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On Mathematical Truth

Wittgenstein and the dissolution of Benacerraf's dilemma

by

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ABSTRACT

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ON MATHEMATICAL TRUTH — WITTGENSTEIN AND THE DISSOLUTION OF BENACERRAF'S DILEMMA

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According to Benacerraf, our most influential theory of meaning leads us to believe that we use mathematical expressions to refer to abstract objects, while our most influential theory of knowledge cannot fathom the idea of a subject knowing about an object that is not causally related to her. So mathematics is either a useful formal linguistic system that we invented, or we can discover causally impassive and inactive objects through intuition. In my view, we only arrive at such a dilemma if we misconceive something along the way. Thus, the goal of this thesis is to dissolve Benacerraf's dilemma. First, I argue for the inadequacy of the dilemma's premises, then I demonstrate the possibility of a non-problematic third alternative. As the argument goes, the dilemma rests on premises that presuppose the truth of the representationalist metasemantics and the causal theory of knowledge. However, these presuppositions promote confusions regarding the expressive function of mathematical language and raise the problematic necessity for postulating abstract elements as regress-stoppers. There are two insights from Wittgenstein at the basis of my criticism: (i) that use is explanatory of meaning, and (ii) that the way we use mathematical statements serves a normative instead of a descriptive function. Then I proceed to develop these insights into an expressivist and pragmatic account of mathematical truth, explained in terms of social practices, lifeforms and their evolutionary history. I argue this view can satisfy Benacerraf's conditions while avoiding his presuppositions and the problems generated by them, thus effectively dissolving this dilemma.

Contents

Declaration of Authorship

I, Aron Barco, declare that the thesis entitled

On Mathematical Truth — Wittgenstein and the dissolution of Benacerraf's dilemma

and the work presented in the thesis are both my own, and have been generated by me as the result of my own original research. I confirm that:

- this work was done wholly or mainly while in candidature for a research degree at this University;
- where any part of this thesis has previously been submitted for a degree or any other qualification at this University or any other institution, this has been clearly stated;
- where I have consulted the published work of others, this is always clearly attributed;
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- I have acknowledged all main sources of help;
- where the thesis is based on work done by myself jointly with others, I have made clear exactly what was done by others and what I have contributed myself;
- none of this work has been published before submission

Signed: ....

Date: 26 July 2018

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[1] In 2013 I was awarded a doctoral scholarship from the CAPES foundation which fully funded my studies at the University of Southampton.

Abbreviations

WORKS BY PAUL BENACERRAF

MT: "Mathematical Truth", in *The Journal of Philosophy* (Seventieth Annual Meeting of the American Philosophical Association Eastern Division), Vol. 70, No. 19, 1973 (pp. 661-679).

WORKS BY LUDWIG WITTGENSTEIN

BT: *The Big Typescript: TS 213*, ed. and trans. by Grant Luckhardt & Maximilian Aue, Oxford: Basil Blackwell, 2005.

LFM: *Lectures on the Foundations of Mathematics*, The University of Chicago Press, 1976.

PI: *Philosophical Investigations*, 4th ed. by G.E.M Ascombe, P.M.S Hacker, and Joachim Schulte, Oxford: Wiley-Blackwell, 2009.

OC: *On Certainty*, Oxford: Basil Blackwell, 1979.

RFM: *Remarks on the Foundations of Mathematics*, 3rd ed. by G. H. Von Wrigth, R. Rhees, , Oxford: Basil Blackwell, 1978.

WLC: *Wittgenstein's Lectures*: *Cambridge, 1932-1935*, ed. by Alice Ambrose, Amherst: Prometheus Books, 2001.

1. Introduction

In *Mathematical Truth* (1973), Paul Benacerraf advanced two seemingly unproblematic requirements for an intelligible account of mathematical truth. According to him, what we seek is a naturalistic explanation of mathematical knowledge, capable of demonstrating the causal connections between us and the objects we claim to know through mathematical activity. Moreover, this account must be subsumed under a global semantic theory of truth, so that our conception of mathematical truth coheres with other instances of truth, amalgamating into a useful consistent concept across all sciences.

However, Benacerraf realized that the satisfaction of one condition seems to preclude the satisfaction of the other. A naturalistic account of mathematical knowledge does not fit with the most influential semantics of our time, the truth-conditional, extensional, reference-based semantic theory. When such account is applied to mathematical statements, we explain their meaning in terms of references to objects such as sets and functions which, unlike material objects, do not participate in causal chains. This supports **platonism**, the philosophical view that mathematical knowledge is about entities whose existence is independent of human thought. Yet, by postulating a mind-independent mathematical reality, this view immediately raises an epistemological problem, for how could we discover casually inert objects?

For Benacerraf, this challenge ends up motivating an opposing constructivist view that departs from the epistemological principle that the condition for mathematical truth is proof, not reference. Whereas platonism takes proofs as a formal way of communicating truths that our intuition discovers while diving in pure reason, **combinatorialism** conceives mathematics itself as a proof activity. As such, no mind-independent referents are introduced to ground discussion of mathematical truth, since mathematical statements are not interpreted as referring directly to abstract objects, but instead to our algorithmic routines for computing (or, constructing) mathematical objects. However, as Benacerraf argues, since this view does not offer objective referents, then the reasons that justify attributing objective truth to mathematics are undermined, as its theories could then be conceived as symbolic games in which we compute the implications of certain sets of conventions (i.e. axioms). This raises a semantic problem, for thus we collapse the concept of truth with that of warranted assertibility, risking the cogency of the notion of mathematical truth.

This philosophical rift also echoes amongst mathematicians. According to a historical investigation by José Ferreirós, since the "crisis" in the foundations of mathematics seen at the beginning of the 20th century, mathematicians have been divided on what regards their methodology; "whether mathematical definitions and proofs have to be constructed from a restricted set of basic objects and operations, or can they be based on free (but consistent) postulates such as the axioms of Infinity and Choice."[1] Broadly speaking then, Benacerraf's dilemma would be a particular point of collision within a larger historical antagonism between postulational and constructivist approaches; the point where their distinct conceptions of meaning, knowledge and truth clash, producing two opposing accounts of mathematical truth.

With this in mind, we may say that Benacerraf diagnosed a philosophical incompatibility. Philosophers of mathematics must choose between two mutually exclusive starting points; either they assume that mathematical expressions refer to abstract objects, explaining mathematical truth under a globally referential account of meaning; or they assume that these expressions denote computational methods instead, grounding mathematical knowledge on an empirically viable causal account. This implies that one of the following is correct: either mathematical structures are naturally occurring and subsequently uncovered by us, or these structures are invented to provide frameworks for description. Philosophical discussion is thus polarized between those that see mathematics as the study of immutable abstract structures, and those that see it as logical deduction of abstract possibilities of transformation in structures that only exist in our minds.

Yet, given the problems that appear in each path, their incompatibility exposes the structure of a dilemma: when the semantic condition is not met, we lose the justification for the claim that mathematical results are objective and necessarily true; and when the epistemological condition is not met, we lack an intelligible account of mathematical knowledge. As Benacerraf puts it, we seem to satisfy one condition at the expense of the other (MT, 662). No matter which approach we chose, we will end caught up in one of the dilemma's horns.

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[1] Ferreirós, J. *The Architecture of Modern Mathematics* (ed. by J. Ferreirós and J.J. Gray, Oxford University Press, 2006), p. 6.

As such, Benacerraf believes his conditions have a devastating effect: "jointly they seem to rule out almost every account of mathematical truth that has been proposed." (MT, 668). The challenges presented have troubled professional philosophers ever since it was published 43 years ago.[2] We may be in agreement with the dominant semantics in scientific discourse and have more explanatory power if we postulate abstract objects, but then we create the problem of explaining our knowledge of such entities and our reliance in such belief. And in case we drop the assumption that mathematics is about external entities existing independently of human cognition, choosing the constructivist path instead, then we create the problem of explaining how arbitrary conventions are necessary to express empirical truths (or, how is that mathematics is more than a game of sign transformation).

1.1. Objectives and methodology

The goal of this thesis is to promote the dissolution of this dilemma, in a Wittgensteinian spirit. I have learned with him that many philosophical problems are a struggle against the bewitchment of our intelligence by means of language (*see* PI, §109). Such bewitchment occurs when people are locked in a single perspective, an inflexible picture of what language does and how it discloses the world to our comprehension, eventually leading to the multiplication of semantic and metaphysical problems.[3]

As I understand him, Wittgenstein's goal was to free human thought from the tyranny of single visions, of being dogmatically devoted to a certain way of picturing the world. He wanted to show the fly the way out of the fly-bottle. And although his way of dissolving philosophical problems was not rigorously defined and packed for the use of future generations, we may say he always worked to show to anyone puzzled by a

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[2] Michael Potter described the influence of Benacerraf's *Mathematical Truth* as: "So standard has it become, in fact, that it is nowadays a painful cliché for articles on mathematical epistemology to begin by stating 'Benacerraf's problem'." ("What is the problem of mathematical knowledge?" in *Mathematical Knowledge*, Oxford University Press, 2007, p. 25).

[3] E.g. the Augustinian picture of language, learned by ostension, may lead one to think that there is an object for every single term in our language. But, as we use expressions of possession such as 'my body', 'your soul', 'our future', 'the electron's charge', 'Saturn's rings', yet we would incur into some incompatible descriptions of the world if we expected that all such uses are similar, expressing possession of objects.

philosophical question that their puzzlement often is the result of a certain entrenched worldview, motivating one to assume all sorts of unwarranted presuppositions to fit that view, imposing an order upon the world. Such presuppositions may serve as bridges for all sorts of false conclusions, generating intellectual conflict, incoherent views, conflicting dualisms, disquietudes, angsts – all sorts of issues that Wittgenstein used to call "mental cramps."

These intellectual disquietudes are not solvable like practical problems, as they steam out of our ways of thinking, not from a lack of knowledge. Thus, to get rid of them, we must reach clarity and perspicuity of thought; reach a synoptic view of the genesis of our angst, commanding a manifold perspective that encircles the original confusion and allow us to avoid it, dispersing the misunderstandings clouding our judgement.

In my view, Benacerraf's dilemma is exactly one such mental cramp. It is the result of bewitchment by a certain picture of mathematical language and knowledge. The philosophical force of this problem, compelling us to choose one of two mutually exclusive accounts of mathematical truth, is a sign that our understanding has taken some unwarranted presuppositions. Therefore, we may diagnose confusions and misconceptions at the roots of our comprehension of mathematics by analysing this dilemma. Exposing its unwarranted presuppositions can function as a guiding thread towards a more fruitful and comprehensible view of mathematical activity.

Since my strategy is to dissolve, I do not attempt to resolve the epistemological issue in favour of platonism, nor the semantic issue in favour of combinatorialism. Instead, I argue that these problems are defused (and thus the dilemma is dissolved) insofar we scrutinize and avoid the presuppositions which motivated Benacerraf's recommendation of a global referential semantics and a causal account of knowledge. Instead of complying with presuppositions that take us to forking paths, I will try to convince the reader of their limitations, because this dilemma is not a problem originated by our ignorance, but an intellectual disquietude born out of a conflicting understanding of mathematics. As Oskari Kuusela explains, the key to a Wittgensteinian approach lies in a transformation of our approach, rather than a theoretical postulation of new theoretical entities:

> […] instead of rushing to find answers to the questions through which philosophical problems are articulated, such questions themselves should be subjected to closer scrutiny. The attempts to answer these questions do not

> reach the roots of the intellectual disquietude they express, ultimately leaving one unsatisfied. Resolving philosophical problems, therefore, calls for a transformation of one's approach.[4]

With this in mind, the bit of Wittgenstein's method which will be most useful to transform our approach is the way he attempted to dislocate old paradigms of thought with examples of possible alternatives. His way was through analogy, not theory. This often involved producing counter-examples that stand against overarching pictures, so we may be free from their single, reductionist vision:

> I may occasionally produce new interpretations, not in order to suggest they are right, but in order to show that the old interpretation and the new are equally arbitrary. (LFM, p. 14)

> […] you may question whether my constantly giving examples and speaking in similes is profitable. My reason is that parallel cases change our outlook because they destroy the uniqueness of the case at hand. For example, the Copernican revolution destroyed the idea that the earth has a unique place in the solar system. (WCL, p. 50)

The *Philosophical Investigations* often follows a seeming structure of a three-staged dialogue: first, a voice evokes a philosophical theory, which then a second voice endorses and expands upon, offering a specific set of circumstances which demonstrates the validity of the theory; then a third deflationary voice evokes scenarios or produce new interpretations that that theory cannot explain, showing how limited were the circumstances considered by the second voice, and also the need to move beyond reductive and artificial restrictions (sometimes even pass the idea that philosophy is in the business of providing theories).[5]

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4 Oskari Kuusela, *The Struggle Against Dogmatism: Wittgenstein and the Concept of Philosophy*, Harvard University Press, 2008, p. 17

5 For a more detailed description of this structure, *see* David G. Stern, "Wittgenstein's critique of referential theories of meaning and the paradox of ostension, *Philosophical Investigations* §§26-48", in *Wittgenstein's Enduring Arguments* (London: Routledge, 2009).

My appropriation of this therapeutic approach employs it to free the philosophical discussion from the impoverish bipolarity of platonism versus combinatorialism. The idea is to operate like the third deflationary voice in Wittgenstein's dialogue, showing the limitations of those views, and then produce a new interpretation, a viable third alternative, in order to demonstrate that it is possible to provide a coherent semantic-epistemological account of mathematical truth without being impaled by one of Benacerraf's horns. We are not reduced to choose between the picture of the mathematician as a scientist-and-explorer, or as an artist-and-historian. We can have our cake and it too — we can provide an understanding of mathematics in which *we invent* techniques *to discover* patterns; in which we develop interconnected networks of concepts in order to talk about logical aspects.

Thus, to expose the misconceptions lending Benacerraf's dilemma its disastrous force, I will examine both horns, making fine-grain distinctions between his conditions and underlying presuppositions that give origin to both horns. **Chapter 2** focuses on platonism and the epistemological horn, while **Chapter 3** focuses on the combinatorial view and the semantic horn. To preview, in my assessment, the major presupposition responsible for creating this dilemma is the interpretation that mathematical statements represent or describe mathematical facts. As we will see, both sides of the dilemma draw comparisons between mathematical and empirical statements, either in terms of justifiability (i.e. the constructivist comparison of empirical evidence with mathematical proof) or in terms of semantic function (i.e. the platonist comparison of empirical and mathematical statements as representing facts). Contrary to these interpretations, I propose there is a radical distinction between uses of mathematical and empirical vocabularies, particularly because the former are responsible for providing criteria of correctness for the latter, criteria we need to be able to tell whether a factual statement that employs mathematical concepts is true.

As it will be argued in **Chapter 4**, these comparisons are based on the metasemantic view known as representationalism. When Benacerraf claims that platonism satisfies his semantic condition at the expense of the epistemological one, he is presupposing a representationalist view of language. That is, by endorsing the platonistic interpretation that mathematical concepts refer directly to abstract objects, he is presupposing the representationalist thesis that concepts represent properties. We must reject this thesis if we want to dissolve Benacerraf's dilemma. The argument starts with the premise that our use of language follows accordingly to socially established norms. Yet, representationalism claims that meanings are given by implicitly and privately formed representations of extra-linguistic features or items. But as so, these representations are also socially unsurveyable — we would not be able to grasp the inferential consequences or circumstances of what an interlocutor claims — and thus generalized scepticism regarding meaning and understanding expressions becomes a possibility, casting a shadow of doubt over communication in general.

Luckily, there are other options for coherently explaining the content of our mathematical statement without having to postulate abstract or mental entities. **Chapter 5** forwards a global expressivist and social pragmatic account of the role that mathematical expressions have in our reasoning, against which the tension exposed by Benacerraf should have no impact. This account explains how semantic relations are instituted in social-normative practices by acts of expression that settle normative standards of correctness for linguistic performances.

From this expressivist perspective, I revisit Benacerraf's comparison of mathematical and empirical statements, albeit this time against the background of Wittgenstein's interpretation of mathematical statements as rules. According to his interpretation, mathematical statements *prescribe* standards of correctness for the use of mathematical concepts in empirical claims. These statements are not to be read as expressing factual-propositional content, since they do not describe anything, but instead as a logical license to a certain conclusion. With this realization, we can finally blunt the epistemological horn of Benacerraf's dilemma, since we no longer expect (or need) mathematical language to represent objects.

At last, **Chapter 6** presents a pragmatic account of mathematical knowledge that is not caught up in the semantic horn. To introduce this view, I first argue why accepting Benacerraf's epistemological presupposition is inadequate. This comes as a consequence of the arguments in previous chapters, for if mathematical statements make no claim to knowledge of facts casually related to the knowing agents, then a causal theory is unsuited to account for this knowledge. In its place, I work with Wittgenstein's conception of *Lebensformen* (form of life) as the root of our logical routines and mathematical practices. From this genetic basis we can build a pragmatic account of mathematics, portraying the character of mathematical knowledge as not simple and homogeneous, but instead as a mosaic of practices, a colourful mix of frameworks, each one normative towards a certain mathematical conceptual neighbourhood.

To finish this introduction, I leave a word of warning for readers seeking a scholarly work on Wittgenstein's philosophy of mathematics. While his remarks certainly provoked my thoughts, this thesis does not offer an exegesis of his philosophy of mathematics. My reading of Wittgenstein constantly intersects with interpretations from scholars and other pragmatically-minded authors inspired by him (of which the most recurrent are Robert Brandom, José Ferreirós, Huw Price, Juliet Floyd, Richard Rorty, and André Porto), and the result is, if anything, my own recipe. The interpretations presented here are not forwarded with the pretension of being final words on Wittgenstein scholarship. His writings are notoriously dense, laconic, and fragmented; their interpretation will remain a controversial matter. My intentions are to take his remarks as guidelines for meditation and methodological reference to what I aim to do: dissolve a philosophical problem.

2. The epistemological horn

To recapitulate, Benacerraf's dilemma is constituted of two conditions whose mutual satisfaction leads to an incompatibility, namely: accounts that can satisfy the semantic condition end up failing the epistemological condition, and vice-versa. This results in a bifurcation of views on mathematical knowledge, each side better equipped to deal with either the semantic or the epistemological challenges raised by Benacerraf's conditions.

In what regards his epistemological conditions (EC), Benacerraf explicitly states that the specification of truth-conditions for a statement must yield conditions that are in principle *possible* for humans to know about;

> ***EC***: The specification of truth-conditions of mathematical statements cannot make it impossible for us to *know* that they are satisfied.[1]

This condition requires that an account of mathematical truth must not make the conditions for said truth *unintelligible* — we must be capable of knowing whether the conditions for the truth of mathematical statements are satisfied[2] — our account must explain how we have come to form the correct beliefs that track the relevant mathematical facts. For Benacerraf this means that an account of mathematical truth must fit with a global epistemology which explains what knowledge in general is, and how we come to reliably acquire it. In this way, this condition put some breaks on the postulation of abstract objects. What Benacerraf requires is, essentially, that such postulation cannot proceed without first passing the test of a suitable epistemology.

Benacerraf contends that the most successful epistemological theory betrays the semantic condition, for explanations of how we come to acquire mathematical knowledge invariably point to the proof activity at its core — they point to the methodology of the

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[1] "[…] the concept of mathematical truth, as explicated, must fit into an over-all account of knowledge in a way that makes it intelligible how we have the mathematical knowledge that we have. An acceptable semantics for mathematics must fit an acceptable epistemology." (MT, 667).

[2] In Benacerraf's own words: "It must be possible to establish an appropriate sort of connection between the truth conditions of *p* (as given by an adequate truth definition for the language in which *p* is expressed) and the grounds on which *p* is said to be known, at least for propositions that one must *come to know* — that are not innate." (MT, 672)

mathematician as the ultimate justifier of his truth claims, but not to the existence of a subject matter.

Of course, this all depends on which epistemological theory is regarded as the most successful, and Benacerraf's preferences are clear, as he is very explicit with where his sympathies lie: he says that in order to have true knowledge about an object, an epistemic subject must be causally connected to it. His reasoning is that the truth-makers of our statements must casually impinge on us a *stimulus* — we must passively recognize their existence, and not actively invent them — so we can recognize their truth.

That is, Benacerraf thinks only the *Causal Theory of Knowledge* (CTK henceforth) provides the terms to specify the truth-conditions of our statements in a universally intelligible manner.[3] As so, besides his explicitly stated epistemological condition, Benacerraf holds an epistemological presupposition (EP):

> ***EP:*** Knowledge must be explained in terms of a causal chain of events connecting the subject to the facts she claims to know.

To remind the reader, Alvin Goldman's original definition of knowledge in CTK may be stated as such: *S knows that p* if and only if the fact *p* is *causally* connected in an "appropriate" way with *S*' s *believing p*.[4] By taking this thesis as a necessary component in any explanation of knowledge, Benacerraf is clearly allowing some unexamined empiricism to infiltrate the epistemological condition of his dilemma. Yet, this may well be the deeper reason why platonism cannot solve his dilemma, for as I have mentioned in the introduction, this account of mathematical knowledge considers it studies casually inert abstract objects which are obviously non-empirical.

[3] "I favor a causal account of knowledge on which for *X* to know that ***S*** is true requires some causal relation to obtain between *X* and the referents of the names, predicates, and quantifiers of *S*. I believe in addition in a causal theory of *reference*, thus making the link to my saying knowingly that *S doubly* causal." (MT, 671).

[4] Goldman, Alvin I. "A Causal Theory of Knowing" *The Journal of Philosophy* 64.12 (1967): 357–372.

2.1. Characterizing the postulational approach

So far, I have treated what Benacerraf calls "standard semantics" as a broadly truth-theoretical referential semantics which interprets declarative sentences as composed of logical connectives and non-logical terms standing for extra-linguistic objects or properties. The fundamental semantic paradigm that animates this sort of account is a representational one, in which language serves to represent the world, singular terms designating particular objects, predicates representing properties, and statements representing possible state of affairs. As so, an interlocutor's grasp of the meaning of a statement would consist in knowing what would have to be the case for it to be true, as specified in terms of truth-conditions.

This is known as an extensional account of predicates and functions. A precise and formal definition of the extensional account is given by Raymond Smullyan.[5] For any predicate P, any number n, and the set $\mathcal{T}$ of truth sentences written in the formal language L, we have that P is true of any number n if $P(n)$ is a true sentence of L. Or, the extension of P is the set of all numbers n that satisfy P. So any set A of numbers is expressible in the language L if A is the extension of some predicate. Now, if we let A designate the extension of P, then $P(n)$ will be a true sentence of L if and only if P expresses the set A of numbers n, as in:

$$P(n) \in \mathcal{T} \leftrightarrow n \in A$$

From this formula, we can infer that a sentence $P(x)$ is true for every object x that satisfy the predicate P. Or we can say $P(x)$ *expresses* the set of all Ps.

What about elementary compositions involving only numbers but no predicates? Could those be true? Still following Smullyan's account, the specification of elementary (or atomic) sentences which serve to pick out particular objects is given as: "An atomic sentence $c_1 = c_2$ (c_1 and c_2 are constant terms) is true iff [if and only if] c_1 and c_2 designate the same natural number"[6]. By "constant term" Smullyan means a term with no variables designating a single natural number. Notice that in this definition of mathematical singular

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[5] Smullyan, R. *Gödel's Incompleteness Theorems* (Oxford University Press, 1992), pp. 14-20.

[6] *Ibidem*, p. 17.

terms, a number-word is a constant designating a number-object, similarly to how in Benacerraf's logical analysis the term '*a*' marks the "name of an element of the universe of discourse of the quantifier" (MT, 663), no matter the particular manners of using that term.

Now, this is just a peek into what a Tarskian definition of truth coupled with a referential semantics would look like for mathematical language, but it may suffice to show what Benacerraf meant when he claimed that the metaphysical view ensuing from such analysis is platonism, an ontological realism about mathematics.[7] The platonist argument has roughly this form:

1. For a declarative sentence to be true, the objects referred by its singular terms must exist and have the properties predicated;
2. There are proved true sentences that refer to non-concrete objects, such as the mathematical;
3. Therefore abstract objects exist.[8]

Thus, platonism is characterized by the claim that the objects of mathematical discourse exist and have properties independently of our considerations, decision procedures or methods of obtainment. Mathematical operations are conceived as our ways of tracking the properties and semantic connections already established between these objects. That is to say that every mathematical question or conjecture already has its result pre-determined before computation, and independently of human thought. In short, mathematics discovers and studies abstract structures, it does not create them.

The reason which justifies considering our use of mathematical terms in math statements as singular terms referring to particular objects is the logical role played by these terms. For the platonist, assessment of which assertions impose on the world some condition that their referents exist turn entirely on a logico-syntactical analysis, as Crispin Wright argues:

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[7] The paradigms of what is customarily regarded as platonistic view are (1) Cantor's defence of power sets and orderings as genuine topic of mathematical study, (2) Frege's exposition of the weaknesses of the psychologist and formalist approaches to the notion of number, and (3) Gödel's discussion of Cantor's Continuum Hypothesis.

[8] Taking cue from Frege's argument for the existence of numbers as self-subsisting objects. I will discuss Frege's influence on the platonist view in detail in §2.3 (*See* Frege, G. *The Foundations of Arithmetic*, 2nd ed, trans by J. L. Austin, Evanston: Northwestern University Press, 1999).

> […] the question whether a particular expression is a candidate to refer to an object is entirely a matter of the sort of syntactic role which it plays in whole sentences. If it plays that sort of role, then the truth of appropriate sentences in which it so features will be sufficient to confer on it an objectual reference; and questions concerning the character of its reference should then be addressed by philosophical reflection on the truth-conditions of sentences of the appropriate kind.[9]

That is, if a term fulfils the syntactical role of a singular term in a declarative sentence, then it must refer. Acceptance of this purely syntactical assessment of reference is a premise in the platonist argument for the existence of abstract objects. The upshot is that any true statement that is not plainly about concrete objects will turn out to be about abstract ones. The ontological thesis about abstract mathematical objects is a consequence of applying a representational view of meaningfulness carried out in a truth-theoretical reference-based semantic analysis (more on the representational view in chapter 4).

From such a view, if we are to justify the claim that mathematical knowledge is objective, then the propositional content of a mathematical statement must represent a self-subsisting mind-independent reality. To be objective, claims of mathematical existence and truth must satisfy some extra-linguistic condition that goes beyond mere intra-linguistic justification by axioms and rules of inference. Thus, mathematical expressions are taken to refer to extra-linguistic objects and their properties.

Apparently, this concern with conceiving mathematical knowledge as objectively true is what brings mathematicians over to platonism. For example, we have Alan Connes, who said:

> Although not all mathematicians recognize it, there exists 'an archaic mathematical reality'. Like the external world, this is *a priori* non-organized, but resist exploration and reveals coherence. Non-material, it is located outside of

[9] Wright, C. *Frege's Conception of Numbers as Objects* (Aberdeen University Press, 1983), p. 51.

> space and time. […] My attitude and that of other mathematicians consists in saying that there exists a mathematical reality that precedes the elaboration of concepts. [10]

And also:

> The prime numbers, for example, which, as far as I'm concerned, constitute a more stable reality than the material reality that surrounds us. The working mathematician can be likened to an explorer who sets out to discover the world.[11]

Another mathematician, V.F.R Jones, argued that mathematical truth is a matter of a "visualization". Jones interestingly brings communication to the fore. Mathematicians often communicate and agree without needing formalized proofs, as if one was uncovering a hidden object only the mind can see. In his words: "If one 'sees' the pictures, then one understands, but otherwise one cannot follow. In principle one could formalize the whole argument, but that would add nothing."[12] Proof cannot exhaust our intuitive grasp of what is right, for proof is a formalization that comes after the intuition. For Jones, the mathematician adopts formalism in order to communicate with others, secure the objective sense of mathematical discourse, but at the end of the day, in the intimacy of his work, the mathematician shares the platonic view of his subject-matter.

From the way Connes and Jones explain their views, we may sense that the appeal of platonism seems to lie in the parallel it sustains between scientific and mathematical discourses, namely: that we are discovering a reality that exists independently of the human

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[10] Alan Connes, "La réalité mathématique archaïque" (*La Recherche* 332: 109, apud Ian Hacking, *Why is there philosophy of mathematics at all?*, Cambridge University Press, 2014, p. 200).

[11] J.-P. Changeux & A. Connes, *Conversations on Mind, Matter and Mathematics* (trans. M.B. DeBovoise, Princeton University Press, Princeton-New Jersey 1995, p. 12) apud Mateusz Hohol, "The Normativity of Mathematics: A Neurocognitive Approach", in *The Many faces of normativity* (eds. J. Stelmach, B. Brożek, M. Hohol, Copernicus Center Press, Kraków, 2013).

[12] Jones, V.F.R. "A credo of sorts", in *Truth in Mathematics* (Oxford University Press, 2005), p. 213.

mind.[13] The allure of the platonist picture resides precisely in respecting the independence of mathematical truth from human culture or conventions. From a platonistic outlook, we can explain the seemingly necessary character of mathematical truth as an indication that its statements capture necessary relations between objects that pre-exist our effective acknowledgement of them. This would explain why previously unexpected inventions in mathematics show themselves time and time again so useful to predict and model nature.

As so, platonism interprets that mathematical statements are descriptive just as ordinary empirical statements. The only important difference is categorical; seen that the entities designated by mathematical singular terms are not found as concrete-physical entities, one assumes they exist in an abstract manner. The platonistic strategy is characterized by this unrestrained adherence to the postulation of abstracta, as motivated by their extensional view of meaning. This strategy, however, which so clearly goes against Occam's Razor and Benacerraf's empiricist inclinations, inevitably raises epistemological questions regarding one's acquaintance with abstract entities.

To tackle this issue, it is worth considering Burges' & Rosen's *A Subject With No Object*, in which the authors stressed that, following on the trail of David Lewis analysis of abstraction, the best way to understand abstract objects is through "the way of negation". That is, we must conceive of these objects as lacking spatial-temporal location, being causally impassive and inactive.[14]

We can see from such a definition the precise departing point of Benacerraf's challenge to platonism: how could we know about such objects if they are supposedly causally inefficacious? As a matter of principle, we could not have discovered such objects. Thus, the very intelligibility of talk about such objects is in question. As Benacerraf noticed, by appealing to the independence of 'mathematical facts' from the human methods of realizing it, the platonist ends up blocking any reasonable explanation of our knowledge of

[13] For Michael Dummett, this analogy is at the very heart of the platonic view: "Platonism, as a philosophy of mathematics, is founded on a simile: the comparison between the apprehension of mathematical truth to the perception of physical objects, and thus of mathematical reality to the physical universe. For the platonist, mathematical statements are true or false independently of our knowledge of their truth-values: they are rendered true or false by how things are in the mathematical realm. And this can be so only because, in turn, their meanings are not given by reference to our knowledge of mathematical truth, but to how things are in the realm of mathematical entities." (*Truth and Other Enigmas*, Harvard University Press, 1978, p. 202).

[14] Burgess & Gideon, *A Subject With No Object* (Cambridge University Press, 1997), p. 20.

those facts. When we disassociate the subject matter from the human methods of studying it, we make it hard to justify belief in the supposedly discovered facts.

So why resort to the platonistic picture of mathematical knowledge when one could choose a milder form of realism? To call the existence of abstract objects into question only works as a counter-argument to interpretations that attribute some *external* truth-condition for mathematical statements. So if a mild realist opts for a semantic analysis that issues only syntactical and not external truth-conditions for math statements, then his claim that numbers exist is epistemologically unproblematic as it does not mean the postulation of self-subsisting abstract objects.[15] A moderate realist could state that mathematical terms refer but pass over in silence about metaphysical questions over the nature of the purported referent.

However, taking the milder strategy is to give up the explanatory power that postulating abstract objects have in specifying the referents which make those statements true. If we do not provide a way to specify *about what* we claim knowledge, it becomes impossible to satisfy EC, a condition which asks for intelligible specifications of truth-conditions for mathematical statements. To be successful at its metaphysical quietism, mild realism has to also be silent about what one has to know in order to refer to mathematical objects. As so, mild realism avoids the kind of epistemological question about how we could know and refer to the definite quadrillionth digit of π when it seems that *in principle* it is *impossible* for us to ever be in the position to know which digit that is.

In other words, to avoid Benacerraf's dilemma, an account of mathematical knowledge must fit an explanation of the meaning of mathematical expressions. After all, this dilemma is about a point of entanglement between semantics and epistemology, in the sense of demanding explanation for expressions of *knowing-that*, or what speakers claim to understand by use of mathematical terms. Thus meaning and understanding are already taken as coordinate concepts by the framing of this dilemma, as an explanation of one will invariably involve the other.

[15] A defender of this sort of mild realism is William Tait (see *The Provenance of Pure Reason*, Oxford University Press, 2005, p. 91).

2.2. The computational issue with arbitrary functions

Why is that the platonistic account is caught up in the epistemological horn of Benacerraf's dilemma? To start, platonism clearly cannot satisfy EP: by definition, there is *no* causal relationship with abstract objects, and thus believing in their existence cannot justify a truth claim from the point of view of CTK. If the truth or falsity of mathematical statements was to depend on the existence of these objects, then no mathematical statement would be falsifiable, because there is nothing to discover.

However, even if we drop CTK, platonism still runs into trouble in satisfying the lighter EC as it is. That is because the notion of discovering abstract objects seems in principle unintelligible. According to Crispin Wright's and Hartry Field's canonical explanations of the issue:

> The fundamental problem is not how, given that mathematical statements are about abstract objects, we could know them to be true, but how they could intelligibly be about such objects in the first place.[16]

> Benacerraf's challenge—or at least, the challenge which his paper suggests to me—is to provide an account of the mechanisms that explain how our beliefs about these remote entities can so well reflect the facts about them. The idea is that *if it appears in principle impossible to explain this*, then that tends to *undermine* the belief in mathematical entities, *despite* whatever reasons we might have for believing in them.[17]

It appears to be in principle impossible to explain how our beliefs can reflect the facts about causally inactive objects lying outside space-time. On one hand, it is trivial to state that numbers exist and are objects of study of arithmetic, just like it would be to state that pain exists and is an object of medical study. But on the other hand, it is borderline unintelligible to say numbers are objects in analogy to objects of empirical discourse. Unlike abstracta, a concrete object does not cause responses exclusively on sapient

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[16] Wright, C. "Benacerraf's Dilemma Revisited" (in: *The European Journal of Philosophy*, 10:1, pp. 101–129, 2002), p. 20.

[17] Field, H. *Realism, Mathematics and Modality*, Blackwell, 1989, p. 26 (emphasis in the original).

creatures thinking and talking about them, they cause responses on other lifeforms and even simple sensitive devices (e.g. scales, thermostats, etc.). Even cultural items, such as individual works of music or film, need a physical base to take form, or else they would be nothing more than plans and intentions. Compared with these ordinary ways in which we talk of objects, the notion of an abstract one runs the danger of becoming an empty analogy; these distinct uses of 'object' have nothing in common besides satisfying a syntactical criterion for identifying a singular term.

In the platonist dictum, attributions of truth do not answer to epistemological principles, as propositions are true by virtue of representing how things are in the extra-linguistic reality, not because we sanction them with proofs. So a well-formed mathematical formula either accurately represents the mathematical facts or it does not, there is no middle term — it does not matter if our methods are insufficient to verify whether certain statements are true, for the platonist, their truth-value is already determined by the mathematical facts.

From conceiving such a gap between possible structures that can obtain physically and those that can obtain mathematically, platonism ends up introducing an *ontological dualism* of a physical reality of concrete objects and a formal-mathematical reality of abstract objects. If I say "There are three books on the desk", I am asserting a physical fact and distinguishable mathematical fact, as Demopoulos explains:

> […] the 'mathematical fact' is that the number of books on the desk is three (say), and the 'physical fact' is that there are three books on the desk. The proof of the above theorem for the case n = 3 shows how the two [facts] are connected, and thus answers the question of how arithmetic applies to the physical world.[18]

On one side we have the contingent features of the physical realm, on the other the fundamental necessities of the mathematical realm, and somehow we uncover truths about

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[18] Demopoulos, W.; Clark, P., "The Logicism of Frege, Dedekind, and Russell", in *The Oxford Handbook of Philosophy of Mathematics and Logic* (Oxford University Press, 2005), p. 137.

the first realm by investigating the second. [19] Yet, if the reference to mathematical objects is decided solely with basis on syntactical considerations, could not these considerations allow for reference to objects that would outrun possibilities of physical instantiation?[20] Can we say all possible adumbrations of mathematical systems are potentially present in our current theories, even if in principle no computational rule can be provided to compute these consequences? This topic is important for the debate surrounding Benacerraf's dilemma because it touches the matter of necessity as a modal notion and how it relates to logical consequence. We ask what grounds that a certain result follows from a certain calculation because there seems to be a lack of understanding of the notions of necessity and determination.

Undoubtedly, what motivated the separation of formal-mathematical from physical possibilities is the notion of *arbitrary functions*. By mid-19th century, Dirichlet introduced these functions as given by an arbitrary totality, in extension, according to the rule:

$$\forall x \in A \; \exists! y \in B \; (x, y) \in F$$

For every object x that belongs to set A there is an object y in set B, such that there is a function from A to B forming the arbitrary subset $\{x, y\}$. A function from set A to set B

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[19] Perhaps even more than Plato, the father figure of this view may well be Pythagoras. Pythagoreans thought a number is the sign of a ratio, the relation that puts everything under proportion. And for the Pythagoreans, everything there is, is in a proportion to something else. Thus numerical relations were deemed the most important properties, even sacred. And this conception of mathematical knowledge implies a metaphysical thesis according to which mathematical language expresses relations that are fundamentally hard-wired into the fabric of the universe, almost as if by discovering new mathematical relations we were uncovering a deeper structure of reality. Max Tegmark is a name that comes to mind as an example of a contemporaneous physicist who believes in the Pythagorean picture. Indeed, he even offers a retelling of the Pythegorean metaphysics. In his view, mathematics describes the most fundamental structures: "Our reality isn't just described by mathematics — it is mathematics […] Not just aspects of it, but all of it, including you." He proposes the Mathematical Universe Hypothesis according to which "external physical reality is a mathematical structure." (See *Our Mathematical Universe*, 2007, at: <http://arxiv.org/pdf/0704.0646.pdf>).

[20] From our current understanding, the spacetime continuum forms a set of points with cardinality $2^{\aleph_0}$, way 'smaller' than the higher cardinalities studied in Set Theory, which can reach an infinite variety of infinities. (*See* Parsons, *Mathematics in Philosophy*, Cornell University Press, 1983, p. 191.)

forms an arbitrary subset *F*. These functions are dubbed *arbitrary* because they can be introduced without a computational rule (i.e. without a definition by a formula or algorithm). In this way functions figure no symbolism for intention and are thus reduced to arbitrarily chosen sets of ordered pairs, as Mathieu Marion explains:

> Adopting Dirichlet's notion involved no loss in existing mathematics but gains in constructions. For example, the new approach ultimately led to an easy generalization of analysis to function spaces. Thus, with Dirichlet mathematicians moved from an intensional notion of function-as-a-rule to a purely extensional conception. [...] This is in fact the origin of the calculus of set theory, within which the notion of function as a set of ordered pairs was developed over the years.[21]

The introduction of arbitrary functions was the origin of the idealistic interpretation of existence claims $\exists x\, P(x)$ as:

$\neg \forall x\, \neg P(x)$ *or* "It is contradictory that P(*x*) be false for every *x*"

And the disjunction $P \vee Q$ as:

$\neg(\neg P \wedge \neg Q)$ *or* "It is contradictory that both *P* and *Q* be false"

In this way, a statement P(*x*) is not proved for each and every case of *x*, but through reduction to absurdity. When platonist-leaning mathematicians accepted this shift from an intensional notion of function-as-a-rule to a purely extensional one, the result was the

[21] Mathieu Marion, *Wittgenstein, Finitism, and the Foundations of Mathematics* (Oxford: Clarendon Press, 1998), p. 7.

unconstrained universalization of a syntactical principle; thus the principle of bivalence became the law of the excluded middle.[22]

However, it is a problematic move to assume all possible mathematical statements are determinately true or false, because of conjectures and independent statements.[23] As so, Marion assesses, "the door is open to arbitrary subsets or arbitrary functions for which no computation rule (i.e. no definition by a formula) can be given."[24] The issue Marion is calling attention to is that, under such a conception of mathematical function, we scrap the requirement for a computational rule to yield a certain result. We are allowed to posit arbitrary functions that yield arbitrary subsets, and thus conclusions can be established even for cases to which we have not devised effective algorithmic procedures. This motivates the postulation of way more results (albeit platonists would prefer to call these 'objects', e.g. sets of transfinite cardinalities) than what we can compute. No computational rule can be given in such cases because, generally, the computational power that these results require extrapolate our computational capacity.

Turing-Church Thesis states that "the effectively calculable number theoretic functions are exactly those functions whose values are computable in Gödel's equational calculus, i.e., the general recursive functions."[25] As Turing clarified, the computable numbers "may be described as the real numbers whose expressions as a decimal are calculable by finite means."[26] These general recursive functions are the limit to what can be

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22 The debate over the adoption of the law of the excluded middle is still pretty much alive, given the issues with undecidable statements. Generally, the universalization of the principle indicates a platonist viewpoint of mathematics, while the negation indicates a constructivist viewpoint.

23 Conjectures are mathematical statements to which we have *not yet* found proof, so they are not explicitly derived from the basic principles of a system of calculus. In a way, we can grasp their sense, but their meaning remains undefined until someone produces a proof which demonstrates how that statement is a consequence of other accepted statements and basic principles. Independent statements (aka. undecidable statements) are those which neither it nor its negation are provable in a given axiomatic system.

24 Marion, M. *Wittgenstein, Finitism, and the Foundations of Mathematics* (Oxford: Clarendon Press, 1998), p. 8.

25 Wilfred Sieg, 'On Computability', in Andrew Irvine (ed.) *Philosophy of Mathematics (Handbook of the Philosophy of Science)*, North Holland—Elsevier, 2009, p. 527.

26 Alan Turing, "On computable numbers, with an application to the Entscheidungsproblem", in *Proceedings of the London Mathematical Society* (Ser. 2, Vol. 42, 1937), p. 230.

computed by a Turing machine, and thus, as far as we know, remain the limit to what can be computed in general. As Juliet Floyd explains:

> Through the Universal Machine's self-symbolizing capacity, Turing was able to show the significance to formal logic of the idea of an *effective* "mode of operation". There is no diagonalizing out of the class of Turing machines, because the fundamental notion is that of a not-everywhere-defined, partial procedure. The very idea of formalized logical consequence is thereby robustly or "absolutely" analyzed and shown to be marked by a general undecidability.[27]

The existence of a ceiling to what is computable shows us it is impossible to use the diagonalization method on the class of partial recursive functions,[28] and without diagonalization one cannot construct a class of arbitrary functions to determine a set with cardinality higher than that of the natural numbers, as Cantor did in his diagonal argument. We may postulate sets of transfinite cardinalities, but we have no effective procedure to actually yield them in computation.

Yet, platonism posits the existence of these entities *regardless* of the possibility of proving the statements that assert their existence. The attribution of truth to a statement or reference to a singular term is thus seen as epistemologically unconstrained — any singular term can refer to an object and any statement can accurately describe a possible world, regardless of our capacity to know about the posited referents. All these claims of knowledge of pure mathematical possibilities that would outrun the possibilities of physical instantiation are based on *unsurveyable* functions, to which we cannot assign a computational rule. However, performing computations is the way in which computing agents (human and computer alike) demonstrate they can follow the inferential steps of mathematical reasoning in practice (i.e. in paper or computer memory). Therefore, if there is no rule to

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27 "*Lebensformen*: Living Logic" (to appear in Christian Martin, ed., *Language, Form(s) of Life and Logic: Invesgitations after Wittgenstein*, available at: <Academia.edu [link]>), p. 24.

28 The class containing zero, successor, and projection functions, closed under composition, primitive recursion and a search operator (*see* Epstein & Carnielli, *Computability, computable functions, logic, and the foundations of mathematics,* Socorro: Advanced Reasoning Forum, 2008, p. 124).

compute a function, we do not know *how* an input x relates with the output $f_{(x)}$. This is the issue with arbitrary functions whose result obtains in abstract: they are fruits of postulations with no regard for how we go about our mathematical practices.

This takes us to the conclusion that platonism fails EC not because it lacks a causal explanation of what we claim to know with mathematical statements — the issue runs at a more fundamental level than that — platonism fails EC for conceiving truth claims as epistemically unconstrained (or, as Putnam says, "radically nonepistemic").[29] In the platonist picture, the issue of whether a statement is true or not has nothing to do with our intellectual capacities, so to demand *any* epistemic constraints from this approach is to block it from the start. What makes the platonistic account be impaled by the epistemological horn is its belief in the existence of mathematical objects to which we possess no reasonable epistemological story for, as we cannot provide one, by virtue of the computational limitations of the computing agents that practice mathematics.

2.3. Frege and the Julius Caesar Problem

To recapitulate, Benacerraf's epistemological condition (EC) calls for a naturalistic account of the epistemology of mathematics which makes clear the causal connections between the knowing agent and the objects known. But the surface syntax of mathematical statements makes it seem that its singular terms refer to abstract entities. The difficulty lies in providing an intelligible explanation, taking into account our cognitive faculties, of how we could come to know causally impassive and inactive objects.

One ingenious attempt at explaining our knowledge of abstracta is Frege's abstraction principles. It is no surprise that Frege's project of providing foundations for mathematical knowledge in formal logics is pivotal for the platonistic account. Thus, in

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[29] Putnam reached the same conclusion in his paper *Realism and Reason*, saying the metaphysical realist was pushing for "a view of truth as radically nonepistemic" (apud Rorty, *Philosophy and the Mirror of Nature*, p. 294).

what follows, I will approach his account from a comparison between the references made with 'New York' and '17', the terms used in Benacerraf's examples:

(ES) There are at least three large cities older than New York;

(MS) There are at least three perfect numbers[30] bigger than 17;

According to Frege, to count as a singular term, a term must designate a non-empty domain (so we know that the purported object exists) and it has to pick out only the one particular object. So what counts as an object is either:

(a) the referent of a proper name;
(b) what predicates are true or false of; or
(c) the elements that compose the ranges of the individual variables which can be bound by quantifiers.

Furthermore, Frege states the condition for a term to qualify as a singular term properly naming an object is to be associated with an *identity criterion*: "If we are to use the symbol *a* to signify an object, we must have a criterion for deciding in all cases whether *b* is the same as *a,* even if it is not always in our power to apply this criterion."[31] This condition is in no way alien to Tarski's recursive condition in his theory of truth. To understand why, we just have to recall that Frege had Leibniz's identity principle in mind, in both its logical and ontological take: that an *a* and a *b* are identical if and only if they are mutually interchangeable without loss of meaning.

The interesting bit is that we should be able to ask meaningful questions to establish the identity of the object referred to by the use of a singular term in *any* statement involving this term. So, let us first look at the case of identifying the object named as "New York" through the reference of the following statements:

"New York is the fourth oldest large city in the USA"

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[30] A perfect number is a positive integer that is equal to the sum of its divisors less than itself. The first is 6 (6 = 3 + 2 + 1), the next is 28 (28 = 14 + 7 + 4 + 2 + 1), then we get to increasingly larger values, as the next one is 496, then 8128, and so forth.

[31] *The Foundations of Arithmetic* (2nd ed, trans by J. L. Austin, Evanston: Northwestern University Press, 1999), §62, p. 73.

"New York is the most populous city in the USA"

"New York is situated in the Manhattan Isle"

In Frege's dictum, these statements should denote the same object, fixing a criterion of identity for the name that saturates the argument places of these statements, thus fixing their truth-value. However, New York was not the most populous city in the USA in all its history; and although the city started there, it is certainly not constrained anymore to Manhattan Isle. So, obviously, the truth-value of these statements will vary depending on the time period and location under consideration. This goes for every empirical statement, as these are contingent on the events they describe. Therefore, determinations of time and space should be part of their hidden logical structure, in order for truth-conditions to be fixed before use.

With that, let us advance for the case that interests us the most here, the identity criterion for a mathematical concept such as '17'. As Demopoulos observes, in the Fregean spirit the "first step toward providing an account of our arithmetical knowledge is the successful explanation of our reference to the numbers"[32]. And our reference to numbers occurs, generally, within statements of quantity. One of Frege's fundamental principles to approach the matter is the context principle: our reference to numbers should always be seen in the context of a proposition, for "only in a proposition have the words really a meaning".[33] So the task at hand is "to define the sense of a proposition in which a number occurs."[34]

To clear the way for his account, Frege first discusses a series of previous attempts, all loosely based around the notion that number is a property of a thing. While discussing John Stuart Mill's philosophy of mathematics in his *The Foundations of Arithmetic*, Frege argued that number could not be a property (be it of physical bodies or aggregate of bodies), because such conception fails to acknowledge that applying a pivot-concept is a

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[32] William Demopoulos. "Our Knowledge of Numbers as Self-Subsistent Objects" (in: *Dialectica,* Vol. 59, N° 2, 2005, pp. 141–159), p. 141.

[33] Frege, G. *The Foundations of Arithmetic*, (2nd ed, trans by J. L. Austin, Evanston: Northwestern University Press, 1999, §60.

[34] *Ibidem*, §62.

necessary step to count. Frege works his way to such realization by showing that we can make various correct statements of number for the same object — a bundle of books can be described as 5 books, 1500 pages, billions of molecules, etc. — and thus the number associated with a physical agglomerate of bodies will depend on what one is counting.[35] While statements that predicate properties of objects (e.g. 'The leaves of that tree are red') have a specific value-range (the set of leaves that fall under the properties of belonging to 'that tree' and of being red), statements of number deal instead with concepts and can only make sense once it is clear what is being counted.

To use Frege's own example, the statement "Jupiter has four moons" is an assertion about the concept 'Jupiter's moons.' In order to uncover the logical form of the proposition, Frege rewrote it as "The number of Jupiter's moons is the number four, or 4", which is of the form 'the number of Fs is x.' Dirk Greimann explained the reason for this rewriting as such: "This semantic approach to explaining our access to the numbers seems to be based on the idea that the numbers are to be given to us by means of the *senses* of sentences of the form 'the number of *F*s = *a*.'"[36]

Following on Frege's reasoning, the number zero is given by the sense of an assertion about a concept that has no objects falling under it. We can generalize from this and say that a number *n* is identified as the collection of all collections with *n* members (e.g. "2" is the class of all pairs). A cardinal number *n* would be the common referent of all equinumerous collections of cardinality *n*. Or, more simply, a number is an equivalence class of sets.

However, as it is well known today, Frege's attempt at providing a criterion of identity for numbers is plagued with severe difficulties. The first already appears in Frege's own *Foundations of Arithmetic*, §57, where the philosopher realizes that fixing the sense of

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[35] In Frege's words: "While looking at one and the same external phenomenon, I can say with equal truth both 'It is a copse' and 'It is five trees', or both 'Here are four companies' and 'Here are 500 men'. Now, what changes here from one judgment to the other is neither any individual object, nor the whole, the agglomeration of them, but rather my terminology. [...] This suggests as the answer [...] that the content of a statement of number is an assertion about a concept." (*The Foundations of Arithmetic*, 2nd ed, Evanston: Northwestern University Press, 1999, §60).

[36] Greimann, D. "What is Frege's Julius Caesar Problem?" In: *Dialectica,* Vol. 57, No 3, 2003, p. 268.

assertions of number as a second order predication about a concept does not yet fix the criteria of identity of a numeral, so we cannot say yet what use of a numeral refers to:

> It is only an illusion that we have defined 0 and 1; in reality we have only fixed the sense of the phrases "the number 0 belongs to", "the number 1 belongs to"; but we have no authority to pick out the 0 and 1 here as self-subsistent objects that can be recognized as the same again.[37]

Fixing the sense of "the number 0 belongs to" is not yet to fix the sense of "the number 0", and thus the attempt at answering the challenge of identifying numbers as self-subsistent objects through the assertion of quantity will not work, as it does not specify which objects the number-words are.

For argument's sake, let us accept Frege's separation of levels of abstraction principles and the reserved place number-words have in second-order predication. Now, how do we define numerals as objects? In his own words:

> But the question is: How do we apprehend logical objects? And I have found no other answer to it than this: We apprehend them as extensions of concepts, or more generally, as ranges of values of functions.[38]

So it is but a consequence of his philosophy of language that Frege could not avoid reading numerals as the objective yet non-physical objects of arithmetic. It is not merely because we can turn "Jupiter has four moons" into "the number of Jupiter's moons is the number four" that one should think numerals are singular terms. In Frege's analysis, the focus is not on the intention of who uttered these sentences, but on which logical role the words fulfil. So we ask, first and foremost, of *all* concepts: is it saturated or unsaturated? If numbers occur saturating functions, then, in Frege's view of language, numbers must be objects:

[37] Frege, G. *Foundations of Arithmetic* (2nd ed, trans by J. L. Austin, Evanston: Northwestern University Press, 1999), §57.

[38] Passage of a letter to Russell, July 28, 1902, *apud* Zalta, Edward N., "Frege's Theorem and Foundations for Arithmetic", *The Stanford Encyclopedia of Philosophy* (Winter 2015 Edition), available at: <http://plato.stanford.edu/archives/win2015/entries/frege-theorem/>.

> In the sentence "there is at least one square root of 4", we have an assertion, not about (say) the definite number 2, nor about - 2, but about a concept, *square root of* 4; *viz.*, that it is not empty. But if I express the same though thus: "The concept *square root of* 4 is realized", then the first six words form the proper name of an object, and it is about this object that something is asserted.[39]

I call this a triangular reading of equations because both sides of an equation name a third, essential object (e.g. what the morning star and the evening start refer to). For Frege equations are epistemologically flat and static, no result is being calculated, no *process of obtainment* taking place in time. This is because both sides are intersubstitutable without loss of meaningfulness: to state $\sqrt{4} = 2$ is to say there is a number with the property of giving the result 4 when multiplied by itself; these are equivalent expressions to name the same abstract object. For contrast, think of the constructivist tradition, whereas equations would be given an operational reading and thus would be seen as processes. This is seen, for example, when one explains a function as a black box with an input and an output value.

What is of special interest to the present discussion is how Frege takes concepts equally, in case they belong to the same semantic category, be it in ordinary language or in arithmetic. The logical structure of $\sqrt{4} = 2$ would be thus similar to that of 'Nico is Rubia's cat': the former asserts the true identity between '$\sqrt{4}$' and '2', and the latter asserts the true identity between the cat owned by Rubia and Nico. That is to say that Frege's reading of arithmetical equations does not discriminate between *types* of result nor between types of objects.

This is when the Julius Caesar Problem strikes first. Numbers, as self-subsistent objects, should be distinguished from other kinds of objects, yet his system does not provide any means for that. Mixed-identity statements are possible, and so there is no way to preclude assertions such as "The number of books is Julius Caesar" or "The object 2 is Julius Caesar". Thus, in order to unambiguously refer to numbers as objects, Frege offered a *principle of abstraction* as a contextual definition of numbers. He dubbed it Hume's Principle:

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[39] Frege, G. "On Concept and Object", in: *Mind* (vol. 60, no. 238, 1951), p. 174.

The number of *Fs* is identical to the number of *Gs* if and only if there is a one-to-one correlation between the *Fs* and the *Gs*.

That is, if we have a class ***a*** of objects that fall under concept *F*, and a class ***b*** of objects that fall under concept *G*, then, if ***a*** and ***b*** can be correlated one-to-one, we may identify the cardinality of ***a*** and ***b*** as an equivalency of these classes. The principle can be formally defined as:

$$F(a) = G(b) \leftrightarrow R_{eq}(a,b)$$

Where R_{eq} is a relation of equivalency. With this general form, we can also form sentences like "the direction of line *a* is identical to the direction of line *b* if and only if line *a* is parallel to line *b*". As I read him, Frege's point in introducing this principle was that counting already exposes the criteria of correctness for the process: one can count correctly *because* one can compare the cardinality of sets. Grasping this sense of equinumerability is thus epistemologically prior to grasping the proper use of numerals. As so, Hume's Principle provides a criterion for the use of numbers, functioning as Frege's tool to define the integers and then reduce them to the logical notion of one-to-one correlation.

What changed from the previous attempt is the recognition that no object is identified by defining the sense of a statement of quantity in terms of a second-level predication, but we can identify numbers as classes of concepts. Bob Hale and Crispin Wright have argued that taking Hume's Principle as an implicit definition of numerical terms (more precisely, a non-logical axiom) is enough to fix the truth-condition of arithmetical statements and satisfy EC.[40] Yet, obviously, to hit the goal of grounding

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[40] "[…] the case for the existence of numbers can be made on the basis of Hume's Principle, and it is important to the neo-Fregean that this should be so, precisely because it provides for a head-on response to the epistemological challenge posed by Benacerraf's dilemma. Hume's Principle, taken as implicitly defining the numerical operator, fixes the truth-conditions of identity-statements featuring canonical terms for numbers as those of corresponding statements asserting the existence of one-to-one correlations between appropriate concepts." ("Logicism in the Twenty-First Century" in *The Oxford Handbook of Philosophy of Mathematics and Logic,* Oxford University Press, 2005, p. 172).

arithmetic in logic in this way, Frege's followers have to argue that the concept of number is not assumed in the derivation of the correlation in Hume's Principle.[41] But even with the aid of Hume's Principle, the Julius Caesar Problem bites back, for we have no means of determining whether or not 'Julius Caesar' is the object that satisfies the equivalence class of all sets of two. To see why, let us call *x* the number-object we assert about the concept 'Jupiter's moons.' To be the referent of 'The number of Jupiter's moons', *x* must be:

(i) equal to its pair *y*, the purported referent of "2 + 2",

or

(ii) it must be possible for "2 + 2" and "the number of Jupiter's moons" to be in a one-to-one correlation.

As one can readily see, these conditions are not able to rule out some queer choice of objects as referents, for any singular term is a possible choice here. Syntactically, we write correctly "2 + 2 = 4", but the principle does not bar us from stating that the object referred to by the expression "2 + 2" is Julius Caesar. So how do we unambiguously identify the object to which our number-words supposedly stand for? Which symbolism is more representative of the number-entity? The arabic numeral? The set theoretical notation? Or as the successor of the successor of zero? It seems we have a deeper problem here. As Shapiro explains:

> Any small, moveable object can play the role of (i.e., can be) black queen's bishop. Similarly, and more generally, anything at all can 'be' 2 — anything can occupy that place in a system exemplifying the natural-number structure. The

41 For instance, George Boolos has demonstrated that, *if* we *assume* there are number-objects, the informal arithmetic devised in Frege's *Foundations*, based in axiomatic second-order logic, can be regarded as consistent. Boolos, however, does not try to assert "the truth of the foundations", for this or even his argument for the consistency of Frege's first system are dependent on the assumption that numbers are self-subsistent objects. Moreover, the reader should be aware that this does not entail a consistency proof for Peano-Dedekind arithmetic (PA), as its axioms are based on first-order logic. A first-order proof of the consistency of PA still remains a challenge. (*See* Boolos, G. "The consistency of Frege's *Foundations of Arithmetic*", in W. D. Hart *The Philosophy of Mathematics*, Oxford University Press, 1996).

Zermelo 2 ({{∅}}), the von Neumann 2 ({∅, {∅}}), and even Julius Caesar can each play that role.[42]

Of course, adding explicit definitions to block the confusion would be a warranted course of action at this point, and Frege considers it in §68 of *The Foundations of Arithmetic*, but it is only in *The Basic Laws of Arithmetic* that he rigorously fixated the meaning of such expressions in terms of value-ranges. That was when Frege introduced a principle to fix the connection between concepts and extensions as a functional correlation, the Basic Law V, formally defined as:

$$\{x : Fx] = \{x : Gx\} \equiv \forall x(Fx \equiv Gx)$$

This logical law asserts that the sets *F* and *G* are identical if and only if they are coextensional. From it, we can derive: (i) the law of extensions, which asserts that an object is only a member of a concept's extension if it falls under that concept, and (ii) the naïve comprehension axiom, which asserts that for every concept (or property) defined by an open formula $\varphi(x)$, where '*x*' is a free variable, there is an extension consisting exclusively of objects falling under that concept (or having that property).[43]

However, as it went down famously in the history of mathematics, this later solution was stopped at its tracks by Russell-Zermelo's Paradox. I will not get into details about the paradox because there is more than one way to derive it from Frege's laws and because the details of the derivation are not relevant to our present discussion. Suffices to say that a contradiction can be derived from the requirements introduced by Basic Law V by considering the set of all sets that are not members of themselves.[44]

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42 Stewart Shapiro, *Philosophy of Mathematics: Structure and Ontology* (Oxford: Oxford University Press, 1997), p. 80.

43 *See* Demopoulos, W.; Clark, P. "The Logicism of Frege, Dedekind, and Russell". In: *The Oxford Handbook of Philosophy of Mathematics and Logic.* (Oxford University Press, 2005), p. 133.

44 Or, as Russell explained, we may consider a barber who shaves all and only those who do not shave themselves — does he shave himself?

In short, the downfall of Frege's programme boils down to the adoption of principles of set formation that allow the derivation of Russell-Zermelo's Paradox. Shapiro tries to vindicate the idea in another manner: given Boolos argument that the *Foundations'* account of arithmetic is consistent, Shapiro claims that one can "consistently identify numbers with extensions, as long as one does not maintain that every open formula determines an extension and that two formulas determine the same extension if and only if they are coextensive."[45] That is, so as long as one abandons Basic Law V, it is possible to interpret numbers as in an ante-rem structuralism.

Yet there is still another subtler problem with this account: the identification of meaning with sets given in extension. As I read him, this is Quine's point with the example of 'creature with a kidney' and 'creature with a heart'.[46] Extensionally, both sets are supposed to be identical. Intentionally though, they could be discrete, since there is nothing impossible about a creature with a heart but no kidney. If we take the meaning of these expressions to be identical to the truth condition, assuming it refers to a totality of creatures given in extension, then we may end up confusing this conceptual connection, assuming every creature with a heart necessarily has a kidney as well.[47]

With this, I conclude that the challenge to identify numbers as objects is not motivated by logical principles of abstraction, but by the doctrine of representationalism that promotes the extensional interpretation of mathematical expressions. When posited with the potential to go beyond concrete possibilities of assertibility and with no constraints on verifiability, the reference to mathematical objects becomes utterly inscrutable. For those like Frege, caught in the epistemological horn of Benacerraf's dilemma, the challenge is to explain how can we know of and state the facts of an abstract

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[45] Stewart Shapiro, *Philosophy of Mathematics: Structure and Ontology* (Oxford: Oxford University Press, 1997), p. 78n.

[46] Quine, W.V.O., *From a Logical Point of View* (Harvard University Press, 1971), p. 31.

[47] Frege himself ended up lamenting his earlier approach on this issue: "One feature of language that threatens to undermine the reliability of thinking is its tendency to form proper names to which no objects correspond. [...] A particularly noteworthy example of this is the formation of a proper name after the pattern of 'the extension of the concept a,' e.g. 'the extension of the concept star.' Because of the definite article, this expression appears to designate an object; but there is no object for which this phrase could be a linguistically appropriate designation. From this has arisen the paradoxes of set theory which have dealt the death blow to set theory itself. I myself was under this illusion when, in attempting to provide a logical foundation for numbers, I tried to construe numbers as sets." ("Sources of Knowledge of Mathematics and the Mathematical Natural Sciences", in *Posthumous Writings*, H. Hermes et al, eds, P. Long and R. White [trans.], Chicago: University of Chicago Press, 1979, p. 269).

reality that supposedly outruns our capacities to know about it. This challenge confronts the ontological commitments they advance as motivated by logical principles of abstraction, since these seem to be introduced for the sole reason of providing an ideal of mathematical practice that is purified from external, contingent and empirical matters, and thus it is hard to explain why application of abstraction principles should entail the existence of a mind-independent mathematical reality.

3. The semantic horn

Akin to the epistemological horn, the semantic horn of Benacerraf's dilemma is also set by a condition that an account of mathematical truth must fulfil, namely:

> ***SC***: The account of mathematical truth must be subsumed under a global semantic theory of truth.[1]

This condition implies that particular instances of truth, such as the mathematical, should be defined in the terms provided by some general theory of truth. Its underlying assumption is that there is only one truth property that must be defined univocally. That is, SC implies that an analysis of truth-conditions *must not* discriminate between vocabularies or contexts of utterance. There should be no separation of discursive practices in an analysis of truth-conditions since these are presumably independent of variances of tone or context; this is, after all, the very point of analyzing propositions in terms of logical form.

Even though Benacerraf admits from the outset of *Mathematical Truth* to be "indulging here in the fiction that *we have* semantics for 'the rest of language'" (MT, 661f), he still raises the requirement for semantic homogeneity, resting his case on the assumption that mathematical language is a part of "language as a whole". He even makes "a plea that the semantical apparatus of mathematics be seen as part and parcel of that of the natural language in which it is done." (MT, 666).

The justification for raising such a condition is that logical analysis reveals that mathematical statements are composed of the regular suspects — quantifiers, singular terms, and predicates — seemingly paralleling the structural composition of ordinary empirical statements. It is no surprise that such a compositional account is given for mathematical statements; many philosophers take *compositionality* as a crucial characteristic of language that enables formulation of novel sentences and creation of new meanings. Or

[1] "For present purposes we can state it as the requirement that there be an over-all theory of truth in terms of which it can be certified that the account of mathematical truth is indeed an account of mathematical *truth*. […]Another way to put this first requirement is to demand that any theory of mathematical truth be in conformity with a general theory of truth — a theory of truth theories, if you like — which certifies that the property of sentences that the account calls 'truth' is indeed truth." (MT, 666).

so it goes in Benacerraf's comparison of an empirical statement (ES) and a mathematical statement (MS) at the heart of the semantic horn of his dilemma:

> (ES) There are at least three large cities older than New York;
> (MS) There are at least three perfect numbers bigger than 17;

Which under a standard analysis of predicate calculus are taken to be modeled by the following logical form:

> (LF) There are at least three *FG*s that bear *R* to *a*.

As the argument goes, this logico-syntactical similarity supposedly indicates more than just an accidental coincidence: it informs a deep grammatical similarity which in turn calls for the same semantic analysis. The meaning of these sentences can be explained as a function of the composition of two-place predicates ('older than' and 'bigger than') and some singular terms (for cities and numbers).[2] And as so Benacerraf considers that the mechanisms which warrant talk of truth for each must be the same,[3] for the truth-conditions of ES are satisfied by the set {London, Paris, Madrid} and the truth-conditions of MS are satisfied by the set {28, 496, 8128}.

To accept the globalization of this view, as Benacerraf recommends, means to read *all* declarative sentences as semantically homogeneous: quantified predications possessing equivalent structures of assertibility and verifiability conditions, determined by virtue of their logical form. As so, there should be no substantial differences between the function of a mathematical and an empirical statement.[4] Taken at *face value*, as literally talking about the things it evokes, both kinds of statement function in exactly the same way. They serve to predicate properties of particular objects, to *represent* possible states of affairs. We may

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[2] Such pairing depends on Frege's semantic equivalence of the role of predicates as they appear in ordinary sentences with the role of functions as they appear in mathematical formulae (see 2.3).

[3] In his words: "A theory of truth for the language we speak, argue in, theorize in, mathematize in, etc., should by the same token provide similar truth conditions for similar sentences". (MT, 662).

[4] "So, to some extent, the question posed in the previous section—how are truth conditions for [MS] to be explained?—may be interpreted as asking whether the sublanguage of English in which mathematics is done is to receive the same sort of analysis as I am assuming is appropriate for much of the rest of English." (MT, 669).

consider mathematical and scientific discourses as equally done against a background domain of objects that guarantee the truth of their statements.

A straight consequence of this face value reading is to charge any account that fails to take into account the semantic parallels between MS and ES as betraying their surface grammar. According to Benacerraf, such accounts would be disentangling what consists in knowing the truth of a mathematical statement from what consists in knowing other instances of truth. In this case, we would have two distinct truth properties, and thus the logical connectives could not be used in the way they currently are to make compound propositions. For instance, the only occasion in which the conjunction *p & q* is true is when both premises are true. So the validity of the conjunction is preserved with the statements "$2H_2 + O_2 \rightarrow 2H_2O$" and "There is water in Mars", expressing one truth of chemistry and one of planetary geology, both about water. However, where the account of mathematical truth does not parallel the account of empirical truth, *p & q* cannot be true, for the premises will not be *equally* true. This could potentially blur the concept and make scientific uses of the word "true" ambiguous.[5] So it seems that if we let this concept to be sprinkled around in uses that make no clear connection to an existing referent, we will stretch the concept into vagueness and eventually lose sight of what we are trying to communicate with it.

Since the satisfaction of the semantic condition depends on the acceptance of a general theory of truth, it is only natural to expect from Benacerraf some orientation with respect to which property is universally shared by statements we deem true. But what property could essentially define all instances of truth? In an unexamined *face value* sense, any statement someone might disagree with will look like a truth-apt[6] statement. Also, every statement can be traced as a correct or incorrect derivation of others, possibly offered as reasons for it.

[5] This is akin to a passage in *Some Remarks on Logical Form* where Wittgenstein realised that his symbolism in the *Tractatus* did not had the appropriate multiplicity to avoid the statement that there are two kinds of brightness, "which is obviously absurd" (in *Proceedings of the Aristotelian Society, Supplementary Volumes — Knowledge, Experience and Realism*, vol. 9, 1929, p. 167).

[6] A truth-apt statement is, unlike a question or command, a statement evaluated in terms of truth or falsity in relation to a certain context.

Unfortunately, Benacerraf's own work does not help much with this task. He points to Tarski's theory of truth as the only candidate, for being a semantic theory which elucidates what truth claims essentially are in a systematic way.[7] However, he does not explain how a logico-mathematical theory such as Tarski's could motivate one or other philosophical views of mathematical activity, or about the nature of its subject matter.[8] Tarski's theory is praised for exploring the consequences of a metalinguistic view of truth for interpreted formal languages, constituting the basis from which other mathematicians brought up Model Theory, but not for resolving philosophical issues regarding the property of truth.[9] Actually, one of the most celebrated achievements of his theory is its capacity of defining truth whilst avoiding stepping into metaphysical speculations. As Field puts it:

> Tarski succeeded in reducing the notion of truth to certain other semantic notions; but that he did not in any way explicate these other notions, so that his results ought to make the word "true" acceptable only to someone who already regarded these other semantic notions as acceptable.[10]

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[7] "I suggest that, if we are to meet this requirement [SC], we shouldn't be satisfied with an account that fails to treat [ES] and [MS] in parallel fashion, on the model of [LF]. There may well be *differences*, but I expect these to emerge at the level of the analysis of the reference of the singular terms and predicates. I take it that we have only one such account: Tarski's, and that its essential feature is to define truth in terms of reference (or satisfaction) on the basis of a particular kind of syntactico-semantic analysis of the language, and thus that any putative analysis of mathematical truth must be an analysis of a concept which is a truth concept at least in Tarski's sense." (MT, 667)

[8] This critique was also expressed by William Tait, who wrote: "It is difficult to understand how Tarski's 'account' of truth can have any significant bearing on any issue in the philosophy of mathematics. For it consists of a definition in mathematics of the concept of truth for a model in a formal language *L*, where both the concept of a formal language and of its models are mathematical notions" (*The Provenance of Pure Reason*, Oxford University Press, 2005, p. 66).

[9] As Tarski himself says: "I hope nothing which is said here will be interpreted as a claim that the semantic conception of truth is the 'right' or indeed the 'only possible' one. I do not have the slightest intention to contribute in any way to those endless, often violent discussions on the subject: 'What is the right conception of truth? […] Disputes of this type are by no means restricted to the notion of truth […] and therefore are in vain." ("The Semantic Conception of Truth: and the Foundations of Semantics", in *Philosophy and Phenomenological Research*, Vol. 4, No. 3, 1944, p. 355).

[10] Field, H. "Tarski's Theory of Truth", in *The Journal of Philosophy* (Vol. 69, No. 13, 1972), p. 347.

Field praises Tarski's theory for defining truth in terms of more palatable semantical relations, such as designation, definition, and satisfaction. Tarski himself set this reduction as a principle when he wrote: "I shall not make use of any semantical concept if I am not able to previously reduce it to other concepts".[11] But as Field remarks, the application of Tarski's theory of truth to any discursive practice will require supplementation by a philosophical account of the supporting semantic notions.

The Disquotational Schema introduced in Tarski's *Convention T* serves to define the structure of true sentences, and according to it, a true sentence must be "materially adequate", in the sense that singular terms and predicates must be satisfied by the elements of the Universe Set.[12] This is the set containing all objects referred to by a formal language, and thus upon which operations within that language are defined. Applying this notion to Tarski's favoured example, we say "Snow is white" is a sentence of some object language which we can ascend to a metalanguage by attributing truth to it, but only in case there are some elements in the underlying universe set which satisfy the predication, thus the truth predicate will be coextensive[13] with the sentential function (i.e. both predicates, the one in the sentence and the truth predicate, indicate the same set).

On reflection one may see that, by itself, Tarski's theory does not entail the claim that true sentences *must* refer to existing external objects, abstract or concrete. The Universe Set can be populated by anything, including mental constructs or fictional entities. In Tarski's own words: "the true sentences may be defined as those sentences which are satisfied by an *arbitrary* sequence of objects".[14] This allows the theory to work for many different universes of discourse without ever having to consider the particularities of the objects in question, as it is not up to the formal system to specify the differences between possible objects.

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[11] Tarski, A. *Logic, Semantics, Metamathematics* (Oxford University Press, 1956), p. 153.

[12] *Ibidem*, p. 188.

[13] I'm following Putnam here in calling this relation between meta and object language 'coextensive', but this choice of vocabulary is not universal. Field, for example, prefer to call them 'coreferential' (cf. Field, "Tarski's Theory of Truth", in *The Journal of Philosophy*, Vol. 69, No. 13, 1972, p. 355).

[14] Tarski, A. *Logic, Semantics, Metamathematics* (Oxford University Press, 1956), p. 215 (my italics).

What Benacerraf seems to find so attractive in Tarski's theory and wants us to apply to every instance of truth is how he conceived semantic satisfaction as a relation obtaining between sentential formulas and assignments of objects which can be extensionally characterized in terms of reference. Benacerraf recommends that we supplement Tarski's theory of truth with a reference-based semantic because, for him, to leave reference out of the picture would be begging the question.[15] *Reference* is the key semantic concept for Benacerraf because it can be used to clarify the ancient Artistotelic insight (also respected by Tarski) that the essential property of true sentences is "saying of what is, that it is", or simply, corresponding to facts. In his words: "Reference is what is presumably most closely connected with truth." (MT, 662f). Therefore, his reasoning is that the objectivity of truth can only be explained by supplementing the Disquotational Schema with a referential semantics.

Clearly, besides raising a demand for a globally consistent semantic account of truth, Benacerraf *presupposes* that such explanation has to be pursued along the lines of a referential theory:

> [The analysis of mathematical truth], it seems to me, can be done only on the basis of some general theory for at least the language as a whole (I assume that we skirt paradoxes in some suitable fashion). Perhaps the applicability of this requirement to the present case amounts only to a plea that the semantical apparatus of mathematics be seen as part and parcel of that of the natural language in which it is done, and thus that whatever *semantical* account we are inclined to give of names or, more generally, of singular terms, predicates, and quantifiers in the mother tongue include those parts of the mother tongue which we classify as mathematese. / I suggest that, if we are to meet this requirement [of SC], we shouldn't be satisfied with an account that fails to treat

[15] "What would be missing, hard as it is to state, is the theoretical apparatus employed by Tarski in providing truth definitions, i.e., the analysis of truth in terms of the 'referential' concepts of naming, predication, satisfaction, and quantification. A definition that does not proceed by the customary recursion clauses for the customary grammatical forms may not be adequate, even if it satisfies Convention T. The explanation must proceed through reference and satisfaction and, furthermore, must be supplemented with an account of reference itself." (MT, 677).

[ES] and [MS] in parallel fashion, on the model of [LF]. There may well be *differences*, but I expect these to emerge at the level of the analysis of the reference of the singular terms and predicates.

Reminiscent of Quine when he elected the vocabulary of fundamental Physics as privileged to tackle the ontological question,[16] Benacerraf evokes the idea of "medium-sized physical objects" as examples of truth-bearers that could anchor the reference of our sentences. Apparently, for him, truth can only be encapsulated by conceiving it in an essentially *referential* connection between language and the domain of what is *physically possible*.

Therefore, I maintain that underneath his explicit semantic condition, Benacerraf implicitly holds a semantic presupposition:

> ***SP:*** The truth property must be decomposable into the satisfaction of semantic relations of representation, such as reference or denotation.

SP certainly makes for a stronger requirement than the condition SC stated above, not only because it demands a certain interpretation of semantic satisfaction, but especially because it carries an implicit metaphysical step: once the truth is explained via the satisfaction of references, thus understood under a representational paradigm, then our conception of truth is that of a substantial property implying the actual existence of the referents as described.

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16 In his words: "[…] nothing happens in the world, not the flutter of an eyelid, not the flicker of a thought, without some redistribution of microphysical states. It is usually hopeless and pintless to determine just what microphysical states lapsed and what ones supervened in the event, but some reshuffling at that level there had to be; physics can settle for no less. If the physicist suspected there was any event that did not consist in a redistribution of the elementary states allowed for by his physical theory, he would seek a way of supplementing his theory. Full coverage in this sense is the very business of physics, and only of physics. (Quine, W.V.O. "Goodman's Ways of Worldmaking", in *New York Review of Books*, 1978, *apud* Mary Leng, *Mathematics & Reality*, Oxford University Press, 2010, p. 37).

3.1. Characterizing the constructivist approach

Benacerraf's calls the view of mathematical truth that satisfies EC and fails SC as combinatorial, a term exclusive to his article.[17] I have checked all the bibliography available to me and, apart from studies on Benacerraf's *Mathematical Truth*, have not found any other writer using "combinatorial view" for a kind of anti-platonism. There are no self-proclaimed combinatorialists, no first proponents of this view. Thus, since this label has no apparent roots in a historically well-defined tradition, such as platonism has, some attention should be given to identifying the authors and philosophies that constitute this view.

Benacerraf attributes this view's parentage to a mix of Hilbert's and Ayer's philosophies. More specifically, to the claim that mathematical elements may be ideal elements that do not refer to reality (Hilbert's idea)[18], and that all systems of calculation are based on linguistic conventions (Ayer's). These two initial commitments have direct implications to the epistemology and semantics of the combinatorial view. Yet this will not suffice to characterize combinatorialism. Firstly, because there are little obvious connections between their thoughts which could set up a whole intricate epistemological and semantical account of mathematics. And secondly, because Benacerraf ascribes the combinatorialist label to all philosophers whose account of mathematical truth agrees with the general principle that mathematical truth is established by an algorithmic procedure of proof, not reference. Such accounts are typical of constructivist views such as advanced by L.E.J. Brouwer, Errett Bishop, Gerhard Gentzen, Michael Dummett, Dag Prawitz, Per Martin-Löf, among others.

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[17] "For lack of a better term and because they almost invariably key on the syntactic (combinatorial) features of sentences, I will call such views 'combinatorial' views of the determinants of mathematical truth." (MT, 665).

[18] In *On the Infinite*, Hilbert proposed mathematics is composed of both real-finite, and ideal-infinite propositions, and in order to know the truth of any, we would need to prove the consistency of this mixture. As Benacerraf reminds us, a good criterion to separate these notions is through the distinction of finitary methods of proof from infinitary methods Hilbert's connection of the *real* with the *finite*, aside the finite and general, may be read as an acceptance that we are limited to "truth conditions whose satisfaction or non-satisfaction mere mortals can ascertain", borrowing Benacerraf's line. (*See* David Hilbert, "On the infinite", in Benacerraf, P. & Putnam, H. *Philosophy of Mathematics: Selected Readings*, 2nd edition, Cambridge University Press, 1983, pp. 183-202).

Luckily, it is easy to show that Benacerraf had some variation of constructivism in mind when he coined 'combinatorialism'. As Benacerraf defines it, the central idea of combinatorialism is that proof is necessary for the determination of the truth-value of mathematical statements. In this way, the postulational approach of platonism gives way to a prioritization of epistemology, of verification and evidence before existence. According to him, it is well established among mathematicians that, regardless of their philosophical view, the truth of a mathematical statement (including conjectures) is only verified when proved by canonical means.[19] As so, an account that would satisfy the epistemological condition (EC) would also make truth relative to proof, and this is precisely what the constructivist tradition does.

We can clearly see this in in the way he has set up EC, where the demand raised is fairly similar to the *knowability principle*, a guiding principle of constructivist approaches introduced by Dummett. Compare:

> ***EC:*** The specification of truth-conditions of mathematical statements cannot make it *impossible* for us *to know* that they are satisfied. (MT, 667)
>
> ***Knowability principle*:** If a statement is true, then it can in principle be known to be true.[20]

It follows from the knowability principle that if the conditions for the truth of a mathematical statement cannot in principle be known to be satisfied, then we cannot attribute a truth-value to it. Now compare this to EC, which claims that if it is *impossible to know* whether the truth-conditions are satisfied, then our account of truth is unsuitable. As we can see, both EC and the principle raise the same requirement for an account of

[19] Even a platonist, for whom "truth" has necessarily a metaphysical depth to it, would undertake proof as a human device for acquiring *certainty* about intuitively known facts, hidden from clear reason until that point. Proofs serve to socially establish and formalize what was previously known merely by means of intuition.

[20] The principle is important for constructivism because, if truth is always knowable and demonstrable in principle, then a true statement is justified by a verified relation with a fact. This is the basis of the verificationist variety of constructivism. *See* Dummet, M. "What is a theory of meaning?", in: *Truth and Meaning,* ed. G. Evans and J. McDowell, Oxford: Clarendon Press, 1976, pp. 67-137.

mathematical truth; namely, that a theory of truth must yield *knowable* truth-conditions in its analysis of mathematical statements.

As so, Benacerraf´s combinatorialism can be read as a loose label for some version of a constructivist philosophy of mathematics. Now we need to specify with better precision which version that is. So let us review how these versions compare and differ.

The constructivist philosophy of mathematics is characterized by the prioritization of epistemological principles over semantic ones. Constructivist accounts focus on methods of obtainment, in a clear attempt to answer epistemological questions first and foremost. The fundamental opposition they exert against platonism lies in the rejection of the idea that mathematical statements have immutable self-standing truth-makers. For a constructivist, possibility is conceptually prior to actual existence — to exist is first and foremost to be constructible. This epistemological preoccupation drives constructivism towards the non-classical interpretation of the quantifiers, a.k.a. Brouwer-Heyting-Kolmogorov interpretation (BHK). In it, existential propositions are reinterpreted not as being directly about objects, but as about what would constitute proof of the claim. Thus existential statements of the form:

$$(\exists x \in \mathrm{D})\, P(x)$$

are explained as asserting knowledge of a canonical proof which assures us that $P(a)$ is true just in case there is some inhabitant a in the domain D.[21] For a constructivist, the essence of a mathematical object is that it is constructible; it must be possible to obtain it by means of an effective algorithmic procedure, rather than assume it is given by an arbitrary function. As Martin-Löf explains:

> In accordance with the intuitionistic interpretation of the existential quantifier, the rule of ∑-introduction may be interpreted as saying that a (canonical) proof of $(\exists x : A)\, B(x)$, is a pair (a, b), where b is a proof of the fact that a satisfies B.[22]

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[21] Futhermore, in a BHK interpretation, an implication $(A \rightarrow B)$ is not taken as equivalent to a disjunction $(\neg A \vee B)$, nor to a negated conjunction $(\neg(A \wedge \neg B))$.

[22] Martin-Löf, P. *Intuitionistic Type Theory* (Naples: Bibliopolis, 1984), p. 42.

Or, as André Porto puts it, in constructivism "we reduce (objectual) existence to *possibility of concrete instantiation*. An object exists if we *could construct it*, i.e., if we have a *method* of obtaining it."[23] Not only the possibility of obtaining the mathematical object is guaranteed by the method, but possession of the method implies that the object *exists in potentiality* — if existence is constructability, then knowledge of an effective algorithmic procedure of construction warrants a truth claim about the mathematical object designated by the result of that operation.

If the theory in which a statement is embedded cannot provide a method for the construction of its purported objects (perhaps due to syntactical limitations, or on pain of contradiction), then the object considered cannot be in the domain of that theory. In other words, only objects designated by computations in principle achievable within the limits of our effective methods can be said to properly exist. Or as Douglas S. Bridges carved the point: "The key feature of constructive mathematics is the identification Existence ≡ Computability."[24] We can draw from this that for the kind of constructivist Benacerraf seemed to have in mind the only mathematical objects that exist are the ones designated by general recursive functions, in accordance to Turing-Church thesis (*see* pp. 21-22).

Yet, there are distinct philosophical views of mathematical truth within the constructivist tradition, such as intuitionism, finitism, and verificationism. Although Krockner before him already gave key emphasis to the computational aspect of mathematics, Brouwer's intuitionism was the first example of a systematic constructivist philosophy of mathematics. It conceives mathematical concepts as products of the human mind and mathematical truths as the mental constructions of those objects. In this view, we must imagine the ordered structures which demonstrate the veracity of the statement; a mathematical object exists insofar its construction can be demonstrated to human minds by the formal mathematical proofs — which here are equivalent to clever psychological tricks designed to instigate the same sequence of thoughts on everyone else — showing the steps in the construction of mental structures.

Whereas Brouwer's intuitionism has no qualms in declaring mathematical objects are mental entities that we create, the finitist strain of constructivism, most notably linked to

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[23] André Porto, "Rule-following and Functions" (*O que nos faz pensar*, Vol. 22, N° 33, 2013), p. 81.

[24] Bridges, Douglas S. "Constructive truth in practice", in *Truth in Mathematics* (Oxford University Press, 2005), p. 53.

Hilbert and the middle Wittgenstein, is likely the most averse to the postulation of abstract entities. They accept that mathematics is, in its entirety, a human creation; albeit these creatures do not live only in our imaginations, as they are algorithmic in nature. As so, the objects of mathematics are finite strings of computations, and mathematical truth is a matter of reaching the final clause in an algorithmic procedure, of concluding an empirical process of obtainment involving computations. Hence why finitism is defined by its opposition to the postulation of completed infinities. As an implication, a finitist would never accept the idea of a unproven mathematical statement having a determined truth-value. Finite algorithmic procedures are necessary, not only at verifying the truth-value of mathematical statements but also at determining it. So there is no motivation here for postulating abstract realities to justify our mathematical claims.

A third major constructivist view of truth is the verificationism developed by Dummett and Prawtiz. This strain of constructivism does not conceive the objects of mathematics as purely mental entities, nor as finite strings of calculations, but as the potential itself — the possibility — of concluding such strings. Seen in this way, mathematics is the study of all potential implications of our routines and algorithmic procedures. The truth of a mathematical statement would thus not be determined by reaching the concluding clause of a computation, but instead by the possibility of reaching such conclusion. Therefore, unlike the finitists, for verificationists all objects of which we possess an algebraic rule or algorithmic procedure for their construction exist in an abstract sense.

For example, for a platonist, the standard of correctness for Goldbach's Conjecture is the abstract structure of integers; so we do not know the answer to this conjecture yet because we have not explored the structure well enough. Whereas verificationism sees the activity as first considering and then establishing these intensional objects as unfinished processes of which we have a limited knowledge. Verificationists only limit themselves to not making statements about features that were not revealed *so far*. Their take on the Goldbach's Conjecture is that its correctness is given by a potential proof; so the statement will remain undecidable until we figure out whether a canonical proof lies within the possibilities constrained by our basic assumptions and axioms.

Now, faced with all these varieties of constructivism, the question then is how should the reader approach Benacerraf's terminology. Is combinatorialism a child of intuitionism, finitism, or verificationism? I reckon the best strategy to understand combinatorialism is to examine what kind of opposition it exerts against platonism.[25] Paying attention to the way Benacerraf sets the debate, the central disagreement regards the conceptions of truth and proof. The departing point of combinatorialism is the claim that mathematical truth is determined in virtue of canonical proof. Or, to put it simply, *truth is provability*.

Now, if we compare this with the constructivist views, we may see that combinatorialism is akin to verificationism. To make proof a precondition for truth is the same move as making assertibility conditions the criteria for correctness. That is, the basic principle of the combinatorialist view lives up to the verificationist conception of truth, the idea that truth claims must be grounded in evidence, and by extension, it implies the verificationist semantics.

We may understand the assertibility conditions of a sentence as the set of premises that warrant use of that sentence; a statement is meaningful if it can be derived from basic premises, allowing the verification of what was claimed. Assertibility conditions restrain what can be said by way of social norms, axioms and syntactical rules of derivation. As so, reference to a mind-independent state of affairs becomes a superfluous condition for meaningfulness, and truth ends up being identified with the conditions for warranted assertibility in that specific discourse.

The only point holding combinatorialism from being entirely subsumed under verificationism is Benacerraf's mention of Ayer's influence. That is, his combinatorialism undertakes a conventionalist thesis regarding the epistemic origins of our methods of proof, while the most exponent verificationists say that only our techniques are constituted through conventions. For them, the inferential steps of a canonical proof would be so tightly determined ("gap-free") that they could be followed by any sapient being. Thus, canonical proofs would not be constituted through conventions but rather construed

[25] As a guiding frame, Stuart Shapiro characterizes two fronts of opposition to platonism in the philosophy of mathematics: anti-realism in truth-value, or in ontology. The first one is characterised by the denial that statements are true or false by virtue of corresponding or not to mind-independent facts, in clear opposition to a truth-functional semantics. The second does not accept the ontological dichotomy of concrete and abstract because it denies that there are abstract objects, a move characteristic of contemporary nominalism. The combinatorial view harbours only the first of Shapiro's denials, denying the complete determination of truth-values of every possible mathematical statement. Combinatorialism does not attack the idea of a truth-theoretical semantics, nor the idea that language represents extra-linguistic items; it is only the demand for reference that is abandoned and replaced by a demand for proof. (*See* 'Philosophy of Mathematics and its Logic', in *The Oxford Handbook of Philosophy of Mathematics and Logic* Oxford University Press, 2005, p. 6-7.)

through the development of new mathematical routines and the consequent historical progress of new proofs correcting old proofs.

Thus we may conclude that combinatorialism is an amalgamation of a conventionalist epistemology with a verificationist semantics. For Benacerraf, this is the best account we have of *how* we come to know mathematical truths, although he also thinks it does not properly explain *what* we come to know through mathematical activity. Being an essentially epistemic treatment of truth, combinatorialism introduces an exceptional, constructivist reading of the quantifiers (i.e. BHK interpretation), thus breaking with a globally homogeneous reference-based semantics and its classical reading of the quantifiers. As such, the combinatorial conception of truth as provability cannot satisfy Benacerraf's semantic condition, for mathematical truth would inevitably contrast with scientific truth. Unlike the scientific method, the canonical methods of mathematical proof obviously do not require nor command an experimental element. While a scientific statement is verified when we gather empirical evidence not invented by humans, a mathematical statement can only be verified with formal evidence that we *must* have invented. With this strategy, combinatorialism ends up impaled in the semantic horn, for it is incapable of explaining what is common between the meaning of a scientific claim and a mathematical one. For instance, from this view, we cannot exclude an analogy between formal systems and games, for the activity of proving theorems in an axiomatic system would be cognitively similar to that of demonstrating the possibility of a play in chess.

3.2. The conventionalist element

According to Benacerraf, a conventionalist account of truth is characteristically combinatorial. This is because combinatorialism conceives that the rules of a proof activity, such as mathematics, are always settled by convention. Take, for instance, the views of the main example of a combinatorialist; Hilbert never explicitly endorsed a conventionalist account of truth, but he did endorse a conventionalist account of axioms. For Hilbert we cannot claim that axioms are self-evidently true because we cannot assert anything meaningful about our system without presupposing those axioms. For him, the analysis of intuition is tantamount to the choice of axiomatic system. He saw the axioms as having a different role: the axioms *define* the very subject-matter; although not in the sense of 'definition' addressed by Quine in *Truth by Convention*, as the introduction of a shorthand term (*definiendum*) equated with another (*definiens*) that only contains terms already defined. Hilbert's view on axioms as defining the subject matter of the theory amounts to what today we would call *implicit definitions*. This is evidently in contrast with a platonic view which considers axioms as general truths that already map all the terrain for the theory, and the theorems derived from such axioms would thus point to specific truths.

The connection between combinatorialism and conventionalism is made clear at the ending pages of *Mathematical Truth,* where Benacerraf employs an argument in Quine's *Truth by Convention* against A. J. Ayer's conventionalist view of logic. In this section, I will sketch the conventionalist account of mathematical truth, Quine's argument against it, and then give my assessment of the importance of this discussion to the dilemma. Let us start with Ayer's own explanation of his conventionalism:

> We cannot deny [logical and mathematical truths] without infringing the conventions which are presupposed by our very denial, and so falling into self-contradiction. And this is the sole ground of their necessity […] There is nothing mysterious about the apodeictic certainty of logic and mathematics. Our knowledge that no observation can ever confute the proposition '7 + 5 = 12' depends simply on the fact that the symbolic expression '7 + 5' is synonymous with '12', just as our knowledge that every oculist is an eye doctor depends on the fact that the symbol 'eye-doctor' is synonymous with 'oculist'. And the same explanation holds good for every other a priori truth.[26]

The idea here seems to be that logical and mathematical truths are *analytic*, construed on linguistic habits and conventions solidified by communities of speakers throughout history. It is through such conventions that interlocutors get to know how the symbols connect. Thence it would be trivial to explain why we say the mathematical axioms are *self-*

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[26] Ayer, A. J. *Language, Truth and Logic* (New York: Dover Publications, 1952), p. 85.

evident — they clearly express the ground rules for every rational agent following the same conventions.[27]

Conventions settle how words are supposed to be used by fixing their semantic neighbourhood — every word has synonyms, antonyms, a grammatical class — and thus, as long as these conventions hold, humans will manage to communicate with minimal noise. What traditionally philosophers call 'a priori truth' would be nothing but a consequence of these linguistic connections with no correspondent part in the world. As so, all that a truth of a priori knowledge could possibly express is restricted to knowledge of the semantic relation between certain terms, knowledge of a part of the language's semantic web. In Ayer's view, logic and mathematics have their foundations as secure as they can possibly be in these conventions. We are not losing rigorousness nor threatening the established mathematical knowledge by illuminating the linguistic nature of analyticity.

Now, before we advance to Quine's argument, it is important to remember that his main interlocutor was not Ayer, it was Carnap. The dissonance between master and student occurred regarding their versions of confirmation holism. Both agreed that a theory composes a framework whose rules ground the meaning of the hypotheses that are consistent with the theory. But as so, a statement not derived from the meaning-giving kernel of a theory cannot have its truth-value determined by this theory's decision procedure. The collateral effect of this view is that whenever a hypothesis is tested against experience, it is not alone — the whole chain of implications leading to that hypothesis is tested as well, and as so, what is under evaluation is the practical value of working within that theoretical framework. The main disagreement between Quine and Carnap emerges because the later thought that the universalisation of the ontological question 'What there is?' is illegitimate. For Carnap, questions about the existence of an object can only be meaningful against the theoretical framework that introduced this object in the first place.

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[27] Although we should mind that even these deep seated agreements would leave margin for politics, as evidenced by the ever-lasting debates on which set of axioms should be taught as the foundations for all mathematics, which sometimes occur even inside the same school or research group. Amongst the most noteworthy examples are the debates over changing Zermelo-Fraenkel axiom system by substituting the axiom of choice for the axiom of determinacy, or the recent arguments in favour of replacing the set theoretical basis of our current foundational theories altogether for a category theory basis.

Thus, the statement that numbers exist is trivially true for the context of arithmetic, and what is meant by it is by no means equivalent to say that Higgs bosons exist. Moreover, for Carnap, the only basis there is to justify choosing one theoretical framework over another is their usefulness towards knowledge growth. Quine, on the other hand, thinks there is a legitimate way to address the ontological question in absolute terms, independently of a theoretical framework.[28]

It is this concern for an ultimate foundation for knowledge that prompted Quine to accuse conventionalism of vicious regress. Quine's argument is that, in order to derive logical truths from linguistic conventions (and form a logical system), one would have to presuppose the truth of logical consequence (that the correct conclusion is the consequence of the set of premises). He claims that it is not clear what the conventionalist picture would achieve. That is because "we can only succeed" in making everyone follow a set of conventional inference rules and arbitrary definitions for the logical constants (the *if*-idiom, the *not*-idiom, the *every*-idiom) "if we are already conversant with the [logical] idiom."[29] To stipulate the rules of logic, we would have to make general statements composed of logical constants, so we would have to understand the logical constants. Therefore, according to Quine, the idea that mathematical and logical truths are the result of following certain linguistic conventions must be wrong, since we already need inference rules in order to infer specific statements from general ones.

The problem with Quine's argument is that this conception that logic is intuitively and naturally grasped demands privileging a certain philosophy of logic. But if so, then which interpretation of the quantifiers is the natural one, the classical or the intuitionistic? Could we also say that humans have an intuitive grasp of the principle of non-contradiction? Quine's argument seems to beg the question regarding the nature of logics: Is it the study of arbitrary patterns of linguistic conventions, or are these conventions based on natural structures that one cannot avoid if looking for a sound argument?

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[28] I'm following the comparative exposition made by Mary Leng in the §2.1.2 of hers *Mathematics & Reality* (Oxford University Press, 2010).

[29] See Quine, W.V.O. *The Ways of Paradox and Other Essays* (New York: Random House, 1966), p. 96-97.

If we so push logics down the epistemological sieve, we will inevitably reach some version of Agrippa's Trilemma regarding the foundations of this discipline, as is the very structure of justification within the boundaries of human reason that locks us down to one of these options: foundationalism, coherentism, or infinisitm.[30] With this in mind, I argue that the conventionalist view on the regress of justifications is only vitiated to foundationalist eyes.

Conventionalism is evidently antithetical to a foundationalist order of explanation. A conventionalist would not see a *vicious* regress in Ayer's idea, she would see an *explanatory regress* from logical calculus to linguistic conventions, and then from there to the human culture that determines these conventions, then from culture to its grounding in our biological constitution, our evolutionary history, and so on. The regress of justifications may be non-repeating and potentially infinite insofar we agree there is a substantial difference between the logical calculus built on top of conventions and what these conventions aim to make explicit. So Quine's argument does not have the force to reject conventionalism, it has force only to raise the epistemological question about the foundations of logics.

Moreover, this argument does not serve to reject conventionalism about mathematical truth. A conventionalist about mathematical truth does not necessarily need to accept conventionalism about logical rules. One may claim that mathematical equations are linguistic conventions and still hold that inferential rules are not instituted through convention — they are not laws arbitrary stipulated by anyone — but instead through the historical decantation of socio-normative practices.

And while there is room to argue there is a circularity in presupposing logical consequence to derive logical rules and a definition of the logical constants, there is no

[30] For a brief reminder of the three:

i) *Foundationalism*: there are first principles, fundamental truths that are self-evident (i.e. it is given to us, readily available to the intellect) and thus automatically justified. In this view, knowledge claims are based on a singular (often sensory) foundation that stops the regress of justifications, and the justifications form a non-repeating finite chain;

ii) *Coherentism*: justification is not found as a given, nor is it simple. It lies instead in the particular arrangement of the set of relations between accepted statements. This makes justification into a matter of internal consistency within the theory, forming repeating finite chains of justifications;

iii) *Infinitism*: no claim is self-evident; every claim needs a reason. So there are no ultimate justifications that provide foundational reasons, every claim of *know-of* has a potentially infinite chain of reasons. This forms potentially infinite chains of non-repeating justifications.

circularity whatsoever in affirming that we presuppose a logical system in order to derive mathematical truths. Actually, the realization that one amongst other logical systems must be presupposed by a mathematical theory characterizes the contemporaneous axiomatic approach to mathematics which starts by explicitly putting forward a regulated logical syntax (e.g. first-order, paraconsistent, intuitionistic logics, etc.) to allow for the construction of proofs for mathematical theorems.

Nevertheless, I think Quine's argument still poses a challenge to any view that considers linguistic conventions as one of the fundamental elements in the constitution of mathematical truths. If it is indeed the case that our understanding of logical consequence is formed through the repetition of socio-cultural practices, then conventionalism must explain how is it that we depart from vacillating commitments towards linguistic conventions and manage to arrive at a complex family of precise and determinate systems of calculation that are crucial for our theorizations about the natural world. The conventionalist must explain the sense in which calculating according to conventions yields an intrinsically necessary result that is exactly what was required to model a natural structure.

3.3. The principle of verification

The semantic thesis that the correct application of the concept of truth and falsehood to sentences depends entirely on the adjustment of one's linguistic behaviour towards evidence is known as verificationism. Opposing the idea that sentences have their correctness fixed by the extensions they represent, verificationism says their correctness is fixed through the possibility of proof. To be in the possession of an effective decision procedure becomes a condition for truth, and as such verificationism does not allow the split between truth and assertibility, as the conditions for assertibility already limit the set of true propositions down to the provable ones. "Intuitionistically, truth of a proposition is analysed as existence of a proof; a proposition is true if there exists a proof of it", defines

Martin-Löf.[31] As so, a statement does not need to refer to an extension in order to be meaningful.[32] Verificationism follows Dummett in denying any place of pride for reference, replacing it instead with the notion of verification of evidence. As such, the principle of verification translates to a semantic key the combinatorialist prioritization of evidence before truth-claims.

According to Prawitz, endorsement of the principle of verification marks the shift from model-theoretical to a proof-theoretical semantics. In his article *Logical consequence: a constructivist view*, where he tackles the classical philosophical question about the understanding and correct recognition of logical consequence, Tarski is the point of reference from which Prawitz wants to move away. In Tarski's account of logical consequence, given a set of premises and a conclusion, if all sentences of the set of premises are true under the assigned interpretations of the non-logical terms, then so is the conclusion, under the same interpretations. Prawitz's account differs in that, for him, the meaning of the non-logical terms must be disentangled from the assignment of truth conditions, because to define meaning one must take truth as an undefined primitive:

> […] truth conditions cannot simultaneously do service both in a definition of truth and in an explanation of the meaning of the sentences in question. […] — this would be like solving two unknowns, given only one equation. […] We

[31] "A Path from Logic to Metaphysics" (in *Congresso Nuovi problemi della Logica e della Filosofia dela Scienza*, 1991, p. 141) *apud* André Porto, 'Wittgenstein and Mathematical Identities', in *Disputatio* (vol. IV, n° 34, 2012), p. 774.

[32] Constructivists generally adopt the intensional conception of function as a formula or rule of derivation, contrary to platonist-leaning mathematicians who accept a purely extensional conception of functions as given in terms of arbitrary subsets (see 2.2).

> must conclude that truth conditions can serve as meaning explanations only if we have a grasp of truth.[33, 34]

But we do not have such an intrinsic grip on what is true. If we had it, everyone would be capable of discerning truth from falsity at every instance and then agree over it. Yet, as Prawitz writes elsewhere: "something does not become correct in mathematics because we hold it to be correct".[35] Instead, in his view, truth may only be known through the application of a method of verification.

Following Dummett on this account, Prawitz understands this as constituting an objection to truth-conditional semantics. To avoid begging the question about the definitions of meaning and truth as based on their relation, verificationists take *evidence* as their primitive unexplained explainer element, for "evidence or what it is to acquire knowledge must be taken as a more fundamental concept than truth — truth may then be defined as the potential existence of evidence."[36] Having evidence should come naturally before believing in the existence of an entity and making truth-claims about it.

From this basis, verificationism conceives the meaning of a statement as making manifest the conditions in which it would be effectively verified. The basic principle of a verificationist semantic is to understand the content of a declarative sentence as identified with the procedure of verification itself, thought off as being a mind-independent method

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33 Prawitz, D. "Logical consequence from a constructivist point of view" in: *The Oxford Handbook of Philosophy of Mathematics and Logic* (Oxford University Press, 2005), p. 674.

34 The following passage from Dummett may also exemplify this point: "The correspondence theory fails as an account of truth because it attempts to characterize the application of the predicate "truth" uniformly for all sentences; since the truth-value of a sentence evidently depends upon its sense, this assumes that the sense of a sentence can be given in advance of a specification of its truth-conditions, but in such a manner that its truth-conditions can then be derived from a knowledge of its sense. […] there can be no uniform account of the conditions under which a sentence is true, the sense of the sentence being taken as already known, any more than there can be a uniform account of what it is to win a game, it being assumed that it is already known what the game is." (*Frege's philosophy of language*, New York: Harper & Row, 1973, pp. 463-464.)

35 Prawitz, D. "Truth and objectivity from a verificationist point of view", in *Truth in Mathematics* (Oxford University Press, 2005), p. 48.

36 Prawitz, D. "Logical consequence from a constructivist point of view", in: *The Oxford Handbook of Philosophy of Mathematics and Logic* (Oxford University Press, 2005), p. 681.

to obtain or construe whatever the sentence declares true. Or, as better said Dummett: "a grasp of the conditions under which the sentence is true may be said to be manifested by a mastery of the decision procedure."[37]

This conception seems to conceive meaning as being algorithmic in nature, in the sense that it may be characterized as step-by-step instructions that demonstrate what is expressed. It conceives meaning as a form of know-how: if one knows how to find-construct-check what was said, then one understands the message. Meaning is processed in parts, through the steps of an algorithm. All potential implications of a statement are determined within the margins of this intensional connection between the expression and its meaning. And, of course, an interlocutor can only understand a claim insofar she knows an algorithmic cognitive method (knowledge-how) for decoding (or computing) these implications.

For the mathematical case specifically, understanding a mathematical statement is having the correct know-how to compute/construct the objected represented. These statements do not state the actual independent existence of abstract objects; they state the abstract existence of a method of construction of a purported object. The possibility of construction is primordial, it grounds the whole sense of attributing truth to a statement. 'Constructing' an object means providing a proof for the existentially quantified statement involving it.

Now, if a sentence has meaningful content only if it is provable, then how should we deal with hypothesis and conjectures? Or what fixes criteria of correctness for future calculations, such as cases which involve so many steps that humans have never computed 'that far'? — Prawitz offers the following answers:

> From a verificationist point of view the natural way to take the conjecture is to understand it as saying that it is provable that it is provable that there are infinitely many primes. This may also be expressed by saying that there exists a proof of the proposition that there are infinitely many twin primes, where 'exist' is to be taken in a tenseless sense, not as implying that a proof has already been construed by us.[38]

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[37] Dummett, M. *Truth and Other Enigmas* (Harvard University Press, 1978), p. 224-225.

[38] Prawitz, D. "Truth and objectivity from a verificationist point of view", in *Truth in Mathematics* (Oxford University Press, 2005), p. 47.

The potential existence of the method to obtain the result we conjecture about is a prerequisite for the meaning of the conjecture. For the conjecture to be meaningful, it must be provable *in principle*; its sense enough to ground the certainty that a canonical proof could be eventually found. A conjecture just is the entertaining of a possibility; it is this potentiality that is meant by the conjecture — after all, since truth conditions have been put aside in favour of assertibility conditions, what fixes the meaning of a sentence is the verification, through the means it should provide, that the scenario it portrays can be potentially obtained. So it is only *after* a method of verification appropriate for that type of statement is identified that we open up space for a proof, and thus for the truth or falsehood of the conjecture to be verified and demonstrated.

Now, because of such account of conjectures, Prawitz has to introduce a distinction between actual and potential existence of proofs. Examples of actual proofs are the historically significant canonical ones, those that show us what we must accept as a conclusion that follows necessarily from a certain set of premises and rules. Proofs that potentially exist, though, are a whole different matter; Prawitz says: "the question of whether something is a proof is fixed when the meanings are given"[39], and so it is how the conjecture is put that ultimately determines if it is provable or not, because the meaningful conjecture is provable in principle. In the same page, he talks about the existence of these proofs as being "tenseless", and having an "abstract existence." It is the prospect of finding, the space to do so, that guarantees that a proof will be found someday:

> Therefore, that a sentence is provable is here to mean simply that there is a proof of it. It is not required that we have actually constructed the proof or that we have a method for constructing it, only that there exists a proof in an abstract, tenseless sense; we may call it *potential existence* to use a term proposed in this connection by Martin-Löf.[40]

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[39] Prawitz, D. "Truth and objectivity from a verificationist point of view", in *Truth in Mathematics* (Oxford University Press, 2005), p. 50.

[40] Prawitz, D. "Comments on the Papers", in *Theoria — Swedish Journal of Philosophy*, Vol 64 (2-3), 1998, p. 285.

The idea seems to be this: if it is possible to construe the mathematical object, then eventually it will be construed. So long we have an effective method to compute an operation, we may say it gives potential existence and determination to its result, even in cases where the practical implementation of the operation is impossible because reaching a concluding clause would take infinitely many steps. In comparison, whilst for the platonist we discover the properties of abstract objects, for the verificationist we discover the abstract methods who, in turn, have the potential to construct the objects. Thus, for the verificationist, these methods are what ultimately justify our talk about mathematical objects. What that means for mathematics is that proof, as the method of obtainment of mathematical objects, is the guarantee of objectivity. For Prawitz, it is not a matter of convention to regard proof as a prerequisite of truth; in reverse, provability is the only truly mind-independent element in this story. [41]

As a consequence, not unlike other philosophies in the intuitionistic-constructivist tradition, Prawitz has to operate with the notion of 'constructive procedure' as primitive, an unexplained explainer. Some method of obtainment has to be defined as the constructive procedure for objects of a certain domain, and what else could be identified as the procedures to construct mathematical objects if not the proof of the theorems where these objects figure?

> It seems that the notion of constructive procedure used here must be taken as a primitive notion. For instance, as perhaps first pointed out by [Rozsa] Peter, it is not possible to define it as a Turing Machine that always yields a value when applied to an argument; the quantifier in this definition must then be understood intuitionistically and this means that to understand the definition we must already know what such as constructive procedure is.[42]

[41] In Prawitz words: "Hence it should be clear that it is not our treating it as a proof that makes it a proof. This seems to be a reasonable claim. It makes something a proof in virtue of the meaning of the expressions involved, which is also reasonable. But it also seems to imply that the question of whether something is a proof is fixed when the meanings are given, that is, when it is given what counts as a canonical proof. From this it is natural to conclude that already, before a proof of a sentence is found, it is determined that there is such a proof. Provability, which I want to identify with truth, becomes this way something objective" ("Truth and objectivity from a verificationist point of view", in *Truth in Mathematics*, Oxford University Press, 2005, p. 50).

[42] *See* Prawitz, D. "Meaning and Proof: on the conflict between classical and intuitionistic logic." (in *Theoria*, 1977), *apud* André Porto, "Rule-following and functions" (*O que nos faz pensar*, Vol. 22, n° 33, 2013), p. 82

From this point of view, the existence of a method is not reducible or explainable in other terms. Again, this is why for them meaningful statements are those which are in principle verifiable. The relation between meaning and verification is basic, as to know what a statement means is to know how to verify it. So understanding a mathematical statement consists in knowing how to perform the operation and construct the object described.

However, how can we be certain that we possess the correct proof for a certain mathematical theorem if we are not allowed to assume the axioms are self-evident truths, as they cannot be proven? Without the postulation of mathematical objects, there seems to be nowhere else for Prawitz to look for in order to offer that certainty. Prawitz's answer lies with the notion of *canonical* proof: "[…] once we have laid down what counts as canonical proofs, it is a factual matter whether an alleged proof amounts to such a canonical proof"[43]. That is, not only every meaningful statement can be proved or demonstrated as unprovable, but this can be done in a *canonical* way. As long as a given proof is not decisive and ultimately compelling, then it is not a canonical proof. And until the canonical proof is found, there will be at least one coherent way to reject that statement.

A canonical proof is not merely a historically significant proof, nor the most famous, or easiest to follow. Actually, that a proof can be significant or easy are only consequences of the clarity established by canonical methods. The canonical proof is the most perspicuous and surveyable form of a proof, one whose inferential steps are so precisely determined that all that is left for us to do is to follow them. It is composed solely of valid and "gap-free" inferences, and so, it would look like 'a logical conclusion' for anyone capable of following inferences.

To fully explore the logical depth of this notion, Prawitz's also advances an explanation of logical consequence. Given a language on which the logical forms and categories of non-logical terms have been specified by way of Gentzen's introduction

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[43] Prawitz, D. "Truth and objectivity from a verificationist point of view", in *Truth in Mathematics* (Oxford University Press, 2005), p. 49.

rules,[44] for all substitutions *S* of non-logical terms with other non-logical terms of the same category, if there is a valid argument for a conclusion from a set Γ^S of premises and hypotheses on which the reasoning depends, then there is a proof of a conclusion A^S from that set.[45] A conclusion is proved canonically when we show there is one continuous string of valid inferences from the premises to it, no gaps, like a tree branching out. The proof shows that the theorem is a consequence of *n* valid inferences forming the canonical argument from a set of premises.

Prawitz even compares his account of canonical proof with Imre Lakatos' notion of progress through new proofs overriding old proofs. That is, he concedes Lakatos' point that mathematical knowledge is acquired through its practice, and it evolves organically by means of a dialectics of proofs and refutations. We start with attempts to prove a statement which will be then improved by refutations until we get a definitive canonical proof either of a possibility or an impossibility of construction. From this view, so long as we engage in the search for canonical proofs, all mathematical problems will eventually be resolved. And if given time we have not found a way to prove an undecidable statement, then this should be taken as evidence that this statement was not meaningful after all.

3.4. Objects springing into being in response to our probing

As we have seen, the accounts offered by Dummett and Prawitz are challenged by the semantic horn of Benacerraf's dilemma to explain how the practice of discerning the valid derivations in systems of calculation could elicit us to claim truths in the objective sense. Benacerraf's semantic condition does not ask for which criteria must be met in order to warrant a mathematical statement, it asks for which criteria must be met in order to guarantee that these statements are semantically satisfied by a self-subsisting mind-independent structure. For Benacerraf, if we are not to beg the question, an account of truth must have something to do with reality in the physical sense of *res extensa*. As he puts

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[44] Gentzen introduced a form of Natural Deduction System, a class of proof system based solely on inference and inference rules. One of the pivotal criteria for such a system is to characterize a set of primitive rules for introduction and elimination of logical constants.

[45] I am making use of Prawitz's own choice of symbolism to draw the point here. For the symbolism, *See* "Logical consequence from a constructivist point of view", in: *The Oxford Handbook of Philosophy of Mathematics and Logic* (Oxford University Press, 2005), p. 693.

it, if we stick to our "customary grammatical forms", then the truth conditions of a declarative sentence can only be specified by analysing its references, within the form of an epistemic agent stating something about certain objects. Or, in order words, Benacerraf's point is that if the meaning of a statement does not depend on referring to existing objects, then there can be no truth to it.

As so, the main problem faced by the combinatorialist view is that their conception of mathematical truth and meaning is given by virtue of intra-systemic features that do not need to correspond to anything external. Here truth gives way for consistency as the justification for asserting the existence of an object. But in this way, Benacerraf says, "they avoid what seems to me to be the necessary route to an account of truth: through the subject matter of the propositions whose truth is being defined" (MT, 678), and:

> The account should imply truth conditions for mathematical propositions that are evidently conditions of their truth (and not *simply*, say, of their theoremhood in some formal system). This is not to *deny* that being a theorem of some system can be a truth condition for a given proposition or class of propositions. It is rather to require that any theory that proffers theoremhood as a condition of truth also *explain the connection between truth and theoremhood.* (MT, 666).

That is, we should not assume that theoremhood implies truth; proving a statement in a formal system is not yet establishing "the truth of what it says". This premise closes the door on constructivist accounts, for their accounts of mathematical meaning and truth are based precisely on considerations of proof and evidence, of mathematical knowledge as a proof activity that is therefore limited to our finitary methods. As so, combinatorialism rejects the unrestricted notion of arbitrary function that runs free in the platonist conception. This sort of account starts with the denial that the truth-value of mathematical statements is directly determined by the satisfaction of their truth-conditions. Truth and meaning are still related, though *only* through the intermediation of an effective procedure of verification.

This is the root problem with this view, for if truth only obtains once we are in condition to acknowledge it, then one may conclude that the mathematical facts are, at least partially, creations of our own minds. By relieving the requirement for reference to actual existing things, combinatorialism endangers emptying away the content of mathematical statements, making mathematical truth akin to logical validity, as if a true-value were just a sign that a formal derivation from a set of axioms is valid. The meaning of a mathematical expression is then taken to be encapsulated by the calculus where it figures, and the calculus, in its turn, defined by a choice of axioms.

This would close explanations of mathematical necessity in terms of its truth, as it would block the consideration of axioms as being self-evident. Axioms would be considered arbitrary starting points, chosen specifically for their usefulness in the derivation of the desired theorems. In this way, our view of mathematical knowledge would boil down to knowing how to perform formal derivations, like the study of the possibilities of play in a game. It is difficult to accept the products of such a practice as true knowledge, for the activity looks like an exercise in aesthetical rigour. How could the game-like activity of creating rule-governed systems of calculation turn out to be *essential* for our understanding of physical systems despite having no constraints nor requirements of empirical evidence? How could this account explain the compelling power that only a proper canonical mathematical proof has, especially when the necessity of the logical inferences is taken to be grounded on implicit definitions and not self-evident truths?

We may reconstruct Benacerraf's challenge to the constructivists in this manner: if we accept the premise that truth-aptness is provability, then we lose the sense in which one says that what is true holds regardless of our believing in it, breaking down the distinction between theoremhood and truth. Unless there is a way to directly assess and establish the truth of the axioms, then this view is not going to overcome Benacerraf's dilemma. Platonism does not face these difficulties in satisfying Benacerraf's semantic condition because it conceives axioms as self-evident truths that are intuitively given. Platonism sees every true statement as ultimately grounded by the ontology it represents and whose existence predates our methods of knowing about it.

Perhaps somewhat inspired by what seems correct in platonism — the idea that we do not get to arbitrarily choose how mathematics is developed — Dummett proposed that constructivism had to make a "minimal concession to realism" for us to reach a middle-ground:

> It seems that we ought to interpose between the platonist and the constructivist picture an intermediate picture, say of objects springing into

being in response to our probing. We do not make the objects but must accept them as we find them (this corresponds to the proof imposing itself on us); but they were not already there for our statements to be true or false of before we carried out the investigations which brought them into being.[46]

Dummett suggests that mathematical objects are products of our inquiry after mathematical truth; not created arbitrarily, with no consideration for our practical application of methods of reasoning, but neither were they discovered as natural structures pre-existing mathematical activity. In the same vein, his verificationist semantics does not entail that abstract mathematical structures *actually* exist, but it does entail that such structures exist in the abstract potential of a canonical method of construction. If it is possible to provide an effective procedure to construct a mathematical object, then it must exist in an abstract way, even in case the construction is not achievable given our current computational capacity. It is as if the mathematical object was first encountered as an undefined shadow, waiting to be probed into the light and get its existence properly recognised. The mathematician would be thus an inventor of abstract forms, constructing effective procedures of computation endowed with abstract potentials which guarantee the objectivity of the results.

Although Dummett portrays this connection as indirect, he is still affirming that human investigations are responsible for bringing about the supposedly independent and abstract mathematical objects into being. He is inadvertently mixing up claims that make the platonist picture insoluble with the constructivist picture — one cannot coherently maintain that the objects are brought to being by human investigation and that such objects exist independently of humans. This is not an attempt at clarifying the nature of mathematics, but an attempt at ending a disagreement by having it both ways. So despite sounding like he is offering a way out of the dilemma, Dummett's way undertakes the very presuppositions that create the dilemma.

[46] Dummett, *Truth and Other Enigmas* (Harvard University Press, 1978), p. 185. Prawitz quote this passage as well in "Truth and objectivity from a verificationist point of view", p. 49.

At first glance, one may think that the core issue in Benacerraf's dilemma is that of reconciling a referential account of mathematical truth with a causal account of mathematical knowledge.[47] Yet on close examination, Benacerraf cleaves platonism and combinatorialism along the lines of what they say about the correctness of mathematical statements — whether these are based on reference to existing abstract structures, or on the possibility of verification via a proof method. So either characterized as describing relations between sets of objects, or as making manifest the conditions in which what is described would be effectively verified, *the essential controversy* regards the characterization of the truth-makers of our mathematical statements. The forking of these paths is caused by a theoretical incompatibility regarding whether truth is an epistemically unconstrained property of sentences, or whether truth is the desired result of an epistemic effective process of obtainment. These views are at odds with each other because they are motivated by the incompatible semantic consequences of their epistemological or metaphysical principles.

Yet, in order to set up this antagonism, one must undertake Benacerraf's presuppositions. To remind the reader, the semantic presupposition (SP) is that there is a *face value* semantic equivalency between mathematical and empirical statements, and the epistemological presupposition (EP) is that mathematical knowledge *must* be explained in causal terms in order to be intelligible. Taken together, SP + EP ask that statements of mathematical knowledge be explained in terms of semantic satisfaction (e.g. via reference) to systems and process that are caused by natural phenomena.[48] But why should mathematical knowledge be considered illegitimate unless it *referred* to *causal* processes? These presuppositions can only be maintained if one already expects all instances of knowledge to be true justified beliefs in which the truth component is fulfilled by reference to a feature or item caught up in a natural order of causation.

If we are going to participate in a debate that is set from the start to have discovery or invention as mutually exclusive conceptions of how we acquire mathematical

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[47] Philip Kitcher, for instance, interprets Benacerraf's dilemma as entailing that "Given that we know some mathematics, it follows that either our best theory of mathematical truth (Platonism) or our best theory of knowledge (a causal theory of knowledge) is mistaken." (*The Nature of Mathematical Knowledge*, Oxford University Press, 1984, pg. 103).

[48] In Benacerraf's words: "I favor a causal account of knowledge on which for *X* to know that ***S*** is true requires some causal relation to obtain between *X* and the referents of the names, predicates, and quantifiers of *S*. I believe in addition in a causal theory of *reference*, thus making the link to my saying knowingly that *S doubly* causal." (MT, 671).

knowledge, then we should at the very least re-evaluate the clarity of those metaphors. Do they help us understand the phenomenon in question? If no, then why do we insist in such dichotomy? That such confusions sprung out of Dummett's attempted solution may be taken as a strong indication that perhaps in order to find the third option we should step back and try to understand the spirit in which one says that mathematics is either invented or discovered, then give philosophical treatment to the worldviews involved.

Inspired by Wittgenstein, Juliet Floyd once made a remark which I would like to take as a point of departure to the treatment of the invented-or-discovered dichotomy:

> If we insist on speaking of *discovery* in mathematics, Wittgenstein suggests that we ought to allow ourselves to think in terms of an analogy with technological discoveries, which are perhaps better conceived of as *inventions*, like the steam engine or the wheel or the decimal notation or the computer.[49]

One could frame the phenomenon of electricity in a similar dichotomous way, for although humanity have witnessed electromagnetic phenomena before devising a theoretical explanation for their occurrence, electricity as something generated in power plants and channelled to our homes only exists because humans once probed the universe in a specific way and discovered how one of its mechanisms works; so we could say, as Dummett did for mathematics, that electricity did not *exist* or had its properties *determined* "for our statements to be true or false of before we carried out the investigations which brought them into being".

My point is that the question of invented-or-discovered is but an apparent conundrum. Mathematical language is a human creation, yet the infinitely many implications of the systems of calculation we have set up with these concepts are discovered. There is both creation and discovery in mathematics, so there is nothing to be gained by insisting that the mathematician is either an artist or a scientist.

Dummett's attempt at a metaphysical middle ground perpetuates the form of this unprofitable dichotomy. Going back to Juliet Floyd's commentary, if we must choose one horn, perhaps the best analogy for the mathematician is that of the inventor-*cum*-discoverer:

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[49] Floyd, J. "Wittgenstein's Philosophy of Logic and Mathematics", in *The Oxford Handbook of Philosophy of Mathematics and Logic* (Oxford University Press, 2005), p. 112.

> […] the mathematician is an inventor, not in the sense of making up truth willy-nilly as he or she goes along, as a pure conventionalist would suppose, but in the sense of engaging in the activities of fashioning proofs, diagrams, notations, routines, or algorithms that allow us to *see* and *accept* (understand, apply) results as answering to what does and does not make sense to us. We 'make' mathematics in the sense that we make history: as actors within it.[50]

I entirely agree with Floyd here — if there is any sense to be rescued from this dichotomy, it is that mathematics is a human creation which enables us to understand and apply its results to explain and better deal with our world. There is discovery in mathematics, as mathematicians do not get to choose how we evolved our cognitive faculties nor the consequences of the formal systems based on our mathematical practices. Yet 'discovery' here is in no way analogous to scientific contexts, as there is no unknown element external to our culture being discovered. In mathematics, we discover unforeseen consequences of the systems of calculation that we set up according to principles of reasoning.

Thus, while I agree with the general direction of Benacerraf's conditions for semantic consistency and epistemic intelligibility, I overtly reject his presuppositions. That is, albeit I agree we should aim for a global and epistemologically intelligible treatment for the concepts of truth and meaning (in SC and EC), he is wrong in presupposing that only a representational account of meaningfulness could offer that, or in presupposing that only a causal explanation of what is expressed in a truth claim could be intelligibly called knowledge.

Floyd's remark does not provide a definite answer to the question regarding what is the subject matter of mathematical inquiry, but it does tell us where to look for one — and it is not in postulations motivated by abstraction principles, but in an examination of our use of mathematical concepts. In my view, this should be carried out as a philosophical investigation of which practices and abilities one must know-how to perform in order to be able to use and understand mathematical vocabulary in a meaningful and effective way.

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[50] Floyd, J. "Wittgenstein's Philosophy of Logic and Mathematics", in *The Oxford Handbook of Philosophy of Mathematics and Logic*, pp. 112-113.

As so, I reject both horns. To choose one means to accept the view that mathematical knowledge is either of abstract facts or of abstract possibilities of proof, yet neither are clear pictures of mathematical activity. This polarization of views regarding mathematical truth deserves a philosophical treatment, in Wittgenstein's style, a therapy of the picture that bewitched philosophers and does not let us think outside of its borders.

4. The metasemantic frame of the dilemma

In the previous chapters, I have shown what distinguishes the two traditions caught up in the horns of Benacerraf's dilemma. The main fissure between platonism and combinatorialism was located in their incompatible conceptions of truth: whether it should be regarded as an epistemically unconstrained property of well-formed propositions, or the desired result of an epistemic effective process of obtainment.

Now I want to show how these two sides actually share the same view regarding the nature of meaningfulness. This chapter is thus entirely focused on metasemantics, in characterizing and criticising the foundational theory of meaning presupposed in Benacerraf's conditions, the premises of his dilemma.

A metasemantics is an account of what linguistic meaning is in itself, of language's function in communication. This is a theory about the nature of meaning in general, as opposed to a theoretical analysis of the meaning of particular linguistic expressions,[1] which semantic theories such as verificationism provide. Thus, every semantic analysis presupposes a metasemantic thesis, as to explain the meaning of any particular sentence, one must employ some notion of how language works.

Now, we already know that platonism and combinatoralism are motivated by the application of, respectively, a referential or a verificationist semantics, resulting in some disagreements about the appropriate semantic analysis of mathematical statements. Yet, when we compare the resulting pictures of mathematical language, we realize that neither platonism or combinatorialism would concede that our use of mathematical terms may not designate self-subsisting objects.

In my examination, this appeal to an underlying objectivity is a residue of an unexamined metasemantic assumption. But in order to bring the connection forward, we must compare the extensional and the intensional accounts of mathematical objects, compare how platonists and combinatorialists conceive mathematical meaning. A good starting point is how meanings are fixed and truth-values determined, since both philosophies would consider that this statement:

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[1] A remarkable employment of this distinction forms the basis of Alexis Burgess' *Metasemantics: New Essays on the Foundations of Meaning* (Oxford University Press, 2014).

$$\text{“The quadrillionth digit in the expansion of } \sqrt{2} \text{ is 8”}$$

is determinately true or false, regardless of whether we have executed the computations required to check it. Its truth does not depend in any way on the cognitive capacities of a computing agent. What was said there is determined by no one, at no particular time or place, and independently of the practical execution of an algorithm. Remember that verificationists say that the abstract possibility of finding an effective procedure to verify a statement already determines its meaning. The mere possibility of an algorithm is enough to determine a truth-value. Hence the abstraction of the subject matter.

To be more specific, based on the referential theory, platonism reads the result as a completed and ordered infinite set, something we can refer to as:

$$\{(0, 4), (1, 1), (2, 4), (3, 2), (4, 1), \ldots\}$$

The object thus represented is considered to be self-subsistent and given as a completed whole in extension. So how could this compare to a combinatorialist reading which does not accept the postulation of a completed infinite extension?

By virtue of its endorsement of semantic verificationism, combinatorialism conceives the result as abstractly pre-determined by a self-subsistent effective procedure which can effectively compute the operation. So combinatorialism would also consider that the decimal expansion of $\sqrt{2}$ is abstractly pre-determined. Their reading differs from the platonistic interpretation in what concerns the nature of this object, and how we come to know about it, but it does not dispute the point that the subject matter of mathematical discourse is objective and abstract.

The extensional reading favoured by platonists is put aside in favour of an intentional reading. As so, the decimal expansion of an irrational number is taken to be given intentionally,[2] on condition of the application of an algorithmic procedure. The object in question is an intensional entity,[3] describable as such:

$$[\lambda x : \mathbb{N}.(\sqrt{2})_{(x)}]$$

This intensional object could not be given in a clear-cut semantic relation, it can only be uncovered by following the correct steps that culminate in the representation of the intended object; it can only be constructed by a mind following the correct instructions. The verificationist view motivates the idea that the objects of mathematical discourse are the algorithmic procedures we execute. As Dummett explains: "to grasp an infinite structure is to grasp the process which generates it, to refer to such a structure is to refer to that process, and to recognize the structure as being infinite is to recognize that the process will not terminate." [4] These objects

[2] In Dummett's words: "The description by means of which a mathematical object is given must always be such as to enable it to be distinguished from other objects of the same kind. However, since mathematical objects are mental constructions, and the mental construction is expressed by means of the description in terms of which the object is given, the objects of intuitionistic mathematics must, in general, be considered as intensional objects; that is to say, that criterion of identity which is given together with the manner in which the object is presented relates to the identity of the description. Thus, for example, if an effectively calculable function is thought of as given by means of a rule of computation, different rules will determine intensionally distinct functions, even if these functions are extensionally equivalent." (Dummett, *Elements of Intuitionism,* Oxford: Claredon, 2nd ed., 2000, p. 16-17).

[3] An intensional entity is one represented by a concept or proposition that fails the principle of extensionality, so these can have the same extension but mean different things. The classical example is Frege's discussion of the concepts 'morning star' and 'evening start' (or alternatively, Quine's discussion of 'creatures with a kidney' and 'creatures with a heart'), both of which can be understood as intensional entities because despite naming the same object, these are different concepts. The intensional element is to be contrasted with the extensional, which in this case is the set of members with the property of being the 'morning star' or being the 'evening start'. Since both sets are satisfied only by Venus, we conclude by saying they are identical in extension but differ in intension.

[4] Dummett, M. *Elements of Intuitionism* (Oxford: Claredon, 2nd ed., 2000), p. 56.

have, in Prawitz words, a "tenseless" or "potential existence".[5] They exist so long there is an algorithm to construct it — the effective procedure of verification which Dummett considers capable of "probing" those potential entities so they "spring into existence."[6]

The idea is that a linguistic expression has a meaning insofar as there is a possibility to specify an effective procedure to verify it. Yet, verificationists know that this cannot be the whole story in understanding what someone else means with words. This is because they accept that, even when meaning is obscure because we do not yet know an effective procedure to verify it (as is the case with conjectures), the expression is still partially understandable, we have a sense of 'where it goes'. After all, in order to know which verification procedure to apply, one needs to know what one will be looking for.

In other words, we can specify such procedures only because we know where they are heading; and since this view conceives mathematical objects as mental entities, then this can only mean that when we make claims about something, we must have a mental representation of the intended object. We must use this representation to produce step-by-step instructions (i.e. algorithms) for others to construct the same mental object. Thus we may say that in a verificationist conception, algorithms and programs must contain a representation of all their computational steps and concluding clauses, in a potential sense, somewhat like a *blueprint*.

As we have seen in 3.3, Prawitz is fully aware that verificationism requires a metaphysically-pregnant conception of possibilities; what is possible must be understood as a way of existing just as legitimate as what is actual. So Prawitz claims that verificationists must assume that there is a self-subsistent culture-independent constructive procedure for every mathematical structure, a verification procedure for every assertible statement. These are primitive unexplained explainers for the

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5 To quote Prawitz again: "[…] that a sentence is provable is here to mean simply that there is a proof of it. It is not required that we have actually constructed the proof or that we have a method for constructing it, only that there exists a proof in an abstract, tenseless sense; we may call it *potential existence* to use a term proposed in this connection by Martin-Löf." ("Comments on the Papers", in *Theoria — Swedish Journal of Philosophy*, Vol 64 (2-3), 1998, p. 285.)

6 Dummett, M. *Truth and Other Enigmas* (Harvard University Press, 1978), p. 185.

verificationist. Thus, speakers must have some grasp of what constitutes evidence for a claim before assessing which procedure is effective at verifying it, otherwise, they would not know even where to begin such verification.

Through verificationism, the objectivity of mathematics would be based on the potential that these procedures have to reach their concluding clause. Mathematics would thus become the study of abstract potentials — not so distant from the platonistic perspective which sees mathematics as the study of abstract structures.

The point of this comparison is that, even though verificationism and truth-conditional referentialism provide us with distinct analyses of meaning, both agree on what words do: represent other words or non-linguistic objects. While referentialism conceives representational purport as a direct word-to-world relation, verificationism conceives it as an indirect relation mediated by a method. One takes the extensional, the other the intensional path, still both meet at the end, understanding linguistic meaning in terms of what in the world is *represented* by our expressions.

One view assumes self-subsistent objects in extension, the other assumes them as pure intentionality (i.e. mental constructions). The first motivates belief in the existence of a represented abstract realm, the second motivates belief that our techniques cast incredible shadows that reach way beyond our cognition ever did. Yet, along the way, they conceive the meaning of the expression as inexplicably transcending its use.

A similar critical comparison was made by Pasqualle Frascolla, who criticises this metasemantics for promoting misunderstandings about the normativity of meaning:

> The mentalist view of meaning as process capable of performing all the steps before they have been made and the platonist view of an ideal world of necessary connections that are pre-existent to our effective acknowledgement are, respectively, the variant 'towards the inner' and 'towards the outer' of one and the same misunderstanding on the nature of grammar rules: that an independent reality corresponds to these

> norms, to these conceptual connections, and that our acknowledgement of their existence is justified inasmuch as it is able to mirror that reality.[7]

To explain how the correctness of a mathematical operation could be fixed ahead of its practical computation — thus justifying the necessity of the result — platonism and combinatorialism recur to distinct poles of the same tactic. That is, the verificationist move of introducing potentials as abstract warrants of assertibility is an equivalent move to the platonist introduction of extensions as meaning-anchors, abstract standards of correction. Both strategies accept that the logical structure of mathematical operations determines the extravagant existence of intensional or extensional abstract entities.

In conclusion, the premise that licenses the claim that mathematical discourse regards abstract objects or possibilities is the representationalist metasemantic thesis, as it compels us to look for non-linguistic represented objects in order to explain our linguistic behaviour.

4.1. Representationalism

In a nutshell, represenationalism is the philosophical view that language's essential function is to represent states of affairs or possible worlds, thus it claims that linguistic concepts either stand for sets of non-linguistic objects (internal or external to the mind) or their properties (intrinsic or extrinsic). Meaning is explained in terms of what our expression purport to represent.

This view has been prevalent in philosophical accounts of linguistic meaning ever since Descartes drew the distinction between mind and extension; it can be found in the philosophies of Kant, Schopenhauer, Russell, Tarski, and Fodor, and also in Wittgenstein's *Tractatus*, just to name a few. Brandom calls it the "master concept of Enlightenment epistemology and semantics", as it "reigns not only in the whole spectrum of analytically pursued semantics, from model-theoretic, through possible worlds, directly counterfactual, and informational approaches to

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[7] Pasquale Frascolla, *Wittgenstein's Philosophy of Mathematics* (London: Routledge, 1994), p. 119.

teleosemantic ones, but also in structuralism inheriting the broad outlines of Saussure's semantics."[8] What these have in common is this philosophical outlook of semantics:

> Representationalism […] is motivated by a designational paradigm: the relation of a name to its bearer. In one standard way of pursuing this direction of explanation, one must then introduce a special ontological category of states of affairs, thought of as being represented by declarative sentences in something like the same way that objects are represented by singular terms.[9]

This is not the weak thesis that only empirical-descriptive vocabularies play a representational role in social reasoning, or that some linguistic models invoke a level of semantic representation.[10] This is the hard thesis that the normal function of vocabularies is to perform a representative role in social reasoning (except for connectives which evidently serve to connect the representative bits).

We may say that language represents the world to the extent that *we believe* that statements '*F(a)*' are true if and only if there is an object denoted by '*a*' with the property denoted by *F*. That is, in linguistic communication, one combines sub-sentential elements to compose a representation for our interlocutors. Our statement can produce the intended representations because they can be articulated in accordance with how the world is articulated (even our use of connectives could be interpreted as an attempt to track the spatiotemporal causal relations between the objects denoted by singular terms). We try to mimic the articulations of the state of

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[8] Brandom, R. *Articulating Reasons* (Harvard University Press, 2001), p. 7; p. 9-10.

[9] *Ibidem*, p. 14.

[10] Metasemantic representationalism should not be confused with representationalist stances in linguistics or cognitive science. A metasemantic thesis is not a scientific theory, but the core claim of a philosophical view of language. While the latter finds utility working with a concept of internal representation to explain how the brain processes information about its surroundings, the former conceives linguistic meaning in terms of representations in order to understand how language works.

affairs with articulations of singular terms and predicates; as a form of mapping of the world. The manifoldness of language would serve to track, or to model, the manifoldness of things in non-linguistic non-logical spaces, be it the external world, or the 'internal world' of the mind.[11] As Alexis Burgess explains:

> [It] seems manifest that declarative utterances of sentences of our language—even those involving vague language—do represent the world as being some way and that they do so in virtue of the representational properties of their sub-sentential constituents. Given a declarative utterance, such as 'John is tall', there are certain ways that things could be that are clearly sufficient for the truth of the utterance, and certain ways that things could be that are clearly incompatible with its truth. Moreover, the representational properties of, for example, 'John is tall', 'Sarah is tall', 'John is clever' and 'Sarah is clever', seem to be systematically related in a manner that is best explained by holding that these sentential representational properties are determined by the representational properties of the constituents 'John', 'Sarah', 'tall', 'clever'. Somehow, then, it seems that our use of language must serve to determine representational contents for the lexical items, phrases and declarative sentences of our language.[12]

That is, the semantic relation that obtains between sub-sentential items and extra-linguistic objects or features is fundamental and primitive. The semantic relation between a concept and what it represents is the simplest component of linguistic meaning, there is no more fundamental explanation of what a sentence

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[11] For more on this topic, *see* Michael P. Lynch, *Truth as One and Many* (Oxford University Press, 2009), "Correspondence and Representation", pp. 21-32.

[12] Alexis Burgess, *Metasemantics: New Essays on the Foundations of Meaning* (Oxford University Press, 2014), p. 144-145.

means than breaking it down to representative bits, a designation of which objects and properties its individual components represent.

> We use language to tell our community and our later selves how things are. Telling how things are requires representational devices, structures that somehow effect a partition in the possibilities. For we say how things are by saying what is ruled in and what is ruled out.[13]

In order to assert a meaningful sentence, we would have to intentionally impose conditions or properties upon the world. Concepts are here understood as epistemic intermediaries necessary for thought to fit the world; forms to be contrasted with the unconceptualized matter that provides rough content to our judgements. Hence why Jackson considers a set of directions to a location as an example of language representing a reality:

> Although it is obvious that much of language is representational, it is occasionally denied. I have attended conference papers attacking the representational view of language given by speakers who have in their pockets pieces of paper with writing on them that tell them where the conference dinner is and when the taxis leave for the airport. How could this happen? I surmise that it is through conflating the obviously correct view that much of language is representational with various controversial views.[14]

Otherwise, if language did not represent, if there was no correspondence possible, one would not be able to encode the correct information about one's conference dinner. It is not the pragmatic aspects in particular instances of use that determine meaning, but the syntactical composition of our sentences that traces the manifold of real state of affairs. Again with Stanley:

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[13] Jackson, F. *From metaphysics to ethics: A defence of conceptual analysis* (Oxford: Clarendon Press, 1998), p. 53.

[14] Jackson, F. "Naturalism and the Fate of the M-Worlds II" (*Proceedings of the Aristotelian Society* Supplementary Volume 71, 1997), p. 270.

> On the simple explanation of the source of our intuitions about the truth-conditions of utterances of sentences we understand, it is due primarily to a compositional process of interpretation. Our knowledge of meaning, together with our knowledge of relevant contextual facts, allows us to assign meanings to the parts of a sentence, and the intuitive truth-conditions of an utterance of that sentence are what results from combining these values.[15]

However, when the minimal unit of significance determining truth-conditions is conceived as intrinsic semantic relations of representation, then an explanation of our conceptual acquisition has to start with humans intuitively grasping these freestanding language-to-world relations, prior to ever engaging in communication. Therefore, speakers would have to possess some *tacit knowledge* of the meanings of concept-words in order to use them correctly. An intuitive faculty for 'grasping' the relation between concept and the extension it represents. As such, concept formation is taken to be something innate to one's consciousness, instead of learned from one's culture; concept-words are understood as translations from pure thought to one's native language.

This does not mean that when all speakers claim to know what a concept represents, they would have a fool-proof criterion for the correct use of that word, as there is still room for ambiguity. Insofar it is a metasemantic thesis, representationalism cannot tell us how to specify the correct application of an expression. What it tells us is that a word such as 'cat' gets a meaning because it *stands for* those felines in the logical space of our linguistic communication. It does not tell us anything useful about which particular cat is mentioned in any sentence. For that,

[15] Jason Stanley, "Semantic in Context", in: Gerhard Preyer & Georg Peter (eds.), *Contextualism in Philosophy: Knowledge, Meaning, and Truth* (Oxford University Press, 2005), p. 221.

one needs context and an analysis of meaning, which may be given in terms of reference, or verification procedures, or inferential connections, etc.

In the remainder of the present chapter, I will be taking aim at this metasemantics. The argument I present is not novel, is a montage made out of an array of pragmatic critiques[16] of the foundational theory of meaning as based on the notion of representation. What is novel about this thesis is turning the resulting montage into a way out of Benacerraf's horns, demonstrating we have an alternative capable of satisfying Benacerraf's conditions and escape their dilemmatic consequences at the same time.

4.2. Rejecting Benacerraf's semantic presupposition

So far I have proposed that referentialism and verificationism are cousins since both theories work from a representationalist viewpoint. Yet, there is more to be said about how the representationalist thesis ended up at the core of Benacerraf's dilemma. We know Benacerraf expected that a philosophical account of mathematical truth would have to fit with a referential theory of meaning. This is because, as he argued, mathematical statements have a logical form similar to empirical statements. So it seems quite crucial to me that one should examine how an apparent similarity of logical form could justify an appeal to global referentialism. Are such comparisons enough reason to suppose that all areas of discourse stand in need of a universally homogeneous referential theory of meaning? Could it not be the case that face value appearances are misleading and the meaning of these statements will only be properly understood once we examine the pragmatic features of their utterances?

Benacerraf's departing point seems to be that an analysis of logical form shows that all declarative sentences — mathematical or empirical alike — are usually composed of the regular suspects: quantifiers, singular terms, and predicates. This

[16] Exposed not only by the later Wittgenstein, as this thesis made evident, as similar arguments were also voiced by the first American pragmatists, James and Dewey; the hermeneutical philosophers Heidegger and Gadamer; and the modern pragmatists Sellars, Rorty, Brandom, Price, and Blackburn. This thesis is indebted to them.

universal *compositionality* of language indicates that a referential truth-theoretical approach can be worked out for all Language, including, of course, the mathematical. As so, taken at *face value*, every declarative sentence should be read as a composition of predicates quantifying over objects of a certain domain named by the singular terms, and what is predicated of these objects constitutes either a true or false representation of a state of affairs (or of a possible world).

It was the interest to understand, classify and determine the features and structure of good argumentation what motivated philosophers to conceive logical forms. We, *qua* logicians, are concerned with specifying the form of valid arguments to better discern them from invalid ones. And, to define the form of valid arguments, we have to analyse the function of each semantic component, the logical role of the parts at the sub-sentential level. By comparing syntactically similar sentences, we can see the threads of shared logical features in the reappearing compositions. So, the determination of logical forms is a necessary step in defining the logical role of sub-sentential components of language.

However, I think Benacerraf reads too much out of a superficial similarity of logical form. Just because '17' figures in a syntactically equivalent position to the proper name 'New York' it is not enough reason to conclude both terms serve the same function in communication. We should not assume that because a word can play the logical role of a singular term, it would automatically play the expressive function of referring to an extra-linguistic item.

The problem with balancing too much on an analysis of logical form is that one may end up attributing to words performances that they could not in principle expose. Singular terms do not refer by themselves; speakers use their *know-how* to refer by using words.[17] Reference is done by agents when these deploy terms in a manner that plays the expressive role of making explicit the subject of discussion. If we are not fully aware of that, we may be misled into thinking words could perform expressive functions all by themselves, without the intentionality of a speaker behind it.

Furthermore, the translation of sentences to logical vocabulary supposedly serves the purpose of shedding some clarity unto the proposition that is muddled by the ambiguity of its ordinary expressions, yet the resulting formal expression does not give us any deeper insight into what follows from the original sentence, what justifies it, what counts as evidence for it, what other consistent interpretations it may have, and so on. But all of these should be knowable through inferences from the meaning of the sentence. So, clearly, finding the logical form of a sentence cannot help in the analysis of its meaning.

The kind of pragmatism I endorse and associate with Wittgenstein understands as *insufficient* the attempt to specify the meaning of a sentence by paraphrasing it in logical vocabulary. When we give centre stage to relations between use and meaning, instead of formal-structural relations between natural and formal vocabularies, we will see a variety of uses in which we employ declarative sentences to perform various functions beyond the descriptive.

We may call the functions for which expressions can be used for *expressive roles* in social reasoning. By 'social reasoning' I mean a kind of exchange involving more than one interlocutor giving or asking for reasons for each other's claims or actions. When the reasoning is social, our interlocutors have a pivotal role in the premises we consider and the conclusions we draw. The elementary example is that of a facial

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[17] As Strawson argues: "'Mentioning', or 'referring', is not something an expression does; it is something that someone can use an expression to do. Mentioning, or referring to, something is a characteristic of a *use* of an expression, just as 'being about' something, and truth-or-falsity, are characteristics of *a use* of a sentence." (Strawson, P.F. "On Referring", in *Mind*, New Series, vol. 59, nº 235, 1950, p. 326). A point that was also developed by Brandom: "In order to use an expression as a name, to refer to or pick out an object with it, one must be able to use the name to *say* something (paradigmatically, to assert something) about the object referred to, indicated, or named. The significance of taking or treating something as a name, as purporting to refer to an object, consists in how one takes it to be proper to use the expression, and the use of expressions as names is unintelligible except in the context of using expressions containing them as sentences" (Brandom, R. *Making it Explicit*, Harvard University Press, p. 82).

expression functioning as a way of letting others know of one's pain or joy. In the same way, a hand gesture may serve to express approval, and a predication may describe what is the case.

On these terms, the debate I am setting up regards the functional roles of our *linguistic* expressions, and representationalism seem to be giving us a reductive answer: that language use only serves one expressive role. Vocabulary items can be divided into those that purport to represent (singular terms and predicates) and those that connect the representing bits (logical connectives).

Yet, looking in terms of expressive roles, could it not be the case that linguistic expressions play other roles besides representing? As Wilfrid Sellars notes:

> Once the tautology 'The world is described by descriptive concepts' is freed from the idea that the business of all non-logical concepts is to describe, the way is clear to an *ungrudging* recognition that many expressions which empiricists have relegated to second-class citizenship in discourse are not *inferior*, just *different*.[18]

To think the use of all non-logical concepts as conveying thoughts that purport to represent the world is too narrowly reductive. Some declarative sentences may be similar on the syntactical surface, yet a close examination may very well reveal they are quite dissimilar at the practical core; the practices from which emerged these ways of talking, and the contexts which warrant the use of those expressions, can be demonstrably different.

An instance of distinction of expressive roles suggested by Sellars comes from the realization that some of our concepts play an *explicative role* in respect to description, thus serving to make explicit features of the framework which makes

[18] Sellars, W. "Counterfactuals, Dispositions, and the Causal Modalities", in *Minnesota Studies in the Philosophy of Science*, *Volume II: Concepts, Theories, and the Mind-Body Problem* (ed. Herbert Feigl, Michael Scriven, and Grover Maxwell (Minneapolis: University of Minnesota Press, 1958, pp. 225-308), §79, p. 282.

empirical description possible.[19] These may be called explanatory-metalinguistic uses whereas agents seek to express the implicit norms of correctness and the features of discursive practices. Implicit norms guiding our linguistic behaviour are instituted by social-normative practices where communication is involved, which are not limited to properly linguistic expressions, but extended to include other performances, such as those of tone and pragmatic force, and non-linguistic forms such as facial expressions and "body language." The expressive role these expressions play *must* be a non-descriptive one, for they are necessary to make description (including scientific explanation) possible and intelligible.

We can extend Sellars' point to also cover modal, intentional, metasemantic, and ontological-categorical vocabularies, which share this broad metalinguistic role that should not be confused with the descriptive role of ordinary empirical vocabulary. As so, we have at least two clearly distinct expressive roles, the descriptive and the metalinguistic-explanatory. Nevertheless, since we never stop coming up with new ways of doing things with words, it seems we could have many more.

[19] In Sellars own words: "It is only because the expressions in terms of which we describe objects, even such basic expressions as words for perceptible characteristics of molar objects, locate these objects in a space of implications, that they describe at all, rather than merely label. The descriptive and explanatory resources of language advance hand in hand." (*apud* Brandom, R. *From Empiricism to Expressivism*, Harvard University Press, 2014, p. 40.)

For Wittgenstein, the function of words can be as diverse as the function of the tools in a toolbox (PI, §11).[20] A hammer and nails may be used for the same function as a screwdriver and screws, such as to attach wooden boards to each other. But if we have in mind all the diversity of tools — if we think also about the wrench, pliers, a carpenter's square (*norma*) or chisel — we realize there are a plethora of possible functions we could assign them, and this is the whole point of the analogy. It is not only the shape and mechanism of the tools what determines their function, but mainly how we put them to use.

The diversity of shapes and mechanisms of our tools increases our possibilities; they are multi-functional forms, just like declarative sentences, that can be used to convey commands, give advice, request, pronounce someone guilty, or married, etc. [21] In these uses, the speaker is not committing herself to descriptions of the world, but is trying to cause her interlocutor to do something, or to understand a situation under a different light, or announce a change in social status, etc. As with tools, it is not the form of the sentence what determines which expressive role it is playing — the form constrains its possible roles, for sure, but what determines one is the act of use, the *way* in which the speaker deploys it.

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[20] Charles Taylor also argues that Wittgenstein showed us that our linguistic expressions encompass a variety of pragmatic and expressive functions: "But Wittgenstein takes us well beyond this, because he sees that making judgments, cast in the form of sentences, is only one among many language-games. More accurately, there is a family of such games, which have in common that they put in play 'propositional contents', combinations of reference and predication, which can be used to make empirical claims ('Sam smokes'), to ask questions about how things are ('Does Sam smoke?'), and to give commands ('Sam, smoke!'). But lots of other things are going on in language. We also establish intimacy or distance; open contact and close it off; cry for, and give or withhold sympathy; disclose the beauty of the world, or the depths of our feelings, or the virtues of the good life, or the nature and demands of God or the gods, and so on. […] the disclosive power of our words in poetry is plainly something we often can't bring to light just by fixing clearly the reference and predication of the judgments we can identify (sometimes, indeed, the force depends on the very uncertainty attending these)." ("Language not mysterious?", in *Reading Brandom*, ed. Weiss, B. & Wanderer, J., Oxford University Press, 2010, p. 33.)

[21] Some possible examples of expressive roles are: descriptive, narrative, explanatory, categorical (categorizing an item in terms of species, genus, etc.), metalinguistic (using words to talk of words), regulative (expressing a rule), normative (manifesting a normative stance), amongst many others. I do not pretend to have an exhaustive list of the functions of deploying a vocabulary, nor am I after one, for what matters here is to establish the point that our use of language is open to a variety of functions, and that to determine what is the particular function of a speech act, we have to comprehend the practical purpose of deploying a vocabulary.

It is not because "I have a pen in my pocket" and "I have a pain in my heart" have similar logical forms that both 'pen' and 'pain' have the role of designating particular objects existing in a certain place. The examination of the syntactical-logical form of a saying cannot determine its expressive role, only the circumstances that warrant talk of pens or pains and the consequences that follow from such talk can determine which roles those terms are playing. It is certainly not the ability to form declarative sentences what determines the consequences of asserting one, such as the consequence that certain objects *must* exist if the sentence is true. The expressive role of a linguistic expression cannot be defined solely on basis of an analysis of logical form or any other purely syntactical criteria, as such analyses do not concern *how* those expressions are to be *used* by agents *in practice*. To know of such consequences, we need to look beyond the logical form of sentences and into the social context.

However, as a counter-argument, one could consider that discovering the logical form of "I have a pen in my pocket" and "I have a pain in my heart" as '$\exists x\ Px$' already *explains* what is the expressive role of each component in these sentences. We can maintain a homogenous referential semantic analysis and still distinguish the referred objects in terms of domains of discourse. So what is the problem in calling 'pains' and 'pens' objects of particular domains of discourse existing in certain places? So long we can distinguish the spaces in which one can find pains or pens as objects, everything should work fine.

One may start to see the problem by considering that pain is not something one *finds* about in the environment, it is something one attributes to oneself or another sentient being based on observations of behaviour and expressions (as someone could be suffering pain and still dissimulate it). As so, why take both those singular terms as having the function of picking out particular objects? It seems to me the only available reason is the one stated by Benacerraf: the desire for global semantic homogeneity under a paradigm that already works to explain scientific discourse. Yet, as soon as this paradigm is globalized and all concepts and singular terms are understood as representing extra-linguistic elements, we raise epistemological challenges to explain what is represented — and how one could access the represented — for non-descriptive, non-empirical, or non-measurable concepts, such as pain.[22]

The notion that we can derive conditions of correct use from assigned meanings presupposes that our uses of expressions are of the same general kind. As so, properties of use can be derived from the process of codifying the expressions used into a logical vocabulary. Yet the expressive roles of distinct vocabularies cannot be captured by such codification because practices of using vocabularies have a dynamic character — we frequently *project*[23] discursive practices to cover new domains, due to changes in the practitioners' lives, culture, environmental circumstances, amongst others relevant factors.

For instance, we have projected the practice of talking of 'having a pen in one's pocket' to the talk of 'having a pain in one's heart' as with practice of 'having gold in one's teeth' to 'having pain in one's teeth'.[24] I would like to say that the deployment of 'having… in' in each of these cases serves a different expressive role that is not picking out particular things one finds in certain places, as from a claim of 'having pain' I cannot infer someone possesses an object I can see and weight, such as with gold or a pen. The inferences I can make from the claim of having pain and the claim of having gold are wildly distinct, so we must be deploying the same 'having in' in different ways. It would be a mistake to disregard this dynamicity and homogenize the expressive roles of these deployments as roughly all the same.

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[22] Huw Price offers a similar defence of the multi-functionality of assertion: "Thinking of the function of assertions as uniformly representational misses important functional distinctions — distinctions we can't put back in just by appealing to differences in what is represented. To get the direction of explanation right, we need to begin with pragmatic differences, differences among the kinds of things that the assertions in question *do* (or more accurately, differences among the kinds of things that their underlying psychological states *do,* for complex creatures in a complex environment). And to get the unity right, we need to note that in their different ways, all of these tasks are tasks whose verbal expressions appropriately invoke the kind of multi-purpose tool that assertion in general is. To say this, we need to say what kind of tool it is — what general things we do with it that we couldn't do otherwise. If the answer is in part that we expose our commitments to criticism by our fellows, then the point will be that this may be a useful thing to do, for commitments with a range of different functional roles (none of them representation as such)." (*Naturalism without mirrors*, Oxford University Press, 2011, p. 223).

[23] This notion of projection of discursive practices will be fundamental for my explanation of the relation between mathematical formulae and uses of mathematical vocabulary in empirical statements in 6.4.

[24] The second example was coined by Brandom in *Between Saying and Doing* (Oxford University Press, 2008), p. 6.

If we are not careful in discerning what is mere appearance, we may confuse the function of a linguistic expression and misunderstand the whole point of talking in a certain way, as Wittgenstein explains:

> What kind of misunderstandings am I talking about? They arise from a tendency to assimilate to each other expressions which have very different functions in the language. We use the word 'number' in all sorts of different cases, guided by a certain analogy. We try to talk of very different things by means of the same schema. This is partly a matter of economy; and, like primitive peoples, we are much more inclined to say, 'All these things, though looking different, are really the same' than we are to say, 'All these things, though looking the same, are really different.' Hence I will have to stress the differences between things, where ordinarily the similarities are stressed, though this, too, can lead to misunderstandings. (LFM I, p. 15)

> We don't notice the enormous variety of all the everyday language-games, because the clothing of our language makes them all alike. (PI, §335).

The "common clothing" in the case of Benacerraf's semantic horn is, of course, the form of a declarative sentence, and the Wittgensteinian point to draw here is that such a similarity of form hides pragmatic differences in the processes according to which these expressions acquire meaning, because declarative sentences can be used for more than one expressive role. In this other passage, Wittgenstein writes precisely about the misunderstanding arising "from a tendency to assimilate to each other expressions which have very different functions in the language" that interests us here:

> The attempt to define the number 3 is like the attempt to define time. When we see that time cannot be defined as the movement of celestial bodies, we seek for another definition. Similarly for the king of chess. Since it cannot be defined as a piece of wood, we then ask, 'What is it?'

> The craving for a definition of number is also prompted by the fact that in saying mathematics treats of *numbers*, and therefore of *the number 3*, as contrasted with using numerals in such contexts as '3 apples', the italicized expressions are *substantives*. (WLC, p. 152)

The examples of time, numbers and the chess king indicate that even though some words may fulfil the logical role of singular terms, our use of them may not serve to represent any natural object or phenomena. Still, use of these terms is necessarily connected to normative architectures laid down by human practices — of treating a piece of wood as the chess king, using number-words to explicitly count, using 'time' to account for the movement of celestial bodies as it unfolds in a certain proportion to our heartbeats — is through participation in these practices that agents learn how to use those terms, and also how to project them into other language-games.

By narrowing down all available expressive roles into representation or logical connection, a global referential theory would not only ignore Wittgenstein's point about the range of expressive roles that our language is capable of, but more importantly, it *puts aside* the very important question of whether non-predicative or non-descriptive declarative sentences could be mistakenly taken as predicating or describing because of mere superficial similarities. Such strategy leads to metaphysical puzzlement regarding which entities would be represented by our non-descriptive, non-predicative, and/or non-empirical claims, as Brandom explains:

> This model inevitably leads to metaphysical extravagance. For there are lots of different kinds of sentences because there are many different ways of using sentences (things one can do with them). Pretty soon one must worry about logical facts and states of affairs (including negative and conditional ones), modal facts and states of affairs, probabilistic ones, normative ones, semantic and intentional ones, and so on, and corresponding kinds of properties to articulate each of them. One of the

> motivations for various local expressivisms is precisely to avoid such extravagance.[25]

Often, we mistakenly assume that the typical background practices involved in the use of a certain vocabulary would be similar to the application of other vocabulary, such as when we assimilate use of mathematical concepts with the use of empirical descriptive concepts. I see Wittgenstein trying to dispel this sort of metaphysical puzzlement regarding the nature of things such as pains, minds or numbers, by asking us to focus on the discursive practices in which these things are evoked. If we can explain how people teach and learn these practices, then there should be no puzzlement over what we use those vocabularies for, because vocabularies acquire meaning according to practices of use (more on this in 4.3). It follows that if we are able to communicate by deploying these vocabularies, then any puzzlement that may arise regarding which objects are represented by our discourse can only be the product of misunderstandings regarding the role of that vocabulary in our lives:

> The paradox disappears only if we make a radical break with the idea that language always functions in one way, always serves the same purpose: to convey thoughts — which may be about houses, pains, good and evil, or whatever. (PI, §304)

We are bound to fall into confusion if we conceive the relation between name and bearer as something that can be made universally explicit by a pre-established semantic structure, in abstraction from how such connections are traced in the everyday practices related to the vocabulary used. The act of referring cannot be mechanically programmed or syntactically fixed for all cases.

The attempt to construe this relation as an abstract freestanding one, independent of a context of utterance, presupposes that we can describe it from a privileged point of view, over and above the dimension of familiar language use. So

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[25] Robert Brandom, "Global anti-representationalism?", in *Expressivism, Pragmatism, Representationalism* (Cambridge University Press, 2013), p. 98.

Wittgenstein's antidote seems to be a recommendation to insert the context of practice back in.

This contextual embedding may be described in terms of a *language-game*: an artificial device to explore the connections between meaning and action. When we conceive a language-game, we draw contextual boundaries that rearrange the interconnection between the items of our vocabulary. Words can be used in one language-game and be dropped out of another; have a certain meaning in one game, another in the other.

Yet we must realize these boundaries were put there by posthumous analysis, and are as artificial as the lines of a chessboard or football field. We introduce the concept of 'language-game' in order to see which *plays*, which possibilities of expression are open for speaking agents in a given context. So, as is the case with every other game, when we attempt to specify the mechanism of a particular language-game, what we do is to clearly state the rules that make up this game space.[26] But this is in no way a simple task, since the norms we follow in everyday speech are often not rigid, imprecise, and incommensurable.

The important lesson to take out of this is that various language-games can be made with the same set of words; the same naming-term may figure in a potentially infinite variety of language-games. Would it be functioning as a name for the same bearer in every case? We have every reason to think not.

If new games are played in new contexts, and every contextual difference produces a difference in expressive function and even meaning, then this introduces the possibility of an endless variance. It is as Wittgenstein says: "When language-games change, then there is a change in concepts, and with the concepts the meanings of words change." (OC, §65). Each language-game is characterized by the interconnections gradually established by the social exchanges of its expressions (relations of implication, justification, antonym, etc.), resulting in a particular configuration of the logical space of conditions and implications, premises and

[26] When we try to frame a section of living conversation within the boundaries of a language-game, we have to make explicit the previously implicit norms of sense and meaningfulness. These norms will have to be stated as rules, and may be separated in Gentzen's style into introduction rules, which govern when a move (which could be a statement or a commandment, for example) may be made (asserted), and elimination rules, which govern what follows from making one correctly (as in which beliefs are being attributed due to the assertion, or which actions fulfil the commandment).

conclusions. And since this inner structure is what gives function to each expression, it is impossible to fix the meaning of one once and for all in abstract, disassociated from its genetic context, that in which the expression can be used to achieve the desired social effect. Therefore, there is no particular object rigidly connected to any term. If meanings change according to the associated practice, then linguistic expressions do not have one comprehensive essence independently of a hosting discursive practice.

In conclusion, the structural similarities pointed out by Benacerraf are not sufficient to sustain the argument that empirical and mathematical statements are used for the same function of *describing*. So long as our view on meaningfulness reduces every possibility of meaning to a possibility of description, we are going to be inevitably inclined to tie together distinct speech acts and the expressive roles they may play. Against this tendency to only see similarities in everything we call a proposition, I argue we must recognize that there is a semantic difference between statements which fulfil different roles in our social reasoning. We may think of them as moves in different games, performed to achieve different purposes. So every pair of practice-and-vocabulary standing in a meaning-use relation will have to be examined by the lights of its own performances. Understanding claims in this practice will depend on proficiency with the abilities at the practical substructure and on following a normative superstructure. These differences in the underlying practices according to which expressions acquire meaning entail differences in the range of functions an expression can be used for. Formal-syntactical similarities cannot override the differences in the practices-and-abilities that are required to meaningfully deploy empirical and mathematical vocabulary.

To summarise, the argument presented in this section has the following steps:

1. In a global representationalist semantics, logical role defines expressive role, and thus linguistic expressions either function as representations of objects (singular terms and predicates), or as connections of the representative bits (logical connectives);
2. However, determining the logical role a term plays is not the same as determining its function in reasoning, its expressive role, since expressions can

be used for more functions than just naming and connecting, just as certain uses can be *projected* into different discursive practices;

3. Therefore, representationalism cannot be globally applied, on pain of mistakenly homogenizing different practices of language use, blinding us to the multiplicity of functions speakers can give to their expressions.

4.3. An argument against the conception of meaning as representation

My point of departure is the premise that we use linguistic expressions according to certain standards of correctness, norms of linguistic behaviour with which we distinguish good from bad performances. There are circumstances which warrant certain uses, consequences to ensuing uses, and we judge performances of deploying a vocabulary accordingly. Or, simply put, meaningfulness is normative.

Held against this premise, it is not clear how the representationalist thesis suffices to explain linguistic meaning, the understanding of circumstances that elicit a conceptualization, nor the consequences that follow from doing so. If we use words to represent non-linguistic items, how are sure of which word to use in order to represent a certain thing to someone else? How do we make explicit to all interlocutors the connection between the use of a word and the intended representation? It seems we would need a specification of its implications and premises, yet that would require the representation to be publicly surveyable: like pointing to something in the visual field of every interlocutor and describing it, leaving no doubt as to what your sentences intent to represent.

But then we have cases of relations like that of the symbol $\sqrt{2}$ and its decimal expansion which may be interpreted classically in set-theoretical terms, or it can be interpreted constructively, in terms of an intuitionistic type theory or lambda calculus. Is $\sqrt{2}$ an ambiguous expression, or is one of these interpretations incoherent? If we hear someone else use this term, how could we know that our understanding of it represents the same thing as their use?

In order to have enough information to specify this, we must have socially determined criteria of correctness for our linguistic expressions; a certain inferential stability gained by using expressions under similar circumstances, expecting the same consequences. Thus, criteria of correctness for public discourse could not have been fixed by self-subsistent life-independent abstract elements, they must have their meanings determined from within the human form of life, by past use of a community.

Let us look at a historical case to illustrate this point. If we look closely at the history of algorithms devised to compute the decimal notation of irrational numbers, we notice that there is a discrepancy in the values yielded. Algorithms to compute π have changed drastically from Archimedes' formulation, passing through to Leibniz's sum formula, ending in the nested algorithms we use today.[27] Does this mean that the ancient Greeks use of π meant something different than what Leibniz meant or we mean today? Archimedes' algorithm establishes an identity between a recursive series of circumscribed polygons and the ratio of circumference to diameter of the circumcircle, and because of this recursive character, it is a method to yield increasingly more precise approximations of the decimal value of π.[28]

What the geometer accomplished here was the invention of an effective procedure with which we can judge claims about the proportion of circumference to the diameter of a circle. Yet his algorithm has long been superseded as the most effective procedure to compute this ratio. The introduction of these new effective procedures provided us with better criteria to judge the correctness of *empirical measurements* of circles, pushing past attempts into the background. Sometimes for being less rigorously defined or yielding a less precise standard, like Archimedes' one. Sometimes simply for being slower to converge to the series, as what happened with Leibniz's algorithm.

If people living in Archimedes' time did not meant something completely different with π than what we mean today, then our use of π *cannot* be designating an intensional function which determines all the steps to be taken. The digits of π are not pre-determined; we only *determine* the series through the application of an

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[27] Archimedes algorithm recursively computes a function of the length of the sides of series of hexagons, which doubles the numbers of sides at every moment. Leibniz's one is a sum function. These days, there are many possible formulations in programming languages, these are calculated faster by computers, but are harder to read. *See* André Porto, "Wittgenstein e a medida da circumferência" in: *Philósophos*, Vol 12 (2), pp. 55-85, 2007.

[28] Such that, with this technique, Archimedes was able to narrow the value of π down to the range $3\frac{10}{71} < \pi < 3\frac{1}{7}$.

algorithm, as in order to have criteria of correctness for having performed any computation, we need an effective procedure to compute it. With the introduction of new algorithmic procedures, the decimal notation of π is further refined, and thus more precise standards are given. New formulations of effective procedures fix new criteria of correctness with which we can judge the mistakes of past attempts. Still, the only way to be able to point out where previous attempts went wrong is through comparison with a more precise system or effective algorithm. Until we devised more effective procedures, the answer we used to get from Archimedes was the *best* available criterion of correctness for having measured the circumference of a circle with radius 1. This means our knowledge of π is not a fixed representation, it was *refined* through the years. For long, Archimedes held the formulation which most successfully made *explicit* — in the form of an expression of a mathematical rule — an implicit norm of our practices of drawing and thinking in terms of shapes and structured spaces.

Now, if we do not make explicit such implications of what one means with symbols such as π or $\sqrt{2}$, how else could we realize the possibility of distinct uses of these terms? Even if we presuppose some form of intuition of essences (e.g. Husserl's *Wesensschau*), it can still be asked how could we be sure that one's *implicit* grasp of the abstract (entity or potential) is the same as his fellow interlocutors, that we are indeed thinking and talking of the same thing.

This line of questioning brings us close to familiar objections to platonism. Echoing the difficulties that Frege faced with the Julies Caesar Problem, when a theory takes it as a logical principle that all referring terms are connected to referent extensions, such a theory is bound to find difficulties in locating the referents in non-empirical cases. For, after all, how should we proceed when the factual *relatum* of a semantic relation is not rigidly nor clearly designated? This difficulty seems to be attenuated in the case of mathematics, as it is particularly hard to clearly perceive *which particular objects* would be unambiguously represented by mathematical vocabulary.

If it is not controversial to take Frege's definition of singular terms as the default representationalist approach, then, to count as a singular term, a term must designate a non-empty domain (so we know that the purported object exists) and it has to pick out only the one particular object. Thus, what counts as an object is either:

(a) the referent of a proper name;
(b) what predicates are true or false of;
(c) the elements that compose the ranges of the individual variables which can be bound by quantifiers.

Now, how can we settle the uniqueness of the object supposedly represented by '2' from these principles? If the correct use of '2' must refer to something objective, it can only be to a multitude of objects, for, as Shapiro puts it: "[…] anything can occupy that place in a system exemplifying the natural-number structure. The Zermelo 2 {{∅}}, the von Neumann 2 {∅, {∅}}, and even Julius Caesar can each play that role."[29] Just in set-theoretical terms we already have at least two canonical ways of defining '2'. And if we think in terms of Frege's *Grundlagen* definition of numbers, the referent of '2' could well be Julius Caesar.[30] So how could we get the real '2' to stand up?

We may picture these freestanding semantic relations as ladders connecting the world at the ground level to language at the first level. That is, just as we climb a semantic ladder ascending from using certain terms (first level) to mentioning those terms (second level), we may climb it down, going from the terms to the objects so represented. But how are we supposed to proceed when the feet of our ladder are not fixed?

When a theory falls short of explicitly specifying the worldly *relatum* on the receiving side of its singular terms, it will inevitably face some version of Julius

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[29] *See* Stewart Shapiro, *Philosophy of Mathematics: Structure and Ontology* (Oxford: Oxford University Press, 1997), p. 80.

[30] *See* Dirk Greimann, "What is Frege's Julius Caesar Problem?" in *Dialectica* (Vol. 57, No 3, 2003).

Caesar Problem, for there is no obvious place to look for those objects.[31] This issue is directly connected to the problem of ostension (in the manner after Wittgenstein in the *Philosophical Investigations* §§26-48). That is, if there are indeed individual particular objects that are picked out by singular terms for each and every kind of discourse, then we should be able to define some of these objects by ostension, as in pointing to the bearer of the name, or attaching a name-tag to an object's body. However, as Wittgenstein claims: "an ostensive definition can be variously interpreted in *every* case." (PI, §28).

Quine's allegory of radical translation provides an extreme version of the problem with ostension, demonstrating that a freestanding reference is inscrutable. As the story goes, a rabbit runs by and the native, pointing to the rabbit, says "Gavagai!"; the English translator takes note "Gavagai = Rabbit", yet he cannot be sure whether that is the native's word for rabbit, or a running rabbit, or the rabbit's whiteness, or maybe a rabbit's part, or even maybe the feeling of joy or luck that those people get when they see a rabbit.[32] That is, even literally pointing to the purported object is *not enough* to determine one is using that expression for the function of naming, even less to fix a criterion of identity for interlocutors not acquainted with all supporting collateral premises and hypotheses. This is an extreme case, for sure, but the issue is not exclusive to cases of radical translation. Even pointing to something and saying "I mean *that* thing!" to clarify the use of a word is only helpful if the interlocutor already understands the practice of ostensively defining a word. So it may be helpful to *remind* an interlocutor of a criterion of identity, but the ostensive act all by itself is not near enough to *fix* the criterion.

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[31] For Huw Price, we cannot make our way from first positing a semantic relation to only then ask for the referred entity, the semantic relation must be demonstrated since the introduction of the signifier: "In order to decide what relation reference is, we need to be able to examine typical cases. In other words, we need to be able to study the various relationships that obtain between words or thoughts on the one side, and the items to which they (supposedly) refer on the other. But how can we do this in the case of "belief" and "desire," while it is up for grabs whether these terms refer to anything? In order to know where to look, we'd have to know not only *that* they refer, but also *to what*. To put this in terms of the location problem: If we need reference to locate belief and desire, we'll find that we need to locate reference, before we can put it to work. Yet we can't locate reference until we've located its worldy relata." (*Naturalism without mirrors*, Oxford University Press, 2011, p. 275).

[32] Quine, W.M.O, *Word and Object* (MIT Press, 1960), §7, §12, pp. 53, 80.

Returning the discussion back to numbers, Wittgenstein once said in a lecture: "We can explain the use of the words 'two', 'three', and so on. But if we were asked to explain what the reality is which corresponds to 'two', we should not know what to say." (LFM, p. 248). After that, he then raised two fingers in the air while asking "this?" Clearly, the meaning of a word cannot be tied to one single application. Someone could frame a picture of two raised fingers and treat it as the standard of correct use of the symbol '2', but that would be also a picture of five fingers, or of one hand, and so forth; and thus, we would also need a reminder that this picture is the canonical representation of '2' and not '5' or '1'. However, having such a rule defeats the purpose of the picture in the first place. If the picture does not make the standard of use explicit for everyone, only for those who already understand its purpose, then it lost its point.

When asked to specify which particular object secures one's true use of a number, one can only answer by producing an interpretation, like that picture of two fingers raised, Zermelo's representation of the set of the set of the empty set, $\{\{\emptyset\}\}$, amongst others. These may serve to guide an application of the concept, yet they cannot guide *every* application, for they cannot constrain which future performances will constitute meaningful use of the term. These can only show one standard of use, yet these standards vary with context. For instance, Zermelo's representation does not give us a standard of correction for nominal uses of numbers, such as in team sports, where numbers are used to assign name-tags for players.

The upshot I draw from Quine's allegory and Wittgenstein's remarks is that no extra-linguistic point of reference can pin down the meaning of an expression forever, and for all interlocutors. The meaning of any word cannot be exhaustively defined by anyone object we may experience. And this implies that meaningful use of "I'm in pain!" or "There are two cars parked outside" does not depend on (nor does it motivate) the postulation of an entity called pain or an entity called '2'.

This idea that an independent reality corresponds to our linguistic norms points one to Richard Rorty's attack against thinking the mind as a mirror of nature.[33] He claims that analytic philosophy is riddled with Cartesian presuppositions of "epistemically privileged representations" that can stop the regress of justifications at some cognitively transparent and intuitively given meaning.[34] He calls them "privileged" because the Cartesian tradition assumes them to be "automatically and intrinsically accurate",[35] precisely what is expected of a direct mirrored reflection of a worldly counterpart.

This epistemic privilege has a double sense: one's privileged access to one's consciousness, and the mind's privileged access to abstract objects. The first sense remits to the assumption that the self is the most reliable witness for its own sensations and experiences; the latter sense remits to the assumption that being aware of external objects is like having mirrored images of them passively imprinted in one's mind. Or, in the Cartesian jargon, is to have their idea immediately and clearly perceived by the intellect.

This process is entirely passive and personal, implying that one does not need to discuss with other people before having all that is necessary for forming a representation, as it is a faculty of our minds. When representations are intrinsically accurate and privately acquired, one would not need to be in a conversation with someone else to *know-of* something. There is an implicit assumption that true knowledge is perfectly attainable individually (hence the "cogito ergo sum").

Also, it means that one does not need to intentionally try to form a representation in order to have one, all that is necessary is for one to be subject to stimuli. This would be the case for all categories of objects, in all domains of discourse, thus begging questions regarding what it would be like to be conscious of an abstract object, or what criteria must be followed in order to constitute a correct representation of one.

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[33] "The picture which holds traditional philosophy captive is that of the mind as a great mirror, containing various representations — some accurate, some not — and capable of being studied by pure, non-empirical methods." (Rorty, R. *Philosophy and the Mirror of* Nature, Princeton University Press, 1979, p. 12).

[34] Rorty, R. *Philosophy and the Mirror of Nature* (Princeton University Press, 1979), Chapter IV.

[35] *Ibidem*, p. 170.

It follows from this that just by being conscious of something, one would have an intrinsically accurate knowledge of the thing, and thus an automatic and infallible access to what one means. When the mind grasps an item external to it, it forms a conceptual representation of it, and this representation is what determines what is true and what is false to claim of that item. So, when one hears someone else, what one understands is her own personal representation of the standard meanings assigned to each word. Therefore, one could *never* be wrong about what one means, because here meaning is not understood as an effect of social reasoning over the vocabulary used, but rather as an effect of using words according to their representative function. Of course, in cases where the represented is a body in our fields of vision, then it is hard to generate disagreement regarding the correct way of representing the state of affairs. This is not the case when we look at topics of conversation that are *not* open to view to all, such as mental states, feelings, or purported abstract objects.

As we cannot compare our representational models of what is out there, we can never be sure what motivated the use of certain words. While visible bodies make good examples of represented objects, talk of implicit, emergent and conceptual topics would lack the proper social criteria of correctness. That is to say, when meaning is explained in terms of representations, it is very easy to generate disagreement and confusion regarding the correct way of representing, say, freedom. No one is in position to reject a claim of someone who possesses (or is convinced of possessing) a clear and immediate perception of freedom. They may not be able to fully describe their representation, but they are certain they had it. For a representationalist, that is enough to qualify as meaningful. Having an insight is enough to satisfy the criteria of correctness. If someone said they had an insight in how to solve Riemann's conjecture, the representationalist is not sure whether that is true or false, but she is sure it is meaningful, as she understands the conjecture has a determined truth-value, we just do not know it until we discover a solution.

However, this imaginative exploration varies too much from agent to agent. The models of the world that speakers carry in the privacy of their minds could and would have whatever content they do regardless of what other inferentially related claims purport to represent. Each representation is free-standing and autonomous; it requires no support from other representations whatsoever — merely having one already counts as knowing something.

This entails that speakers would have to *tacitly know* which external feature a certain concept purport to represent in order to bring about knowledge of the truth conditions of sentences. Yet we still have no explanation as through which processes this 'bringing about' would take place; tacit knowledge of a concept's content is *implicit*, and therefore a speaker could not have grasped inferential implications that would only appear from adopting content made explicit in an interlocutor's claim. Another person's tacit and *a priori* grasp (or intuition of essence) of a concept could only be her own implicit business. But how could one peer into the mind of his interlocutor to know which representation she has in mind when deploying a certain term? How could tacit knowledge of implicitly given representations turn out to constitute the social standards according to which we judge and understand each other's claims?

The fundamental flaw in erecting a semantic superstructure on a basis of isolated personal representations is that there will be no way of cashing out the intrinsic normative authority and social accountability involved in deploying concepts. According to representationalism, knowing the correct use of a concept is a personal matter, dependent on which representations the individual agent associates with that concept. However, interlocutors could be deploying the same words but not be aware that their use aims at distinct representations, and thus follow different criteria of correctness. As such we could not employ each other's claims as premises to understand the implications of applying a concept. If this was the case, one could never rely on the inferential implications he thinks follows from another's claims in order to understand what this interlocutor means.

This breach in our capacity to nail down the norms guiding each other's linguistic behaviour is just a step away from semantic pessimism or scepticism. Think of Saul Kripke's sceptic, who asks: How can we tell whether or not half of the world employs one paradigm of division and the other half employs another, but we have not discovered the discrepancy yet because we have not expanded these series far enough?

When the criteria of correctness cannot be made manifest to the participants of a communication exchange, we cannot rule out situations where someone uses a familiar concept in a way that only superficially resembles ours, but after close examination reveals the attribution of a different function to that same expression. This would allow us to put into question whether we all have slightly different standards for calling something green or for saying we are in pain. Extrapolate these doubts to cover all discursive practices and all concepts, and what we get is global scepticism about meaning. As Kripke puts it: "[…] if the sceptic is right, the concepts of meaning and intending one function rather than another will make no sense. […] if this is correct, there can of course be no fact about which function I meant."[36] And if there is no disputable fact, nothing that could be misapprehended, that one could agree or disagree over, then there are no criteria of correctness, no way of telling which function was intended or whether it was applied correctly. If the sceptic is right, Kripke concludes: "There can be no such a thing as meaning anything by any word. […] any present intention could be interpreted so as to accord with anything we may choose to."[37, 38] Whereas one would think there is mutual understanding and communication of ideas, our reality would actually be closer to a global undiagnosed cacophony.

As such, the representationalist thesis casts a major doubt over the possibility of communication in general. That is, if the representationalist explanation is correct, if our concepts are intrinsically accurate representations implicitly and passively acquired by a mind mirroring nature, then communication starts looking like an *accidental handshake* between private vocabularies. We could be talking past each other, assuming that by making certain conceptualizations we are producing an intended model or representation in interlocutors' minds, when they would actually interpret such concepts based on *their own* private representations and models.

Following on the steps of Dummett's criticism of platonism, John McDowell reached the same conclusion when assessing the platonist insistence on the principle of bivalence in the case of undecidable statements:

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[36] Kripke, S. *Wittgenstein on Rules and Private Language* (Harvard University Press, 1982), p. 13.

[37] *Ibidem*, p. 55.

[38] It is worth mentioning that these arguments against representationalism are based on Wittgenstein's discussion of our expressions of sensation. Roughly, the point is this: if meaning is thought of in terms of representations, then understanding an interlocutor's complaint about a pain will remain forever out of reach for me; how could I know another's use of 'pain' reflects the same reality as my use? Under representationalism, the meaning of talking of pain is sublimated, becoming whatever one wishes to report as pain.

> Affirmation of the principle of bivalence without restriction, or conformity with classical methods of inference, will not serve; the platonist wanted these practices to be *justified* by an *independently* acceptable account of the relation between meaning and truth. So platonism apparently makes no concession to the thought that someone's understanding of a sentence must be able to be made fully overt in his use of it. The trouble is that this leaves it a mystery how one person can know another person's meaning.[39]

According to McDowell, to insist in such cases where we are in no epistemic condition or state of information to justify the assertion or negation of a mathematical statement that nevertheless the statement has a definite truth-value would imply that "we can no longer plausibly identify what it would be for a sentence to be true with what it would be for one of the relevant decidable circumstances to obtain."[40] To insist there is a clear meaning here is to insist that the intentionality we manifest in our linguistic behaviour must be capable of signifying way more than what the inferential implications of our claims would allow. That is, the meaning of expressions would inexplicably transcend their use.

So long the standards of correctness are not publicly and explicitly manifest, then this issue will be looming over our heads. In order to avoid generalized scepticism about communication, meaning must be taken as overtly manifest in use. Conceptual understanding and linguistic intentionality *must be manifest* in the linguistic behaviour of interlocutors. We only understand a saying if the speaker deploys words according to the social norms of the practices in which that vocabulary is normally used. The grounds for our knowing the referents of words cannot be tacitly inborn, they must be created and kept alive by a community, made explicit for all interlocutors to learn.

Moreover, if an explanation of meaning was exhaustible as the revelation of a pre-existing language-to-world relation determining the correct way of using that

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[39] McDowell, J. *Meaning, Knowledge and Reality* (Harvard University Press, 1998), p. 347

[40] *Ibidem*, p. 349.

expression, then this relation should also determine the inferential role of that expression, the inferential circumstances and consequences of its application. Yet, unless one knows the other relevant concomitant commitments, collateral premises, and auxiliary hypotheses, then the inferential circumstances and consequences are not going to be readily specifiable.

The representationalist picture portrays the human capacity to mean and understand linguistic expressions as based on an automatic access to the conceptual, not so different from some special access to platonic forms. Yet, unless all concepts users have the same intuitions about the application of concepts, then these matters must have been learned socially first. Concepts are not determined regardless of use, as if given in an ideal world of necessary connections that are pre-existent to our effective acknowledgement of them. The assumption of this world is insufficient to explain how such relations could have provided public criteria of correctness for the use of concepts.

If we accept Wittgenstein's recommendation to start at the context of practice, then it is only at the level of discursive practices — as opposed to the level of individual words or isolated sentences — that we are going to find the appropriate normative structure capable of providing criteria of identity to fix a naming relationship; something that has to be checked case-by-case, since words can always appear anew, applied in an unexpected context with a borrowed sense. We cannot specify a naming relation in general for a term from an analysis of logical form. No syntactical criteria could ever determine whether a naming relation holds for every possible instance of use, for syntactical features cannot account for the work done by pragmatic features in determining the expressive role of an expression.

To account for meaning, we have to consider the little social games we play with language, where naming relations are introduced and used. And the unspoken norms that give a leeway of meaningfulness to our expressions are forever changing according to social reasoning. Hence why thorough specification of meaning requires asking for reasons, producing layers of explanations by interlocutors inclined to comprehend one another.

So long our practices which make meaning manifest are not taken into account by our philosophy, the finer differences in deploying mathematical and empirical vocabulary will remain obscured. If deploying distinct vocabularies requires proficiency with distinct sets of practices-and-abilities, then semantic differences should be evaluated by what sort of know-how one must possess in order to meaningfully deploy a vocabulary to manifest a knowledge-that. That is, if we are going to explain how applications of numerals in '2 + 2 = 4', 'Jupiter has 4 moons', and '√4' are very unlike each other,[41] we must examine the know-how, the practices required for one to count as meaningfully deploying '4' for each particular function.

To summarise, the argument presented in this section has the following steps:

1. Language use is normative; i.e. we distinguish good and bad performances;
2. According to representationalism, the criteria for correct use of a concept is the representation of extra-linguistic objects or properties;
3. Because representations are privately and implicitly formed by individuals, they are also *socially unsurveyable* — I do not have direct access to the representation you intended, so I cannot sanction your use of concepts;
4. If we think language works as so, then we enable generalized semantic scepticism, casting a shadow of doubt over the possibility of communication in general;
5. However, since we do in fact communicate and understand each other's claims, then the representational relation cannot be taken as primitive, on pain of *mistakenly* conceiving meaning as something unsurveyable, above and beyond instances of use, from which we could not derive socially explicit criteria of correctness for the use of expressions.

4.4. Wittgenstein's critique of conceiving meaning as anticipating reality

Now that we have covered reasons to abandon the thesis that meaning is a species of representation, we may relate the argument back to the comparison of a platonist

[41] Namely, the first deploys numerals as instances of integer (or as isomorphic items of the linguistic category of integer) and thus requires proficiency in a mathematical practice, namely knowing how to operate sums with them; the second deploys numerals to count applications of the pivot concept 'Jupiter's moon' and thus requires empirical knowledge of how many of the roundish things orbit the gas giant; and the third deploys a numeral to designate an irrational number and thus requires understanding numbers as functions.

and a combinatorialist interpretation of mathematical operations at the beginning of this chapter. To remind the reader, in these interpretations, a mathematical operation has its result determined by the properties of the represented objects; the inferential implications of using a symbol are already determined by the objective facts, we just have to acknowledge them in predication.

However, following the argument above, to claim that all steps towards the concluding clause of an algorithmic routine are pre-determined independently of a practical execution of the algorithm has the effect of turning mathematical meaning into a mystery. How could anyone use a symbol such as π in accordance with all its implications, including an infinite series of integers? By merely deploying the symbol π, practitioners would be effectively meaning way more than they could possibly grasp.

Taking a cue from Wittgenstein, I see an issue with this view of semantic relations that are merely acknowledged by use, rather than created by gradual regularities of use. It portrays mathematical operations as abstract mechanisms pairing domains and codomains at no particular time or place, and the mathematical object as an abstraction awkwardly divorced from the practices and routines of the living organism that defines it. There are numerous writings in which Wittgenstein broached this topic,[42] yet perhaps none other is so effective at unveiling the analogy of mathematical operations and machines as his discussions regarding rule-following in the *Philosophical Investigations*:

> Here I'd like to say first of all: your idea was that this *meaning the order* had in its own way already taken all those steps: that in meaning it, your mind, as it were, flew ahead and took all the steps before you physically arrived at this or that one. / So you were inclined to use such expressions as "The steps are *really* already taken, even before I take them in writing or in speech or in thought". And it seemed as if they

[42] For instance: "We have then a rule for dividing, expressed in algebraic or general terms, — and we have also *examples*. One feels inclined to say, 'But surely the rule points into infinity — flies ahead of you – determines long before you get there what you ought to do.' 'Determines' — in that it leads you to do so-and-so. But this is a mythical idea of a rule-flying through the whole arithmetical series." (LFM XIII, p. 124).

> were in some *unique* way predetermined, anticipated in the way that only meaning something could anticipate reality. (PI, §188)

Clearly, Wittgenstein takes target at the idea that meaning an object would imply grasping its computational implications all at once. It is like saying that just by having the blueprint of a machine you could pre-determine all consequences of turning it on before even seeing it in practice. Or that intentional use of the symbol for an irrational number would suffice to determine all decimal places in its expansion. This idea seems to presuppose some mysterious potency of the mind, whose intentionality mysteriously flies ahead of the algorithmic routine and determines the value of each decimal place up to the concluding clause, even though no computational routine could reach it. A whole order of inferential steps that no one has ever followed through. The idea seems to be that if a mechanism was built to get there, it will eventually, so we might as well already consider it as a completed sequence.

From such considerations, Wittgenstein soon realized that much of the debate between platonists and constructivists turns around this metaphysically pregnant conception of possibility, either on the rails of pre-existent necessary connections or as steps pre-determined ahead of practical performance.

> The machine, I might say for a start, seems already to contain its own mode of operation. What does that mean? — If we know the machine, everything else — that is the movements it will make — seem to be already completely determined.
>
> We talk as if these parts could only move in this way, as if they could not do anything else. Is this how it is? Do we forget the possibility of their bending, breaking off, melting, and so on? Yes; in many cases we don't think of that at all. We use a machine, or a picture of a machine, as a symbol of a particular mode of operation. For instance, we give someone such a picture and assume that he will derive the successive movements of the parts from it. (Just as we can give someone a number by telling him that it is the twenty-fifth in the series 1, 4, 9, 16,…)

> […] But when we reflect that the machine could also have moved differently, it may now look as if the way it moves must be contained in the machine *qua* symbol still more determinately than in the actual machine. As if it were not enough for the movements in question to be empirically predetermined, but they had to be really — in a mysterious sense — already *present*. And it is quite true: the movement of the machine *qua* symbol is predetermined in a different way from how the movement of any given actual machine is. (PI, §193)

We may interpret the "machine" in this scenario as an algorithm we may use to compute the twenty-fifth member in the series x^2. In this case, the possibilities of movement of our machine are specified by the set of squared integers. As such, considering a "machine *qua* symbol" is equivalent to considering the well-formed formulae in which we write the squaring algorithm. Or, even, considering a computer program capable of running this algorithm and operating x^2.

With this in mind, we can distinguish two senses of 'pre-determination' discussed by Wittgenstein:

(i) a prediction of what will be the case if agents follow the norms of mathematical practice;
(ii) a determination of possibilities, of what *can only happen* as an implication of the rules (like a rigid physical mechanism that only moves in one pre-determined way, such as a piston).

The first sense is an empirical claim that does not assume modal or metaphysical claims. When we wish to determine what will happen, we are merely guessing or prophesizing the future. This is the sense in which we say that, under normal conditions, the machine will move like so-and-so; or that an agent will get 625 as the twenty-fifth member in the series x^2, if he follows the arithmetical rules correctly.

The second sense is the opposite; akin to trying to measure the shadows of the possible movements of a machine, or to exhaustively determine all possible outcomes of performing an algorithm. Talking in this sense, we are engaged in metaphysics. Some thesis regarding the modality of mathematical objects will be required to justify any such pre-determination of the inferential steps, or this treatment in analogy with mechanical steps. This is why we see the postulation of abstract entities; they serve to justify the algorithm's potency to pre-determine its inferential path.

It is this second sense that interests those who presuppose that mathematical discourse is about objective entities. Not concerned with what computing agents can and may do, but rather with what the algorithm can achieve in ideal conditions. Computing agents eventually die or break down, so they cannot help to justify why this statement: "The quadrillionth digit in the expansion of $\sqrt{2}$ is 8" is determinately true or false.

That is, both options in our dilemma, platonism and combinatorialism, do not accept human practice as the basis of mathematical truth. The argument against maverick approaches who emphasize history and community is that, if what compels a computing agent to arrive at a result is merely a rational obligation to a culture (in which agents are brought to use formulae by training in arithmetic), then we have lost the objectivity of mathematics. A socio-normative criterion is insufficient to secure the objectivity and universality required to explain the success that our mathematical systems enjoy at correctly quantifying, measuring and modelling nature. It portrays mathematical properties as a subjective feature, and mathematical language as just another human artefact.

They think the mathematical possibilities as all pre-determined by necessary connections or procedural constructive methods capable of performing all the steps before they have been practically made. As in the quote above, mathematics is portrayed as if "it were not enough for the movements in question to be empirically predetermined, but they had to be really — in a mysterious sense — already *present*."

But how is that "the machine *qua* symbol" contains the ways the machine could move *more determinedly* than the actual functioning machine? How is that just using or employing a function could already determine how all steps in the computation of this function will be resolved? How could the formulae or program represent how the steps of the algorithm will be resolved before the concrete performance of those steps?

To secure the claim that mathematical truth is grounded on non-linguistic entities, one must see a potency hidden in the mathematical formulae or programs in which we write algorithmic routines somehow determining all inferential steps ahead of the inferring agents. It is as if algorithmic routines could self-perform and reach a conclusion by themselves, like automatons. Or as if the blueprint could pre-determine all possibilities of movement for the designed machine. Yet, they forget computing agents and machines eventually break down.

These interpretations inevitably *create the need* for abstractly given standards of use, because something with intentional capacity must fill in the role that practical use has in defining meaning. The attempt to understand meaning in terms of a Cartesian *privileged representation*, not dependent on social reasoning, requires some measure of abstraction from the ever-changing particularities of context and speaker's intentions. In the lack of computing agents following norms of practice, something else has to do that job. The theorist must posit the mechanisms of meaningfulness as being set independently of everyday linguistic practice. But once the contextual variances of *use* are conceived as secondary to the syntactical-logical form of the term, there is no other way to explain meaningfulness that does not appeal to an abstract element endowed with a potential to run ahead of effective instances of application of a concept and fix its meaning.

A good example of this tendency is drawn in the *Philosophical Investigations*, when an interlocutor says that the meaning of a sign can only be fixed by an implicit abstract standard, "the psychical thing":

> How does it come about this arrow → *points*? Doesn't it seem to carry in it something besides itself?—"No, not the dead line on paper; only the psychical thing, the meaning, can do that." — That is both true and false. The arrow points only in the application that a living creature makes of it. This pointing is *not* a hocus-pocus that can be performed only by the mind. (PI, §454)

Wittgenstein's second voice[43] is here playing an analogous role to the platonist: for him, if the arrow on the paper points to anything, it is only because we know an arrow 'stands for' the abstract concept of *pointing-to*, something we apprehend early in life by watching people drawing arrows to point. The postulation of a "psychical element" is welcomed because it guarantees objective meaningfulness for every instance of use of the arrow. Following this, Wittgenstein immediately inserts a therapeutic deflationary voice telling us that any meaning the arrow may have for us is only in effect of an intentional application by a living creature (which, evidently, must follow our normative practices of using symbols to point), and that our minds do not go from reading an instance of arrow to grasping the transparent meaning of every use of arrows in general, the abstract concept of *pointing-to*. Meaning is not an effect of whatever one "has in mind", it is an effect of how words are used.

The point is that the postulation of an inscrutable standard as the original meaning (a platonic form, a meaning-body [*Bedeutungskörper*]) is an irrelevant device to explain meaningfulness because meaningful use of an expression is not governed by which object a person has in their mind when uttering the words:

> One can't shake oneself free of the idea that using a sentence consists in imagining something for every word. One fails to bear in mind the fact that one calculates, operates, with words, and in due course transforms them into this or that picture. — It is as if one believed that a written order for a cow, which someone is to hand over to me, always had to be accompanied by a mental imagine of a cow if the order was not to lose its sense. (PI §449).

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[43] I am referring to David G. Stern's interpretation of the method behind *Philosophical Investigations*' dialogues as a 3-stage setting, with three distinct voices. To recapitulate, the 1st evokes a philosophical thesis, doctrine or position; the 2nd describes a specific set of circumstances to which the first is an appropriate explanation; then a 3rd deflationary voice shows how limited those circumstances were, and then tries to move beyond them, sometimes even pass the idea that philosophy is in the business of providing thesis or explanations (see "Wittgenstein's critique of referential theories of meaning and the paradox of ostension, *Philosophical Investigations* §§26-48", in *Wittgenstein's Enduring Arguments*, London: Routledge, 2009).

As in one of Wittgenstein's examples, the criterion of identity for the chess king is neither fixed by our pointing to a wooden figurine nor by talking of an essence or ideal type of chess king. You do not need a specifically carved piece of wood in your visual field or in your imagination to assert a meaningful sentence about it. All you need is to know the king's role in the game. Having a representation of a chess king in one's mind is quite irrelevant for understanding what a chess king *does*; for that one has to understand the rules of chess.

Or, consider PI §40 and §80, where one of the voices stresses that when someone dies, use of the name to refer to the deceased remains understandable, for only the bearer of the name dies, not the use to achieve that meaning. The explanation is that meaning is not determined by any extra-linguistic entity we may associate with the use of an expression, but by how we use it. So long there is a community following on the social norms of use for a term, keeping it alive, it does not matter whether or not an associated object *has ever existed*.

Likewise, no abstract standard is needed to represent the result of mathematical operations, or to contain all inferential steps within its logical form. We do not need to postulate abstract standards to explain meaningfulness when the derivation of criteria of identity from the basic principles of mathematics is all that counts in mathematical practice. When a new theorem is proved, establishing a new identity between mathematical concepts, we are fixing a new criterion of correctness for having performed the relevant operation. The inferential relations between '$x \circ y$' and the result '$= z$' must be demonstrated as the consequence of the basic premises through a canonical method of proof, opening the way for the employment of algorithms to compute '$x \circ y$', whose success will then be judged according to the standard '$x \circ y = z$'.

The mathematical theorem is not true because it mirrors a real feature of the world, but instead because it functions as the rule according to which we judge good performances involving its concepts. As Wittgenstein explains, this social-normative criterion of manifestation is what really matters:

> What a geometrical proposition means, what kind of generality it has — all of this must show itself when we see how it is applied. For even if someone *were to mean* something inaccessible by it, it wouldn't help him,

> because he can only apply it in a way that's completely open and intelligible to everyone. (BT, §137, p. 494-5).

Mathematical language exists to communicate, and as the argument above goes, it is hard to see how could we successfully communicate if what we are doing is exchanging intuitively acquired representations of non-linguistic stuff. Hence, when we considered whether it is determined that the quadrillionth decimal place in the expansion of $\sqrt{2}$ is a digit 8 or not, the question should not be interpreted as turning on whether one will always or never *find* an 8, but instead on whether one *ought to write* '8' as the quadrillionth remainder of the ratio of circumference to diameter, according to the rules of calculation. It is a matter of rational obligation, not of causal necessitation. And as such, we must answer that is not determined; not yet at least, for we know of no case in which a computing agent has reached the quadrillionth step in the infinite task of computing the decimal expansion of $\sqrt{2}$.

From a viewpoint that departs from an examination of the practices that bestow meaning upon our expressions, the sense of our number-talk could not have been based on implicit grasp of a pre-determined infinity of numbers, as number-talk is what allows us to specify the number of iterations of the application of a concept in the first place. What pins down the correct way to use a vocabulary are the norms given by the history of the relevant associated practices.

Recent developments in cognitive science have shown that humans and other animals can *subitize*: the capacity to discern quantities of a certain cardinality just by observation, without counting. According to a research conducted by Lakoff and Núñez, the human ceiling for this faculty is the discernment of 8 units.[44] What is more surprising, however, is the conclusion of Frank *et al.*, that even though we all have the capacity to subitize, some human cultures have not developed any conception of number whatsoever.[45] The team of researchers studied the discursive practices of the Pirahã people. Isolated at the heart of the Brazilian Amazon, the Pirahã did not import foreign words and concepts related to numerosity. They do use words roughly equivalent to the English 'more' and 'less', but they have no

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[44] Lakoff G.; Núñez R., *Where Mathematics Comes From* (New York: Basic Books, 2000) p. 19.

[45] *See* Frank, M.; Everett, D.; Fedorenko, E.; Gibson, E. "Number as a cognitive technology: Evidence from Pirahã language and cognition", in *Cognition* (No 108, 2008), pp. 819–824.

definite number-words, not even for one. And still, as the researchers point out: "A total lack of exact quantity language did not prevent the Pirahã from accurately performing a task which relied on the exact numerical equivalence of large sets." Their conclusion is that numbers are better thought of as cognitive tools rather than names for abstract things that all humans would know about:

> The case of Pirahã suggests that languages that can express large, exact cardinalities have a more modest effect on the cognition of their speakers: They allow the speakers to remember and compare information about cardinalities accurately across space, time, and changes in modality. […] However, the use of a discrete, symbolic encoding to represent complex and noisy perceptual stimuli allows speakers to remember or align quantity information with much higher accuracy than they can by using their sensory short-term memory. Thus, numbers may be better thought of as an invention: A cognitive technology for representing, storing, and manipulating the exact cardinalities of sets. [46]

Number-words are cultural artefacts, a cognitive technology we created and use for remembering, manipulating and communicating quantities. Not unlike standards of measurement, number systems are inventions which offer us standards with which we can answer questions of the sort "How many?" or "How much?".[47]

Saying that true use of the numeral '2' is one which refers to the abstract number '2' is like saying that true use of the metric system is that which refers to 'The Metre'. As in the case of numbers, our talk of lengths in metres could not have been based on tacit knowledge of *that specific length*, as it was the introduction of the metre-standard what allowed us to specify lengths in terms of a universal system of proportions to one metre. That is, without having arbitrarily chosen a standard, we would not have a system for comparing lengths.

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[46] Frank, M.; Everett, D.; Fedorenko, E.; Gibson, E. "Number as a cognitive technology: Evidence from Pirahã language and cognition", in *Cognition* (No 108, 2008), p. 823.

[47] *See* Dummett, M. *Frege's Philosophy of Language* (New York: Harper & Row, 1973), p. 483.

In both cases, what we have is a man-made rule-governed system which offers us linguistic structures (e.g. the natural numbers $\mathbb{N}$, and the metric system) that allows us to specify something we *do* in social-normative practices involving objects (counting and measuring). As these are regulated by social norms, these systems present standards (the natural number sequence, and a system of proportions to one metre) which demonstrate the correct way of using its terms (integers and metre). So to insist that uses of the numeral '2' are not only counting objects but also referring to an abstract 2-object, or to insist that the meter standard[48] has the length of 1 meter, is applying the standard to itself. But how can we judge the correctness of a performance when the performance itself is setting the standard of correction?

Counting the natural numbers is not akin to using the numbers to count items, rather it is just repeating the succession of integers as one has learned it, spelling out the standard of correctness, the criteria according to which we judge performances of counting. Applying a ruler with metric markings to measure something *is* providing a standard of correction for comparisons of length, as with the ruler we can tell what is longer than what. Likewise, applying integers as a logical form in which to count things *is* providing a standard of correction for counting. The standards of the natural number sequence and the metric system give us criteria with which to judge the correctness and goodness of performances of counting or measuring.

It is a self-defeating move to look for the particular object that a number is, just as it is self-defeating to look for the particular length of 'The Metre' — these are standards, not things, and any particular thing can be used as the standard, even Julius Caesar. It a source of misunderstandings to assume that their use is supposed to represent particular objects. Instead, we use standards to compare properties of objects. More specifically, the numbers and the metre are cognitive tools of frameworks that we employ to specify quantities and lengths of objects.

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[48] For instance, one could have in mind the platinum-iridium bar which was taken as standard for a metre. Albeit, originally, the meter was defined as the unit in the division of the distance between the North Pole and the Equator line in 10,000,000 units. Nowadays the meter is defined as the distance travelled by light in a certain fixed fraction of a second.

5. The pragmatic expressivist alternative

Wittgenstein taught that certain intellectual problems originate in misunderstood language, and thus are not in need of a solution; rather, these problems are dispelled when we reach clarity regarding the function and pragmatic significance of the misunderstood expressions. To avoid such misunderstandings, we must comprehend the possible roles our expressions may play in reasoning. Thus, we must analyse the various ways in which the community of speakers puts expressions to use.

The premise of this thesis is that Benacerraf's dilemma is one such example of an intellectual problem originated in misunderstood language. That is, this dilemma only troubles those philosophies bewitched by a single vision of language and of the knowledge we express through it. Once we expand our horizons and see the possibilities lying beyond the small scope of Benacerraf's premises, the dilemma loses its philosophical force.

To demonstrate these possibilities, the remainder of this thesis will introduce an alternative account of mathematical truth. As mentioned in the introduction, my intention in producing a new interpretation is not theoretical, but therapeutic: to demonstrate that one can think outside of the representationalist and empiricist boxes, and that there are other ways of thinking mathematics that do not get caught in Benacerraf's dilemma. A pragmatic and expressivist outlook is particularly useful for this goal because it does not presuppose that language's essential function is representation, nor that every knowledge claim must be traced back to a causal chain of events as a product of a passive acquaintance with naturally given structures. It explains mathematical meaning without postulations of abstract entities, and explains mathematical knowledge as springing from our active engagement with the world, concerned with communicating its patterns and structures with precision. This chapter focuses on the semantic aspect of this view, while its epistemological ramifications are covered in the next chapter.

The pragmatic expressivist metasemantic claims that linguistic meaningfulness is an effect of the social use of expressions. By 'expression' we mean the turning of something previously veiled or implicit into something socially explicit, cognoscible to all interlocutors. Think of a facial expression making explicit one's previously implicit feelings, or a linguistic expression making explicit one's implicit thoughts and intentions. Our expressive acts gradually construct meaning. Given time and involvement, we dress many actions, gestures, facial expressions, sounds, symbols, and bodies surrounding us, with meaning. As these relate to our practices, our routines and goals, they get caught up and embedded in our culture.

The way we use vocabulary items is key in explaining meaning, as linguistic performances clearly have an expressive function: speakers use words to make explicit in *sayings* what previously was only implicit in their *doings*. Expressing oneself with linguistic items is turning *implicit* feelings, intentional stances, doxastic commitments, practical standards, amongst others, *into the explicit* content of a sentence.

Contrasting with the passivity of representations, expressions are actively performed; it is a bringing about, the creation of a *second nature* in order to understand nature. And unlike with representations (which are rather opaque unless introduced matching external *relata*), with expressions we can see immediately how correctness and goodness are evaluated based on the propriety or impropriety of the act to the circumstances of use. Already when we teach our linguistic practices and techniques we say "Do not say *that*, say *this*… instead" or "It is not pronounced… it is …" indicating the ways of one *ought* to use expressions in order to be understood by others. The difference between correct and incorrect linguistic expressions is no more mysterious than for any other action, like running or swimming — they are instituted by a history of use, and *all* can be equally decomposed into basic practical or rational principles learned in their respective social-normative practices.

The linguistic meaning lies in the way we connect practices with vocabularies. The pragmatic meaning-use relation of vocabulary with hosting social practices just is the process according to which utterances acquire meaning. The idea is to explain meaningfulness as the social effect of such performances; meaning as the result of an expressive act following the norms of a discursive practice, of what agents are required to *do* in order to be understood by fellow interlocutors; or how they must rectify their linguistic behaviour in order to be part of the conversation. So for every instance in which a speaker uses a vocabulary, there is a set of practices-and-abilities that, if and when performed, are sufficient to manifest a meaning to his fellow interlocutors. Grasping these practices is grasping what sort of *practical know-how* one needs to make a meaningful claim to *know-that* with a certain vocabulary. Or, making explicit a previously implicit practical knowledge by elaborating and codifying it into an algorithmic routine (e.g., elaborating knowledge of how to cook a Paella in a step-sequence of instructions).

These pragmatic meaning-use connections we keep alive in everyday life gradually sediment throughout a history of use, constituting intra-linguistic relations which determine the roles that linguistic expressions may play in social reasoning. And histories of use have a normative weight. With time and repetition, these practices grow roots, they are institutionalized and become the normal use.

As so, this is a social pragmatic version of a global expressivism. In such approach, all normative matters of authority and responsibility for what one claims are regulated in and by social practices, and thus are dependent on culture. Unlike representational language-to-world relations, all semantic relations are intra-linguistic. As so, a social pragmatic view has no need to postulate an independent reality to correspond to our norms of discourse, or to the structure of our conceptual networks.

Another distinctive characteristic of a social pragmatic form of expressivism is its commitment to an agent-based *subject naturalism* which departs from a naturalistic conception of the speaking agent. This pragmatic version of naturalism was introduced by Huw Price, for the reason that "philosophy needs to begin with what science tells us *about ourselves*."[1] That is, unlike a full-blown object naturalism, the pragmatic version does not presume a fact about the whole universe, such that every existing entity is in principle knowable through the scientific method. Subject naturalism respects what seems correct about naturalism – starting our reflections in an acknowledgement of fact about the nature of the reflecting agent – while rejecting assumptions regarding how the world should be described.

In comparison, the objective version of naturalism is a reductive view, as for it, all that exists is specifiable in a privileged naturalistic vocabulary. An entity that cannot be described naturalistically must not exist. So, for instance, physicalism would be a modern example of object naturalism, the view that only physical entities exist, or that we should only take into consideration what is describable by a physical theory. According to Price, object naturalism entices us to always look after objective truth-makers for our claims, therefore motivating the representational picture of language. To abandon representationalism, we must also abandon the assumption that every truth-apt discourse relates back to naturally given objects.

[1] Price, H. *Naturalism Without Mirrors* (Oxford University Press, 2011), p. 186.

> Insofar as our claims are representational, it seems plausible to assume that they are uniformly representational, whatever the subject matter—in other words, that representation is a univocal notion, in this sense. But if representation is viewed as relation to our natural environment, univocity leads to the placement problem in an acute form. The problem is solved by abandoning the external notion of representation in favour of an internal notion; by recognizing that the grip of the alternative picture rests in large part on the disquotational platitudes; and by insisting that we theorize about our relations to our natural environment in a different, non-semantic vocabulary. So long as we practice our naturalism in another key—in the pragmatic, functional dimension that opens up when we abandon Representationalism—we retain univocity where it matters, while avoiding the placement problems altogether.[2]

To practice naturalism in another key is to shift our view of language and knowledge away from those matching-correspondence object-oriented approaches, as these inherently lead to placement problems: the problem of coherently 'placing' all kinds of truth in a natural world.[3] As Price says: "If all reality is ultimately natural reality, how are we to 'place' moral facts, mathematical facts, meaning facts, and so on?"[4] That is, if non-logical linguistic expressions can only represent extra-linguistic features, then the semantic analysis of vocabularies with no obvious worldly *relata* will invariably issue semantic and epistemological problems similar to Benacerraf's dilemma.[5]

The claim that linguistic meaning is a species of representation presupposes a substantial ontological distinction between representing minds and represented objects. It purports to a metaphysical dualism. Thus, such a view opens the possibility of metaphysical extravagances such as the ontological distinction of physical and mathematical facts, giving away the impression that mathematics talks about an abstract realm that does not coincide with our physical reality.

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[2] Price, H. *Naturalism Without Mirrors* (Oxford University Press, 2011), p. 32-33.

[3] *Ibidem*, p. 6.

[4] *Ibidem*, p. 187.

[5] As mentioned in 4.2, ethical, modal, normative, ontological-categorical, and historical vocabularies. It seems that in all these cases, problems similar to those presented by Benacerraf could be raised.

Price's subject naturalism departs from a smaller assumption which only concerns the thinking subject, instead of concerning every thinkable thing. This assumption lies in an empirical attitude towards the human subject and its practices of accounting for what there is and should be. Such approach seeks to specify what natural creatures like us *do* when employing a vocabulary *as* fundamental to disclose the facts. Thus, naturalism in a pragmatic key focuses on understanding the role that vocabularies have in our lives, instead of looking for objective features that would supposedly correspond to the structural implications of our symbolic systems.

This strategy is exactly what we need to avoid the horns of Benacerraf's dilemma. As an example, in the following passage, Brandom explains how the subject naturalistic approach would help us in avoiding the trappings of Benacerraf's epistemological horn:

> The subject naturalist's question is how to understand the practices of counting and doing arithmetic in virtue of which (natural) number talk means what it does. If we can explain, in naturalistically acceptable terms, how it is possible to teach and learn to count and calculate using numerals, ontological difficulties of the sort that exercise the object naturalist should be taken at most to throw doubt on the aptness of this sort of discourse to the kind of representationalist semantic treatment that can then be seen to be the source of those difficulties.[6]

We can move past the ontological difficulties that hinder the representationalist view by understanding the pragmatic role that vocabularies have in our lives, analysing the sufficient and the necessary practices and abilities for using a vocabulary, instead of trying to justify its use with represented structures. From such perspective, the key mechanics for the determination of meaning is no longer which existing features or properties a concept represents, but rather which expressive role a conceptualization plays in social reasoning. Whereas representationalism is reductive of expressive functions, narrowing all properly assertoric discourse to the function of describing a possible world, pragmatic expressivism sees multiple expressive functions, of which description is but one. The first paints a homogeneous portrait of language use, the latter paints a multi-functional mosaic.

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[6] Brandom, R. *From Empiricism to Expressivism* (Harvard University Press, 2014), p. 92.

Moreover, this undertaking of a subjective version of naturalism also has the implication of making pragmatic expressivism antithetical to ontological speculation. Subject naturalism implies no ontological commitments towards some narrative about *what there is* or *could be*. This view is not interested in practising First Philosophy, in the sense of an inquiry into the universal foundations of knowledge in general. In this classical sense of metaphysics, all that can be known, said or thought about must be described by or expressed in those universal terms; otherwise, it is regarded as unknowable, unintelligible or unthinkable. Pragmatic expressivism abandons this sort of philosophical enterprise by rejecting the notion that we have a privileged vocabulary which discloses the essence of all things. This rejection follows on the steps of Brandom's argument which, drawing influence from Rorty and Wittgenstein, claims that such privileging of a naturalistic vocabulary is but a forceful imposition of a single view upon the world:

> What is wrong with the metaphysical sort of privileging of vocabularies is that it requires the idea of some vocabulary being necessarily privileged by *how things are*—God's vocabulary, or Nature's, or even Mind's, or Meaning's vocabulary—quite apart from our contingent projects and attitudes. […] any such program will turn out, upon examination, to have been motivated by a philosophical anxiety that can be traced to some relatively specific misleading philosophical picture of what knowledge, mind, meaning, or reality *must* be like—on pain of some Bad Consequence.[7]

Brandom seems to be here following Wittgenstein's method of examining the underlying pictures of the metaphysical tradition, considering traditional metaphysical programmes as being motivated by philosophical anxieties, instead of concrete practical issues. By the lights of such consideration, metaphysical problems do not require solution — as they do not have one — they require a therapeutic dissolution, a treatment of the misunderstandings at their genesis.

The pragmatic expressivist view adopts the same attitude towards classical metaphysics, and as so, it gives us the necessary leeway to avoid the horns of Benacerraf's dilemma. From this perspective, an account of mathematical truth does not proceed

[7] Brandom, R. *Between Saying and Doing* (Oxford University Press, 2008), p. 222-223.

through the postulation of abstract entities, or the attribution of potency to algorithms. Thus, this view can provide an intelligible and universal treatment of truth claims without the metaphysical extravagance that follows from the platonistic view, or the semantic heterogeneity that follows a constructivist approach.

5.1. Concepts as nodes in networks of inferential relations

As we have seen above, expressivism explains meaning in terms of a social effect that follows from using words to make explicit to reasoning what was previously implicit. Expressing oneself with language is employing concepts, conceptualizing a subject matter, and as such, ascending that content into social reasoning. [8] This explanation of the conceptual starts at the level of practical processes. Instead of *assuming* an agent's grasp of concepts that represent existing features and items, pragmatic expressivism tries to explain how our linguistic performances could mean anything by examining what one has to *do* in order to be counted as saying something. Our concepts are, therefore, fruits of the social-pragmatic function of expression.

A social pragmatic and globally expressivist view of language understands that humans have an *active* role, not only at grasping the conceptual but also at devising it. We are not born knowing how to use concepts. Even conceding Chomsky's point that our grammatical capacity is tacit or inborn, still, expressivism insists we can only understand conceptual content (and thus also how to properly use concepts) once we partake in *social* discursive practices.[9] Of course, one can still make self-standing assertions, without participating in any discursive practice; but one cannot learn the use of concepts alone. The speaker has to acquire the proper *knowledge-that*, and the only explanation available for this knowledge which does not have to postulate abstract entities proceeds through the process

[8] As Brandom puts it: "The process of explicitation is to be the process of applying concepts: conceptualizing some subject matter." (*Articulating Reasons*, Harvard University Press, 2000, p. 8).

[9] Chomsky himself is no friend of a reference-based approach. *See* Noam Chomsky, *New Horizons in the Study of Language and Mind* (Cambridge University Press, 2000, p. 102f). In the passage, Chomsky says that, insofar we understand language use, it is possible that language only has a semantics in the sense of a pragmatic study of its use, and not in the sense of studying a dimension of representational relations.

of learning: acquiring the relevant knowledge as to *how* a concept can be used to make a meaning manifest to our interlocutors.

As so, concept formation is necessarily tied to language learning, whereas one masters the use of concept-words. Seemingly, one acquires this at a young age, observing adults expressing articulations of concepts and reliably responding accordingly. Since we are of the same form of life (see 6.2) and thus have similar ways of reasoning, this observation can lead the child to notice the invisible architecture of social commitments and entitlements adults are manifesting with their claims. One perceives, imitates, and train; assess results and then train some more – an intentional cycle of learning how to do things with words. Learning how to express oneself with language is figuring out ways of taking one's intentional stance towards something and turning it into a manifestation of linguistic intentionality, bestowing intentional content unto one's expressions (imagine how a child learns to ask for her favourite toy by calling its name in place of showing that desire by babbling and/or pointing at it). In order to acquire proficiency in thinking with and applying concepts, one has to engage in communication and learn by example, imitation, and training, all in order to *see* the interconnections between circumstances in which agents apply concepts and the consequences (in their actions, attitudes and stances) of doing so.

As such, a sapient agent's command over a concept could not have been passively formed through an intuition of essences. Rather, this is an intellectual ability demonstrated when the agent reliably recognizes instances of the concept and can classify these under the same linguistic category, expressing familiar aspects in distinct instances (e.g. light, lightness, lit, bright). Understanding someone, or manifesting a meaning, depends on the practical mastery of the sort of circumstances that entitles (or commits) one to a certain concept, and what follows from applying it.

Or, in other words, a conceptualization can make a certain meaning manifest *because* of the way in which that concept is historically connected to the application of other concepts; what determines a concept's role in any given discursive practice is relations to other concepts associated with that practice. Yet this is not an automatic function of meaning, as in the representationalist picture; it is rather a *social effect* that one *may* achieve when using concepts in accordance to the norms of the discursive practice.

We articulate our expressions by chaining linguistic forms together, presenting conceptual connections. With time, these chains gradually sediment into complex intertwined networks (or, semantic webs) which enable speakers to understand and see how one conceptualization is warranted by a previous one, and how one may license others. These networks can be noticed in the way our discursive practices allow us to give and ask for reasons for our claims and actions,[10] showing how the use of a concept is inferentially linked to the use of others.

That is, in order to provide motives and reasons, answer questions regarding how one may act, how a claim may be falsified, and so on, one needs to understand the inferential potential that follows from the concepts applied, the conditions and implications of a conceptualization. Therefore, since these connections define the uses of our concepts, then the conceptual content of our claims is determined by the correctness of our inferences.[11]

With this in mind, we may define concepts as inferential roles,[12] nodes in networks of inferential relations drew by our expressive practices. The overall semantic web constituted by the interplay of inferentially linked conceptualizations determines the circumstances that elicit and the consequences that follow the use of a concept.

As it can be seen, the pragmatic expressivist metasemantics is built on top of a first-level inferentialist semantics, since the first explains meaning as an effect of using interlinked concepts, and such internal connections are fixed by our inferential practices. In an inferentialist analysis, the meaning of a sentence is explained in terms of the inferential circumstances and implications of that sentence; as such, a sentence does not function as a representation of non-linguistic items, it instead evokes certain conceptual links, a certain semantic neighbourhood. Unlike reference-based or verification-based semantics, there is no connection between sayings and the environment, only between sayings, actions and thoughts.

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[10] The practice of giving and asking for reasons is taken by Brandom as showing the inferential architecture of language use. For more on this, see *Making it Explicit* (Harvard University Press, 1994), p. 48; and *Articulating Reasons* (Harvard University Press, 2000), pp. 10-15.

[11] *See* Brandom, R. *Articulating Reasons* (Harvard University Press, 2000), p. 52.

[12] *See* Brandom, R. *Making It Explicit* (Harvard University Press, 1994), p. 618.

Inference is such a key notion (although not the master concept[13]) in the analysis of meaning because being capable of inferential articulation is what distinguishes *sapient* concept users from mimics. According to Brandom, the difference between the former and the latter, which could be a parrot or an automatic thermostat, is that the sapient knower has "the practical know-how to situate that response in a network of inferential relations."[14] While the parrot or the thermostat may give a reliable response to the appropriate situation, uttering "That's red!" when seeing a red surface or turning on the heating when room temperature hits below a certain mark, they cannot draw further inferences and tell what follows from something being red or the room being cold, or what does not follow, what is an evidence for it, and so on. Thus we say neither creature or machine are sapient. The parrot may be sentient and possess a certain level of intelligence, but it is not conscious of how linguistic concepts are interlinked.

In other words, being capable of partaking in inferential practices is an attribute of our human form of life. We can recognize and think in terms of networks of conceptual relations. As such, the inferentialist semantics entails **semantic holism**: we can only explain the meaning of a sentence by means of others; explain the use of a word with other words. Thus, as Brandom claims, "one cannot have *any* concepts unless one has *many* concepts."[15] In order to know how to use any concept, one must already know how to use many others, grasping the internal connections that hold a vocabulary together with a practice. Semantic holism also means that speakers cannot use vocabulary items in whatever way they wish. In order to be understood, speakers can only work within the possibilities of their inherited semantic horizon. One cannot be understood if she deploys vocabulary in a manner that fellow interlocutors cannot root back to some historically established discursive practice. Words are like worn coins silently placed in one's hand; their meaning is tied to socially inherited norms of trade, a history one must recur to when setting up new uses. The meaningfulness of new applications is inevitably linked to old ones; they maintain a certain familiarity, a resemblance marked by overlapping similarities in use.

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13 It is important to notice, as it will come about later in an preventive answer to criticism in Appendix I, that inference is further reduced and explained in terms of the pragmatic distinction of undertaking and attributing a commitment, and the distinction between the normative stances we express of commitments and entitlements.

14 Brandom, R. *Articulating Reasons* (Harvard University Press, 2000), p. 162.

15 *Ibidem*, p. 15.

From this metaphor, we may see how the expressivist outlook opens up the possibility of explaining communication as the attribution of, and attempt to change, each other's commitments and entitlements. This is Brandom's deontic scorekeeping model which represents communication exchanges as language-games in which interlocutors keep track of each other's rational licenses and obligations, and also attempt to change these through assertions of commitment and attributions of entitlement, as if we were keeping track and attempting to change a scoreboard of deontic attitudes.[16] As so, communication is structured by our holding and licensing responsibility for linguistic expressions that we use to give or ask for reasons for what we think and do.

The primitive and fundamentally explanatory elements of the scorekeeping model come from the fundamental pragmatic distinction between undertaking or attributing deontic attitudes to oneself or to others. These are the actions we count as changing the scores. The basic linguistic 'move' capable of altering deontic scores is one with the pragmatic force of an assertion. Assertions can play such role because they are acts of making commitments explicit, manifesting a normative stance, expressing what others may hold us accountable for (as we are rationally responsible for the claims we make).

Most of our discursive practices are assertoric in this manner, as they present a 'move' (a saying or a doing) that has the pragmatic force necessary to play a role as a premise or conclusion in social reasoning. The assertoric-inferential architecture of our discursive practices allows us to alter each other's practical and rational commitments, duties, entitlements, licenses, etc. Every assertion, even the free-standing ones that are not part of a dialogue, presupposes this inferential architecture to conclude from premises and serve as reasons for further judgements.

Furthermore, with the elementary pragmatic distinction between attributing or undertaking deontic attitudes, doxastic and practical commitments and entitlements, we can map relations of **compatibility**, in order to distinguish good and bad inferences. Because our application of concepts in inferentially articulated chains makes manifest to our interlocutors the commitments we undertake, it also draws a set of compatible commitments (which may include concomitant commitments implicitly undertaken), and a set of incompatible ones. Thus, every conceptualization entitles one to further compatible claims and precludes incompatible ones.

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16 *See* Brandom, R. *Articulating Reasons* (Harvard University Press, 2000), p. 142.

These sets of relations of compatibility define a logical space of implications that follow from applying certain interconnected concepts. As so, the possibility of chaining certain concepts in discourse can be understood as a possibility of movement within the logical space of inferential relations. Each composition of concepts in our sayings opens up the possibility of continuing one's reasoning through a certain conceptual neighbourhood, going through certain nodes in the network of inferential relations, while also closing the possibility of proceeding through incompatible routes.

5.2. Anaphoric deflationism about truth and reference

To say that a proposition *P corresponds* to a state of affairs is interpreted through expressivist lenses as a claim that the space of implications drawn from a certain predication *P* (i.e. the sets of commitments entitled or precluded by asserting it) is compatible with all other concomitant commitments. Or simply, *P* does not clash with any other socially accepted description of what is the case. In such treatment, assessing whether or not a statement is true amounts to evaluating whether it can be held as a premise in reasoning without generating incompatibilities.

As such, a good strategy to explain our use of 'true' and 'refers' from an expressivist perspective is to recur to the anaphoric interpretation, which offers a theory about the expressive role of these terms. Its goal is to provide an account of how one properly uses the concepts of truth and reference, and what significance such uses have. *Anaphora* means semantic dependency on previous uses. The anaphoric interpretation treats 'true' and 'refers' as *proform-forming operators*, that is, as devices that substitute previously deployed words or sentences, much like pronouns. Their application forms *prosentences*, sentences whose meaning is anaphorically dependant on previously asserted sentences. Thus this theory explains these devices in terms of anaphoric relations their use maintains with ancestor expressions, constituting anaphoric chains that provide meaning to current uses.

Let us start with **truth**. Expressions containing "… is true" are prosentences that inherit their content from antecedent locutions, and thus their meaning is recoverable from context. The difference between treating truth as a prosentential operator instead of a predicate is easy to demonstrate, take for example:

(1) *P*: "Force is necessary and sufficient to accelerate mass."

(2) *P* is true.

(3) '*P* is true' is true.

Reading (1) as a modal statement about relations between the concepts of force and mass, if 'is true' is understood as a predicate, then (2) is about the first modal statement, and (3) is about the statement about the modal statement. That is, there is a difference of meaning which can be described in Tarskian terms as a semantic ascension, first from object language to meta-language (or semantic level), and then from that to a metasemantic level.

On the other hand, if 'is true' is understood as a prosentence-forming operator, all three sentences have exactly the same content — the succession is explained as an anaphoric chain inheriting meaning from the original assertion in (1). The anaphoric account does not explain truth in terms of correspondence or coherence, but in the same terms of its anaphoric antecedent. So the truth of "Force is necessary and sufficient to accelerate mass" is to be explained in terms of force, mass and acceleration. What before was explained as the attribution of a controversial property (e.g. correspondence to reality) to mysterious meaning-entities which would bear it (propositions), is now explained as the deployment of a linguistic device, a proform prosentential-operator, whose application is semantically dependent on its antecedents.

"Could somebody accept (2) without understating *P*? If yes, then how are these supposed to share their meanings?" — Well, people often accept unjustified truth claims. It is perfectly possible to accept a prosentence without being able to recover its antecedent, as an unjustified (or blind) endorsement. That is, someone may nod in approval when another claims "What the professor said was true" even without knowing that the professor claimed "It is true that force is necessary and sufficient to accelerate mass". This entails that, if understood as a proform-forming operator, 'true' cannot play a primitive and fundamentally explanatory role of our use of words, as it is dependent on such usage.

Compared with the theory that truth depends on reference to existing entities, the anaphoric interpretation has the advantage of being modest, since truth is explained in terms of the platitude that to assert '*P* is true' is just to assert *P*. This theory is controversial only in denying that truth is a property, as the anaphoric role of the truth operator keeps it from playing the sort of *explanatory* role that a representational truth-theoretical semantics assigns to a truth predicate. What characterises a predicate in an inferentialist analysis is that it conceptualises a subject matter, playing an inferential role as premise or conclusion; this is not something that the truth operator does, as there are no particular inferential circumstances and consequences related to claiming that '*P* is true' that are not already related to the claim that *P*. Thus, if truth-talk can be explained in terms of anaphora, then 'is true' should not be read as predicate denoting a property. [17]

The anaphoric interpretation offers an analysis of both the meaning and the pragmatic features of truth claims as that of acknowledging antecedent premises or endorsing a conclusion. Therefore, the truth operator has two characteristic expressive roles, that of:

(i) re-asserting previously asserted claims, and
(ii) generalizing claims.

According to Paul Horwich's expressivist analysis, this second role is due to the truth operator allowing a speaker to generalize valid forms of expression (e.g. "Every statement of the form '$p \lor \neg p$' is true"), to endorse or reject claims when these cannot be explicitly stated by the present interlocutors (e.g. "You can trust him, he only tells the truth"), or even saying something that all interlocutors should undertake in their reasoning (e.g. "The speed of light is a universal constant").[18]

In this manner the anaphoric account makes justice to an insight present in the writings of Frege, Tarski, and even in performative accounts of truth, such as Strawson's; namely, that to attribute true to *P* has the same force — the same pragmatic significance — as simply asserting that *P*. The intentional attitude of taking a sentence as true is an acknowledgement of the assertional commitment expressed by that sentence. That is, taking a statement as true amounts to nothing more than the platitude of asserting it. As Strawson explains:

[17] All ordinary uses of the truth operator can be accounted for in this model, including the quantificational and embedded ones evaluated by Geach in his counter-argument against an expressivist semantics (See Annex I).

[18] *See* Horwich, P. *Truth* (2nd ed., Oxford University Press, 1998), pp. 4-5 and 136-137.

> The sentence 'What the policeman said is true' has no use except to confirm the policeman's story; but […] does not say anything further about the policeman's story. […] It is a device for confirming the story without telling it again. So, in general, in using such expressions, we are confirming, underwriting, agreeing with, what somebody has said; but […] we are not making any assertion additional to theirs; and are never using 'is true' to talk about something which is what they said, or the sentences they used in saying it.[19]

The anaphoric interpretation follows Strawson in denying that an attribution of truth attributes a particular property to a declarative sentence. As such, the notion that the truth operator is a disquotational device is captured in the anaphoric sense; Tarski's condition of adequacy (Condition T) is satisfied in the anaphoric theory, since every claim within a particular vocabulary will yield disquotational truth conditions. From this, we can infer that to assert is to present as true, and assertoric content has a negation that also possesses assertoric content.

Yet the departure of the anaphoric interpretation from Tarski's intentions becomes clear when we realize that, whereas he asked for a *material* interpretation of his adequacy conditions (preferably in terms of correspondence to reality), the anaphoric interpretation delivers only a deflated notion of the adequacy conditions — it does not specify the content of a truth claim in terms of sets of objects with their relations and properties, but instead in terms of particular arrangements of concomitant statements which entitle or warrant the claim to be taken as a valid premise or conclusion.

What we *do* when we endorse someone else by attributing truth to their statement is that we are socially attributing a doxastic commitment to that person, and expressing our own commitment to it. Instead of saying 'We know $A \rightarrow B$ is true, and we have proved A, so B is true', one can always say 'We have assumed $A \rightarrow B$, and we have proved A, so we may conclude B.' Reproducing or just accepting a claim as an expression of true knowledge is the same as judging it a fit conclusion for a set of concomitant premises, and thus as a fit premise for further inferences. Or, as Wittgenstein once wrote: "The *truth* of my statements is the test of my *understanding* of these statements" (OC, §80). To assess a claim as an

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[19] Strawson, P. "Truth", in: *Analysis* (Vol 9: 83-97, 1949), p. 93.

expression of true knowledge is judging it as a fit premise for further inferences, or a fit conclusion for concomitant premises; it is endorsing the expressed doxastic commitment towards what is the case, undertaking it and acknowledging it as a fact. Talk about truth is talk of one's intentional attitude towards an expression of *knowing-that*.

Doxastic commitments and beliefs are worth having to the extent that they play a mediating role effectively: the role of facilitating inference in ways that help us communicate and cope with intellectual anxieties or practical problems. We deem as true those expressions of doxastic commitments that effectively play a role in enabling and supporting inferences, particularly in the solving of practical problems, as they enable us to share information we would not otherwise possess, helping us anticipate scenarios (e.g. "It is cloudy and dark outside; so if you do not want to get wet, better take an umbrella with you.")

As such, success at coping with practical problems and intellectual anxieties replaces the notion of a 'correspondence to the facts' as the characteristic of truth attributions. In Brandom's words: "Talk about the cardinal importance of concern with truth is a dispensable *façon de parler*. What actually matters is the pragmatic attitude of *taking*-true or putting forward *as* true, that is, judging or asserting."[20] The application of the truth operator sanctions the commitment thus expressed to be socially generalized — to be accepted as a viable premise or conclusion for all rational interlocutors.

Now let us talk of **reference**. The anaphoric account of reference also treats 'refers' as a pronoun-forming operator, usually employed to form anaphorically indirect definite descriptions. Just like "That is true" used as a response to "Chomsky wrote a book on social inequality" inherits its meaning from its antecedent, a saying of the like "I'm referring to the one who wrote 'Requiem for the American Dream'" inherits its meaning from an anaphoric chain of antecedent expressions whose expressive role is to indirectly describe the author. Thus making a reference is not the same thing as meaning something; instead, every reference inherits its meaning from antecedent uses. It is a *pronomial anaphoric reference relation*. So the expression "The one referred to as 'Chomsky'" is an indirect description whose content is inherited from antecedent applications of the term 'Chomsky', not the American philosopher itself nor a mental representation of him.

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[20] Brandom, R. *Making it Explicit* (Harvard University Press, 1994), p. 82.

Reference thus understood is not based on a *causal* relation between linguistic expression and referent object. Just like truth turns out not to be a substantial property of statements, reference turns out not to be a substantial property either, as it is not in the business of representing extra-linguistic items. There is no extra-linguistic word-to-world relation of reference from an inferentialist semantics, the only notion of reference at play here is *intra-linguistic*, based on chains of anaphorically dependant singular terms. As with the truth operator, these anaphorically dependent singular terms inherit their content from other uses, and thus their meaning is recoverable from context. This is a relation between uses of words which form equivalence classes of inter-substitutable expressions which indirectly describe the same subject matter.

For instance, imagine Ray has two cars and he says to James: "My car took a knock last night and I have had to leave it in the garage for repair." To which James replies: "Which car are you referring to?" And Ray says: "The blue Astra." Here the anaphoric chain starts with the first use of 'My car' and continues as:

'My car'… 'it'…'car'…'The blue Astra'

At first sight, one may think the first 'My car' could not be referring to the blue Astra, as it comes first and it is not a definite description. But there is nothing wrong in saying that the first use of 'My car' referred to the correct car, precisely because it belongs to the same anaphoric chain as the later definite description. An anaphoric chain can be initiated by an indefinite description (e.g. 'That mountain', 'The house', 'My glasses') and be continued by use of more indefinite descriptions, pronouns, or move on to definite descriptions. Yet, of course, if the chain only has indefinite descriptions, then it may fail to play the expressive role of making explicit the subject of discussion, thus failing to be a proper reference.

To make a reference is the same as starting or inserting a new link in an anaphoric chain. Brandom botanises the roles in these chains in three distinctions: (1) between anaphoric initiating expressions and the dependent of these, (2) between intersubstitutable and non-intersubstitutable types,[21] and (3) between lexically complex and simple types. Examples of anaphoric links that are invariant under substitution and lexically simple are proper names. An invariant and lexically complex example would be "The last King of France." Examples of non-intersubstitutable lexically simple initiators are 'this' or 'that'. And a non-intersubstitutable lexically complex example would be like "A member of the British parliament." All these can be used as *initiators* of anaphoric chains, and then pronouns, direct and indirect definite descriptions, may follow as *dependent* links in the anaphoric chain.

Reference is thus a linguistic performance with the expressive role of defining that which one holds commitments about, marking the subject of discussion. We do it typically by deploying vocabulary items such as 'of', 'about' or 'that' — terms that are particularly useful in making explicit one's intentional inclination towards an extra-linguistic item. Or, as Brandom says, it is a species of *de re* attribution of propositional attitude:

> […] assessment of what people are talking and thinking *about,* rather than what they are saying about it, is a feature of the essentially *social* context of *communication.* Talk about representation is talk about what it is to secure communication by being able to use one another's judgments as reasons, as premises in our own inferences, even just hypothetically, to assess their significance in the context of our own collateral commitments.[22]

Expressions such as "This word corresponds to..." are thus not a descriptive, but actually a normative move: they are used to make explicit in what context it is appropriate to use the word. So, for instance, if I am watching chess being played with non-standard tokens and I recognize one of them being moved in an L-manner, I am entitled to affirm "This one corresponds to the knight!" because I know the rules that determine the knight's possibilities of movement. Once I have introduced that token as playing the knight's role,

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[21] I.e. those which if applied to substitute other links along the anaphoric chain will change the meaning of the complex sentence. For example, compare: (i) "A member of the British parliament threatened to vote against the Carbon Emission Reductions Bill, yet seeing a fierce public opposition, he decided to retract from this decision" with (ii) "A member of the British parliament threatened to vote against the Carbon Emission Reductions Bill, yet seeing a fierce public opposition, a member of the British parliament decided to retract from this decision." — Clearly, the substitution executed in (ii) gives this chain a different meaning than the one presented in (i).

[22] Brandom, R. *Articulating Reasons* (Harvard University Press, 2000), p. 167-168.

then I will be held accountable to other concomitant beliefs, such as that this is the only piece that can jump over others.

5.3. The expressivist account of objective representation

In an expressivist point of view, we abandon the picture that language maps reality or possible worlds in favour of the picture that discourse and world emerge as the poles of practice; language and world are intrinsically related through our routines and practices. Linguistic communication should not be understood on the model of exchanging pictures of the world; there is no *causal* relation between the logical form of sentences and the structure of the world, in any direction. All attempts at expressing this structure are already filtered through human perception and cognition, then limited by the expressive power of our vocabularies.

Even in cases of direct observational reports, when we use words in a manner that is *not specifiable* apart from consideration of the external facts and objects that responsively bring about or are brought about by their use, still linguistic expressions do not get their meaning from representing extra-linguistic features. Non-inferential circumstances of application of concepts are *subsidiary* to their inferential applications, as these inferential networks are what define the concept's role in reasoning. What interaction or effects human language could have in the world can only be intermediated by humans, cognitive systems moved by words, or perhaps machines moved by our codes. Unlike merely sentient creatures, we sapient agents are capable of interacting with the environment according to explicitly stated reasons, given and examined in our inferential discursive practices. Such agents are the only ones who are moved by linguistic expressions into changing what they 'think' or into carrying out actions. As so, words do not fulfil an inferential role for representing objects or properties of the external world. An inferential role is fixed by the particularly articulated ways in which we use words in speech and writing. No statement is capable of mirroring reality or *corresponding* in any meaningful way to a possible world or state of affairs.

Still, none of this means that the notion of representation is to be expunged from philosophical conversation. Talk of representation does point towards an important aspect of intentional attitudes, namely that they aim to fulfil a norm of correctness for the application of concepts that is responsible to how things are in the external environment, not only how we wish they were. When one utters a statement with the function of reporting or describing, one may be applying concepts for non-inferential reasons, aiming at correctly representing a feature in a system that is external to the inferential superstructure of language, the purpose of which is to change scores of commitments and entitlements of one's fellow interlocutors.

The use of this sort of conceptual neighbourhood, in particular of ordinary empirical descriptive vocabulary, is not specifiable unless one can consider the circumstantial external facts that warrant their application.[23] For instance, let us consider the observational report "Oh look, Patricia brought us tea" — we could embed this statement in a conditional, drawing out the inferential entitlements that warrant the assertion of this sentence, but the claim itself does not require any previous empirical justification to have a meaning, since it was a *direct observation* of environmental circumstances what responsively brought about a warrant for using those words. The meaning of claims like this one is not specifiable apart from consideration of their non-inferential circumstances of use, as our discursive practices are caught up in this to-and-fro with objects we perceive and interact with, and thus expressive performances that make moves from action to language, and language to action.[24]

In a pragmatic expressivist outlook, we repurpose the concept of *representation* to the level of metasemantics: a representation is a composition of descriptions. A representative model of reality is based on a bundle of conceptualizations, of logical forms chained together through a discursive practice. If it was made explicit in declarative sentences, it would have to be into a set of compatible sentences, such as a narrative, or possibly an argument. To have a representation of a state of affairs or of a possible word is an effect of

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[23] Or, as Brandom puts it: "All our concepts are what they are in part because of their inferential links to others that have non-inferential circumstances or consequences of application — concepts, that is, whose proper use is not specifiable apart from consideration of the facts and objects that responsively bring about or are brought about by their application. The normative structure of authority and responsibility exhibited by assessments and attributions of reliability in perception and action is causally conditioned." (*Making it Explicit*, Harvard University Press, 1994, p. 331).

[24] Non-inferential relations between practical commitments and a state of affairs can be catalogued as language-entry transitions (from perception to testimonies or claims), and language-exist transitions (from claims to actions).

undertaking commitment to a set of *compatible descriptions* which, when taking into consideration how we use each other's claims as premises for our own claims, may very well be identified with a consensus in social reasoning; or may generate debate, in case someone argues that one or more threads in the mesh of claims constituting a representation are wrong or unjustified — this would be equivalent to reorganizing our web of commitments, and for that the debater must produce supporting claims to substitute the faulty inferential links in a socially endorsed argument.[25] Therefore, the notion of representation indicates the social dimension of propositional content.

As such, no freestanding sentence or singular term would suffice to constitute a representation. Linguistic representation is an effect of groups of transactions, not the currency of exchange. An effect of undertaking a multitude of compatible commitments embedded in a worldview. And this goes for use of 'representation' in the mathematical discursive practices as well, as Brandom explains:

> '$x2 + y2 = 1$' and '$x + y = 1$' do not *resemble* the circle and line that they represent. They represent those figures in virtue of the facts relating the whole *system* of equations to the whole *system* of extended figures, in virtue of which, for instance, one can compute the number of points of intersection between the figures by simultaneously solving the corresponding equations. This original understanding of representation in terms of global isomorphism is an essentially holistic one.[26]

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[25] I find this passage by Brandom is a particularly helpful explanation of this point: "If being a consumer of representational purport, taking something as a representation of something, is understood as believing of it that it correctly represents (or equally if the purport is understood as intending that it do so), then an infinite explanatory regress is generated by the possibility of querying the nature of the representational purport ('that ...') and success ('of ...') such a belief exhibits. There must be some way of understanding something as a representation that consists not in interpreting it (in terms of something else understood as a representation) but in taking, treating, or using it in practice *as* a representation. To understand what it is for red dots on a map to purport to represent cities and wavy blue lines to purport to represent rivers, the theorist must look to the practice of using a map to navigate. If such purport is to provide a model applicable to representational purport in general, that practice must admit of construals that do not appeal to the formation of propositionally contentful beliefs. The practice must be intelligible in terms of what counts as following it or going against it in what one actually does: the way it guides the behavior of those who can use maps." (*Making it Explicit*, Harvard University Press, 1994, p. 74).

[26] Brandom, R. "Global Anti-Representationalism?", in: *Expressivism, Pragmatism*, *Representationalism*, p. 9.

These mathematical formulae can be said to *represent* circles and lines in general not because they *resemble* them, but rather because these formulae deploy concepts already caught up in conceptual systems which we apply to think of and talk about spatial structures. The representation is said to be a global isomorphism because it is a product of a system of interrelated concepts developed in an interplay of practices. So to say that a formula represents a circle or a line would not make sense if we had not developed and projected older practices of using numbers as points on a line, then employed the Cartesian plane to draw figures unto the interpretation of those algebraic formulae. Thus, we should not understand one who says "'$x2 + y2 = 1$' represents the circle" as claiming that this equation *alone* represents an abstract standard which all concrete circles strive to resemble.

From employing the anaphoric interpretation and harvesting its anti-metaphysical consequences, the inferentialist semantics suggests a novel understanding of the traditional semantic notion of representation as a *semantic meta-vocabulary*: a vocabulary that let us specify the conceptual content (i.e. meaning) of our claims. For instance, when one says something along the lines of "Gold represents wealth", one is making explicit the inferential role of certain uses of the word 'gold' (i.e., that it should be treated as playing the role of 'wealth').

The use of semantic meta-vocabulary 'represents' in "'$x2 + y2 = 1$' represents the circle" is only concerned with the meanings expressed, and not with the performance of the practices-and-abilities that were required to achieve meaningfulness. This is why, when we talk of what the formula represents, we only say it represents a circle and omit the underlying practices an agent must know-how to perform in order to understand uses of '$x2 + y2 = 1$'. That is, whenever 'represents' is used, we have our eyes set on the objects related to the practices in which we use that expression, not on the practices themselves. To specify what are the practices required to say something, we need to employ a *pragmatic meta-vocabulary* (see 6.4).

5.4. An analysis of meaning in terms of use

In order to provide an expressivist-inferentialist analysis of meaning in terms of use, we must first characterize the kinds of inference that hold our semantic networks together. Inferences are bifurcated in two essential kinds. First, there are the *materially valid*, when our application of concepts is articulated according to a perceived order of causation. These are appropriate inferences by virtue of a material connection between environmental and practical circumstances that warrant the deployment of those concepts. They run on observations and explanations of cause-and-effect and thus are not held on basis of formal validity. That is, the authorial responsibility one undertakes by asserting one of these is not going to be justifiable with basis solely on logical rules; to sanction one of these inferences, we need to observe empirical-material relations holding between the particular instances that fall under application of those concepts. [27]

The kind of goodness of inference that interests us the most here are those expressed in conditionals that are valid despite having nothing in common with causes-and-effects. These traffic on *inferential licenses* between *deontic* and *practical* commitments and entitlements which justify the validity of inferences expressed in conditionals such as:

> "He was declared *guilty*; thus he shall be *punished*."
> "If you *go to* Florence, you *must see* cathedral"
> "Purple *comes after* blue in the colour wheel, thus purple is *closer* to blue *than* yellow"
> "If you *multiply* 5 by 5 items, you *ought to get* 25 items."

These inferential connections are based on norms of practice which, as with any practice, are related to material objects (people, cathedrals, coloured things, countable units, etc.), yet the practices themselves (of keeping justice, recommending, configuring a colour wheel, or giving a mathematical rule) are not trying to track relations of cause-and-effect — we do not say that seeing Santa Maria del Fiore is a causal effect of one having gone visit Florence — the inferences related to these practices are not valid for being materially good. Agents are entitled to these claims regardless of their capacity to perceive and reliably respond to

[27] Examples are: "Where there is *smoke*, there is *fire*"; "I saw *lightning*, so I will be hearing *thunder* soon"; "If I had *dropped* this pen, it would have *fallen* to the ground"; "If London is *east* of Southampton, then Southampton is *west* of London".

non-linguistic stimuli, solely by virtue of rational ruling and concomitant or collateral inferential entitlements from the respective discursive practices.

If I assert any of the conditionals above, fellow interlocutors should not hold me committed to a belief about an extra-linguistic state of affairs; I should not be held accountable to a picture of reality. Nor can an interlocutor take one of the above conclusions as a premise to something material, such as "If he shall be punished by the legal system, then he must be a psychopath" which is unjustified and thus an invalid inference. These two kinds of inferential validity are constituted from the elementary expressions of doxastic and deontic attitudes of commitment and entitlement. An expression of a deontic commitment ("shall be punished") cannot serve as a premise for a doxastic commitment ("is a psychopath"), because an expression of responsibility or duty does not justify an expression of belief or a description of environmental circumstances, and vice-versa. In short, the validity of the second kind of inference is not answerable to environmental circumstances and thus does not require an epistemological story explaining the agent's reliable discriminative reporting disposition in using those words, nor they imply claims incompatible with some description of what is the case in the environment.

Keeping in mind this differentiation of kind of inferences, we will have to operate a distinction of *circumstances* and *consequences* of *use*. Following Dummett, I distinguish these as two aspects of the use of linguistic expressions: inferential and non-inferential circumstances which justify or entitle one to saying something, and social consequences of doing so.[28] From this bifurcation, we can treat verification-like assessments of assertibility as regarding the circumstances warranting a saying, and referential-like assessments of truth as regarding the consequences of a saying.

One kind of responsibility is to the inferential order among concepts (we inherit these structures as set by culture and history), and the other is to a horizon of knowledge of environmental circumstances (set by our perception of the world, based on our understanding of everything non-human). These two norms should not be collapsed into one, for some statements are going to be assessed as true-or-false although they may be inferentially unjustified or even unjustifi*able*, such as with: "There are exactly 1,934,285,433 galaxies above my head" — it may be a correct description, but humans do not currently know (and likely *never* will) of a method to accurately count each and every galaxy in that

[28] As it appears in Dummett, M. *Frege's Philosophy of Language*, (New York: Harper & Row, 1973), p. 453.

loosely defined direction, so there are no compatible commitments that could work as premises entitling this claim. Moreover, some other statements may turn out to be inferentially justified, yet not epistemologically justified, as it goes in the classic example of "Look, a barn!" exclaimed by someone driving by a *fake* barn façade.

Moreover, these two kinds of responsibilities shift from context to context for different reasons. For instance, it is inappropriate to conclude "It's because the sun is not shinning" after someone said "It is cold in here", as that claim does not follow from this premise, and thus it does not satisfy the inferential circumstances. Yet, that claim still can meet all potential reasonable epistemological circumstances — the sun could indeed not be shining — and as so it could be used to correctly describe the state of affairs at that moment. But if we change the inferential circumstances and make our interlocutor say "It is suddenly so dark", then the original conclusion "It's because the sun is not shinning" is now inferentially warranted, as the compound of both statements expresses a materially correct inference, and it could, on condition of getting the environmental circumstances right, also be the correct empirical reason for why it is cold. Which norm one has to satisfy, and how, is a matter that depends entirely on the vocabulary deployed and for what expressive role.

One way of operating an analytical distinction between these two kinds of normative correctness is following Huw Price's strategy of separating the concept of representation into two kinds, i- and e-representation. Each one tracks the success of different norms of correctness. *I-representation* tracks what Price calls "in-game answerability", a normative constraint given by the implicit norms of a particular language-game that is equivalent to an assertibility warrant; it is what we look for when exploring the internal consequences of an inferential architecture, assessing whether some claim counts as an inferential consequence of others. And *E-representation* tracks "environmental answerability", a normative inter-subjective assessment of objective representation;[29] it is what we aim at when we intend

[29] *See* Price, H. *Naturalism without mirrors* (Oxford University Press, 2011), pp. 20-23. *See* also "Two expressivist programmes, two bifurcations", in *Expressivism, Pragmatism, Representationalism* (Cambridge University Press, 2013), p. 164.

some feature in a representing system to match the manifoldness and vary in parallel with some feature of a represented system.[30]

While I do not wish to use the term 'representation' here, for the reason that I am already using 'representation' as an effect of social reasoning rather than a norm of correctness, I consider that Price's split does the same work as my intended distinction of inferential and epistemological circumstances and consequences of use. This is because we can project Price's distinction unto Dummett's one, allowing the analysis of the differences between internal and external circumstances and consequences of use.

On the input side of circumstances, we have the first bifurcation of *internal-inferential* licenses (i-conditions), and *external-environmental* entitlements (e-conditions), according to which one is justified to undertake or attribute a commitment or action. On the output side of consequences, we have a second bifurcation of implied commitments and endorsements, further divided into *deontic* on one side (those related to duty), and *doxastic* on the other (those related to the coherence of our beliefs).

	***Circumstances* of use**	***Consequences* of use**
Intra-linguistic	Inferential licenses and entitlements (equivalent to assertibility conditions)	*Deontic* commitments and endorsements (equivalent to truth conditions)
Extra-linguistic	Epistemological-environmental conditions (language-entry transitions: perception, reliable dispositions to respond differentially)	*Doxastic* commitments and endorsements (language-exist transitions: attributing attitudes, actions, beliefs)

[30] In some cases the specification of the inferences available from the application of a concept *coincides* with an specification of the truth conditions for the stated proposition. A good example of this happens with the ordinary empirical statement "It is raining", which allows an inference to the conclusion "The streets will be wet", a true and inferentially warrant consequence of that premise. But our issue is, of course, with those cases in which they do not coincide, and in particular with whether the inferential potential of empirical and mathematical statements coincide. And this adds up another reason to keep the analysis of the satisfaction of these two norms a separate business.

Moreover, an analysis of meaning must also take into consideration the function of use. Thus, to the i- and e-clauses in our analysis of meaning we shall add a third *functional clause* which serves to investigate the act of expression itself, determining the expressive role for which the speaker deployed those words. Thus, we may resume the three clauses[31] of an analysis of meaning in terms of use as such:

i-clause (material-inferential, intra-linguistic, in-game) — concerns the inferential connections of the concepts applied; it captures the normativity of meaning by making explicit (in conditionals) the implicit norms of inference according to which our expressions acquire meaning.

e-clause (epistemological-environmental) — concerns relevant dispositions or attitudes towards how things are in the world despite our will which may entitle one to certain claims or actions involving the relevant concept. Assessments of *objective representation* are assessments of groups of statements that make explicit doxastic commitments and endorsements.

f-clause (function or expressive role) — concerns the functions that linguistic expression may perform in making explicit previously implicit contents, given their particular i- and e- constraints.

On the one hand, when we rectify our linguistic behaviour to satisfy intra-linguistic circumstances, we are *justifying* our deontic commitments. On the other hand, when rectify our linguistic behaviour to satisfy extra-linguistic circumstances, we are *endorsing* a doxastic commitment. The first case is adequacy to internal criteria of correctness that regulate the application of concepts within a discursive practice, whereas the second is adequacy to external criteria that are inter-subjectively given by social reasoning. As such, the i- and e-clauses can be clearly uneven from case to case; certain statements may not satisfy their e-circumstances, failing to inform on the state of non-linguistic items, even though they may

[31] The idea of analysing meaning in these three clauses originates in Michael Williams' paper "How pragmatists can be local expressivists", which appears as a discussion with Huw Price (who separates i- and e-representation) in *Expressivism, Pragmatism, Representationalism* (Cambridge University Press, 2013), p. 134-135.

meet all the relevant i-circumstances. In other words, there is a hierarchy of commitments and entitlements which is fundamental for meaningfulness. To be meaningful, every saying must be first and foremost answerable to the i-clauses of their hosting discursive practice, licensed by intra-linguistic circumstances, as it is according to it that vocabulary items acquire a role in reasoning. A linguistic performance that fails its i-clause will sound meaningless or nonsensical.

Nevertheless, a specification of meaning is not complete without the e- and f-clauses, for even if defined by its inferential position, a vocabulary item can be wrongly applied to the environmental circumstances, or the same item can be used for more than one expressive role. For instance, numerals may be used in the cardinal sense to count, or in the ordinal sense to assign a sequence-order, or even in a non-standard nominal sense to assign names. Therefore, in order to fully understand someone's use of a specific numeral, it is not enough to be aware of its inferential connections, we must also grasp for which function the speaker manifestly used that numeral, and so, whether or not it attempts to conform to environmental circumstances.

Yet not all are environmentally warranted, as they are not required to. After determining the inferential role of an expression, we can further determine whether it demands an observation of environmental circumstances. In general, one will be held accountable to the empirical facts whenever one makes an assertion about extra-linguistic environmental features (e.g. observational reports, empirical claims, scientific hypotheses).[32] A linguistic performance that raises and fails an e-clause is considered false (which, in an inferentialist breakdown, means that the claim is *incompatible* with reliably formed supporting claims held by the community of speakers), yet not meaningless, for it is the satisfaction of its i-clause what fixates the meaning of the performance.

Nothing similar to falsity or meaninglessness happens in the f-clause, for this one does not answer to clear-cut criteria of correction. This is because a linguistic performance *cannot fail* to fulfil its expressive role, as it is the act of performance itself that determines an expressive role for the expression. At most, an expression may fail an intended role; someone may use certain terms for a certain expressive role, yet end up being understood

[32] For instance, if a speaker were to make an observational report using the verb 'running', a condition for correct use of this verb would be that the associated action must be observable in the environment by some agent, but not necessarily the speaker herself. So a speaker may report that her partner is out running without watching the action unfold, just by inferring it from the observation that his running tennis are not in their usual place in the closet.

as fulfilling a different role. Still, this is not a failure on the speaker's part in recognizing the relevant entitling circumstances, but rather that the interlocutors misunderstood the manner in which the speaker had used those terms.

As it can be seen, specification of expressive role is invariably tied to the context of utterance, for linguistic expressions have ranges of possible roles. One can always use the same vocabulary for more than one function, or use distinct vocabularies for the same function, as long as the vocabularies have an overlapping expressive power. That is, the expressive function of an expression is something plastic and malleable, constrained only by the inferential vicinity of the concepts therein chained.

Remitting back to the argument in 4.2, this means that the determination of which expressive role a statement is performing depends on pragmatic properties, not on syntax. The first constraint to expressive role is given by the *i-clause* (which defines the leeway of meaning for the concepts employed); secondly, by the *e-clause* (which defines the environmental context of application); and last but not least, on the intentional authority of the speaker in commanding the use of that expression, an authority that is only cognoscible against the background of a social exchange of responsibilities, commitments and duties.[33] In short, which expressive role a concept plays is something that is constrained by how people intentionally put the concept to use within historically inherited semantic margins.

5.5. Mathematical identities as two-way inferential rules

Now, we are ready to put all the theoretical apparatus to work in the task of understanding the assessment of mathematical statement in terms of truth and falsity, and whether it is consonant with distinctions of truth and falsity in what regards empirical statements. The first difficulty to consider was already spelt out by Wittgenstein; it concerns the superficial similarity that all assertions seem to possess when compared in terms of their truth-aptness:

> Should we not shake our heads, though, when someone shewed us a multiplication done wrong, as we do when someone tells us it is raining, if it is not raining?—Yes; and here is a point of connexion. But we also make gestures

[33] *See* Brandom, R. *Making it Explicit* (Harvard University Press, 1994), p. xii.

> to stop our dog, e.g. when he behaves as we do not wish. We are used to saying "2 times 2 is 4", and the verb "is" makes this into a proposition, and apparently establishes a close kinship with everything that we call a 'proposition'. Whereas it is a matter only of a very superficial relationship. (RFM III, §4).

Indeed, in an unexamined *face value* sense, every assertion that someone might disagree with will look like the assertion of a truth-apt proposition. Furthermore, from the anaphoric interpretation of truth, every asserted statement can be embedded in the prosentence-forming operator '… is true'. In this account, *attributing truth* to a claim entails undertaking a commitment or attributing it to an interlocutor. Yet, this tells us nothing of significance about the possible distinctions in function that '…is true' may have when accompanying mathematical or empirical statements, as it does not deliver an analysis of *which* commitments one is expressing with these statements, not even which species of commitment one holds (whether deontic, doxastic or practical). Therefore, in order to examine the deeper differences between these statements, we will have to assess the inferential and non-inferential circumstances and consequences of their use, and their expressive role.

The central question we must deal with regards what is it that one must know how to *do*, which practices-and-abilities must an agent be able to perform, in order to count as meaningfully asserting that an operation $a \circ b$ yields a particular result c. Furthermore, we must inquire whether this doing is pragmatically distinct from what one must know how to do in order to claim that a certain object x possess a property, as in $P(x)$. That is, when we embed the equation $a \circ b = c$ in the '… is true' operator, does it serve to fulfil the same pragmatic function as in $P(x) = True$.?

As it happens, we do not have to dig much deeper to notice dissimilarities between mathematical and empirical statements creeping out. For instance, we can analyse the attribution of truth to Benacerraf's examples in terms of the 3-clauses, comparing their original formulation with prosentences formed with the truth operator. (To make it visually less messy, I will refer separately to the prosentence T(x) and the sentence x in each case);

***p:* "There are at least 3 large cities older than New York"**

T(*p*): "It *is true* that there are at least 3 large cities older than New York"

	***Circumstances* of use**	***Consequences* of use**
i-clause (intra-linguistic)	Inferences from *p* to T(*p*) are always good, and *p* is entitled by sets of inferentially related collateral premises regarding the age of large American cities.	T(*p*) attributes a deontic commitment to the claim that *p*, and it entitles interlocutors to take the claim *p* as a suitable premise.
e-clause (extra-linguistic)	T(*p*) is justifiable by the verifiable ages of cities and how many are there.	T(*p*) attributes a doxastic commitment to what is described by *p* as a suitable premise or conclusion, given the known collateral premises and auxiliary hypotheses.
f-clause (expressive role)	T(*p*) has the role of generalizing *p* as a suitable claim in social reasoning and communication. And *p* itself has the referential role of identifying anaphoric antecedent singular terms to which the claim *r:* "Older than New York" applies.	

***q:* "There are at least 3 perfect numbers bigger than 17"**

T(*q*): "It *is true* that there are at least 3 perfect numbers bigger than 17"

	***Circumstances* of use**	***Consequences* of use**
i-clause (intra-linguistic)	Inferences from *q* to T(*q*) are always good, and *q* is entitled by sets of inferentially related collateral premises regarding integers and explicit formulaic definitions of perfect and natural numbers.	By claiming T(*q*) the speaker undertakes a commitment to the claim that *q* and entitles interlocutors to claim *q*.
e-clause (extra-linguistic)	T(*q*) is justifiable by rational principles specified in the i-clause;	T(*q*) attributes an endorsement of rational principles specified in the i-clause.

f-clause (expressive role)	T(*q*) serves to generalize *q* as a suitable claim in social reasoning and communication. And *q* itself has the metalinguistic-explanatory role of specifying one incompletely determined consequence from the space of implications (or inferential entitlements) that follows from the definitions of perfect and natural numbers.

In the first case, we have ordinary descriptive statements involving only empirical concepts. When one utters a statement articulating this sort of conceptual neighbourhood for a descriptive function, one is endorsing an expression of how things are, or stand in relation to each other, in a way that does not depend on the inferential articulations of language. We submit an empirical claim to particular norms of epistemic appraisal specified by the *e-clause*, those that evaluate epistemological correctness. We assess the intellectual virtues of a subject who claims knowledge of a state of affairs by taking as criteria how well she noticed the environmental circumstances, and how aware she is of the consequences of her claim.

On the second case, things are not so. When one asserts a mathematical equation in a declarative sentence, we submit it to quite different norms of appraisal, as we do not use mathematical statements to make doxastic endorsements or to change the doxastic commitments attributed to an interlocutor. This is not their function because:

(i) the conceptual relations that mathematical statements serve to establish are *impersonal* (i.e., what is asserted obtains for everyone, everywhere and all the time);

(ii) mathematical statements are not constrained by the state in which we find things in the natural-causal world.

Mathematical statements do not regard what is the case in the external environment, but what *ought to be* in our practices. They are not truths within a timeframe, as are statements with e-clauses to fulfil, which are always contingent on time and place. As so, even though we may utter mathematical statements in the form of declarative sentences which expose a cognitive structure that may be denied or misapprehended, there is no consequence of this use that entails the existence of non-linguistic entities, as their use does not draw rational responsibilities towards external-environmental circumstances (or, is not answerable to the conditions of an *e-clause*).

Overall, the expressivist analysis shows that mathematical statements are distinct from empirical statements in most levels except one. For the differences we have: they acquire meaning in relation to completely distinct practical processes, and thus are mostly used for different expressive roles, drawing distinct use conditions. The only similarity seems to be that both can take the form of declarative sentences, and thus can be asserted in order to attribute commitments and entitlements. Yet an examination of *which* commitments, and *which* entitlements, will demonstrate how semantically inequivalent these expressions are.

Mathematical statements only have inferential circumstances of use, and thus are only evaluated according to the practical consequences of abiding by them. Equations may be said to "correspond to reality" only insofar as our social practices are guided by the mathematical rules. The only reality that "corresponds" to our use of mathematical vocabulary is the particular inferential circumstances which entitles and justifies it, and the practical consequences of following a mathematical rule in order to draw a valid inference. And such inferential networking is relevant only for those who employ the techniques involved and partake in that social practice.

This would effectively make mathematical statements into *analytic statements*, if by 'analytic statement' we understand one that concerns logic or language use and does not provide information about the world, so we do not submit it to norms of epistemic appraisal as specified by an *e-clause*. Its overall meaning can be grasped simply by knowing the inferential role of its component parts.[34]

As flagged out earlier, my interpretation finds its roots in Wittgenstein's idea that mathematical formulae function like grammatical rules. That is, we use these linguistic forms to prescribe criteria of correctness for operations with mathematical concepts. In his words:

[34] That is *not* to say that a mathematical identity is trivially analytic, as its statement is not self-explanatory. '5 × 5 = 25' does not simply tells one "Whatever you call 'multiplying 5 by 5', you ought to call '25'". Unlike with 'bachelor' and 'unmarried man', where if you understand the concepts you know that when one applies the other applies as well, it is possible to first count 5 columns, then 5 rows, but then only 24 units altogether. That is, one can only understand that '25 = 5 × 5' when one knows *how* computing 5 cycles of 5 items *is as* counting 25 items. Acquisition of this sort of practical knowledge is detrimental for the meaningful use of this vocabulary, since the rule only expresses *knowledge-that* '5 × 5 = 25', which cannot inform agents of how to perform the operation.

> The question arises, what we take as criterion of going according to the rule. Is it for example a feeling of satisfaction that accompanies the act of going according to the rule? Or an intuition (intimation) that tells me I have gone right? Or is it certain practical consequences of proceeding that determine whether I have really followed the rule?—In that case it would be possible that 4 + 1 sometimes made 5 and sometimes something else. It would be thinkable, that is to say, that an experimental investigation would shew whether 4 + 1 always makes 5.
>
> It is not supposed to be an empirical proposition that the rule leads from 4 to 5, then *this*, the result, must be taken as the criterion for one's having gone by the rule.
>
> Thus the truth of the proposition that 4 + 1 makes 5 is, so to speak, overdetermined. Overdetermined by this, that the result of the operation is defined to be the criterion that this operation has been carried out. (RFM VI—16, p. 319)

As I read him, Wittgenstein's idea is that formulae serve a normative function which is not to be confused with that of descriptive statements. An equation's function is not to predicate properties of objects, but to prescribe criteria for the correct performance of certain operations.[35] We may say, in a Wittgensteinian fashion, that the surface grammar of equations is misleading. Since we lend ourselves to the form of declarative sentences to express them, they may appear to be similar to bipolar true-or-false empirical statements, but in practice we follow 2 + 3 = 5 *as* commanding "Let 2 + 3 be 5", or "By definition, 3 + 2 is 5", and sometimes even as a step in an algorithm: "Compute 2 + 3 to get 5". It is

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[35] As Wittgenstein considers in this passage, for example: "The justification of the proposition 25 × 25 = 625 is, naturally, that if anyone has been trained in such-and-such a way, then under normal circumstances he gets 625 as the result of multiplying 25 by 25. But the arithmetical proposition does not assert *that*. It is so to speak an empirical proposition hardened into a rule. It stipulates that the rule has been followed only when that is the result of the multiplication. It is thus withdrawn from being checked by experience, but now serves as a paradigm for judging experience. // If we want to make practical use of a calculation, we convince ourselves that it has been 'worked out right', that the correct result has been obtained. And there can be only one correct result of (e.g.) the multiplication; it doesn't depend on what you get when you apply the calculation. Thus we judge the facts by the aid of the calculation and quite differently from the way in which we should do so, if we did not regard the result of the calculation as something determined once for all." (RFM VI—23, p. 325)

not a description of a state that some objects happen to be in, but as a prescription of a two-way criterion of correctness, both for what counts as a correct computation of 2 + 3 and for what counts as counting up to 5. Formulae prescribe what one *ought to* conclude in case one computes some operations correctly.

Based on Wittgenstein's idea, we may say that the expressive role of mathematical formulae is a *metalinguistic-explanatory* one, of making explicit implicit inferential relations forged in social normative practices (e.g. practices associated with a theoretical framework, or system of calculation defined in a mathematical theory), according to which we use the items of mathematical vocabulary. Their role is to specify the interconnections between these linguistic expressions by prescribing that some operation of such and such *is as* — is *corrected by* and *corrects* — some other operation of such and such. The equation symbol connects the application of these concepts,[36] licensing an agent to substitute the symbols on one side for those on the other, on condition of computing the operations on each side.

As a general case, we may say the operation '$a \circ b$' is the criterion of correction for the operation '$c \circ d$', and vice-versa[37]:

$$a \circ b \rightleftharpoons c \circ d$$

Mathematical identities function as *inferential return-tickets*, or two-way inferential transitions, licensing a computing agent to substitute the items on one side for the other, or to make a claim involving them, provided that the agent executes the relevant operations with an effective procedure. As such, the expression of a mathematical identity in the logical form of an equation fixes a range of inferential licenses, determining the inferential potential that follows from the applicability of a mathematical concept. Equations trace out and regulate the transit through inferential networks in which math concepts are caught up.

This claim does not imply that mathematical vocabulary serves exclusively a metalinguistic-explanatory role. After all, mathematical vocabulary is also used in descriptions, such as in "Jupiter has 4 moons", "Proceed to gate 42", or in "Is the second

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[36] I say mathematical formulae make explicit *conceptual* articulations, in place of merely symbolic transformations, because when one knows that 2 + 3 = 5 establishes the criterion of correctness for the computation of 2 + 3, one also knows that II + III = V equally expresses that standard. The mathematical rule draws the inferential potential from the applicability of instances of the concept of integer instead of merely their particular symbols.

[37] This interpretation of Wittgenstein is masterfully presented by André Porto in his "Wittgenstein and Mathematical Identities", in *Disputatio*, vol. IV, n° 34, 2012.

door on the right”. The claim only states that formulae fulfil this expressive role, only those expressions of identity that obey the syntactical rules of mathematical theories.

Nevertheless, it does follow from this claim that the descriptive applications of mathematical vocabulary acquire meaning insofar they respect the inferential role of those items in the mathematical practices. For the use of numeral to be intelligible in expressions such as “4 moons”, “gate 42” and “2nd”, these uses *must* be in accordance to all inferentially related equations in which figure the terms 4, 42 and *SS'* 0 (i.e. the successor of the successor of zero). All meaningful deployment of mathematical vocabulary is tributary to that item’s inferential role in the mathematical discourse.

As an implication, in my reading, asserting the compound statement “I have two pounds in my left pocket and three in my right pocket, therefore I have five pounds in total” is **not** semantically equivalent to asserting ‘2 + 3 = 5’. The compound statement is about the amount of coins in one’s pockets, whereas the equation is about a metalinguistic connection between the operation of 2 + 3 and the operation of counting up to 5 items. From left to right, it gives *the only criterion* of correctness for having performed the sum 2 + 3, and from the right to the left, it gives *a criterion* of correctness for the employment of the number 5.

An equation serves to express what *ought* to follow, inferentially, when one performs such-and-such operations on such-and-such concepts — so our attitude towards it is to follow it like the rule for what one ought to obtain just in case one has calculated correctly. To assert an equation is to put forward criteria of correctness for operations on certain mathematical concepts, something according to which we judge the correctness of assertions involving those concepts. As Wittgenstein would put it, rules of inference are constitutive of the criteria according to which we judge statements as true or false:

> We can conceive the rules of inference — I want to say — as giving the signs their meaning, because they are rules for the use of these signs. So that the rules of inference are involved in the determination of the meaning of the signs. In this sense rules of inference cannot be right or wrong. (RFM VII, §30)

The meaning of a rule of inference does not depend on any particular state of affairs we may discover, but on the very intelligibility of the concepts applied, as it makes explicit the correlations in which the objects that fall under those concepts normally stand in. I may correctly count 25 peanuts in front of me and then state that they could be rearranged into 5 rows and 5 columns. But when I offer the math statement '$5 \times 5 = 25$' as the reason behind my doing, I am not showing you the result of an experiment I made by performing some permutations on the set of peanuts, I am stating that the performance of the operation on the left-side constitutes the criterion of correctness for the operation on the right-side, regardless of the counting unit, thus normalizing the implementation of that operation by any computing agent.

It is through the practical performance of operations that we realize an identity as the rule that makes explicit the implicit norm of correlation between applications of concepts. The fact that every rational interlocutor endorses a claim such as '$2 + 3 = 5$' indicates that this expression succeeds in making explicit what we *do* in practice with those concepts — it successfully expresses a two-way inferential connection which operates as a norm on how we use those concepts. As so, the equation offers necessary criteria which the inference made in that compound statement had to fulfil in order to be intelligible. If I had said instead: "I have two pounds in my left pocket and three in my right pocket, therefore I have **six** pounds in total" I would stand in need of correction, for, as Wittgenstein explains: "The rule doesn't express an empirical connection but we make it because there is an empirical connection" (LFM, p. 292). When you say you have counted two and three items, you ought to say you have five items in total because this is how these concepts relate according to the rational principles and normative practices which underpin the human culture of counting.

The mathematical rules provide *relative definitional constraints to* the *use* of math concepts, not explicit definitions.[38] Mathematical vocabulary is usually defined from within symbolic frameworks and the associated practices and theories, or implicitly by the axioms of a system. And the derivation of new theorems (new mathematical rules) enriches the meaning of the old terms by constraining their correct employment.

We may conceive the mathematical identity as an expression of *knowledge-that* which codifies the norm for performances of mathematical operations, and as such it only governs what you *ought* to write at the end. The knowledge as to *how* will depend on the effective procedure (e.g. the algorithm or diagram) implemented by the computing agent. If

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[38] The idea of treating mathematical rules as providing "relative definitional constrains" for the application of mathematical concepts in empirical contexts is due to André Porto. See his "Wittgenstein on Mathematical Identities", in *Disputatio*, vol. IV, n° 34, 2012.

the computing agent has the appropriate *know-how* to perform that operation, this agent ought to get said result. And it is on this condition that the sense of the mathematical statement is fixed: in function of the practical performance of a computing agent that carries out those calculations to see how performing one *is as* performing the other.

I take that *operation* as a primitive notion which defines the grammar of mathematical statements. By 'operation', I mean transformations, morphisms, and functions operated on distinct mathematical concepts. The symbol for a mathematical operation designates a performance that must be completed, a *computation* to be carried out by a computing agent using an effective procedure, such as an algorithm or diagram. And the reason to make it into a primitive is that it is the practical effective computation of operations on mathematical concepts what allow us to establish the non-trivial identities that we state by means of equations. It is their practical performance that establishes metalinguistic connections between instances of concepts which in turn act as relative definitional constraints for any employment of that concept.

Furthermore, by prescribing that carrying out the operations on one side *is as* carrying out the operations on the other, a mathematical identity also provides constraints that regiment the use of the symbols and linguistic categories figuring in it. It is according to these rules that we judge the correctness of empirical statements applying those mathematical concepts. That is, the equation regulates the employment of its concepts, both within and outside of mathematics, whenever the symbolic framework to which it belongs is employed. In this way, mathematical statements determine criteria according to which we qualify or disqualify empirical claims regarding quantities, proportions or spaces. Or, as better says Wittgenstein:

> To be practical, mathematics must tell us facts. — But do these facts have to be mathematical facts? Why should not mathematics instead of 'teaching us the facts' create the forms of what we call facts? (RFM VII, §18)

> You might say: Mathematics and logic are part of the *apparatus* of language, not part of the application of language. It is the whole system of arithmetic which makes it possible for us to use '900' as we do in ordinary life. It *prepares* '900' for the work it has to do. (LFM, p. 250)

My way of developing his ideas take a cue from Brandom's examination of the expressive roles of normative, logical and modal vocabularies. He categorizes them as pragmatic metalanguages, as they have the expressive resources to talk about certain ways of *using* language and the proprieties that govern each. My interpretation is that mathematical vocabulary plays a similar role in relation to empirical vocabulary. A mathematical statement makes explicit *some* inferential relations related to the application of certain math concepts, including ordinary empirical descriptive statements employing that mathematical vocabulary. The mathematical statement captures a norm of practice according to which we use the concept, functioning as a rule prescribing certain interconnections between concepts.

Now that we have set out an expressivist interpretation of mathematical statements, we may conclude this chapter by contemplating that attributing truth to a mathematical statement should not lead to the epistemological and metaphysical problems that Benacerraf draws from it. Correct use of mathematical vocabulary does not require nor involve representation of extra-linguistic mind-independent objects. In fact, it does not depend on any condition that is not practical or inferential. The use of 'truth' in this sense of *i-correctness* is not equivalent to a substantially mind-independent conception of truth. Therefore, mathematical statements are not true in the sense that matters for Benacerraf, as a norm that tracks reference to an external reality, a norm for marking the success of an external-environmental representation.

Yet, still, we can satisfy Benacerraf's semantic condition, maintaining semantic homogeneity under an anaphoric account of truth claims. Moreover, the anaphoric interpretation empties away all pretensions to the ontology of a truth claim, as there will be no property to connote the existence of underlying objects. Taking a statement as true does not entail its content represents real existing things, only that what it says can aptly play a role in reasoning as premise or conclusion.

Futhermore, the expressivist account is immune to the argument used against combinatorialism, since it does not entail the collapse of a specification of truth conditions unto assertibility conditions, and this distinction is replaced by that of circumstances and consequences of use.

In sum, the pragmatic expressivist perspective offers an alternative to both combinatorialist and platonist options which can accommodate what is right in both analyses: that statements deploying empirical vocabulary in order to describe raise a rational obligation towards how things are in the environment, and statements deploying mathematical vocabulary in order to establish metalinguistic connections raise a rational obligation towards inferential coherence.

6. Mathematical knowledge in a pragmatic key

To recapitulate, I have argued that the epistemological horn of Benacerraf's dilemma is blunted if we reject a representationalist view of language. We must step outside the representationalist view of language and pragmatically examine the practical circumstances and consequences related to the use of linguistic expressions. In its place, a philosophical account can proceed via a social pragmatic subject naturalist approach which is capable of satisfying Benacerraf's epistemological condition while working around his epistemological presuppositions. This approach offers a homogeneous treatment of meaning across all discursive practices while not treating attributions of truth as designating an epistemically unconstrained property such as correspondence to reality.

However, the semantic horn still challenges us to explain why we have the feeling that mathematics deals with truths independent of the human mind. What is that makes applying mathematical frameworks so successful and profitable for scientific inquiry? If our mathematical theorems are rules of inference, then would this not undermine the objectivity of scientific knowledge based on mathematical techniques? That is, would not a creature that thinks and infers differently from us follow different mathematical theorems?

Benacerraf assumed that platonism provides the better answer: the compulsion and effectiveness of mathematics would be due to the necessary connections between the self-subsisting abstract entities it studies. What we mean with the use of mathematical statements holds for every possible world, as we cannot *conceive* any scenario in which they would not. So the correct solution for a mathematical operation holds *inexorably* — it could not have been otherwise, as its computation cannot be yielding *accidental* results and still count as a mathematical operation — or, as one may say borrowing Popper's terms, mathematical theorems are unfalsifiable.[1]

Imagine we witness a carpenter measuring the internal angles of a flat triangular-shaped table with his trusty protractor, only to turn around and claim those angles added up to 170º. Now, would you say his experiment falsifies the geometrical theorem that the sum of the internal angles of a triangle in a flat surface adds up to 180º?

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[1] In Popper's terminology, an unfalsifiable statement is also non-scientific and non-empirical, for it cannot be tested nor verified in experiments.

Probably not. Most likely, you would not take this as the result of a proper application of Euclidean geometry, nor as a proper computation of the sum of the internal angles of a triangle. Way before doubting the geometrical principles we use when reasoning about shapes, one would probably suppose a mistake was made by the carpenter (perhaps he does not fully understand the geometry, or his instruments were inadequate for the task); we do not know precisely what caused the mistake, but we would surely never put the theorem in doubt because of it. After all, what use would we have for a rigid system of calculation if the practical implementation of these operations yielded elastic results which in some weird days identify the sum of the internal angles of a triangle as more or less than 180º? Mathematical statements are not like that, they are not about operations that yield accidental results; the content asserted is impersonal, the result of the operation is necessarily so, always, everywhere, for everyone.

This purported necessity of mathematical operations is usually explained in terms of independence from human inferences, as a kind of *a priori* knowledge of abstract entities or potentials. As we have seen in Chapter 4, platonism and combinatorialism portray mathematical operations as designating abstract processes determining their correct result independently of any practical performance. The difference for each side being that, in platonism, the inexorability of the statement reflects the immutability of the facts of a mathematical reality, whereas in combinatorialism the inexorability of the statement reflects the unilateral abstract potential that the effective procedure has to determine that fact.

As I see the matter, the real philosophical issue here lies in what to make of this inexorability, how to comprehend such compulsion. In Chapter 5, coming from the semantic side of the matter, I have proposed to read mathematical formulae as inferential licenses that prescribe two-way criteria of correctness for having performed math operations. Now, in the epistemological side, my argument is that this proposal naturally extends to a pragmatic account of knowledge.

The epistemological wing of representationalism is the philosophical view that one's conscious experience is made of mental representations of external reality that are filtered by one's sensory organs. The world of our experience is not the world in itself, but a replica, mirrored on the properties of the real one — a miniature film produced by (and played for) a lonely brain. The mind gets to know objective reality by creating some sort of mental map of its states, and is able to communicate with other minds by evoking these mappings.

In a pragmatist outlook we replace the traditional pictures of mathematical activity as the scientific investigation of abstract mind-independent structures, or of the abstract potentials of proof methods — the two conceptions that brought us to the horns of Benacerraf's dilemma — with a picture of mathematical activity as a social-normative practice that serves the expressive and pragmatic purposes of humans.

Pragmatism typically starts with the rejection of Cartesian dualism and the tradition that inspired such "spectator theories of knowledge".[2] The first pragmatists, William James, John Dewey, and Charles Peirce, all rejected Descartes' principles for substituting a description of what is readily liveable by anyone (i.e. thinking) with an artificial analytical abstraction (such as "I think therefore I am"). As Bruce Wilshire explains, the pragmatic movement rejected intellectual rationalizations of experience, because it departs from the principle that living experience is not in need of theoretical foundations:

> Thinking that there are discrete mental contents or elements results from an initial reflection and analysis that forgets itself. It smuggles itself in and falls asleep. Mental contents—or so-called "sense data"—are not the building blocks of our minding life, the pragmatists maintained. Rather, they are the by-products, the artifacts, of the analysis that forgets itself.[3]

Endorsing the pragmatist's point, throughout this thesis I have been calling practitioners of mathematics as *agents* instead of subjects. The reason for this vocabulary choice is now evident: just as the shift from representationalism to pragmatic expressivism is marked by abandonment of a *passive* picture of our role in the creation of meaning in favour of an *active* one, so is the shift from "naturalized epistemologies" and "spectator theories of knowledge" to a pragmatic conception of knowledge.

The issue with these theories is similar to the misconceptions of the representationalist account of meaningfulness: ignoring the agent's pivotal role in the constitution of knowledge. This is shown by how these theories hold on to the idea that we can contemplate matters with absolute impartiality, from a disengaged perspective, as if our minds were disembodied. As so, the particular biological constitution, intellectual virtues,

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[2] *apud* Josh Whitford & Francesco Zirpoli, "Pragmatism, practice, and the boundaries of organization" in *Organization Science* (published online), June 24, 2014.

[3] Wilshere, B. *Fashionable Nihilism* (State University of New York Press, 2002), p. 11.

and biases of the knowing agent are disregarded as subjective and thus not explicitly concerning the objects known. However, perception and cognition are not passive reflections of nature, as these are actions conducted for implicit or explicit reasons, according to a purpose.

Pragmatists abandon the conception of knowledge as the result of subjects passively observing the world in favour of the conception of knowledge as the result of agents actively engaging with the world; probing, testing, and dressing the world's furniture with meaning, then discussing about things with words, assigning these words inferential roles to play in our explanations of how the world works — knowledge as the product of sapient agents engaging in social practices of inquiry, investigation, reflection and explanation.

As such, pragmatism has no use for the platonic distinction of true knowledge from mere opinion (separating *episteme* from *doxa*), or between appearance and reality, is corroded in the pragmatic approach, for there will be no absolutely correct method to know the truth. Each and every methodology operates within a cultural and historical horizon. As times and cultures change, so too our tools to see and engage with the world change, and thus our agreement over the basic facts — the self-evident; or what our community understands as logical simplicity — shifts ever so slightly.

This lack of a rigid boundary between objective truth and subjective opinion implies that pragmatists do not regard Truth as the only worthwhile goal of philosophical reflection. To know the state of the world is detrimental to our survival, yet it is only one amongst other ways in which we can cognitively enrich ourselves. Pragmatists worry just as much about understanding diverging possibilities of expression, or grasping distinct possibilities of explanation. For, after all, truth is just an anaphoric device to endorse and generalize statements; it is thus more relevant to understand how we arrive at the statement, what one must do to conceive of it, than its truth-value. Thus, pragmatists generally eschew epistemological foundationalism in favour of a hermeneutical approach, one which seeks to understand all possibilities of expression, and not just those of a privileged naturalistic vocabulary. The goal of a hermeneutic philosophy is not some mythical absolute truth, but *edification* (or *Bildung*, in Gadamer's sense)[4, 5] of individuals and communities.

In such a view, a claim that an agent possesses knowledge should not be regarded as describing a mental state, but as placing that agent's commitments (or justification of commitments) in the logical space of implications, in which interlocutors can demand justification, reason, or evidence, if and when required. It follows from this that an expression of knowledge is considered *justified* when it coalesces with sets of compatible claims. To break down what an agent knows, we have to examine her understanding of how intra-linguistic-inferential and environmental-epistemological circumstances may entitle her claims and actions. This may be taken from a pool of antecedent claims, since the inferential practice is social discursive; that is, in the practice of articulating premises and conclusions as related to expressions of attitudes of entitlement or commitments, we learn to take each other's claims and sayings in general as reasons for other claims or sayings.

The paradigm definition of knowledge as *justified true belief* is understood as an *inferentially entitled commitment*. Thus I am conceiving knowledge as a comprehensive system of thought formed from exchanges of claims in social reasoning. The idea is, in Rorty's words, to "see knowledge as a matter of conversation and of social practice, rather than as an attempt to mirror nature."[6] Generating and acquiring knowledge are synonymous with gaining wisdom and refining one's intellectual virtues.

Now that we know of the possibility of a pragmatic account of knowledge, we must ask: why did Benacerraf consider that *only* a causal account of knowledge could provide us with the desired universal coherency and intelligibility regarding the concept of knowledge?

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4 *See* Rorty's "From Epistemology to Hermeneutics", chapter VII of *Philosophy and the Mirror of Nature*. He brings Gadamer's concept of *Bildung* to the pragmatist fold translating it to *edification*. For both, it means the gradual constitution of one's knowledge, character and worldview through education, contact with past wisdom, cultural practices, living experiences, and also through expanding a synoptic view of the possibilities of thought and expression (i.e. of the indefinitely many ways there are to explain something or describe oneself).

5 *See* Hans-Georg Gadamer, *Truth and Method*, 2nd ed., translation by Joel Weinsheimer and Donald G. Marshall, London: Continuum, 2004.

6 Rorty, R. *Philosophy and the Mirror of Nature* (Princeton University Press, 1979), p. 171.

6.1. Rejecting Benacerraf's epistemological presupposition

To remind the reader, according to the Causal Theory of Knowledge (CTK): "*S knows that p* if and only if the fact *p* is *causally* connected in an 'appropriate' way with *S*' s *believing p*."[7] So, what is the reason for Benacerraf demanding a causal explanation of mathematical knowledge? Perhaps that is due to the allure of CTK at the time, right before major issues were found in the theory.[8] Yet, judging by what Benacerraf wrote, this seems to be justified by this thought: In order to objectively recognize what is true, truth-makers must casually impinge on us a stimulus; we must passively recognize their existence and properties, and not actively invent them. So perhaps instead of revising the arguments against the particular theory that he endorses, we should focus on the *genre* of this theory.

According to Shapiro, CTK may be seen as "an instance of a widely held genre called 'naturalized epistemology,' whose thesis is that the human subject is a thoroughly natural being situated in the physical universe."[9] On the lines of Price's distinction of subject and object naturalisms (*see* introduction to chapter 5), this would fit squarely with the latter, a view according to which the only entities that exist are those specifiable in a semantically privileged naturalistic vocabulary which in this case are the vocabularies of causation and fundamental physics.

So Benacerraf thought that in order to be considered knowledge, the produce of our mathematical practices must be explainable through the same routes and mechanisms as our acquisition of empirical-scientific knowledge. Otherwise, those practices would not be producing knowledge, only human artefacts; they would be closer to an art form than a

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7 Alvin Goldman, "A Causal Theory of Knowing" *The Journal of Philosophy* 64.12, 1967, pp. 357–372.

8 Gettier problems have exposed the fragility of the justification clause for knowledge, when knowledge is understood as justified true belief. Since then, theories which rely heavily on a connection between knowing subject and object, as CTK does, have been seriously challenged. A close examination of CTK in particular shows that this theory cannot account for expressions of knowledge of future events, e.g. the sunset of tomorrow, as there is no causal chain from it to our current state, so we cannot claim to know when it is going to happen or how it is going to look like. Besides future events, knowledge based on universalisations and generalizations would also pose a similar problem for the same reason of lacking a causal connection.

9 Shapiro, S. *Philosophy of Mathematics: Structure and Ontology* (Oxford: Oxford University Press, 1997), p. 110.

science. Benacerraf acknowledges mathematical knowledge only insofar as its acquisition can be related and explained through a connection with naturalistic empirical methods.

In my assessment, mathematical knowledge is severely misconceived when seen through the lenses of CTK because math formulae are not expressions of belief or mental states caused by knowledge of mind-independent facts. *One does not believe* '2 + 3 = 5' as one would believe a testimony or a description, one rather *follows it like a rule* — the difference is crucial to understand the role of a mathematical formula. The driving force behind my rejection of employing object-naturalistic epistemologies to explain mathematics is Wittgenstein's remarks concerning the normativity of rule-following behaviour in general.

The original locus of these is around PI §185,[10] in the case of the student tasked with continuing a series [$\lambda n.\ n + 2$] for the natural numbers ℕ, who, however, after reaching 1000, continues the series as '1004, 1008, 1012'. So when the student was given that rule of calculation to continue, was she at that moment linked to an abstract process pointing to the correct outcome of that exercise, inevitably foreshadowing the yielding of the correct series? Or, in other words, does the necessity of the result comes from a determination of what *can only* happen, or of what *should* happen? The first cannot be the answer since no one would be *causally compelled* to write the correct answer. By itself, a rule cannot move a computing agent, it can only inform the agent of which result ought to obtain just in case the agent has performed the described operations correctly.

To explain the reasons for the student's deviant behaviour we have to understand that [$\lambda n.\ n + 2$] is not an expression of knowledge of something causally necessitated, as for one to understand and operate with this expression, one must grasp a rational obligation. This is because mathematical formulae do not serve to express what is *casually* determined to happen in a given situation with the force of an 'is'. A formula does not serve to express what must be — in the sense of a natural-metaphysical necessity — it instead expresses *how one ought to proceed*, which *telos* a computing agent should aim at when performing that operation in order to count as participating in the mathematical discursive practice. v That

[10] Wittgenstein writes: "We say to him: 'Look what you've done!' — He doesn't understand. We say: 'You were meant to add 2 look how you began the series!' — He answers: 'Yes, isn't it right? I thought that was how I was meant to do it.' — Or suppose he pointed to the series and said: 'But I went on in the same way.' — It would now be no use to say: 'But can't you see....?' — and repeat the old examples and explanations for him again. In such a case, we might perhaps say: this person finds it natural, once given our explanations, to understand our order as *we* would understand the order: "Add 2 up to 1000, 4 up to 2000, 6 up to 3000, and so on." (PI, §185).

one ought to write 1002 after 1000 when continuing [λ*n*. *n* + 2] is not due to causal necessitation, it is a practical commitment to a norm of use and a rule of reasoning. Formulae serve to attribute *practical* commitments and entitlements with the force of an 'ought', in the sense of what sapient agents are *rationally* obliged to be capable of doing in order to be counted as using mathematical language to think and say something.

So to say that I understand how to proceed in calculating with a formula is to say that I am able to perform operations according to the rules expressed by the relevant mathematical statements. This means there is always room for deviation from a series of inferences, for even in simple cases such as the student above "somebody may reply like a rational person and yet not be playing our game." (RFM I, §115). That is, the student could well have responded to her teacher: "However many rules you give me — I give a rule which justifies *my* employment of your rules." (RFM I, §113). So the result of a math operation could not be pre-determined by force of mind-independent natural mechanisms.

Wittgenstein's rule-following considerations show us that the performance of mathematical operations is not set on rails — there are no metaphysical grounds for the correct continuation of a series of inferences, no natural necessity raising a factual obligation for our student to continue that series normally. To think that mathematical formulae concern what necessarily must happen is either a metaphysically charged way of thinking about our very own *expectations* or, at most, to confuse the termination clause of an algorithmic procedure with metaphysical predetermination.

To better explain this latter point, let us return to the carpenter's example above. Why is it that the most likely response to the carpenter's mistake is to look for an error in his execution, instead of an error in the geometrical theory? My answer is that the expression of the geometrical rule sets the criterion of correctness for the sum of a triangle's internal angles, and it is according to this criterion that we judge whether or not a measurement of a triangle's internal angles was good; we *do not* judge the theorem according to measurements of triangular-shaped objects. To doubt this theorem based on a carpenter's measurements would indicate lack of understanding of what a flat triangle is, as the table he was cutting could not *conceivably* have this shape and its internal angles not add up to 180º.

An important lesson to take out of this case is that there is no amount of empirical evidence that would falsify a criterion of correctness for the application of a mathematical concept. While the truth of empirical statements that involve the application of 'triangle' may be checked for a possible misuse of the terms or misapprehension of the facts (as it happened with our carpenter), mathematical statements regarding the properties of triangles are not subjectable to empirical test or verification. The mathematical statement is insulated from contingencies of causal change, as from the moment it is uttered to fulfil a metalinguistic-explanatory expressive role, it is passed beyond such tests and is hardened into a rule. Followed as a rule, the mathematical statement then plays a role in constituting a frame for possible descriptions which will be judged as correct or incorrect in accordance to this frame.

One is only correct in calling a shape 'triangular' if the internal sum of its angles follows the rule that says it *ought* to be identical to 180º, as these rules define the role of what we call 'triangle' in the framework of Euclid's geometry. So triangle and the property of being triangular are theoretical constructs which provide relative definitional constraints for talking about triangular-shaped objects. At no moment a consideration of triangles as abstract objects is required in order to report that an object is of a triangular shape since grasp of the role that the word 'triangular' has in reasoning does not require the satisfaction of an extra-linguistic condition (or, it has no e-clause).[11]

By saying statements which express inferential licenses are not based on a metaphysical mind-independent causal structure, I am following, Wittgenstein[12], Kant[13] and

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[11] Sellars made the same point in this passage: "[…] while my ability to use 'triangular' understandingly involves an ability to use sentences of the form '—is triangular' in reporting and describing matters of physical, extralinguistic fact, my ability to use 'triangularity' understandingly involves no new dimension of the reporting and describing of extralinguistic fact—no scrutiny of abstract entities—but constitutes, rather, my grasp of the adjectival role of 'triangular'." (Sellars, W. "Grammar and Existence: A Preface to Ontology", §XIV, *apud* Brandom, R. *From Empiricism to Expressivism*, Harvard University Press, 2014. p. 238).

[12] "The only correlate in language to an intrinsic necessity is an arbitrary rule. It is the only thing which one can milk out of this intrinsic necessity into a proposition" (PI, §372). And: "The avowal of adherence to a form of expression, if it is formulated in the guise of a proposition dealing with objects (instead of signs) must be 'a priori.' For its opposite will really be unthinkable, inasmuch as there corresponds to it a form of thought, a form of expression that we have excluded." (Wittgenstein, Ts 220, §911; apud Oskari Kuusela, *The Struggle Against Dogmatism: Wittgenstein and the Concept of Philosophy*, Harvard University Press, p. 104).

[13] Brandom explains that: "[…] by 'necessary' Kant means 'in accord with a rule'. […] So for Kant, concepts are to be understood by the theorist in terms of the *rules* that make them explicit, rules that specify how the concepts are *properly* or *correctly* applied and otherwise employed." (Robert Brandom, *Making it Explicit*, p. 10).

Rorty,[14] for whom there is nothing more to claiming that the application of a concept necessarily follows from the application of another than to say that it follows according to a rule. Statements whose truth is deemed necessary and whose negation is nonsensical, such as the mathematical, indicate what follows from what with such certainty that we may think they are descriptions of mind-independent connections that preceded our means to effectively know about them. As Wittgenstein pointed out, these statements do not give room for the possibility of things being otherwise, their sense opens no possibility for falsity.[15] As so, these sentences do not present us with two mirrored possibilities as the case (either *P* or ¬*P*) for us to consent or disagree with, but with forms of presentation we may use in our sayings. So the conditions for the negation of a "necessary statement" are inconceivable because the normative constraints asserted by that statement are constraints on reason itself. These are frames for possible pictures, not pictures themselves. They do not serve to state something testable, but instead to state a rule of testing.

So we may comprehend the non-temporality, impersonality, and inexorability of stating the result of a mathematical operation as that of a rule, making explicit certain implicit connections between the application of concepts in practice. The meaning of such a statement is determined entirely by its inferential circumstances and consequences of use, it does not require consideration for anything external to the mathematical calculi. So there are no particular mind-independent objects being studied in mathematical activity, and thus no epistemological difficulties in specifying the use conditions of a mathematical statement.

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[14] "The idea of 'necessary truth' is just the idea of a proposition which is believed because the 'grip' of the object upon us is ineluctable. Such a truth is necessary in the sense in which it is sometimes necessary to believe that what is before our eyes looks red—there is a power, not ourselves, which compels us. The objects of mathematical truths will not *let* themselves be misjudged or misreported." (Rorty, *Philosophy and the Mirror of Nature,* Princeton University Press, 1979, p. 157-158).

[15] In Wittgenstein own words: "A proposition which it is supposed to be impossible to imagine as other than true has a different *function* from one for which this does not hold" (RFM IV—4). And: "I can't imagine the opposite of this" does not mean: my powers of imagination are unequal to the task. These words are a defence against something whose form makes it look like an empirical proposition, but which is really a grammatical one." (PI, §251). Furthermore, I refer to this explanation by Luiz Henrique dos Santos: "To characterize the proposition as bipolar is to understand that the essence of propositional representation resides on this choice, in the privilege that is attributed by means of the proposition to one of the poles of one alternative in prejudice of another." (Luiz Henrique dos Santos. 'A Essência da Proposição e a Essência do Mundo', introduction to the Brazilian edition of Wittgenstein's *Tractatus Logico-Philosophicus* (São Paulo: Edusp, 1994), p. 55.

It follows from this that the only possible source to *guarantee* a mathematical result obtains as the same every time can only be *how* we engage with the world while thinking of it in mathematical terms from a frame constituted beforehand by our form of life and culture. There is nothing else standing behind the utterance of the formula than the training and mastering of techniques for the effective computation, whose understanding is shown when one puts the relevant technique to use in the appropriate context.

The inexorability and reliability of mathematical statements — or, the reason why we want to call them "necessary truths" — is, first and foremost, the fruit of our adherence to them as rules capable of expressing the norms we think and live by. And secondly, the fruit of us not allowing exceptions to them because we cannot think the concepts they govern outside of the boundaries of sense prescribed by these rules. So a causal explanation of mathematical knowledge is inappropriate, as it would misconceive this kind of knowledge that has nothing to do with causes-and-effects. The compulsion of math statements is due to normative constraints we adopt as a community of practitioners of the mathematical practices, to regulate our use of mathematical vocabulary.

To summarise, the argument presented in this section has the following steps:

1. According to CTK, a subject possess knowledge just in case her beliefs are causally related to the facts;
2. However, mathematical formulae are not expressions of belief in what *is*, *was* or *will be*, rather these are expressions of what a computing agent *ought to* conclude from certain operations. It is not natural causation, but rational obligation what constitutes the inexorability of mathematical formulae.
3. Therefore CTK is unsuited to explain the rule-following aspect of our mathematical practices.

6.2. Mathematical activity as a manifestation of the human form of life

In Chapter 5 I have developed Wittgenstein's idea (that mathematical statements function like grammatical rules) into a full-blown expressivist thesis: the expressive role of a mathematical formula is to make explicit — in a form that can be manipulated systematically in a calculus — the inferential articulations in between applications of mathematical concepts. Now it is time to ground this expressivist thesis in a suitable pragmatic explanation of mathematical knowledge. Let us start by clarifying the fundamental pragmatic distinction between norms and rules.

Rules are statements which serve to make norms explicit; they evoke an explicit kind of rule-following, where agents are capable of reading and interpreting the rule so they may act accordingly. There is no distinction between an expression having the status of a rule and our application of it — a statement only gets *treated as* a rule of inference because of application of the normative practices which it governs. Or, the expression of a rule only is taken as the rule because it makes explicit the norm of how we act. Having a rule in this sense amounts to behaving or acting according to the norms of social custom. Or, as Wittgenstein puts elsewhere: "To follow a rule, to make a report, to give an order, to play a game of chess, are *customs* (usages, institutions)" (PI, §199). We follow formulae as rules prescribing criteria of correctness for the involved operations, so long as these rules properly express the norms guiding our behaviour and practices.

Norms, on the other hand, can be lived by without being spelt out explicitly by or to an agent, for a norm is the result of fixing a normal or standard performance for an activity. One may say "You should not have done that" indicating the breakage of a norm, yet that does not give a clear indication of what should be done in that situation. Hence why we need rules — so we can shed light and debate the norms guiding our social practices — as there are many norms guiding our linguistic behaviour of which we are not fully conscious of (e.g. regarding tone, gestures, facial expressions, etc.). Norms appear whenever agents engaging in a practice *act* consistently and coherently *according* to the commitments they undertake or hold others accountable for (akin to Brandom's deontic scorekeeping model of communication).

Contrasting with a regulist interpretation, according to which our linguistic behaviour while making judgements is governed by explicit rules of conduct, the pragmatic interpretation understands that the validity of rules is not based on more conventions or stipulated principles, but rather on a constancy of conduct, a normality that emerges from repeated cycles of performance and correction. Norms are socio-cultural patterns, regularities that emerge from the customs and habits of a linguistic community. Our own grasp of the norms for participation in our traditional discursive practices remains mostly *intuitive*, i.e. not open to rationalization until we express them in an *assertoric form* proper for the inferences of an autonomous discursive practice.

As so, the standards prescribed by mathematical statements are not followed because of some established convention, or because of the authority of mathematicians in these matters. These are the rules, in place of any others, because they appropriately make explicit what seems to us the 'normal' way to think and communicate transformations of quantity, structure, position, rates of change, amongst other concepts that we employ in claims about the empirical world. As Severin Schroeder puts it: "The rule's usefulness depends on its continued empirical appropriateness."[16] There is a primitive correctness of performance found in a history of practice that must be captured by the rules.

This pragmatic distinction of norms and rules points directly to the main disagreement between pragmatism and constructivism: the latter does not ground the legitimacy of mathematical theorems on the norms of our ways of living, but instead on canonical methods of construction that are necessarily non-subjective, independent of the relativities of human cultural practices.[17] Of course, this is not like the regulist interpretation of rules, because for them math theorems do not function as inferential rules, but instead as instructions to arrive at the representation of certain mental objects.

From a pragmatic point of view, we have no reason to think our rules of inference are universal, that any other intelligent lifeform would acknowledge them and thus be capable of correctly following the steps of our proofs. Long before we arrive at the clarity of a proof, we must be in agreement over what qualifies as clarity of thought. So for a pragmatist, the existence of proof is not the fundamental reason why we accept and undertake certain mathematical theorems. Proofs are certainly sufficient (for a human interlocutor), but not necessary for mathematical truth; what is necessary is a historically established normativity underlying the behaviour of the agents in the social practices which call for the use of mathematical vocabulary. Successful proofs *rely on* a common linguistic behaviour between interlocutors. Their normative authority only exists and moves those sapient agents who partake in the human mathematical practices. Clear proofs must navigate within the confines of a social-historically established language structure.

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16 "Mathematical Propositions as Rules of Grammar", in *Grazer Philosophische Studien (nº* 89, 2014), p. 31.

17 To review, all variants of constructivism endorse the principle that the truth-value of a mathematical statement turns on its proof, and agree that what counts as proof is an algorithmic method to construct the intended mathematical structure. This is what defines a constructivist approach.

Proofs demonstrate connections between mathematical concepts for all capable of following their inferences. In pragmatic terms, their role is to establish canonical paths from premises to conclusions. A proof tells us that, for those who infer like us, if assuming a set of assumptions so-and-so, they *ought to* undertake such-and-such conclusions. What is thus established is not the abstract potential of an effective algorithmic procedure to reach a concluding clause, but a pattern of behaviour across all social practices involving that particular computational method. This implies that whatever glues our inferential practices together is more relevant to the truth of a mathematical theorem than the formulas describing the steps of its proof.

As so, pragmatism rejects the constructivist view of mathematical truth which turns on providing a canonical and objective recipe for construction, an effective method for verifying existence. Pragmatism sees truth-talk as a coping mechanism for practical issues, made possible because of a *pre-cognitive agreement* over how we infer. To prove is to argue, convincingly, how the practice ought to go on, to demonstrate what ought to follow next for those who partake in this activity; a sort of convincing that could only emerge in the deepest of agreements: that of sharing the human form of life. A proof-theoretical activity presupposes conceptual stability, a communal agreement on a normal way to chain logical forms together.[18]

Computability is not a culture-independent property of algorithms, but a property of routines established in the human form of life. To quote two mathematicians on this one: "Mathematics is not an activity performed by a computer in a vacuum."[19] Computational methods are based on norms of inferential practices, which are formed when we reason in community, not in the privacy of a mathematician's mind. These methods exist because they serve practical and theoretical needs of an intelligent lifeform attempting to model, predict, and overall cope with its lifeworld.

Any attempt at expressing a rule of calculation will only be adopted and followed as a proper mathematical rule if it suits the community's ways of judging and acting, up to the point where the rule seems to point "the natural way to go", as in Wittgenstein's remark:

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[18] This is also why Quine's argument against conventionalism does not trouble a pragmatic conception, since the latter does not claim that the premises of our inferences are based on explicit social conventions. Pragmatism understands that our basic theoretical principles are based on regularities and norms ("the normal way to go") gradually established through social practices aimed at modelling, predicting, and coping with our lifeworld.

[19] Stewart, I.; Tall, D. *The Foundations of* Mathematics (Oxford University Press, 1977), p. 3.

> Mathematical truth isn't established by their all agreeing that it's true — as if they were witnesses of it. Because they all agree in what they do, we lay it down as a rule, and put it in the archives. Not until we do that have we got to mathematics. One of the main reasons for adopting this as a standard, is that it's the natural way to do it, the natural way to go — for all these people. (LFM, p. 107)

To better understand what Wittgenstein meant with "all agree in what they do", we need to read this in comparison with these paragraphs of PI:

> "So you are saying that human agreement decides what is true and what is false?" — What is true or false is what human beings *say*; and it is in their *language* that human beings agree. This is agreement not in opinions, but rather in form of life. (PI, §241)

> It is not only agreement in definitions, but also (odd as it may sound) agreement in judgments that is required for communication by means of language. This seems to abolish logic, but does not do so. — It is one thing to describe methods of measurement, and another to obtain and state results of measurement. But what we call 'measuring' is in part determined by a certain constancy in results of measurement. (PI, §242).

By putting agreement in judgement as a precondition for communication (and thus as a prerequisite for discursive practices), Wittgenstein goes against some long-standing epistemological views according to which we first grasp the meaning of an utterance to then assess it as true or false. In his view, in order for interlocutors to understand each other, they need agreement on how they judge what is. Inferential rules are not instituted by explicit conventions; we do not have to discuss and agree on the normal ways to infer before we start chaining linguistic expressions together. So the level of agreement that is necessary for us to infer in the same ways could not be given by *ad hoc* definitions, it must be already present, as a shared form of life.

Arbitrary as this may look at first, it does not imply the meltdown of logical consequence; logic is about the consequences of meaning, and as so, it presupposes a community of sapient agents agreeing on the forms of *how* to judge, so we can play the same language-games. Logic is *of* life, thus it presupposes a lifeform, it requires practitioners immersed in certain possibilities of how to proceed. So, in order for a community of agents to bind commitments and entitlements to others, bind linguistic expressions to other expressions, employ mathematical chains of reasoning to solve material-practical problems — in sum, to be able to connect all sorts of logical forms in our expressive range to draw out chains of reasoning with interlocutors — there must be a shared form of life, including here a pre-cognitive agreement on how they infer.

As Wittgenstein once wrote, "[…] to imagine a language is to imagine a form of life." (PI, §19). To imagine a language is to imagine one possibility of logical structuring of a living being, "one that shows an aspect of life, draws out procedures we have and concoct *in* life."[20] As Floyd explains, *form-of-life* is what lies in between biological form and cultural life, between wirings and traditions. The form of a living being is shaped by evolutionary processes, and it provides the leeway of possibilities of configuration for its life. A form of life shows itself in the routines of a living being, in the possibilities of action and expression within:

> Wittgenstein's transposition of *Welt* into *Form* moves concertedly away from an older idea of life and meaning understood in terms of necessities inherent in organic unities or wholes (worlds, totalities, systems, biological individuals or kinds, societies given through organic configurations of persons, cultures, peoples, histories, nations) to a contrastingly evolutionary, modular, piecemeal, diverse, fabricated, multi-aspectual, procedural, and dynamically interwoven conception of *possibilities* in life, logic, language, environment, experience and philosophy itself.

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[20] Juliet Floyd, "Chains of Life: Turing, *Lebensform*, and the Emergence of Wittgenstein's Later Style" (in *Nordic Wittgenstein Review* 5 (2), 2016), p. 62.

> […] By means of *Lebensform* Wittgenstein has reconstrued the traditional notion of "form", turning it toward regularities and norms of procedure *in* life, these lodged in a world where contingency and partiality (*Regelmässigkeiten*) are moved to the fore. Culture and human community are to be recovered, not analyzed: they are inherited, argued with, sought and fashioned, never simply "given"— any more than logic itself is.[21]

Such mutual understanding and concordance is an empathetic recognition of other sapient agents operating and behaving as I do (as we are of the same species, this come easily for us; we tend to naturally agree in how we do things, in a very general sense). Making inferences and judging them as valid or invalid are practices founded on this agreement. Without it, there would be no notion of inferential entitlement or logical validity. It is not that what we call a 'valid inference' (e.g. "If smoke, then fire") would now be invalid, but actually that there would be no grounds on which to compare one's claims and actions as entitling other agent's claims and actions.

The same goes for the example of measurement, as agreeing on what counts as measuring and what counts as a ruler are prerequisites for accepting a claim as stating the result of a measurement. As the technique is performed time and time again, we draw out the correct and incorrect ways of measuring, in accordance to a constancy of results, which retroactively plays a role in characterizing the practice of measuring and in what we recognize as a successful application of its techniques.

Our similarity of form of life allows the experience of the tacit, of having something being understood the same way across a social group without explicit statement. It is the reason why we can reach agreements in judgement while engaging in social reasoning — since our thought flows from the same lifeform, we can easily agree on how to proceed (forming norms of inference) and not remain in disagreements regarding how to go about the simplest structuring of life (such as in the purpose and method of measuring). Our form of life is the root *cause* of our common sense. It forms the tapestry of our everyday life, the universal context against which humans can distinguish simples. *Simplicity* in the sense of those background considerations that we ourselves never thought through, but inherited wholesome while learning to behave as adults; considerations digested and pass

[21] Floyd, Juliet. "*Lebensformen*: Living Logic" (to appear in Christian Martin, ed., *Language, Form(s) of Life and Logic: Invesgitations after Wittgenstein*, available at: <Academia.edu [link]>), p. 19.

over by past generations, yet that still play an active role in our present lives, determining our view of what there is, and how one could know about it. We may call this a naïve, unexamined, or instinctive metaphysics.

In what concerns mathematical truth, the point is not that the constitution of our particular lifeform explains why we follow, say, Euclidean or Dedekind-Peano axioms when operating with geometrical or arithmetical concepts — to establish this point we would need more than philosophy, we would also need an evolutionary theory that explained how we evolved to think in those forms — rather, the point is that this following is *typical* of humans, it characterizes our lifeform. Our everyday agreement in what we *do* with mathematical formulae reveals a characteristically shared nature in the way we reason.[22] This essential agreement in form of life provides us with a common horizon in which we can collectively teach and learn with each other the same technical practices.

In conclusion, by enlisting Wittgenstein's idea that agreement in form of life is a prerequisite for discursive practices, we find an answer to the question regarding the compulsion of the mathematical rules which does not get caught in the epistemological horn of Benacerraf's dilemma. The practices licensed by mathematical formulae are not constituted through conventions; for sure, conventions play a role in our mathematical activities, and knowledge of them is often necessary for an agent to properly *do* mathematics. But conventions are not the basis of mathematical knowledge, for they are only relevant for an agent once she partakes on the proto-mathematical practices-and-abilities. Without grasping these practices, an agent cannot even begin to understand our use of mathematical vocabulary. What configures and characterizes the fundamental practices we engage in mathematical activity is, first and foremost, our agreement in form of life, which offers a leeway of mutual understanding for us to share the same practices. And secondly, is our culture, which offers us common ground to settle the *normal ways* of performing a technique, as in a community's way of doing things that may be instituted and underpinned by conventions, but not created by them.

[22] Or, as José Ferreirós puts it: "We are talking about a form of intersubjectivity and the associated 'relative necessity', so to speak, of mathematical results. Indeed, its detailed analysis makes it possible to claim that this is a peculiarly *strong form of intersubjectivity* — very likely, the strongest there is for humans." (*Mathematical Knowledge and the Interplay of Practices*, Princeton University Press, 2016, p. 160).

6.3. Ferreirós' pragmatic agent-based account of theoretical mathematics

The most important passage in the pragmatic account of mathematical knowledge is that from the normative practices we share because of an organic agreement as a lifeform, to the theoretical practices of making explicit the norms and principles according to which we think and act. This is how we build theoretical mathematical knowledge. In this section, I will elaborate on how this transition works.

In his *Mathematical Knowledge and the Interplay of Practices*, José Ferreirós recently developed a pragmatic agent-based account of mathematical knowledge that is symbiotic to the social pragmatic expressivist outlook.[23] In his account, mathematics is like an edifice based on the practices and routines of our form of life. The superstructure is composed of mathematical theories (e.g. set theory, group theory, differential geometry, topology, etc.) and it rests on a basis of practical knowledge associated with the wide application of mathematical techniques and concepts in social reasoning. Use and understanding of mathematics *depend* on this technical and practical background, as its expressions and techniques are cognitively linked to non-mathematical practices.

Ferreirós explains that "at the basis of mathematical knowledge and understanding, there lie other practices that are not mathematical, properly speaking: the techniques (technical practices) of counting, measuring, and drawing geometrical shapes."[24] He calls these *proto-mathematical practices*, as they are fundamental for the mathematical enterprise and yet agents can perform these techniques without having to know how to apply symbolic frameworks (i.e. agents can learn some technique of counting without knowing how to perform arithmetic operations and manipulate its symbols).

The reason why Ferreirós regards counting, measuring, and drawing shapes as fundamental for mathematics proper is not that these are the only technical practices in which mathematicians engage while pursuing their lines of inquiry — this list could be expanded to include techniques of combination and separation, for instance — but Ferreirós stops at these three because these are the most *pregnant* notions. These are called

[23] Not unlike the pragmatic expressivist explanation of meaning, so to in Ferreirós' pragmatic account we have a division based on 'implicit in doing' and 'explicit in saying' — that between the fundamental interplay of practices whose implicit norms form a basis of practical knowledge upon which the mathematical theories are built.

[24] Ferreirós, J. *Mathematical Knowledge and the Interplay of Practices* (Princeton University Press, 2016), p. 42.

proto-mathematical because these are the practical cognitive antecedents upon which we base the central mathematical disciplines of arithmetic, geometry, and hybrids such as analysis.

From the technique of **counting** we can introduce number-words as counting devices, and then, by examining and expressing the norms implicit in our technique of counting (e.g. norms for ordering and matching the numerals) we arrive at the basic rules of calculation, which may be done from within the symbolic framework of arithmetic. Finally, from this basis, we allow the construction of the natural number system $\mathbb{N}$, whose internal inferential relations constitute the rules according to which we talk of quantities of physical objects. The same goes for the other two techniques. The technique of **measurement** is necessary for one to introduce fractions as ratios and then a theory of proportion. By examining and making explicit the norms implicit in how we perform measurements, we can express the rules of proportion, and thus construct the rational number system $\mathbb{Q}$. At last, from the technique of **drawing shapes**, we can introduce regular shapes (e.g. polygons) and soon make explicit the norms of construction of those polygons in terms of a geometrical theory, from which we construct the real number system $\mathbb{R}$.[25]

Technical practices	→	**Symbolic calculi**	→	**Structuring principles**
Counting practices	→	Reckoning arithmetic	→	Peano-Dedekind axioms, $\mathbb{N}$ structure;
Measuring practices	→	Fraction arithmetic	→	Proportion theory, $\mathbb{Q}$ structure;
Practical geometry	→	Euclidean geometry	→	Cartesian geometry, $\mathbb{R}$ structure.

When we are engaged in the normative practices of counting, measuring and drawing, we are also engaged with the physical world. And through the performance of these practices, we observe regular empirical correlations between certain transformations which we conceptually understand as changes in quantity or shape of particulars.

[25] For more on these relations, see Ferreirós, J. *Mathematical Knowledge and the Interplay of Practices* (Princeton University Press, 2016), pp. 38-39, and p. 113 onwards.

Yet, we do not get Mathematics proper just by observing empirical patterns of correlation. Mathematics is constituted of more than socially learned techniques. We have not gotten Mathematics until our attitude shifts from engaging in these practices for common-or-garden needs to *theorizing* about these practices,[26] characterized by the rigorous study of the correctness and exactness of particular[27] mathematical results from a symbolic framework.

This shift is marked by the differences between practical techniques (counting, measuring, and drawing shapes) and theoretical practices (calculating, deducing measurements from proportions, demonstrating the construction of regular polygons from basic rules, etc.). The first set of practices does not involve employing a whole symbolic framework, just the performance of the technique by application of a standard. Whereas in the second set of practices, the criteria of correctness for performances are made explicit with the expressive tools of symbolic frameworks, which allows agents to codify and store information, and thus calculate with much higher accuracy than we could simply from memory, without these rule-governed systems of calculation.

This takes us to a final and fundamental difference between these sets of practices: with the introduction of symbolic systems that allow for codification and storing of information, we can introduce methods of proof. That is, from the platform of a symbolic system, we can introduce methods to demonstrate (specify and explain) how is that application of a concept follows from another.

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[26] For Ferreirós, the passage from proto-mathematical practices to theoretical mathematics is mainly represented by a change of purposes — for instance, from using number to count towards wonder for a *precise* definition of π — in his words: "For practical purposes, reliable counting, robust storing of the data, and simple, reliable procedures of calculation (by pen or machine, say the abacus) are essential in the world of numbers. But number theory is a different issue, guided by the search for precise results. When the Pythagoreans, around 500 BCE, became fascinated with the fact that all numbers fall under the odd/ even dichotomy, they were already in the business of "contemplation" as the Greeks said, of theoretical thinking." (*Mathematical Knowledge and the Interplay of Practices*, Princeton University Press, 2016, p. 115).

[27] Mathematicians do not study the consequences of *all* frameworks, or of *any* set of axioms. They dedicate themselves to particular cases (e.g. the field ℝ, or the cumulative hierarchy of sets) which they see as more fundamental, useful, surprising, elegant, or perhaps even more beautiful than other frames or systems.

According to Ferreirós, a mathematical symbolic framework encompasses the quadruple <*S, R, Q, L*>:

- *S*, a set of accepted statements, such as basic norms or axioms;
- *R*, a conventional method of reasoning used by the agents to deduce the implications of *S*;
- *Q*, a set of questions and hypotheses related to the intellectual curiosity of the time;
- *L*, a formal language. [28]

Symbolic frameworks are designed to scrutinize and articulate mathematical concepts; it is in accordance with these forms of reasoning that the canonical forms of proofs are defined. That is, a mathematician needs to be working from a framework to devise a theory or construct a system of calculation. Once we have a canonical proof of a statement about the result of carrying out certain operations as defined in a framework, we have a demonstration of how, according to an *R*, that is the last in a gap-free sequence of inferences from the set *S* of that framework. Moreover, new frameworks often offer theoretical innovations that may improve perspicuity or rigour of our mathematical language, or perhaps establish new conceptual connections, introducing whole new areas of research.

Now, what is of main interest for us here is the possibility of accounting for this theoretical mathematical knowledge on basis of systematic links amongst the underlying practices. To understand these links, mind that practices can easily borrow from one another, be projected unto another, or even be coordinated together.[29] From multiple conjoined or coordinated applications, these might form families of inter-connected practices, possibly sharing methodology, common interests, goals, and more interestingly,

[28] *Mathematical Knowledge and the Interplay of Practices* (Princeton University Press, 2016), p. 26.

[29] E.g. projection of the practice of talking about having gold in one's teeth to talk about having pain in one's teeth (see 4.2). And as an example for coordination, the ability to walk in equilibrium is latter employed in cultivating the more complex abilities of riding a bicycle or slacklining. That is, a technical practice developed at a time to solve some specific concrete problems could be latter employed as a subroutine in a more complex set of techniques. As techniques of carving knives out of rocks for protection and hunting might have been projected into techniques of carving statues for art, or pillars and blocks as building blocks, and then vice-versa.

frameworks. This is the case with the elementary technical practices of counting, measuring and drawing shapes, as these are shared across many domains, from the hard sciences to home finances, from engineering to the arts. It is thanks to the unending applicability of the proto-math techniques that we are in possession of systematic ways to cut across multiplicities of phenomena and frame them for the patterns and structures that they share.

Ferreirós' thesis is that the ample sharing of these techniques and abilities forges interconnections that support future mathematical advances. The interconnectedness of the proto-mathematical techniques constitutes a tapestry of implicit norms of practice which guides and constraints future developments. In other words, theoretical postulations and hypotheses in mathematics are constrained by systematic interconnections between the proto-mathematical practices.[30] Interconnections between how we count and how we talk about quantities, how we measure and talk of proportions, how we draw shapes and talk of structured spaces. These interconnections constraint the range of admissible results for an application of a mathematical technique.

Or, as Ferreirós prefers to put it, we have an *interplay of practices* and working knowledge strata that "guides the constitution of meaningful concepts and restricts admissible principles."[31]

> We have *working knowledge of several different practices and strata of knowledge, together with their systematic interconnections.* This causes links that restrict the admissible — for instance, when a new framework is being developed — and that are responsible for much of the objectivity of mathematical results and developments. The interplay of practices acts as a constraint and a guide.[32]

For instance, the realization of systematic interconnections in the practice of counting acts as guide and constraint for the construction of a calculus with numbers, but once we have deduced a theorem that connects those concepts — that is, correctly performing the operation of multiplication on the numerical concepts 10^{100} and 10 *is as*

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[30] Such as links between proto-mathematical practices *and* our form of life; the implicit norms of these practices *and* the explicit rules of axiomatic systems; the role of proto-mathematical practices in science *and* the role of mathematical frameworks in the scientific modelling of phenomena.

[31] *Mathematical Knowledge and the Interplay of Practices* (Princeton University Press, 2016), p. 159.

[32] *Ibidem,* p. 39.

counting up to 10^{101} items — then any empirical claim that does not respect this connection is immediately disqualified. An agent will likely never observe a particular empirical correlation between correctly computing '$10^{100} \times 10$' and counting up to 10^{101} items, yet still, she would know one fixes a criterion of correctness for the other because she can infer it from the rules of the arithmetical calculus. Thus the mathematical frameworks make available systems of calculation and results that are *reliable* because they are consistently rigid in only concerning relations between mathematical concepts, something intra-linguistic that holds by virtue of practice, independently of any particular concrete correlation to which we may employ those concepts.

These connections present us with a normative base-structure that gets hardened into rules through the theoretical mathematical practice, as professional mathematicians zealously explore the consequences of those systematic interconnections, only bestowing theoremhood to those that can be meticulously showed to consistently belong to a system set by basic rules.[33]

From the tapestry formed by these links, we advance from proto-mathematics to the theoretical practice of mastering canonical methods of argumentation and proof, making sense of mathematical problems and solving them, and so on. The result is the formation of theories whose concepts may hold many (sometimes unexpected) intricate implicit connections with concepts regimented by other theories, by virtue of their interconnected basis. These connections are studied and made explicit in formulae by mathematicians as they work to prove new theorems. Thus Ferreirós concludes:

> The whole history of mathematics can be presented as the gradual development of a network of links connecting different core notions that initially lie separate, i.e., as the creation of a delicate tapestry or spider web

[33] For Ferreirós, there is nothing more to claiming that mathematics is objective than to say that it is constrained by the links between our practices: "Form our viewpoint, the legendary objectivity of mathematics is, more than anything, a strongly constrained form of intersubjectivity. The constraints range from roots in basic human cognition, at the elementary level, to intra-mathematical connections, and extend to include links with the sciences, with technology, and even with the arts." (*Mathematical Knowledge and the Interplay of Practices,* Princeton University Press, 2016, p. 257).

> establishing bridges between the discrete and the continuum, number and geometry, algebra and analysis — and, later, topology — etcetera. The process has been aided by moves of generalization, by steps of identification of significant hybrid notions […], by the introduction of new hypotheses, and so on.[34]

In a picture, mathematical knowledge is like an edifice gradually built with a collection of different methods related to their practical basis. As I see it, Ferreirós' account seems to compliment Wittgenstein's view of mathematics as a colourful mix of techniques of proof.[35] That is, both philosophers point towards the mosaic character of mathematical practices — or the *interplay* in which math practices are caught up — as the source of the ubiquitousness of its vocabulary and multi-usefulness of its frameworks.

Now, with the above considerations, we may define mathematical knowledge as an agent-based working knowledge of:

- *Effective procedures of computation:* algorithms, diagrams and graphs that, if carried out appropriately by a computing agent considering both methodology and computational capacities, entitle inference from application of certain mathematical concepts to the applications of others;
- *Methods of proof and refutation*: canonical methods used by the community of mathematicians for demonstration and argumentation;
- *Concept engineering***:** with the possession of the above, the mathematician can achieve innovation: develop a new theory, engineer a new framework to supply for a certain expressive demand, or propose novel questions and conjectures that push the boundaries of previous knowledge.

It follows from this account that mathematical knowledge cannot be fully formalized in a grand meta-theory, in the style of Russell's and Whitehead's *Principia Mathematica*. A pragmatic account of mathematical knowledge resists formalization because of the

[34] *Mathematical Knowledge and the Interplay of Practices,* Princeton University Press, 2016, p. 40

[35] In Wittgenstein's words: "I'm inclined to say that mathematics is a colorful mix [*ein buntes Gemisch*] of techniques of proof. — And upon this is based its manifold applicability and its importance" (RFM III—46).

interplay of practices involved in the application of its frameworks. That is, some inferences allowed in the construction of mathematical knowledge are justified implicitly by practice instead of a formally expressed premise (e.g. many grasp the $\mathbb{N}$ structure directly from elaboration-by-training from the ability to count, before or quite independently of learning the principles of set construction). This know-how cannot be learned simply by following the rules of a meta-mathematical theory, as it requires participation in a culture via engagement in social-normative practices, from where agents may master the practical techniques and proper use of the vocabulary of a mathematical framework.[36]

Notwithstanding that much of these techniques can be algorithmically decomposed into instructions that a machine can execute, so long as this machine does not *learn* the inferential connections between the use of concepts,[37] it will only be able to follow the rules instead of stating them. To run our frameworks instead of devising them. Mechanical computing agents may compute using rules and symbols, yet only sapient concept users can create mathematical frameworks, introducing new rules and symbols. The capacity to solve conjectures, suggest new ones or create new frameworks is beyond mere mechanical mimics because only sapient concept users can create (and live within their own) meaning.

6.4. Mathematical vocabulary as a pragmatic meta-vocabulary

With Ferreirós' idea of an interplay of practices upon which theoretical mathematical knowledge is based, we can explore in detail the connection between empirical and mathematical statements as seen from the lenses of pragmatic expressivism. The broad use of mathematical vocabulary and techniques establish such systematic links between mathematical and scientific theories. Thus, theoretical mathematical knowledge would *not* be appropriate to model and license empirical predictions if it did not serve to express the norms of our practices directed towards real objects.

In order to better analyse this intricate relation, I will employ the meaning-use diagrams that Brandom introduced in *Between Saying and Doing* to analyze the expressive role of normative vocabulary. These diagrams demonstrate the connections between practices

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[36] It seems like we have found another face of the phenomenon of incompleteness in mathematics, only now from a pragmatic consideration of this knowledge.

[37] This would mean, essentially, that the machine has passed Turing's Test.

of use and vocabularies. More specifically, they show which practices an agent must *know-how* to perform in order to be counted by fellow interlocutors *as* deploying a certain vocabulary. The elementary meaning-use relations (MUR) are:

- *PV-sufficiency* [practice-to-vocabulary]: when acquisition and performance of a set of practices-and-abilities are sufficient to count as meaningfully deploying a vocabulary.
- *VP-sufficiency* [vocabulary-to-practice]: when a vocabulary is sufficient to specify a set of practices-and-abilities.
- *PP-sufficiency* [practice-to-practice]: when a set of practices-and-abilities can be elaborated into another, either by:
  (a) affinity in the performances; as sometimes being proficient at a doing ends up facilitating another practice (e.g. when one can be brought to play the bass simply by elaborating on practices-and-abilities that one already possessed in order to play the guitar);
  (b) training, (e.g. by elaborating unto basic kinesthetic abilities one can be brought to play tennis or dance salsa);
  (c) algorithmic decomposition into programmable subroutines, allowing for implementation and replication of the practice or ability by a machine.

Surely, MUR are not only sufficiency relations. For each of the above, there is also a *necessary* version. The variance is simple: whenever a practice or vocabulary is said to be sufficient to deploy or to specify a target practice or vocabulary, in the necessary version, we may say the former *presupposes* the first. In other words, a *PV-necessity* claim is nothing but the pragmatic claim that to be able to participate in certain language-games, or to deploy a particular vocabulary, one must be able to engage in the underlying related practices. For example, in order to deploy an item of observational empirical descriptive vocabulary and say that something is red in the observational way of making non-inferential reports, one *must* be able to reliably respond to a variety of sensorial stimuli.

Now, the MUR that interest us here are between mathematical vocabulary (V math), mathematical practices (P math), proto-mathematical practices (P proto-math), and a subset of empirical vocabulary concerning expressions of *quantity*, *proportions* or *structured spaces* (V QPS). I have limited the set of empirical statements under consideration to QPS because of my endorsement of Ferreirós thesis that the mathematical practices are based on the proto-math practices. Moreover, I cannot claim that mathematical rules make explicit the inter-connections between the applications of *all* empirical concepts, for there are plenty of those whose use does not require knowledge of how to perform mathematical operations. QPS vocabulary is composed of terms and expressions such as 'amount', 'number', 'many', 'more than', 'double', 'halved', 'ratio', 'square', 'round', 'parallel', 'angle', amongst many others. These terms are usually used to operate *on* a *pivot concept* or standard unit (e.g., Jupiter's moon), whose instances of application may be counted (4 moons), or put in a proportional ratio in order to measure a magnitude (Io has approximately 41,910.000 km^2)[38] or to tell the shape of the objects thus conceptualized (Io is oval).

MUR 1) PV-sufficiency: In order to *say that* an object has a certain mass or speed, measure one of its properties, tell its shape, its temperature, or even its position in a coordinate system — to meaningfully make and also understand these claims, one *should know how* to count, measure and draw shapes. That is to say that there is a set of practices-and-abilities whose apprehension and performance is sufficient to give meaning to our mixed statements through which we talk of quantities, measurements and the shape-structure of items of empirical vocabulary. So my first MUR claim is that the proto-mathematical practices are PV-sufficient to deploy the subset QPS.

MUR 2) PP-necessity: Proper mathematical practices are elaborated by training (El. Tr.) from the proto-mathematical practices.[39] This is a claim that having the latter is PP-necessary for developing the former, so anyone capable of performing the proto-math technical practices-and-abilities can be *brought by training* to develop the proper mathematical practices-and-abilities. This is a PP-necessity claim because mathematical practices *presuppose* the proto-math ones — if one does not know how to count, then one will lack the basic abilities to understand how to calculate addition and subtraction, consequently missing the point of using the symbol $\mathbb{N}$. If one does

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[38] Information taken from the Wikipedia article on Io.

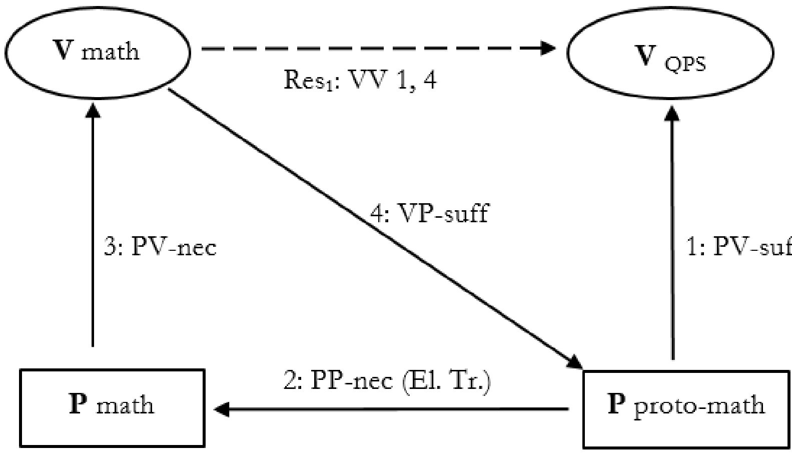
[39] Apart from elaboration by training from proto-math practices (which is the way our children learn mathematical practices-and-abilities related with arithmetic and geometry), mathematical practices may be acquired through algorithmic elaboration of the proto-math practices-and-abilities into subroutines, as evidenced by machines capable of implementing those practices.

not know how to measure, one will not understand what a proportion is, and thus will not be able to learn how to operate fractions, or understanding numbers in the ordinal sense, as successor functions. Lastly, if one has no training in drawing and manipulating shapes (i.e. playing with spatially structured forms), then one will not know how to operate geometrical transformations or constructions and consequently will not comprehend the geometries.

MUR 3) PV-necessity: To understand the proper use of a mathematical term, one must be able to tell the difference between good and bad inferential relations in the underlying practical performances for which that term is used. To understand the use of such an item, one *must know how* to resolve a computation involving that item, master the systems of calculation in which it plays a role. The claim, then, is that the mathematical practices are PV-necessary for the meaningful deployment of mathematical vocabulary. One cannot meaningfully deploy any item of this vocabulary without knowing the role of the concept in computations. And to know that role, one must grasp the position of that term in the inferential network draw by the mathematical practices. For instance, in order to know how to use integers, one must be capable of knowing how differently elicited instances of deployment of integers are practically related. This can be made explicit by equations: the practical knowledge of how counting 5 cycles of 3 things makes for a total of 15 things can be made explicit as $3 \times 5 = 15$. The reason why this equation serves to make explicit a good inference is that the application of those integers is already implicitly correlated in the proto-math practice of counting.

MUR 4) VP-sufficiency: Mathematical vocabulary is sufficient to specify what we do in the proto-mathematical practices. Talk of integers in the number-line is sufficient to specify how we think quantities or sequences in our practices of counting and measuring; talk of fractions is sufficient to specify measurements as proportions (the "comparison" of a counting-unit, pivot concept or standard against an extension); and geometrical vocabulary is sufficient to specify and formalize what we do when we draw and manipulate shapes or structured spaces.

Now, MURs 1 and 4 can be composed into a complex vocabulary-to-vocabulary relation, a description of the relationship between mathematical and empirical vocabulary that we have been looking for. To see this, we must draw the meaning-use diagram resulting from the above four claims:



From the composition of MUR 1 and 4, we draw Resolution$_1$, according to which mathematical vocabulary functions to specify the proto-mathematical practices that are sufficient for meaningful claims about quantity, proportion or space. That is, mathematical vocabulary allows one to specify what one must do to construct a regular pentagon with ruler and compass, or bisect an angle, or find a proportional ratio, amongst others. In Brandom's terms, this makes mathematical vocabulary into a *pragmatic meta-vocabulary* for QPS vocabulary: one whose expressive resources allow us to talk about the *use* of another vocabulary, and the proprieties that govern this use. That is, when one is able to count, measure and draw shapes, one already possess a sufficient set of practices-and-abilities to use vocabulary that lets one *say* what one must *do* to talk of quantities, proportions or structured-spaces.

Thus, by making explicit the implicit norms of our computational practices, the mathematical rules end up regimenting social reasoning on these matters, constituting part of the grammar of empirical statements.

7. Conclusion

This thesis claims that Benacerraf's dilemma owes its philosophical force and disastrous implications to representationalist and empiricist presuppositions concerning language and knowledge. The dilemma only emerges within the scope of a philosophy that unreflectively embraces these views, and therefore, it dissolves away once we examine and abandon such presuppositions.

As we have seen, the main traditions in the philosophy of mathematics are caught in the horns of this dilemma because of their representationalist and empiricist assumptions regarding mathematical meaning. Albeit recognizing that platonism and combinatorialism posit different modalities of abstracta to function as standards of correction for mathematical operations (one in actuality, the other in potential), nonetheless both views recur to the same representationalist metasemantics, which motivates the conception of mathematical correctness as having an abstract nature, given and determined independently of human practices. This is because the representationalist view conceives linguistic norms as corresponding to semantic relations that are established objectively, independently of our use of linguistic expressions. Meaningful use of an expression is one that refers, or can be verified as, representing the intended object. And herein lies the root of the misunderstandings.

Notwithstanding that such relations are posit as implicit manifestations of meaning, revealed to us privately and passively through the formation of representations of objects and properties, they are still regarded as effective at guiding the linguistic behaviour of concrete agents in social discourse. In other words, representationalism presumes that we grasp the same semantic relations, without explaining how intended representations could make manifest the criteria for the correct use of an expression.

This strategy overlooks the pragmatic aspect, the necessary know-how one must have to meaningfully apply expressions and participate in discursive practices. And as so, representationalist strategies, such as reference-based platonism or the verification-based combinatorialism, dissociate mathematical expressions from their underlying object-directed practices (such as counting, measuring, and drawing shapes), obscuring the processes according to which mathematical expressions acquire meaning and function in social reasoning.

When we examine the use of mathematical expressions as an anthropological phenomenon, implicit representational standards of correction drop out of consideration as irrelevant in the face of actual practice. What actually gives these expressions a cognitive function is the way they have been historically used to make certain information explicit in inferentially articulated conceptualizations. These inferential relations constitute the web of language, whose network structure makes it possible to manifest a meaning in social reasoning when we employ the correct inferential chains of concepts.

The semantic horn of his dilemma only troubles those who think that meaningful use of concepts represents something objective. When we discard the notion that language use is representational, we defuse the semantic horn of the dilemma. In order to avoid the trap, we must recognize that our meaning (mathematical or otherwise) is not universally cognoscible, nor tied directly to the environment. Meaning is something intelligent life creates and sustains. What we mean is an effect of certain social interactions, and as so, it only exists for and within the human form of life. Therefore, we must abandon the preconception that mathematics deals with objective and universal structures, that our computations have their results abstractly determined beforehand by necessary truths.

In a similar vein, Benacerraf's empiricist presupposition that only a causal theory of knowledge could explain mathematics in a way that coheres with an account of scientific knowledge is based on a misunderstanding of what we *do* with mathematical statements. Even though mathematical discourse is related to empirical discourses and their causal objects, we do not unveil such objects with mathematical statements, but instead make explicit the rules of reasoning for empirical practices that involve talk of quantities, proportions, and structured spaces. Mathematics does not talk of any object whatsoever; instead, it tells us the right way to think of objects in general.

This thesis takes the goal of defusing Benacerraf's dilemma as a guiding thread towards a clearer understanding of mathematics. Its *leitmotiv* is to foment a change in the debate, moving away from these metaphysical speculations and endless discussions over the nature of mathematical objects. In this spirit, I offer an alternative to postulational and constructivist approaches, based on pragmatic and expressivist principles. A pragmatic expressivist explanation of the inexorability of mathematics and effectiveness of mathematical theories proceeds through an examination of the norms of social reasoning in the practices of counting, measuring and drawing shapes, which are typical of our form of life.

Thus mathematical knowledge concerns the methods of reasoning which *allow* us to see and understand certain natural structures. Mathematics offers us the linguistic structures to understand and talk clearly about properties, patterns, and aspects of objects in general. It is not a "useful fiction", but rather, a useful grammar to make explicit the forms and structures of human thought. I defer to Floyd's explanation once more:

> The equation's "truth," if we wish to speak this way, holds as much in virtue of our own, contingently evolved commitments to certain methods of representation and ongoing communal practices and needs as it does in virtue of the nature of things. Like a system of measurement, mathematics (like logic) is for Wittgenstein a complex human artefact, situated and created in and for an evolving natural world, and its claims to objectivity and applicability ultimately turn upon our human ability to find one another in sufficiently constant agreement about results of its application to make the practice prove its worth.[1]

To have mathematical practices is to be part of a certain form of life. As such, as we have seen in the case of the Amazon's Pirahã people (p. 139), our mathematical practices are not independent of culture or history. Isolated from the intellectual developments happening in Eurasia, this community developed very different mathematical practices. The Pirahã can perform cognitive tasks involving quantities with precision, yet they have no written system to demonstrate the computation; they developed no terms to make explicit the algorithm they have used to perform those tasks. The Pirahã do not have linguistic or written expressions which would translate to our numbers, thus they have no socially established system to order expressions of quantity, and so nothing equivalent to our arithmetic. Their case shows how distant human cultures may develop different mathematical practices, or none at all. They may not share our enthusiasm with precision, or perhaps they do not associate concepts with the technique of counting.

It follows from this that we can never be sure whether a non-human sapient lifeform would develop similar mathematical theories, or even if they would find any value in our theories. Not only could they have no interest in mathematical frameworks as we know

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[1] Floyd, J. "Wittgenstein's Philosophy of Logic and Mathematics", in *The Oxford Handbook of Philosophy of Mathematics and Logic*, p. 108.

them, but they could not engage in something similar to our mathematical activities. Aliens may not recognize these matters as being worth of attention. They may have no theoretical interests, or no theoretical interests for precision, exactness, coherence, or consistency of results, or for making explicit structuring principles of a system of calculation.

To exemplify this point, I refer to Wittgenstein allegories of communities of sapient agents whose practices may somewhat resemble our own mathematical practices, but their purpose, their application of the somewhat similar techniques, is nowhere near ours. He imagined, for instance, a community that defines the price of wood based on the area occupied by the logs instead of their total weight (RFM I, §§143-150). It may sound like stupidity for us, but it is not a self-evident truth that the price of the wood stacks must be directly proportional to the weight. Perhaps those people care far more for trading and maintaining good diplomatic relations with their neighbours than profit. Or perhaps they just have too much wood and want to give some away. In the end, it does not matter how we justify their behaviour, the point is that the rationale behind their use can only be grasped by practitioners in similar conditions and form of life. That is, the upshot of these allegories is that the usefulness of mathematical techniques and their frameworks depend on the practical needs and the form of life of the practitioners.[2]

Thus, the practices of calculating, applying interconnected systems of measurement, constructing polygons, demonstrating structuring principles, resolving conjectures, and so on, are all human practices. *Mathematics is not the language of nature*, it is a human language which serves to express structures of our own reasoning, norms of our own practices, both of which inevitably involve nature. The mathematician is a conceptual artist, fashioning algorithms, demonstrating conceptual connections, examining structural relations, and creating the frameworks which allow us to see and understand these connections. The mathematician inaugurates conceptual spaces latter employed in the development of novel theories or crafts.

Mathematical vocabulary serves to add expressive power to our natural languages. With it, we can create intricate systems of calculation (arithmetic, geometries, etc.) that give form to our talk of quantities, positions, shapes, morphisms, and so on. We devise them to

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[2] This is not the only example. Wittgenstein also pictured a community measuring lengths with a rubber ruler which expands in hot days and shrink in cold ones (RFM I, §5). A group sharing nuts according to a weird calculation rule that results in nuts disappearing (RFM I, §137). And a community writing their mathematical proofs on wallpapers to adorn their walls (LFM, pp. 34, 36, 40).

give form to our thoughts and communicate the details of increasingly more complicated systems we discover in nature. Mathematicians provide frameworks to reason with. We should not confuse the glasses that allow us to better see the world with the objects we discover out there.

The impression that it is a miracle that the physical environment fits in our mathematical models is merely a side effect of the role of mathematics as provider of the frames within which descriptions of quantities, proportions and shapes in the environment become possible.[3] Our understanding of these aspects of the world would be severely limited without the expressive power of mathematical language. Examining its application from a historical perspective that emphasizes agents exerting their practices in accordance to certain conceptual frameworks, it is undeniable that the development of mathematical techniques contributed for our descriptions of natural phenomena to be increasingly more precise. Engaging the world through the lenses of mathematics has enabled us to deepen our comprehension of its natural mechanisms.

Mathematics is a useful toolbox in the hands of scientists, helping us express empirical patterns and aspects, and also to examine them in models. Or, in a Quinean sense, we may say mathematical techniques are *indispensable tools* for the expression and modelling of certain aspects of reality, allowing us to understand and predict natural phenomena. Mathematical vocabulary plays such a unique role in our lives because it secures communication about these features of our practices that are inevitably involved with the furniture of the world. Mathematicians design linguistic categories that enable us to understand the manifoldness of objects in general; categories which increase the expressive power of our natural languages by enabling us to examine with precision and perspicuity each and every matter involving quantities, proportions, or spatial structures.[4]

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[3] I am referring to Eugene's Wigner comment that the appropriateness with which the mathematical systems come to model and allow the explanation of physical phenomena is a miracle "which we neither understand nor deserve." ("The unreasonable effectiveness of mathematics in the natural sciences", in: *Communications in Pure and Applied Mathematics*, vol. 13, No. I, 1960, p. 9).

[4] Compare with this remark by Wittgenstein: "I'm inclined to say that mathematics is a colorful mix [ein buntes Gemisch] of techniques of proof. — And upon this is based its manifold applicability and its importance." (RFM III—46)

Appendices

Appendix I: Pragmatic expressivism and Geach's objection

Traditional expressivist accounts of meaning face a notorious difficulty raised by Peter Geach. According to him, semantic expressivism analyses meaning by specifying a criterion for a string of words to count as an expression of a proposition *p*. Thus a locution may be counted as committing the speaker to the claim that *p* when it satisfies some criterion for *expressing p*. So for instance, only when one deploys the word 'good' to recommend something that he satisfied the criteria for expressing the property of goodness. However, as Geach noticed, someone who asserts *p* is not engaging in the same speech act as someone asserting ¬*p*, so does this mean that both uses do not express the same proposition?

The problem is not only with negation; it pops up every time we embed *p* with logical connectives. People do not try to commend something by asking "Is this good?" or by inferring that "If this is good, then that is good as well". These uses of 'good' seem to carry a different meaning than the assertion that "This is good". If expressivism tells us that these non-descriptive uses should be considered also as non-truth-apt, then how are we to interpret one such statement when it figures as a premise in a truth-apt conditional if the overall truth-value depends on the truth-value of the premise? The expressivist interpretation must be wrong then, for according to it in claiming that *p*, denying that *p*, and taking *p* as premise, one is **not** using *p* in the same manner, and thus these uses are not going to have the same meaning – when they obviously should have, for otherwise, we lose grounds to maintain that *p* is incompatible with ¬*p*. Thus, as Geach's objection goes, expressivism cannot account for the composition of complex sentences with logical connectives.

Although, Geach's argument works both ways: not only as an objection to an expressivist account but to reference-based and verification-based semantics as well. In his response to Geach, Richard Hare argued that every semantic project has to face some version of the task of explaining the composition of meaning in complex sentences. For instance, according to a standard referential truth-conditional analysis, the ordinary empirical descriptive statement "This is red" does not have the same truth table than "This is not red", and that has not stopped these semanticists from holding that the empirical vocabulary item 'red' means the same in both cases, even though they also claim that the meaning of these statements is given via a specification of its truth conditions.[1]

Most importantly, Geach's objection does not apply to my thesis, as it is directed at an expressivist semantics, and my application of the expressivist thesis is at the level of a meta-semantics. Pragmatic expressivism does not claim that meaning *is* an expressed intentional state, but rather that meaning is the role that a linguistic performance plays in social reasoning, a role determined according to the practices underpinning the vocabulary deployed and to how it was deployed. So linguistic performances, including making statements, can play a variety of expressive roles in our lives beside the representational.

Furthermore, an expressivist meta-semantics does not take truth as a primitive element that explains meaning and logical consequence. Instead, it points to the inferential articulations of undertaking and attributing commitments and entitlements as the key to understand meaning and logical consequence. As such, in this theory, the validity of conditionals in arguments and proofs is guaranteed not by a preservation of truth, but by the goodness of inferences that obtain through the correlations of practical and factual-environmental circumstances and consequences that warrant the application of concepts. So conditionals are not read as compositions of truth-functions articulated by logical connectives, they are read as logical rules that serve the pragmatic function of drawing out inferential connections between conceptualizations.[2]

Now, stepping down to the level of semantic analysis, according to the inferentialist semantics, embedding *p* in a conditional will not change the content of the expression in relation to claiming that *p*. This is because none of these uses tries to change the position of the concept applied in relation to its inferential vicinity (that is what linguistic norms and metalinguistic statements about concepts serve for). So since both expressions just employ a concept and none asserts something about the concept itself, then both equally acquire meaning according to the role of that concept in discursive practices. The difference here is that the conditional does not assert *p*, it makes explicit some entitlement or endorsement

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1 See Mark Schroeder, 'What is the Frege-Geach Problem?' in *Philosophy Compass* 3/4 (2008): 703–720, p. 706.

2 What is behind this reading is Brandom's logical expressivist thesis, which claims that logical vocabulary is universally elaborated from and explicative of inferential connections. (*Between Saying and Doing*, Oxford University Press, 2008, pp. 44-45).

that is a consequence of asserting it. To be more specific, let us look at an example relevant to the subject of this thesis:

"If 11 is prime, then it cannot be divided without remainder by any other number."

The expressive function of this conditional is to make explicit that to say "11 is prime" is to issue a ticket licensing a computing agent to infer, when given any another number *n* greater than 1 and not 11, that *n* does not divide 11 without remainder.[3]

Does "11 is prime" means the same when standing alone and when used as the premise of that conditional? Yes, "11 is prime" *must* mean the same in both cases; not because the same words are deployed to express the same proposition, but rather because the same practices underpin how both these sentences acquire meaning and for which function they are used. In order for someone to be able to deploy the items '11', 'prime', 'divided', 'remainder', this person would have to at least *know how* to count, and arguably also know how to operate with integers, know an algorithm to compute divisions, and know the definition of prime number.[4] In short, knowing how to perform the appropriate mathematical practice is *necessary* to be able to use mathematical vocabulary meaningfully.

Thus, embedding "11 is prime" in a conditional cannot change the content of the expression, for that is fixed by the position of the concepts applied in an inferential network. It only subtracts the pragmatic element of assertion: by asserting the conditional, one is not asserting that "11 is prime", but is instead drawing the inferential neighbourhood of the concept 'prime', since the job of the conditional is to draw a piece of the inferential potential that is consequence the application of the concept 'prime'. And this role is practically relevant because mastery of the use of this concept can only be achieved

[3] Pasquale Frascolla offers a similar interpretation of the concept 'prime' based on Wittgenstein's remarks as well. It goes: "Take a statement such as "11,003 is prime". Once this is proven, a grammar rule, excluding as senseless certain empirical descriptions of the form "such-and-such outcome has been obtained by a correct application of the decision procedure for the property of being prime" – and thus providing a new criterion for correctness of the operations of dividing 11,003 –, is adopted. Then, the attribution to the predicate "prime" of a meaning transcending the extension acknowledged up to the moment of the proof amounts to the assumption that the *general rules* of the method of checking the property are able normatively to condition the process leading to the adoption of such *a particular rule*." (*Wittgenstein's Philosophy of Mathematics*, London: Routledge, 1994, p. 57).

[4] I have only put 'counting' as a minimal criterion to remind the reader that if one already knows-how to count, then one possesses a set of abilities that can be elaborated through training and implementation of algorithms into the proper mathematical practice of dividing.

by knowing the possibilities of inferential articulation available from the application of that concept.

Appendix II: Pragmatic expressivism and Category Theory

A crucial piece of this thesis is the realization that we use mathematical formulae for a non-descriptive, metalinguistic-explanatory function, which traces out relations between linguistic categories of mathematical expressions that are essential for our talk of quantities, proportions, and spaces. But it is important to make clear that this metalinguistic aspect is not in the model-theoretical sense of a higher-order term that refers to a lower-order expression. Rather, it is in the sense of establishing normative connections amongst concepts which regulate the inferential traffic between premises and conclusions that deploy those concepts. To better explain this aspect and provide a bridge between this thesis and the actual mathematical practices, I will hereby introduce two possibilities of formalization in Category Theory for mathematical rules and concepts:

(i) Mathematical rules are *morphisms* between categories;[1]

(ii) Mathematical concepts are *linguistic categories.*[2]

According to Category Theory, a category is defined as a quadruple **C** = (O, hom, *id*, ∘) consisting of:

(I) A class of particular instances of **C** (a.k.a. **C**-points or **C**-objects) *A*, *B*, *C*,…

(II) For each pair of particulars, a set hom(*A*, *B*) whose members are morphisms $f: A \to B$

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[1] This suggestion was first made by André Porto in his "Wittgenstein on Mathematical Identities", in *Disputatio*, vol. IV, n° 34, 2012.

[2] Category theory was the fruit of Felix Klein's *Erlanger Programm*, a framework of research focused on projective geometry and symmetries of groups. The theory was introduced to facilitate the study and characterization of mathematical structures in terms of their structure-preserving transformations, especially the covariant and contravariant ones (See Awodey, S. *Category Theory,* Oxford: Clarendon, 2006, p. 1-2.). For more general information on categories, see <https://ncatlab.org/nlab/show/category>.

Examples of categories are sets, classes, ordinal numbers, groups, rings, fields, binary algebras, modules, monoids, distributive lattices, closure spaces, convergence spaces, matrices, metric spaces, topological spaces, vector spaces, and so on. For a comprehensive explanation of why these concepts are categories, see Mac Lane, S. *Categories for the Working Mathematician* (2nd ed, New York: Springer, 1971); Adamek; Herrlich; Strecker *Abstract and Concrete Categories* (Mineola: Dover Publications, 2009).

(III) For each particular, an identity morphism such as $id_A : A \rightarrow A$

(IV) And a rule of composition associating for any two **C**-morphisms, e.g. $f: A \rightarrow B$ and $g: B \rightarrow C$, a composite $f \circ g: A \rightarrow C$ subject, according to the following conditions:

(a) *associativity*: for the morphisms $f: A \rightarrow B$, $g: B \rightarrow C$, and $h: C \rightarrow D$ the equation $h \circ (g \circ f) = (h \circ g) \circ f$ holds.

(b) *unity*: identity morphisms may also be composed, as in for $f: A \rightarrow B$, we have $f \circ id_A = f = id_B \circ f$.

If we allow the interpretation that mathematical concepts are categories, then the metalinguistic connections between concepts may be conceived as *morphisms* between the particular instances of a category, such as the **C**-objects defined above. Just as we learn what is the role in reasoning of a particular instance of a concept by comprehending its position in an inferential network, we learn the role of a **C**-object by comprehending its morphisms to other categorical objects. And similarly to how we grasp concepts by mastering which inferences are precluded and licensed by the employment of that concept in predication, we grasp the structure of a mathematical category by comprehending the interconnections between its particular instances and the intra-linguistic relations it maintains with other categories (i.e. its functors to other categories).

Furthermore, as some category theorists have maintained: "In category theory it is the morphism, rather than the objects, that have the primary role."[3] and "It's the arrows that really matter!"[4] – In this theory, the "objects" are not self-subsistent, but particular instances of a category, defined by their morphisms to other instances, and not by a list of properties, as in set theory. In this sense, they are better defined for their *role* in the system of morphisms than as objects grouped together in a set for fulfilling a property.

It is this emphasis on variance and invariance under transformation, focused on the role the particular instance has in relation to the many transformations it undergoes, that makes the language of category theory such a powerful tool to find expression for what is peculiar about mathematical statements: that by prescribing the correct result of certain

[3] Adamek; Herrlich; Strecker *Abstract and Concrete Categories* (Mineola: Dover Publications, 2009), pgs. 28-29.

[4] Awodey, S. *Category Theory* (Oxford: Clarendon, 2006), pg. 8.

operations, they establish transformations (morphisms) between linguistic categories (concepts).

In particular, when I said that arithmetical equations establish a metalinguistic connection between instances of numbers, this may now be further specified. An equation between certain operations on integers are *isomorphisms* between particular instances of the category of integer. An isomorphism is a structure-preserving map, provided there are the morphisms $f: A \rightarrow B$ and $g: B \rightarrow A$, then $g \circ f = id_A$ and $f \circ g = id_B$. Given these conditions, the **C**-objects A and B are said to be isomorphic, which just means that they are identical.[5]

Other examples of such intra-categorical connections between operations on instances of the same concept are arithmetical equations, geometric formulae, and any other statement in which the concepts figuring in it are either introduced by definitions or implicitly defined by the basic premises or axioms of a singular system.

Yet, these are not the only connections drawn. There are also inter-categorical connections between operations carried out on distinct concepts. To visualize what I mean, think how the most powerful way to compute a system of linear equations such as:

$$x - 2y + z = 1$$
$$2x - y + z = 2$$
$$4x + y - z = 1$$

Is to separate the coefficients of the system as matrices:

$$\begin{matrix} A & \circ & X & = & B \end{matrix}$$

$$\begin{pmatrix} 1 & -2 & 1 \\ 2 & -2 & 1 \\ 4 & 1 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$$

Which one can then solve by finding the inverse of matrix A, then computing
$X = A^{-1} \circ B$.

So we may formulate inter-categorical connections in the language of category theory as *functors* between categories. A functor is a set of structure-preserving functions between categories that maps every morphism as it is. That is, a functor F with domain in category **C** and codomain in category **D**, denoted as $F : \mathbf{C} \rightarrow \mathbf{D}$, assigns to each **C**-object A a **D**-object $F(A)$, and for each **C**-morphism $f : A \rightarrow B$ a **D**-morphism $F(f) : F(A) \rightarrow F(B)$.

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[5] *See* Adamek; Herrlich; Strecker *Abstract and Concrete Categories* (Mineola: Dover Publications, 2009), p. 29.

Other examples of inter-categorical connections include but are not limited to: the algebraic solutions introduced by Descartes and Viète, who showed that equations can be a pivotal constraint and guide in solving geometrical problems; mappings between integers and real numbers; the connection between geometrical definite integrals to algebraic indefinite integrals established by the first fundamental theorem of calculus; Felix Klein's use of complex functions to solve problems in the theory of rotation groups; Andrew Willey's proof of Fermat Last Theorem, which had to draw connections between arithmetic and the algebraic concept of elliptic curves. As a rule of thumb, any math statement expressing articulations between concepts defined in two different systems is drawing an inter-categorical connection.

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