HOMOTOPY DECOMPOSITIONS OF THE CLASSIFYING SPACES OF POINTED GAUGE GROUPS

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Abstract. Let G be a topological group and let $\mathcal{G}^*(P)$ be the pointed gauge group of a principal G-bundle $P \longrightarrow M$. We prove that if G is homotopy commutative then the homotopy type of the classifying space $B\mathcal{G}^*(P)$ can be completely determined for certain M. This also works p-locally, and valid choices of M include closed simply-connected four-manifolds when localized at an odd prime p. In this case, an application is to calculate part of the mod-p homology of the classifying space of the full gauge group.

1. Introduction

Let G be a topological group and let M be a pointed space. Let $P \longrightarrow M$ be a principal G-bundle over M. The *gauge group* $\mathcal{G}(P)$ is the group of G-equivariant automorphisms of P that fix M. The *pointed gauge group* $\mathcal{G}^*(P)$ is the subgroup of $\mathcal{G}(P)$ that fixes the fibre over the basepoint in M. Gauge groups are of wide interest due to their prominent role in both mathematical physics, Donaldson theory, and the study of semi-stable holomorphic vector bundles and their related moduli spaces. Important problems are to calculate the mod-p homology and cohomology of the classifying spaces $B\mathcal{G}(P)$ and $B\mathcal{G}^*(P)$ for a prime p when M is a closed simply-connected four-manifold, and to determine the integral homotopy types of various spaces related to $B\mathcal{G}^*(P)$ when M is an orientable closed Riemann surface.

In this paper, assume that the topological groups have the homotopy type of connected, finite type CW-complexes. We show that if G is homotopy commutative then for certain spaces M there is a homotopy decomposition of $B\mathcal{G}^*(P)$ as recognizable factors. This also works p-locally. Two applications are given. The first is in the case when G is a simply-connected, simple compact Lie group and M is a closed simply-connected four-manifold. For appropriate primes p, a p-local homotopy decomposition of $B\mathcal{G}^*(P)$ holds and this is used to determine a large split subalgebra of the mod-p cohomology of the full gauge group $B\mathcal{G}(P)$. The second is in the case when G is the infinite unitary group and M is a closed orientable Riemann surface. A homotopy decomposition of $B\mathcal{G}^*(P)$ is used to determine the homotopy type of the space $\mathrm{Hom}(\pi_1(\Sigma_g), U)$ of homorphisms from the fundamental group of the Riemann surface to the infinite unitary group.

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The key result is a decomposition of certain pointed mapping spaces. Consider adjunction spaces of the form

$$N = (\bigvee_{i=1}^{m} \Sigma A_i) \cup_a e^n$$

where $\bigvee_{i=1}^{m} \Sigma A_i$ is a CW-complex of dimension strictly less than n, $a\colon S^{n-1} \longrightarrow \bigvee_{i=1}^{m} \Sigma A_i$ is the attaching map of the n-cell, and $m \geq 2$. For $1 \leq i \leq m$, let $\iota_j : \Sigma A_j \longrightarrow \bigvee_{i=1}^{m} \Sigma A_i$ be the inclusion of the j^{th}-wedge summand. Let $\mathcal{N}$ be the collection of all such adjunction spaces N with the additional property that the attaching map a factors through a map a' which is a wedge sum of some of the Whitehead products $\Sigma A_j \wedge A_k \xrightarrow{[\iota_j,\iota_k]} \bigvee_{i=1}^{m} \Sigma A_i$.

Observe that there is a cofibration

$$\bigvee_{i=1}^{m} \Sigma A_i \xrightarrow{b} N \xrightarrow{q} S^n$$

where b is the inclusion and q collapses $\bigvee_{i=1}^{m} \Sigma A$ to a point. Let G be a topological group and let BG be its classifying space. Then the cofibration sequence induces a fibration sequence

$$\text{(1)} \qquad \mathrm{Map}^*(N, BG) \xrightarrow{b^*} \mathrm{Map}^*(\bigvee_{i=1}^{m} \Sigma A_i, BG) \xrightarrow{a^*} \mathrm{Map}^*(S^{n-1}, BG).$$

Theorem 1.1. *Let $N \in \mathcal{N}$ and let G be a topological group whose multiplication is homotopy commutative. Then the map b^* in (1) has a right inverse and there is a homotopy equivalence*

$$Map^*(N, BG) \simeq Map^*(\bigvee_{i=1}^{m} \Sigma A_i, BG) \times Map^*(S^n, BG).$$

A p-local version of Theorem 1.1 also holds if the multiplication on G is only homotopy commutative at p. This is particularly relevant since James and Thomas [9] showed that no simply-connnected, simple compact Lie group has its standard multiplication being homotopy commutative, but McGibbon [15] showed that after localizing at an odd prime there are cases when the multiplication is homotopy commutative and he classified these. The classification is given in Section 2.

The connection with gauge groups comes from work of Gottlieb [5] or Atiyah and Bott [2]. They showed that if M is a pointed space and $P \longrightarrow M$ is a principal G-bundle then there is a homotopy equivalence $B\mathcal{G}^*(P) \simeq \mathrm{Map}^*_P(M, BG)$, where $\mathrm{Map}^*_P(M, BG)$ is the component of $\mathrm{Map}^*(M, BG)$ that contains the map inducing P. Consider two cases. First, let M be a closed simply-connected four-manifold and let G be a simply-connected simple compact Lie group. By [16], M is homotopy equivalent to a CW-complex $(\bigvee_{i=1}^{m} S^2) \cup_a e^4$. Second, let M be an orientable closed Riemann surface of genus g and let $G = U(n)$. Classically (see [6] for instance), M is homotopy equivalent to a CW-complex $(\bigvee_{i=1}^{2g} S^1) \cup_a e^2$. In either case, $[M, BG] \cong \mathbb{Z}$ so there is a component of $\mathrm{Map}^*(M, BG)$ for each integer k, and this integer determines a corresponding equivalence class of principal G-bundles $P \longrightarrow M$. Write P_k for the equivalence class corresponding to k and let $\mathcal{G}^*_k(M) = \mathcal{G}^*(P_k)$.

Let $\Omega^3_0 G$ be the component of $\Omega^3 G$ containing the basepoint. Write $X_{(p)}$ for a space X localized at the prime p.

Corollary 1.2. *Let M be a closed simply-connected Spin four-manifold with m two-cells, $m \geq 2$, and let G be a simply-connected simple compact Lie group whose multiplication is homotopy commutative when localized at p. Then there is a p-local homotopy equivalence*

$$B\mathcal{G}_k^*(M)_{(p)} \simeq (\prod_{i=1}^{m} \Omega G_{(p)}) \times \Omega_0^3 G_{(p)}.$$

In the second case, stabilize by considering the infinite unitary group U. Since U is an infinite loop space its loop multiplication is homotopy commutative. Write Σ_g for the surface of genus g, and let $\Omega_0 U$ be the component of ΩU containing the basepoint.

Corollary 1.3. *Let Σ_g be a closed orientable closed Riemann surface of genus $g \geq 1$. Then there is an integral homotopy equivalence*

$$B\mathcal{G}_k^*(\Sigma_g) \simeq (\prod_{i=1}^{2g} U) \times \Omega_0 U.$$

Corollaries 1.2 and 1.3 are the first systematic decompositions of the classifying spaces of pointed gauge groups. In the context of Corollary 1.2, Masbaum [14] proved the $G = SU(2)$ case earlier but by using different methods that depended on the specific group. Also, while a great deal of work has been done recently to identify the p-local homotopy types of gauge groups [10, 12, 13, 19] and study their properties [11], nothing has been done for their classifying spaces.

Applications of these decompositions to the mod-p homology of gauge groups and the homotopy type of $\mathrm{Hom}(\pi_1(\Sigma_g), U)$ will be discussed in the final section of the paper.

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2. Preliminary homotopy theory

In this section we discuss some notions from homotopy theory involving Whitehead products and the homotopy commutativity of topological groups. As we are building towards a strictly commutative diagram in (6) rather than a homotopy commutative diagram, some extra care will be taken along the way.

Let G be a topological group and let

$$ev\colon \Sigma\Omega BG \longrightarrow BG$$

be the evaluation map. Let $i_L\colon \Sigma\Omega BG \longrightarrow \Sigma\Omega BG \vee \Sigma\Omega BG$ and $i_R\colon \Sigma\Omega BG \longrightarrow \Sigma\Omega BG \vee \Sigma\Omega BG$ be the inclusions of the left and right wedge summands respectively and let

$$[i_L, i_R]\colon \Sigma\Omega BG \wedge \Omega BG \longrightarrow \Sigma\Omega BG \vee \Sigma\Omega BG$$

be the Whitehead product of i_L and i_R. By [1] there is a homotopy equivalence

$$(\Sigma\Omega BG \vee \Sigma\Omega BG) \cup_{[i_L, i_R]} C(\Sigma\Omega BG \wedge \Omega BG) \simeq \Sigma\Omega BG \times \Sigma\Omega BG$$

where $C(\Sigma\Omega BG \wedge \Omega BG)$ is the reduced cone on $\Sigma\Omega BG \wedge \Omega BG$. Let t be the composite

$$t\colon \Sigma\Omega BG \vee \Sigma\Omega BG \xrightarrow{ev \vee ev} BG \vee BG \xrightarrow{\nabla} BG$$

where ∇ is the folding map and let

$$[ev, ev]\colon \Sigma\Omega BG \wedge \Omega BG \longrightarrow BG$$

be the Whitehead product of ev with itself. Note that $[ev, ev]$ is homotopic to $\nabla \circ [i_L, i_R]$. The following proposition connects the homotopy commutativity of G to the existence of a certain extension.

Proposition 2.1. *Let G be a topological group. Then the following are equivalent:*

(a) *G is homotopy commutative;*
(b) *the Whitehead product $[ev, ev]$ is null homotopic;*
(c) *there is a strictly commutative diagram*

$$\begin{array}{ccc} \Sigma\Omega BG \vee \Sigma\Omega BG & \xrightarrow{\quad t \quad} & BG \\ \downarrow & \nearrow_{e} & \\ (\Sigma\Omega BG \vee \Sigma\Omega BG) \cup_{[i_L, i_R]} C(\Sigma\Omega BG \wedge \Omega BG) & & \end{array}$$

for some map e. □

Proof. The equivalence of parts (a) and (b) was proved by James and Thomas [8] and the equivalence of parts (b) and (c) was proved by Arkowitz [1]. □

Remark 2.2. It should be noted that the homotopy commutativity condition in Proposition 2.1 is fairly restrictive. For example, there are no simply-connected, simple compact Lie groups which are homotopy commutative [9]. However, obstructions to homotopy commutativity may vanish when localized at a prime p (see [7] for a good discussion of localization). McGibbon [15] classified those simply-connected, simple compact Lie groups G which are homotopy commutative at p. To describe these, recall that G is rationally homotopy equivalent to a product of spheres, $G \simeq_{\mathbb{Q}} \prod_{i=1}^{l} S^{2n_i - 1}$. The *type* of G is defined to be $\{n_1, \ldots, n_l\}$. The loop multiplication on G is homotopy commutative when localized at p in precisely the following cases:

$$p > 2n_l; \qquad G = Sp(2) \text{ and } p = 3; \qquad G = G_2 \text{ and } p = 5. \tag{2}$$

On the other hand, Bott periodicity implies that the infinite matrix groups U, SU, SO and Sp are all infinite loop spaces and so are integrally homotopy commutative.

Next, we generalize the (a) implies (c) part of Proposition 2.1. Let $X_1, \ldots, X_m$ be path-connected, pointed spaces and consider the wedge $\bigvee_{i=1}^m \Sigma X_i$. For $1 \leq j \leq m$, let $\iota_j \colon \Sigma X_j \longrightarrow \bigvee_{i=1}^m \Sigma X_i$ be the inclusion of the j^{th} wedge summand. Let

$$f\colon \bigvee_{1\leq j<k\leq m} \Sigma X_j \wedge X_k \longrightarrow \bigvee_{i=1}^m \Sigma X_i$$

be the wedge sum of the Whitehead products $[\iota_j, \iota_k]$. Let

$$T(\Sigma X_1, \ldots, \Sigma X_m) = \Big(\bigvee_{i=1}^m \Sigma X_i\Big) \cup_f C\Big(\bigvee_{1\leq j<k\leq m} \Sigma X_j \wedge X_k\Big).$$

Observe that there is a homotopy equivalence

$$T(\Sigma X_1, \ldots, \Sigma X_m) \simeq \bigcup_{1\leq j<k\leq m} \Sigma X_j \times \Sigma X_k.$$

To be clear, $T(\Sigma X_1, \ldots, \Sigma X_m)$ is a subspace of $\Sigma X_1 \times \cdots \times \Sigma X_m$, each $\Sigma X_j \times \Sigma X_k$ in the union is regarded as including into the (j,k) coordinates of $\Sigma X_1 \times \cdots \times \Sigma X_m$, and intersections are identified.

This construction is natural. Suppose that there are maps $g\colon \Sigma A \longrightarrow Z$, $h\colon \Sigma B \longrightarrow Z$ and $t\colon Z \longrightarrow Z'$. Represent the homotopy class $[g,h]$ as the adjoint of the Samelson product $\langle g', h'\rangle$, where $g'\colon A \longrightarrow \Omega Z$ and $h'\colon B \longrightarrow \Omega Z$ are the adjoints of g and h respectively. The Samelson product is defined by the pointwise commutator in ΩZ, which commutes with any loop map $\Omega Z \xrightarrow{\Omega t} \Omega Z'$. Thus we obtain $t \circ [g,h] = [t \circ g, t \circ h]$ on the nose. Hence, given maps $f_i\colon \Sigma X_i \longrightarrow \Sigma X_i'$ for $1 \leq i \leq m$, we obtain a strictly commutative diagram

$$\begin{array}{ccc} \bigvee_{i=1}^m \Sigma X_i & \xrightarrow{\bigvee_{i=1}^m f_i} & \bigvee_{i=1}^m \Sigma X_i' \\ \downarrow & & \downarrow \\ T(\Sigma X_1, \ldots, \Sigma X_m) & \xrightarrow{T(f_1,\ldots,f_m)} & T(\Sigma X_1', \ldots, \Sigma X_m'). \end{array} \tag{3}$$

In our case, for $1 \leq i \leq m$, let $X_i = \Omega BG$. Write $T(\Sigma\Omega BG)$ for $T(\Sigma\Omega BG, \ldots, \Sigma\Omega BG)$. Let t_m be the composite

$$t_m\colon \bigvee_{i=1}^m \Sigma\Omega BG \xrightarrow{\bigvee_{i=1}^m ev} \bigvee_{i=1}^m BG \xrightarrow{\nabla_m} BG$$

where ∇_m is the m-fold folding map. By Proposition 2.1, if G is homotopy commutative then the restriction of t_m to any pair $\Sigma\Omega BG \vee \Sigma\Omega BG$ extends to a map

$$(\Sigma\Omega BG \vee \Sigma\Omega BG) \cup_{[i_L, i_R]} C(\Sigma\Omega BG \wedge \Omega BG) \longrightarrow BG.$$

Construct an extension for all pairs of wedge summands indexed by (j,k) for $1 \leq j < k \leq m$. Observe that the extensions are compatible because they intersect only on the wedge summands. Thus they may be assembled to produce a map $T(\Sigma\Omega BG) \longrightarrow BG$ extending t_m. This is recorded as follows.

Lemma 2.3. *Let G be a topological group whose loop multiplication is homotopy commutative, Then there is a strictly commutative diagram*

$$\begin{array}{ccc} \bigvee_{i=1}^{m} \Sigma\Omega BG & \xrightarrow{t_m} & BG \\ \downarrow & \nearrow_{e_m} & \\ T(\Sigma\Omega BG) & & \end{array}$$

for some map e_m. □

We close this section with one more observation about $T(\Sigma X_1, \ldots, \Sigma X_m)$. Let $X \xrightarrow{E} \Omega\Sigma X$ be the suspension map, defined by sending $x \in X$ to the loop ω_x on ΣX, where ω_x is characterized by $\omega_x(t) = (t, x)$. The evaluation map $\Sigma\Omega Y \xrightarrow{ev} Y$ is defined by sending (s, ω) to $\omega(s)$. The definitions imply that the composite $\Sigma X \xrightarrow{\Sigma E} \Sigma\Omega\Sigma X \xrightarrow{ev} \Sigma X$ is the identity map on ΣX. Now suppose that there is a map $f \colon \Sigma X \longrightarrow Y$. The naturality of the evaluation map implies that there is a strictly commutative diagram

$$\begin{array}{ccccc} & & \Sigma\Omega\Sigma X & \xrightarrow{\Sigma\Omega f} & \Sigma\Omega Y \\ & {}^{\Sigma E}\nearrow & \downarrow{\scriptstyle ev} & & \downarrow{\scriptstyle ev} \\ \Sigma X & = & \Sigma X & \xrightarrow{f} & Y. \end{array}$$

Thus, if $\overline{f} = (\Sigma\Omega f) \circ \Sigma E$, then we obtain a lift

$$\begin{array}{ccc} & & \Sigma\Omega Y \\ & {}^{\overline{f}}\nearrow & \downarrow{\scriptstyle ev} \\ \Sigma X & \xrightarrow{f} & Y. \end{array} \tag{4}$$

Combining this with (3) we obtain the following.

Lemma 2.4. *Suppose that for $1 \leq i \leq m$ there are maps $f_i \colon \Sigma X_i \longrightarrow Y$. Then there is a strictly commutative diagram*

$$\begin{array}{ccc} \bigvee_{i=1}^{m} \Sigma X_i & \xrightarrow{\bigvee_{i=1}^{m} \overline{f}_i} & \bigvee_{i=1}^{m} \Sigma\Omega Y \\ \downarrow & & \downarrow \\ T(\Sigma X_1, \ldots, \Sigma X_m) & \xrightarrow{T(\overline{f}_1, \ldots, \overline{f}_m)} & T(\Sigma\Omega Y, \ldots, \Sigma\Omega Y). \end{array}$$

□

3. The class $\mathcal{N}$

Recall from the Introduction that $\mathcal{N}$ is the class of adjunction spaces

$$N = (\bigvee_{i=1}^{m} \Sigma A_i) \cup_a e^n$$

where $\bigvee_{i=1}^{m} \Sigma A_i$ is a CW-complex of dimension strictly less than n, the attaching map a factors through a map a' which is a wedge sum of some of the Whitehead products $\Sigma A_j \wedge A_k \xrightarrow{[\iota_j, \iota_k]}$

$\bigvee_{i=1}^m \Sigma A_i$, and $m \geq 2$. The factorization condition on a can be restrictive. In the context of gauge groups, one typically wants to work with an N that is homotopy equivalent to a manifold. Most manifolds do not satisfy the attaching map condition. However, there are some very interesting families of manifolds that do. For example:

(1) if M is a simply-connected Spin 4-manifold with $H^2(M;\mathbb{Z})$ of rank $m \geq 2$ then M is homotopy equivalent to a CW-complex $(\bigvee_{i=1}^m S^2) \cup_a e^4 \in \mathcal{N}$;

(2) if Σ_g is a closed orientable surface of genus $g \geq 1$ then Σ_g is homotopy equivalent to a CW-complex $(\bigvee_{i=1}^{2g} S^1) \cup_a e^2 \in \mathcal{N}$;

(3) if M is a simply-connected Spin 5-manifold then M is homotopy equivalent to a CW-complex $(\bigvee_{i=1}^m \Sigma A_i) \cup_a e^5$ where each ΣA_i is either S^2, S^3 or a mod-p^r Moore space of dimension 3, and if $m \geq 2$ then this CW-complex is in $\mathcal{N}$.

The CW-structure for M in (1) is due to Milnor [16]; the CW-structure for Σ_g in (2) is commonly known, one reference is [6]; the CW-structure for M in (3) is given in [18]. Other examples exist, such as certain $(n-1)$-connected $2n$-dimensional manifolds [20] and the connected sum of products of two spheres.

The property that is needed for the spaces in $\mathcal{N}$ is the following. Recall that there is a homotopy cofibration $S^{n-1} \xrightarrow{a} \bigvee_{i=1}^m \Sigma A_i \xrightarrow{b} N$ where b is the inclusion.

Lemma 3.1. *Let $N \in \mathcal{N}$. Then there is an extension*

$$\begin{array}{ccc} \bigvee_{i=1}^m \Sigma A_i & \xrightarrow{b} & N \\ \downarrow & \swarrow_{e_N} & \\ T(\Sigma A_1, \ldots, \Sigma A_m) & & \end{array}$$

for some map e_N.

Proof. Since $N = (\bigvee_{i=1}^m \Sigma A_i) \cup_a e^n$, to show that the extension e_N exists it is equivalent to show that the composite $S^{n-1} \xrightarrow{a} \bigvee_{i=1}^m \Sigma A_i \longrightarrow T(\Sigma A_1, \ldots, \Sigma A_m)$ is null homotopic. By definition, $T(\Sigma A_1, \ldots, \Sigma A_m)$ is the adjunction space formed from coning off the sum of all the Whitehead products $[\iota_j, \iota_k]$ for $1 \leq j < k \leq m$. In particular, each composition $\Sigma A_j \wedge A_k \xrightarrow{[\iota_j,\iota_k]} \bigvee_{i=1}^m \Sigma A_i \longrightarrow T(\Sigma A_1, \ldots, \Sigma A_m)$ is null homotopic. Thus, as a factors through a wedge sum of some of the Whitehead products $[\iota_j, \iota_k]$, the composite $S^{n-1} \xrightarrow{a} \bigvee_{i=1}^m \Sigma A_i \longrightarrow T(\Sigma A_1, \ldots, \Sigma A_m)$ is also null homotopic. □

4. A decomposition of Map$^*(N, BG)$

Let $N \in \mathcal{N}$. In the sequence of maps

$$S^{n-1} \xrightarrow{a} \bigvee_{i=1}^m \Sigma A_i \xrightarrow{b} N \xrightarrow{q} S^n$$

the maps a and b form a homotopy cofibre sequence, while b and q form a cofibre sequence on the nose. If G is a topological group then there is an induced sequence

$$\mathrm{Map}^*(S^n, BG) \xrightarrow{q^*} \mathrm{Map}^*(N, BG) \xrightarrow{b^*} \mathrm{Map}^*(\bigvee_{i=1}^{m} \Sigma A_i, BG) \xrightarrow{a^*} \mathrm{Map}^*(S^{n-1}, BG) \tag{5}$$

where the maps q^* and b^* form a fibre sequence on the nose while b^* and a^* form a homotopy fibre sequence. We will show that if the multiplication on G is homotopy commutative then the map b^* has a right inverse.

Let $f\colon \bigvee_{i=1}^m \Sigma A_i \longrightarrow BG$ be a pointed map. Universally, a map out of a wedge is determined by its restrictions to the wedge summands, so $f = \bigvee_{i=1}^m f_i$ where $f_i\colon \Sigma A_i \longrightarrow BG$ is the restriction of f to ΣA_i. By (4), each f_i lifts through $\Sigma\Omega BG \xrightarrow{ev} BG$ to a map $\overline{f}_i = (\Sigma\Omega f_i) \circ \Sigma E$. So if $N \in \mathcal{N}$ and the multiplication on G is homotopy commutative, we may combine the diagrams in Lemmas 2.3, 2.4 and 3.1 to obtain a strictly commutative diagram

$$\begin{array}{ccccccc} & & \bigvee_{i=1}^m \Sigma A_i & \xrightarrow{\bigvee_{i=1}^m \overline{f}_i} & \bigvee_{i=1}^m \Sigma\Omega BG & \xrightarrow{t_m} & BG \\ & \swarrow^{b} & \downarrow & & \downarrow & \nearrow_{e_m} & \\ N & \xrightarrow{e_N} & T(\Sigma A_1,\ldots,\Sigma A_m) & \xrightarrow{T(\overline{f}_1,\ldots,\overline{f}_m)} & T(\Sigma\Omega BG). & & \end{array} \tag{6}$$

By the definitions of t_m and each $\overline{f}_i$, we have $t_m \circ (\bigvee_{i=1}^m \overline{f}_i) = \bigvee_{i=1}^m f_i$. So (6) lets us define a map

$$\theta\colon \mathrm{Map}^*(\bigvee_{i=1}^{m} \Sigma A_i, BG) \longrightarrow \mathrm{Map}^*(N, BG)$$

by $\theta(f) = \theta(\bigvee_{i=1}^m f_i) = e_m \circ T(\overline{f}_1, \ldots, \overline{f}_m) \circ e_N$. We wish to show that θ is continuous and that $b^* \circ \theta$ is the identity map.

Lemma 4.1. *The map θ is continuous.*

Proof. The map θ is defined as the composite of the continuous maps e_m and e_N and the continuous functor $T(\overline{f}_1, \ldots, \overline{f}_m)$. Note that if Y is a locally compact Hausdorff space then the composition $\mathrm{Map}^*(Y, Z) \times \mathrm{Map}^*(X, Y) \longrightarrow \mathrm{Map}^*(X, Z)$ is continuous with respect to the compact open topology. Therefore θ is continuous. □

Lemma 4.2. *The composite of continuous maps*

$$Map^*(\bigvee_{i=1}^{m} \Sigma A_i, BG) \xrightarrow{\theta} Map^*(N, BG) \xrightarrow{b^*} Map^*(\bigvee_{i=1}^{m} \Sigma A_i, BG)$$

is equal to the identity map.

Proof. By definition, b^* sends a map $\phi\colon N \longrightarrow BG$ to the composite $\bigvee_{i=1}^m \Sigma A_i \xrightarrow{b} N \xrightarrow{\phi} BG$. Therefore, by definition of θ, we have

$$b^* \circ \theta(f) = b^* \circ \theta(\bigvee_{i=1}^{m} f_i) = b^*(e_m \circ T(\overline{f}_1, \ldots, \overline{f}_m) \circ e_N) = e_m \circ T(\overline{f}_1, \ldots, \overline{f}_m) \circ e_N \circ b.$$

By (6) and the definition of t_m, we have

$$e_m \circ T(\overline{f}_1, \ldots, \overline{f}_m) \circ e_N \circ b = t_m \circ (\bigvee_{i=1}^{m} \overline{f}_i) = \bigvee_{i=1}^{m} f_i = f.$$

Thus $b^* \circ \theta(f) = f$. $\square$

Proof of Theorem 1.1. In general, suppose that $\Omega B \xrightarrow{\partial} F \xrightarrow{r} E \xrightarrow{s} B$ is a homotopy fibration sequence and r has a right homotopy inverse $t\colon E \longrightarrow F$. Then s is null homotopic because: (i) $r \circ t \simeq 1_E$ implies that $s \simeq s \circ r \circ t$, and (ii) $s \circ r$ is null homotopic as it is the composition of two consecutive maps in a homotopy fibration. The null homotopy for s implies that $F \simeq E \times \Omega B$. In our case, consider the homotopy fibration sequence (5). By Lemma 4.2, the map b^* has a right inverse. Therefore there is a homotopy equivalence

$$\mathrm{Map}^*(N, BG) \simeq \mathrm{Map}^*(\bigvee_{i=1}^{m} \Sigma A_i, BG) \times \mathrm{Map}^*(S^n, BG).$$

$\square$

To illustrate Theorem 1.1 we consider two cases of interest. Note that $\mathrm{Map}^*(S^t, BG) \simeq \Omega^{t-1}G$.

Example 4.3. Let M be a simply-connected Spin 4-manifold with m two-cells, where $m \geq 2$. As in Section 3, there is a homotopy equivalence $M \simeq (\bigvee_{i=1}^{m} S^2) \cup_a e^4$. Let G be a simply-connected, simple compact Lie group listed in (2), whose multiplication is homotopy commutative when localized at p. By [7], p-localization commutes with mapping spaces in the context of simply-connected (and more generally, nilpotent) spaces, so we have $\mathrm{Map}^*(M, BG)_{(p)} \simeq \mathrm{Map}^*(M_{(p)}, BG_{(p)})$. Thus Theorem 1.1 implies that there is a homotopy equivalence

$$\mathrm{Map}^*(M, BG)_{(p)} \simeq (\prod_{i=1}^{m} \Omega G_{(p)}) \times \Omega^3 G_{(p)}.$$

Example 4.4. Let Σ_g be a close orientable surface of genus $g \geq 1$. As in Section 3, $\Sigma_g \simeq (\bigvee_{i=1}^{2g} S^1) \cup_a e^2 \in \mathcal{N}$. Let $G = U$, the infinite unitary group. Since U is an infinite loop space it is homotopy commutative so by Theorem 1.1 there is a homotopy equivalence

$$\mathrm{Map}^*(\Sigma_g, BU) \simeq (\prod_{i=1}^{2g} U) \times \Omega U.$$

We close this section by proving Corollaries 1.2 and 1.3.

Proof of Corollary 1.2. Recall from the Introduction that if G is a simply-connected simple compact Lie group, M is a simply-connected four-manifold and $P_k \longrightarrow M$ is a principal G-bundle induced by the homotopy class in $[M, BG] \cong \mathbb{Z}$ corresponding to k, then there is a homotopy equivalence $B\mathcal{G}_k^*(M) \simeq \mathrm{Map}_k^*(M, BG)$. By Example 4.3, there is a p-local homotopy equivalence $\mathrm{Map}_k^*(M, BG)_{(p)} \simeq (\prod_{i=1}^{m} \Omega G_{(p)}) \times \Omega_k^3 G_{(p)}$, where $\Omega_k^3 G$ is the connected component of $\Omega^3 G$ that contains the map $S^3 \longrightarrow G$ of degree k in the third homology group. Since $\pi_0(\Omega^3 G)$ is a group, there is a homotopy equivalence $\Omega^3_k G \simeq \Omega^3_0 G$. Therefore $B\mathcal{G}^*_k(M)_{(p)} \simeq (\prod_{i=1}^m \Omega G_{(p)}) \times \Omega^3_0 G_{(p)}$, as asserted. $\square$

Proof of Corollary 1.3. Again, recall from the Introduction that if $G = U$, Σ_g is a closed orientable surface of genus g and $P_k \longrightarrow \Sigma_g$ is a principal G-bundle induced by the homotopy class in $[\Sigma_g, BU] \cong \mathbb{Z}$ corresponding to k, then there is a homotopy equivalence $B\mathcal{G}_k(\Sigma_g) \simeq \mathrm{Map}^*_k(\Sigma_g, BU)$. By Example 4.4, there is a homotopy equivalence $\mathrm{Map}^*_k(\Sigma_g, BU) \simeq (\prod_{i=1}^{2g} U) \times \Omega_k U$ where $\Omega_k U$ is the connected component of ΩU that contains the map $S^1 \longrightarrow U$ of degree k in the first homology group. Since $\pi_0(\Omega U)$ is a group, there is a homotopy equivalence $\Omega_k U \simeq \Omega_0 U$. Therefore there is a homotopy equivalence $B\mathcal{G}_k(\Sigma_g) \simeq (\prod_{i=1}^{2g} U) \times \Omega U$, as asserted. $\square$

5. Applications

In this section we give two applications, one to the calculation of the mod-p homology or cohomology of the classifying space of certain full gauge groups, and the other to the homotopy type of a certain group of homomorphisms.

First, return to the case when G is a simply-connected simple compact Lie group, M is a simply-connected four-manifold, and $P_k \longrightarrow M$ is a principal G-bundle induced by the homotopy class in $[M, BG] \cong \mathbb{Z}$ corresponding to k. By [2] there is a homotopy commutative diagram

$$\begin{array}{ccc} B\mathcal{G}^*_k(M) & \longrightarrow & B\mathcal{G}_k(M) \\ \downarrow{\scriptstyle \psi^*} & & \downarrow{\scriptstyle \psi} \\ \mathrm{Map}^*_k(M, BG) & \longrightarrow & \mathrm{Map}_k(M, BG) \end{array} \tag{7}$$

where ψ^* and ψ are homotopy equivalences. Observe also that there is a fibration

$$\mathrm{Map}^*_k(M, BG) \longrightarrow \mathrm{Map}_k(M, BG) \xrightarrow{ev} BG$$

where ev evaluates a map at the basepoint of M. Stated in terms of gauge groups, up to homotopy equivalences, there is a fibration

$$B\mathcal{G}^*_k(M) \longrightarrow B\mathcal{G}_k(M) \longrightarrow BG.$$

Take homology and cohomology with mod-p coefficients. Corollary 1.2 immediately implies that if G is homotopy commutative when localized at p then there is a coalgebra isomorphism

$$H_*(B\mathcal{G}^*_k(M)) \cong (\otimes_{i=1}^m H_*(\Omega G)) \otimes H_*(\Omega^2_0 G)$$

and an algebra isomorphism

$$H^*(B\mathcal{G}^*_k(M)) \cong (\otimes_{i=1}^m H^*(\Omega G)) \otimes H^*(\Omega^2_0 G).$$

We aim to prove the following.

Theorem 5.1. *Let M be a closed simply-connected Spin four-manifold and let G be a simply-connected simple compact Lie group whose multiplication is homotopy commutative when localized at p. Then the composite of coalgebras*

$$\otimes_{i=1}^{m} H_*(\Omega G) \longrightarrow H_*(B\mathcal{G}_k^*(M)) \longrightarrow H_*(B\mathcal{G}_k(M))$$

has a left inverse, and the composite of algebras

$$H^*(B\mathcal{G}_k(M)) \longrightarrow H^*(B\mathcal{G}_k^*(M)) \longrightarrow \otimes_{i=1}^{m} H^*(\Omega G)$$

has a right inverse.

For example, let $G = SU(2)$, in which case G is homeomorphic to S^3 and $H^*(\Omega S^3)$ is well known. This case is of key interest in Donaldson theory and a major open problem is to calculate the mod-p homology of $B\mathcal{G}_k(M)$. As $SU(2)$ is homotopy commutative when localized at primes $p \geq 5$, Theorem 5.1 applies for any such prime, giving significant information about $H_*(B\mathcal{G}_k(M))$.

To prove Theorem 5.1, we begin by recalling some general facts about mapping spaces. Let $X_1, \ldots, X_m$ and Y be Hausdorff spaces, and let $\coprod_{i=1}^{m} X_i$ be their disjoint union. Then there is a homemorphism

$$\mathrm{Map}\,(\coprod_{i=1}^{m} X_i, Y) \cong \prod_{i=1}^{m} \mathrm{Map}\,(X_i, Y).$$

Further, if each of $X_1, \ldots, X_m$ and Y are pointed, then there is a homeomorphism

$$\mathrm{Map}^*(\bigvee_{i=1}^{m} X_i, Y) \cong \prod_{i=1}^{m} \mathrm{Map}^*(X_i, Y).$$

These two decompositions are compatible in the following sense. There is a quotient map

$$\mathfrak{q}\colon \coprod_{i=1}^{m} X_i \longrightarrow \bigvee_{i=1}^{m} X_i$$

which identifies the basepoints in each space X_i to a common point. So there is an induced map

$$\mathfrak{q}^*\colon \mathrm{Map}\,(\bigvee_{i=1}^{m} X_i, Y) \longrightarrow \mathrm{Map}\,(\coprod_{i=1}^{m} X_i, Y).$$

The two homeomorphisms above are compatible via a strictly commutative diagram

$$\begin{array}{ccccc}
\mathrm{Map}^*(\bigvee_{i=1}^{m} X_i, Y) & \xrightarrow{\mathrm{incl}} & \mathrm{Map}\,(\bigvee_{i=1}^{m} X_i, Y) & \xrightarrow{\mathfrak{q}^*} & \mathrm{Map}\,(\coprod_{i=1}^{m} X_i, Y) \\
\downarrow\cong & & & & \downarrow\cong \\
\prod_{i=1}^{m} \mathrm{Map}^*(X_i, Y) & & \xrightarrow{\prod_{i=1}^{m}\mathrm{incl}} & & \prod_{i=1}^{m} \mathrm{Map}\,(X_i, Y).
\end{array} \tag{8}$$

Returning to the case of interest, as in Section 3, if M is any closed simply-connected Spin 4-manifold then there is a space $N = (\bigvee_{i=1}^{m} S^2) \cup_a e^4 \in \mathcal{N}$. The inclusion $\bigvee_{i=1}^{m} S^2 \xrightarrow{b} N$ induces a commutative diagram

$$(9)\qquad \begin{array}{ccc} \mathrm{Map}^*(N,BG) & \longrightarrow & \mathrm{Map}\,(N,BG) \\ \downarrow{\scriptstyle b^*} & & \downarrow{\scriptstyle b^*} \\ \mathrm{Map}^*(\bigvee_{i=1}^m S^2,BG) & \longrightarrow & \mathrm{Map}\,(\bigvee_{i=1}^m S^2,BG). \end{array}$$

Localizing at p, the fact that mapping spaces commute with localization of nilpotent spaces [7] implies that there is a homotopy commutative diagram

$$(10)\qquad \begin{array}{ccc} \mathrm{Map}^*(M,BG)_{(p)} & \longrightarrow & \mathrm{Map}\,(M,BG)_{(p)} \\ \downarrow{\scriptstyle \simeq} & & \downarrow{\scriptstyle \simeq} \\ \mathrm{Map}^*(N,BG)_{(p)} & \longrightarrow & \mathrm{Map}\,(N,BG)_{(p)} \end{array}$$

Juxtaposing the diagrams (7), (8), (9) and (10) we obtain a p-local homotopy commutative diagram

$$\begin{array}{ccc} B\mathcal{G}_k^*(M)_{(p)} & \longrightarrow & B\mathcal{G}_k(M)_{(p)} \\ \downarrow{\scriptstyle \psi^*} & & \downarrow{\scriptstyle \psi} \\ \mathrm{Map}_k^*(M,BG)_{(p)} & \longrightarrow & \mathrm{Map}\,_k(M,BG)_{(p)} \\ \downarrow{\scriptstyle \simeq} & & \downarrow{\scriptstyle \simeq} \\ \mathrm{Map}_k^*(N,BG)_{(p)} & \longrightarrow & \mathrm{Map}\,_k(N,BG)_{(p)} \\ \downarrow{\scriptstyle b^*} & & \downarrow{\scriptstyle b^*} \\ \mathrm{Map}^*(\bigvee_{i=1}^m S^2,BG)_{(p)} & \longrightarrow & \mathrm{Map}\,(\bigvee_{i=1}^m S^2,BG)_{(p)} \\ \Big\downarrow{\scriptstyle \cong} & & \downarrow{\scriptstyle \mathfrak{q}^*} \\ & & \mathrm{Map}\,(\coprod_{i=1}^m S^2,BG)_{(p)} \\ & & \downarrow{\scriptstyle \cong} \\ \prod_{i=1}^m \mathrm{Map}^*(S^2,BG)_{(p)} & \xrightarrow{\prod_{i=1}^m \mathrm{incl}} & \prod_{i=1}^m \mathrm{Map}\,(S^2,BG)_{(p)}. \end{array}$$

By Lemma 4.2, the map b^* has a right inverse. Lifting this, up to homotopy, through the homotopy equivalences $B\mathcal{G}_k^*(M)_{(p)} \xrightarrow{\psi^*} \mathrm{Map}_k^*(M,BG)_{(p)} \xrightarrow{\simeq} \mathrm{Map}_k^*(N,BG)_{(p)}$, we obtain the following.

Lemma 5.2. *Let M be a closed simply-connected Spin four-manifold and let G be a simply-connected simple compact Lie group whose multiplication is homotopy commutative when localized at a prime p. Then there is a homotopy commutative diagram*

$$\begin{array}{ccccc} Map_k^*(\bigvee_{i=1}^m S^2,BG)_{(p)} & \longrightarrow & B\mathcal{G}_k^*(M)_{(p)} & \longrightarrow & B\mathcal{G}_k(M)_{(p)} \\ & \searrow^{\simeq} & \downarrow & & \downarrow \\ & & \prod_{i=1}^m Map^*(S^2,BG)_{(p)} & \xrightarrow{\prod_{i=1}^m incl} & \prod_{i=1}^m Map\,(S^2,BG)_{(p)}. \end{array}$$ □

Lemma 5.2 is used to extract information about $H_*(B\mathcal{G}_k(M))$ and $H^*(B\mathcal{G}_k(M))$.

Proof of Theorem 5.1. Consider the map $\mathrm{Map}^*(S^2, BG) \xrightarrow{\text{incl}} \mathrm{Map}\,(S^2, BG)$ whose p-localization appears in the bottom row of the diagram in Lemma 5.2. The inclusion is the fibre of the evaluation map $\mathrm{Map}\,(S^2, BG) \xrightarrow{ev} BG$ which sends a map $f\colon S^2 \longrightarrow BG$ to $f(*)$. Also, we have $\mathrm{Map}^*(S^2, BG) = \Omega G$. So there is a fibration

$$\Omega G \longrightarrow \mathrm{Map}\,(S^2, BG) \xrightarrow{ev} BG. \tag{11}$$

By (2), the cases when the multiplication on G is homotopy commutative when localized at p are known. In each such case, $H^*(G)$ is an exterior algebra on odd degree generators, so by [3] $H^*(BG)$ is a polynomial algebra on even degree generators. Since cohomology is with mod-p coefficients, we can dualize to see that $H_*(BG)$ is also concentrated in even degrees. Further, by [4] the integral cohomology of ΩG is concentrated in even degrees, and therefore so is the mod-p cohomology. Therefore the homology Serre spectral sequence for the fibration (11) collapses at the E^2-term and there are no extension issues. Hence

$$H_*(\mathrm{Map}\,(S^2, BG)) \cong H_*(BG) \otimes H_*(\Omega G).$$

Consequently, taking homology for the diagram in Lemma 5.2, we see that the composite

$$\otimes_{i=1}^m H_*(\Omega G) \longrightarrow H_*(B\mathcal{G}_k^*(M)) \longrightarrow H_*(B\mathcal{G}_k(M))$$

has a left inverse.

Similarly,

$$H^*(\mathrm{Map}\,(S^2, BG)) \cong H^*(BG) \otimes H^*(\Omega G)$$

and the composite

$$H^*(B\mathcal{G}_k(M)) \longrightarrow H^*(B\mathcal{G}_k^*(M)) \longrightarrow \otimes_{i=1}^m H^*(\Omega G)$$

has a right inverse. □

We now turn to the second application. Let K and L be topological groups, and let $\mathrm{Hom}(K, L)$ be the set of homomorphisms from K to L, topologized as a subspace of the mapping space $\mathrm{Map}(K, L)$. If BK, BL are the classifying spaces of K and L respectively, there is a natural map

$$B\colon \mathrm{Hom}(K, L) \longrightarrow \mathrm{Map}^*(BK, BL).$$

This map has been a subject of intense study due to its connections with the Sullivan conjecture in homotopy theory, to the moduli space of representations in algebraic geometry, and to the space of flat connections modulo gauge equivalence in Yang-Mills theory. Consider the special case

$$\mathrm{Hom}(\pi_1(\Sigma_g), U(n)) \longrightarrow \mathrm{Map}^*(B\pi_1(\Sigma_g), BU(n)).$$

Since the universal cover of Σ_g is contractible there is a homotopy equivalence $\Sigma_g \simeq B\pi_1(\Sigma_g)$. So up to a homotopy equivalence we may regard the preceding map as

$$\mathrm{Hom}(\pi_1(\Sigma_g), U(n)) \longrightarrow \mathrm{Map}^*(\Sigma_g, BU(n)).$$

Ramras [17, Theorem 3.4] used gauge theoretic methods to show that this map is an injection on π_0 and an isomorphism on π_m for $m \leq 2g(n-1)+1$. Stabilizing to the infinite unitary group, we obtain a map

$$\mathrm{Hom}(\pi_1(\Sigma_g), U) \longrightarrow \mathrm{Map}^*(\Sigma_g, BU)$$

which is an injection on π_0 and an isomorphism on π_m for every $m \geq 1$. Thus if $\mathrm{Hom}_I(\pi_1(\Sigma_g), U))$ is the component of $\mathrm{Hom}(\pi_1(\Sigma_g), U))$ containing the identity map, from Corollary 1.3 we obtain homotopy equivalences

$$\mathrm{Hom}_I(\pi_1(\Sigma_g), U)) \xrightarrow{\simeq} \mathrm{Map}^*_0(\Sigma_g, BU) \xrightarrow{\simeq} (\prod_{i=1}^{2g} U) \times \Omega_0 U$$

which lets one easily identify $\pi_m(\mathrm{Hom}(\pi_1(\Sigma_g), U))$ for $m \geq 1$.

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