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# An improved AFS phase for $AdS_3$ string integrability

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ABSTRACT

We propose a number of modifications to the classical term in the dressing phase for integrable strings in $AdS_3 \times S^3 \times S^3 \times S^1$, and check these against existing perturbative calculations, crossing symmetry, and the semiclassical limit of the Bethe equations. The principal change is that the phase for different masses should start with a term $Q_1 Q_2$, like the one-loop $AdS_3$ dressing phase, rather than $Q_2 Q_3$ as for the original $AdS_5$ AFS phase.



## 1. Introduction

The central object in the integrable picture of planar $AdS_5/CFT_4$ is the all-loop S-matrix, and the Bethe ansatz equations which follow from this [1]. Its nontrivial dependence on the 't Hooft coupling $\lambda$ comes from the dressing phase, and expanding at strong coupling this has the form

$$\sigma_{\mathrm{BES}}(x, y) = \exp\left[ i \frac{\sqrt{\lambda}}{2\pi} \sum_{r,s\geq 2} c_{r,s}(\lambda)\, Q_r(x) Q_s(y) \right] \tag{1}$$

where $c_{r,s} = \left(\delta_{r+1,s} - \delta_{r,s+1}\right) + a_{r,s}/\sqrt{\lambda} + \mathcal{O}(1/\lambda)$. The first term was introduced by Arutyunov, Frolov and Staudacher (AFS) in [2] as a correction needed to match classical strings in $AdS_5 \times S^5$. The coefficients $a_{r,s}$ are the extension to one-loop strings of [3], and this was later extended to all loops in [4].

The dressing phase for $AdS_3$ backgrounds is different, and is now understood quite well at one loop [5–8]; see also [9,10]. However we believe that the classical part of the dressing phase has been treated incorrectly in the literature. This is the subject of our Letter.

A new feature of strings in $AdS_3 \times S^3 \times S^3 \times S^1$ is that there are excitations (above the BMN state) of mass 1, $\alpha$, $1-\alpha$ and 0 [11], rather than just one mass in $AdS_5 \times S^5$ or two in $AdS_4 \times CP^3$. The bosonic modes of mass $s_1 = \alpha$ and $s_3 = 1-\alpha$ are excitations in the two $S^3$ factors (which have different radii), and there are two such excitations in each sphere, one in the left copy of the algebra (labelled 1, or 3) and one in the right ($\bar{1}$, or $\bar{3}$). These and their superpartners are the elementary particles in the Bethe ansatz description of [12], which gives the spectrum as

$$\Delta - J = \sum_{\ell} \sum_{k=1}^{K_\ell} E_\ell(p_{\ell,k}),$$
$$E_\ell(p_{\ell,k}) = \sqrt{s_\ell^2 + 4h^2 \sin^2 \frac{p_{\ell,k}}{2}} \tag{2}$$

where the allowed $p_{\ell,k}$ are constrained by equations of the form $e^{ip_{\ell,k}L} = \prod_{j\neq k} S(p_k, p_j)$, using the S-matrix of [13]. This must include (for the first time[1]) a dressing phase for the scattering of particles of different mass.

The first classical phase for two particles of different mass was written down by Borsato, Ohlsson Sax and Sfondrini [12], who gave

$$\sigma_{\mathrm{BOS}}(x, y) = \left( \frac{1 - \frac{1}{x^+ y^-}}{1 - \frac{1}{x^+ y^+}} \, \frac{1 - \frac{1}{x^- y^+}}{1 - \frac{1}{x^- y^-}} \right)^{i \frac{h}{W_{xy}} \left(x + \frac{1}{x} - y - \frac{1}{y}\right)} \left( \frac{1 - \frac{1}{x^- y^+}}{1 - \frac{1}{x^+ y^-}} \right) \tag{3}$$

where the masses $s_x$, $s_y$ enter explicitly through

$$W_{xy} = \frac{4 s_x s_y}{s_x + s_y} = \begin{cases} 2 s_x, & s_x = s_y \\ 4 s_x s_y, & s_x + s_y = 1. \end{cases}$$

When $s_x = s_y = 1$, this is exactly the original AFS phase used in $AdS_5$. A similar phase was used in [18] when comparing to tree-

[1] In the $AdS_4 \times CP^3$/ABJM correspondence there are particles of mass 1 and $\frac{1}{2}$, but only the latter appear in the Bethe equations, and hence in the AFS phase. The heavy particles are composite objects, mirror bound states [14,15]. The entire dressing phase for this correspondence is simply half the BES phase [16,17].

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level amplitudes, but with the exponent of the first factor replaced by[2]

$$\frac{ih}{W_{xy}}\Big(x^+ + \frac{1}{x^+} - y^- - \frac{1}{y^-}\Big) = \frac{ih}{W_{xy}}\Big(x + \frac{1}{x} - y - \frac{1}{y}\Big) - \frac{s_x + s_y}{W_{xy}}.$$

The last term here has no effect on tree-level worldsheet scattering.

Our first proposal is that the correct generalisation of the AFS phase to particles of different mass is instead:

$$\sigma_{\text{AFS}}(x,y) = \left(\frac{1-\frac{1}{x^+y^-}}{1-\frac{1}{x^+y^+}}\,\frac{1-\frac{1}{x^-y^+}}{1-\frac{1}{x^-y^-}}\right)^{i\frac{h}{W_{xy}}\left(x+\frac{1}{x}-y-\frac{1}{y}\right)} \times \left(\frac{1-\frac{1}{x^+y^+}}{1-\frac{1}{x^-y^-}}\right)^{\frac{s_x-s_y}{W_{xy}}} \left(\frac{1-\frac{1}{x^-y^+}}{1-\frac{1}{x^+y^-}}\right)^{\frac{s_x+s_y}{W_{xy}}}. \tag{4}$$

This follows from changing the original definition, the first term of (1), by an overall factor:

$$\sigma_{\text{AFS}}(x,y) = \exp\Big\{ i\frac{h}{W_{xy}}\sum_{r=2}^{\infty}\left[Q_r(x)Q_{r+1}(y) - Q_{r+1}(x)Q_r(y)\right]\Big\}. \tag{5}$$

However this change alone will break the agreement with tree-level worldsheet scattering seen in [18], as we discuss below. This leads us to suggest two further modifications, which we parameterise by $\beta, \delta, \Delta$, in addition to $\gamma, \Gamma$ in [12]. Of these five parameters, three will be fixed by tree-level scattering, and one more by a semiclassical limit of the Bethe equations.

- In the one-loop dressing phase, an important difference from the $AdS_5$ case is that the sum starts with $a_{1,2}Q_1Q_2$ [10,5,6], rather than $a_{2,3}Q_2Q_3$ as in (1). It seems natural to wonder if this should apply to the classical phase too, and thus our second proposal is to include a factor

$$\sigma_{\text{one}}(x,y) = \exp\Big\{ i\frac{h}{W_{xy}}\left[p_x\, Q_2(y) - p_y\, Q_2(x)\right]\Big\}. \tag{6}$$

  We use $\sigma_{\text{one}}^{\beta}\sigma_{\text{AFS}}$ as the classical phase for different-mass scattering only, with power $\beta = 1$ most natural.
- The S-matrix derived in [13] contains a number of unfixed scalars $S^{\ell m}$, each of which should include the dressing phase. An ansatz for the remaining factors was given in [12], and our third proposal is that this should be slightly modified, introducing a phase like the one needed for the string frame, but with an arbitrary power. Explicitly, we set

$$S^{11}(x,y) = \left(\frac{x^-y^+}{x^+y^-}\right)^{\frac{1}{2}+\gamma+\delta} \times \left[\frac{1-\frac{1}{x^+y^-}}{1-\frac{1}{x^-y^+}}\,\sigma_{\text{AFS}}^2(x,y)\right]^{1+2\gamma} \sigma_{LL}^2(x,y)$$

$$S^{13}(x,y) = \left(\frac{x^-y^+}{x^+y^-}\right)^{\Gamma+\Delta} \times \left[\frac{1-\frac{1}{x^+y^-}}{1-\frac{1}{x^-y^+}}\,\sigma_{\text{one}}^{2\beta}(x,y)\,\sigma_{\text{AFS}}^2(x,y)\right]^{1+2\Gamma} \times \sigma_{LL}^2(x,y). \tag{7}$$

The expressions in [12] have unfixed $\gamma$ and $\Gamma$ but $\delta = \Delta = 0$, while going to the string frame would normally mean increasing $\delta$ and $\Delta$ by $\frac{1}{2}$. We write the one-loop dressing phase $\sigma_{LL}$ outside the power of $1+2\gamma$, as it was the total phase which was calculated by semiclassical means in [6]. We omit the two-loop and higher phases.

## 2. Tree-level BMN scattering

Let us now test this against the results of Sundin and Wulff [18], who computed tree-level Feynman diagrams in the worldsheet theory. To do this we must take the BMN limit, writing $p = \tilde{p}/h$ with $\tilde{p}$ order 1 and $h \gg 1$. Then we can expand

$$x^{\pm} = \frac{s_x + \omega_x}{\tilde{p}_x} \pm \frac{i(s_x+\omega_x)}{2h} + \mathcal{O}\Big(\frac{1}{h^2}\Big),$$
$$\text{where}\quad \omega_x \equiv \sqrt{s_x^2 + \tilde{p}_x^2} = E_x(p_x) + \dots.$$

The charges used above are $Q_1(x) \equiv p_x = -i\log(x^+/x^-)$ and, for $n > 1$,

$$Q_n(x) \equiv \frac{i}{n-1}\left[\frac{1}{(x^+)^{n-1}} - \frac{1}{(x^-)^{n-1}}\right] = \frac{\tilde{p}_x}{h}\Big(\frac{\omega_x - s_x}{\tilde{p}_x}\Big)^{n-1} + \frac{0}{h^2} + \mathcal{O}\Big(\frac{1}{h^3}\Big).$$

Apart from obvious phases, the other expansions we will need for this limit are

$$\frac{1-\frac{1}{x^+y^-}}{1-\frac{1}{x^-y^+}}\left[\sigma_{\text{one}}^{\beta}\,\sigma_{\text{AFS}}\right]^2 = 1 + \frac{i}{2h}\Big[-\tilde{p}_x(\omega_y - s_y)\Big(\frac{1}{s_x} - \frac{4\beta}{W_{xy}}\Big) + \tilde{p}_y(\omega_x - s_x)\Big(\frac{1}{s_y} - \frac{4\beta}{W_{xy}}\Big)\Big] + \dots$$

$$\frac{x^+ - y^-}{x^- - y^+} = 1 + \frac{i}{2h}\left[\tilde{p}_x - \tilde{p}_y + \frac{\alpha(\tilde{p}_x + \tilde{p}_y)^2}{\omega_x\tilde{p}_y - \omega_y\tilde{p}_x}\right] + \mathcal{O}\Big(\frac{1}{h^2}\Big),$$
$$s_x = s_y = \alpha \text{ only}.$$

Consider two bosons from the left sector of the theory, “1” of mass $\alpha$ and “3” of mass $1-\alpha$. As in [18], and in [20,14], we should allow for some unknown gauge dependence through $\tilde{a}$ in addition to the spin-chain S-matrix. However for the mixed-mass case we allow two parameters $\tilde{b}, \tilde{c}$ (and expect them to be equal at $\alpha = \frac{1}{2}$). Thus we write the scattering amplitudes as

$$A^{11}(x,y) = \exp\Big[-\frac{i\tilde{a}}{h\alpha}(\omega_x\tilde{p}_y - \omega_y\tilde{p}_x)\Big]\frac{x^+ - y^-}{x^- - y^+}\,S^{11}(x,y),$$
$$s_x = s_y = \alpha$$
$$A^{13}(x,y) = \exp\Big[-\frac{i}{h}\Big(\tilde{c}\frac{\omega_x\tilde{p}_y}{\alpha} - \tilde{b}\frac{\omega_y\tilde{p}_x}{1-\alpha}\Big)\Big]\,S^{13}(x,y),$$
$$s_x = \alpha,\ s_y = 1-\alpha. \tag{8}$$

[2] The variables $x^{\pm}$ depend on the mass $s_x$ through

$$x^{\pm} + \frac{1}{x^{\pm}} = x + \frac{1}{x} \pm i\frac{s_x}{h}$$

where $h = \sqrt{\lambda}/2\pi + c + \mathcal{O}(1/\sqrt{\lambda})$ is the Bethe coupling, normalised as in [13,19,12,18].

The corresponding worldsheet results are in Eq. (3.2) of [18]. These depend on the AFZ gauge parameter [21] which is $a=\frac{1}{2}$ for the simplest light-cone gauge[3]:

$$A^{11}_{\rm WS}(\tilde p_x,\tilde p_y)=1+\frac{i}{2h}\frac{\alpha(\tilde p_x+\tilde p_y)^2}{\omega_x\tilde p_y-\omega_y\tilde p_x}+\frac{i}{2h}(1-2a)\left[\omega_x\tilde p_y-\omega_y\tilde p_x\right]+\mathcal{O}\Big(\frac{1}{h^2}\Big)$$

$$A^{13}_{\rm WS}(\tilde p_x,\tilde p_y)=1+\frac{i}{2h}(1-2a)\left[\omega_x\tilde p_y-\omega_y\tilde p_x\right]+\mathcal{O}\Big(\frac{1}{h^2}\Big).$$

Matching $A^{11}=A^{11}_{\rm WS}$ and $A^{13}=A^{13}_{\rm WS}$, and demanding that $\Gamma\neq-\frac{1}{2}$, we find that

$$\beta=1,\qquad \delta_{\rm SF}=\frac{1}{2}\qquad \Delta_{\rm SF}=-\frac{1}{2}-2\Gamma \tag{9}$$

and

$$2\tilde a=1+2\gamma+(2a-1)\alpha,\quad 2\tilde b=-(1+2\Gamma)+(2a-1)(1-\alpha),$$
$$2\tilde c=-(1+2\Gamma)+(2a-1)\alpha.$$

We write $\delta_{\rm SF}$ to indicate that these are the parameters in the string frame; in the spin chain frame we have $\delta=0$ and $\Delta=-1-2\Gamma$ instead.

The comparison performed in [18] used $\sigma_{\rm BOS}$ for $A^{13}$ (and for $A^{11}$, $\sigma_{\rm BOS}=\sigma_{\rm AFS}$). If we repeat this allowing arbitrary parameters (including $\beta$, and demanding $\alpha\neq\frac{1}{2}$, $\Gamma\neq-\frac{1}{2}$) we find that

$$\beta=0,\qquad \delta_{\rm SF}=\Delta_{\rm SF}=\tfrac{1}{2}$$

and $2\tilde a=1+2\gamma+(2a-1)\alpha$, $2\tilde b=1+2\Gamma+(2a-1)(1-\alpha)$, $2\tilde c=1+2\Gamma+(2a-1)\alpha$. Setting $\gamma=\Gamma=0$ returns precisely the phases used in [18].

We can similarly check agreement for scattering with a "$\bar 1$" or "$\bar 3$" particle in the right sector, using the same gauge phases $\tilde a$, $\tilde b$, $\tilde c$ as before with the appropriate $\hat S$ matrix elements from [13]:

$$A^{1\bar 1}(x,y)=e^{-\frac{i\tilde a}{h\alpha}(\omega_x\tilde p_y-\omega_y\tilde p_x)}\,\frac{\sqrt{1-\frac{1}{x^+y^+}}\sqrt{1-\frac{1}{x^-y^-}}}{1-\frac{1}{x^+y^-}}\,S^{1\bar 1}(x,y)$$

$$A^{1\bar 3}(x,y)=e^{-\frac{i}{h}(\tilde c\frac{\omega_x\tilde p_y}{\alpha}-\tilde b\frac{\omega_y\tilde p_x}{1-\alpha})}\,\frac{\sqrt{1-\frac{1}{x^+y^+}}\sqrt{1-\frac{1}{x^-y^-}}}{1-\frac{1}{x^+y^-}}\,S^{1\bar 3}(x,y).$$

The phases $S^{\ell\bar m}$ should be modified from those of [12] by the same factors $\delta$, $\Delta$, i.e.

$$S^{1\bar 1}(x,y)=\left[\frac{1-\frac{1}{x^+y^-}}{1-\frac{1}{x^-y^+}}\right]^{-\frac{1}{2}}S^{11}(x,y),$$
$$S^{1\bar 3}(x,y)=\left[\frac{1-\frac{1}{x^+y^-}}{1-\frac{1}{x^-y^+}}\right]^{+\frac{1}{2}}S^{13}(x,y) \tag{10}$$

(and $\sigma_{LL}$ is replaced with $\sigma_{LR}$) and the worldsheet results are [18]

$$A^{1\bar 1}_{\rm WS}(\tilde p_x,\tilde p_y)=A^{11}_{\rm WS}(\tilde p_x,\tilde p_y)-\frac{i}{2h}\frac{4\alpha\tilde p_x\tilde p_y}{\omega_x\tilde p_y-\omega_y\tilde p_x}+\mathcal{O}\Big(\frac{1}{h}\Big),$$
$$A^{1\bar 3}_{\rm WS}(\tilde p_x,\tilde p_y)=A^{13}_{\rm WS}(\tilde p_x,\tilde p_y).$$

Clearly we obtain no new constraints from these.

[3] These are $A^{(22)}$ and $A^{(23)}$ in the notation of [18], where the particle of mass $\alpha$ is "2". We have also restored a factor $1/h$.

## 3. Crossing relations

We can obtain a check on the phases described above from crossing symmetry [22]. If we stay in the BMN limit there is nothing to learn, since (by construction) we have not changed the results. But if we take the semiclassical limit without small momentum ($h\gg 1$, $p\sim 1$) then we obtain a nontrivial check which in fact mixes the classical and one-loop phases. The relevant equations from [12] for the scalars $S^{\ell m}$ (7) and $S^{\ell\bar m}$ (10) are

$$S^{11}(x,y)S^{1\bar 1}(x,\bar y)=\frac{x^--y^+}{x^--y^-}\sqrt{\frac{x^+}{x^-}}\sqrt{\frac{x^--y^-}{x^+-y^+}}=i\,e^{i(p_x-p_y)/4}\frac{1-e^{i(p_x+p_y)/2}}{1-e^{i(p_x-p_y)/2}}+\mathcal{O}\Big(\frac{1}{h}\Big)$$

$$S^{13}(x,y)S^{1\bar 3}(x,\bar y)=\frac{x^+-y^-}{x^--y^-}\sqrt{\frac{x^+}{x^-}}\sqrt{\frac{x^--y^-}{x^+-y^+}}=-i\,e^{i(3p_x-3p_y)/4}\frac{1-e^{i(p_x+p_y)/2}}{1-e^{i(p_x-p_y)/2}}+\dots. \tag{11}$$

Here $\bar y$ indicates that the argument has been moved $y^\pm\to 1/y^\pm$. On the right we use $x^\pm=e^{\pm ip_x/2}+\mathcal{O}(1/h)$, and separate two factors: a phase and a trigonometric part. (There are two more crossing equations, for $S^{11}(x,\bar y)S^{1\bar 1}(x,y)$ and $S^{13}(x,\bar y)S^{1\bar 3}(x,y)$. These can be treated almost identically.)

In this $h\gg 1$ limit we can write the complete dressing phase as

$$\sigma^\beta_{\rm one}\,\sigma_{\rm AFS}\,\sigma_{LL}\,\sigma_{\text{higher-loop}}=\exp\left[ih(\beta\theta_{\rm one}+\theta_{\rm AFS})+i\theta_{LL}+\mathcal{O}(1/h)\right]$$

with each $\theta$ of order 1. Considering (11) at order $h$ in the exponent, the cancellation is very simple from (5) and (6), because $Q_n(1/y^\pm)=-Q_n(y^\pm)+\mathcal{O}(1/h)$. At order $h^0$ it's easier to use form (4) for the AFS phase. The exponent $\frac{ih}{W_{xy}}(x+\frac{1}{x}-y-\frac{1}{y})$ has terms at order $h$ and $h^{-1}$ but not $h^0$, so this first factor does not contribute. The other two factors give

$$\sigma_{\rm AFS}(x^\pm,y^\pm)\times\sigma_{\rm AFS}\Big(x^\pm,\frac{1}{y^\pm}\Big)=\left(\frac{1-\frac{1}{x^+y^+}}{1-\frac{1}{x^-y^-}}\times\frac{1-\frac{y^+}{x^+}}{1-\frac{y^-}{x^-}}\right)^{\frac{s_x-s_y}{W_{xy}}}\left(\frac{1-\frac{1}{x^-y^+}}{1-\frac{1}{x^+y^-}}\times\frac{1-\frac{y^+}{x^+}}{1-\frac{y^-}{x^-}}\right)^{\frac{s_x+s_y}{W_{xy}}}=\exp\left[i\frac{2p_xs_y}{W_{xy}}+\mathcal{O}\Big(\frac{1}{h}\Big)\right].$$

At the same order there is also a contribution from (6). Using $Q_2(1/y^\pm)=-Q_2(y^\pm)-2s_y/h+\mathcal{O}(1/h^2)$ we see that it exactly cancels the last equation if $\beta=1$:

$$\sigma_{\rm one}(x^\pm,y^\pm)\,\sigma_{\rm one}\Big(x^\pm,\frac{1}{y^\pm}\Big)=\exp\Big(-i\frac{2p_xs_y}{W_{xy}}+\dots\Big). \tag{12}$$

Note that if $\beta=0$, it is difficult to imagine what would cancel the phase $e^{ip_x/2\alpha}$ from $\sigma_{\rm AFS}$ in the $S^{13}S^{1\bar 3}$ case at generic $\alpha$.[4]

For the remaining factors in $S^{\ell m}$ (7) and $S^{\ell\bar m}$ (10), the contribution is

[4] If we used (3) instead, the power would be an integer: $\sigma_{\rm BOS}(x^\pm,y^\pm)\,\sigma_{\rm BOS}(x^\pm,\frac{1}{y^\pm})=e^{ip_x}+\mathcal{O}(1/h)$ in both the $S^{11}S^{1\bar 1}$ and $S^{13}S^{1\bar 3}$ cases.

$$S^{11}S^{1\bar{1}}: \quad \left(\frac{x^- y^+}{x^+ y^-}\right)^{\frac{1}{2}+\gamma+\delta} \left[\frac{1-\frac{1}{x^+ y^-}}{1-\frac{1}{x^- y^+}}\right]^{1+2\gamma} \times \left(\frac{x^- y^-}{x^+ y^+}\right)^{\frac{1}{2}+\gamma+\delta} \left[\frac{1-\frac{y^-}{x^+}}{1-\frac{y^+}{x^-}}\right]^{-\frac{1}{2}+2\gamma} = i\,\exp\left[-ip_x(7/4+4\gamma+2\delta)+ip_y/4\right]+\mathcal{O}(1/h).$$

Combined with $(e^{ip_x})^{2(1+2\gamma)}$ from $\sigma_{\rm AFS}$, and using coefficients (9) with the spin-chain-frame $\delta=0$, we get $e^{ip_x/4}$ as in (11). For the mixed mass case, the remaining contribution is instead

$$S^{13}S^{1\bar{3}}: \quad -i\,\exp\left[-ip_x(5/4+4\Gamma+2\Delta)-ip_y/4\right]+\mathcal{O}(1/h)$$

which combined with $\sigma_{\rm one}\sigma_{\rm AFS}$ gives $e^{i3p_x/4}$. In both cases the power of $e^{ip_y}$ does not yet match (11).

At order $h^0$ there will also be a contribution from the one-loop phase. The semiclassical calculation of this in [6] gave the following final answer for left–left scattering:

$$\theta_{LL}(x^\pm,y^\pm)=\chi(x^+,y^+)-\chi(x^+,y^-)-\chi(x^-,y^+)+\chi(x^-,y^-) = \left(I_{yx}-I_{xy}\right),$$

$$I_{yx}=\sum_\pm \frac{\mp 1}{16\pi}\int_{U_\pm} dz\,\frac{\partial G(z,y^\pm)}{\partial z}\,G(z,x^\pm) \tag{13}$$

and for left–right scattering:

$$\theta_{LR}(x^\pm,y^\pm)=\tilde\chi(x^+,y^+)-\tilde\chi(x^+,y^-)-\tilde\chi(x^-,y^+)+\tilde\chi(x^-,y^-) = \left(\widetilde{I}_{yx}-\widetilde{I}_{xy}\right),$$

$$\widetilde{I}_{yx}=\sum_\pm \frac{\mp 1}{16\pi}\int_{U_\pm} dz\,\frac{\partial G(z,y^\pm)}{\partial z}\,G\left(\frac{1}{z},x^\pm\right)$$

where

$$G(z,x^\pm)\equiv -i\log\left(\frac{z-x^+}{z-x^-}\right)-\frac{p_x}{2}.$$

Notice that $G(\frac{1}{z},x^\pm)=G(z,1/x^\pm)$. Then it is easy to see that $\theta_{LL}(x^\pm,y^\pm)+\theta_{LR}(x^\pm,1/y^\pm)=0$, and thus there is no contribution to crossing from evaluating at $1/y^\pm$. However in moving $y^\pm\to 1/y^\pm$ we move some poles across contours.

Let us focus on the effect on the term $\tilde\chi(x^+,y^+)$. The only pole in the integrand at $z=y^+$ comes from $\partial_z G(z,y^\pm)$ in $\widetilde{I}_{yx}$. Moving the pole to $z=1/y^+$ pulls it across $U_+$ anti-clockwise, and the final pole has residue $-iG(y^+,x^\pm)$. The contribution is then

$$\Delta\tilde\chi(x^+,y^+)=\frac{i}{8}\left[-\log(y^+-x^+)+\frac{1}{2}\log x^+\right]$$

There is a similar contribution from $\widetilde{I}_{xy}$, from the log cut. Together these give the remainder of (11):

$$\sigma_{LL}^2(x,y)\sigma_{LR}^2(x,\bar y)=\frac{\sqrt{x^+-y^-}\sqrt{x^--y^+}}{\sqrt{x^+-y^+}\sqrt{x^--y^-}} = e^{-ip_y/2}\frac{1-e^{i(p_x+p_y)/2}}{1-e^{i(p_x-p_y)/2}} \tag{14}$$

## 4. Semiclassical limit of Bethe equations

Another check of the phases is to look at the semiclassical limit of the Bethe equations, which should reproduce the finite-gap equations. This calculation was also done in [12], so we do not show much detail. But the result is changed by using our phase: [12] found $\Gamma=\gamma+\frac{1}{2}$.

It suffices to look at the left sector, with $K_1\neq 0$ and $K_3\neq 0$ only. Then Eqs. (4.5) and (4.7) of [12] become

$$\begin{aligned}
\frac{2\pi n_{1,k}}{2\alpha} &= \frac{-x}{x^2-1}\Big\{\Big[L+K_1(\tfrac{1}{2}+\gamma+\delta)+K_3(\Gamma+\Delta)\Big] \\
&\quad + Q_{1,2}[1+(1+2\gamma)] + Q_{3,2}\left[(1+2\Gamma)\frac{1-\alpha-\beta}{\alpha}\right]\Big\} \\
&\quad + \frac{-1}{x^2-1}\frac{(1+2\Gamma)}{\alpha}\Big[\alpha Q_{1,1}+(\beta-\alpha)Q_{3,1}\Big] \\
&\quad + 2\fint dy\,\frac{\rho_1(y)}{x-y} - \frac{(1+\Gamma)}{\alpha}\Big[\alpha Q_{1,1}+(1-\alpha)Q_{3,1}\Big] \\
\frac{2\pi n_{3,k}}{2(1-\alpha)} &= \frac{-x}{x^2-1}\Big\{\Big[L+K_1(\Gamma+\Delta)+K_3(\tfrac{1}{2}+\gamma+\delta)\Big] \\
&\quad + Q_{3,2}[1+(1+2\gamma)] + Q_{1,2}\left[(1+2\Gamma)\frac{\alpha-\beta}{1-\alpha}\right]\Big\} \\
&\quad + \frac{1}{x^2-1}[\text{winding}] + 2\fint dy\,\frac{\rho_3(y)}{x-y} + [\text{constant}]
\end{aligned} \tag{15}$$

where $Q_{\ell,n}$ is the total charge $Q_n$ of particles of type $\ell$ (and of course $Q_1$ is momentum, $Q_2$ an energy). Define $\mathcal{E}_\ell$ to be the curly brackets above (i.e. $-\frac{1}{2}$ the sum of the residues at $x=\pm 1$, divided by the mass).

If we set $\mathcal{E}_1=\mathcal{E}_3$ (which in the language of [23] means working above the $\zeta=\phi$ vacuum) we find

$$\beta=1, \qquad \gamma+\Gamma=-\frac{3}{2}, \qquad \delta-\Delta=1+2\Gamma. \tag{16}$$

We have derived these constraints on the parameters independent of the near-BMN comparison, (9), but the two are clearly compatible. Using both (i.e. using (16) and $\delta=0$) we get

$$\begin{aligned}
2\pi\mathcal{E}_1 &= 2\pi\mathcal{E}_3 \\
&= L-(1+\Gamma)(K_1+K_3)-(1+2\Gamma)\left(Q_{1,2}+Q_{3,2}\right).
\end{aligned}$$

## 5. Conclusion

In summary, we suggest three alterations to the classical dressing phase given in [12] for strings in $AdS_3\times S^3\times S^3\times S^1$, when scattering particles of different mass:

1. Preserve the AFS phase's form $\theta_{\rm AFS}=ih/W\sum_{r=2}^\infty\left[Q_rQ'_{r+1}-Q_{r+1}Q'_r\right]$, which gives (4).
2. Start this sum from $r=1$, giving one more term, (6) with $\beta=1$.
3. Add an extra string frame-like phase, as in (7), with $\Delta=-1-2\Gamma$.

Testing these against the tree-level near-BMN scattering [18], we find that given the first point, the other two are obligatory. And all parameters but $\gamma$ and $\Gamma$ are then fixed. The crossing equations (up to one-loop order) give a similar constraint; in particular the first point requires the second. Finally the semiclassical limit of the Bethe equations gives another, compatible constraint which also relates $\gamma$ and $\Gamma$.

This leaves one free parameter. We conjecture that this is $\gamma = 0$, and thus $\Gamma = -\frac{3}{2}$, because known string solutions can be placed in one or both $S^3$ factors, and this fact must be reflected in the Bethe equations. As $\alpha \to 0, 1$ we approach $AdS_3 \times S^3 \times T^4$ with a unit radius sphere, and thus should recover the usual $su(2)$ equation.[5] At $\alpha = \frac{1}{2}$ we can place exactly the same solution in each $S^3$, and the situation is very similar to that studied in $AdS_4 \times CP^3$ in [16], where it was necessary to scale the coupling $h$ by the mass of the particles.

The S-matrix has been compared to one-loop worldsheet scattering only for massive modes at $\alpha = 1$, when the background is $AdS_3 \times S^3 \times T^4$ [18,7,24]. This is only sensitive to the equal-mass phase $S^{11}$, and is thus unaffected by our proposal.[6]

In the case of $AdS_3 \times S^3 \times T^4$ with mixed NS–NS and R–R flux, some issues of how to correctly define the AFS phase were discussed in [25]. In that case, the dispersion relation is $E(p) = \sqrt{M^2 + 4h^2(1-\chi^2)\sin^2(p/2)}$ with $M^2(p) = (1 \pm \chi hp)^2$, differing for left and right sectors (with $\chi = 0$ for pure R–R). But no differences from the earlier proposal of [26] are claimed at tree level.

The dressing phase also matters a great deal in the quantum Bethe equations; this is of course how the one-loop phase was discovered [3]. Comparisons of such results against one-loop energy corrections to spinning strings have been published in [10,27], and (unlike $AdS_5 \times S^5$) they do not yet see perfect agreement.

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[5] In particular we expect the usual AFS phase. This is the reason for not allowing some power of $\sigma_{\text{one}}$ in $S^{11}$ (7).

[6] But note aside that both (6) and (5) are zero at order $1/h^2$ in the BMN limit.