tag{13.17}$$

Two-Form Potential The 2-form potential is given by

$$C_\alpha = C_{mn\,\alpha} dy^m \wedge dy^n. \tag{13.18}$$

The new basis for the 2-forms will be given by the following six 2-forms

$$\{dw_1, dw_2, \sigma_1\} \wedge \{\sigma_2, \sigma_3\} \tag{13.19}$$

The expression for C_α in terms of this basis is rather complicated, but reduces to a manageable expression in the $t \to 1$ limit, which we will report later in Section 13.2.

Four-Form Potential The 4-form potential is given by

$$C = \mathring{C} + \frac{1}{4!\,\xi^2}(f_1 d_4^1 + f_2 d_4^2 + f_3 d_4^3) \tag{13.20}$$

with the coefficients

$$f_1 = \frac{-12 w_2\, t^4 (1+\lambda t^2)^2 (1-\lambda t^4)^2}{(1-t^2)^4 (1-\lambda^2 t^6)^2} \tag{13.21}$$

$$f_2 = \frac{12 w_1\, t^4 (1-\lambda t^2)^2 (1+\lambda t^4)^2}{(1-t^2)^4 (1-\lambda^2 t^6)^2} \tag{13.22}$$

$$f_3 = \frac{48 w_1 w_2\, \lambda t^6}{(1-t^2)^3 (1-\lambda^2 t^6)} \tag{13.23}$$

and the 4-forms

$$\begin{aligned} d_4^1 &= \epsilon^{ijm}\epsilon^{kln}(u^m u^n - v^m v^n) du^i \wedge du^j \wedge dv^k \wedge dv^l && (13.24)\\ &= \frac{\zeta^{1/2}}{4}\frac{(w_1-1)}{(w_1+1)}\, dw_1 \wedge dw_2 \wedge \sigma^2 \wedge \sigma^3 \\ &\quad + \frac{1}{2}(1+w_1^2-w_2^2)\, dw_1 \wedge \sigma^1 \wedge \sigma^2 \wedge \sigma^3 + w_1 w_2\, dw_2 \wedge \sigma^1 \wedge \sigma^2 \wedge \sigma^3 \\ d_4^2 &= \epsilon^{ijm}\epsilon^{kln}(u^m v^n + v^m u^n) du^i \wedge du^j \wedge dv^k \wedge dv^l && (13.25)\\ &= \frac{\zeta^{1/2}}{4}\frac{w_2}{(w_1+1)}\, dw_1 \wedge dw_2 \wedge \sigma^2 \wedge \sigma^3 \\ &\quad + w_1 w_2\, dw_1 \wedge \sigma^1 \wedge \sigma^2 \wedge \sigma^3 + \frac{1}{2}(1-w_1^2+w_2^2)\, dw_2 \wedge \sigma^1 \wedge \sigma^2 \wedge \sigma^3 \\ d_4^3 &= du^i \wedge du^j \wedge dv^i \wedge dv^j && (13.26)\\ &= \frac{\zeta^{1/2}}{4}\frac{1}{w_1+1}\, dw_1 \wedge dw_2 \wedge \sigma^2 \wedge \sigma^3 \\ &\quad + \frac{w_1}{2}\, dw_1 \wedge \sigma^1 \wedge \sigma^2 \wedge \sigma^3 + \frac{w_2}{2}\, dw_2 \wedge \sigma^1 \wedge \sigma^2 \wedge \sigma^3 \end{aligned}$$

The combined 4-form reads

$$C = \mathring{C} + \frac{t^4}{4\xi^2(1-t^2)^4(1-\lambda^2t^6)^2}\Big[\quad \tilde{f}_1\, dw_1 \wedge dw_2 \wedge \sigma^2 \wedge \sigma^3 + \tilde{f}_2\, dw_1 \wedge \sigma^1 \wedge \sigma^2 \wedge \sigma^3 + \tilde{f}_3\, dw_2 \wedge \sigma^1 \wedge \sigma^2 \wedge \sigma^3 \Big] \tag{13.27}$$

with

$$\begin{aligned}
\tilde{f}_1 &= \frac{\zeta^{1/2}}{2}\frac{w_2}{1+w_1}(1+\lambda t^2)^2(1-\lambda t^4)^2 \qquad (13.28)\\
\tilde{f}_2 &= -w_2\left[(1+\lambda t^2)^2(1-\lambda t^4)^2 - w_1^2(1-\lambda t^2)^2(1+\lambda t^4)^2 - w_2^2(1+\lambda t^2)^2(1-\lambda t^4)^2\right]\\
\tilde{f}_3 &= w_1\left[(1-\lambda t^2)^2(1+\lambda t^4)^2 - w_1^2(1-\lambda t^2)^2(1+\lambda t^4)^2 - w_2^2(1+\lambda t^2)^2(1-\lambda t^4)^2\right].
\end{aligned}$$

Six-Form Potential The non-vanishing components of the 6-form potential are the following

$$\begin{aligned}
C_{\mu\nu\rho\sigma\tau\, m}{}^{\alpha} &= \omega_{\mu\nu\rho\sigma\tau}\, \Xi_m{}^{\alpha}\,,\\
C_{\mu\nu\rho\sigma,\, mn}{}^{\alpha} &= \omega_{\mu\nu\rho\sigma\tau}\, g^{\tau\lambda}\Xi_{\lambda\, mn}{}^{\alpha}\,.
\end{aligned} \tag{13.29}$$

We can transform the 1-forms Ξ^α and 2-forms Ξ^α_λ into the Pilch-Warner basis and rewrite some of the differentials in terms of vector harmonics. The result for the 1-forms is

$$\Xi^\alpha = \frac{\sqrt{3}}{2}\frac{(\lambda t^3+t)\left(\lambda^3t^{12}+\lambda^3t^{10}+\lambda^2t^8-3\lambda(\lambda+1)t^6+\lambda t^4+t^2+1\right)}{(t^2-1)^2\left(\lambda^2t^6-1\right)^2}\begin{pmatrix} Y_{\mathbf{5}}^{(1,1)} \\ -Y_{\mathbf{5}}^{(1,0)}\end{pmatrix}. \tag{13.30}$$

For the 2-forms we get

$$\begin{aligned}
\Xi^\alpha_y = {}& \frac{\sqrt{3}}{\zeta}\frac{t\left(\lambda t^2-1\right)\left(\lambda t^4+1\right)}{(t^2-1)\left(\lambda^2t^6-1\right)}\begin{pmatrix} (1+w_1)\sqrt{\zeta}\sigma^1 \\ -\frac{w_1w_2}{1+w_1}dw_1-(1-w_1)dw_2+(1-w_1)\sqrt{\zeta}\sigma^1\end{pmatrix}\wedge\begin{pmatrix} Y_{\mathbf{5}}^{(1,0)} \\ Y_{\mathbf{5}}^{(1,1)}\end{pmatrix}\\
&+\frac{\sqrt{3}}{\zeta}\frac{t(\lambda^2t^6+3\lambda t^4-3\lambda t^2-1)}{(t^2-1)\left(\lambda^2t^6-1\right)}\begin{pmatrix} -\frac{1}{2}\frac{\zeta}{1+w_1}dw_1+w_2\sqrt{\zeta}\sigma^1-w_1dw_1-w_2dw_2 \\ -\frac{1}{2}\frac{\zeta}{1+w_1}dw_1+w_2\sqrt{\zeta}\sigma^1\end{pmatrix}\wedge\begin{pmatrix} Y_{\mathbf{5}}^{(1,1)} \\ Y_{\mathbf{5}}^{(1,0)}\end{pmatrix}
\end{aligned} \tag{13.31}$$

Note that there is further t-dependence in the volume form $\omega_{\mu\nu\rho\sigma\tau}$ and the inverse metric $g^{\tau\lambda}$. In the basis where t is used for the radial coordinate they are given by

$$g^{\lambda\tau} = \text{diag}(-e^{-2\phi(t)}, e^{-2\phi(t)}, e^{-2\phi(t)}, e^{-2\phi(t)}, t^2) \tag{13.32}$$

$$e^{2\phi(t)} = \frac{1}{t^2}(1-t^2)(1-\lambda^2t^6)^{1/3}e^{2C_1} \tag{13.33}$$

$$\omega_{\mu\nu\rho\sigma\tau} = \frac{1}{t} e^{4\phi(t)} \epsilon_{\mu\nu\rho\sigma\tau} = \frac{1}{t^5} \left(1 - t^2\right)^2 \left(1 - \lambda^2 t^6\right)^{2/3} e^{4C_1} \epsilon_{\mu\nu\rho\sigma\tau}. \tag{13.34}$$

13.2 Singularity

In this section we discuss the behaviour of the ten-dimensional solution as we approach the position where the five-dimensional solution has a curvature singularity.

As we reviewed in Section 9.1, in five dimensions we have a metric coupled to two real scalars m and σ, which have a domain-wall profile along the radial direction. The complete solution is given in (9.14) and contains two integration constants C_1 and C_2, which parametrise the mass deformation and the gaugino condensate, respectively, as discussed in Section 9.1. The geometry is singular as the radial coordinate y approaches either C_1 or C_2, as one can verify by computing curvature invariants. As in [54] we parametrise the location of the singularity by defining

$$t = \exp\left(-(y - C_1)\right), \qquad \chi = 2\sqrt{1-t} \tag{13.35}$$

The singularity of the $5d$ metric is then located at $t \to 1$ or equivalently $\chi \to 0$, and the curvature scalar and Kretschmann invariant are given by

$$R = -\frac{16}{\chi^4} + O(\chi^{-2}), \qquad R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma} = \frac{640}{\chi^8} + O(\chi^{-6}) \tag{13.36}$$

Since the scalar fields are also singular in this limit one may wonder if there is a different conformal frame than the Einstein frame, where the geometry is regular or at least less singular. It turns out that this is not the case. We will see later that the situation is different in 10 dimensions. We also define (again following [54])

$$\lambda = e^{-3(C_1 - C_2)}. \tag{13.37}$$

It was argued in [23] that $C_1 \geq C_2$ and this translates to $\lambda \leq 1$ with the equality corresponding to the case where the singularities in $m(y)$ and $\sigma(y)$ coincide. The singularity structure of the $10d$ solution depends on whether $\lambda < 1$ or $\lambda = 1$ and we will discuss the two cases separately.

13.2.1 The $\lambda < 1$ Case

We have computed the curvature scalar of the $10d$ solution and it has a limit which is regular in the radial coordinate as $\chi \to 0$:

$$\lim_{\chi \to 0} R^{(\text{full})} = \frac{\sqrt{2}}{1-\lambda^2} \frac{3(1+\lambda^2) - w_1^2(1 - 4\lambda + \lambda^2) - w_2^2(1 + 4\lambda + \lambda^2)}{(1 - w_1^2 - w_2^2)^{5/4}}. \tag{13.38}$$

However, the Ricci scalar is now singular at

$$\zeta \equiv 1 - w_1^2 - w_2^2 = 0 \tag{13.39}$$

which is precisely the ring singularity discussed in [54]. The metric near the singularity is given by

$$\begin{aligned} ds^2 = \frac{\zeta^{1/4}}{\sqrt{2}} \Bigg\{ & (1+\chi^2 f_0)(2e^{2C_1}(1-\lambda^2)^{1/3}\eta_{\mu\nu}dx^\mu dx^\nu + d\chi^2) + \frac{\chi^2}{4}\left(\frac{1}{\zeta}Y_{\mathbf{5}}^{(0,0)\,2} + \sigma_2^2 + \sigma_3^2\right) \\ & + \frac{1}{2\zeta}\left(\frac{1-\lambda}{1+\lambda}dw_1^2 + \frac{1+\lambda}{1-\lambda}dw_2^2\right) + \frac{\chi^2}{8\zeta(1-\lambda^2)}\omega_\parallel + \frac{\chi^2}{16\zeta^2(1-\lambda^2)^2}\omega_D \Bigg\} + \mathcal{O}(\chi^4) \end{aligned} \tag{13.40}$$

where $Y_{\mathbf{5}}^{(0,0)}$ is an SO(3) vector harmonic, the expression for which can be found in Chapter 11. The coefficient f_0 and the differentials $\omega_\parallel$ and ω_D are given by

$$f_0 = \lambda \frac{w_1^2 - w_2^2}{4\zeta(1-\lambda^2)} \tag{13.41}$$

$$\omega_\parallel = (1+\lambda^2)(2\zeta+1)2e^{2C_1}(1-\lambda^2)^{1/3}\eta_{\mu\nu}dx^\mu dx^\nu + (2+(3\zeta-1)(1-\lambda^2))d\chi^2 \tag{13.42}$$

$$\begin{aligned} \omega_D = & \left[(1-\lambda)^2 d(w_1^2) + (1+\lambda)^2 d(w_2^2)\right]^2 \\ & - \left[(1+\lambda^2)(3-2\zeta) - 2\lambda(w_1^2 - w_2^2)\right]\left[(1-\lambda)^2 dw_1^2 + (1+\lambda)^2 dw_2^2\right] \\ & + 8\lambda(\zeta+1)\left[(1-\lambda)^2 dw_1^2 - (1+\lambda)^2 dw_2^2\right] \end{aligned} \tag{13.43}$$

Pilch and Warner in [54] also computed the near-singularity metric. Their metric is reproduced by setting $f_0 = \omega_\parallel = \omega_D = 0$ and $\frac{1}{\zeta}Y_{\mathbf{5}}^{(0,0)} \to \sigma^1$ (modulo a typo in one of the coefficients of dw_1^2). Note that $d\chi^2 + \frac{1}{4}\chi^2(\sigma_1^2 + \sigma_2^2 + \sigma_3^2)$ is just the flat metric on $\mathbb{R}^4$ and the terms in the first line of (13.40) combine to give the eight-dimensional Minkowski spacetime. This was interpreted in [54] as evidence that the singularity is associated with 7-branes. We cannot however ignore the terms with $f_0, \omega_\parallel, \omega_D$ and $\frac{1}{\zeta}Y_{\mathbf{5}}^{(0,0)}$ because they are of the same order as $\frac{1}{4}\chi^2(\sigma_1^2 + \sigma_2^2 + \sigma_3^2)$. Taking these terms into account, we find no evidence for 7-branes in the near-singularity structure of the metric. At the position of the $5d$ singularity, $\chi = 0$, the $10d$ metric is of co-dimension 4:

$$ds^2 = \frac{\zeta^{1/4}}{\sqrt{2}}\left(2e^{2C_1}(1-\lambda^2)^{1/3}\eta_{\mu\nu}dx^\mu dx^\nu + \frac{1}{2\zeta}\left(\frac{1-\lambda}{1+\lambda}dw_1^2 + \frac{1+\lambda}{1-\lambda}dw_2^2\right)\right). \tag{13.44}$$

Note that the limit $\chi \to 0$ is not a decoupling limit, i.e. the metric in (13.40) does not solve the bulk equations of motion and the curvature of this metric does not agree with (13.38). To properly account for (13.38) one needs to keep higher order terms in χ. First consider the ten-dimensional metric G_{MN}. One can check that in the expansion in the radial coordinate χ around $\chi = 0$ the lowest order in χ that occurs is the constant order χ^0, which is also manifest in (13.40). The same analysis performed on the full inverse metric G^{MN} shows that its lowest order in χ is the order χ^{-2}. Given this information we can deduce to which orders we need to expand G_{MN} and G^{MN} in order to obtain results consistent with the full computation of the Ricci scalar in the limit $\chi \to 0$. Schematically the Riemann tensor and the Ricci scalar are given in terms of the metric and its inverse as follows:

$$R_{MNRS} \sim \partial^2 G + G^{-1}\partial G \partial G \tag{13.45}$$

$$R \sim G^{-1}G^{-1}(\partial^2 G + G^{-1}\partial G \partial G). \tag{13.46}$$

Since $G^{-1} \sim 1/\chi^2$ to get the correct constant term in the Ricci scalar the Riemann tensor needs to be at least of order χ^4. Then, from the second term in (13.45) we infer that $\partial G \partial G$ has to be at least of order χ^6, and since a derivative with respect to χ lowers the order in χ by 1, G has to be at least of order χ^7. Similarly, one can deduce that one needs to keep terms at least up to order χ^4 in the inverse metric G^{-1}. We have explicitly checked that keeping the metric and inverse metric to these orders one indeed obtains a curvature scalar consistent with (13.38).

Similarly, one can study the order to which one has to keep the other fields in order for the bulk equations to be satisfied, order by order in χ^2. In general, one cannot truncate this series at some fixed order and have the field equations satisfied, as different orders contribute to different terms in the field equations.

We now provide the near-singularity behaviour of the warp factor and all other fields.

Warp-Factor Following [54] we define

$$\xi^2 = \Delta^{-8/3} \tag{13.47}$$

Then the warp factor has the following leading behaviour as $\chi \to 0$,

$$\xi = \frac{8\zeta^{1/2}}{\chi^4} + \mathcal{O}(\chi^3) \tag{13.48}$$

Axion/Dilaton In the limit $\chi \to 0$ the axion/dilaton matrix $m_{\alpha\beta}$ is regular and takes the following form:

$$m_{\alpha\beta} = \zeta^{-1/2} \begin{pmatrix} 1 + w_1 & w_2 \\ w_2 & 1 - w_1 \end{pmatrix} + \mathcal{O}(\chi^2) \tag{13.49}$$

Two-Form Potential In the limit $\chi \to 0$ the 2-form reduces to the following expression,

$$C^1 = \frac{\sqrt{3}}{\zeta}\left(Y_{\mathbf{5}}^{(0,0)} \wedge Y_{\mathbf{5}}^{(1,1)} - \frac{1}{2}dw_1 \wedge Y_{\mathbf{5}}^{(1,0)} - \frac{1}{2}dw_2 \wedge Y_{\mathbf{5}}^{(1,1)}\right) + \mathcal{O}(\chi^2) \tag{13.50}$$

$$C^2 = \frac{\sqrt{3}}{\zeta}\left(-Y_{\mathbf{5}}^{(0,0)} \wedge Y_{\mathbf{5}}^{(1,1)} + \frac{1}{2}dw_1 \wedge Y_{\mathbf{5}}^{(1,1)} - \frac{1}{2}dw_2 \wedge Y_{\mathbf{5}}^{(1,0)}\right) + \mathcal{O}(\chi^2) \tag{13.51}$$

where the $\mathrm{SO}(3)_{\mathrm{diag}}$ invariant vector harmonics $Y_{\mathbf{5}}^{(k,m)} = Y_{\mathbf{5}\,n}^{(k,m)} dy^n$ are 1-forms on the S^5 cotangent bundle, and k and m are integers that label the harmonics, see Chapter 11 for more details.

Four-Form Potential The limit $\chi \to 0$ for the 4-form is regular and gives the following result

$$C = \mathring{C} + \frac{1}{4}\left(\frac{1+w_1}{2}\right)^{-1} \frac{w_2}{16\sqrt{\zeta}}\, dw_1 \wedge dw_2 \wedge \sigma^2 \wedge \sigma^3 + \frac{1}{16}(w_1 dw_2 - w_2 dw_1) \wedge \sigma^1 \wedge \sigma^2 \wedge \sigma^3 + \mathcal{O}(\chi^2) \tag{13.52}$$

Six-Form Potential We get

$$C_{(6)} \to \frac{\sqrt{3}}{2} e^{4C_1}(1-\lambda^2)^{2/3} d\tau \wedge dx^1 \wedge dx^2 \wedge dx^3 \wedge (\chi d\chi) \wedge \begin{pmatrix} Y_{\mathbf{5}}^{(1,1)} \\ -Y_{\mathbf{5}}^{(1,0)} \end{pmatrix} + \mathcal{O}(\chi^2). \tag{13.53}$$

The coordinate τ denotes the time. Note that there are some powers of χ that come from the volume form $\omega_{\mu\nu\rho\sigma\tau}$ and cancel some divergences. See equation (13.34) for the expansion of $\omega_{\mu\nu\rho\sigma\tau}$ in terms of the radial coordinate.

13.2.2 The $\boldsymbol{\lambda = 1}$ Case

We now set $\lambda = 1$ first and then take the $\chi \to 0$ limit. The Ricci scalar then becomes

$$R^{(\lambda=1)} = \frac{1}{6\sqrt{3}}\left(\frac{8}{\chi^2} - 1\right) \frac{(8 + w_1^2 - 8w_2^2)(10 - w_1^2 - 10w_2^2)}{(4 - w_1^2 - 4w_2^2)^{9/4}} + \mathcal{O}(\chi). \tag{13.54}$$

Thus in this case the $10d$ metric is still singular at $\chi = 0$, though diverging at slower rate than the $5d$ solution. In addition, the metric is singular at $(w_1, w_2) = (0, \pm 1)$ (which corresponds to $(\theta, \phi) = (\pi/4, \pi/2 \pm \pi/2)$), but there is no ring singularity anymore. The metric itself takes the following form

$$ds_5^2 = \Omega^{1/2}\left[12^{1/3}e^{2C_1}\eta_{\mu\nu}dx^\mu dx^\nu\chi^{2/3} + \left(\frac{3}{8}\chi^2+1\right)d\chi^2 + \frac{1}{24\hat{\Omega}^2}\left(\frac{8}{\chi^2}-1\right)dw_2^2 + \frac{\chi^2}{4\zeta}\delta_2\right] + \mathcal{O}(\chi^{7/3}) \tag{13.55}$$

The differential δ_2 is given by

$$\delta_2 = Y_{\mathbf{5}}^{(0,0)\,2} + \frac{3}{4}dw_1^2 - \frac{1}{4}dw_2^2 - \frac{\zeta}{48\hat{\Omega}^2}dw_2^2 - \frac{\zeta(1+w_2^2)}{18\hat{\Omega}^4}dw_2^2 + \frac{2-w_1^2}{6\hat{\Omega}^2}dw_2^2 + \frac{w_1w_2}{2\hat{\Omega}^2}dw_1dw_2 + \frac{w_1}{3\hat{\Omega}^2}Y_{\mathbf{5}}^{(0,0)}dw_2 + \frac{9}{2+w_1}Y_{\mathbf{5}}^{(1,0)\,2} + \frac{2+w_1}{\hat{\Omega}^2}\left(\frac{2w_2}{2+w_1}Y_{\mathbf{5}}^{(1,0)} - Y_{\mathbf{5}}^{(1,1)}\right)^2, \tag{13.56}$$

and $\hat{\Omega}$ is defined as

$$\hat{\Omega} = \frac{1}{3}\sqrt{4-w_1^2-4w_2^2}. \tag{13.57}$$

The leading order terms in this metric reproduce the result found by Pilch and Warner [54], but we have additional subleading terms.

Warp-Factor The warp factor has the following leading behaviour under $\lambda \to 1$, then $\chi \to 0$

$$\xi = \frac{16\hat{\Omega}}{\chi^4} + \mathcal{O}(\chi^{-3}) \tag{13.58}$$

Axion/Dilaton The axion/dilaton matrix $m_{\alpha\beta}$ is regular and takes the following form:

$$m_{\alpha\beta} = \frac{1}{3\hat{\Omega}}\begin{pmatrix}2+w_1 & 2w_2\\ 2w_2 & 2-w_1\end{pmatrix} + \mathcal{O}(\chi^2). \tag{13.59}$$

Two-Form Potential The limit of the 2-form may be written in terms of wedge products of the vector harmonics found in Chapter 11:

$$C^1 = \frac{\sqrt{3}}{\zeta}\left(Y_{\mathbf{5}}^{(0,0)} - \frac{2\zeta-3w_1}{18\hat{\Omega}^2}dw_2\right)\wedge Y_{\mathbf{5}}^{(1,1)} - \frac{\sqrt{3}}{2\zeta}\left(dw_1 + \frac{w_1w_2}{3\hat{\Omega}^2}\right)\wedge Y_{\mathbf{5}}^{(1,0)} + \mathcal{O}(\chi^2) \tag{13.60}$$

$$C^2 = \frac{\sqrt{3}}{\zeta}\left(-Y_{\mathbf{5}}^{(0,0)} - \frac{2\zeta+3w_1}{18\hat{\Omega}^2}dw_2\right)\wedge Y_{\mathbf{5}}^{(1,0)} + \frac{\sqrt{3}}{2\zeta}\left(dw_1 + \frac{w_1w_2}{3\hat{\Omega}^2}\right)\wedge Y_{\mathbf{5}}^{(1,1)} + \mathcal{O}(\chi^2) \tag{13.61}$$

Four-Form Potential The limit of the 4-form is regular, and is given by

$$\begin{aligned} C = \mathring{C} &+ \frac{w_2\zeta^{1/2}}{72(1+w_1)\hat{\Omega}^2}\, dw_1 \wedge dw_2 \wedge \sigma^2 \wedge \sigma^3 - \frac{w_1}{48\hat{\Omega}^2} dw_2 \wedge \sigma^1 \wedge \sigma^2 \wedge \sigma^3 \\ &+ \frac{1}{16}(w_1 dw_2 - w_2 dw_1) \wedge \sigma^1 \wedge \sigma^2 \wedge \sigma^3 + \mathcal{O}(\chi^2) \end{aligned} \tag{13.62}$$

$$\begin{aligned} = \mathring{C} &+ \left(\frac{1+w_1}{2}\right)^{-1} \frac{w_2\zeta^{1/2}}{144\hat{\Omega}^2}\, dw_1 \wedge dw_2 \wedge \sigma^2 \wedge \sigma^3 - \frac{w_1}{48\hat{\Omega}^2} dw_2 \wedge \sigma^1 \wedge \sigma^2 \wedge \sigma^3 \\ &+ \frac{1}{16}(w_1 dw_2 - w_2 dw_1) \wedge \sigma^1 \wedge \sigma^2 \wedge \sigma^3 + \mathcal{O}(\chi^2) \end{aligned} \tag{13.63}$$

Six-Form Potential For the 6-form we get

$$C_{(6)} \to -\frac{7e^{4C_1}}{2^{2/3}3^{5/6}} d\tau \wedge dx^1 \wedge dx^2 \wedge dx^3 \wedge (\chi^{7/3} d\chi) \wedge \begin{pmatrix} Y_{\mathbf{5}}^{(1,1)} \\ -Y_{\mathbf{5}}^{(1,0)} \end{pmatrix} + \mathcal{O}(\chi)^{10/3}, \tag{13.64}$$

where τ is the time coordinate. Again, as in the case $\lambda < 1$, there are some powers of χ that come from the volume form $\omega_{\mu\nu\rho\sigma\tau}$ and cancel some divergences. See equation (13.34) for the expansion of $\omega_{\mu\nu\rho\sigma\tau}$ in terms of the radial coordinate.

13.2.3 Different Frames

Since the solution involves non-trivial scalars there is an intrinsic ambiguity in the definition of the spacetime metric: one can rescale the metric with powers of the scalars. Different probes see different metrics and different conformal frames carry different physical meaning. For example, supergravity probes see the Einstein frame metric and strings see the string frame metric. In some cases singular geometries are regular in a different frame. For example, the geometry of non-conformal Dp branes is singular in the Einstein and string frame but it is regular in the "dual frame" [113] and this is also the frame best suited for holography [18, 114]. Here we want to analyse the dependence of the singularity on the choice of frame.

Usually one uses the dilaton when discussing different frames[1]. Since our solution has both an axion and a dilaton we will consider a general rescaling by both: $g_{\hat{\mu}\hat{\nu}} \to \tilde{g}_{\hat{\mu}\hat{\nu}} = \Omega^2 g_{\hat{\mu}\hat{\nu}}$ with the scaling factor $\Omega = e^{x\Phi} C_0^z$ given by some powers of the dilaton e^Φ and the axion C_0 parametrised through constants x and z. Given the definition of the axion/dilaton matrix $m_{\alpha\beta}$

$$m_{\alpha\beta} = \frac{1}{\Im\tau}\begin{pmatrix} |\tau|^2 & -\Re\tau \\ -\Re\tau & 1 \end{pmatrix}, \qquad \tau = C_0 + ie^{-\Phi}, \tag{13.65}$$

[1] One reason for this is that the axion is more properly viewed as a 0-form potential and has an associated gauge invariance.

we can write the rescaling parameter as $\Omega = m_{22}^x(-m_{12}/m_{22})^z$. To compute the effect of the rescaling on the Ricci scalar we can use the standard formula for the Weyl rescaling of the Ricci scalar (see for example [101]), which we also discussed in Chapter 10:

$$\tilde{R} = \Omega^{-2}\left[R - 18g^{\hat{\mu}\hat{\nu}}\nabla_{\hat{\mu}}\nabla_{\hat{\nu}}\log\Omega - 72g^{\hat{\mu}\hat{\nu}}(\nabla_{\hat{\mu}}\log\Omega)(\nabla_{\hat{\nu}}\log\Omega)\right]. \tag{13.66}$$

The $\lambda < 1$ case After the rescaling the Ricci scalar takes the following form:

$$\tilde{R}^{(\lambda<1)} = \frac{\mathcal{P}^{(\lambda<1)}(w_1, w_2)}{(1 - w_1^2 - w_2^2)^{\frac{5}{4}-x} w_2^{2+2z}(1 - w_1)^{2+2x-2z}} + \mathcal{O}(\chi), \tag{13.67}$$

where $\mathcal{P}^{(\lambda<1)}(w_1, w_2)$ is a polynomial in w_1 and w_2 with coefficients containing x, z, and λ. After a careful inspection it is evident that there is no choice of x and z that removes the denominator. One can also show that the numerator $\mathcal{P}^{(\lambda<1)}(w_1, w_2)$ is non-zero for any choice of x, z, and $\lambda < 1$, therefore the singularity in the curvature cannot be completely removed.

One can now study what type of singular behaviour the terms in the denominator entail. The term $(1 - w_1^2 - w_2^2)$ is just the original ring singularity along the circle $w_1^2 + w_2^2 = 1$. The term w_2 leads to singularities on parts of the ring corresponding to $\theta \in \{0, \pi\}$ or $\phi = \pi/2$, while the term $(1 - w_1)$ reduces the singularity to a single point $(w_1, w_2) = (1, 0)$, which is equivalent to the value $\theta = 0$. Thus we see that the least singular behaviour that we can get is achieved by choosing $x \geq 5/4$ and $z \leq -1$ which leads to a singularity of type $(1 - w_1)^a$ with $a \geq 9/4$, i.e. in this case we only have a singularity at a single point. It would be interesting to understand the meaning of these frames.

The $\lambda = 1$ case We can now repeat the same analysis for the case $\lambda = 1$. The transformed Ricci scalar has the form

$$\tilde{R}^{(\lambda=1)} = \left(\frac{8}{\chi^2} - 1\right)\frac{\mathcal{P}^{(\lambda=1)}(w_1, w_2)}{(4 - w_1^2 - 4w_2^2)^{\frac{9}{4}-x} w_2^{2z}(2 - w_1)^{2+2x-2z}} + \mathcal{O}(\chi). \tag{13.68}$$

First of all, also in this case it can be shown that $\mathcal{P}^{(\lambda=1)}(w_1, w_2)$ cannot be identically zero for any choice of x and z. This means that the singularity χ^{-2} in the radial coordinate can never be removed. Notice however that the term $(2 - w_1)$ in the denominator is never zero since $-1 \leq w_1 \leq 1$, and therefore we can arrange that the singularity in the angular directions is removed completely by choosing $x \geq 9/4$ and $z \leq 0$.

Chapter 14

Identification of D-Branes

One of the most intriguing and controversial features of the GPPZ solution [23] is the singularity that one finds as one follows the radial direction from the conformal boundary towards the inside of the bulk. On the field theory side the flow along the radial direction can be interpreted as the renormalisation group flow from the UV to the IR, and the appearance of the singularity in the bulk suggests that the strongly coupled IR physics on the field theory side cannot be accounted for by the five-dimensional dual. By uplifting the five-dimensional solution to ten dimensions one is hoping that the features of that theory might suffice to shed light on the singularity and provide physical input to the field theory counterpart. If this is the case then the singularity in five dimension should be removed, or at least improved, in higher-dimensional physics. One mechanism by which this could come about is that the singularity that one sees in five dimensions is a remnant of a brane source in ten dimensions. If this is indeed the case, then one should be able to see the source appearing in the equations of motion of the ten-dimensional form-fields that couple to the corresponding branes. In the GPPZ solution both scalars that are turned on correspond to components of the ten-dimensional 3-form flux F_3 and its dual F_7, which couple electrically and magnetically to D1 and D5 branes. One might wonder how this type of branes might appear in a set-up in which one had only D3 branes to begin with. One possible explanation was suggested by Polchinski and Strassler [48] in which they argue that the transverse F_7 flux can polarise D3 branes, which makes them acquire dipole momenta under the F_7. The fact that such polarisation processes should be possible was first observed and described by Myers [50]. We saw that after the uplift various other $F_{(p+2)}$ fluxes get turned on, which may or may not lead to polarisations of the corresponding Dp branes. For example it was suggested by Pilch and Warner after examining the ten-dimensional metric near the singularity that 7-branes might be present in the system too. If such claims are to be true, then it should be possible to find the corresponding delta function source in the equations of motion of the field strength that the branes couple to. Such a delta function source would appear because in the presence of branes the action has to be extended to contain the DBI part, which after variation becomes the source in the equations of motion.

Finding a delta function source in the equations of motion is certainly not easy. As one knows from the simple example of a point source in electrodynamics one needs to carefully examine the equations of motion at points at which the field strength is singular and decide whether a delta function appears. In what follows we would like to start with the example of a point charge and develop an efficient method for point source detection which avoids the manual examination of the equations of motion.

14.1 Looking for a Point Charge

In this section we would like to study the simple example of a static electric point charge in four dimensions, and how it can be described via a source term in the action. We will see how the delta function source appears in the equations of motion, and how it can also be detected by integrating the electric flux. Thus let us consider the free theory of a U(1) gauge potential $C_1 = A_\mu dx^\mu$. Up to an irrelevant overall rescaling the action reads

$$S \sim \int \mathrm{d}^4x \; F_{\mu\nu}F^{\mu\nu} \sim \int F_2 \wedge (\star F_2). \tag{14.1}$$

In the last term we have written the field strength as a 2-form $F_2 = dC_1$ which is the differential of the potential C_1. To obtain the equations of motion we need to vary the action with respect to A_μ, which gives the familiar result

$$\partial_\mu F^{\mu\nu} = 0. \tag{14.2}$$

Before solving this equation of motion we may choose a gauge, which we set to the Coulomb gauge by fixing $\partial_\mu A^\mu = 0$. It is now straightforward to show that the following potential and field strength indeed solve the equation of motion (14.2):

$$A^\mu = \delta^{\mu 0}\frac{q}{4\pi}\frac{1}{r} \qquad\qquad F^{0i} = -\partial_i A^0 = \frac{q}{4\pi}\frac{x^i}{r^3}. \tag{14.3}$$

In solving the equation of motion we have introduced the integration constant q, which, as we know, is the charge of the point particle located at the centre of the coordinate system. However, it turns out that we were too quick to conclude that the equations of motion are solved, and what really is the case is that we have solved the equations of motion only *almost* everywhere. Indeed, it is true that for $r = 0$ both the potential and the field strength diverge, and we can verify the equation of motion only where the field strength is finite.

Therefore, the precise statement is:

$$\partial_i F^{0i} = \frac{q}{4\pi}\left(\frac{\delta^i_i}{r^3} - \frac{x^i x^i}{r^5}\right) = 0 \qquad (r \neq 0). \tag{14.4}$$

One may wonder what physical process the divergence at $r = 0$ corresponds to, and how one could quantify it. One standard way is to regularise the divergence at $r = 0$, and to show that $\partial_\mu F^{\mu\nu}$ indeed produces a delta function, which one interprets as a source. A more systematic and cleaner way to arrive at the same conclusion is to integrate the equation of motion against a sufficiently nice test function. If there is no source then the integral should vanish, otherwise one will obtain a non-vanishing result. For our purposes the Gaussian $G(x) = e^{\frac{1}{2}r^2}$ is a good enough test function, not least because it falls off exponentially at infinity and therefore allows integration by parts without picking up additional boundary terms. One may object that our choice of the Gaussian centred at the coordinate origin is biased by the knowledge that there should be a source at exactly that location, however, any shifts of the Gaussian would lead to exactly the same results. The integration of the equations of motion against the Gaussian test function can be performed analytically and yields the following result:

$$\int \mathrm{d}^3x\; \partial_i F^{0i} G(x) = \int \mathrm{d}^3x\; x_i F^{0i} e^{\frac{1}{2}r^2} = q\int_0^\infty \mathrm{d}r\, r^2 \frac{1}{r}\; e^{\frac{1}{2}r^2} = q. \tag{14.5}$$

The fact the the result is non-vanishing shows that the left-hand side of the equation of motion, $\partial_i F^{0i}$, cannot have been zero everywhere. Moreover, since we know that $\partial_i F^{0i} = 0$ everywhere except at $r = 0$ we can conclude that $\partial_i F^{0i}$ must be proportional to a delta function, and its integration against the Gaussian should give the Gaussian evaluated at the origin times the coefficient in front of the delta function. Since $G(0) = 1$ we thus find

$$\partial_i F^{0i} = q\delta^3(x). \tag{14.6}$$

This result can be viewed from two different angles. Taking one point of view one might say that the solution in equation (14.3) is not a solution at all, which is a perfectly acceptable claim. Taking a different perspective, however, one might say that the solution is correct, but the equations of motion are missing the source on the right hand side. To obtain such a source term the action has to be extended in such a way that upon variation with respect to the potential A_μ a source term appears. It is straightforward to check that such an extension would have to be of the following form:

$$S \to S + \delta S \quad \text{with} \quad \delta S = \int \mathrm{d}^4x\; A_\mu J^\mu \sim \int C_1 \wedge (*J), \tag{14.7}$$

where we have defined the source 1-form $J_1 = J_\mu dx^\mu$, and $J^\mu = q\delta^3(x)\delta^{\mu 0}$. We can think of this source as a 0-brane, and the source part of the action as an integral over the world-sheet of that brane, which is a string in 4 dimensions located at the origin in space and extending infinitely in the time direction. This can be made manifest by rewriting the new source term in the action in the following way:

$$\delta S = \int C_1 \wedge (*J) = q \int A_0(0,t)\,\mathrm{d}t = q \int_{\text{brane world-sheet}} C_1. \tag{14.8}$$

In summary, by not discarding the invalid solution we found an extended version of our theory which allows for electrically charged particles. The change in the action that produces this extension is the integral of the potential that couples to that source over the source world-sheet. Moreover we saw that such sources can appear in solutions to the equations of motion that originate from an action in which the source was not accounted for and that it can be detected by integrating a test function across the part of the space-time that is transverse to the world-sheet of the source. The same logic generalises seamlessly to higher-dimensional sources called p-branes, which shall be explored in the following section.

14.2 Sources for p-Branes in Flat Space

Analogously to the previous section we can start with an action for a 3-from field F_3. If it does couple to any source then it will be through its potential C_2. This potential has to be integrated over the world-volume of a two-dimensional source object, a 1-brane, and the two-dimensional source shall be denoted by J_2. Thus the 1-brane action we would like to write is of the following form:

$$\begin{aligned} S &\sim \int F_3 \wedge (*F_3) + C_2 \wedge (*J_2) \\ &\sim \int F_3 \wedge (*F_3) + q \int_{\text{1-brane world-sheet}} C_2. \end{aligned} \tag{14.9}$$

After varying with respect to C_2 we obtain the equation of motion for the field strength F_3 with a source term on the right-hand side:

$$\partial_\mu F^{\mu\rho\sigma} = J^{\rho\sigma}. \tag{14.10}$$

The source $J^{\rho\sigma}$ describes a two-dimensional object, a string extending along one direction in space and another in time. For a frame in which the string extends spatially along the x^1 direction and sits at the origin of the (x^2, x^3) plane the source would be given by

$$J^{\mu\nu} \sim q\delta(x^2)\delta(x^3)\delta^{[\mu\nu]}_{01}. \tag{14.11}$$

The tensorial indices of the source necessarily span the part of the space-time tangent space that is parallel to the brane. This means that the equations of motion will exhibit a source term only in the case where the free tensor indices of the field strength are oriented along the source, which in our adapted frame reads

$$\partial_\mu F^{\mu 01} = J^{01} = q\delta(x^2)\delta(x^3). \tag{14.12}$$

As in the previous section we can detect the presence of the string by integrating the source over the space transverse to the string against a Gaussian test function $G(x) = \exp\left[\frac{1}{2}((x^2)^2 + (x^3)^2)\right]$. The result is as in the case of the point particle the charge, and by using the equation of motion and integrating by parts we see that

$$\begin{aligned} q = \int_{(x_2,x_3)\text{-plane}} \mathrm{d}^2x\ J^{01}G(x) = -\int \mathrm{d}^2x\ F^{\mu 01}\partial_\mu G(x) = \int \mathrm{d}^2x\ F^{\mu 01}(\delta_{\mu 2}x^2 + \delta_{\mu 3}x^3)G(x) \\ = \int \mathrm{d}^2x\ x_\mu F^{\mu 01} G(x). \end{aligned} \tag{14.13}$$

We observe that the charge is obtained by integrating the quantity $x_\mu F^{\mu\nu_1,\nu_2}$ over the space transverse to the brane and with the free indices ν_i pointing along the brane. We can generalise to arbitrary frames by introducing a projector $P_\parallel^{\rho\sigma}$ onto the parallel directions on the tangent space and write

$$q = \int \mathrm{d}^2x_\perp\ P_{\parallel\rho\sigma}\, x_\mu F^{\rho\sigma\mu}\ G(x_\perp). \tag{14.14}$$

It is clear that there is nothing special about choosing a 1-brane and exactly the same computation can be carried out for arbitrary p-branes in flat space. In a realistic setting the branes carry some mass and energy and therefore back-react onto the geometry warping it in the vicinity of the source. We will investigate in the following section how the detection of such sources can be approached.

14.3 Realistic Branes

Before studying the brane sources first note that even though form-fields can couple both electrically and magnetically to branes, only the electric coupling needs to be studied. This is because one can always pass to a dual description in which a magnetic source is viewed as an electric source for the dual potential.

Let us now study the brane sources in curved space-time. Consider a $(p+1)$ potential A_{p+1} that couples electrically to a stack of N Dp-branes located at the origin of the transverse space. The resulting geometry was described in detail in literature, and we will follow the conventions used in [115]. The metric and the potential A_{p+1} are described in terms of a warp factor H, which is given by

$$H = 1 + \frac{Q}{r^{(7-p)}}, \qquad p < 7, \tag{14.15}$$

where r is the radial coordinate in the space transverse to the brane, and the charge Q is given by [115]

$$Q = \frac{(2\pi l_s)^{7-p} N g_s}{(7-p)\omega_{9-p}}, \qquad \omega_k = \frac{2\pi^{k/2}}{\Gamma(k/2)}. \tag{14.16}$$

The symbol ω_k denotes the surface area of a $(k-1)$-sphere S^{k-1} embedded in a k-dimensional Euclidean space $\mathbb{R}^k$. The warp factor H appears with complementary powers in front of the parallel and transverse parts of the metric, while the components of the potential parallel to the brane contain the inverse of H:

$$ds^2 = H^{-1/2} ds^2_{(p,1)} + H^{1/2} ds^2_{(9-p)} \tag{14.17}$$

$$A_{01\dots p} = H^{-1} - 1. \tag{14.18}$$

The Yang-Mills part of the action of this set-up is of the standard form $S \sim \int \sqrt{-g} F_{p+2} \wedge (\star F_{p+2})$. Note, however, that we now have to include the non-trivial metric contribution in the measure of the integral. The variation of this action leads to the sourceless equation of motion for the field strength F_{p+2}. Because of the non-trivial metric the partial derivatives from the previous section are replaced by covariant derivatives:

$$0 = \nabla_{\mu_1} F^{\mu_1\dots\mu_{p+2}} = \frac{1}{\sqrt{-g}} \partial_{\mu_1} \left(\sqrt{-g} F^{\mu_1\dots\mu_{p+2}} \right). \tag{14.19}$$

The last equality in the equation of motion above is due to the anti-symmetry of the indices in F_{p+2}: the Christoffel symbols corresponding to some free index ν in $\nabla_\mu F^{\mu\dots\nu\dots}$ will appear contracted as $\Gamma^\nu_{\mu\rho} F^{\mu\dots\rho\dots}$, and since $\Gamma^\nu_{\mu\rho}$ is symmetric in its lower indices the whole contribution vanishes. Thus, in sum the covariant derivative in $\nabla_{\mu_1} F^{\mu_1\dots\mu_{p+2}}$ is really only covariant with respect to the contracted index, and the identity follows from the definition of the Christoffel symbols in terms of the metric.

Given the solution for the potential A_{p+1} in (14.18) it is easy to derive the field strength $F_{p+2} = dA_{p+1}$. The non-vanishing components are given by $F_{i0\dots p} = \partial_i A_{0\dots p} = -\partial_i H/H^2$, where the index i can be any of the transverse directions, and after raising the indices using the inverse of the metric in (14.17) we obtain

$$F^{i0\dots p} = -H^{p/2} F_{i0\dots p} = H^{p/2-2} \partial_i H = \frac{\partial_i H}{\sqrt{-g}}. \tag{14.20}$$

It is noteworthy that the result it proportional to $1/\sqrt{-g}$. This factor cancels exactly the equivalent contribution in the covariant derivative in (14.19), so that the equation of motion takes a very simple form:

$$\nabla_i F^{i0\ldots p} = \frac{\partial_i \partial_i H}{\sqrt{-g}} \equiv \frac{\Box_\perp H}{\sqrt{-g}}. \tag{14.21}$$

Further, given the expression for H in (14.15) it is easy to see that $\partial_i \partial_i H$ is identically zero away from the origin of the transverse space, so its support is at most the surface corresponding to $r = 0$. Since the term $\sqrt{-g}$ has no zeroes away from the origin of the transverse space we can conclude that the equation of motion $\nabla_i F^{i0\ldots p} = 0$ is true almost everywhere, except at $r = 0$, where further investigation is necessary. This observation is completely analogous to the sources in flat space discussed earlier. As before we can decide whether $\nabla_i F^{i0\ldots p}$ is identically zero or proportional to a delta function source by integrating against a flat Gaussian test function $G(x_\perp) = e^{-\frac{1}{2}x^i x^i} = e^{-\frac{1}{2}r^2}$. The integral is elementary, and to carry it out we integrate the covariant derivative by parts to get

$$\begin{aligned}\int \mathrm{d}^{9-p}x \, \sqrt{-g} \, \nabla_i F^{i0\ldots p} \, G(x_\perp) &= \int \mathrm{d}^{9-p}x \, \sqrt{-g} \, x^i F^{i0\ldots p} \, G(x_\perp) \\ &= -(7-p)\omega_{9-p} Q = -(2\pi l_s)^{7-p} N g_s.\end{aligned} \tag{14.22}$$

Thus we do find the delta function source indeed, and the correct equation of motion therefore reads

$$\nabla_i F^{i0\ldots p} = \frac{-1}{\sqrt{-g}} (2\pi l_s)^{7-p} N g_s \delta^{9-p}(x_i) \tag{14.23}$$

This is the result we were looking for: we see that the equations of motion are solved almost everywhere in the 10-dimensional space, apart from the points that are on the brane. As before one can take this as a reason to discard the solution, or extend the action to include the source that we just found by adding a DBI part to the action.

One can easily generalise this procedure to a stack of N Dp branes located at an arbitrary point $\vec{x}_0$ in the transverse space. The only change in the solution enters through a coordinate change in the warp factor:

$$H = 1 + \frac{Q}{|\vec{x} - \vec{x}_0|^{7-p}}. \tag{14.24}$$

Using $\partial_i H = -(7-p) Q \frac{(\vec{x}-\vec{x}_0)_i}{|\vec{x}-\vec{x}_0|^{9-p}}$ the integration against the transverse Gaussian leads to the following expression

$$\int \mathrm{d}^{9-p}x \, \sqrt{-g} \, x^i F^{i0\ldots p} \, G(x) = -(7-p) Q \int \mathrm{d}^{9-p}x \, \frac{\vec{x}(\vec{x}-\vec{x}_0)}{|\vec{x}-\vec{x}_0|^{9-p}} \, G(x). \tag{14.25}$$

We can carry out this integral analytically by shifting the integration variable to $\vec{x} \to \vec{x} + \vec{x}_0$, and writing the integration in adapted spherical coordinates where the angle between $\vec{x}$ and $\vec{x}_0$ is one of the angle coordinates θ so that $\vec{x}\vec{x}_0 = x x_0 \cos\theta$. This gives

$$-(7-p) Q G(\vec{x}_0) \int \mathrm{d}^{8-p}\Omega \int_0^\infty \mathrm{d}x \, (x + x_0 \cos\theta) \exp\left(-\frac{1}{2}x^2 - x x_0 \cos\theta\right). \tag{14.26}$$

The integrand is a total derivative so that the radial integration is equal to unity. The angular integration gives the surface area of an $(8-p)$-sphere, which we denote by ω_{9-p} so that the final result reads

$$-(2\pi l_s)^{7-p} N g_s G(\vec{x}_0). \tag{14.27}$$

This result is equivalent to the previous case of a delta function source at the origin, and the only difference is the shift in the transverse direction. A further generalisation is to k stacks of N_i Dp-branes located at $\vec{x}_i$, with $\sum_i N_i = N$. In this case the warp-factor is given by [116]

$$H = 1 + \sum_{i=1}^{k} \frac{Q_i}{|\vec{x}-\vec{x}_i|^{7-p}}, \qquad Q_i = \frac{N_i}{N} Q. \tag{14.28}$$

Due to the linearity of H the computations performed for one stack carry over straightforwardly to the k-stack case and we get

$$\int \mathrm{d}^{9-p}x \, \sqrt{-g} \, x_i F^{i0\ldots p} \, G(x) = -(2\pi l_s)^{7-p} N g_s \sum_i \frac{N_i}{N} G(\vec{x}_i). \tag{14.29}$$

For a continuous distribution $\sigma(\vec{x})$ of Dp branes, $\int \sigma(\vec{x}) = 1$, the warp factor is given by

$$H(\vec{x}) = 1 + Q \int \mathrm{d}^{9-p}x' \, \frac{\sigma(\vec{x}')}{|\vec{x}-\vec{x}'|^{7-p}}. \tag{14.30}$$

For the same reasons of linearity the generalisation is again straightforward and we get

$$\int \mathrm{d}^{9-p}x \, \sqrt{-g} \, x_i F^{i0\ldots p} \, G(x) = -(2\pi l_s)^{7-p} N g_s \int \mathrm{d}^{9-p}x \, \sigma(\vec{x}) G(\vec{x}), \tag{14.31}$$

which is the continuous version of all the results obtained so far, and can be reduced to the discrete case by an appropriate choice of the distribution $\sigma(\vec{x})$.

Yet another generalisation that one could think of is the case of intersecting branes. As an example we will consider stacks of D5 and D3 branes intersecting on a two-dimensional surface, which in the notation of [115] shall be denoted by $(2|\mathrm{D5} \perp \mathrm{D3})$. We can choose the branes to be embedded in the 10-dimensional space as follows

	t	1	2	3	4	5	6	7	8	9
D3	$\times$	$\times$	$\times$	$\times$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$
D5	$\times$	$\times$	$\times$	$\cdot$	$\times$	$\times$	$\times$	$\cdot$	$\cdot$	$\cdot$

The solution for this system can be expressed in terms of two warp factors H_3 and H_5 defined as

$$H_3 = 1 + \frac{Q_3}{r}, \qquad H_5 = 1 + \frac{Q_5}{r}, \tag{14.32}$$

where Q_3 and Q_5 are some charges with a particular value [117, 118] that is of no relevance to the current discussion. Note that both warp factors have exactly the same behaviour with respect to the radial coordinate r, even though they correspond to different branes. The radial coordinate r in this case shall only refer to the three-dimensional part of the transverse space that is shared by the D3 and D5 branes. This is a general feature and for any system of intersecting branes with the common transverse space of dimension k all warp factors will be of the form $H = 1 + Q/r^{k-2}$ so that $\Box_\perp H \sim \delta(x_\perp)$. The solution for the metric is given by

$$ds^2 = (H_3 H_5)^{1/2}\Big[(H_3H_5)^{-1}(-dt^2 + dx_1^2 + dx_2^2) \\ + H_3^{-1}dx_3^2 + H_5^{-1}(dx_4^2 + dx_5^2 + dx_6^2) + (dx_7^2 + dx_8^2 + dx_9^2)\Big], \quad (14.33)$$

which reduces to the one stack case considered earlier by setting $Q_3 = 0$ or $Q_5 = 0$. The field strengths expressed through the warp factors take exactly the same form as before:

$$F^{(5)}_{i0123} = -\frac{\partial_i H_3}{H_3^2}, \qquad F^{(7)}_{i012456} = -\frac{\partial_i H_5}{H_5^2}, \quad (14.34)$$

and as before we can use the inverse of the metric above to raise the indices of the field strengths. With $\sqrt{-g} = H_3^{1/2} H_5^{-1/2}$ we obtain

$$F^{(5)i0123} = \frac{\partial_i H_3}{\sqrt{-g}}, \qquad F^{(7)i012456} = \frac{\partial_i H_5}{\sqrt{-g}}. \quad (14.35)$$

In complete analogy to the previously considered cases the divergences of the field strengths read

$$\nabla_i F^{(5)i0123} = \frac{\Box_\perp H_3}{\sqrt{-g}}, \qquad \nabla_i F^{(7)i012456} = \frac{\Box_\perp H_5}{\sqrt{-g}}. \quad (14.36)$$

Given that $\Box_\perp H_i \sim \delta^\perp(x_\perp)$ the detection of the branes can be performed through an integration of the divergences $\nabla_i F^{(5)}$ and $\nabla_i F^{(7)}$ against a Gaussian test function across the common transverse space.

14.4 D-Branes in the Uplift

14.4.1 General Considerations

In this section we would like to apply the knowledge about the brane sources just developed to our uplift setting and try to integrate various ten-dimensional form-fields in order to investigate the presence of branes. There is one big difference between the toy examples of the previous sections and the search of the branes in the uplift, namely that in the staged settings we knew exactly which direction were parallel to the branes, and which transverse. In our setting we cannot infer such knowledge, and so it is not possible a priori to write an integral with respect to the transverse coordinates only. Note however, that in the previous examples one could in fact have integrated over the whole space rather than just the transverse directions since the field strength did not depend on the parallel coordinates $x_\parallel$. Thus the integral $\int \mathrm{d}x_\parallel$ would just give the infinite world volume of the brane. To regularise it we can extend the Gaussian to cover the parallel directions as well, so that each would contribute a factor of $\int_{\mathbb{R}} \mathrm{d}x\, e^{-\frac{1}{2}x^2} = \sqrt{2\pi}$. This extension to the integral over the whole space would also eliminate the problem that $\mathrm{d}^{9-p}x_\perp\sqrt{-g}$ is not an invariant measure, while $\mathrm{d}^{10}x\sqrt{-g}$ is.

Another issue one might wonder about is how to deal with the angular coordinates that we use in the uplift and whether or not they need to be treated differently in the integration. One observation that one might make is that since angular coordinates describe compact domains no regularisation via the Gaussian is necessary. Thus with respect to our coordinate system where $x^\mu = \{x^1, x^2, x^3, x^4, y, \alpha_1, \alpha_2, \alpha_3, \theta, \phi\}$, with x^1 being the time coordinate, we can take the Gaussian to be

$$G(x) = \exp\left[-\frac{1}{2}(x_1^2 + x_2^2 + x_3^2 + x_4^2 + y^2)\right]. \tag{14.37}$$

Since we do not know a priori which directions on the tangent space correspond to transverse and which to parallel directions we will need to integrate all components of the field strength. Thus the integrals of the equations of motion that we would like to evaluate will be of the following form

$$\begin{aligned}\int &\mathrm{d}^{10}x\sqrt{-g}\;\nabla_{\mu_1}F^{\mu_1\ldots\mu_p}\;G(x)\\ &= \int_{\partial\mathrm{bulk}}\mathrm{d}^9\Sigma_{\mu_1}\sqrt{-g}\,F^{\mu_1\ldots\mu_p}\;G(x) - \int\mathrm{d}^{10}x\;\sqrt{-g}\,F^{\mu_1\ldots\mu_p}\;\partial_{\mu_1}G(x).\end{aligned} \tag{14.38}$$

This is just partial integration written in terms of the product rule. As long as the bulk boundary corresponds to $\{x^1,\ldots,x^4\} \to \pm\infty$ or $y \to \infty$ the Gaussian $G(x)$ will make sure that the contribution from the boundary integral $\int \mathrm{d}^9\Sigma_\mu(\ldots)$ vanishes. However, if there are other horizons in the bulk, for example if the space-time is cut off at some finite value of the radial coordinate $y = y_0$, then there would be non-zero contributions, which require a careful analysis. If there are no such horizons and therefore $\int \mathrm{d}^9\Sigma_\mu(\ldots) = 0$ we see that the integration by parts does not lead to any problems related to the compact coordinates and we obtain

$$\int\mathrm{d}^{10}x\sqrt{-g}\;\nabla_{\mu_1}F^{\mu_1\ldots\mu_p}\;G(x) = -\int\mathrm{d}^{10}x\;\sqrt{-g}F^{\mu_1\ldots\mu_p}\;\partial_{\mu_1}G(x). \tag{14.39}$$

We can further simplify by first noting that $\partial_\mu G(x) = -x^\mu G(x)$ for $x_\mu \in \{x^1,\ldots,x^4,y\}$, and $\partial_\mu G(x) = 0$ whenever x^μ is one of the angular coordinates. Furthermore, because we know that our specific solution does not depend on the coordinates $\{x^1, x^2, x^3, x^4\}$ the contribution where x^μ is one of these coordinates vanishes by parity. So we see that the only contribution that we get is when $x^\mu = y \equiv x^5$, and in absence of boundary terms we can write

$$\int \mathrm{d}^{10}x\sqrt{-g}\;\nabla_{\mu_1}F^{\mu_1\cdots\mu_p}\;G(x) = \int \mathrm{d}^{10}x\;\sqrt{-g}(\delta^5_{\mu_1}y)F^{\mu_1\cdots\mu_p}\;G(x). \tag{14.40}$$

The integration formulas developed so far allow us to perform a check on the uplifted form-fields. Since in the uplift all possible fluxes are turned on, it should be in principle checked for each one of them whether or not they contain an electric D-brane source. The most plausible attempts at explaining the GPPZ singularity are based on D5 branes [48] and D7 branes [54], which is why we would like to start by focussing our attention on these two cases.

14.4.2 Looking for D5 Branes

The D5 branes couple electrically to the C_6 form, which produces an F_7 flux, therefore in order to detect the presence of D5 branes we need to analyse the F_7 equations of motion, which in their source-less form are given by

$$\nabla_{\hat{\mu}_1}(m_{\alpha\beta}F^{\hat{\mu}_1\cdots\hat{\mu}_7\;\beta}) = 0. \tag{14.41}$$

The presence of D5 branes with electric coupling to F_7 would produce a delta-function source on the right-hand side of (14.41). Before applying the integration prescription in (14.40) we recall that in principle we need to perform the integration for all possible configurations of indices $\mu_2 \ldots \mu_7$. In order to reduce the amount of data to analyse it is possible to reduce the number of free indices in (14.41) by passing from F_7 to its dual F_3. In our conventions their components are related by

$$F^{\hat{\mu}_1\cdots\hat{\mu}_7\;\beta} = \frac{1}{6}(-g)^{-1/2}\,\epsilon^{\hat{\mu}_1\cdots\hat{\mu}_{10}}m^{\beta\gamma}F_{\hat{\mu}_8\hat{\mu}_9\hat{\mu}_{10}\;\gamma}. \tag{14.42}$$

Note that whenever we write $\sqrt{-g}$ we mean the determinant of the full ten-dimensional metric. The dualisation transforms the equation of motion of F_7 into the Bianchi identity for the F_3, which is given by

$$\partial_{[\hat{\mu}_1}F_{\hat{\mu}_2\hat{\mu}_3\hat{\mu}_4]\;\alpha} = 0. \tag{14.43}$$

Recall that sources that are electric with respect to F_7 would appear as magnetic sources in the Bianchi identity for its dual F_3. To check for the presence of electric D5 source we will integrate the equation of motion (14.41) over the bulk, as described in the previous subsection. After substituting the equation of motion (14.41) into (14.40), and changing the integrand to the dual description we arrive at the following expression:

$$\begin{aligned}\int \mathrm{d}^{10}x\sqrt{-g}\,\nabla_{\hat{\mu}_1}(m_{\alpha\beta}F^{\hat{\mu}_1\ldots\hat{\mu}_7\,\beta})\,G(x) &= \frac{1}{6}\epsilon^{\mu_1\ldots\mu_{10}}\int \mathrm{d}^{10}x\,(y\delta^5_{[\mu_1}F_{\mu_8\mu_9\mu_{10}]\alpha})\,G(x)\\ &\equiv \frac{1}{6}\epsilon^{\mu_1\ldots\mu_{10}}\mathcal{I}^{(7)}_{\mu_1\mu_8\mu_9\mu_{10};\alpha}.\end{aligned} \tag{14.44}$$

As explained above, none of the fields in our solution depend on any of the coordinates $\{x^1, x^2, x^3, x^4\}$, and they enter in the integral $\mathcal{I}^{(7)}$ only through the Gaussian $G(x)$. Therefore the integration with respect to these coordinates is trivial and contributes a factor of $4\pi^2$.

We can carry out the integration with respect to the SO(3) angles $\{\alpha_1, \alpha_2, \alpha_3\}$. Since they do not appear in the warp factor it turns out that the integrand in $\mathcal{I}^{(7)}$ is just a polynomial in trigonometric functions in α_i, and therefore the integration with respect to these angles can be carried out analytically. After this is done many components in $\mathcal{I}^{(7)}$ vanish by parity, and only the following six components remain non-zero:

$$\mathcal{I}^{(7)}_{5,6,9,10;\alpha}\quad \mathcal{I}^{(7)}_{5,7,8,9;\alpha}\quad \mathcal{I}^{(7)}_{5,7,8,10;\alpha}\quad \text{with}\quad \alpha\in\{1,2\}. \tag{14.45}$$

Next the integration with respect to the other two angles θ and ϕ can be studied. Since they appear explicitly in the warp factor the integrand becomes a fraction with polynomials in trigonometric functions in θ and ϕ in the numerator and the denominator, and it seems that in general an analytic evaluation would be impossible to perform. However, it is still easy to check whether or not some of the integrals vanish by parity. It turns out that three out of the six components of $\mathcal{I}^{(7)}$ in (14.45) are set to zero, and the remaining non-zero components are

$$\mathcal{I}^{(7)}_{5,6,9,10;1}\quad \mathcal{I}^{(7)}_{5,7,8,9;2}\quad \mathcal{I}^{(7)}_{5,7,8,10;1}. \tag{14.46}$$

To make progress we can compute the remaining three integrals in y, θ, and ϕ numerically. To do so, we need to give numerical values to all parameters entering in the integrals, which in our case are the two constants C_1 and C_2. The numerical values we will choose are

$$C_1 = 1.3 \qquad \text{and} \qquad C_2 = 1.0. \tag{14.47}$$

Given the singular behaviour of the solution at $y = C_1$ we would like to regularise the integration by setting the domain for the radial coordinate to $y \in [C_1 + c, \infty]$, where c is a cut-off parameter. We will evaluate the integral for various values of c in order to check to which value the series of integrals with decreasing c converges. In [54] it was found that as one approaches the singularity along the radial direction the solution starts exhibiting singular behaviour along the boundary of the (θ, ϕ) domain. This singularity was referred to by the authors of [54] as the ring singularity, and we are able to confirm its presence. In view of this we will also regularise the angular integration with the same cut-off parameter

c. The integration domains for y, θ, and ϕ will be therefore taken to be

$$y \in [C_1, \infty] \to y \in [C_1 + c, \infty] \tag{14.48a}$$

$$\theta \in [0, \pi/2] \to \theta \in [c/2, \pi/2 - c/2] \tag{14.48b}$$

$$\phi \in [0, \pi] \to \phi \in [c, \pi - c]. \tag{14.48c}$$

Note that we evaluate the radial integral over the domain $[C_1, \infty]$ rather than $[-\infty, \infty]$. The reason is that it seems that the integrand is badly behaved in the interval $y \in [C_2, C_1]$, and numerical integration fails there. This certainly requires a more thorough investigation, and one might need to conclude that the space-time terminates at the radial value $y = C_1$ thus introducing a new horizon. If this is the case then then one will need to also include the boundary integration as described in equation (14.38). Leaving this issue for a thorough investigation in future we evaluated the integrals $\mathcal{I}^{(7)}_{5,6,9,10;1}$, $\mathcal{I}^{(7)}_{5,7,8,9;2}$, and $\mathcal{I}^{(7)}_{5,7,8,10;1}$ in the regularised range as described in equation (14.48). The numerical integration was performed using the software Mathematica [81]. We used various built in integration strategies in order to ensure numerical stability. An example for such an integration performed for the component $\mathcal{I}^{(7)}_{5,6,9,10;1}$ is shown in Figure 14.1. One can see that different integration methods all give very similar results and converge to a non-zero value for vanishing cut-off $c \to 0$. However, at this stage we cannot yet claim with certainty that this indeed corresponds to D5 brane sources. As noted above it is still not clear whether or not one needs to include a horizon at $y = C_1$ and the corresponding boundary integral. The exploration of this issue is subject of an ongoing investigation.

14.4.3 Looking for D7 Branes

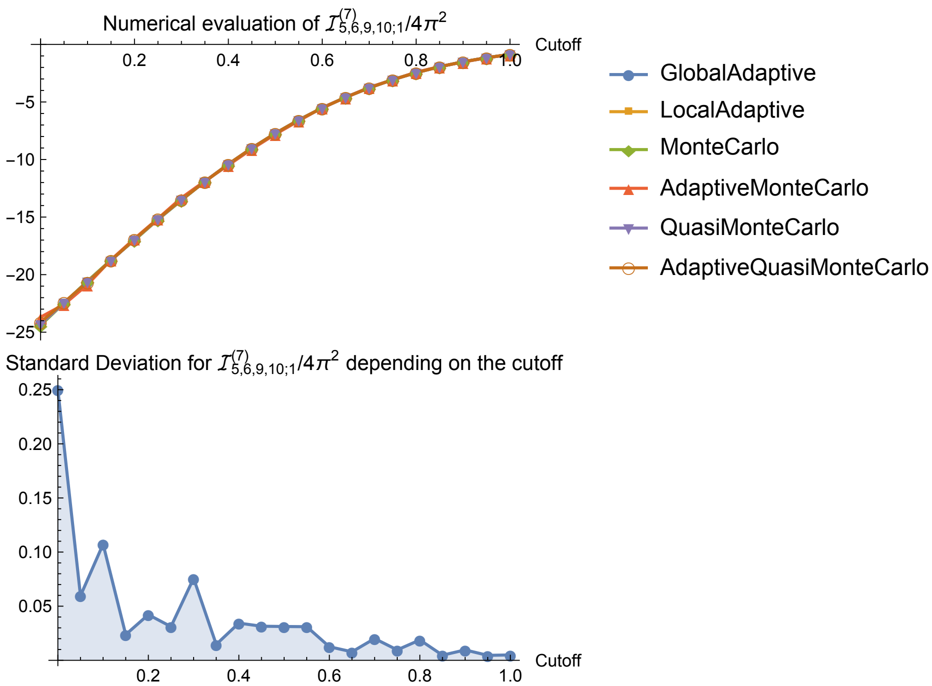
The case of D7 branes can be processed similarly to the D5 brane case. The potential electric sources for these branes will be found in the equations of motion for the F_9 form-field, which is given by

$$\nabla_{\hat{\mu}_1} F^{\hat{\mu}_1 \ldots \hat{\mu}_9}{}_{\alpha}{}^{\beta} + j_{\hat{\mu}_1}{}_{\gamma}{}^{\beta} F^{\hat{\mu}_1 \ldots \hat{\mu}_9}{}_{\alpha}{}^{\gamma} = 0. \tag{14.49}$$

The term $j_{\hat{\mu}}{}_{\gamma}{}^{\beta}$ appearing in this equation is the axion-dilaton current defined by $j_{\hat{\mu}}{}_{\gamma}{}^{\beta} = \partial_{\hat{\mu}} m_{\gamma\alpha} m^{\alpha\beta}$. If there are any D7 branes to which F_9 electrically couples then the corresponding source terms will appear on the right-hand side of equation (14.49). To find these sources we will proceed similarly to the previous subsection and integrate the F_9 equation of motion over the bulk. Again it proves useful to change to the dual form, which is essentially the axion-dilaton current. The precise relation between the components is the following:

$$F^{\hat{\mu}_1 \ldots \hat{\mu}_9}{}_{\alpha}{}^{\beta} = (-g)^{-1/2} \, \varepsilon^{\hat{\mu}_1 \ldots \hat{\mu}_{10}} \, j_{\hat{\mu}_{10}\alpha}{}^{\beta}. \tag{14.50}$$

Figure 14.1: Numerical evaluation of $\mathcal{I}^{(7)}_{5,6,9,10;1}$ for a cut-off $c \in \{0,1\}$, and the radial integration performed in the interval $y \in [C_1 + c, \infty)$. The upper graph shows the value of the integral produced by different numerical integration methods, the lower graph shows the standard deviation computed from the values of the integral corresponding to different numerical integration methods.



As in the case of F_7, the equation of motion of F_9 transforms into the Bianchi identity for its dual and reads

$$\partial_{[\hat{\mu}_1} j_{\hat{\mu}_2]\,\alpha}{}^{\beta} + j_{[\hat{\mu}_1\,\gamma}{}^{\beta} j_{\hat{\mu}_2]\,\alpha}{}^{\gamma} = 0. \tag{14.51}$$

After performing similar steps as for the F_7 in the previous section we arrive at the following expression for the bulk integral which we would like to evaluate:

$$\begin{aligned} &\int \mathrm{d}^{10}x \sqrt{-g}\, \nabla_{\hat{\mu}_1} F^{\hat{\mu}_1 \ldots \hat{\mu}_9}{}_{\alpha}{}^{\beta} + j_{\hat{\mu}_1\,\gamma}{}^{\beta} F^{\hat{\mu}_1 \ldots \hat{\mu}_9}{}_{\alpha}{}^{\gamma}\, G(x) \\ &\qquad = \epsilon^{\mu_1 \ldots \mu_{10}} \int \mathrm{d}^{10}x\, \left(y \delta^5_{[\mu_1} j_{\hat{\mu}_{10}]\,\alpha}{}^{\beta} + j_{[\hat{\mu}_1\,\gamma}{}^{\beta} j_{\hat{\mu}_{10}]\,\alpha}{}^{\gamma}\right) G(x) \\ &\qquad \equiv \epsilon^{\mu_1 \ldots \mu_{10}} \mathcal{I}^{(9)}{}_{\mu_1 \mu_{10};\alpha}{}^{\beta}. \end{aligned} \tag{14.52}$$

Note that the axion-dilaton matrix $m_{\alpha\beta}$ is SO(3)-invariant and therefore does not depend on the three SO(3) angles α_i. Consequently also the axion-dilaton current $j_{\mu;\alpha}{}^{\beta}$ does not depend on α_i, and all components of the Bianchi-identity (14.51) which refer to these angles vanish. This descends down to the bulk integral (14.52) in which the integrand does not depend on α_i, and therefore the integration with respect to these angles is trivial and gives a numerical factor of $4\pi^3$. Since as in the previous subsection the integrand does not depend on the coordinates $\{x^1, x^2, x^3, x^4\}$, we can also evaluate the corresponding integrals, which give a contribution equal to $4\pi^2$. In this way seven of the ten integrals are trivial and give a cumulative contribution of $16\pi^5$. The remaining non-vanishing components of ${\mathcal{I}^{(9)}}_{\mu_1\mu_{10};\alpha}{}^{\beta}$ are given by

$$ {\mathcal{I}^{(5)}}_{5,9;\alpha}{}^{\beta} \qquad {\mathcal{I}^{(5)}}_{5,10;\alpha}{}^{\beta} \qquad {\mathcal{I}^{(5)}}_{9,10;\alpha}{}^{\beta} \qquad \forall\{\alpha,\beta\}\in\{1,2\}. \tag{14.53} $$

After also taking into account the parity in θ and ϕ only half of the components above turn out to be non-vanishing, namely the following:

$$ {\mathcal{I}^{(5)}}_{5,9;1}{}^{1} \qquad {\mathcal{I}^{(5)}}_{5,10;1}{}^{2} \qquad {\mathcal{I}^{(5)}}_{9,10;1}{}^{2} \tag{14.54} $$
$$ {\mathcal{I}^{(5)}}_{5,9;2}{}^{2} \qquad {\mathcal{I}^{(5)}}_{5,10;2}{}^{1} \qquad {\mathcal{I}^{(5)}}_{9,10;2}{}^{1}. \tag{14.55} $$

These are the integrals that have to be evaluated numerically. We applied to same strategy as for the D5 branes and employed different integration methods to evaluate the regularised integral over the domain described in (14.48) while gradually decreasing the cut-off. The emerging results are qualitatively similar in that the integration seems to converge to a non-zero value for a vanishing cut-off. However, the numerical stability of the integration worsens as one approaches the singularity, and different numerical integration methods spread significantly close to the cut-off value $c = 0$, which should be taken as a sign that it might be necessary to employ more sophisticated integration methods. We hope the the ongoing investigation will be able to improve the results of the numerical analysis as well as clarify the precise topology of the space-time and the existence or non-existence of a space-time horizon at the critical radial value.

Chapter 15

Summary and Conclusion

In this thesis we set off by reviewing and recapitulating the $\mathcal{N} = 4$ super Yang-Mill theory in four dimensions and aspects related to it. Since the particular application of the holographic principle used in the text is restricted to the case of the correspondence between four-dimensional gauge theories and their gravitational duals, it proved useful to set the stage by introducing the field theory in the conventions that were used throughout the rest of the text. We devoted a substantial time to writing down the supersymmetry transformations and checking explicitly that they are indeed a global symmetry of the action. This was important for several reasons. On the one hand the superconformal multiplets that are used later in the text are built by repeated application supersymmetry transformations on primary operators, which is why we had to make sure that the transformations that we used were indeed correct and consistent. On the other hand, even though the $\mathcal{N} = 4$ super Yang-Mills theory is one of the most used field theories in the context of holography, maybe even *the* most used theory, it was not possible to find an explicit check of the invariance of the action in literature in the way we performed it. While we are sure that such a computation has been carried out by others, and might exist in a written down form, we find that going through some non-trivial steps as they appear in matching the cancellations between various terms might well contribute to the common knowledge of the community and be henceforth available as a useful resource. A handy side-effect of reviewing the supersymmetry transformations of the $\mathcal{N} = 4$ super Yang-Mills theory was the recollection and derivation of certain sigma matrix identities. While various versions and listings for $\mathfrak{so}(1,3)$ sigma matrices exist in literature, of which we chose and reviewed a particular one, identities for $\mathfrak{so}(6)$ sigma matrices are more difficult to come by. We derived and checked an extensive list of such identities, and highlighted the group theoretical aspect of some of them in one of the appendices.

A part of the revision of the $\mathcal{N} = 4$ theory was the construction of short superconformal multiplets and their relation to the dual gravitational modes. Even though this material is considered well-known we believe that we found a way to present some aspects of the duality in a new light. We pointed out that the mapping of the supergravity modes to operators in short multiplets exhibits interesting patterns; after the mapping to the short multiplet, different tensorial components of the same ten-dimensional field fall into columns, and Kaluza-Klein towers together with their twins “grow” perpendicularly to the multiplets sweeping across multiplets of all orders in a skewed way. We also found that the masses of twinned Kaluza-Klein modes are related in a simple way when expressed in terms of the order of the short multiplet that they correspond to.

The next part of the thesis built upon the idea that renormalisation group flows in field theory can be holographically related to domain wall solutions in supergravity. While this relation per se is widely known the novel idea that we developed is that for some classes of operators the beta-functions are classical and can be therefore computed exactly. After invoking the holographic dictionary it is possible to relate these exact beta-functions to domain-wall solutions in the bulk, which, in turn allows us to exactly determine the scalar potential to which these modes are subjected. We were able to confirm these findings by comparison to known potentials that were derived non-holographically.

The renormalisation group flows we considered are triggered by certain deformations of the $\mathcal{N} = 4$ super Yang-Mills theory which share the common property of preserving an $\mathcal{N} = 1$ symmetry. This property rendered the exact computation of the beta-function possible in the first place, and served as an inspiration to try and classify all possible $\mathcal{N} = 1$ preserving deformations. What has emerged from this idea is a systematic study of the $\mathcal{N} = 1$ decomposition of the $\mathcal{N} = 4$ short multiplets, and a complete understanding of the operators that would appear in the action. We understood that each F-term chiral operator is always accompanied by an additional chiral operator which turns on an extra bulk mode. One might eliminate this operator by carefully choosing the tensorial structure of the corresponding coupling.

During the study of F-term operators and the similar non-scalar operators corresponding to top components of chiral superfields it became clear that what was found to be the so-called twin operators can be modelled as D-terms, or more generally as top-components of certain real superfields. On the field theory side the two types of operators can be grouped by writing a deformation containing both twins in a unified manner. We expanded further on the supersymmetric deformation by top components of compound superfields and studied what a general deformation containing operators from an arbitrary number of short multiplets of different orders would look like. It was not surprising to find that chiral superfields can be generalised to arbitrary holomorphic functions thereof, since this is the well-known superpotential formulation. A similar analysis performed on real superfields shows that they can be generalised to harmonic functions, which neatly matches the harmonic functions that were independently found [43] to appear in the bulk description of the deformations we considered.

A prototypical field theory deformation that our investigation built upon is the GPPZ mass term, which can be found in the lowest order short multiplet and corresponds to a mode in the graviton multiplet in the bulk. An additional attractive feature of the GPPZ deformation is the presence of a non-trivial gaugino condensate. This feature is quite similar to what happens in real-life strongly coupled field theories, which develop condensates as well. The holographic description of condensates follows a similar pattern as true field theory deformations in the sense that for each operator that condenses a dual bulk mode has to be turned on. While we can classify the corresponding operators in the same scheme we developed for true deformations it seems that our understanding of the precise relationship between the energy dependence of the condensates and the bulk physics needs to be further understood. Even though in the GPPZ case the application of the beta-function formalism to the condensate yields the expected results, one needs to further investigate if any amendments are necessary in the general case. In particular we would like to understand the subtle role of non-perturbative contributions, as well as anomalies, which have been seen appearing in such settings. A similar disclaimer also applies to the true deformations that were considered. While $\mathcal{N} = 1$ non-renormalisation theorems can be used at their full power to exclude certain quantum corrections, the absence of the anomalous dimension for operators in short multiplets is not expected to be upheld once the the $\mathcal{N} = 4$ symmetry is broken down to $\mathcal{N} = 1$. Thus it would be desirable to investigate and quantify under which conditions the absence of an anomalous dimension can be maintained.

Apart from understanding the condensates in the framework of the holographic beta function there are various other natural research directions that present themselves. One limitation of our beta-function description is that it requires a one-scalar truncation in the bulk and correspondingly a one-operator deformation in the field theory. Moreover since the bulk radial direction can be related to the energy scale of the field theory the deformation operator must not mix with other operators along the RG flow, otherwise additional modes in the bulk would be turned on. Thus a natural extension seems to find descriptions that involve multiple operators and bulk modes. These could either be pure deformation operators like the mass term we saw earlier, or contain both true deformations and vacuum expectation values that place the theory in a different vacuum, similarly to the case of the GPPZ.

Yet another attractive application of the formalism we introduced is holography in cosmology and inflation, aspects of which were introduced and discussed in [34, 119]. Domain wall solutions in AdS spaces similar to those that we used for modelling of field theory RG flows can be analytically continued to FLRW cosmologies that interpolate between a static Einstein Universe and a de Sitter vacuum [34] with the potential of the theory obtained from that of the AdS theory by inverting the overall sign. Some applications to the study of the inflationary universe were made by McFadden and Skenderis [120, 121] which confirmed that perturbative quantum field theories can be used to describe cosmological inflation holographically, and to obtain quantitative results. In doing so we can reverse the logic we were using in the holographic-beta function computation presented in this thesis. One can start with the gravity potential with the properties that are required by inflation or other processes that we would like to study in cosmology, and then try to determine the form of the beta-function that could generate such a potential. From this one could attempt to construct a field theory dual that could be used to further study the gravity side and with it the cosmology, thus closing the loop.

The second project discussed in detail in this thesis is the uplift of the GPPZ solution from five to ten dimensions. Even though it was constructed almost twenty years ago, to this day the GPPZ solution remains one of the most interesting applications of the holographic duality. It shares many properties with field theories used to describe the Standard Model of particles, therefore making it an excellent candidate for providing a holographic handle on the treatment of otherwise hardly accessible strongly coupled non-perturbative physics of the QCD. The original GPPZ solution was constructed by relating the $\mathcal{N} = 4$ super Yang-Mills theory deformed by a chiral mass term with the maximal gauged supergravity in five dimensions. This theory exhibits such features as a running coupling constant, gaugino condensation, and confinement in the infra-red, but as one descends down the energy scale a singularity in the dual bulk description appears, indicating that the supergravity approximation has reached the boundary of its possibilities. In a collaboration we generalised and upgraded the supergravity construction by uplifting the five-dimensional solution to ten dimensions, and initiated a study of the new features that come with it which were invisible from the five-dimensional point of view.

The uplift of the GPPZ solution was made technically possible due to ideas of generalised geometry and double field theory which were used by Samtleben and Hohm to formulate the so-called exceptional field theory construction [58, 122, 123]. This framework was further specialised to prove that any solution of five-dimensional gauged supergravity lifts to IIB supergravity in ten dimensions [57], and provided explicit formulas how to construct such uplifts. We used precisely these formulas to obtain a full ten-dimensional version of the GPPZ solution including all non-trivial fluxes. Along the way it became clear that it makes sense to uplift a generalised version of the GPPZ solution in which the complex phases of the modes corresponding to the mass deformation and the gaugino condensate are kept, thereby doubling the number of uplifted degrees of freedom from two to four. In fact, a fifth degree of freedom needs to be included in form of a U(1) gauge field to make the uplift consistent. The complex angles just mentioned enter completely covariantly in the ten-dimensional uplift and can be related to two distinct local U(1) symmetries. One of these U(1) symmetries can be identified as the counterpart of the $\mathrm{U}(1)_{\mathrm{R}}$ symmetry of the dual $\mathcal{N} = 1$ field theory. The other U(1) might be part of the local composite symmetry that appears in the coset formulation of the supergravity scalar fields, and is necessary to maintain a previously fixed gauge. However, the details of this additional U(1) still need to be better understood, and we are hoping to find a precise quantitative way of putting it in the right context.

Once the full uplift was obtained we initiated various studies and tests of the solution. After spending some time on numerically checking that the uplift indeed satisfies all equations of motion we proceeded with the analysis of the asymptotic structure of the fields and the metric near the conformal boundary. Since the GPPZ solution was obtained by a deformation by relevant operators, the contribution of which vanishes in the deep UV, one necessary condition for the consistency of the solution is that it asymptotes to the $AdS_5 \times S^5$ vacuum near the conformal boundary. One way of thinking of this property is as boundary conditions under which the solution was constructed in the first place. After looking at subleading orders of the axion-dilaton matrix we found perfect agreement with the zero temperature limit of the asymptotic analysis of Freedman and Minahan [62], but differences to the analysis of Polchinski and Strassler [48].

For an expansion to subleading orders it is worth to remember that the solution can be seen as a deformation of a background containing an S^5 part. Thus a natural basis for a systematic expansion of fields is in terms of S^5 harmonics. The deformation breaks the SO(6) isometry group of the five-sphere to an SO(3) subgroup, which is why the expansion only contains those harmonics that are invariant under this symmetry. We devoted some effort to the construction of such harmonics using their defining differential equation, as well as by using group theory techniques. We derived the full set of scalar harmonics with this property, an infinite subset of vector harmonics, and the lowest tensor harmonics. In the further course of this project it will be useful to continue this derivation and complete the number of the SO(3) invariant harmonics to a full set.

While the asymptotic analysis is very useful as a consistency check of the holographic dictionary, as well as for a comparison with the asymptotics found by other authors, the expansion of our solution near the singularity is expected to provide new clues about its nature and physical interpretation. The behaviour of the metric close to the singularity was constructed by Pilch and Warner [54], and an interpretation was given that the near singularity geometry that emerges is that of D7 branes. However, we find that if all terms of the same order are kept in the expansion then this interpretation can no further be maintained. In fact, from this new point of view it seems that the geometry might be much rather that of D5 branes. This would be consistent with the dielectric interpretation of Polchinski and Strassler [48], where D3 branes are polarised to D5 branes. In the same article Polchinski and Strassler also derive an asymptotic form of the metric close to the singularity. However, we were not able to match the terms in our expansion with those arising in the asymptotic expansion in their article.

The question about the presence of D-branes, and the possibility that they source the singularity is of central importance to the uplift and was in fact one of the motivations for this project. The evidence available so far, some examples of which were discussed earlier, is based on the asymptotic expansion of fields and the metric, and is therefore of indirect and ambiguous nature. Thus it seems indispensable to find a better and more direct way to detect the presence or absence of D-branes. By their coupling to form fields D-branes would appear as delta-function sources in the equations motion, and even though the equations of motion have been verified to hold without the source terms, the possibility for the presence of branes should not be considered as ruled out. We use an example of a classical point charge to show that the delta function sources typically appear upon differentiation of some fields, which, if not treated with extreme care are easily interpreted to give no source terms. Thus it appears that the manual search for source terms in complicated equations of motion can prove to resemble the search for a needle in a haystack, which is why a better method for the source detection is necessary. As we showed it is possible to fold the equations of motion with certain test functions over the space-time manifold, with the result being either exactly zero, or non-zero depending on whether or not sources are present. However, there are important caveats to be aware of. It is crucial to understand the geometry of the space-time that one integrates over, in particular all of its horizons and boundaries, which have to be taken into account. Additionally one needs to check for contributions from the singularity. The integration itself might be non-trivial, and require special algebraic care in order to yield the correct results. As for now we have numerical evidence that certain D-branes are indeed in the system, but to make the definitive claim further analysis is necessary. We hope to complete this stage of our study in future.

Part IV

Appendix

Appendix A

Notation, Conventions, Identities

A.1 General Conventions

In this appendix we would like to list the conventions and identities that we used throughout this thesis. For Lorentzian space-times of any dimension we will use the "mostly positive" metric. In the four-dimensional Minkowski space it reads

$$\eta_{\mu\nu} = \mathrm{diag}(-,+,+,+) \tag{A.1}$$

$$\eta = \det(\eta_{\mu\nu}) = -1\,. \tag{A.2}$$

We fix the conventions for the the two-dimensional and four-dimensional Levi-Civita tensors in such a way that when written with all indices up they have the canonical positive sign. Because the determinant of the four-dimensional Minkowski metric is negative, the four-dimensional Levi-Civita tensor with all indices down acquires an additional sign. We also define the two-dimensional Levi-Civita tensors with an additional minus sign so that the SU(2) indices can be raised an lowered by contracting with the ϵ-tensors from the left, which is the standard convention.

$$\epsilon^{12} = \epsilon^{\dot{1}\dot{2}} = \epsilon^{1234} = 1 \qquad\qquad \epsilon_{12} = \epsilon_{\dot{1}\dot{2}} = \epsilon_{1234} = -1. \tag{A.3}$$

The generalised Kronecker delta with n upper and n lower indices is defined as a completely anti-symmetrised tensor product of n ordinary delta functions:

$$\delta^{i_1\dots i_n}_{j_1\cdots j_n} = \delta^{[i_1}_{j_1}\cdots\delta^{i_n]}_{j_n}. \tag{A.4}$$

Note that by anti-symmetrising the upper indices, we automatically anti-symmetrises the lower indices. A contraction of two epsilon tensors using the Minkowski metric $\eta_{\mu\nu}$ can be expressed in terms of the generalised delta-function as follows:

$$\epsilon_{i_1\ldots i_n k_1\ldots k_m}\epsilon^{j_1\ldots j_n k_1\ldots k_m} = \eta\, n!m!\,\delta^{j_1\ldots j_n}_{i_1\ldots i_n} \tag{A.5}$$

The definitions for the Riemann tensor, the Ricci tensor, and Ricci scalar are chosen to be identical to those used by Freedman and van Proyen [63]:

$$\begin{aligned} R_{\mu\nu\rho}{}^{\sigma} &= (-\partial_\mu\Gamma^\sigma_{\nu\rho} + \partial_\nu\Gamma^\sigma_{\mu\rho} + \Gamma^\tau_{\rho\mu}\Gamma^\sigma_{\nu\tau} - \Gamma^\tau_{\rho\nu}\Gamma^\sigma_{\mu\tau}) \\ &= 2(-\partial_{[\mu}\Gamma^\sigma_{\nu]\rho} + \Gamma^\tau_{\rho[\mu}\Gamma^\sigma_{\nu]\tau}) \end{aligned} \tag{A.6}$$

$$R_{\mu\nu} = R_{\mu\rho\nu}{}^{\rho} \tag{A.7}$$

$$R = g^{\mu\nu}R_{\mu\nu} \tag{A.8}$$

A.2 Summary of Conventions for the Uplift

This subsection is a plain list of various definitions that we used in this text, which serves as an easy look up, irrespective of similar definitions that may exist elsewhere in literature.

$$m(y) = \frac{\sqrt{3}}{2}\log\left(\frac{1+e^{-(y-C_1)}}{1-e^{-(y-C_1)}}\right) = \sqrt{3}\,\mathrm{arctanh}(e^{-(y-C_1)}) \tag{A.9}$$

$$\sigma(y) = \frac{1}{2}\log\left(\frac{1+e^{-3(y-C_2)}}{1-e^{-3(y-C_2)}}\right) = \mathrm{arctanh}(e^{-3(y-C_2)}) \tag{A.10}$$

$$\begin{aligned} \phi(y) &= \frac{1}{2}\log(2\sinh(y-C_1)) + \frac{1}{6}\log(2\sinh(3(y-C_2))) \\ &= y - \frac{C_1+C_2}{2} - \log\cosh\frac{m(y)}{\sqrt{3}} - \frac{1}{3}\log\cosh\sigma(y). \end{aligned} \tag{A.11}$$

$$C_1 \leq C_2 \tag{A.12}$$

$$m_0 = \sqrt{3}e^{C_1}, \qquad \sigma_0 = \frac{1}{2}e^{3C_2}. \tag{A.13}$$

$$m(y) = m_0\,e^{-y} + \cdots \tag{A.14}$$

$$\sigma(y) = 2\sigma_0\,e^{-3y} + \cdots = 2\langle\lambda\lambda\rangle e^{-3y} + \cdots \tag{A.15}$$

$$\phi(y) = y + \cdots \tag{A.16}$$

$$\xi^2 = \Delta^{-8/3}. \tag{A.17}$$

$$t = e^{-(y-C_1)}, \qquad \chi = 2(1-t)^{1/2}, \qquad \lambda = e^{-3(C_1-C_2)} \leq 1 \tag{A.18}$$

AdS boundary:	$y = \infty$	$t = 0$	$\chi = 2$	(A.19)
Singularity:	$y = C_1$	$t = 1$	$\chi = 0$	(A.20)

$$\mu = e^{\sigma} = \sqrt{\frac{1+\lambda t^3}{1-\lambda t^3}} \qquad \nu = e^{m/\sqrt{3}} = \sqrt{\frac{1+t}{1-t}}. \tag{A.21}$$

$$\mathbb{R}^6 \supset S^5 \ni \vec{y} = (y_1, \ldots y_6) = (u_1, u_2, u_3, v_1, v_2, v_3) = (\vec{u}, \vec{v}). \tag{A.22}$$

$$\vec{y}^{\,2} = \vec{u}^{\,2} + \vec{v}^{\,2} = 1 \tag{A.23}$$

$$\vec{u} = R \begin{pmatrix} 0 \\ 0 \\ \cos\theta \end{pmatrix} \equiv R.\vec{u}_0 \qquad \vec{v} = R \begin{pmatrix} 0 \\ \sin\theta\sin\phi \\ \sin\theta\cos\phi \end{pmatrix} \equiv R.\vec{v}_0 \tag{A.24}$$

$$R \in \mathrm{SO}(3), \quad \theta \in [0, \pi/2], \quad \phi \in [0, \pi] \tag{A.25}$$

$$w_1 = 2\,\vec{u}^{\,2} - 1 = 1 - 2\,\vec{v}^{\,2} = \vec{u}^{\,2} - \vec{v}^{\,2} = \cos(2\theta) \tag{A.26}$$

$$w_2 = 2\,\vec{u}.\vec{v} = \sin(2\theta)\cos\phi \tag{A.27}$$

$$-1 \leq w_1, w_2 \leq 1 \tag{A.28}$$

$$\begin{aligned} \zeta &= 1 - w_1^2 - w_2^2 \\ &= 4\sin^2\theta\cos^2\theta\sin^2\phi \\ &= \sin^2(2\theta)\sin^2\phi \end{aligned} \tag{A.29}$$

$$\begin{aligned}\hat{\Omega} &= \frac{1}{3}\sqrt{4 - w_1^2 - 4w_2^2} \\ &= \frac{1}{3}\sqrt{4(1 - w_1^2 - w_2^2) + 3w_1^2} \\ &= \frac{1}{3}\sqrt{4\zeta + 3w_1^2} \\ &= \frac{1}{3}\sqrt{4\sin^2(2\theta)\sin^2\phi + 3\cos^2(2\theta)}\end{aligned} \tag{A.30}$$

A.3 Fields and the Radial Coordinate in GPPZ and PW

In this section we introduce the notation used by Girardello, Petrini, Porrati, and Zaffaroni (GPPZ) [23] and by Pilch, Warner (PW) [54], and point out their similarities and differences. The definitions that we decided to use in this text are listed above in Appendix A.2.

The GPPZ solution [23] is given as

$$\phi(y) = \frac{1}{2}\log(2\sinh(y - c_1)) + \frac{1}{6}\log(2\sinh(3y - c_2)) \tag{A.31}$$

$$m(y) = \frac{\sqrt{3}}{2}\log\left(\frac{1 + e^{-(y-c_1)}}{1 - e^{-(y-c_1)}}\right) \tag{A.32}$$

$$\sigma(y) = \frac{1}{2}\log\left(\frac{1 + e^{-3(y-c_2/3)}}{1 - e^{-3(y-c_2/3)}}\right). \tag{A.33}$$

We have replaced the capital integration parameters C_1 and C_2 by the small-lettered c_1 and c_2, otherwise the solution is as in equations (38)–(40) in [23]. The authors argue that on physical grounds one should assume that $c_2 \leq 3c_1$. The radial coordinate y used in GPPZ is such that the AdS metric takes the form

$$ds^2 = e^{2y}\eta_{\mu\nu}dx^\mu dx^\nu + dy^2. \tag{A.34}$$

The AdS boundary is at $y = \infty$, and corresponds to the UV of the field theory. The asymptotics of the fields in the near boundary limit $y \to \infty$ are the following:

$$m(y) = \sqrt{3}\,e^{-(y-c_1)} + \mathcal{O}\left(e^{-2(y-c_1)}\right) \tag{A.35}$$

$$\sigma(y) = e^{-3(y-c_2/3)} + \mathcal{O}\left(e^{-6(y-c_2/3)}\right). \tag{A.36}$$

The fields $m(y)$ and $\sigma(y)$ can also be rewritten as follows

$$m(y) = \sqrt{3}\,\text{arctanh}\left(e^{-(y-c_1)}\right) \tag{A.37}$$

$$\sigma(y) = \text{arctanh}\left(e^{-3(y-c_2/3)}\right). \tag{A.38}$$

Using these expressions it is easy to see that $\phi(y)$ can be brought to the following form:

$$\begin{aligned}\phi(y) &= y - \frac{c_1 + c_2/3}{2} + \frac{1}{2}\log(1 - e^{-2(y-c_1)}) + \frac{1}{6}\log(1 - e^{-6(y-c_2/3)}) \\ &= y - \frac{c_1 + c_2/3}{2} - \log\cosh\frac{m(y)}{\sqrt{3}} - \frac{1}{3}\log\cosh\sigma(y).\end{aligned} \tag{A.39}$$

The metric of the GPPZ domain-wall solution is given by

$$ds^2 = e^{2\phi(y)}\eta_{\mu\nu}dx^\mu dx^\nu + dy^2 \tag{A.40}$$

and because $m \to 0$ and $\sigma \to 0$ as one approaches the boundary at $y = \infty$ it is clear from equation (A.39) that in this limit the AdS geometry of (A.34) is recovered.

We can connect the GPPZ notation with the notation in Section 7 in PW. First the authors define

$$\mu(y) = e^{\sigma(y)} \tag{A.41}$$

$$\nu(y) = e^{m(y)/\sqrt{3}}. \tag{A.42}$$

Note that there is a typo above the equation (7.1) in [54] where there is a factor of 2 in the definition of $\nu(y)$ that should not be there. The correct definition is in their equation (6.4). The radial coordinate r used in PW is equivalent to the radial coordinate y used in GPPZ up to rescaling. With L being the AdS radius the precise correspondence is

$$y = \frac{r}{L}. \tag{A.43}$$

Let us denote the integration constants used in PW by capital letters C_1 and C_2 to distinguish them from the integration constants c_1 and c_2 used in GPPZ. Their identification is as follows:

$$C_1 = c_1 \tag{A.44}$$

$$C_2 = c_2/3. \tag{A.45}$$

The radial coordinate y and one of the integration constant C_i can be traded for new variables t and λ defined as follows in PW:

$$t = e^{-(y-C_1)} \tag{A.46}$$

$$\lambda = e^{-3(C_1-C_2)}. \tag{A.47}$$

The fields μ and ν expressed in terms of these new variables take a simple form:

$$\mu(t) = \sqrt{\frac{1+\lambda t^3}{1-\lambda t^3}} \tag{A.48}$$

$$\nu(t) = \sqrt{\frac{1+t}{1-t}}. \tag{A.49}$$

Yet another definition of the radial coordinate introduced in PW is the following

$$\chi = 2(1-t)^{1/2}. \tag{A.50}$$

Furthermore Pilch and Warner define the field $A(t)$ that controls the domain-wall profile of the metric:

$$\begin{aligned} A(t) &= \frac{1}{6}\log(t^{-3} - \lambda^2 t^3) + \frac{1}{2}\log(t^{-1} - t) + C_1 \\ &= y + \frac{1}{6}\log(1 - (\lambda t^3)^2) + \frac{1}{2}\log(1 - t^2) \\ &= y - \frac{1}{3}\log\frac{\mu + \mu^{-1}}{2} - \log\frac{\nu + \nu^{-1}}{2} \\ &= y - \frac{1}{3}\log\cosh\sigma - \log\cosh\frac{m}{\sqrt{3}}. \end{aligned} \tag{A.51}$$

After comparing the definition of $\phi(y)$ in (A.39) and $A(t)$ in (A.51) we see that they are related by

$$A = \phi + \frac{C_1 + C_2}{2}. \tag{A.52}$$

A shift of $\phi(y)$ by a constant drops out in the first order equations of motion so that $A(y)$ is also a solution. This shift freedom is equivalent to an integration constant that we could have called C_3 that appears after solving the equations of motion for $\phi(y)$. Note that any shift of $\phi(y)$ by a finite constant leads to exactly the same asymptotic behaviour near the boundary because $\phi(y) \to y$ as y becomes large and the constant shift can be neglected.

Finally PW define the following constants

$$m_0 = \frac{\sqrt{3}}{2}e^{C_1}, \qquad \sigma_0 = \frac{1}{3}e^{3C_2}. \tag{A.53}$$

Given the asymptotics in equations (A.35)–(A.36) these definitions seem somewhat unnatural as the asymptotic behaviour of the fields becomes

$$m(y) = 2m_0\, e^{-y} + \dots \tag{A.54}$$

$$\sigma(y) = 3\sigma_0\, e^{-3y} + \dots \tag{A.55}$$

Generically, solutions of such type can represent both a deformation of the dual field theory by an operator $\mathcal{O}$ and/or different vacua of the same theory characterised by a vacuum expectation value $\langle\mathcal{O}\rangle$. The behaviour of the solution in the asymptotic AdS region, $y \to +\infty$, discriminates between the two options. For $y \to +\infty$, the asymptotic behaviour consist of a non-normalisable part and a normalisable one

$$\varphi \underset{y\to+\infty}{\sim} e^{(\Delta-d)y}(A+\cdots)+e^{-\Delta y}(B+\cdots)\,, \tag{A.56}$$

where Δ is the conformal dimension of the dual operator and the dots in the leading non-normalisable part are local functions of A while the dots in the normalisable part are functions of both A and B. The coefficient A of the non-normalisable solution is interpreted as a deformation of the Lagrangian, while the coefficient B of the normalisable solution is related to the vacuum expectation value of the operator $\mathcal{O}$ dual to φ by $B = \frac{1}{2\Delta-d}\langle\mathcal{O}\rangle$ [82]. In this light it is more natural to define

$$m_0 = \sqrt{3}e^{C_1}, \qquad \sigma_0 = 2e^{3C_2}, \tag{A.57}$$

which leads to the asymptotics

$$m(y) = m_0\, e^{-y} + \ldots \tag{A.58}$$

$$\sigma(y) = \frac{\sigma_0}{(2\Delta-d)}e^{-3y} + \ldots \tag{A.59}$$

In our problem we have $\Delta = 3$ and $d = 4$, so that $2\Delta - d = 2$. In this way m_0 and σ_0 represent directly physical quantities, namely the mass deformation and the gaugino condensate $\sigma_0 = \langle\lambda\lambda\rangle$.

A.4 Coordinates on the S^5

For the internal manifold we think of the round sphere as embedded in $\mathbb{R}^6$ described by the coordinates $y_1, \ldots y_6$, so that on the sphere $\vec{y}^2 = 1$. The six coordinates can be thought of as split into two triplets

$$\vec{y} = (y_1, \ldots y_6) \to (u_1, u_2, u_3, v_1, v_2, v_3) \equiv (\vec{u}, \vec{v}). \tag{A.60}$$

This definition is in agreement with that used by Pilch and Warner [54]. By definition it is true that $\vec{u}^2 + \vec{v}^2 = 1$ and the diagonal SO(3) acts on u^i and v^j simultaneously in the vector representation. In [54] the authors show that $\vec{u}$ and $\vec{v}$ can be written as

$$\vec{u} = R.\begin{pmatrix}0\\0\\\cos\theta\end{pmatrix} \equiv R.\vec{u}_0 \qquad\qquad \vec{v} = R.\begin{pmatrix}0\\\sin\theta\sin\phi\\\sin\theta\cos\phi\end{pmatrix} \equiv R.\vec{v}_0 \tag{A.61}$$

with $\theta \in [0, \pi/2]$, $\phi \in [0, \pi]$ and $R = R(\alpha_1, \alpha_2, \alpha_3)$ a generic SO(3) matrix parametrised by three Euler angles α_i. In [54] the authors further define

$$w_1 = 2\,\vec{u}^2 - 1 = \cos(2\theta) \tag{A.62}$$

$$w_2 = 2\,\vec{u}.\vec{v} \qquad = \sin(2\theta)\cos\phi \tag{A.63}$$

so that the internal manifold is described by the coordinates $\{\alpha_1, \alpha_2, \alpha_3, w_1, w_2\}$. To write form-fields in terms of these coordinates it is useful to compute the differentials du^i and dv^i. In the coordinates just introduced this translates to

$$d\vec{u} = dR.\vec{u}_0 + R.d\vec{u}_0 = R.(R^{-1}dR.u_0 + d\vec{u}_0). \tag{A.64}$$

As explained in Appendix D the Maurer-Cartan form $R^{-1}dR$ can be decomposed into left-invariant 1-forms σ^i so that the differentials du^i can be written as

$$d\vec{u} = R(i\sigma^i T^i.u_0 + du_0), \tag{A.65}$$

and analogously for dv^i. Since in the quantities in which we are interested the SO(3) indices are always contracted the overall factor of R in the differentials drops out. With the fact that the SO(3) generators are given by $(T^j)^{ik} = i\epsilon^{ijk}$, in SO(3)-invariant expressions we can substitute

$$du^i \to \epsilon^{ijk} u_0^j \sigma^k + du_0^i \tag{A.66}$$

$$dv^i \to \epsilon^{ijk} v_0^j \sigma^k + dv_0^i. \tag{A.67}$$

PW define the following two useful variables which we also used in this thesis:

$$\zeta = 1 - w_1^2 - w_2^2 \tag{A.68}$$

$$\hat{\Omega} = \frac{1}{3}\sqrt{4(1 - w_1^2 - w_2^2) - 3w_1^2} = \frac{1}{3}\sqrt{4 - w_1^2 - 4w_2^2}. \tag{A.69}$$

More precisely, they will turn out to be the coefficients of the leading terms of the warp factor in the $t \to 1$ and $\lambda \to 1$, $t \to 1$ limits.

A.5 Supersymmetry

The $\mathfrak{so}(1,3)$ sigma matrices σ^μ and $\bar{\sigma}^\mu$ that appear in calculations involving four-dimensional Weyl fermions are defined in Appendix B. The two-dimensional epsilon tensors, which are introduced in Appendix A.1 are defined in such a way that the SU(2) doublet index is raised and lowered by contractions with the epsilon tensor from the left:

$$\psi^{\alpha} = \epsilon^{\alpha\beta}\psi_{\beta} \qquad \psi_{\alpha} = \epsilon_{\alpha\beta}\psi^{\beta} \qquad \psi^{\dot{\alpha}} = \epsilon^{\dot{\alpha}\dot{\beta}}\psi_{\dot{\beta}} \qquad \psi_{\dot{\alpha}} = \epsilon_{\dot{\alpha}\dot{\beta}}\psi^{\dot{\beta}}. \tag{A.70}$$

The convention for the contraction of two four-dimensional Weyl spinors follows the standard that is recognised in most of the literature:

$$\eta\chi \equiv \eta^{\alpha}\chi_{\alpha} \qquad\qquad \bar{\eta}\bar{\chi} \equiv \bar{\eta}_{\dot{\alpha}}\bar{\chi}^{\dot{\alpha}}\,. \tag{A.71}$$

The tensor product of two equal Weyl spinors is anti-symmetric in the spinor index, and is therefore proportional to the two-dimensional epsilon tensor. With the contraction definitions as above this leads to the following identities

$$\psi_{\alpha}\psi_{\beta} = \frac{1}{2}\epsilon_{\alpha\beta}\psi^2 \qquad\qquad \bar{\psi}^{\dot{\alpha}}\bar{\psi}^{\dot{\beta}} = \frac{1}{2}\epsilon^{\dot{\alpha}\dot{\beta}}\bar{\psi}^2 \tag{A.72}$$

$$\psi^{\alpha}\psi^{\beta} = -\frac{1}{2}\epsilon^{\alpha\beta}\psi^2 \qquad\qquad \bar{\psi}_{\dot{\alpha}}\bar{\psi}_{\dot{\beta}} = -\frac{1}{2}\epsilon_{\dot{\alpha}\dot{\beta}}\bar{\psi}^2. \tag{A.73}$$

It might be easier to remember where the minus sign is by raising or lowering one of the spinor indices. If the spinor indices end up in the same positions as for the contraction convention in equation (A.71) then the sign is positive, otherwise it is negative:

$$\psi^{\alpha}\psi_{\beta} = \frac{1}{2}\delta^{\alpha}_{\beta}\psi^2 \qquad\qquad \bar{\psi}_{\dot{\alpha}}\bar{\psi}^{\dot{\beta}} = \frac{1}{2}\delta^{\dot{\beta}}_{\dot{\alpha}}\bar{\psi}^2 \tag{A.74}$$

$$\psi_{\alpha}\psi^{\beta} = -\frac{1}{2}\delta^{\beta}_{\alpha}\psi^2 \qquad\qquad \bar{\psi}^{\dot{\alpha}}\bar{\psi}_{\dot{\beta}} = -\frac{1}{2}\delta^{\dot{\alpha}}_{\dot{\beta}}\bar{\psi}^2. \tag{A.75}$$

Using these identities we can derive another useful identity involving four Grassmann variables θ and $\bar{\theta}$ and two sigma-matrices:

$$(\theta\sigma^{\mu}\bar{\theta})(\theta\sigma^{\nu}\bar{\theta}) = \frac{1}{2}\theta^2\bar{\theta}^2\eta^{\mu\nu}\,. \tag{A.76}$$

In this thesis we used the definitions for the chiral derivatives and chiral coordinates that are equivalent to the definition in Wess and Bagger [66] and read

$$D_{\alpha} = \frac{\partial}{\partial\theta^{\alpha}} + i(\sigma^{\mu}\bar{\theta})_{\alpha}\partial_{\mu} \qquad\qquad \bar{D}_{\dot{\alpha}} = -\frac{\partial}{\partial\bar{\theta}^{\dot{\alpha}}} - i(\theta\sigma^{\mu})_{\dot{\alpha}}\partial_{\mu} \tag{A.77}$$

$$y^{\mu} = x^{\mu} + i\theta\sigma^{\mu}\bar{\theta} \qquad\qquad \bar{y}^{\mu} = x^{\mu} - i\theta\sigma^{\mu}\bar{\theta} = (y^{\mu})^{\dagger}. \tag{A.78}$$

Appendix B

Sigma and Gamma Matrices

B.1 SO(1, 3) Sigma Matrices

Our definition of the sigma matrices and their compounds differs from the standard reference [66] but is fairly common otherwise. Therefore we provide an extensive list of definitions and identities that we used throughout this thesis.

The Pauli matrices are defined as usual:

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} , \tag{B.1}$$

and we define the extended Pauli matrices by

$$\sigma^\mu_{\alpha\dot\alpha} = (\mathbf{1}, \sigma^i) \tag{B.2}$$

$$\bar\sigma^{\mu\,\dot\alpha\alpha} = \epsilon^{\dot\alpha\dot\beta}\epsilon^{\alpha\beta}\sigma^\mu_{\beta\dot\beta} = (\mathbf{1}, -\sigma^i) \tag{B.3}$$

As usual, the two-dimensional sigma-matrices can be used to define the four-dimensional gamma matrices. Note that there is a minus sign in the Clifford algebra since we chose the Minkowski metric to have a minus sign in the time component. Some authors rescale the gamma matrices by an imaginary unit $\gamma^\mu \to i\gamma^\mu$ in order to restore the canonical form of the anti-commutator, but we prefer to keep the minus sign:

$$\gamma^\mu = \begin{pmatrix} 0 & \sigma^\mu \\ \bar\sigma^\mu & 0 \end{pmatrix} \qquad \{\gamma^\mu, \gamma^\nu\} = -2\eta^{\mu\nu}\mathbf{1} . \tag{B.4}$$

Throughout this thesis we will work in the two-dimensional Weyl basis, therefore the properties of the four-dimensional gamma matrices are not really needed at this point. For the sake of completeness let us just remark that apart from the symmetric combination of two gamma matrices, which gives the anti-commutator of the Clifford algebra one can also form an anti-symmetric combination. With the correct pre-factor these matrices furnish a four-dimensional (Dirac) spinor representation for the Lorentz algebra:

$$\gamma^{\mu\nu} = \frac{i}{2}\gamma^{[\mu}, \gamma^{\nu]} = \frac{i}{2}\left(\gamma^{\mu}\gamma^{\nu} + \eta^{\mu\nu}\right) \tag{B.5}$$

$$[\gamma^{\mu\nu}, \gamma^{\rho\sigma}] = i(\eta^{\mu\rho}\gamma^{\nu\sigma} + \eta^{\nu\sigma}\gamma^{\mu\rho} - \eta^{\mu\sigma}\gamma^{\nu\rho} - \eta^{\nu\rho}\gamma^{\mu\sigma}). \tag{B.6}$$

B.1.1 Identities

This subsection is a collection of useful formulas involving the two-dimensional sigma matrices. We tried to cover most of the formulas found in [66], adapted their form to our conventions, and checked the validity of all formulas using computer software.

The symmetric product of two sigma-matrices is just the anti-commutator of the Clifford algebra:

$$\sigma^{(\mu}\bar{\sigma}^{\nu)} = -\eta^{\mu\nu}\mathbf{1} \tag{B.7}$$

$$\bar{\sigma}^{(\mu}\sigma^{\nu)} = -\eta^{\mu\nu}\mathbf{1}. \tag{B.8}$$

It is useful to compute various traces over the space-time and SU(2) indices:

$$\mathrm{tr}(\sigma^{\mu}\bar{\sigma}^{\nu}) = -2\eta^{\mu\nu} \tag{B.9}$$

$$\sigma^{\mu}_{\alpha\dot{\alpha}}\bar{\sigma}^{\dot{\beta}\beta}_{\mu} = -2\delta^{\beta}_{\alpha}\delta^{\dot{\beta}}_{\dot{\alpha}}. \tag{B.10}$$

An important definition are the anti-symmetric products of the sigma matrices:

$$\sigma^{\mu\nu}{}_{\alpha}{}^{\beta} = \frac{i}{2}(\sigma^{[\mu}\bar{\sigma}^{\nu]})_{\alpha}{}^{\beta} = \frac{i}{2}\sigma^{[\mu}_{\alpha\dot{\gamma}}\bar{\sigma}^{\nu]\dot{\gamma}\beta} \tag{B.11}$$

$$\bar{\sigma}^{\mu\nu\,\dot{\alpha}}{}_{\dot{\beta}} = \frac{i}{2}(\bar{\sigma}^{[\mu}\sigma^{\nu]})^{\dot{\alpha}}{}_{\dot{\beta}} = \frac{i}{2}\bar{\sigma}^{[\mu\,\dot{\alpha}\gamma}\sigma^{\nu]}_{\gamma\dot{\beta}}. \tag{B.12}$$

They are both traceless, self-dual and anti-self-dual, and each furnish a two-dimensional (Weyl) spinor representation of the Lorentz algebra

$$\mathrm{tr}(\sigma^{\mu\nu}) = \mathrm{tr}(\bar{\sigma}^{\mu\nu}) = 0 \tag{B.13}$$

$$\frac{1}{2}\epsilon_{\mu\nu\rho\sigma}\sigma^{\rho\sigma} = i\sigma_{\mu\nu} \tag{B.14}$$

$$\frac{1}{2}\epsilon_{\mu\nu\rho\sigma}\bar{\sigma}^{\rho\sigma} = -i\bar{\sigma}_{\mu\nu} \tag{B.15}$$

$$[\sigma^{\mu\nu}, \sigma^{\rho\sigma}] = i(\eta^{\mu\rho}\sigma^{\nu\sigma} + \eta^{\nu\sigma}\sigma^{\mu\rho} - \eta^{\mu\sigma}\sigma^{\nu\rho} - \eta^{\nu\rho}\sigma^{\mu\sigma}) \tag{B.16}$$

$$[\bar{\sigma}^{\mu\nu}, \bar{\sigma}^{\rho\sigma}] = i(\eta^{\mu\rho}\bar{\sigma}^{\nu\sigma} + \eta^{\nu\sigma}\bar{\sigma}^{\mu\rho} - \eta^{\mu\sigma}\bar{\sigma}^{\nu\rho} - \eta^{\nu\rho}\bar{\sigma}^{\mu\sigma}) \tag{B.17}$$

The tensors $\sigma^{\mu\nu}$ and $\bar{\sigma}^{\mu\nu}$ with both spinor indices raised or lowered are symmetric in these spinor indices

$$\begin{aligned} \sigma^{\mu\nu}{}_{\alpha\beta} &= \sigma^{\mu\nu}{}_{(\alpha\beta)} & \qquad \bar{\sigma}^{\mu\nu}{}_{\dot{\alpha}\dot{\beta}} &= \bar{\sigma}^{\mu\nu}{}_{(\dot{\alpha}\dot{\beta})} \\ \sigma^{\mu\nu\,\alpha\beta} &= \sigma^{\mu\nu\,(\alpha\beta)} & \qquad \bar{\sigma}^{\mu\nu\,\dot{\alpha}\dot{\beta}} &= \bar{\sigma}^{\mu\nu\,(\dot{\alpha}\dot{\beta})}. \end{aligned} \tag{B.18}$$

By combining the equations (B.7) and (B.8) for the symmetric products with the definitions for the anti-symmetric products in (B.11) and (B.12) one arrives at formulas that allow to expand products of two sigma matrices:

$$\sigma^{\mu}\bar{\sigma}^{\nu} = -\eta^{\mu\nu}\mathbf{1} - 2i\sigma^{\mu\nu} \tag{B.19}$$

$$\bar{\sigma}^{\mu}\sigma^{\nu} = -\eta^{\mu\nu}\mathbf{1} - 2i\sigma^{\mu\nu}. \tag{B.20}$$

The following block of equations covers all possible expansions for the tensor products of two sigma matrices, i.e. the cases where no SU(2) indices are contracted.

$$\sigma^{[\mu}_{\alpha\dot{\alpha}}\sigma^{\nu]}_{\beta\dot{\beta}} = i\,\sigma^{\mu\nu}_{\alpha\beta}\epsilon_{\dot{\alpha}\dot{\beta}} - i\,\bar{\sigma}^{\mu\nu}_{\dot{\alpha}\dot{\beta}}\epsilon_{\alpha\beta} \tag{B.21}$$

$$\sigma^{(\mu}_{\alpha\dot{\alpha}}\sigma^{\nu)}_{\beta\dot{\beta}} = -\frac{1}{2}\eta^{\mu\nu}\epsilon_{\alpha\beta}\epsilon_{\dot{\alpha}\dot{\beta}} + 2\eta_{\rho\sigma}\sigma^{\mu\rho}_{\alpha\beta}\bar{\sigma}^{\nu\sigma}_{\dot{\alpha}\dot{\beta}} \tag{B.22}$$

$$\sigma^{\mu}_{\alpha\dot{\alpha}}\sigma^{\nu}_{\beta\dot{\beta}} = -\frac{1}{2}\eta^{\mu\nu}\epsilon_{\alpha\beta}\epsilon_{\dot{\alpha}\dot{\beta}} + 2\eta_{\rho\sigma}\sigma^{\mu\rho}_{\alpha\beta}\bar{\sigma}^{\nu\sigma}_{\dot{\alpha}\dot{\beta}} + i\,\sigma^{\mu\nu}_{\alpha\beta}\epsilon_{\dot{\alpha}\dot{\beta}} - i\,\bar{\sigma}^{\mu\nu}_{\dot{\alpha}\dot{\beta}}\epsilon_{\alpha\beta} \tag{B.23}$$

$$\bar{\sigma}^{\dot{\alpha}\alpha}_{[\mu}\bar{\sigma}^{\dot{\beta}\beta}_{\nu]} = -i\,\sigma^{\alpha\beta}_{\mu\nu}\epsilon^{\dot{\alpha}\dot{\beta}} + i\,\bar{\sigma}^{\dot{\alpha}\dot{\beta}}_{\mu\nu}\epsilon^{\alpha\beta} \tag{B.24}$$

$$\bar{\sigma}^{\dot{\alpha}\alpha}_{(\mu}\bar{\sigma}^{\dot{\beta}\beta}_{\nu)} = -\frac{1}{2}\eta_{\mu\nu}\epsilon^{\dot{\alpha}\dot{\beta}}\epsilon^{\alpha\beta} + 2\eta^{\rho\sigma}\sigma^{\alpha\beta}_{\mu\rho}\bar{\sigma}^{\dot{\alpha}\dot{\beta}}_{\nu\sigma} \tag{B.25}$$

$$\bar{\sigma}^{\dot{\alpha}\alpha}_{\mu}\bar{\sigma}^{\dot{\beta}\beta}_{\nu} = -\frac{1}{2}\eta_{\mu\nu}\epsilon^{\dot{\alpha}\dot{\beta}}\epsilon^{\alpha\beta} + 2\eta^{\rho\sigma}\sigma^{\alpha\beta}_{\mu\rho}\bar{\sigma}^{\dot{\alpha}\dot{\beta}}_{\nu\sigma} - i\,\sigma^{\alpha\beta}_{\mu\nu}\epsilon^{\dot{\alpha}\dot{\beta}} + i\,\bar{\sigma}^{\dot{\alpha}\dot{\beta}}_{\mu\nu}\epsilon^{\alpha\beta} \tag{B.26}$$

$$\sigma^{[\mu}_{\alpha\dot{\alpha}}\bar{\sigma}^{\nu]\,\dot{\beta}\beta} = -i\,\delta^{\dot{\beta}}_{\dot{\alpha}}\sigma^{\mu\nu}{}_{\alpha}{}^{\beta} + i\,\delta^{\beta}_{\alpha}\bar{\sigma}^{\mu\nu\,\dot{\beta}}{}_{\dot{\alpha}} \tag{B.27}$$

$$\sigma^{(\mu}_{\alpha\dot{\alpha}}\bar{\sigma}^{\nu)\,\dot{\beta}\beta} = -\frac{1}{2}\eta^{\mu\nu}\delta^{\beta}_{\alpha}\delta^{\dot{\beta}}_{\dot{\alpha}} + 2\eta_{\rho\sigma}\sigma^{\mu\nu}{}_{\alpha}{}^{\beta}\bar{\sigma}^{\mu\nu\,\dot{\beta}}{}_{\dot{\alpha}} \tag{B.28}$$

$$\sigma^{\mu}_{\alpha\dot{\alpha}}\bar{\sigma}^{\nu\,\dot{\beta}\beta} = -\frac{1}{2}\eta^{\mu\nu}\delta^{\beta}_{\alpha}\delta^{\dot{\beta}}_{\dot{\alpha}} + 2\eta_{\rho\sigma}\sigma^{\mu\nu}{}_{\alpha}{}^{\beta}\bar{\sigma}^{\mu\nu\,\dot{\beta}}{}_{\dot{\alpha}} - i\,\delta^{\dot{\beta}}_{\dot{\alpha}}\sigma^{\mu\nu}{}_{\alpha}{}^{\beta} + i\,\delta^{\beta}_{\alpha}\bar{\sigma}^{\mu\nu\,\dot{\beta}}{}_{\dot{\alpha}} \tag{B.29}$$

Next we list various products of three sigma matrices.

$$\frac{1}{2}(\sigma^\mu\bar{\sigma}^\nu\sigma^\rho - \sigma^\rho\bar{\sigma}^\nu\sigma^\mu) = -i\epsilon^{\mu\nu\rho\sigma}\sigma_\sigma \quad \text{(B.30)}$$
$$\frac{1}{2}(\bar{\sigma}^\mu\sigma^\nu\bar{\sigma}^\rho - \bar{\sigma}^\rho\sigma^\nu\bar{\sigma}^\mu) = i\epsilon^{\mu\nu\rho\sigma}\bar{\sigma}_\sigma \quad \text{(B.31)}$$

$$\frac{1}{2}(\sigma^\mu\bar{\sigma}^\nu\sigma^\rho + \sigma^\rho\bar{\sigma}^\nu\sigma^\mu) = \eta^{\mu\rho}\sigma^\nu - \eta^{\mu\nu}\sigma^\rho - \eta^{\nu\rho}\sigma^\mu \quad \text{(B.32)}$$
$$\frac{1}{2}(\bar{\sigma}^\mu\sigma^\nu\bar{\sigma}^\rho + \bar{\sigma}^\rho\sigma^\nu\bar{\sigma}^\mu) = \eta^{\mu\rho}\bar{\sigma}^\nu - \eta^\mu\nu\bar{\sigma}^\rho - \eta^{\nu\rho}\bar{\sigma}^\mu \quad \text{(B.33)}$$

After adding the symmetric and anti-symmetric cases we get expressions for general products of three sigma matrices:

$$\sigma^\mu\bar{\sigma}^\nu\sigma^\rho = \eta^{\mu\rho}\sigma^\nu - \eta^{\mu\nu}\sigma^\rho - \eta^{\nu\rho}\sigma^\mu - i\epsilon^{\mu\nu\rho\sigma}\sigma_\sigma \quad \text{(B.34)}$$
$$\bar{\sigma}^\mu\sigma^\nu\bar{\sigma}^\rho = \eta^{\mu\rho}\bar{\sigma}^\nu - \eta^{\mu\nu}\bar{\sigma}^\rho - \eta^{\nu\rho}\bar{\sigma}^\mu + i\epsilon^{\mu\nu\rho\sigma}\bar{\sigma}_\sigma. \quad \text{(B.35)}$$

We can anti-symmetrise the last two indices in equations (B.34) and (B.35) to derive products of σ^μ and $\bar{\sigma}^\mu$ with $\sigma^{\nu\rho}$ and $\bar{\sigma}^{\nu\rho}$:

$$\sigma^\mu\bar{\sigma}^{\nu\rho} = -i\eta^{\mu[\nu}\sigma^{\rho]} + \frac{1}{2}\epsilon^{\mu\nu\rho\sigma}\sigma_\sigma \quad \text{(B.36)}$$
$$\bar{\sigma}^\mu\sigma^{\nu\rho} = -i\eta^{\mu[\nu}\bar{\sigma}^{\rho]} - \frac{1}{2}\epsilon^{\mu\nu\rho\sigma}\bar{\sigma}_\sigma. \quad \text{(B.37)}$$

Finally, using (B.34) and (B.35) again and the trace identity (B.9) on obtains the trace of a product of four sigma matrices:

$$\frac{1}{2}\,\mathrm{tr}(\sigma^\mu\bar{\sigma}^\nu\sigma^\rho\bar{\sigma}^\sigma) = \eta^{\mu\nu}\eta^{\rho\sigma} - \eta^{\mu\rho}\eta^{\nu\sigma} + \eta^{\mu\sigma}\eta^{\nu\rho} + i\epsilon^{\mu\nu\rho\sigma} \quad \text{(B.38)}$$
$$\frac{1}{2}\,\mathrm{tr}(\bar{\sigma}^\mu\sigma^\nu\bar{\sigma}^\rho\sigma^\sigma) = \eta^{\mu\nu}\eta^{\rho\sigma} - \eta^{\mu\rho}\eta^{\nu\sigma} + \eta^{\mu\sigma}\eta^{\nu\rho} - i\epsilon^{\mu\nu\rho\sigma}. \quad \text{(B.39)}$$

After symmetrising the space-time indices pairwise and scaling by appropriate factors one obtains yet another set of useful identities

$$(\sigma^{\mu\nu})^{\alpha\beta}\,(\sigma^{\rho\sigma})_{\alpha\beta} = -\eta^{\mu[\rho}\eta^{\sigma]\nu} + \frac{i}{2}\epsilon^{\mu\nu\rho\sigma} \quad \text{(B.40)}$$
$$(\bar{\sigma}^{\mu\nu})^{\dot{\alpha}\dot{\beta}}\,(\bar{\sigma}^{\rho\sigma})_{\dot{\alpha}\dot{\beta}} = -\eta^{\mu[\rho}\eta^{\sigma]\nu} - \frac{i}{2}\epsilon^{\mu\nu\rho\sigma}. \quad \text{(B.41)}$$

B.1.2 Self-Duality Projector

In this section we will show that the matrices ${\sigma^{\mu\nu}}_\alpha{}^\beta$ and ${\bar{\sigma}^{\mu\nu\,\dot{\alpha}}}_{\dot{\beta}}$ defined in equations (B.11) and (B.12) not only furnish the spinor representations $\mathbf{4}$ and $\bar{\mathbf{4}}$ of the SO(1,3), as was shown in equations (B.16) and (B.17), but can also be used as projectors onto the self-dual and anti-self-dual parts of anti-symmetric SO(1,3) two-tensors, such as the field strength $F_{\mu\nu}$.

Given an anti-symmetric two-tensor like field strength tensor $F_{\mu\nu}$ its Hodge dual is defined as

$$\tilde{F}_{\mu\nu} = \frac{1}{2}\epsilon_{\mu\nu\rho\sigma}F^{\rho\sigma}. \tag{B.42}$$

Because of the minus sign in the inverse relation $\frac{1}{2}\epsilon_{\mu\nu\rho\sigma}\tilde{F}^{\rho\sigma} = -F_{\mu\nu}$ we can split the field strength into its self-dual and anti-self-dual parts by writing

$$F_{\mu\nu} = \frac{1}{2}(F_{\mu\nu} - i\tilde{F}_{\mu\nu}) + \frac{1}{2}(F_{\mu\nu} + i\tilde{F}_{\mu\nu}) = F^+_{\mu\nu} + F^-_{\mu\nu}. \tag{B.43}$$

The self-duality and anti-self-duality are checked by taking the Hodge dual

$$(\star F^+)_{\mu\nu} = \frac{1}{2}\epsilon_{\mu\nu\rho\sigma}F^{+\,\mu\nu} = iF^+_{\mu\nu} \tag{B.44}$$

$$(\star F^-)_{\mu\nu} = \frac{1}{2}\epsilon_{\mu\nu\rho\sigma}F^{-\,\mu\nu} = -iF^-_{\mu\nu}. \tag{B.45}$$

We will now show that that ${\sigma^{\mu\nu}}_{\alpha\beta}$ and ${\bar{\sigma}^{\mu\nu}}_{\dot{\alpha}\dot{\beta}}$ are exactly the projectors on the self-dual part $F^+_{\mu\nu}$ and anti-self-dual part $F^-_{\mu\nu}$ of the field strength tensor. The key insight needed for the proof are the relations (B.14) and (B.15). They are used in a straightforward way to show that

$${\sigma^{\mu\nu}}_{\alpha\beta}F^-_{\mu\nu} = {\bar{\sigma}^{\mu\nu}}_{\dot{\alpha}\dot{\beta}}F^+_{\mu\nu} = 0. \tag{B.46}$$

Thus given the decomposition $F_{\mu\nu} = F^+_{\mu\nu} + F^-_{\mu\nu}$ we see that

$${\sigma^{\mu\nu}}_{\alpha\beta}F_{\mu\nu} = {\sigma^{\mu\nu}}_{\alpha\beta}F^+_{\mu\nu} \equiv F^+_{\alpha\beta} \tag{B.47}$$

$${\bar{\sigma}^{\mu\nu}}_{\dot{\alpha}\dot{\beta}} = {\bar{\sigma}^{\mu\nu}}_{\dot{\alpha}\dot{\beta}}F^-_{\mu\nu} \equiv F^-_{\alpha\beta}. \tag{B.48}$$

Finally, we can use the relation in (B.21) to obtain the splitting into F^+ and F^- in the same equation:

$$\sigma^{[\mu}_{\alpha\dot{\alpha}}\sigma^{\nu]}_{\beta\dot{\beta}}F_{\mu\nu} = i\,F^+_{\alpha\beta}\epsilon_{\dot{\alpha}\dot{\beta}} - i\,F^-_{\dot{\alpha}\dot{\beta}}\epsilon_{\alpha\beta} \tag{B.49}$$

From the group theory perspective this is the manifestation of the fact that under the identification of the (complexified) algebra of the SO(1,3) with that of the SU(2) × SU(2), the $\mathbf{6}$ of the SO(1,3) becomes the $(\mathbf{3},\mathbf{1}) + (\mathbf{1},\mathbf{3})$ of the SU(2) × SU(2):

$$\mathbf{6} \to (\mathbf{3},\mathbf{1}) + (\mathbf{1},\mathbf{3}). \tag{B.50}$$

One can easily convince oneself by a simple counting that the field strength $F_{[\mu\nu]}$ carries 6 independent components, and is therefore the **6** of the SO(1, 3). As we saw in equations (B.18) the matrices ${\sigma^{\mu\nu}}_{\alpha\beta}$ and ${\bar{\sigma}^{\mu\nu}}_{\dot{\alpha}\dot{\beta}}$ are both symmetric in their SU(2) indices and so corresponds to the **3** of the left and right SU(2) factors in the SU(2) × SU(2). Finally the anti-symmetric tensors $\epsilon_{\alpha\beta}$ and $\epsilon_{\dot{\alpha}\dot{\beta}}$ are the singlets **1**. To summarise we have

$$F_{\mu\nu} = \mathbf{6} \text{ of the SO}(1,3) \tag{B.51}$$

$$F^{+}_{\alpha\beta}\epsilon_{\dot{\alpha}\dot{\beta}} = (\mathbf{3},\mathbf{1}) \text{ of the SU}(2)\times\text{SU}(2) \tag{B.52}$$

$$\epsilon_{\alpha\beta}F^{-}_{\dot{\alpha}\dot{\beta}} = (\mathbf{1},\mathbf{3}) \text{ of the SU}(2)\times\text{SU}(2). \tag{B.53}$$

B.2 SO(6) Sigma Matrices

The six-dimensional sigma-matrices for the SO(6) can be introduced in a complete analogy to the SO(1, 3) case, and we define them following [72] in terms of the 't Hooft symbols η^i_{AB} and $\bar{\eta}^i_{AB}$. Note that the SU(4) epsilon tensor ϵ^{ABCD} used in this subsection has nothing to do with the SO(1, 3) group and therefore its components are given by $\epsilon^{1234} = \epsilon_{1234} = 1$.

$$\Sigma^{I,AB} = \{\eta^1_{AB}, \eta^2_{AB}, \eta^3_{AB}, i\bar{\eta}^1_{AB}, i\bar{\eta}^2_{AB}, i\bar{\eta}^3_{AB}\} \tag{B.54}$$

$$\overline{\Sigma}^I_{AB} = \{\eta^1_{AB}, \eta^2_{AB}, \eta^3_{AB}, -i\bar{\eta}^1_{AB}, -i\bar{\eta}^2_{AB}, -i\bar{\eta}^3_{AB}\} \tag{B.55}$$

$$\eta^i_{AB} = \epsilon_{iAB4} + \delta_{iA}\delta_{B4} - \delta_{iB}\delta_{A4} \tag{B.56}$$

$$\bar{\eta}^i_{AB} = \epsilon_{iAB4} - \delta_{iA}\delta_{B4} + \delta_{iB}\delta_{A4}. \tag{B.57}$$

By their definition the matrices $\Sigma^{I,AB}$ and $\overline{\Sigma}^I_{AB}$ are anti-symmetric in their spinor indices:

$$\Sigma^{I,AB} = -\Sigma^{I,BA} \qquad\qquad \overline{\Sigma}^I_{AB} = -\overline{\Sigma}^I_{BA}. \tag{B.58}$$

The symbols $\Sigma^{I,AB}$ are also equivalent to their four-dimensional counterparts $\sigma^\mu_{\alpha\dot{\alpha}}$ in the sense that they can be interpreted as Clebsch-Gordan coefficients for tensor products of two spinor representations. There are two spinor representations in the SO(6), the **4** and the $\bar{\mathbf{4}}$, and the sigma matrices project onto the real **6**, which is the anti-symmetric part of the tensor products of two spinors of the same chirality. This is part of a more general statement that any representation in a tensor product of any two spinors can be projected out by an anti-symmetric product of a number of sigma matrices. The same is true for Dirac fermions with sigma matrices replaces by the gamma matrices. For SO(6) the tensor products of spinors are

$$\mathbf{4}\otimes\mathbf{4} = \mathbf{6} + \mathbf{10} \tag{B.59}$$

$$\bar{\mathbf{4}}\otimes\bar{\mathbf{4}} = \mathbf{6} + \overline{\mathbf{10}} \tag{B.60}$$

$$\mathbf{4} \otimes \bar{\mathbf{4}} = \mathbf{1} + \mathbf{15}. \tag{B.61}$$

To cover all representations in the tensor product of two SO($2N$) spinors one needs to take anti-symmetric products of $0, 1, 2, \ldots, N$ sigma matrices. Anti-symmetric products of more than N sigma matrices can be reduced to products of fewer than N sigma matrices by contractions with the $2N$-dimensional ϵ-tensor of the SO($2N$). Thus for SO(6) we need to take anti-symmetric products of up to three sigma matrices:

$$\mathbf{1}_4 = \mathbf{1} \tag{B.62}$$
$$\Sigma^I = \overline{\Sigma}^I = \mathbf{6} \tag{B.63}$$
$$\Sigma^{[I}\overline{\Sigma}^{J]} = \overline{\Sigma}^{[I}\Sigma^{J]} = \mathbf{15} \tag{B.64}$$
$$\Sigma^{[I}\overline{\Sigma}^{J}\Sigma^{K]} = \overline{\mathbf{10}} \tag{B.65}$$
$$\overline{\Sigma}^{[I}\Sigma^{J}\overline{\Sigma}^{K]} = \mathbf{10}. \tag{B.66}$$

Moreover, we recall that the $\mathfrak{so}(6)$ and the $\mathfrak{su}(4)$ algebras are isomorphic to each other. Under this identification the spinor representation **4** of the $\mathfrak{so}(6)$ becomes the fundamental representation of the $\mathfrak{su}(4)$, and the sigma matrices Σ^I and $\overline{\Sigma}^I$ can be viewed as a way to represent the real **6** of the $\mathfrak{su}(4)$ by an anti-symmetric pair of fundamental indices.

The identities that hold for $\Sigma^{I,AB}$ and $\overline{\Sigma}^I_{AB}$ are structurally very similar to those for their four-dimensional counterparts $\sigma^\mu_{\alpha\dot{\alpha}}$ and $\bar{\sigma}^{\mu,\dot{\alpha}\alpha}$, however sometimes they differ slightly, and sometimes one finds new relations due to the altered index structure. Below we provide an extensive list of identities and properties of the six-dimensional sigma matrices, all of which have been checked using a computer algebra system.

First of all we note that $\Sigma^{I,AB}$ and $\overline{\Sigma}^I_{AB}$ are related to each other both by complex conjugation and the contraction with the ϵ-tensor. These properties combined lead to the reality conditions in equation (B.71) and (B.72) which ensure that the sigma matrices in fact carry six real degrees of freedom, and not six complex ones. This is in agreement with the fact that the **6** of the $\mathfrak{su}(4)$ is a real representation.

$$\left(\Sigma^{I,AB}\right)^* = \overline{\Sigma}^I_{AB} \tag{B.67}$$
$$\left(\overline{\Sigma}^I_{AB}\right)^* = \Sigma^{I,AB} \tag{B.68}$$

$$\tfrac{1}{2}\epsilon_{ABCD}\Sigma^{I,CD} = \overline{\Sigma}^I_{AB} \tag{B.69}$$
$$\tfrac{1}{2}\epsilon^{ABCD}\overline{\Sigma}^I_{CD} = \Sigma^{I,AB} \tag{B.70}$$

$$(\Sigma^{I,AB})^* = \frac{1}{2}\epsilon_{ABCD}\overline{\Sigma}^I_{CD} \tag{B.71}$$
$$(\overline{\Sigma}^I_{AB})^* = \frac{1}{2}\epsilon^{ABCD}\overline{\Sigma}^I_{CD} \tag{B.72}$$

Various contraction of the vector and spinor indices lead to trace-like relations similar to the four-dimensional case

$$\Sigma^{I,AB}\overline{\Sigma}^J_{AB} = 4\delta^{IJ} \tag{B.73}$$
$$\Sigma^{I,AB}\overline{\Sigma}^I_{CD} = 4\delta^{[A}_C\delta^{B]}_D \tag{B.74}$$
$$\Sigma^{I,AB}\Sigma^{I,CD} = 2\epsilon^{ABCD} \tag{B.75}$$
$$\overline{\Sigma}^I_{AB}\overline{\Sigma}^I_{CD} = 2\epsilon_{ABCD}. \tag{B.76}$$

Also similarly to the four-dimensions the symmetric product of two sigma matrices is proportional to the identity

$$\Sigma^{(I,AC}\overline{\Sigma}^{J)}_{CB} = -\delta^{IJ}\delta^A_B \tag{B.77}$$
$$\overline{\Sigma}^{(I}_{AC}\Sigma^{J),CB} = -\delta^{IJ}\delta^B_A \tag{B.78}$$

Again, while the symmetrisation of six-dimensional indices of Σ^I and $\overline{\Sigma}^I$ can be reduced, as we just saw in equation (B.77), the anti-symmetrisation cannot. We therefore define symbols Σ^{IJ} and $\overline{\Sigma}^{IJ}$ as follows:

$$\Sigma^{IJ\,A}{}_B = \frac{i}{2}\Sigma^{[I,AC}\overline{\Sigma}^{J]}_{CB} \tag{B.79}$$
$$\overline{\Sigma}^{IJ}{}_A{}^B = \frac{i}{2}\overline{\Sigma}^{[I}_{AC}\Sigma^{J]CB}. \tag{B.80}$$

They have the property of being traceless and can be transformed into the negatives of each other by swapping the spinor indices. Additionally one also finds an interesting trace property:

$$(\Sigma^{IJ})^A{}_A = 0 = (\overline{\Sigma}^{IJ})_A{}^A \tag{B.81}$$
$$(\overline{\Sigma}^{IJ})_A{}^B = -(\Sigma^{IJ})^B{}_A \tag{B.82}$$
$$\mathrm{tr}\left(\Sigma^{IJ}\Sigma^{KL}\right) = (\Sigma^{IJ})^A{}_B(\Sigma^{KL})^B{}_A = \delta^{IK}\delta^{JL} - \delta^{IL}\delta^{JK}. \tag{B.83}$$

Finally, as should be expected, the symbols Σ^{IJ} furnish the spinor representation **4** of the SO(6), this is in complete analogy to the sigma matrices $\sigma^{\mu\nu}$ in the case of the Lorentz group. Since it holds that $\overline{\Sigma}^{IJ} = -(\Sigma^{IJ})^*$, the symbols $\overline{\Sigma}^{IJ}$ correspond to the conjugate spinor representation $\bar{\mathbf{4}}$. One can check that Σ^{IJ} and $\overline{\Sigma}^{IJ}$ indeed represent the algebra by computing their commutators:

$$[\Sigma^{IJ},\Sigma^{KL}] = i\delta^{IK}\Sigma^{JL} + i\delta^{JL}\Sigma^{IK} - i\delta^{IL}\Sigma^{JK} - i\delta^{JK}\Sigma^{IL} \tag{B.84}$$

$$[\overline{\Sigma}^{IJ},\overline{\Sigma}^{KL}] = i\delta^{IK}\overline{\Sigma}^{JL} + i\delta^{JL}\overline{\Sigma}^{IK} - i\delta^{IL}\overline{\Sigma}^{JK} - i\delta^{JK}\overline{\Sigma}^{IL}\,. \tag{B.85}$$

We can construct new sigma matrices by contractions with the ϵ-tensor followed by an anti-symmetrisation. Unlike in four dimensions, the ϵ-tensor that we need to use has four indices so that we not just raise or lower an index but end up with four of them:

$$(\Sigma^{IJ})^{ABCD} = (\Sigma^{IJ})^{[ABCD]} = \tfrac{1}{2}\left\{(\Sigma^{IJ})^{A}{}_{E}\epsilon^{EBCD} - (\Sigma^{IJ})^{B}{}_{E}\epsilon^{EACD}\right\} \tag{B.86}$$

$$(\overline{\Sigma}^{IJ})_{ABCD} = (\overline{\Sigma}^{IJ})_{[ABCD]} = \tfrac{1}{2}\left\{(\overline{\Sigma}^{IJ})_{A}{}^{E}\epsilon_{EBCD} - (\overline{\Sigma}^{IJ})_{B}{}^{E}\epsilon_{EACD}\right\}. \tag{B.87}$$

As before it is useful to expand various products of pairs of $\Sigma^{I\,AB}$ and $\overline{\Sigma}^{I}_{AB}$ in which the vector indices are either symmetrised or anti-symmetrised. By adding the symmetrised and anti-symmetrised identities one also obtains the tensor products of two sigma-matrices without any symmetrisation.

$$\Sigma^{[I\,AB}\Sigma^{J]\,CD} = 2i(\Sigma^{IJ})^{ABCD} \tag{B.88}$$

$$\overline{\Sigma}^{[I}_{AB}\overline{\Sigma}^{J]}_{CD} = 2i(\overline{\Sigma}^{IJ})_{ABCD} \tag{B.89}$$

$$\Sigma^{[I,AB}\overline{\Sigma}^{J]}_{CD} = 4i(\Sigma^{IJ})^{[A}{}_{[C}\delta^{B]}{}_{D]} \tag{B.90}$$

$$\Sigma^{(I\,AB}\Sigma^{J)\,CD} = \frac{3}{4}\delta^{IJ}\epsilon^{ABCD} + (\Sigma^{IK})^{[A}{}_{E}(\Sigma^{JK})^{B]}{}_{F}\epsilon^{EFCD} \tag{B.91}$$

$$\overline{\Sigma}^{(I}_{AB}\overline{\Sigma}^{J)}_{CD} = \frac{3}{4}\delta^{IJ}\epsilon_{ABCD} + (\overline{\Sigma}^{IK})_{[A}{}^{E}(\overline{\Sigma}^{JK})_{B]}{}^{F}\epsilon_{EFCD} \tag{B.92}$$

$$\Sigma^{(I\,AB}\overline{\Sigma}^{J)}_{CD} = \frac{3}{2}\delta^{IJ}\delta^{[A}_{C}\delta^{B]}_{D} + 2(\Sigma^{IK})^{[A}{}_{[C}(\Sigma^{JK})^{B]}{}_{D]} \tag{B.93}$$

$$\Sigma^{I\,AB}\Sigma^{J\,CD} = \frac{3}{4}\delta^{IJ}\epsilon^{ABCD} + 2i(\Sigma^{IJ})^{ABCD} + (\Sigma^{IK})^{[A}{}_{E}(\Sigma^{JK})^{B]}{}_{F}\epsilon^{EFCD} \tag{B.94}$$

$$\overline{\Sigma}^{I}_{AB}\overline{\Sigma}^{J}_{CD} = \frac{3}{4}\delta^{IJ}\epsilon_{ABCD} + 2i(\overline{\Sigma}^{IJ})_{ABCD} + (\overline{\Sigma}^{IK})_{[A}{}^{E}(\overline{\Sigma}^{JK})_{B]}{}^{F}\epsilon_{EFCD} \tag{B.95}$$

$$\Sigma^{I\,AB}\overline{\Sigma}^{J}_{CD} = \frac{3}{2}\delta^{IJ}\delta^{[A}_{C}\delta^{B]}_{D} + 4i(\Sigma^{IJ})^{[A}{}_{[C}\delta^{B]}{}_{D]} + 2(\Sigma^{IK})^{[A}{}_{[C}(\Sigma^{JK})^{B]}{}_{D]} \tag{B.96}$$

All these identities involving two sigma-matrices are various manifestations of the group theoretical fact that in SO(6) the product of the fundamental representations decomposes as $\mathbf{6} \otimes \mathbf{6} = \mathbf{1} + \mathbf{15} + \mathbf{20}'$. We can contract two of the spinor indices to obtain another useful identity

$$\Sigma^{I\,AC}\overline{\Sigma}^{J}_{AB} = \delta^{IJ}\delta^{C}_{B} + 2i(\Sigma^{IJ})^{C}{}_{B}. \tag{B.97}$$

After adding a third sigma matrix and contracting various indices one arrives at some useful properties for products of three sigma matrices

$$\frac{1}{2}\left(\Sigma^{I}\overline{\Sigma}^{J}\Sigma^{K} + \Sigma^{K}\overline{\Sigma}^{J}\Sigma^{I}\right) = \delta^{IK}\Sigma^{J} - \delta^{IJ}\Sigma^{K} - \delta^{JK}\Sigma^{I} \tag{B.98}$$
$$\frac{1}{2}\left(\overline{\Sigma}^{I}\Sigma^{J}\overline{\Sigma}^{K} + \overline{\Sigma}^{K}\Sigma^{J}\overline{\Sigma}^{I}\right) = \delta^{IK}\overline{\Sigma}^{J} - \delta^{IJ}\overline{\Sigma}^{K} - \delta^{JK}\overline{\Sigma}^{I} \tag{B.99}$$

$$\frac{1}{2}\left(\Sigma^{I}\overline{\Sigma}^{J}\Sigma^{K} - \Sigma^{K}\overline{\Sigma}^{J}\Sigma^{I}\right) = \Sigma^{[I}\overline{\Sigma}^{J}\Sigma^{K]} = -\frac{i}{3!}\epsilon^{IJKLMN}\Sigma^{L}\overline{\Sigma}^{M}\Sigma^{N} \tag{B.100}$$
$$\frac{1}{2}\left(\overline{\Sigma}^{I}\Sigma^{J}\overline{\Sigma}^{K} - \overline{\Sigma}^{K}\Sigma^{J}\overline{\Sigma}^{I}\right) = \overline{\Sigma}^{[I}\Sigma^{J}\overline{\Sigma}^{K]} = \frac{i}{3!}\epsilon^{IJKLMN}\overline{\Sigma}^{L}\Sigma^{M}\overline{\Sigma}^{N} \tag{B.101}$$

$$\Sigma^{I}\overline{\Sigma}^{J}\Sigma^{K} = \delta^{IK}\Sigma^{J} - \delta^{IJ}\Sigma^{K} - \delta^{JK}\Sigma^{I} + \Sigma^{[I}\overline{\Sigma}^{J}\Sigma^{K]} \tag{B.102}$$
$$\overline{\Sigma}^{I}\Sigma^{J}\overline{\Sigma}^{K} = \delta^{IK}\overline{\Sigma}^{J} - \delta^{IJ}\overline{\Sigma}^{K} - \delta^{JK}\overline{\Sigma}^{I} + \overline{\Sigma}^{[I}\Sigma^{J}\overline{\Sigma}^{K]}. \tag{B.103}$$

It is interesting that in equations (B.100) and (B.101) the anti-symmetrisation of only two indices automatically anti-symmetrises all three of them. The second equality in the same set of equations is due to the fact that for all SO($2N$) groups an anti-symmetric N-tensor is in fact reducible and decomposes into its self-dual and anti-self-dual parts. For SO(6) this corresponds to the reducible $\binom{6}{3} = 20$ which decomposes into $\mathbf{10} + \overline{\mathbf{10}}$. Thus the right-hand side of one of the equations (B.100) and (B.101) is the $\mathbf{10}$ and the other one is the $\overline{\mathbf{10}}$. The equations (B.102) and (B.103) can also be viewed from the group theory perspective and correspond to the decomposition $\mathbf{6} \otimes \mathbf{6} \otimes \mathbf{6} = 3 \times \mathbf{6} + (\mathbf{10} + \overline{\mathbf{10}}) + \mathbf{50} + 2 \times \mathbf{64}$. The first three terms on the right-hand side of (B.102) and (B.103) correspond to the $3 \times \mathbf{6}$ while the remaining term is the $\mathbf{10}$ in one of the equations and the $\overline{\mathbf{10}}$ in the other one. We see that there are no terms corresponding to the $\mathbf{50}$ and $2 \times \mathbf{64}$. The reason is the following. Were the spinor indices on the three sigma matrices on the left-hand side not contracted, then we would have expected that all representations of the tensor product decomposition should appear on the right-hand side. But since we did contract the spinor indices to get the matrix multiplication on the left-hand side, the same index contraction on the right-hand side made all entries in the matrices corresponding to the $\mathbf{50}$ and the $\mathbf{64}$ vanish, which is why they disappeared from the decomposition.

By adding yet another sigma-matrix and taking the traces we obtain

$$\mathrm{tr}\left(\Sigma^I\overline{\Sigma}^J\Sigma^K\overline{\Sigma}^L\right) = 4(\delta^{IJ}\delta^{KL} - \delta^{IK}\delta^{JL} + \delta^{IL}\delta^{JK}) \tag{B.104}$$

$$\mathrm{tr}\left(\overline{\Sigma}^I\Sigma^J\overline{\Sigma}^K\Sigma^L\right) = 4(\delta^{IJ}\delta^{KL} - \delta^{IK}\delta^{JL} + \delta^{IL}\delta^{JK}). \tag{B.105}$$

Let us just for completeness sake mention that also in six dimensions the gamma matrices in Weyl basis can be defined in terms of the sigma matrices. The definition is completely analogous to four dimensions:

$$\hat{\gamma}^I = \begin{pmatrix} 0 & \Sigma^I \\ \overline{\Sigma}^I & 0 \end{pmatrix} \qquad \{\hat{\gamma}^I, \hat{\gamma}^J\} = -2\delta^{IJ}\mathbf{1}\,. \tag{B.106}$$

Also here it is possible to show that properly normalised and anti-symmetrised products of two gamma-matrices represent the $\mathfrak{so}(6)$ algebra:

$$\hat{\gamma}^{IJ} = \frac{i}{2}\hat{\gamma}^{[I}, \hat{\gamma}^{J]} = \frac{i}{2}\left(\hat{\gamma}^I\hat{\gamma}^J + \delta^{IJ}\right) \tag{B.107}$$

$$[\hat{\gamma}^{IJ}, \hat{\gamma}^{KL}] = i(\delta^{IK}\hat{\gamma}^{JL} + \delta^{JL}\hat{\gamma}^{IK} - \delta^{IL}\hat{\gamma}^{JK} - \delta^{JK}\hat{\gamma}^{IL}). \tag{B.108}$$

Appendix C

Quartic Bulk Couplings from Witten Diagrams

In this appendix we would like to demonstrate how the results for the scalar potential in supergravity can be related to quantities on the field theory side. In particular, we will see that the most direct way to probe supergravity couplings is to compute field theory correlation functions of very specific operators.

Recall that in the previous chapter we showed how a specific deformation of the field theory by a chiral primary operator $\mathcal{O}_\Delta$ of conformal dimension Δ allows to relate the beta function for the induced RG flow to the supergravity scalar potential for the modes that holographically correspond to $\mathcal{O}_\Delta$. In particular, we are able to integrate the differential equation for the beta function in the specific case under consideration and obtain an expression for the supergravity scalar potential. This scalar potential, expanded in a power series, provides a prediction for n-point supergravity couplings to all orders.

After having obtained the scalar potential for the supergravity it would be desirable to perform some consistency checks, and confirm that the assumptions that were made in the derivation are indeed correct. To perform the easiest non-trivial test for the supergravity couplings, in the following sections we would like to describe the holographic computation of a generic four point function of chiral primary operators $\mathcal{O}_\Delta$. Later we can specialise to the operators that fulfil the assumptions made in the holographic beta function computation and test the predications made by the calculation.

Let us recall how holography relates field theory correlation functions to modes on the gravity side. The fundamental idea of holography is that the string theory partition function in the bulk should be equivalent to the boundary field theory partition function [13]:

$$Z_{\text{string}} = Z_{\text{FT}} \tag{C.1}$$

On the field theory side we can define the generating functional for connected correlations functions $\Gamma[J]$ by writing $Z_{\mathrm{FT}} = e^{-\Gamma[J]}$, where J collectively denotes all possible sources. Another proposition of holography is that a source J_Δ for a given field theory operator $\mathcal{O}_\Delta$ can be associated with a bulk mode ϕ_Δ. In the supergravity limit, where the 't Hooft coupling λ is assumed to be large, this translates to the fact that operators $\mathcal{O}_\Delta$ in the UV of the boundary theory are sourced by bulk supergravity modes close to the boundary, thus we can identify J_Δ with the boundary values $\phi_{0,\Delta}$ of the supergravity modes ϕ_Δ. In the saddle point approximation the string partition function, that is now actually the supergravity partition function is dominated by the extremal value of the supergravity action, in other words the on-shell action where the equations of motion are solved with boundary conditions $\phi_{0,\Delta}$. In sum, the saddle point approximation allows us to write

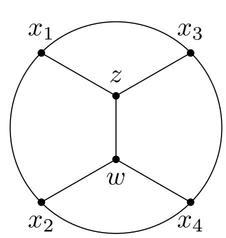
$$\Gamma[\phi_{0,\Delta}] = S^{\text{sugra}}_{\text{on-shell}}[\phi_{0,\Delta}] \tag{C.2}$$

where on the left side we have replaced the sources J by the boundary conditions ϕ_0 as explained above, and the supergravity action on the right side is a functional of the said boundary conditions. Given this identification, and the definition of the generating functional, we see that the field theory correlation functions can be computed by taking functional derivatives of the on shell supergravity action with respect to the sources, which we identified with the boundary conditions ϕ_0 for the supergravity modes [13, 14, 124]:

$$\langle \mathcal{O}_{\Delta_1}(x_1)\mathcal{O}_{\Delta_2}(x_2)\ldots\rangle = \frac{\delta}{\delta\phi_{0,\Delta_1}(x_1)}\frac{\delta}{\delta\phi_{0,\Delta_2}(x_2)}\ldots S^{\text{sugra}}_{\text{on-shell}}[\phi_{0,\Delta}]\Big|_{\phi_0=0} \tag{C.3}$$

The non-trivial part at this point is to find solutions to the supergravity equations of motion given the boundary conditions $\phi_{0,\Delta}$. Given the complexity of the supergravity theory, in the general case the full solutions cannot be obtained. To still be able to obtain the correlators, one solves the equations of motion perturbatively order by order in fields. Because in (C.3) the sources are set to zero after functional derivatives are taken, to get an expression for the n-point function one needs to solve the equations of motion to the n-th order in the corresponding fields.

Very much like for the Feynman diagrams, perturbative solutions of the supergravity equations of motion can be visualised diagrammatically. One such sample diagram, called Witten diagram is shown below



Unlike for Feynman diagrams, however, here the computations are done in position space, and the big circle represents the bulk boundary on which the field theory lives. The internal vertices represent the couplings of the supergravity theory, and we have two types of propagators. The ones that connect internal vertices to points on the boundary are called bulk to boundary propagators $K(z,\vec{x})$. Those incorporate boundary conditions that have to be applied at first order in perturbation theory, for example, if the first order equation for a scalar $\phi(z) = \phi^{(0)}(z) + \ldots$ is given by

$$(\Box - m^2)\phi^{(0)} = 0 \tag{C.4}$$

then it can be solved in terms of the boundary condition $\phi_0(\vec{x})$ by

$$\phi^{(0)}(z) = \int \mathrm{d}^d x \, K(z,\vec{x})\phi_0(\vec{x}) \tag{C.5}$$

with the Green's function $K(z,\vec{x})$ that we call the the bulk to boundary propagator that obeys

$$(\Box_z - m^2)K(z,\vec{x}) = 0 \tag{C.6}$$

$$\lim_{z_0\to 0} z_0^{\Delta-d} K(z,\vec{x}) = \delta(\vec{z}-\vec{x})\,. \tag{C.7}$$

Note that we are using a coordinate system for the Euclidean AdS space in which the metric is given by

$$ds^2 = \frac{1}{z_0^2}(dz_0^2 + d\vec{z}^2) \qquad g_{\mu\nu} = z_0^{-2}\delta_{\mu\nu} \tag{C.8}$$

with $z = (z_0, \vec{z})$, $z_0 = 0$ corresponding to the AdS boundary, and $z_0 = \infty$ to the origin. Field theory amplitudes with Lorentzian signature can be obtained by analytic continuation. The asymptotic behaviour of $K(z,\vec{x})$ near the boundary in (C.7) reflects the fact that the boundary conditions for the fields are imposed such that [11]

$$\phi(z) \overset{z_0\to 0}{\to} z_0^{\Delta-d}\phi_0(\vec{z}) \tag{C.9}$$

As before, d is the dimension of the boundary manifold where the field theory lives, thus the bulk manifold is a $(d+1)$-dimensional AdS space. The operator dimension Δ is related to the mass of ϕ by $m^2 = \Delta(\Delta - d)$. One can show that $K(z,\vec{x})$ is given by [11, 124, 125]

$$K_\Delta(z,\vec{x}) = C_\Delta \left(\frac{z_0}{z_0^2 + (\vec{z}-\vec{x})^2} \right)^{\Delta} \tag{C.10}$$

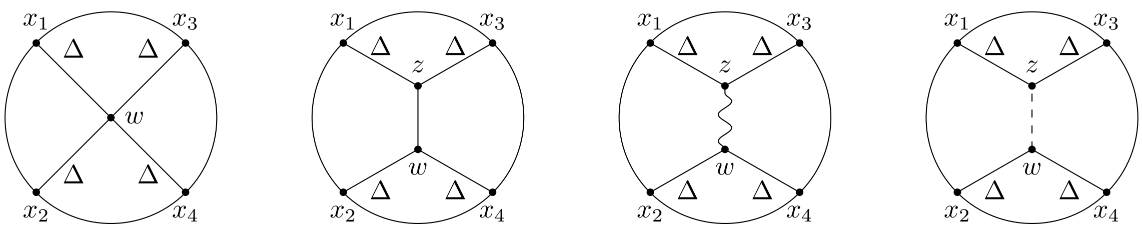
with

$$C_\Delta = \frac{\Gamma(\Delta)}{\pi^{d/2}\Gamma(\Delta - d/2)} \quad \text{for} \quad \Delta > \frac{d}{2}, \qquad C_{d/2} = \frac{\Gamma(d/2)}{2\pi^{d/2}}\,. \tag{C.11}$$

Higher orders in perturbation theory will involve bulk to bulk propagators $G(z,z')$ since equations for higher orders in perturbation theory are not solved in terms of boundary conditions, but in terms of lower order solutions, which are bulk fields, such as $\phi^{(0)}(z)$. These Green's functions are represented by lines in Witten diagrams that connect two internal vertices. Their explicit form is known [125], but is not relevant for us. What is important is the fact that being Green's functions they satisfy

$$-(\Box_z - m^2)G(z,z') = \frac{\delta(z-z')}{\sqrt{g'}} = (z_0')^{d+1}\delta(z-z') \tag{C.12}$$

The problem that the rest of this chapter is devoted to is the holographic computation of the four point function of a chiral primary operator $\mathcal{O}_\Delta(x)$. It is obvious that the computation will involve the evaluation of the following Witten diagrams



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and additionally analogous t and u channel graphs for the exchange diagrams above. The three exchange diagrams above show the contributions from the scalar, vector, and graviton exchange, and it is part of the problem to correctly determine the exchanged particles, at least for the scalar and the vector case. An additional inherent difficulty of the evaluation of exchange diagrams is the fact that they contain two internal vertices which means that two bulk integrals have to be carried out. An elegant way to simplify such diagrams was found in [126], which we will describe in the next section and use for the evaluation.

The results that we will obtain for the exchange diagrams can be summarised as

$$\begin{aligned}&\langle \mathcal{O}_\Delta(x_1)\mathcal{O}_\Delta(x_2)\mathcal{O}_\Delta(x_3)\mathcal{O}_\Delta(x_4)\rangle \propto \\ &\qquad \sum_{l=1}^{l_{\max}} \tilde{C}_l \int_w K_{\Delta-l}(w,\vec{x}_1)K_{\Delta-l}(w,\vec{x}_3)K_\Delta(w,\vec{x}_2)K_\Delta(w,\vec{x}_4) \\ &\qquad\qquad \times \left(D_l^{(s)}(w,\vec{x}_i) + D_l^{(v)}(w,\vec{x}_i) + D_l^{(t)}(w,\vec{x}_i)\right)\end{aligned} \tag{C.13}$$

which involves a finite sum over four point contact bulk diagrams with pre-factors $\tilde{C}_l$ and a various factors for the scalar (s) vector (v) and the tensor (t) exchange contributions that need to be determined. We will explain these building blocks and give their explicit form in the following sections.

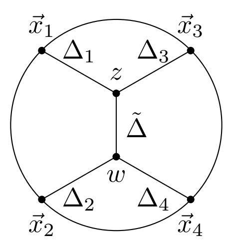
C.1 Reduction of Exchange Diagrams

In this section we would like to discuss the method for the reduction of exchange Witten diagrams for four point amplitudes involving scalar, vector, and graviton exchange to finite sums of four point contact graphs. The general procedure was discussed by [126], here we would like to outline the method, and complement by additional computations. The results will be used in the next section for a concrete case.

As mentioned in the previous section, unlike the contact Witten diagram, which corresponds to one bulk integral, the exchange graphs require the computation of two bulk integrals. In the past the evaluation of one of the two integrals was done by expanding the integrand in a power series and re-summing the terms back in a clever way, as was done in [125, 127, 128] and other papers. The procedure of expansion and summation is quite involved, and given the relatively simple final expressions obtained, the authors of [126] recognised that there could be a simpler method. The general idea is that one can apply $(\Box - m^2)$ to one of the bulk integrals and use the defining property of the bulk to bulk propagators in (C.12) to get a differential equation, all while using the conformal symmetry to reduce the number of degrees of freedom. The resulting differential equation can be solved recursively and the recursion is finite for exactly those sets of parameters that correspond to supergravity.

In what follows we would like to sketch the main steps for the scalar exchange which were described in [126]. The idea for the vector and graviton exchange is similar, although the complexity is higher due to tensorial structures involved. For these cases we will limit ourselves to a summary of the relevant results that we will need for our computation.

Using AdS coordinates as in the previous section, consider the following four point scalar exchange Witten diagram:



As explained in the previous section, the graph contributes to the four point function

$$\langle \mathcal{O}_{\Delta_1}(\vec{x}_1)\mathcal{O}_{\Delta_2}(\vec{x}_2)\mathcal{O}_{\Delta_3}(\vec{x}_3)\mathcal{O}_{\Delta_4}(\vec{x}_4)\rangle \tag{C.14}$$

and represents the following amplitude

$$S(\vec{x}_1,\vec{x}_2,\vec{x}_3,\vec{x}_4) = \int_w \int_z K_{\Delta_1}(z,\vec{x}_1)K_{\Delta_3}(z,\vec{x}_3)G_{\tilde{\Delta}}(z,w)K_{\Delta_2}(w,\vec{x}_2)K_{\Delta_4}(w,\vec{x}_4) \tag{C.15}$$

with the following short-hand notation that we will be using throughout the rest of the chapter

$$\int_z \equiv \int_{\text{AdS}} \mathrm{d}^{d+1} z \sqrt{g} = \int_{\text{AdS}} \mathrm{d}^{d+1} z \frac{1}{z_0^{d+1}}. \tag{C.16}$$

Further, we will usually use $\vec{x}$ to denote boundary coordinates and add a vector arrow to distinguish them from bulk coordinates, which will be denoted by z or w without the vector arrow.

Before we proceed, we would like to discuss some identities and conventions regarding the coordinates and their symmetry transformations, as these will be useful at several points throughout further computations.

First recall that in the coordinates we are using the metric is given by $g_{\mu\nu} = \frac{1}{z_0^2}\delta_{\mu\nu}$ and $g \equiv \det(g_{\mu\nu}) = z_0^{-2D}$, with $D = d+1$ the dimension of the AdS space. As in [126] we would like to write coordinate contractions using $\delta_{\mu\nu}$ rather than $g_{\mu\nu}$, for example $z^2 = \delta_{\mu\nu} z^\mu z^\nu = \sum_\mu z^\mu z^\mu$. One of the key ingredients for the computation is the use of the inversion transformation, which is given by $z^\mu = \frac{z'^\mu}{z'^2}$, in other words, $z^2 = 1/(z')^2$. Consider the transformation of $(z-w)^2$ under inversions:

$$(z-w)^2 = z^2 + w^2 - 2zw = \frac{1}{z'^2} + \frac{1}{w'^2} - 2\frac{z'w'}{z'^2 w'^2} = \frac{w'^2 + z'^2 - 2z'w'}{z'^2 w'^2} = \frac{(z'-w')^2}{z'^2 w'^2}. \tag{C.17}$$

This allows us to define the so-called “chordal distance” u that is now manifestly invariant under inversions:

$$u = \frac{(z-w)^2}{2 z_0 w_0}. \tag{C.18}$$

Note also that because $z = (z_0, \vec{z})$, $w = (w_0, \vec{w})$ we can split off the d-dimensional coordinates in the chordal distance and write it as

$$u = \frac{(z_0 - w_0)^2 + (\vec{z} - \vec{w})^2}{2 z_0 w_0} \tag{C.19}$$

which reveals that the chordal distance is also invariant under the d-dimensional Poincaré subgroup of the conformal group that acts on the components $\vec{z}$ and $\vec{w}$. It is known [125, 129, 130] that the bulk to bulk propagator $G(z,w)$ is in fact a function of the chordal distance only, and therefore we can write $G(z,w) \to G(u)$ and it follows from the properties of the chordal distance that $G(u)$ is invariant under D-dimensional bulk inversions and the d-dimensional boundary Poincaré group.

It is natural to re-interpret boundary coordinates $\vec{x}$ as bulk coordinates with $x = (0, \vec{x})$ and write expressions like $(z - \vec{x})^2$. This allows us to write bulk to boundary propagators as

$$K_\Delta(z, \vec{x}) = C_\Delta \left(\frac{z_0}{(z-\vec{x})^2} \right)^\Delta = C_\Delta \left(\frac{z_0'}{(z' - \vec{x}\,')^2} \right)^\Delta (\vec{x}\,')^{2\Delta} \qquad \vec{x}\,' = \frac{\vec{x}^{\,2}}{|\vec{x}|^2} \tag{C.20}$$

Finally let us show that the bulk integration measure $\mathrm{d}^D z\sqrt{g}$ is invariant under bulk inversions. Given the inversion transformation $z^\mu = \frac{z'^\mu}{z'^2}$, the corresponding Jacobian is given by

$$\mathcal{J}^\mu{}_\nu(z) = \frac{\partial z^\mu}{\partial z'^\nu} = z^2\left(\delta^\mu_\nu - 2\frac{z^\mu z_\nu}{z^2}\right) = z^2 J^\mu{}_\nu(z)\,. \tag{C.21}$$

To compute the Jacobian determinant $\mathcal{J} = \det(\mathcal{J}^\mu{}_\nu)$ observe that $J^\mu{}_\rho J^\rho{}_\nu = \delta^\mu_\nu$, from where it follows that $\mathcal{J} = \sqrt{\det(z^4\delta^\mu_\nu)} = z^{2D}$. Thus we conclude immediately that

$$\mathrm{d}^D z\sqrt{g} = \mathrm{d}^D z\frac{1}{z_0^D} = \mathrm{d}^D z'\frac{\mathcal{J}}{z_0^D} = \mathrm{d}^D z'\left(\frac{z^2}{z_0}\right)^D = \mathrm{d}^D z'\frac{1}{(z_0')^D} = \mathrm{d}^D z'\sqrt{g'}\,. \tag{C.22}$$

Given these transformations it is now not too difficult to follow the steps in [126]. Start with the amplitude defined in (C.15) and perform a coordinate translation on both integration variables by $\vec{x}_1$: $z \to z + \vec{x}_1$, $w \to w + \vec{x}_1$. Because the bulk to bulk propagator is invariant under this shift, the net effect is the shift in the bulk to boundary propagators and we get

$$S(\vec{x}_1,\vec{x}_2,\vec{x}_3,\vec{x}_4) = \int_w\int_z K_{\Delta_1}(z,0)K_{\Delta_3}(z,\vec{x}_{31})G_{\tilde{\Delta}}(u)K_{\Delta_2}(w,\vec{x}_{21})K_{\Delta_4}(w,\vec{x}_{41}) \tag{C.23}$$

with $\vec{x}_{31} = \vec{x}_3 - \vec{x}_1$ and similarly for other coordinates. Now we can split the amplitude into two parts as follows

$$S(\vec{x}_1,\vec{x}_2,\vec{x}_3,\vec{x}_4) = \int_w A(w,\vec{x}_{31})K_{\Delta_2}(w,\vec{x}_{21})K_{\Delta_4}(w,\vec{x}_{41}) \tag{C.24}$$

$$A(w,\vec{x}_{31}) = \int_z K_{\Delta_1}(z,0)K_{\Delta_3}(z,\vec{x}_{31})G_{\tilde{\Delta}}(u) \tag{C.25}$$

and convert $A(w,\vec{x}_{31})$ into an expression without integrals, which reduces the expression for the whole amplitude to only one bulk integration. First, apply inversion to $\vec{x}_{31}$, z, and w. Using the identities for coordinate inversions above it is easy to check that this leads to

$$A(w,\vec{x}_{31}) = |\vec{x}_{31}|^{-2\Delta_3}C_{\Delta_1}C_{\Delta_3}I(w' - \vec{x}_{31}') \tag{C.26}$$

$$I(w) = \int_z G_{\tilde{\Delta}}(u)(z_0)^{\Delta_1}\left(\frac{z_0}{z^2}\right)^{\Delta_3}. \tag{C.27}$$

The normalisation constants C_Δ for the bulk to boundary propagators were defined in (C.11), mind also the difference in the argument of I in the equations above. It is easy to see that $I(w) = \lambda^{\Delta_{13}}I(w/\lambda)$, $\Delta_{13} = \Delta_1 - \Delta_3$, for any real number λ. Choose $\lambda = w_0$ such that

$$\frac{w}{w_0} = \left(1, \frac{\vec{w}}{w_0}\right) \tag{C.28}$$

and by the fact that $I(w)$ is invariant under rotations in $\vec{w}$ we realise that $I(w/w_0)$ only depends on $\frac{|\vec{w}|}{w_0}$, or equivalently on

$$t = \frac{w_0^2}{w^2} = \frac{1}{1 + \frac{|\vec{w}|}{w_0}} \,. \tag{C.29}$$

This allows us to reduce the number of degrees of freedom in $I(w)$ further, and we can write

$$I(w) = (w_0)^{\Delta_{13}} f(t) \,. \tag{C.30}$$

Given that $I(w)$ contains the bulk to bulk propagator, which, in turn, satisfies its defining differential equation (C.12), one can apply the corresponding differential operator to $f(t)$, which results in a differential equation. Imposing asymptotic conditions on the behaviour of $f(t)$ selects one solution, and the differential equation can be solved recursively resulting in a finite sum

$$f(t) = \sum_{k=k_{\min}}^{k_{\max}} a_k t^k \qquad k_{\min} = \frac{\tilde{\Delta} - \Delta_{13}}{2} \quad k_{\max} = \Delta_3 - 1 \,. \tag{C.31}$$

This is only true if Σ^+ is a positive integer, where

$$\Sigma^{\pm} = \frac{\Delta_1 + \Delta_3 \mp \tilde{\Delta}}{2} \,, \tag{C.32}$$

in which case it is also true that $k_{\min} \leq k_{\max}$. In the cases which we will be considering this condition is fulfilled up to the case $\Sigma^+ = 0$, this extremal case will be commented on in the next section.

The coefficients a_k were given in [126] in a recursive form. We can re-sum the recursion to obtain the following expression for $k_{\min} \leq k \leq k_{\max}$:

$$a_k = \frac{1}{4} \frac{\Gamma(\Sigma^+)\Gamma(\Sigma^- - \frac{d}{2})}{\Gamma(\Delta_3)\Gamma(\Delta_1)} \frac{\Gamma(k)\Gamma(k + \Delta_{13})}{\Gamma(k + \frac{\Delta_{13} - \tilde{\Delta}}{2} + 1)\Gamma(k + \frac{\Delta_{13} + \tilde{\Delta} - d}{2} + 1)} \tag{C.33}$$

Now we can go all steps backwards and re-substitute the expressions obtained into the amplitude $S(x_i)$. As a last step, shift the only remaining integration variable w backwards to $w \to w - \vec{x}_1$ to get

$$S(\vec{x}_1, \vec{x}_2, \vec{x}_3, \vec{x}_4) = \sum_{k=k_{\min}}^{k_{\max}} a_k \frac{C_{\Delta_1} C_{\Delta_3}}{C_k C_{\Delta_{13}+k}} |\vec{x}_{31}|^{2k - 2\Delta_3} \times \tag{C.34}$$

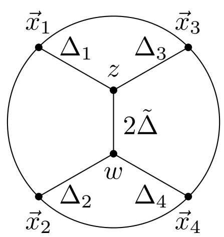
$$\times \int_w K_{\Delta_{13}+k}(w, \vec{x}_1) K_k(w, \vec{x}_3) K_{\Delta_2}(w, \vec{x}_2) K_{\Delta_4}(w, \vec{x}_4) \tag{C.35}$$

We can reduce the number of constant pre-factors by redefining a_k to incorporate the C_i in it, and also shift the summation variable k to $l = \Delta_3 - k$ to render the expression symmetric in $\vec{x}_1$ and $\vec{x}_3$. This leads to

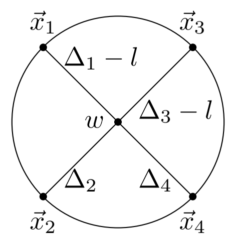
$$S(\vec{x}_1, \vec{x}_2, \vec{x}_3, \vec{x}_4) = \sum_{l=1}^{\Sigma^+} a_l |\vec{x}_{31}|^{-2l} \int_w K_{\Delta_1 - l}(w, \vec{x}_1) K_{\Delta_3 - l}(w, \vec{x}_3) K_{\Delta_2}(w, \vec{x}_2) K_{\Delta_4}(w, \vec{x}_4) \tag{C.36}$$

$$a_l = \frac{1}{4} \frac{\Gamma(\Sigma^+)\Gamma(\Sigma^- - \frac{d}{2})}{\Gamma(\Sigma^+ - l + 1)\Gamma(\Sigma^- - \frac{d}{2} - l + 1)} \frac{\Gamma(\Delta_1 - \frac{d}{2} - l)\Gamma(\Delta_3 - \frac{d}{2} - l)}{\Gamma(\Delta_1 - \frac{d}{2})\Gamma(\Delta_3 - \frac{d}{2})} \tag{C.37}$$

We see that the net result is that a scalar exchange diagram can be reduced to a finite sum over contact graphs, or pictorially:



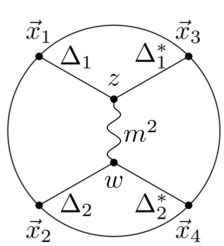
$$= \sum_{l=1}^{\Sigma^+} a_l |\vec{x}_{31}|^{-2l} \tag{C.38}$$



This completes the discussion of the scalar exchange diagram, and we can proceed with the vector and graviton exchange diagrams. The basic idea remains the same: split the amplitude into two parts, convert one of them into a differential equation, and solve it recursively. Some additional complexity is added by the tensorial structures that appear. For this reason, we will abstain from going into the details of the derivation and will list the net outcome, which is similar to the scalar exchange: the diagram reduces to a sum over contact graphs weighted by some coefficients.

For the vector exchange it is assumed that the vector fields couple to conserved currents of the scalars, in other words, a cubic vertex with a vector field will have two scalar fields of the same dimension attached, and we must identify $\Delta_1 = \Delta_3$ and $\Delta_2 = \Delta_4$. This also implies that the scalar that couples to the vector has a corresponding charge and the correlator that the vector exchange diagram contributes to is $\langle \mathcal{O}_{\Delta_1}(x_1)\mathcal{O}_{\Delta_2}(x_2)\mathcal{O}^*_{\Delta_1}(x_3)\mathcal{O}^*_{\Delta_2}(x_4)\rangle$.Thus the amplitude we want to compute corresponds to the following Witten diagram

$$V(\vec{x}_1, \vec{x}_2, \vec{x}_3, \vec{x}_4) =$$



$$= \int_w \int_z K_{\Delta_1}(z, x_1) \frac{\overleftrightarrow{\partial}}{\partial z^\rho} K_{\Delta_1}(z, \vec{x}_3) \times g^{\rho\mu}(z) G_{\mu\nu}(u) g^{\nu\sigma}(w) \times K_{\Delta_2}(w, \vec{x}_2) \frac{\overleftrightarrow{\partial}}{\partial w^\sigma} K_{\Delta_2}(w, \vec{x}_4)\,. \tag{C.39}$$

Splitting this amplitude in the same way as for the scalar exchange and following similar steps one can show that the amplitude is given by

$$V(\vec{x}_1,\vec{x}_2,\vec{x}_3,\vec{x}_4) = \sum_{l=1}^{\Pi^+} a_l |\vec{x}_{31}|^{-2l} \int_w K_{\Delta_1 - l}(w,\vec{x}_1) K_{\Delta_1 - l}(w,\vec{x}_3) K_{\Delta_2}(w,\vec{x}_2) K_{\Delta_2}(w,\vec{x}_4) \times \tag{C.40}$$

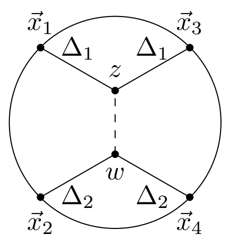
$$\times \left[\frac{(w-\vec{x}_3)_\mu}{(w-\vec{x}_3)^2} - \frac{(w-\vec{x}_1)_\mu}{(w-\vec{x}_1)^2}\right] w_0^2 \left[\frac{(w-\vec{x}_2)_\mu}{(w-\vec{x}_2)^2} - \frac{(w-\vec{x}_4)_\mu}{(w-\vec{x}_4)^2}\right] \tag{C.41}$$

$$a_l = (\Delta_1 - l)\Delta_2 \frac{\Gamma(\Pi^+)\Gamma(\Pi^-)}{\Gamma(\Pi^+ - l + 1)\Gamma(\Pi^- - l + 1)} \frac{\Gamma(\Delta_1 - \frac{d}{2} - l)^2}{\Gamma(\Delta_1 - \frac{d}{2})^2} \tag{C.42}$$

$$\Pi^\pm = \Delta_1 - \left(\frac{d-2}{4} \pm \frac{1}{4}\sqrt{(d-2)^2 + 4m^2}\right) . \tag{C.43}$$

Finally, let us consider the graviton exchange. After making a similar assumption as for the vector, namely that the graviton couples to scalar bilinears of scalars of the same dimension, we would like to evaluate the following diagram:

$$G(\vec{x}_1,\vec{x}_2,\vec{x}_3,\vec{x}_4) = \tag{C.44}$$



$$= \int_w \int_z T^{\mu'\nu'}(z,\vec{x}_1,\vec{x}_2) G_{\mu\nu\mu'\nu'}(u) T^{\mu\nu}(w,\vec{x}_2,\vec{x}_4) \tag{C.45}$$

$$T_{\mu\nu}(w,\vec{x}_2,\vec{x}_4) = D_\mu K_{\Delta_2}(w,\vec{x}_2) D_\nu K_{\Delta_2}(w,\vec{x}_4) - \frac{1}{2} g_{\mu\nu} \times \tag{C.46}$$

$$\times \left[D_\rho K_{\Delta_2}(w,\vec{x}_2) D^\rho K_{\Delta_2}(w,\vec{x}_4) + \Delta_2(\Delta_2 - d) K_{\Delta_2}(w,\vec{x}_2) K_{\Delta_2}(w,\vec{x}_4)\right] \tag{C.47}$$

Applying similar techniques as previously, we obtain the following result for the graviton exchange:

$$G(\vec{x}_1,\vec{x}_2,\vec{x}_3,\vec{x}_4) = \sum_{l=1}^{\Delta_1 + 1 - \frac{d}{2}} a_l |\vec{x}_{31}|^{-2l} \int_w K_{\Delta_1 - l}(w,\vec{x}_1) K_{\Delta_1 - l}(w,\vec{x}_3) K_{\Delta_2}(w,\vec{x}_2) K_{\Delta_2}(w,\vec{x}_4) \times \tag{C.48}$$

$$\times \left[J_{\mu 0}(w-\vec{x}_1) J_{\mu 0}(w-\vec{x}_2) J_{\nu 0}(w-\vec{x}_1) J_{\nu 0}(w-\vec{x}_4) - \frac{d(d-\Delta_2)}{\Delta_2(d-1)}\right] \tag{C.49}$$

$$a_l = -\frac{\Delta_2^2}{2} \frac{\Gamma(\Delta_1 + 1 - \frac{d}{2})}{\Gamma(\Delta_1 + 1 - \frac{d}{2} - l + 1)} \frac{\Gamma(\Delta_1 + 1)}{\Gamma(\Delta_1 - l)} \frac{\Gamma(\Delta_1 - \frac{d}{2} - l)^2}{\Gamma(\Delta_1 - \frac{d}{2})^2} \tag{C.50}$$

C.2 Exchange Diagrams for a Particular Case

In this section we would like start with the results obtained in the previous section and specialise to a particular case. One interesting case is the four point function of chiral primary operators of equal dimension Δ in $d = 4$ super Yang-Mills which is dual to a supergravity theory in AdS_5.

Start with the scalar exchange diagram, and substitute $\Delta_1 = \Delta_3 = \Delta$ into the definition of $\Sigma^{\pm}$:

$$\Sigma^{\pm} = \frac{\Delta_1 + \Delta_3 \mp \tilde{\Delta}}{2} = \Delta \mp \frac{\tilde{\Delta}}{2}\,. \tag{C.51}$$

The dimension $\tilde{\Delta}$ is the dimension of the exchanged scalar operator, and its values are restricted by the form of the trilinear couplings of scalars. We can study this coupling by looking at the underlying group theory. In the cubic vertices under consideration $\tilde{\Delta}$ couples to two superconformal primaries of dimension Δ, the SU(4) representation of which is therefore given by $[0, \Delta, 0]$. Consider for example the case $\Delta = 3$, $[0, \Delta, 0] = [0, 3, 0] = \ydiagram{3,3}$. The tensor product of two such operators decomposes as follows:

$$[0,3,0] \otimes [0,3,0] = \ydiagram{3,3} \otimes \ydiagram{3,3} = \ydiagram{6,6} + \ydiagram{5,5,1,1} + \ydiagram{4,4,2,2} + \ydiagram{3,3,3,3} + \dots \tag{C.52}$$

$$= \ydiagram{6,6} + \ydiagram{4,4} + \ydiagram{2,2} + \bullet + \dots \tag{C.53}$$

$$= [0,6,0] + [0,4,0] + [0,2,0] + [0,0,0] + \dots \tag{C.54}$$

It is easy to see the pattern: the tensor product of two superconformal chiral primaries of dimensions Δ decomposes into a sum that contains dimensions $\tilde{\Delta} \in \{0, 2, \dots, 2\Delta\}$. To get a vertex that is a singlet with respect to SU(4) the third operator has therefore to be of exactly one of these dimensions, and we can therefore write $\Sigma^{\pm}$ as follows:

$$\Sigma^{\pm} = \Delta \mp s\,, s \in \{0, \dots, \Delta\} \tag{C.55}$$

This is consistent with the condition for the termination of the recursive solution for the exchange diagram $\Sigma^{+} > 0$ up to the extremal case where $s = \Delta$. This was also noticed in [126], and the authors suggest that this case should be discussed separately. In the appendix of [125] it is suggested that for such extremal cases the scalar coupling may need to be altered to include derivatives, which seems to lead to sensible results.

Proceeding with the scalar amplitude we insert $\Delta_1 = \Delta_3 = \Delta$ and the parametrisation of $\tilde{\Delta}$

suggested above into (C.36) to get

$$S(\vec{x}_1,\vec{x}_2,\vec{x}_3,\vec{x}_4) = \sum_{l=1}^{\Delta-s} a_l |\vec{x}_{31}|^{-2l} \int_w K_{\Delta-l}(w,\vec{x}_1)K_{\Delta-l}(w,\vec{x}_3)K_{\Delta}(w,\vec{x}_2)K_{\Delta}(w,\vec{x}_4) \quad \text{(C.56)}$$

$$a_l = \frac{1}{4}\frac{\Gamma(\Delta-s)\Gamma(\Delta+s-2)}{\Gamma(\Delta-s-l+1)\Gamma(\Delta+s-2-l+1)}\frac{\Gamma(\Delta-2-l)^2}{\Gamma(\Delta-2)^2} \quad \text{(C.57)}$$

As we will see, the factor

$$\tilde{C}_l = \frac{\Gamma(\Delta-2-l)^2}{\Gamma(\Delta-2)^2} \quad \text{(C.58)}$$

in a_l will also appear in the vector and graviton exchange in exactly the same form, and it makes sense to separate it out:

$$S(\vec{x}_1,\vec{x}_2,\vec{x}_3,\vec{x}_4) = \sum_{l=1}^{\Delta-s} \tilde{C}_l |\vec{x}_{31}|^{-2l} \int_w K_{\Delta-l}(w,\vec{x}_1)K_{\Delta-l}(w,\vec{x}_3)K_{\Delta}(w,\vec{x}_2)K_{\Delta}(w,\vec{x}_4)D_l^{(s)} \quad \text{(C.59)}$$

$$D_l^{(s)} = \frac{1}{4}\frac{\Gamma(\Delta-s)\Gamma(\Delta+s-2)}{\Gamma(\Delta-s-l+1)\Gamma(\Delta+s-2-l+1)} \quad \text{(C.60)}$$

Next, consider the vector exchange, for which we defined the variables

$$\Pi^{\pm} = \Delta_1 - \left(\frac{d-2}{4} \pm \frac{1}{4}\sqrt{(d-2)^2+4m^2}\right). \quad \text{(C.61)}$$

As noted in [126], in AdS_5 supergravity the mass m^2 of Kaluza-Klein vectors is restricted to values $m^2 = p^2 - 1$ with $1 \leq p < 2\Delta_1 - 1$ and p odd. This is enforced by the SU(4) group theory, and the fact that the Kaluza-Klein fields have been worked out explicitly in [42] and one can read off the masses for the vectors that appear. It is convenient to write $p = 2v - 1$ with $v \in \{1,\ldots,\Delta_1 - 1\}$, and with $\Delta_1 = \Delta$ we obtain

$$\Pi^+ = \Delta - v\,, \qquad \Pi^- = \Delta + v - 1\,, \qquad v \in \{1,\ldots,\Delta-1\}\,. \quad \text{(C.62)}$$

We can now substitute this into the vector exchange amplitude found in (C.40) to obtain

$$V(\vec{x}_1,\vec{x}_2,\vec{x}_3,\vec{x}_4) = \sum_{l=1}^{\Delta-v} a_l |\vec{x}_{31}|^{-2l} \int_w K_{\Delta-l}(w,\vec{x}_1)K_{\Delta-l}(w,\vec{x}_3)K_{\Delta}(w,\vec{x}_2)K_{\Delta}(w,\vec{x}_4)\times \quad \text{(C.63)}$$

$$\times \left[\frac{(w-\vec{x}_3)_\mu}{(w-\vec{x}_3)^2} - \frac{(w-\vec{x}_1)_\mu}{(w-\vec{x}_1)^2}\right] w_0^2 \left[\frac{(w-\vec{x}_2)_\mu}{(w-\vec{x}_2)^2} - \frac{(w-\vec{x}_4)_\mu}{(w-\vec{x}_4)^2}\right] \quad \text{(C.64)}$$

$$a_l = \Delta(\Delta-l)\frac{\Gamma(\Delta-v)\Gamma(\Delta+v-1)}{\Gamma(\Delta-v-l+1)\Gamma(\Delta+v-l)}\frac{\Gamma(\Delta-2-l)^2}{\Gamma(\Delta-2)^2}\,. \quad \text{(C.65)}$$

Finally, as for the scalar exchange, we would like separate out the common factor $\tilde{C}_l$ in a_l, and obtain

$$V(\vec{x}_1, \vec{x}_2, \vec{x}_3, \vec{x}_4) = \sum_{l=1}^{\Delta - v} \tilde{C}_l |\vec{x}_{31}|^{-2l} \int_w K_{\Delta - l}(w, \vec{x}_1) K_{\Delta - l}(w, \vec{x}_3) K_\Delta(w, \vec{x}_2) K_\Delta(w, \vec{x}_4) D_l^{(v)} \tag{C.66}$$

$$D_l^{(v)} = \Delta(\Delta - l) \frac{\Gamma(\Delta - v)\Gamma(\Delta + v - 1)}{\Gamma(\Delta - v - l + 1)\Gamma(\Delta + v - l)} \times \tag{C.67}$$

$$\times \left[\frac{(w - \vec{x}_3)_\mu}{(w - \vec{x}_3)^2} - \frac{(w - \vec{x}_1)_\mu}{(w - \vec{x}_1)^2} \right] w_0^2 \left[\frac{(w - \vec{x}_2)_\mu}{(w - \vec{x}_2)^2} - \frac{(w - \vec{x}_4)_\mu}{(w - \vec{x}_4)^2} \right] \tag{C.68}$$

Since there is only one graviton field, no special discussion regarding various couplings is necessary, and we can give the result directly, with the correct dimensions substituted, and the coefficients split off as before:

$$G(\vec{x}_1, \vec{x}_2, \vec{x}_3, \vec{x}_4) = \sum_{l=1}^{\Delta - 1} \tilde{C}_l |\vec{x}_{31}|^{-2l} \int_w K_{\Delta - l}(w, \vec{x}_1) K_{\Delta - l}(w, \vec{x}_3) K_\Delta(w, \vec{x}_2) K_\Delta(w, \vec{x}_4) D_l^{(g)} \tag{C.69}$$

$$D_l^{(g)} = -\frac{\Delta^2}{2} \frac{\Gamma(\Delta - 1)}{\Gamma(\Delta - l)} \frac{\Gamma(\Delta + 1)}{\Gamma(\Delta - l)} \times \tag{C.70}$$

$$\times \left[J_{\mu 0}(w - \vec{x}_1) J_{\mu 0}(w - \vec{x}_2) J_{\nu 0}(w - \vec{x}_1) J_{\nu 0}(w - \vec{x}_4) - \frac{4(4 - \Delta)}{3\Delta} \right] \tag{C.71}$$

It is pleasant to see that all exchange diagrams assume a similar form. In particular, the w-integral corresponds to the same four point contact graph in all three cases. This gives hope that the amplitudes can be summed over to give a sensible expression for the four point correlator. However, several things need yet to be worked out. First of all, the extremal case for the scalar exchange where $\Sigma^+ = 0$ needs to be addressed separately. As mentioned before, it is possible, that the cubic coupling for this case may need to be altered, this would lead to a different expression for the corresponding amplitude which is not yet known. Furthermore, so far we only discussed s-channel exchange diagrams. In the full correlator the t and u channels also contribute and need to be integrated in the sum. In practise nothing new needs to be worked out and we only need to change the labels for the $\vec{x}_i$ and add more terms to the sum. However, changing the labels for the $\vec{x}_i$ means also that we will get different contact graphs that we are summing over. In the best case, the s, t, and u channels can be summed separately by taking into account all corresponding graphs for the scalar, vector, and graviton exchange. Finally, to sum over diagrams, each of them needs to be weighted by the corresponding values for the trilinear couplings. These couplings can be found in literature [109, 131–134], and need to be matched carefully.

Appendix D

Left-Invariant One-Forms for the SO(3)

The left invariant 1-forms are dual to left-invariant vector-fields that span the Lie algebra $\mathfrak{g}$ of a Lie group G. Let $\{T^a\}$ be the set of generators that span $\mathfrak{g}$, and f^{abc} the structure constants so that

$$[T^a, T^b] = if^{abc}T^c. \tag{D.1}$$

To derive the left-invariant 1-forms start by choosing a group element $g \in G$ and compute the corresponding Maurer-Cartan form $\omega_g = g^{-1}dg$. An elementary computation shows that $d\omega_g + \omega_g \wedge \omega_g = 0$. Note that the Maurer-Cartan form is a Lie algebra valued 1-form and one can therefore decompose it with respect to the basis $\{T^a\}$ by writing

$$\omega_g = i\sigma^a T^a. \tag{D.2}$$

The 1-forms σ^a are exactly the left-invariant 1-forms. After inserting the decomposition in (D.2) into $d\omega_g + \omega_g \wedge \omega_g = 0$ we see that the forms σ^a have the following property:

$$d\sigma^a = \frac{1}{2}f^{abc}\sigma^b \wedge \sigma^c. \tag{D.3}$$

Let us find a set of left-invariant 1-forms for the SO(3) rotation group. The $\mathfrak{so}(3)$ structure constants are given by $f^{abc} = \epsilon^{abc}$, and because the fundamental representation of the $\mathfrak{so}(3)$ is equivalent to the adjoint representation of the $\mathfrak{su}(2)$ the generators of the $\mathfrak{so}(3)$ algebra are given by the structure constants, and can be taken to be $(T^b)^{ac} = i\epsilon^{abc}$. The group elements are computed by exponentiation of the algebra elements, $R_a(\alpha) = \exp(i\alpha T^a)$, and are given by the standard rotation matrices

$$R_x(\alpha) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\alpha & \sin\alpha \\ 0 & -\sin\alpha & \cos\alpha \end{pmatrix} \; R_y(\alpha) = \begin{pmatrix} \cos\alpha & 0 & -\sin\alpha \\ 0 & 1 & 0 \\ \sin\alpha & 0 & \cos\alpha \end{pmatrix} \; R_z(\alpha) = \begin{pmatrix} \cos\alpha & \sin\alpha & 0 \\ -\sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{D.4}$$

We can choose the parametrisation of a generic SO(3) rotation by three Euler angles to be $g(\alpha_1, \alpha_2, \alpha_3) = R_z(\alpha_3)R_x(\alpha_2)R_z(\alpha_1)$, where the range of the angles is $0 \leq \alpha_1, \alpha_3 \leq 2\pi$, and $0 \leq \alpha_2 \leq \pi$. Next we compute the Maurer-Cartan form $\omega_g(\alpha_1, \alpha_2, \alpha_3) = g^{-1}dg$, and using the fact that the generators are normalised so that $\mathrm{tr}(T^a T^b) = 2\delta^{ab}$, the left-invariant 1-forms are projected out by $\sigma^a = \frac{1}{2i}\,\mathrm{tr}(\omega_g T^a)$. Thus we obtain the following result:

$$\sigma^1 = \cos\alpha_1 \; d\alpha_2 + \sin\alpha_1 \sin\alpha_2 \; d\alpha_3 \tag{D.5a}$$

$$\sigma^2 = \sin\alpha_1 \; d\alpha_2 - \cos\alpha_1 \sin\alpha_2 \; d\alpha_3 \tag{D.5b}$$

$$\sigma^3 = d\alpha_1 + \cos\alpha_2 \; d\alpha_3. \tag{D.5c}$$

These are the left-invariant 1-forms that we used in this thesis. A different set of left-invariant 1-forms is discussed in the well-known review article by Eguchi, Gilkey, and Hanson [135]. The forms mentioned on page 247 in their article can be constructed using the same rotation matrices as we used above by taking the group element parametrisation to be $g(\psi, \theta, \phi) = R_z(\phi)R_y(\theta)R_z(\psi)$ with $0 \leq \psi, \phi \leq 2\pi$ and $0 \leq \theta \leq \pi$. The derivation by Eguchi et al. was performed for the SU(2) group, which is a double cover of the SO(3), hence the range of the angle ψ is doubled to $0 \leq \psi \leq 4\pi$. However, since both algebras $\mathfrak{su}(2)$ and $\mathfrak{so}(3)$ are isomorphic, the left-invariant 1-forms are exactly the same. With this parametrisation the following 1-forms are determined:

$$\sigma^1 = \sin\psi \; d\theta - \cos\psi \sin\theta \; d\phi \tag{D.6a}$$

$$\sigma^2 = -\cos\psi \; d\theta - \sin\psi \sin\theta \; d\phi \tag{D.6b}$$

$$\sigma^3 = \cos\theta \; d\phi + \; d\psi. \tag{D.6c}$$

To obtain exactly the same normalisation as in Eguchi et al. one needs to further rescale the 1-forms to $\sigma^a \to \frac{1}{2}\sigma^a$.

Appendix E

Explicit Representation Branching

E.1 $\mathrm{SO}(6) \to \mathrm{SO}(3) \times \mathrm{SO}(3) \to \mathrm{SO}(3)_{\mathrm{diag}}$

In this section we tabulate some useful branching rules for the decomposition under $\mathrm{SO}(3) \times \mathrm{SO}(3) \subset \mathrm{SO}(6)$ and $\mathrm{SO}(3)_{\mathrm{diag}} \subset \mathrm{SO}(3) \times \mathrm{SO}(3) \subset \mathrm{SO}(6)$. By the Lie algebra isomorphisms $\mathfrak{so}(6) \cong \mathfrak{su}(4)$ and $\mathfrak{so}(3) \cong \mathfrak{su}(2)$ this is equivalent to studying the special embedding $\mathrm{SU}(2) \times \mathrm{SU}(2) \subset \mathrm{SU}(4)$ [79], and has nothing to do with the regular embedding $\mathrm{SU}(2) \times \mathrm{SU}(2) \times \mathrm{U}(1) \subset \mathrm{SU}(4)$. Note that the conjugate SO(6) representations are computed as $[k, m_1, m_2]^* = [k, m_2, m_1]$. The representations that can be obtained by conjugation are not listed. Note that because all SO(3) representations are real, any SO(6) representation will decompose in exactly the same way as its conjugate. Because the two SO(3) subgroups are embedded symmetrically into the SO(6), all decomposed representations are symmetric under the exchange of the two SO(3) factors. Note also that all SO(6) representations $[k, m_1, m_2]$ for which the sum $m_1 + m_2$ is even, in other words those with an integer spin, decompose into odd SO(3) representations and vice versa.

We used the Mathematica package "LieART" [136] to generate the representations and their branching. Because the algebra embedding $\mathrm{SO}(3) \times \mathrm{SO}(3) \subset \mathrm{SO}(6)$ is special, first one needs to generate a projection matrix by providing the branching of the SO(6) generating spinor representation $\mathbf{4} \to (\mathbf{2}, \mathbf{2})$. This branching can be deduced from the fact that $\mathbf{4} \times \mathbf{4} = \mathbf{6} + \mathbf{10}$ and the fact that taking tensor products commutes with the branching. We can compute the branching of the representations on the right-hand side by realizing that the $\mathbf{6}$ is just the fundamental representation and the $(\mathbf{10} + \overline{\mathbf{10}})$ corresponds to an anti-symmetrized triplet of fundamental indices. The branching of these representations can be done by hand by looking at how an SO(6) index splits into two SO(3) indices. With the branching for the $\mathbf{6}$ and the $\mathbf{10}$ so obtained, the only branching for the $\mathbf{4}$ which is compatible with the tensor product decomposition $\mathbf{4} \times \mathbf{4} = \mathbf{6} + \mathbf{10}$ is $\mathbf{4} \to (\mathbf{2}, \mathbf{2})$.

Finally, under the embedding $SO(3)_{diag} \subset SO(3) \times SO(3)$ the representations branch according to $(\mathbf{r}_1, \mathbf{r}_2) \to \mathbf{r}_1 \otimes \mathbf{r}_2$.

Table E.1: Some representation branching rules under the special subalgebra embedding $SO(3) \times SO(3) \subset SO(6)$

		SO(6)	$\to$	SO(3) $\times$ SO(3)
$[0,1,0]$	$=$	$\mathbf{4}$	$\to$	$(\mathbf{2},\mathbf{2})$
$[1,0,0]$	$=$	$\mathbf{6}$	$\to$	$(\mathbf{3},\mathbf{1})+(\mathbf{1},\mathbf{3})$
$[0,2,0]$	$=$	$\mathbf{10}$	$\to$	$(\mathbf{1},\mathbf{1})+(\mathbf{3},\mathbf{3})$
$[0,1,1]$	$=$	$\mathbf{15}$	$\to$	$(\mathbf{3},\mathbf{1})+(\mathbf{1},\mathbf{3})+(\mathbf{3},\mathbf{3})$
$[1,0,1]$	$=$	$\mathbf{20}$	$\to$	$(\mathbf{2},\mathbf{2})+(\mathbf{4},\mathbf{2})+(\mathbf{2},\mathbf{4})$
$[2,0,0]$	$=$	$\mathbf{20'}$	$\to$	$(\mathbf{1},\mathbf{1})+(\mathbf{3},\mathbf{3})+(\mathbf{5},\mathbf{1})+(\mathbf{1},\mathbf{5})$
$[0,0,3]$	$=$	$\mathbf{20''}$	$\to$	$(\mathbf{2},\mathbf{2})+(\mathbf{4},\mathbf{4})$
$[0,4,0]$	$=$	$\mathbf{35}$	$\to$	$(\mathbf{1},\mathbf{1})+(\mathbf{3},\mathbf{3})+(\mathbf{5},\mathbf{5})$
$[0,2,1]$	$=$	$\mathbf{36}$	$\to$	$(\mathbf{2},\mathbf{2})+(\mathbf{4},\mathbf{2})+(\mathbf{2},\mathbf{4})+(\mathbf{4},\mathbf{4})$
$[1,2,0]$	$=$	$\mathbf{45}$	$\to$	$(\mathbf{3},\mathbf{1})+(\mathbf{1},\mathbf{3})+(\mathbf{3},\mathbf{3})+(\mathbf{5},\mathbf{3})+(\mathbf{3},\mathbf{5})$
$[3,0,0]$	$=$	$\mathbf{50}$	$\to$	$(\mathbf{3},\mathbf{1})+(\mathbf{1},\mathbf{3})+(\mathbf{5},\mathbf{3})+(\mathbf{3},\mathbf{5})+(\mathbf{7},\mathbf{1})+(\mathbf{1},\mathbf{7})$
$[2,1,0]$	$=$	$\mathbf{60}$	$\to$	$(\mathbf{2},\mathbf{2})+(\mathbf{4},\mathbf{2})+(\mathbf{2},\mathbf{4})+(\mathbf{4},\mathbf{4})+(\mathbf{6},\mathbf{2})+(\mathbf{2},\mathbf{6})$
$[1,1,1]$	$=$	$\mathbf{64}$	$\to$	$(\mathbf{3},\mathbf{1})+(\mathbf{1},\mathbf{3})+2(\mathbf{3},\mathbf{3})+(\mathbf{5},\mathbf{1})+(\mathbf{1},\mathbf{5})+(\mathbf{5},\mathbf{3})+(\mathbf{3},\mathbf{5})$
$[0,3,1]$	$=$	$\mathbf{70}$	$\to$	$(\mathbf{3},\mathbf{1})+(\mathbf{1},\mathbf{3})+(\mathbf{3},\mathbf{3})+(\mathbf{5},\mathbf{3})+(\mathbf{3},\mathbf{5})+(\mathbf{5},\mathbf{5})$
$[0,2,2]$	$=$	$\mathbf{84}$	$\to$	$(\mathbf{1},\mathbf{1})+2(\mathbf{3},\mathbf{3})+(\mathbf{5},\mathbf{1})+(\mathbf{1},\mathbf{5})+(\mathbf{5},\mathbf{3})+(\mathbf{3},\mathbf{5})+(\mathbf{5},\mathbf{5})$
$[1,3,0]$	$=$	$\mathbf{84'}$	$\to$	$(\mathbf{2},\mathbf{2})+(\mathbf{4},\mathbf{2})+(\mathbf{2},\mathbf{4})+(\mathbf{4},\mathbf{4})+(\mathbf{6},\mathbf{4})+(\mathbf{4},\mathbf{6})$
$[4,0,0]$	$=$	$\mathbf{105}$	$\to$	$(\mathbf{1},\mathbf{1})+(\mathbf{3},\mathbf{3})+(\mathbf{5},\mathbf{1})+(\mathbf{1},\mathbf{5})+(\mathbf{5},\mathbf{5})+(\mathbf{7},\mathbf{3})+(\mathbf{3},\mathbf{7})+(\mathbf{9},\mathbf{1})+(\mathbf{1},\mathbf{9})$
$[0,1,4]$	$=$	$\mathbf{120}$	$\to$	$(\mathbf{2},\mathbf{2})+(\mathbf{4},\mathbf{2})+(\mathbf{2},\mathbf{4})+(\mathbf{4},\mathbf{4})+(\mathbf{6},\mathbf{4})+(\mathbf{4},\mathbf{6})+(\mathbf{6},\mathbf{6})$
$[2,2,0]$	$=$	$\mathbf{126}$	$\to$	$(\mathbf{1},\mathbf{1})+2(\mathbf{3},\mathbf{3})+(\mathbf{5},\mathbf{1})+(\mathbf{1},\mathbf{5})+(\mathbf{5},\mathbf{3})+(\mathbf{3},\mathbf{5})+(\mathbf{5},\mathbf{5})+(\mathbf{7},\mathbf{3})+(\mathbf{3},\mathbf{7})$
$[1,1,2]$	$=$	$\mathbf{140}$	$\to$	$(\mathbf{2},\mathbf{2})+2(\mathbf{4},\mathbf{2})+2(\mathbf{2},\mathbf{4})+2(\mathbf{4},\mathbf{4})+(\mathbf{6},\mathbf{2})+(\mathbf{2},\mathbf{6})+(\mathbf{6},\mathbf{4})+(\mathbf{4},\mathbf{6})$
$[3,0,1]$	$=$	$\mathbf{140'}$	$\to$	$(\mathbf{2},\mathbf{2})+(\mathbf{4},\mathbf{2})+(\mathbf{2},\mathbf{4})+(\mathbf{4},\mathbf{4})+(\mathbf{6},\mathbf{2})+(\mathbf{2},\mathbf{6})+(\mathbf{6},\mathbf{4})+(\mathbf{4},\mathbf{6})+(\mathbf{8},\mathbf{2})+(\mathbf{2},\mathbf{8})$
$[1,4,0]$	$=$	$\mathbf{140''}$	$\to$	$(\mathbf{3},\mathbf{1})+(\mathbf{1},\mathbf{3})+(\mathbf{3},\mathbf{3})+(\mathbf{5},\mathbf{3})+(\mathbf{3},\mathbf{5})+(\mathbf{5},\mathbf{5})+(\mathbf{7},\mathbf{5})+(\mathbf{5},\mathbf{7})$
$[0,3,2]$	$=$	$\mathbf{160}$	$\to$	$(\mathbf{2},\mathbf{2})+(\mathbf{4},\mathbf{2})+(\mathbf{2},\mathbf{4})+2(\mathbf{4},\mathbf{4})+(\mathbf{6},\mathbf{2})+(\mathbf{2},\mathbf{6})+(\mathbf{6},\mathbf{4})+(\mathbf{4},\mathbf{6})+(\mathbf{6},\mathbf{6})$
$[2,1,1]$	$=$	$\mathbf{175}$	$\to$	$(\mathbf{3},\mathbf{1})+(\mathbf{1},\mathbf{3})+2(\mathbf{3},\mathbf{3})+(\mathbf{5},\mathbf{1})+(\mathbf{1},\mathbf{5})+2(\mathbf{5},\mathbf{3})+2(\mathbf{3},\mathbf{5})+(\mathbf{7},\mathbf{1})+(\mathbf{1},\mathbf{7})+(\mathbf{5},\mathbf{5})+(\mathbf{7},\mathbf{3})+(\mathbf{3},\mathbf{7})$

Table E.2: Some representation branching rules under the special subalgebra embedding $SO(3)_{diag} \subset SO(3) \times SO(3) \subset SO(6)$

		SO(6)	→	$SO(3)_{diag}$
$[0,1,0]$	=	$\mathbf{4}$	→	$\mathbf{1}+\mathbf{3}$
$[1,0,0]$	=	$\mathbf{6}$	→	$2(\mathbf{3})$
$[0,2,0]$	=	$\mathbf{10}$	→	$2(\mathbf{1})+\mathbf{3}+\mathbf{5}$
$[0,1,1]$	=	$\mathbf{15}$	→	$\mathbf{1}+3(\mathbf{3})+\mathbf{5}$
$[1,0,1]$	=	$\mathbf{20}$	→	$\mathbf{1}+3(\mathbf{3})+2(\mathbf{5})$
$[2,0,0]$	=	$\mathbf{20'}$	→	$2(\mathbf{1})+\mathbf{3}+3(\mathbf{5})$
$[0,0,3]$	=	$\mathbf{20''}$	→	$2(\mathbf{1})+2(\mathbf{3})+\mathbf{5}+\mathbf{7}$
$[0,4,0]$	=	$\mathbf{35}$	→	$3(\mathbf{1})+2(\mathbf{3})+2(\mathbf{5})+\mathbf{7}+\mathbf{9}$
$[0,2,1]$	=	$\mathbf{36}$	→	$2(\mathbf{1})+4(\mathbf{3})+3(\mathbf{5})+\mathbf{7}$
$[1,2,0]$	=	$\mathbf{45}$	→	$\mathbf{1}+5(\mathbf{3})+3(\mathbf{5})+2(\mathbf{7})$
$[3,0,0]$	=	$\mathbf{50}$	→	$4(\mathbf{3})+2(\mathbf{5})+4(\mathbf{7})$
$[2,1,0]$	=	$\mathbf{60}$	→	$2(\mathbf{1})+4(\mathbf{3})+5(\mathbf{5})+3(\mathbf{7})$
$[1,1,1]$	=	$\mathbf{64}$	→	$2(\mathbf{1})+6(\mathbf{3})+6(\mathbf{5})+2(\mathbf{7})$
$[0,3,1]$	=	$\mathbf{70}$	→	$2(\mathbf{1})+6(\mathbf{3})+4(\mathbf{5})+3(\mathbf{7})+\mathbf{9}$
$[0,2,2]$	=	$\mathbf{84}$	→	$4(\mathbf{1})+5(\mathbf{3})+7(\mathbf{5})+3(\mathbf{7})+\mathbf{9}$
$[1,3,0]$	=	$\mathbf{84'}$	→	$2(\mathbf{1})+6(\mathbf{3})+5(\mathbf{5})+3(\mathbf{7})+2(\mathbf{9})$
$[4,0,0]$	=	$\mathbf{105}$	→	$3(\mathbf{1})+2(\mathbf{3})+6(\mathbf{5})+3(\mathbf{7})+5(\mathbf{9})$
$[0,1,4]$	=	$\mathbf{120}$	→	$3(\mathbf{1})+7(\mathbf{3})+6(\mathbf{5})+4(\mathbf{7})+3(\mathbf{9})+\mathbf{11}$
$[2,2,0]$	=	$\mathbf{126}$	→	$4(\mathbf{1})+5(\mathbf{3})+9(\mathbf{5})+5(\mathbf{7})+3(\mathbf{9})$
$[1,1,2]$	=	$\mathbf{140}$	→	$3(\mathbf{1})+9(\mathbf{3})+10(\mathbf{5})+6(\mathbf{7})+2(\mathbf{9})$
$[3,0,1]$	=	$\mathbf{140'}$	→	$2(\mathbf{1})+6(\mathbf{3})+7(\mathbf{5})+7(\mathbf{7})+4(\mathbf{9})$
$[1,4,0]$	=	$\mathbf{140''}$	→	$2(\mathbf{1})+8(\mathbf{3})+6(\mathbf{5})+5(\mathbf{7})+3(\mathbf{9})+2(\mathbf{11})$
$[0,3,2]$	=	$\mathbf{160}$	→	$4(\mathbf{1})+8(\mathbf{3})+9(\mathbf{5})+7(\mathbf{7})+3(\mathbf{9})+\mathbf{11}$
$[2,1,1]$	=	$\mathbf{175}$	→	$3(\mathbf{1})+9(\mathbf{3})+11(\mathbf{5})+9(\mathbf{7})+3(\mathbf{9})$

E.2 SU(4) → SU(3) × U(1)

Table E.3: Some representation branching rules under the regular subalgebra embedding $SU(3) \times U(1) \subset SU(4)$

		SU(4)	→	SU(3) × U(1)
$[1,0,0]$	=	$\mathbf{4}$	→	$\mathbf{1}_{-3}+\mathbf{3}_1$
$[0,1,0]$	=	$\mathbf{6}$	→	$\mathbf{3}_{-2}+\overline{\mathbf{3}}_2$

Table E.3: Some representation branching rules under the regular subalgebra embedding SU(3) × U(1) ⊂ SU(4) (continued)

		SU(4)	→	SU(3) × U(1)
[2, 0, 0]	=	**10**	→	$\mathbf{1}_{-6} + \mathbf{3}_{-2} + \mathbf{6}_{2}$
[1, 0, 1]	=	**15**	→	$\mathbf{1}_{0} + \mathbf{3}_{4} + \overline{\mathbf{3}}_{-4} + \mathbf{8}_{0}$
[0, 1, 1]	=	**20**	→	$\mathbf{3}_{1} + \overline{\mathbf{3}}_{5} + \overline{\mathbf{6}}_{1} + \mathbf{8}_{-3}$
[0, 2, 0]	=	**20′**	→	$\overline{\mathbf{6}}_{4} + \mathbf{6}_{-4} + \mathbf{8}_{0}$
[2, 0, 1]	=	**36**	→	$\mathbf{1}_{-3} + \mathbf{3}_{1} + \overline{\mathbf{3}}_{-7} + \mathbf{6}_{5} + \mathbf{8}_{-3} + \mathbf{15}_{1}$
[2, 1, 0]	=	**45**	→	$\mathbf{3}_{-8} + \overline{\mathbf{3}}_{-4} + \mathbf{6}_{-4} + \mathbf{8}_{0} + \mathbf{10}_{0} + \mathbf{15}_{4}$
[0, 3, 0]	=	**50**	→	$\mathbf{10}_{-6} + \overline{\mathbf{10}}_{6} + \mathbf{15}_{-2} + \overline{\mathbf{15}}_{2}$
[1, 2, 0]	=	**60**	→	$\overline{\mathbf{6}}_{1} + \mathbf{6}_{-7} + \mathbf{8}_{-3} + \mathbf{10}_{-3} + \mathbf{15}_{1} + \overline{\mathbf{15}}_{5}$
[1, 1, 1]	=	**64**	→	$\mathbf{3}_{-2} + \overline{\mathbf{3}}_{2} + \overline{\mathbf{6}}_{-2} + \mathbf{6}_{2} + \mathbf{8}_{6} + \mathbf{8}_{-6} + \mathbf{15}_{-2} + \overline{\mathbf{15}}_{2}$
[2, 0, 2]	=	**84**	→	$\mathbf{1}_{0} + \mathbf{3}_{4} + \overline{\mathbf{3}}_{-4} + \overline{\mathbf{6}}_{-8} + \mathbf{6}_{8} + \mathbf{8}_{0} + \mathbf{15}_{4} + \overline{\mathbf{15}}_{-4} + \mathbf{27}_{0}$
[0, 4, 0]	=	**105**	→	$\mathbf{15}'_{-8} + \overline{\mathbf{15}}'_{8} + \mathbf{24}_{4} + \overline{\mathbf{24}}_{-4} + \mathbf{27}_{0}$
[2, 2, 0]	=	**126**	→	$\overline{\mathbf{6}}_{-2} + \mathbf{6}_{-10} + \mathbf{8}_{-6} + \mathbf{10}_{-6} + \mathbf{15}_{-2} + \overline{\mathbf{15}}_{2} + \mathbf{15}'_{-2} + \overline{\mathbf{24}}_{2} + \mathbf{27}_{6}$
[1, 1, 2]	=	**140**	→	$\mathbf{3}_{1} + \overline{\mathbf{3}}_{5} + \overline{\mathbf{6}}_{1} + \mathbf{6}_{5} + \mathbf{8}_{9} + \mathbf{8}_{-3} + \overline{\mathbf{10}}_{-3} + \mathbf{15}_{1} + \overline{\mathbf{15}}_{5} + \overline{\mathbf{15}}_{-7} + \mathbf{24}_{1} + \mathbf{27}_{-3}$
[0, 3, 1]	=	**140′**	→	$\mathbf{10}_{-3} + \overline{\mathbf{10}}_{9} + \mathbf{15}_{1} + \overline{\mathbf{15}}_{5} + \overline{\mathbf{15}}'_{5} + \mathbf{24}_{1} + \overline{\mathbf{24}}_{-7} + \mathbf{27}_{-3}$
[1, 2, 1]	=	**175**	→	$\overline{\mathbf{6}}_{4} + \mathbf{6}_{-4} + \mathbf{8}_{0} + \mathbf{10}_{0} + \overline{\mathbf{10}}_{0} + \mathbf{15}_{4} + \mathbf{15}_{-8} + \overline{\mathbf{15}}_{8} + \overline{\mathbf{15}}_{-4} + \mathbf{24}_{4} + \overline{\mathbf{24}}_{-4} + \mathbf{27}_{0}$
[0, 5, 0]	=	**196**	→	$\mathbf{21}_{10} + \overline{\mathbf{21}}_{-10} + \mathbf{35}_{-6} + \overline{\mathbf{35}}_{6} + \mathbf{42}_{-2} + \overline{\mathbf{42}}_{2}$
[2, 3, 0]	=	**280**	→	$\mathbf{10}_{-12} + \overline{\mathbf{10}}_{0} + \mathbf{15}_{-8} + \overline{\mathbf{15}}_{-4} + \mathbf{15}'_{-8} + \overline{\mathbf{21}}_{-4} + \mathbf{24}_{4} + \overline{\mathbf{24}}_{-4} + \mathbf{27}_{0} + \mathbf{35}_{0} + \mathbf{42}_{4} + \overline{\mathbf{42}}_{8}$
[1, 4, 0]	=	**280′**	→	$\mathbf{15}'_{-11} + \overline{\mathbf{15}}'_{5} + \overline{\mathbf{21}}_{-7} + \mathbf{24}_{1} + \overline{\mathbf{24}}_{-7} + \mathbf{27}_{-3} + \mathbf{35}_{-3} + \overline{\mathbf{35}}_{9} + \mathbf{42}_{1} + \overline{\mathbf{42}}_{5}$
[2, 1, 2]	=	**300**	→	$\mathbf{3}_{-2} + \overline{\mathbf{3}}_{2} + \overline{\mathbf{6}}_{-2} + \mathbf{6}_{2} + \mathbf{8}_{6} + \mathbf{8}_{-6} + \mathbf{10}_{6} + \overline{\mathbf{10}}_{-6} + \mathbf{15}_{10} + \mathbf{15}_{-2} + \overline{\mathbf{15}}_{2} + \overline{\mathbf{15}}_{-10} + \mathbf{24}_{-2} + \overline{\mathbf{24}}_{2} + \mathbf{27}_{6} + \mathbf{27}_{-6} + \mathbf{42}_{-2} + \overline{\mathbf{42}}_{2}$
[2, 2, 1]	=	**360**	→	$\overline{\mathbf{6}}_{1} + \mathbf{6}_{-7} + \mathbf{8}_{-3} + \mathbf{10}_{-3} + \overline{\mathbf{10}}_{-3} + \mathbf{15}_{1} + \mathbf{15}_{-11} + \overline{\mathbf{15}}_{5} + \overline{\mathbf{15}}_{-7} + \mathbf{15}'_{1} + \mathbf{24}_{1} + \overline{\mathbf{24}}_{5} + \overline{\mathbf{24}}_{-7} + \mathbf{27}_{9} + \mathbf{27}_{-3} + \mathbf{35}_{-3} + \mathbf{42}_{1} + \overline{\mathbf{42}}_{5}$
[1, 3, 1]	=	**384**	→	$\mathbf{10}_{-6} + \overline{\mathbf{10}}_{6} + \mathbf{15}_{-2} + \overline{\mathbf{15}}_{2} + \mathbf{15}'_{-2} + \overline{\mathbf{15}}'_{2} + \mathbf{24}_{10} + \mathbf{24}_{-2} + \overline{\mathbf{24}}_{2} + \overline{\mathbf{24}}_{-10} + \mathbf{27}_{6} + \mathbf{27}_{-6} + \mathbf{35}_{-6} + \overline{\mathbf{35}}_{6} + \mathbf{42}_{-2} + \overline{\mathbf{42}}_{2}$

E.3 SU(4) → SU(2) × SU(2) × U(1) → SU(2) × SU(2)

Table E.4: Some representation branching rules under the regular subalgebra embedding SU(2) × SU(2) × U(1) ⊂ SU(4)

		SU(4)	→	SU(2) × SU(2) × U(1)
[1, 0, 0]	=	**4**	→	$(\mathbf{2},\mathbf{1})_{1} + (\mathbf{1},\mathbf{2})_{-1}$
[0, 1, 0]	=	**6**	→	$(\mathbf{1},\mathbf{1})_{2} + (\mathbf{1},\mathbf{1})_{-2} + (\mathbf{2},\mathbf{2})_{0}$

Table E.4: Some representation branching rules under the regular subalgebra embedding SU(2) × SU(2) × U(1) ⊂ SU(4) (continued)

		SU(4)	→	SU(2) × SU(2) × U(1)
[2, 0, 0]	=	**10**	→	$(\mathbf{2},\mathbf{2})_0+(\mathbf{3},\mathbf{1})_2+(\mathbf{1},\mathbf{3})_{-2}$
[1, 0, 1]	=	**15**	→	$(\mathbf{1},\mathbf{1})_0+(\mathbf{2},\mathbf{2})_2+(\mathbf{2},\mathbf{2})_{-2}+(\mathbf{3},\mathbf{1})_0+(\mathbf{1},\mathbf{3})_0$
[0, 1, 1]	=	**20**	→	$(\mathbf{2},\mathbf{1})_1+(\mathbf{2},\mathbf{1})_{-3}+(\mathbf{1},\mathbf{2})_3+(\mathbf{1},\mathbf{2})_{-1}+(\mathbf{3},\mathbf{2})_{-1}+(\mathbf{2},\mathbf{3})_1$
[0, 2, 0]	=	**20′**	→	$(\mathbf{1},\mathbf{1})_4+(\mathbf{1},\mathbf{1})_0+(\mathbf{1},\mathbf{1})_{-4}+(\mathbf{2},\mathbf{2})_2+(\mathbf{2},\mathbf{2})_{-2}+(\mathbf{3},\mathbf{3})_0$
[2, 0, 1]	=	**36**	→	$(\mathbf{2},\mathbf{1})_1+(\mathbf{1},\mathbf{2})_{-1}+(\mathbf{3},\mathbf{2})_3+(\mathbf{3},\mathbf{2})_{-1}+(\mathbf{2},\mathbf{3})_1+(\mathbf{2},\mathbf{3})_{-3}+(\mathbf{4},\mathbf{1})_1+(\mathbf{1},\mathbf{4})_{-1}$
[2, 1, 0]	=	**45**	→	$(\mathbf{2},\mathbf{2})_2+(\mathbf{2},\mathbf{2})_{-2}+(\mathbf{3},\mathbf{1})_4+(\mathbf{3},\mathbf{1})_0+(\mathbf{1},\mathbf{3})_0+(\mathbf{1},\mathbf{3})_{-4}+(\mathbf{3},\mathbf{3})_0+(\mathbf{4},\mathbf{2})_2+(\mathbf{2},\mathbf{4})_{-2}$
[0, 3, 0]	=	**50**	→	$(\mathbf{1},\mathbf{1})_6+(\mathbf{1},\mathbf{1})_2+(\mathbf{1},\mathbf{1})_{-2}+(\mathbf{1},\mathbf{1})_{-6}+(\mathbf{2},\mathbf{2})_4+(\mathbf{2},\mathbf{2})_0+(\mathbf{2},\mathbf{2})_{-4}+(\mathbf{3},\mathbf{3})_2+(\mathbf{3},\mathbf{3})_{-2}+(\mathbf{4},\mathbf{4})_0$
[1, 2, 0]	=	**60**	→	$(\mathbf{2},\mathbf{1})_5+(\mathbf{2},\mathbf{1})_1+(\mathbf{2},\mathbf{1})_{-3}+(\mathbf{1},\mathbf{2})_3+(\mathbf{1},\mathbf{2})_{-1}+(\mathbf{1},\mathbf{2})_{-5}+(\mathbf{3},\mathbf{2})_3+(\mathbf{3},\mathbf{2})_{-1}+(\mathbf{2},\mathbf{3})_1+(\mathbf{2},\mathbf{3})_{-3}+(\mathbf{4},\mathbf{3})_1+(\mathbf{3},\mathbf{4})_{-1}$
[1, 1, 1]	=	**64**	→	$(\mathbf{1},\mathbf{1})_2+(\mathbf{1},\mathbf{1})_{-2}+(\mathbf{2},\mathbf{2})_4+2(\mathbf{2},\mathbf{2})_0+(\mathbf{2},\mathbf{2})_{-4}+(\mathbf{3},\mathbf{1})_2+(\mathbf{3},\mathbf{1})_{-2}+(\mathbf{1},\mathbf{3})_2+(\mathbf{1},\mathbf{3})_{-2}+(\mathbf{3},\mathbf{3})_2+(\mathbf{3},\mathbf{3})_{-2}+(\mathbf{4},\mathbf{2})_0+(\mathbf{2},\mathbf{4})_0$
[2, 0, 2]	=	**84**	→	$(\mathbf{1},\mathbf{1})_0+(\mathbf{2},\mathbf{2})_2+(\mathbf{2},\mathbf{2})_{-2}+(\mathbf{3},\mathbf{1})_0+(\mathbf{1},\mathbf{3})_0+(\mathbf{3},\mathbf{3})_4+(\mathbf{3},\mathbf{3})_0+(\mathbf{3},\mathbf{3})_{-4}+(\mathbf{4},\mathbf{2})_2+(\mathbf{4},\mathbf{2})_{-2}+(\mathbf{2},\mathbf{4})_2+(\mathbf{2},\mathbf{4})_{-2}+(\mathbf{5},\mathbf{1})_0+(\mathbf{1},\mathbf{5})_0$
[0, 4, 0]	=	**105**	→	$(\mathbf{1},\mathbf{1})_8+(\mathbf{1},\mathbf{1})_4+(\mathbf{1},\mathbf{1})_0+(\mathbf{1},\mathbf{1})_{-4}+(\mathbf{1},\mathbf{1})_{-8}+(\mathbf{2},\mathbf{2})_6+(\mathbf{2},\mathbf{2})_2+(\mathbf{2},\mathbf{2})_{-2}+(\mathbf{2},\mathbf{2})_{-6}+(\mathbf{3},\mathbf{3})_4+(\mathbf{3},\mathbf{3})_0+(\mathbf{3},\mathbf{3})_{-4}+(\mathbf{4},\mathbf{4})_2+(\mathbf{4},\mathbf{4})_{-2}+(\mathbf{5},\mathbf{5})_0$
[2, 2, 0]	=	**126**	→	$(\mathbf{2},\mathbf{2})_4+(\mathbf{2},\mathbf{2})_0+(\mathbf{2},\mathbf{2})_{-4}+(\mathbf{3},\mathbf{1})_6+(\mathbf{3},\mathbf{1})_2+(\mathbf{3},\mathbf{1})_{-2}+(\mathbf{1},\mathbf{3})_2+(\mathbf{1},\mathbf{3})_{-2}+(\mathbf{1},\mathbf{3})_{-6}+(\mathbf{3},\mathbf{3})_2+(\mathbf{3},\mathbf{3})_{-2}+(\mathbf{4},\mathbf{2})_4+(\mathbf{4},\mathbf{2})_0+(\mathbf{2},\mathbf{4})_0+(\mathbf{2},\mathbf{4})_{-4}+(\mathbf{4},\mathbf{4})_0+(\mathbf{5},\mathbf{3})_2+(\mathbf{3},\mathbf{5})_{-2}$
[1, 1, 2]	=	**140**	→	$(\mathbf{2},\mathbf{1})_1+(\mathbf{2},\mathbf{1})_{-3}+(\mathbf{1},\mathbf{2})_3+(\mathbf{1},\mathbf{2})_{-1}+(\mathbf{3},\mathbf{2})_3+2(\mathbf{3},\mathbf{2})_{-1}+(\mathbf{3},\mathbf{2})_{-5}+(\mathbf{2},\mathbf{3})_5+2(\mathbf{2},\mathbf{3})_1+(\mathbf{2},\mathbf{3})_{-3}+(\mathbf{4},\mathbf{1})_1+(\mathbf{4},\mathbf{1})_{-3}+(\mathbf{1},\mathbf{4})_3+(\mathbf{1},\mathbf{4})_{-1}+(\mathbf{4},\mathbf{3})_1+(\mathbf{4},\mathbf{3})_{-3}+(\mathbf{3},\mathbf{4})_3+(\mathbf{3},\mathbf{4})_{-1}+(\mathbf{5},\mathbf{2})_{-1}+(\mathbf{2},\mathbf{5})_1$
[0, 3, 1]	=	**140′**	→	$(\mathbf{2},\mathbf{1})_5+(\mathbf{2},\mathbf{1})_1+(\mathbf{2},\mathbf{1})_{-3}+(\mathbf{2},\mathbf{1})_{-7}+(\mathbf{1},\mathbf{2})_7+(\mathbf{1},\mathbf{2})_3+(\mathbf{1},\mathbf{2})_{-1}+(\mathbf{1},\mathbf{2})_{-5}+(\mathbf{3},\mathbf{2})_3+(\mathbf{3},\mathbf{2})_{-1}+(\mathbf{3},\mathbf{2})_{-5}+(\mathbf{2},\mathbf{3})_5+(\mathbf{2},\mathbf{3})_1+(\mathbf{2},\mathbf{3})_{-3}+(\mathbf{4},\mathbf{3})_1+(\mathbf{4},\mathbf{3})_{-3}+(\mathbf{3},\mathbf{4})_3+(\mathbf{3},\mathbf{4})_{-1}+(\mathbf{5},\mathbf{4})_{-1}+(\mathbf{4},\mathbf{5})_1$
[1, 2, 1]	=	**175**	→	$(\mathbf{1},\mathbf{1})_4+(\mathbf{1},\mathbf{1})_0+(\mathbf{1},\mathbf{1})_{-4}+(\mathbf{2},\mathbf{2})_6+2(\mathbf{2},\mathbf{2})_2+2(\mathbf{2},\mathbf{2})_{-2}+(\mathbf{2},\mathbf{2})_{-6}+(\mathbf{3},\mathbf{1})_4+(\mathbf{3},\mathbf{1})_0+(\mathbf{3},\mathbf{1})_{-4}+(\mathbf{1},\mathbf{3})_4+(\mathbf{1},\mathbf{3})_0+(\mathbf{1},\mathbf{3})_{-4}+(\mathbf{3},\mathbf{3})_4+2(\mathbf{3},\mathbf{3})_0+(\mathbf{3},\mathbf{3})_{-4}+(\mathbf{4},\mathbf{2})_2+(\mathbf{4},\mathbf{2})_{-2}+(\mathbf{2},\mathbf{4})_2+(\mathbf{2},\mathbf{4})_{-2}+(\mathbf{4},\mathbf{4})_2+(\mathbf{4},\mathbf{4})_{-2}+(\mathbf{5},\mathbf{3})_0+(\mathbf{3},\mathbf{5})_0$

Table E.4: Some representation branching rules under the regular subalgebra embedding SU(2) × SU(2) × U(1) ⊂ SU(4) (continued)

		SU(4)	→	SU(2) × SU(2) × U(1)
[0, 5, 0]	=	**196**	→	$(\mathbf{1},\mathbf{1})_{10}+(\mathbf{1},\mathbf{1})_{6}+(\mathbf{1},\mathbf{1})_{2}+(\mathbf{1},\mathbf{1})_{-2}+(\mathbf{1},\mathbf{1})_{-6}+(\mathbf{1},\mathbf{1})_{-10}+(\mathbf{2},\mathbf{2})_{8}+(\mathbf{2},\mathbf{2})_{4}+(\mathbf{2},\mathbf{2})_{0}+(\mathbf{2},\mathbf{2})_{-4}+(\mathbf{2},\mathbf{2})_{-8}+(\mathbf{3},\mathbf{3})_{6}+(\mathbf{3},\mathbf{3})_{2}+(\mathbf{3},\mathbf{3})_{-2}+(\mathbf{3},\mathbf{3})_{-6}+(\mathbf{4},\mathbf{4})_{4}+(\mathbf{4},\mathbf{4})_{0}+(\mathbf{4},\mathbf{4})_{-4}+(\mathbf{5},\mathbf{5})_{2}+(\mathbf{5},\mathbf{5})_{-2}+(\mathbf{6},\mathbf{6})_{0}$
[2, 3, 0]	=	**280**	→	$(\mathbf{2},\mathbf{2})_{6}+(\mathbf{2},\mathbf{2})_{2}+(\mathbf{2},\mathbf{2})_{-2}+(\mathbf{2},\mathbf{2})_{-6}+(\mathbf{3},\mathbf{1})_{8}+(\mathbf{3},\mathbf{1})_{4}+(\mathbf{3},\mathbf{1})_{0}+(\mathbf{3},\mathbf{1})_{-4}+(\mathbf{1},\mathbf{3})_{4}+(\mathbf{1},\mathbf{3})_{0}+(\mathbf{1},\mathbf{3})_{-4}+(\mathbf{1},\mathbf{3})_{-8}+(\mathbf{3},\mathbf{3})_{4}+(\mathbf{3},\mathbf{3})_{0}+(\mathbf{3},\mathbf{3})_{-4}+(\mathbf{4},\mathbf{2})_{6}+(\mathbf{4},\mathbf{2})_{2}+(\mathbf{4},\mathbf{2})_{-2}+(\mathbf{2},\mathbf{4})_{2}+(\mathbf{2},\mathbf{4})_{-2}+(\mathbf{2},\mathbf{4})_{-6}+(\mathbf{4},\mathbf{4})_{2}+(\mathbf{4},\mathbf{4})_{-2}+(\mathbf{5},\mathbf{3})_{4}+(\mathbf{5},\mathbf{3})_{0}+(\mathbf{3},\mathbf{5})_{0}+(\mathbf{3},\mathbf{5})_{-4}+(\mathbf{5},\mathbf{5})_{0}+(\mathbf{6},\mathbf{4})_{2}+(\mathbf{4},\mathbf{6})_{-2}$
[1, 4, 0]	=	**280′**	→	$(\mathbf{2},\mathbf{1})_{9}+(\mathbf{2},\mathbf{1})_{5}+(\mathbf{2},\mathbf{1})_{1}+(\mathbf{2},\mathbf{1})_{-3}+(\mathbf{2},\mathbf{1})_{-7}+(\mathbf{1},\mathbf{2})_{7}+(\mathbf{1},\mathbf{2})_{3}+(\mathbf{1},\mathbf{2})_{-1}+(\mathbf{1},\mathbf{2})_{-5}+(\mathbf{1},\mathbf{2})_{-9}+(\mathbf{3},\mathbf{2})_{7}+(\mathbf{3},\mathbf{2})_{3}+(\mathbf{3},\mathbf{2})_{-1}+(\mathbf{3},\mathbf{2})_{-5}+(\mathbf{2},\mathbf{3})_{5}+(\mathbf{2},\mathbf{3})_{1}+(\mathbf{2},\mathbf{3})_{-3}+(\mathbf{2},\mathbf{3})_{-7}+(\mathbf{4},\mathbf{3})_{5}+(\mathbf{4},\mathbf{3})_{1}+(\mathbf{4},\mathbf{3})_{-3}+(\mathbf{3},\mathbf{4})_{3}+(\mathbf{3},\mathbf{4})_{-1}+(\mathbf{3},\mathbf{4})_{-5}+(\mathbf{5},\mathbf{4})_{3}+(\mathbf{5},\mathbf{4})_{-1}+(\mathbf{4},\mathbf{5})_{1}+(\mathbf{4},\mathbf{5})_{-3}+(\mathbf{6},\mathbf{5})_{1}+(\mathbf{5},\mathbf{6})_{-1}$
[2, 1, 2]	=	**300**	→	$(\mathbf{1},\mathbf{1})_{2}+(\mathbf{1},\mathbf{1})_{-2}+(\mathbf{2},\mathbf{2})_{4}+2(\mathbf{2},\mathbf{2})_{0}+(\mathbf{2},\mathbf{2})_{-4}+(\mathbf{3},\mathbf{1})_{2}+(\mathbf{3},\mathbf{1})_{-2}+(\mathbf{1},\mathbf{3})_{2}+(\mathbf{1},\mathbf{3})_{-2}+(\mathbf{3},\mathbf{3})_{6}+2(\mathbf{3},\mathbf{3})_{2}+2(\mathbf{3},\mathbf{3})_{-2}+(\mathbf{3},\mathbf{3})_{-6}+(\mathbf{4},\mathbf{2})_{4}+2(\mathbf{4},\mathbf{2})_{0}+(\mathbf{4},\mathbf{2})_{-4}+(\mathbf{2},\mathbf{4})_{4}+2(\mathbf{2},\mathbf{4})_{0}+(\mathbf{2},\mathbf{4})_{-4}+(\mathbf{5},\mathbf{1})_{2}+(\mathbf{5},\mathbf{1})_{-2}+(\mathbf{1},\mathbf{5})_{2}+(\mathbf{1},\mathbf{5})_{-2}+(\mathbf{4},\mathbf{4})_{4}+(\mathbf{4},\mathbf{4})_{0}+(\mathbf{4},\mathbf{4})_{-4}+(\mathbf{5},\mathbf{3})_{2}+(\mathbf{5},\mathbf{3})_{-2}+(\mathbf{3},\mathbf{5})_{2}+(\mathbf{3},\mathbf{5})_{-2}+(\mathbf{6},\mathbf{2})_{0}+(\mathbf{2},\mathbf{6})_{0}$
[2, 2, 1]	=	**360**	→	$(\mathbf{2},\mathbf{1})_{5}+(\mathbf{2},\mathbf{1})_{1}+(\mathbf{2},\mathbf{1})_{-3}+(\mathbf{1},\mathbf{2})_{3}+(\mathbf{1},\mathbf{2})_{-1}+(\mathbf{1},\mathbf{2})_{-5}+(\mathbf{3},\mathbf{2})_{7}+2(\mathbf{3},\mathbf{2})_{3}+2(\mathbf{3},\mathbf{2})_{-1}+(\mathbf{3},\mathbf{2})_{-5}+(\mathbf{2},\mathbf{3})_{5}+2(\mathbf{2},\mathbf{3})_{1}+2(\mathbf{2},\mathbf{3})_{-3}+(\mathbf{2},\mathbf{3})_{-7}+(\mathbf{4},\mathbf{1})_{5}+(\mathbf{4},\mathbf{1})_{1}+(\mathbf{4},\mathbf{1})_{-3}+(\mathbf{1},\mathbf{4})_{3}+(\mathbf{1},\mathbf{4})_{-1}+(\mathbf{1},\mathbf{4})_{-5}+(\mathbf{4},\mathbf{3})_{5}+2(\mathbf{4},\mathbf{3})_{1}+(\mathbf{4},\mathbf{3})_{-3}+(\mathbf{3},\mathbf{4})_{3}+2(\mathbf{3},\mathbf{4})_{-1}+(\mathbf{3},\mathbf{4})_{-5}+(\mathbf{5},\mathbf{2})_{3}+(\mathbf{5},\mathbf{2})_{-1}+(\mathbf{2},\mathbf{5})_{1}+(\mathbf{2},\mathbf{5})_{-3}+(\mathbf{5},\mathbf{4})_{3}+(\mathbf{5},\mathbf{4})_{-1}+(\mathbf{4},\mathbf{5})_{1}+(\mathbf{4},\mathbf{5})_{-3}+(\mathbf{6},\mathbf{3})_{1}+(\mathbf{3},\mathbf{6})_{-1}$
[1, 3, 1]	=	**384**	→	$(\mathbf{1},\mathbf{1})_{6}+(\mathbf{1},\mathbf{1})_{2}+(\mathbf{1},\mathbf{1})_{-2}+(\mathbf{1},\mathbf{1})_{-6}+(\mathbf{2},\mathbf{2})_{8}+2(\mathbf{2},\mathbf{2})_{4}+2(\mathbf{2},\mathbf{2})_{0}+2(\mathbf{2},\mathbf{2})_{-4}+(\mathbf{2},\mathbf{2})_{-8}+(\mathbf{3},\mathbf{1})_{6}+(\mathbf{3},\mathbf{1})_{2}+(\mathbf{3},\mathbf{1})_{-2}+(\mathbf{3},\mathbf{1})_{-6}+(\mathbf{1},\mathbf{3})_{6}+(\mathbf{1},\mathbf{3})_{2}+(\mathbf{1},\mathbf{3})_{-2}+(\mathbf{1},\mathbf{3})_{-6}+(\mathbf{3},\mathbf{3})_{6}+2(\mathbf{3},\mathbf{3})_{2}+2(\mathbf{3},\mathbf{3})_{-2}+(\mathbf{3},\mathbf{3})_{-6}+(\mathbf{4},\mathbf{2})_{4}+(\mathbf{4},\mathbf{2})_{0}+(\mathbf{4},\mathbf{2})_{-4}+(\mathbf{2},\mathbf{4})_{4}+(\mathbf{2},\mathbf{4})_{0}+(\mathbf{2},\mathbf{4})_{-4}+(\mathbf{4},\mathbf{4})_{4}+2(\mathbf{4},\mathbf{4})_{0}+(\mathbf{4},\mathbf{4})_{-4}+(\mathbf{5},\mathbf{3})_{2}+(\mathbf{5},\mathbf{3})_{-2}+(\mathbf{3},\mathbf{5})_{2}+(\mathbf{3},\mathbf{5})_{-2}+(\mathbf{5},\mathbf{5})_{2}+(\mathbf{5},\mathbf{5})_{-2}+(\mathbf{6},\mathbf{4})_{0}+(\mathbf{4},\mathbf{6})_{0}$

The branchings under SU(2) × SU(2) ⊂ SU(4) can be obtained by removing the U(1) charge from the branchings under SU(2) × SU(2) × U(1) ⊂ SU(4) in the table above.

Table E.5: Some representation branching rules under the regular subalgebra embedding SU(2) × SU(2) ⊂ SU(4)

		SU(4)	→	SU(2) × SU(2)
$[1,0,0]$	$=$	$\mathbf{4}$	$\rightarrow$	$(\mathbf{2},\mathbf{1})+(\mathbf{1},\mathbf{2})$
$[0,1,0]$	$=$	$\mathbf{6}$	$\rightarrow$	$2(\mathbf{1},\mathbf{1})+(\mathbf{2},\mathbf{2})$
$[2,0,0]$	$=$	$\mathbf{10}$	$\rightarrow$	$(\mathbf{2},\mathbf{2})+(\mathbf{3},\mathbf{1})+(\mathbf{1},\mathbf{3})$
$[1,0,1]$	$=$	$\mathbf{15}$	$\rightarrow$	$(\mathbf{1},\mathbf{1})+2(\mathbf{2},\mathbf{2})+(\mathbf{3},\mathbf{1})+(\mathbf{1},\mathbf{3})$
$[0,1,1]$	$=$	$\mathbf{20}$	$\rightarrow$	$2(\mathbf{2},\mathbf{1})+2(\mathbf{1},\mathbf{2})+(\mathbf{3},\mathbf{2})+(\mathbf{2},\mathbf{3})$
$[0,2,0]$	$=$	$\mathbf{20'}$	$\rightarrow$	$3(\mathbf{1},\mathbf{1})+2(\mathbf{2},\mathbf{2})+(\mathbf{3},\mathbf{3})$
$[2,0,1]$	$=$	$\mathbf{36}$	$\rightarrow$	$(\mathbf{2},\mathbf{1})+(\mathbf{1},\mathbf{2})+2(\mathbf{3},\mathbf{2})+2(\mathbf{2},\mathbf{3})+(\mathbf{4},\mathbf{1})+(\mathbf{1},\mathbf{4})$
$[2,1,0]$	$=$	$\mathbf{45}$	$\rightarrow$	$2(\mathbf{2},\mathbf{2})+2(\mathbf{3},\mathbf{1})+2(\mathbf{1},\mathbf{3})+(\mathbf{3},\mathbf{3})+(\mathbf{4},\mathbf{2})+(\mathbf{2},\mathbf{4})$
$[0,3,0]$	$=$	$\mathbf{50}$	$\rightarrow$	$4(\mathbf{1},\mathbf{1})+3(\mathbf{2},\mathbf{2})+2(\mathbf{3},\mathbf{3})+(\mathbf{4},\mathbf{4})$
$[1,2,0]$	$=$	$\mathbf{60}$	$\rightarrow$	$3(\mathbf{2},\mathbf{1})+3(\mathbf{1},\mathbf{2})+2(\mathbf{3},\mathbf{2})+2(\mathbf{2},\mathbf{3})+(\mathbf{4},\mathbf{3})+(\mathbf{3},\mathbf{4})$
$[1,1,1]$	$=$	$\mathbf{64}$	$\rightarrow$	$2(\mathbf{1},\mathbf{1})+4(\mathbf{2},\mathbf{2})+2(\mathbf{3},\mathbf{1})+2(\mathbf{1},\mathbf{3})+2(\mathbf{3},\mathbf{3})+(\mathbf{4},\mathbf{2})+(\mathbf{2},\mathbf{4})$
$[2,0,2]$	$=$	$\mathbf{84}$	$\rightarrow$	$(\mathbf{1},\mathbf{1})+2(\mathbf{2},\mathbf{2})+(\mathbf{3},\mathbf{1})+(\mathbf{1},\mathbf{3})+3(\mathbf{3},\mathbf{3})+2(\mathbf{4},\mathbf{2})+2(\mathbf{2},\mathbf{4})+(\mathbf{5},\mathbf{1})+(\mathbf{1},\mathbf{5})$
$[0,4,0]$	$=$	$\mathbf{105}$	$\rightarrow$	$5(\mathbf{1},\mathbf{1})+4(\mathbf{2},\mathbf{2})+3(\mathbf{3},\mathbf{3})+2(\mathbf{4},\mathbf{4})+(\mathbf{5},\mathbf{5})$
$[2,2,0]$	$=$	$\mathbf{126}$	$\rightarrow$	$3(\mathbf{2},\mathbf{2})+3(\mathbf{3},\mathbf{1})+3(\mathbf{1},\mathbf{3})+2(\mathbf{3},\mathbf{3})+2(\mathbf{4},\mathbf{2})+2(\mathbf{2},\mathbf{4})+(\mathbf{4},\mathbf{4})+(\mathbf{5},\mathbf{3})+(\mathbf{3},\mathbf{5})$
$[1,1,2]$	$=$	$\mathbf{140}$	$\rightarrow$	$2(\mathbf{2},\mathbf{1})+2(\mathbf{1},\mathbf{2})+4(\mathbf{3},\mathbf{2})+4(\mathbf{2},\mathbf{3})+2(\mathbf{4},\mathbf{1})+2(\mathbf{1},\mathbf{4})+2(\mathbf{4},\mathbf{3})+2(\mathbf{3},\mathbf{4})+(\mathbf{5},\mathbf{2})+(\mathbf{2},\mathbf{5})$
$[0,3,1]$	$=$	$\mathbf{140'}$	$\rightarrow$	$4(\mathbf{2},\mathbf{1})+4(\mathbf{1},\mathbf{2})+3(\mathbf{3},\mathbf{2})+3(\mathbf{2},\mathbf{3})+2(\mathbf{4},\mathbf{3})+2(\mathbf{3},\mathbf{4})+(\mathbf{5},\mathbf{4})+(\mathbf{4},\mathbf{5})$
$[1,2,1]$	$=$	$\mathbf{175}$	$\rightarrow$	$3(\mathbf{1},\mathbf{1})+6(\mathbf{2},\mathbf{2})+3(\mathbf{3},\mathbf{1})+3(\mathbf{1},\mathbf{3})+4(\mathbf{3},\mathbf{3})+2(\mathbf{4},\mathbf{2})+2(\mathbf{2},\mathbf{4})+2(\mathbf{4},\mathbf{4})+(\mathbf{5},\mathbf{3})+(\mathbf{3},\mathbf{5})$
$[0,5,0]$	$=$	$\mathbf{196}$	$\rightarrow$	$6(\mathbf{1},\mathbf{1})+5(\mathbf{2},\mathbf{2})+4(\mathbf{3},\mathbf{3})+3(\mathbf{4},\mathbf{4})+2(\mathbf{5},\mathbf{5})+(\mathbf{6},\mathbf{6})$
$[2,3,0]$	$=$	$\mathbf{280}$	$\rightarrow$	$4(\mathbf{2},\mathbf{2})+4(\mathbf{3},\mathbf{1})+4(\mathbf{1},\mathbf{3})+3(\mathbf{3},\mathbf{3})+3(\mathbf{4},\mathbf{2})+3(\mathbf{2},\mathbf{4})+2(\mathbf{4},\mathbf{4})+2(\mathbf{5},\mathbf{3})+2(\mathbf{3},\mathbf{5})+(\mathbf{5},\mathbf{5})+(\mathbf{6},\mathbf{4})+(\mathbf{4},\mathbf{6})$
$[1,4,0]$	$=$	$\mathbf{280'}$	$\rightarrow$	$5(\mathbf{2},\mathbf{1})+5(\mathbf{1},\mathbf{2})+4(\mathbf{3},\mathbf{2})+4(\mathbf{2},\mathbf{3})+3(\mathbf{4},\mathbf{3})+3(\mathbf{3},\mathbf{4})+2(\mathbf{5},\mathbf{4})+2(\mathbf{4},\mathbf{5})+(\mathbf{6},\mathbf{5})+(\mathbf{5},\mathbf{6})$
$[2,1,2]$	$=$	$\mathbf{300}$	$\rightarrow$	$2(\mathbf{1},\mathbf{1})+4(\mathbf{2},\mathbf{2})+2(\mathbf{3},\mathbf{1})+2(\mathbf{1},\mathbf{3})+6(\mathbf{3},\mathbf{3})+4(\mathbf{4},\mathbf{2})+4(\mathbf{2},\mathbf{4})+2(\mathbf{5},\mathbf{1})+2(\mathbf{1},\mathbf{5})+3(\mathbf{4},\mathbf{4})+2(\mathbf{5},\mathbf{3})+2(\mathbf{3},\mathbf{5})+(\mathbf{6},\mathbf{2})+(\mathbf{2},\mathbf{6})$
$[2,2,1]$	$=$	$\mathbf{360}$	$\rightarrow$	$3(\mathbf{2},\mathbf{1})+3(\mathbf{1},\mathbf{2})+6(\mathbf{3},\mathbf{2})+6(\mathbf{2},\mathbf{3})+3(\mathbf{4},\mathbf{1})+3(\mathbf{1},\mathbf{4})+4(\mathbf{4},\mathbf{3})+4(\mathbf{3},\mathbf{4})+2(\mathbf{5},\mathbf{2})+2(\mathbf{2},\mathbf{5})+2(\mathbf{5},\mathbf{4})+2(\mathbf{4},\mathbf{5})+(\mathbf{6},\mathbf{3})+(\mathbf{3},\mathbf{6})$
$[1,3,1]$	$=$	$\mathbf{384}$	$\rightarrow$	$4(\mathbf{1},\mathbf{1})+8(\mathbf{2},\mathbf{2})+4(\mathbf{3},\mathbf{1})+4(\mathbf{1},\mathbf{3})+6(\mathbf{3},\mathbf{3})+3(\mathbf{4},\mathbf{2})+3(\mathbf{2},\mathbf{4})+4(\mathbf{4},\mathbf{4})+2(\mathbf{5},\mathbf{3})+2(\mathbf{3},\mathbf{5})+2(\mathbf{5},\mathbf{5})+(\mathbf{6},\mathbf{4})+(\mathbf{4},\mathbf{6})$

Appendix F

Explicit Branching of Short Multiplets

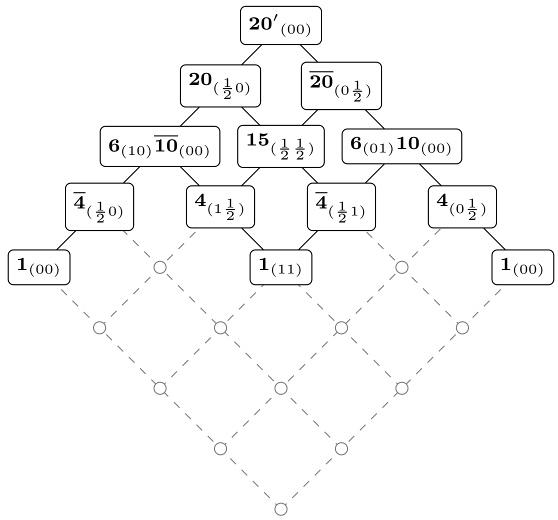
In this appendix we would like to tabulate the explicit fitting of $\mathcal{N}=1$ and $\mathcal{N}=2$ superfields into short $\mathcal{N}=4$ representations for the lowest orders $p \in \{2,3,4,5\}$ in the $\mathcal{N}=1$ case and $p \in \{2,3\}$ in the $\mathcal{N}=2$ case. More details on the way these decompositions were constructed can be found in the main text in Chapters 2 and 7.

F.1 Short Multiplets in the $\mathcal{N}=1$ Description

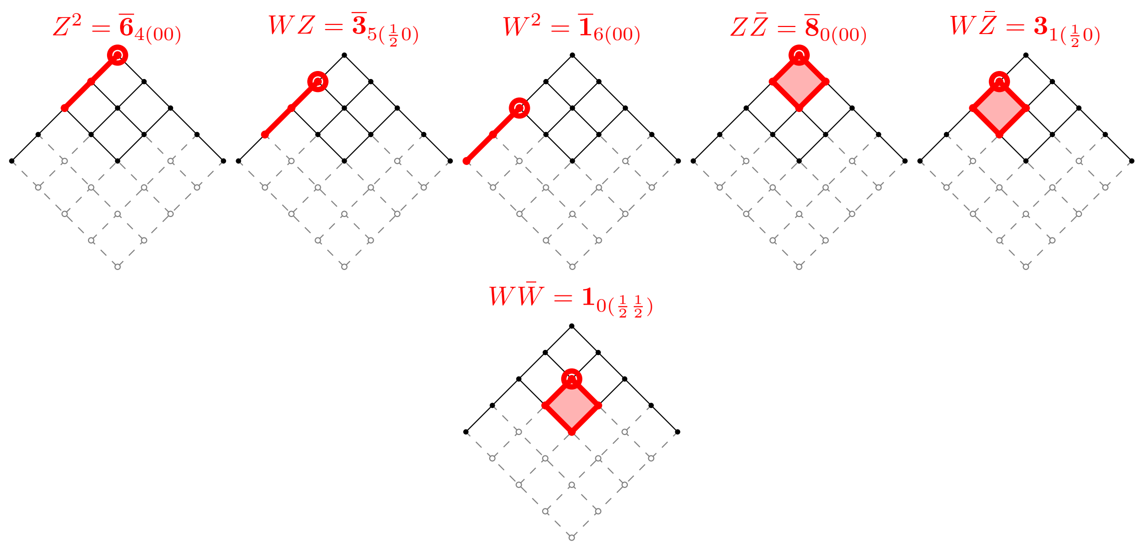
The general rule for the decomposition is that for a given order p short multiplet one should take all possible p-fold products of the field strength superfield W_α, the chiral superfields Z_i, and their conjugates $\overline{W}_{\dot\alpha}$ and $\bar{Z}^i$. The quantum numbers of the composite operators so constructed are computed from the quantum numbers of the superfields, and the representations of the component fields can be deduced by consistency with the transformation properties of the Grassmann coordinates θ.

F.1.1 Case $p=2$

The $p=2$ short multiplet is given by

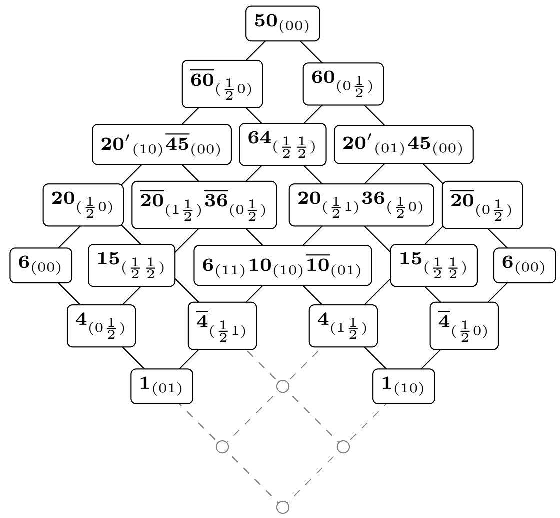


After restriction to an $\mathcal{N}=1$ sub-algebra these representations branch under $\mathrm{SU}(3)\times\mathrm{U}(1)\subset\mathrm{SU}(4)$ as described in Appendix E.2, and can be found in the following $\mathcal{N}=1$ superfields and their conjugates that cover the short $\mathcal{N}=4$ multiplet:

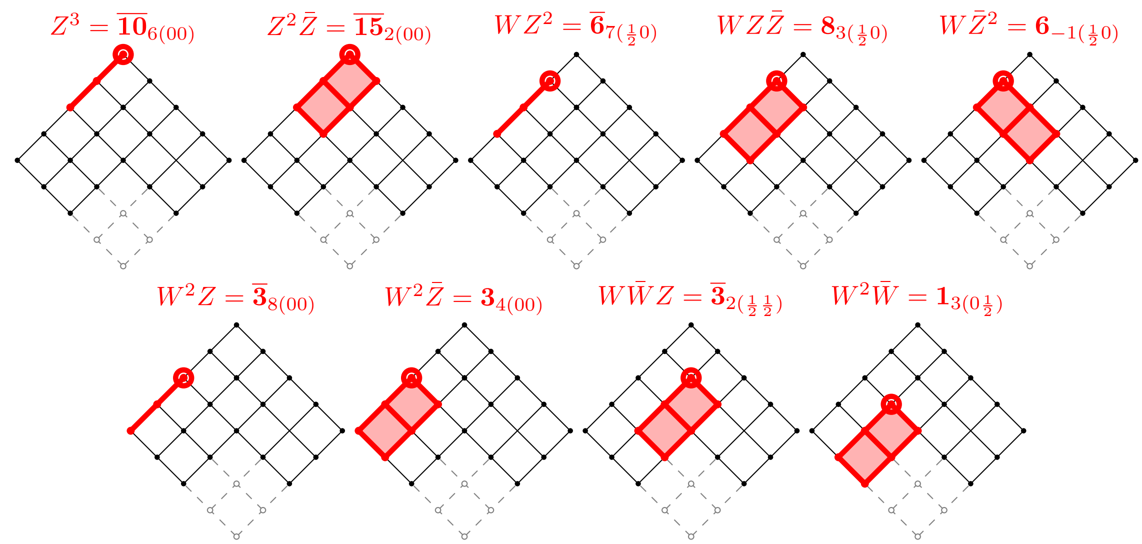


F.1.2 Case $p=3$

The $p=3$ short multiplet is given by

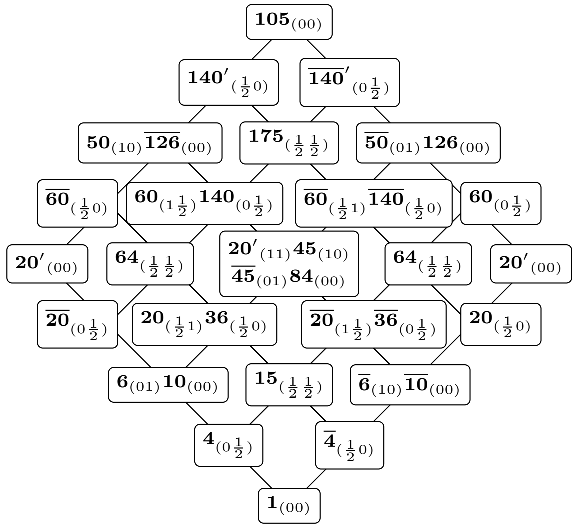


After restriction to an $\mathcal{N}=1$ sub-algebra these representations branch under $\mathrm{SU}(3)\times\mathrm{U}(1)\subset\mathrm{SU}(4)$ as described in Appendix E.2, and can be found in the following $\mathcal{N}=1$ superfields and their conjugates that cover the short $\mathcal{N}=4$ multiplet:

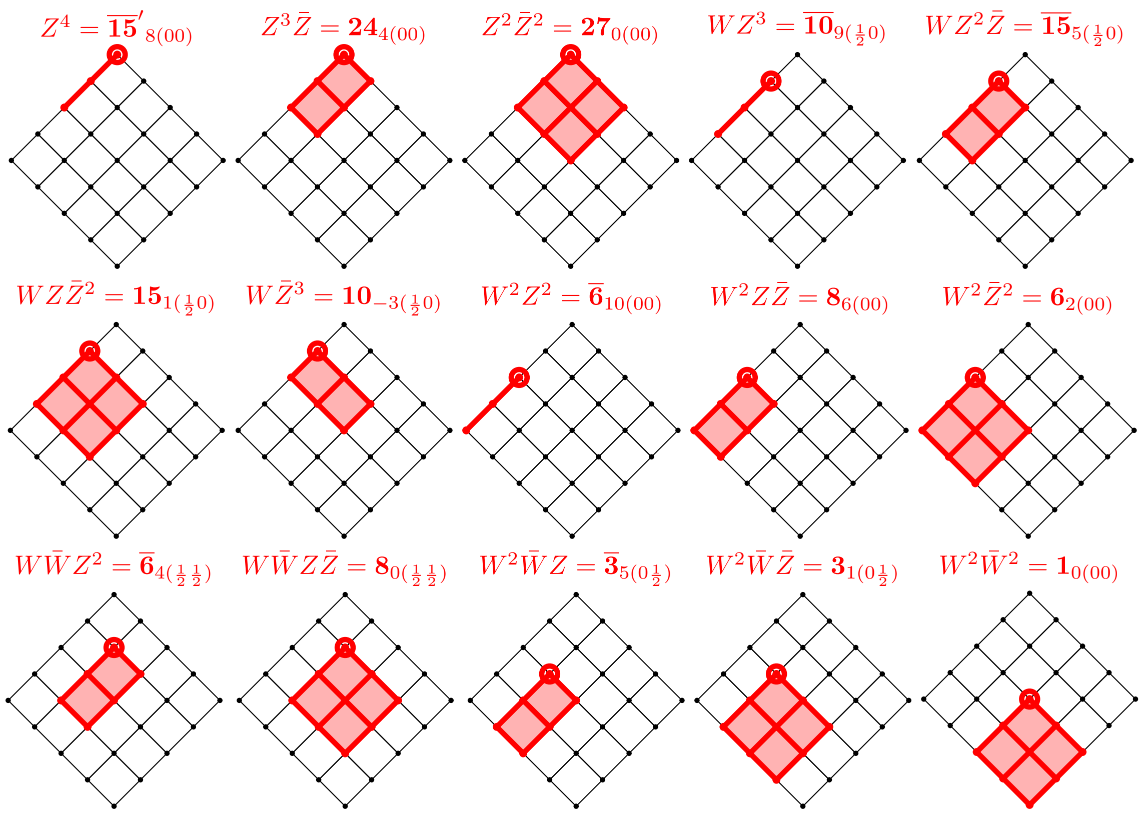


F.1.3 Case $p=4$

The $p=4$ short multiplet is given by

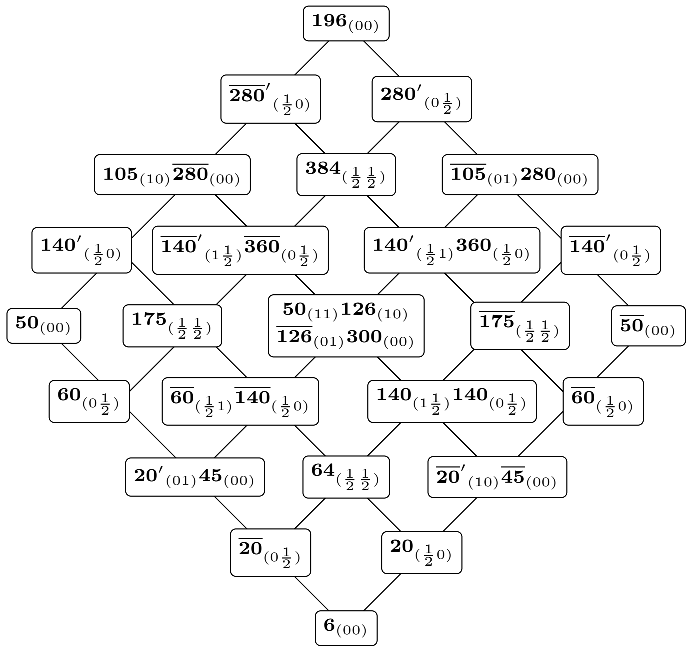


After restriction to an $\mathcal{N}=1$ sub-algebra these representations branch under $\mathrm{SU}(3)\times\mathrm{U}(1)\subset\mathrm{SU}(4)$ as described in Appendix E.2, and can be found in the following $\mathcal{N}=1$ superfields and their conjugates that cover the short $\mathcal{N}=4$ multiplet:

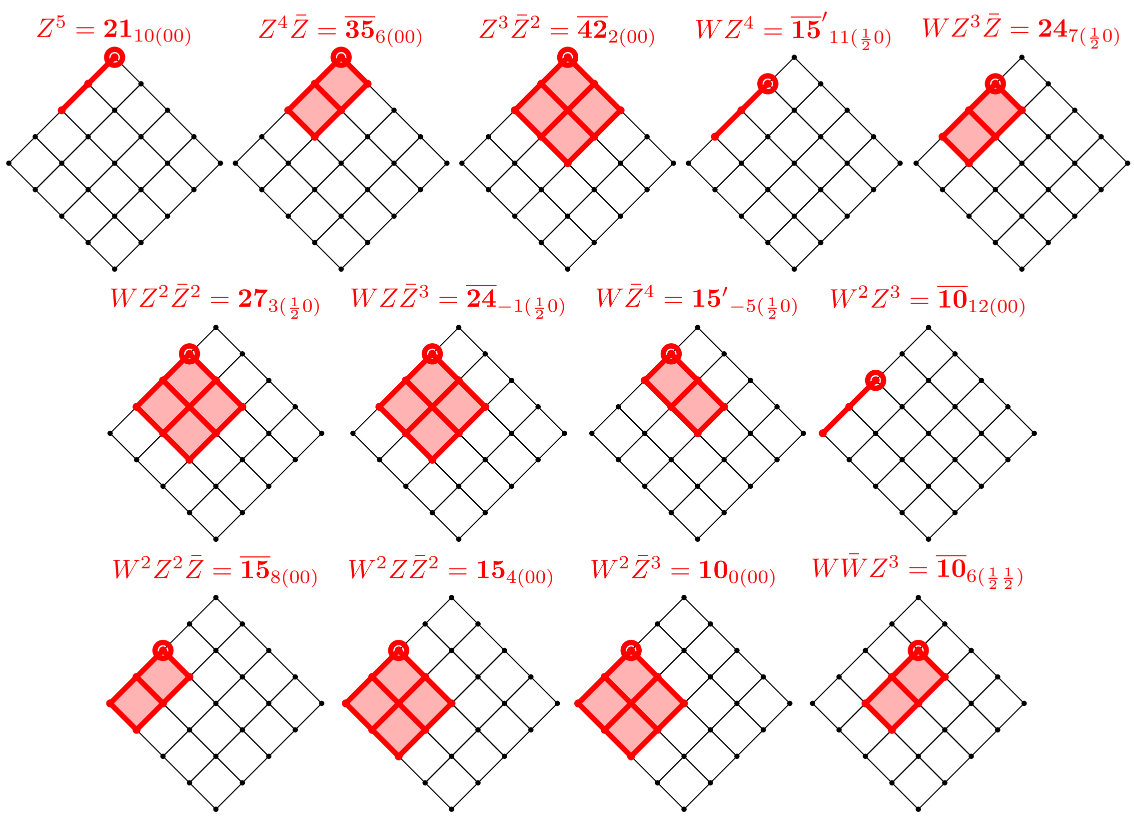


F.1.4 Case $p=5$

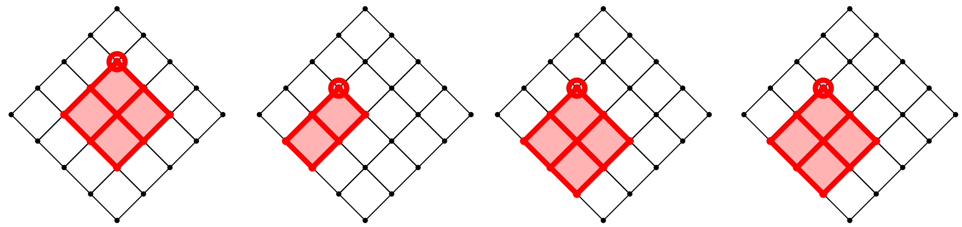
The $p=5$ short multiplet is given by



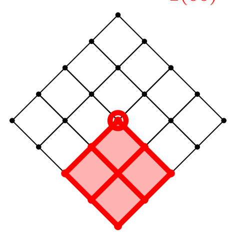
After restriction to an $\mathcal{N} = 1$ sub-algebra these representations branch under $\mathrm{SU}(3) \times \mathrm{U}(1) \subset \mathrm{SU}(4)$ as described in Appendix E.2, and can be found in the following $\mathcal{N} = 1$ superfields and their conjugates that cover the short $\mathcal{N} = 4$ multiplet:



$W\bar{W}Z^2\bar{Z} = \overline{\mathbf{15}}_{2(\frac{1}{2}\frac{1}{2})}$ $W^2\bar{W}Z^2 = \bar{\mathbf{6}}_{7(0\frac{1}{2})}$ $W^2\bar{W}Z\bar{Z} = \mathbf{8}_{3(0\frac{1}{2})}$ $W^2\bar{W}\bar{Z}^2 = \mathbf{6}_{-1(0\frac{1}{2})}$



$W^2\bar{W}^2Z = \bar{\mathbf{3}}_{2(00)}$

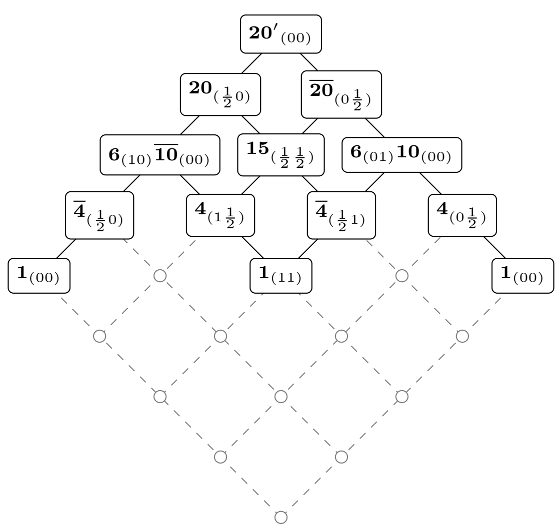


F.2 Short Multiplets in the $\mathcal{N} = 2$ Description

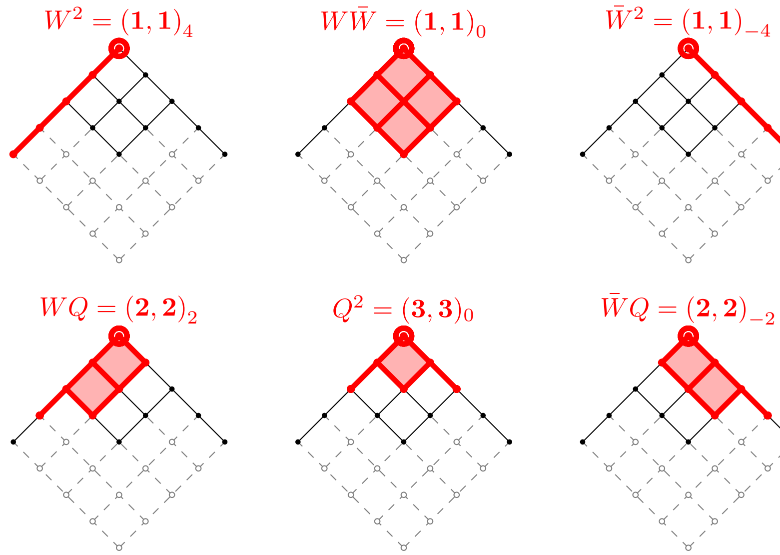
As for the branching under $\mathcal{N} = 1$, the decomposition of the $\mathcal{N} = 4$ short multiplets under $\mathcal{N} = 2$ can be formulated in terms of products of $\mathcal{N} = 2$ superfields. To find the decomposition of the order-p $\mathcal{N} = 4$ short multiplet take all possible p-fold products of the $\mathcal{N} = 2$ superfields Q, W, and $\bar{W}$ that we introduced in the previous section. Their physical components span the area of the $\mathcal{N} = 4$ short multiplet that is covered.

F.2.1 Case $p = 2$

The order-2 $\mathcal{N} = 4$ short multiplet

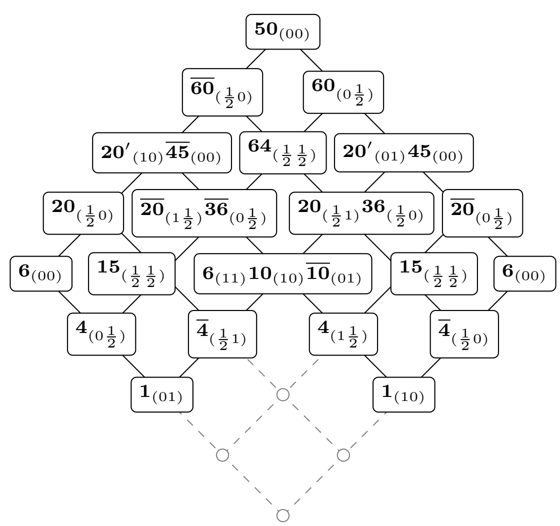


decomposes into $\mathcal{N} = 2$ superfields as follows:

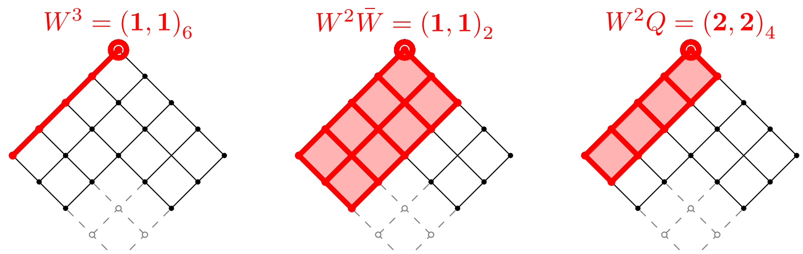


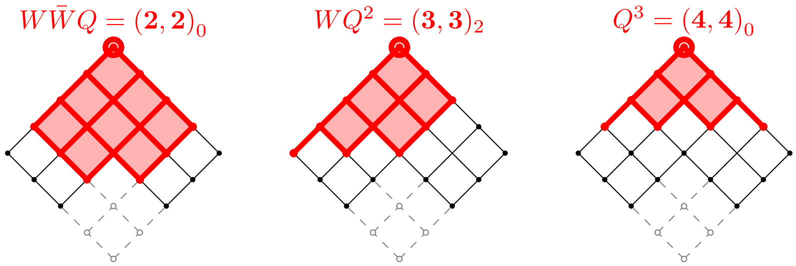
F.2.2 Case $p = 3$

The order-3 $\mathcal{N}=4$ short multiplet



decomposes into $\mathcal{N}=2$ superfields as follows:





Here we have omitted those diagrams that can be obtained from the presented ones by complex conjugation.

Appendix G

Holographic Supergravity Potentials Listing

In this appendix we list explicit expressions for supergravity potentials that were obtained using the formula (5.18) in Chapter 5, which was obtained using holographic RG flow techniques. This formula is valid for supergravity modes that are dual to field theory operators that preserve at least an $\mathcal{N} = 1$ supersymmetry. We limit ourselves to dimensions d =3, 4, and 6, which seem most relevant. Only relevant operators of the field theory with dimensions $\Delta < d$ were considered.

G.1 Dimension $d = 3$ dual to AdS_4

$\Delta = 1$:

$$V = \frac{1}{4\kappa^2}\left[\cosh^2(\sqrt{2}\phi) - 6\cosh(\sqrt{2}\phi) - 7\right] \tag{G.1}$$

$$= \frac{1}{4\kappa^2}\left(\cosh(\sqrt{2}\phi) + 1\right)\left(\cosh(\sqrt{2}\phi) - 7\right) \tag{G.2}$$

$$= \frac{1}{2\kappa^2}\cosh^2\left(\frac{\phi}{\sqrt{2}}\right)\left(\cosh(\sqrt{2}\phi) - 7\right) \tag{G.3}$$

$\Delta = 2$:

$$V = -\frac{1}{4\kappa^2}\left[\cosh^2(\phi) + 6\cosh(\phi) + 5\right] \tag{G.4}$$

$$= -\frac{1}{4\kappa^2}\left(\cosh(\phi) + 1\right)\left(\cosh(\phi) + 5\right) \tag{G.5}$$

$$= -\frac{1}{2\kappa^2}\cosh^2\left(\frac{\phi}{2}\right)\left(\cosh(\phi) + 5\right) \tag{G.6}$$

G.2 Dimension $d = 4$ dual to AdS_5

$\Delta = 1$:

$$V = \frac{3}{4\kappa^2}\left[\cosh^2(\sqrt{2}\phi) - 4\cosh(\sqrt{2}\phi) - 5\right] \quad (G.7)$$
$$= \frac{3}{4\kappa^2}\left(\cosh(\sqrt{2}\phi) + 1\right)\left(\cosh(\sqrt{2}\phi) - 5\right) \quad (G.8)$$
$$= \frac{3}{2\kappa^2}\cosh^2\left(\frac{\phi}{\sqrt{2}}\right)\left(\cosh(\sqrt{2}\phi) - 5\right) \quad (G.9)$$

$\Delta = 2$:

$$V = -\frac{3}{\kappa^2}\left(\cosh\left(\frac{2\phi}{\sqrt{3}}\right) + 1\right) \quad (G.10)$$
$$= -\frac{6}{\kappa^2}\cosh^2\left(\frac{\phi}{\sqrt{3}}\right) \quad (G.11)$$

$\Delta = 3$:

$$V = -\frac{3}{4\kappa^2}\left[\cosh^2\left(\sqrt{\frac{2}{3}}\phi\right) + 4\cosh\left(\sqrt{\frac{2}{3}}\phi\right) + 3\right] \quad (G.12)$$
$$= -\frac{3}{4\kappa^2}\left(\cosh\left(\sqrt{\frac{2}{3}}\phi\right) + 1\right)\left(\cosh\left(\sqrt{\frac{2}{3}}\phi\right) + 3\right) \quad (G.13)$$
$$= -\frac{3}{2\kappa^2}\cosh^2\left(\frac{\phi}{\sqrt{6}}\right)\left(\cosh\left(\sqrt{\frac{2}{3}}\phi\right) + 3\right) \quad (G.14)$$

G.3 Dimension $d = 6$ dual to AdS_7

$\Delta = 1$:

$$V = \frac{5}{2\kappa^2}\left[\cosh^2(\sqrt{2}\phi) - 3\cosh(\sqrt{2}\phi) - 4\right] \quad (G.15)$$
$$= \frac{5}{2\kappa^2}\left(\cosh(\sqrt{2}\phi) + 1\right)\left(\cosh(\sqrt{2}\phi) - 4\right) \quad (G.16)$$
$$= \frac{5}{\kappa^2}\cosh^2\left(\frac{\phi}{\sqrt{2}}\right)\left(\cosh(\sqrt{2}\phi) - 4\right) \quad (G.17)$$

$\Delta = 2$:

$$V = \frac{5}{4\kappa^2}\left[\cosh^2\left(\sqrt{\frac{8}{5}}\phi\right) - 6\cosh\left(\sqrt{\frac{8}{5}}\phi\right) - 7\right] \quad (G.18)$$
$$= \frac{5}{4\kappa^2}\left(\cosh\left(\sqrt{\frac{8}{5}}\phi\right) + 1\right)\left(\cosh\left(\sqrt{\frac{8}{5}}\phi\right) - 7\right) \quad (G.19)$$

$$= \frac{5}{2\kappa^2}\cosh^2\left(\sqrt{\frac{2}{5}}\phi\right)\left(\cosh\left(\sqrt{\frac{8}{5}}\phi\right) - 4\right) \tag{G.20}$$

$\Delta = 3$:

$$V = -\frac{15}{2\kappa^2}\left(\cosh\left(\sqrt{\frac{6}{5}}\phi\right) + 1\right) \tag{G.21}$$

$$= -\frac{15}{\kappa^2}\cosh^2\left(\sqrt{\frac{3}{10}}\phi\right) \tag{G.22}$$

$\Delta = 4$:

$$V = -\frac{5}{4\kappa^2}\left[\cosh^2\left(\frac{2}{\sqrt{5}}\phi\right) + 6\cosh\left(\frac{2}{\sqrt{5}}\phi\right) + 5\right] \tag{G.23}$$

$$= -\frac{5}{4\kappa^2}\left(\cosh\left(\frac{2}{\sqrt{5}}\phi\right) + 1\right)\left(\cosh\left(\frac{2}{\sqrt{5}}\phi\right) + 5\right) \tag{G.24}$$

$$= -\frac{5}{2\kappa^2}\cosh^2\left(\frac{\phi}{\sqrt{5}}\right)\left(\cosh\left(\frac{2}{\sqrt{5}}\phi\right) + 5\right) \tag{G.25}$$

$\Delta = 5$:

$$V = -\frac{5}{2\kappa^2}\left[\cosh^2\left(\sqrt{\frac{2}{5}}\phi\right) + 3\cosh\left(\sqrt{\frac{2}{5}}\phi\right) + 2\right] \tag{G.26}$$

$$= -\frac{5}{2\kappa^2}\left(\cosh\left(\sqrt{\frac{2}{5}}\phi\right) + 1\right)\left(\cosh\left(\sqrt{\frac{2}{5}}\phi\right) + 2\right) \tag{G.27}$$

$$= -\frac{5}{\kappa^2}\cosh^2\left(\frac{\phi}{\sqrt{10}}\right)\left(\cosh\left(\sqrt{\frac{2}{5}}\phi\right) + 2\right) \tag{G.28}$$

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