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Homotopy theory of gauge groups over certain 7-manifolds

by

Ingrid Amaranta Membrillo Solis

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ABSTRACT

FACULTY OF SOCIAL, HUMAN AND MATHEMATICAL SCIENCES
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Doctor of Philosophy

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The gauge groups of principal G-bundles over low dimensional spaces have been extensively studied in homotopy theory due to their connections to other areas in mathematics, such as the Yang-Mills gauge theory in mathematical physics. In 2011 Donaldson and Segal established the mathematical set-up to construct new gauge theories over high dimensional spaces.

In this thesis we study the homotopy theory of gauge groups over 7-manifolds that arise as total spaces of S^3-bundles over S^4 and their connected sums. We classify principal G-bundles over manifolds M up to isomorphism in the following cases:

(1) M is an S^3-bundle over S^4 with torsion-free homology;

(2) M is an S^3-bundle over S^4 with non-torsion-free homology and $\pi_6(G) = 0$;

(3) M is a connected sum of S^3-bundles over S^4 with torsion-free homology and $\pi_6(G) = 0$.

We obtain integral homotopy decomposition of the gauge groups in the cases for which the manifold is either a product of spheres, or a twisted product of spheres, or a connected sum of those. We obtain p-local homotopy decompositions of the loop spaces of the gauge groups in the cases for which the manifold has torsion in homology. Gauge groups of principal G-bundles over manifolds homotopy equivalent to S^7 are classified up to a p-local homotopy equivalence.

Contents

UNIVERSITY OF Southampton

Academic thesis: Declaration of authorship

I, **Ingrid Amaranta Membrillo Solis**, declare that this thesis and the work presented in it are my own and has been generated by me as the result of my own original research.

Homotopy theory of gauge groups over certain 7-manifolds

I confirm that:

1. This work was done wholly or mainly while in candidature for a research degree at this University;
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3. Where I have consulted the published work of others, this is always clearly attributed;
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"En el curso del tiempo he sido muchos, pero ese torbellino fue un largo sueño".
J. L. Borges, Undr.

List of symbols

$\mathbb{R}$	The field of real numbers.
$\mathbb{C}$	The field of complex numbers.
$\mathbb{H}$	The quaternion algebra.
$\mathbb{O}$	The octonion algebra.
$\mathbb{F}P^n$	The n-dimensional projective space over the algebra $\mathbb{F} = \mathbb{R}, \mathbb{C}, \mathbb{H}$ or $\mathbb{O}$.
$\mathbb{Z}_n$	Integers modulo n.
$\mathbb{Z}_{(p)}$	The group of integers localised at a prime p.
(n_1, n_2)	Greatest common divisor of integers n_1 and n_2.
$v_p(m)$	Valuation of an integer m at a prime p.
S^n	The n-sphere.
D^n	The n-disk.
$\pi_n(X)$	The n-th homotopy group of a pointed space X.
$H_n(X)$	The n-th homology group of a pointed space X.
$H^n(X)$	The n-th cohomology group of a pointed space X.
$\mathbb{1}_X$, $\mathbb{1}$	The identity map in a space X.
ι_n	The identity map $\mathbb{1}_{S^n} : S^n \to S^n$, viewed as a generator of $\pi_n(S^n)$.
$\operatorname{im} f$	Image of a function f.
$\ker f$	Kernel of a function f.
$\operatorname{coker} f$	Cokernel of a function f.
$X \sharp Y$	Connected sum of spaces X and Y.
X^n	The n-skeleton of a space X.
$\Omega^n X$	The n-loop space of a pointed space X, that is, $\mathrm{Map}^*(S^n, X)$.
$X \vee Y$	Wedge sum of pointed spaces X and Y.
$X \wedge Y$	Smash product of pointed spaces X and Y.
$\Sigma^n X$	The n-reduced suspension of a pointed space X.
$\Sigma^n f$	The n-reduced suspension of a pointed map f.
$P^n(k)$	The n-Moore space of degree k.
$\mathrm{Map}(X, Y)$	Set of maps between spaces X and Y.
$\mathrm{Map}^*(X, Y)$	Set of pointed maps between pointed spaces X and Y.
$\mathrm{Map}_f(X, Y)$	Path component of $\mathrm{Map}(X, Y)$ containing the map $f : X \to Y$.
$\mathrm{Map}^*_f(X, Y)$	Path component of $\mathrm{Map}^*(X, Y)$ containing the pointed map $f : X \to Y$.
$[X, Y]$	Set of homotopy classes of maps between spaces X and Y.

$[X,Y]_*$	Set of homotopy classes of pointed maps between pointed spaces X and Y.
$Prin_G(X)$	Set of isomorphism classes of principal G-bundles over a space X.
BG	The classifying space of a group G.
$\mathcal{G}_f(X)$	Gauge group of the principal G-bundle with classifying map $f : X \to BG$.
$\mathcal{G}^*_f(X)$	Pointed gauge group of the principal G-bundle with classifying map $f : X \to BG$.
Δ	Diagonal map.
∇	Folding map.
$[f,g]$	The Whitehead product of the maps f and g.
$\langle f,g\rangle$	The Samelson product of the maps f and g.
ev	The evaluation map.

Chapter 1

Introduction

1.1 Motivation

The study of the homotopy theory of mapping spaces treated as topological spaces dates back at least to the first half of the last century (see [FLS10]). The problem of classifying the homotopy types of path components of mapping spaces was introduced for the first time in 1946 by Whitehead. Some results on the classification of mapping spaces include computation of homotopy groups, cellular decompositions, and computation of the homology and cohomology of the path components of mapping spaces. Besides the intrinsic interest that exists from a homotopical point of view, the problem of the classification of mapping spaces is nowadays a subject of extensive research due to the connections to other areas of mathematics. One of these connections is provided by the gauge groups.

Let G be a topological group with classifying space BG. Given a principal G-bundle $p : P \to X$ with a classifying map $f : X \to BG$, the gauge group $\mathcal{G}_f(X)$ is the group of G-equivariant automorphisms $\phi : P \to P$ covering the identity in X. We say $\mathcal{G}_f(X)$ is a gauge group of the bundle $p : P \to X$ or a gauge group over the manifold X. Gottlieb [Got72] showed that there is a weak homotopy equivalence

$$B\mathcal{G}_f(X) \simeq \mathrm{Map}_f(X, BG),$$

where $\mathrm{Map}_f(X, BG)$ is the path component of $\mathrm{Map}(X, BG)$ containing the classifying map f of the principal G-bundle. This result allows us to regard the gauge groups as a link between the homotopy theory of mapping spaces and the more geometrical theory of principal bundles.

One of the biggest interests in understanding the topology of gauge groups lies in mathematical physics. Gauge groups appear in the formulation of the so-called gauge theories, which are physical theories mathematically formulated using the theory of fibre bundles.

A very simplified version of what a gauge theory consists of, from a mathematical perspective, is the following. Let $P \xrightarrow{p} M$ be a smooth principal G-bundle over M, where G is a compact Lie group. The tangent bundle of P, denoted TP, is a fibre bundle over P whose fibres are vector spaces tangent to P at each point. There is a special vector subbundle $V \subset TP$, called the vertical bundle, consisting of all vectors tangent to the fibres of P. A connection of a principal G-bundle is a choice of vector subbundle $A \subset TP$ that is invariant under the action of G and such that $TP = V \oplus A$. Physical phenomena are modelled by means of principal G-bundles P and certain functions $\mathcal{F} : \mathcal{A}_P \to \mathbb{R}$, where $\mathcal{A}_P$ is the space of all connections of TP. The gauge group $\mathcal{G}_P$ of the bundle P acts on the space of connections $\mathcal{A}_P$. The function $\mathcal{F}$ is defined to be invariant under the action of the gauge group, and so induces $\mathcal{F} : \mathcal{A}_P/\mathcal{G}_P \to \mathbb{R}$. The goal pursued in gauge theories is to find the subspace of connections $\mathcal{M}_P \subset \mathcal{A}_P/\mathcal{G}_P$ that contains the critical points of $\mathcal{F}$. Gauge theories have succeeded in modelling several physical phenomena such as electroweak force, however, the description of the dynamical behaviour of elementary particles in a 4-dimensional space-time is still an open question. According to Atiyah [Ati88], there is hope for gauge theories, such as the Yang-Mills gauge theory, to provide an answer to this question.

Application of gauge theories in the study of 4-manifolds has been successful in differential geometry. Using ideas coming from the Yang-Mills theory, Donaldson [Don86] obtained topological information of the classifying spaces of gauge groups of principal $SU(2)$-bundles to define polynomial invariants over 4-manifolds. These invariants have been used to distinguish differentiable structures on homeomorphic manifolds.

Understanding the topology of the gauge groups and their classifying spaces is crucial in the context of gauge theories and their applications in mathematics. A less rigid approximation to the study of these mapping spaces is the one provided by homotopy theory. Therefore, at this point the classification of mapping spaces, such as the gauge groups, becomes a primary problem in the context of homotopy theory.

Classification of gauge groups and their associated classifying spaces up to homotopy equivalence has been an active research area in homotopy theory for at least the last 25 years. For example, in [Mas91] Masbaum studied the homotopy classification of the path-components of $\mathrm{Map}(X, BSU(2))$ where X is a 4-dimensional CW-complex. He showed that there are infinitely many homotopy types amongst these mapping spaces. In contrast, Kono [Kon91] proved the following result regarding the number of homotopy types of the gauge groups of $SU(2)$-bundles over S^4.

Theorem A (to appear as Theorem 4.24). *Let $P_k \to S^4$ be the principal $SU(2)$-bundle over S^4 classified by $k \in \mathbb{Z}$, and let $\mathcal{G}_k$ be its gauge group. Then $\mathcal{G}_k$ is homotopy equivalent to $\mathcal{G}_{k'}$ if and only if $(12, k) = (12, k')$.*

This result shows that there are finitely many homotopy types of gauge groups of principal $SU(2)$-bundles over the 4-sphere. Crabb and Sutherland proved a general result on the number of homotopy types of gauge groups. In [CS00] it is proved that if the base space of the principal G-bundle is a finite CW-complex X and G is compact connected Lie group, then the number of homotopy types of gauge groups of the principal G-bundles over X is finite. As a consequence of this result, considerable attention has been paid to counting the number of homotopy types of gauge groups (see for instance [HK06, KKKT07, CHM08, CS09, The10b, The12, KKT14]).

Even though the theory of principal bundles is not limited by the dimension of the base space of the principal bundles, most work has been carried out in the case of low dimensional spaces. There are at least a couple of reasons that could explain this trend. The first one is related to the fact that the main gauge theories, such as the Yang-Mills theory are defined for principal bundles over low dimensional spaces. The second reason is the big success that the application of gauge theories have had in differentiable geometry. Gauge theories have substantially contributed to give some answers in the classification problem of differentiable 4-manifolds.

New ideas coming from mathematical physics suggest that mathematical modelling of physics of elementary particles might require the use of high dimensional spaces. In [DT98] Donaldson and Thomas introduced some ideas to construct gauge theories where the dimension of the base spaces of the principal G-bundles is higher than 4. A decade later, these ideas were formalised in [DS11], creating in this way a new area of research that has seen an accelerated development within differential geometry in recent years. However, the homotopy theory of principal G-bundles over high dimensional spaces and their gauge groups is largely unknown.

The aim of this work is to explore the homotopy theory of principal G-bundles over certain 7-dimensional manifolds and their gauge groups, which might be of the interest in other areas of mathematics such as mathematical physics. The family of 7-manifolds we are primarily interested in are those arising as the total spaces of S^3-bundles over S^4. We also are interested in the homotopy theory of gauge groups of principal G-bundles over connected sums of the aforementioned sphere bundles. The main results of this work are presented in the next section. We will keep the same numbering of the results as the one that is given in the following chapters.

1.2 Main results

The classification of S^3-bundles over S^4 goes back to the work of Steenrod on the classification of sphere bundles over spheres [Ste44, Ste51]. The following result is the classification of S^3-bundles over S^4.

Proposition B (to appear as Proposition 3.44)**.** *The equivalence classes of S^3-bundles over S^4 are in one-to-one correspondence with elements of $\pi_3(SO(4)) \cong \mathbb{Z} \oplus \mathbb{Z}$.*

Let $M_{l,m}$ be the total space of the S^3-bundle over S^4 classified by $(l,m) \in \mathbb{Z} \oplus \mathbb{Z}$ and let G be a simply connected simple compact Lie group. Let $Prin_G(M_{l,m})$ be the set of isomorphism classes of principal G-bundles over $M_{l,m}$. Our first result is the classification of principal G-bundles over manifolds $M_{l,m}$.

Proposition C (to appear as Proposition 5.6)**.** *Let G be a simply connected simple compact Lie group such that $\pi_6(G) \cong 0$. Then there is a one-to-one correspondence between $Prin_G(M_{l,m})$ and $\mathbb{Z}_m$. More precisely,*

(1) if $m = 0$ then $Prin_G(M_{l,m}) \xrightarrow{1\text{-}1} \mathbb{Z}$;

(2) if $m > 1$ then $Prin_G(M_{l,m}) \xrightarrow{1\text{-}1} \mathbb{Z}_m$.

Moreover, the projection $M_{l,m} \xrightarrow{\pi} S^4$ induces a map

$$\pi_* : [S^4, BG] \to [M_{l,m}, BG]$$

which is a bijection if $m = 0$ and a surjection if $m > 1$.

Proposition C shows that, assuming $\pi_6(G) \cong 0$, the number of isomorphism classes of principal G-bundles is infinite if $m = 0$, and finite otherwise. Once the sets $Prin_G(M_{l,m})$ have been computed the next step is to obtain a result that allows us to classify the homotopy types of gauge groups of principal bundles over manifolds $M_{l,m}$. Let $\mathcal{G}_k(M_{l,0})$ be the gauge group of the principal G-bundle classified by the integer k. The next result concerns this task for the cases where $M_{l,m}$ has torsion-free homology and $M_{l,m} \not\simeq S^7$. It is known that $Prin_G(S^4) = \mathbb{Z}$. Let $\mathcal{G}_k(S^4)$ denote the unpointed gauge group over S^4 classified by $k \in \mathbb{Z}$. Given a map $S^6 \xrightarrow{\xi_l} S^3$, with $\xi_l \not\simeq *$, we denote by Y_l its homotopy cofibre.

Theorem D (to appear as Theorem 5.10)**.** *Let G be a simply connected simple compact Lie group such that $\pi_6(G) \cong 0$ and let $M_{l,m} \to S^4$ be a sphere bundle with cross section. Let $P_k \to M_{l,m}$ be a principal G-bundle classified by $k \in \mathbb{Z}$. There are homotopy decompositions*

$$\mathcal{G}_k(M_{l,0}) \simeq \mathcal{G}_k(S^4) \times \mathrm{Map}^*(Y_l, G).$$

Moreover, if $l \equiv 0 \pmod{12}$ there are homotopy equivalences

$$\mathcal{G}_k(M_{l,0}) \simeq \mathcal{G}_k(S^4) \times \Omega^3 G \times \Omega^7 G.$$

Theorem D implies that the determination of the homotopy type of $\mathcal{G}_k(M_{l,0})$ is reduced to determining that of $\mathcal{G}_k(S^4)$ using Lie groups G with $\pi_6(G) = 0$. These gauge groups have been computed for different groups G. For example, from [The15, Theorem 1.1] we obtain the following corollary. Let (n_1, n_2) denote the greatest common divisor of n_1 and n_2.

Corollary E (to appear as Corollary 5.11)**.** *Suppose M is either $S^3 \times S^4$ or any twisted product $S^3 \tilde{\times} S^4$. Localised rationally or at any prime there are 16 homotopy types of gauge groups of principal $SU(5)$-bundles over M.*

The proof of Theorem D relies on the splitting of the cofibre $C_{l,m}(\pi)$ of the projection map $M_{l,m} \xrightarrow{\pi} S^4$. For the case of manifolds with torsion in homology, it is not clear if analogous splittings exist, however, we are able to obtain a splitting of $\Sigma C_{l,m}(\pi)$. As such, the results in Theorem F are stated in terms of the loops on gauge groups rather than the gauge groups themselves.

The cofibration sequence $S^n \xrightarrow{m} S^n \to P^{n+1}(m)$ induces a fibration sequence

$$\mathrm{Map}^*(P^{n+1}(m), BG) \to \mathrm{Map}^*(S^n, BG) \xrightarrow{m_*} \mathrm{Map}^*(S^n, BG).$$

Let $\Omega^n BG\{m\}$ denote the space $\mathrm{Map}^*(P^{n+1}(m), BG)$. If $M_{l,m}$ has torsion in homology we have the following result on the homotopy decomposition of the loop space of the gauge groups up to localisation at a prime $p \geq 5$. Let $v_p(m)$ denote the valuation of $m \in \mathbb{Z}$ at a prime p.

Theorem F (to appear as Theorem 5.15)**.** *Let $m > 1$ be an integer and $p \geq 5$ be a prime. Let $P_k \to M_{l,m}$ be a principal G-bundle classified by $k \in \mathbb{Z}_m$ where G is a simply connected simple compact Lie group. Suppose all spaces are localised at p. There are p-local homotopy equivalences*

(1) $\mathcal{G}_0(M_{l,m}) \simeq \Omega^7 G \times G$ if $v_p(m) = 0$;

(2) $\Omega\mathcal{G}_k(M_{l,m}) \simeq \Omega^8 G \times X_k$ if $v_p(m) \geq 1$, where X_k fits into a homotopy fibration

$$\Omega^4 G\{m\} \to X_k \to \Omega G.$$

Moreover, if $v_p(m) = r \geq 1$ and $p^r | k$, then $X_k \simeq \Omega G \times \Omega^4 G\{m\}$.

Chapter 5 ends with the homotopy classification of gauge groups of principal G-bundles over total spaces of S^3-bundles over S^4 such that $M_{l,m} \simeq S^7$. In this case G is any simply connected simple compact Lie group, in particular, the restriction to $\pi_6(G) = 0$ is not needed.

Theorem G (to appear as Theorem 5.17)**.** *Let G be a simply connected simple compact Lie group and let $P_k \to S^7$ and $P_{k'} \to S^7$ be principal G-bundles. Then*

(1) for $G = SU(2) \cong Sp(1)$ or $G = G_2$, there is a homotopy equivalence $\mathcal{G}_k(S^7) \simeq \mathcal{G}_{k'}(S^7)$ when localised rationally or at any prime if and only if $(3, k) = (3, k')$;

(2) for $G = SU(3)$, there is a homotopy equivalence $\mathcal{G}_k(S^7) \simeq \mathcal{G}_{k'}(S^7)$ when localised rationally or at a prime $p \geq 3$ if and only if $(3, k) = (3, k')$;

(3) otherwise, the gauge group of the unique principal G-bundle decomposes as

$$\mathcal{G}_0(S^7) \simeq \Omega^7 G \times G.$$

Chapter 6 is an extension of the results of Chapter 5. Let $o(G)$ be the order of $\pi_6(G)$. In the previous chapter all the groups G were such that $o(G) = 1$. This assumption simplifies the computation of the homotopy sets $[M_{l,m}, BG]$, since the action of $\pi_6(G)$ on the homotopy set $[M_{l,m}, BG]$ in this case is trivial. A new strategy is then required working with groups G such that $o(G) > 1$.

Let $M_{l,0}$ be the sphere bundle classified by $(l, 0) \in \mathbb{Z} \oplus \mathbb{Z}$. In the study of gauge groups of principal G-bundles over $M_{l,0}$ when $o(G) \neq 1$, the value of l for each $M_{l,0}$ is crucial. The following result generalises Proposition C for the case of manifolds with torsion-free homology.

Proposition H (to appear as Proposition 6.3)**.** *Let G be a simply connected simple compact Lie group. Then*

$$Prin_G(M_{l,0}) = \mathbb{Z} \times \mathbb{Z}_{(o(G),l)}.$$

Moreover, the projection $\pi : M_{l,0} \to S^4$ induces a bijection $\pi_* : \pi_4(BG) \to [M_{l,0}, BG]$ if $(l, o(G)) = 1$.

For some manifolds $M_{l,0}$ it is possible to extend the strategy used to obtain homotopy decompositions of gauge groups when $o(G) = 1$, to the case when $o(G) > 1$.

Theorem I (to appear as Theorem 6.5)**.** *Let G be a simply connected simple compact Lie group such that $o(G) > 1$. Given a principal G-bundle $P_k \to S^3 \tilde{\times}_l S^4$, if $(l, o(G)) = 1$, then there is a homotopy equivalence*

$$\mathcal{G}^*_k(S^3 \tilde{\times}_l S^4) \simeq \Omega^4 G \times \mathrm{Map}^*(Y_l, G).$$

The case of manifolds obtained as a connected sum of spaces $M_{l,0}$ is also studied in Chapter 6. Let M be a manifold such that

$$M \simeq M_{l_1} \sharp \cdots \sharp M_{l_d}$$

where $M_{l_i} = M_{l_i,0}$.

The manifold M has a cellular structure given by $(\bigvee_{i=1}^d S^3_i \vee \bigvee_{i=1}^d S^4_i) \cup_\Phi e^7$. Define $\Pi : M \to \bigvee_{i=1}^d S^4_i$ as the composite

$$\Pi : M \xrightarrow{\text{pinch}} \bigvee^d M_{l_i} \xrightarrow{\bigvee^d \pi_i} \bigvee^d S^4_i,$$

where the maps $\pi_i : M_{l_i} \to S^4_i$ are projections. The following is a classification of principal G-bundles over connected sums of manifolds M_{l_i}.

Proposition J (to appear as Proposition 6.11)**.** *If $\pi_6(G) \cong 0$ then $Prin_G(M) = \mathbb{Z}^d$. Moreover, the map*

$$\Pi_* : \bigoplus_{i=1}^{d} \pi_4(BG) \to [M, BG]$$

is a bijection.

A homotopy decomposition of gauge groups is also obtained in the case when M is a connected sum of torsion-free S^3-bundles over S^4 and G satisfies $\pi_6(G) \cong 0$.

Theorem K (to appear as Theorem 6.12)**.** *Let $P_K \to M$ be a principal G-bundle over a manifold $M \simeq M_{l_1} \sharp \cdots \sharp M_{l_d}$ classified by $K = (k_1, \ldots, k_d) \in \mathbb{Z}^d$. Suppose G is a simply connected simple compact Lie group such that $\pi_6(G) \cong 0$. Then there exists a homotopy decomposition*

$$\mathcal{G}_K(M) \simeq \mathcal{G}_K(\bigvee^{d} S^4_i) \times \mathrm{Map}^*(C_\Pi, BG)$$

where C_Π is the homotopy cofibre of the map $\Pi : M \to \bigvee^d S^4$.

1.3 Summary of contents

Chapter 2 contains elementary definitions and some well-known results in homotopy theory. The material presented in this chapter is the basis for this work and it will be used throughout the following discussion.

Chapter 3 is dedicated to the discussion of the topology and homotopy theory of S^3-bundles over S^4. The classification of S^3-bundles over S^4 is stated in Proposition 3.44. The works of James and Whitehead [JW54], Sasao [Sas65] and Crowley and Escher [CE03] are discussed in relation to the homotopy classification of the total spaces of S^3-bundles over S^4.

In Chapter 4, the topology and homotopy theory of principal G-bundles and their associated gauge groups are discussed, including some aspects of the classification of principal G-bundles. An important result on the homotopy theory of gauge groups is presented in Theorem 4.16, which will be widely used throughout the subsequent chapters. Some results on the homotopy theory of gauge groups of principal G-bundles over spaces of dimension $n \leq 4$ are included. In the last section of this chapter we lay out some results on gauge groups over high dimensional manifolds inside and outside of homotopy theory.

The last two chapters of this work contain the proofs of the results stated in Section 1.2. Chapter 5 is entirely dedicated to the homotopy theory of gauge groups related to

S^3-bundles over S^4, whereas Chapter 6 contains some additional results for the sphere bundles and the proofs of the results for connected sums.

Chapter 2

Elements of homotopy theory

In this chapter we present elementary definitions and results in homotopy theory that will be used to discuss the homotopy theory of gauge groups. We start by looking at the description of mapping spaces as topological spaces, as well as defining homotopy sets. We also introduce the definition of an H-space and its dual, a co-H-space. Then we define two more constructions, namely, fibrations and cofibrations. In the last two sections of this chapter we discuss some of the more specific constructions that will become important in the study of the homotopy theory of gauge groups. Theorems and propositions presented in this chapter are well-known results and most of the proofs are omitted. The material presented in this chapter is based on [AGP08, Ark11, DK01, Sel08, Whi78].

2.1 Mapping spaces and homotopy sets

Let X and Y be topological spaces. A *map* $f : X \to Y$ is a continuous function between X and Y. We denote by $\mathrm{Map}(X, Y)$ the set of all maps from X to Y. We can endow $\mathrm{Map}(X, Y)$ with the *compact-open topology*: take as a subbasis the family of sets $\omega(K, U)$, for $K \subset X$ compact and $U \subset Y$ open, defined by

$$\omega(K, U) = \{f \in \mathrm{Map}(X, Y) \mid f(K) \subset U\}. \tag{2.1.1}$$

Given maps $f : X \to Y$ and $g : A \to B$, there exists an induced map

$$g^f : \mathrm{Map}(Y, A) \to \mathrm{Map}(X, B)$$

defined by $g^f(\alpha) = g \circ \alpha \circ f$. This way we can obtain two functors $\mathcal{F}, \mathcal{F}' : Top \to Top$ in the category Top of topological spaces and continuous maps. For a topological space

Y, we denote the identity in Y by $\mathbb{1}_Y : Y \to Y$, or by $\mathbb{1} : Y \to Y$ if the space Y is clear from the context.

(1) For a fixed topological space Z, let $\mathcal{F}(-) = \mathrm{Map}(-, Z) : Top \to Top$ be the contravariant functor such that if $g : X \to Y$ is any map then

$$\mathcal{F}(g) = \mathbb{1}_Z^g : \mathrm{Map}(Y, Z) \to \mathrm{Map}(X, Z). \tag{2.1.2}$$

(2) For a fixed topological space X, let $\mathcal{F}'(-) = \mathrm{Map}(X, -) : Top \to Top$ be the covariant functor such that if $h : Y \to Z$ is any map then

$$\mathcal{F}'(h) = h^{\mathbb{1}_X} : \mathrm{Map}(X, Y) \to \mathrm{Map}(X, Z). \tag{2.1.3}$$

We will usually denote $\mathcal{F}(g)$ and $\mathcal{F}'(h)$ by g_* and h_*, respectively, and we will call g_* and h_* the maps *induced* by g and h, respectively.

Definition 2.1. Let X, Y be topological spaces. The *product* of X and Y, denoted $X \times Y$, is a topological space along with two maps $p_X : X \times Y \to X$ and $p_Y : X \times Y \to Y$, called the projections, that satisfies the following universal property: given two maps $f : Z \to X$ and $g : Z \to Y$, there exists a unique map $h : Z \to X \times Y$, denoted by $h = (f, g)$, such that the diagram

$$\begin{array}{ccccc} & & Z & & \\ & \swarrow^{f} & \downarrow{\scriptstyle h} & \searrow^{g} & \\ X & \xleftarrow{p_X} & X \times Y & \xrightarrow{p_Y} & Y \end{array}$$

commutes. The *diagonal* map $\Delta : X \to X \times X$ is defined by $\Delta = (\mathbb{1}_X, \mathbb{1}_X)$, that is, $\Delta(x) = (x, x)$.

Dually, we define the *coproduct* of X and Y, $X \amalg Y$, by the following universal property: there exist maps $i_X : X \to X \amalg Y$ and $i_Y : Y \to X \amalg Y$, called inclusions, such that given two maps $f : X \to Z$ and $g : Y \to Z$, there exists a unique map $h : X \amalg Y \to Z$, denoted by $h = \{f, g\}$, making the diagram

$$\begin{array}{ccccc} & & Z & & \\ & \nearrow^{f} & \uparrow{\scriptstyle h} & \nwarrow^{g} & \\ X & \xrightarrow{i_X} & X \amalg Y & \xleftarrow{i_Y} & Y \end{array}$$

commute. The *folding* map $\nabla = X \amalg X \to X$ is defined by $\nabla = \{\mathbb{1}_X, \mathbb{1}_X\}$.

The following proposition is a consequence of the universal properties of $X \times Y$ and $X \amalg Y$.

Proposition 2.2. *Let X, Y, Z be Hausdorff topological spaces such that X and Y are Hausdorff. There exist homeomorphisms*

(1) $\mathrm{Map}(X \amalg Y, Z) \cong \mathrm{Map}(X, Z) \times \mathrm{Map}(Y, Z)$,

(2) $\mathrm{Map}(X, Y \times Z) \cong \mathrm{Map}(X, Y) \times \mathrm{Map}(X, Z)$. □

Theorem 2.3 (Exponential Law)**.** *Let X, Y, Z be topological spaces such that X and Y are Hausdorff and Y is locally compact. There exists a homeomorphism*

$$\mathrm{Map}(X \times Y, Z) \cong \mathrm{Map}(X, \mathrm{Map}(Y, Z)).$$ □

Definition 2.4. A topological space X is *pointed* if it has a distinguished point $x_0 \in X$ called the *basepoint*. A map $f : X \to Y$ is a *pointed map* if $f(x_0) = y_0$, with x_0 and y_0 the basepoints of X and Y, respectively. We denote by $\mathrm{Map}^*(X, Y)$ the set of pointed maps of X to Y. Given two pointed topological spaces X and Y, we define its *wedge sum*, or simply *wedge*, as the quotient space

$$X \vee Y = X \amalg Y / \{x_0, y_0\}$$

where x_0 and y_0 are the basepoints of X and Y, respectively. Notice that we can identify $X \vee Y$ with a subspace of $X \times Y$ as

$$X \vee Y = \{(x, y) | x = x_0 \text{ or } y = y_0\}.$$

Note that for pointed spaces X and Y, their product $X \times Y$ and wedge sum $X \vee Y$ can be defined in a categorical setting, as in Definition 2.1, if instead of the category of spaces and maps we use the category of pointed spaces and pointed maps.

We may define three maps:

(1) given maps $f : X \to X'$ and $g : Y \to Y'$ between pointed or unpointed topological spaces, define $f \times g : X \times Y \to X' \times Y'$ by

$$(f \times g)(x, y) = (f(x), g(y));$$

(2) given maps $f : X \to X'$ and $g : Y \to Y'$ between pointed topological spaces, define $f \vee g : X \vee Y \to X' \vee Y'$ by

$$(f \vee g)(x, y) = (f(x), g(y));$$

(3) given maps $f : X \to Z$ and $g : Y \to Z$ between pointed topological spaces, define $\{f, g\} : X \vee Y \to Z$ by

$$\{f, g\}(x, y) = \begin{cases} f(x) & \text{if } y = y_0, \\ g(y) & \text{if } x = x_0. \end{cases}$$

Proposition 2.5. *Let X, Y, Z be pointed topological spaces such that X and Y are Hausdorff. There exist homeomorphisms*

(1) $\mathrm{Map}^*(X \vee Y, Z) \cong \mathrm{Map}^*(X, Z) \times \mathrm{Map}^*(Y, Z)$,

(2) $\mathrm{Map}^*(X, Y \times Z) \cong \mathrm{Map}^*(X, Y) \times \mathrm{Map}^*(X, Z)$. □

Let X, Y be pointed topological spaces with basepoints x_0 and y_0, respectively. We define the *smash product* of X and Y as the quotient space

$$X \wedge Y = X \times Y \;/\; X \times \{y_0\} \cup \{x_0\} \times Y.$$

Notice that $X \wedge Y = X \times Y / X \vee Y$.

Theorem 2.6 (Pointed Exponential Law)**.** *Let X, Y, Z be pointed topological spaces such that X and Y are Hausdorff. There exists a homeomorphism*

$$\mathrm{Map}^*(X \wedge Y, Z) \cong \mathrm{Map}^*(X, \mathrm{Map}^*(Y, Z)).$$ □

Definition 2.7. A Haussdorff topological space X is *compactly generated* if it satisfies the following condition: a subset A of X is closed if and only if $A \cap C$ for every compact subset C of X.

Definition 2.7 implies that a space is compactly generated if its topology is the *weak topology* generated by all of its compact subsets.

Most elementary homotopy theory can be developed in an arbitrary category of (pointed) topological spaces. However, it becomes necessary to work in a category that is closed under certain constructions such as product spaces, mapping spaces or identification spaces, for instance. It is also desirable that properties of mapping spaces such as the exponential law hold in complete generality. Thus, for now on we will assume that all spaces are compactly generated. For a further discussion on the convenience of choosing the category of compactly generated spaces and their continuous maps, we refer the reader to [Ste67].

Let I denote the closed unit interval $[0, 1]$. A *path* on X is a map $\gamma \in \mathrm{Map}(I, X)$. Given $x, y \in X$ we say that x is connected with y whenever there exists a path α such that $\alpha(0) = x$ and $\alpha(1) = y$. This definition is an equivalence relation which divides X into subsets called *path components*. The set of path components of X is denoted by $\pi_0(X)$. We say that X is *path connected* if $\pi_0(X)$ has a single element.

Definition 2.8. The maps $f, g : X \to Y$ are *homotopic*, denoted by $f \simeq g$, if there exists a map $H : X \times I \to Y$, called a *homotopy*, such that $H(x, 0) = f(x)$ and $H(x, 1) = g(x)$. If X, Y are pointed spaces we say that the homotopy H *preserves the basepoint* if in addition $H(x_0, t) = y_0$ for all $t \in I$, where x_0 and y_0 are the basepoints of X and Y, respectively.

The concept of homotopy defines an equivalence relation on the set $\mathrm{Map}(X, Y)$. Thus given spaces X and Y we denote by $[X, Y]$ the set of equivalence classes called *homotopy classes of maps* from X to Y. We denote by $[X, Y]_*$ the set of basepoint preserving homotopy classes of maps. If the spaces X, Y are locally compact Hausdorff spaces then we have

$$[X, Y] = \pi_0(\mathrm{Map}(X, Y)).$$

A based set is a set S with a fixed element s_0 called the basepoint. Let S and T be based sets with basepoints s_0, t_0 respectively. Then a function $f : S \to T$ such that $f(s_0) = t_0$ is called a *based function* or *pointed function*. The *kernel* of a based function $f : S \to T$, denoted $\ker f$, is the set

$$\ker f = f^{-1}(t_0) = \{x \in S \mid f(x) = t_0\}.$$

The *image* of a function f is

$$\mathrm{im}\, f = f(S) = \{f(x) \mid x \in S\}.$$

The concept of exact sequence is common in the context of abelian categories. There is a useful generalisation that works in the category of pointed sets. Let A, B, C be pointed sets with basepoints a_0, b_0, c_0, respectively. A sequence of functions

$$A \xrightarrow{f} B \xrightarrow{g} C \tag{2.1.4}$$

such that $f(a_0) = b_0$ and $g(b_0) = c_0$ is called *exact at* B if

$$f(A) = g^{-1}(c_0),$$

that is, $\mathrm{im}\, f = \ker g$. The sequence of based sets and based functions

$$\cdots \longrightarrow S_{i-1} \xrightarrow{f_{i-1}} S_i \xrightarrow{f_i} S_{i+1} \longrightarrow \cdots$$

is *exact* if it is exact at each S_i.

We wish to highlight some subtleties that arise due to the lack of multiplicative structures in pointed sets. In the category of groups for instance, it is well known that a sequence

$$1 \xrightarrow{h} A \xrightarrow{f} B \tag{2.1.5}$$

is exact if and only if f is injective. For pointed sets and pointed functions, exactness of (2.1.5) does not necessarily imply that the map f is injective. We ilustrate this point with two examples.

Example 2.9. Consider the sequence (2.1.5) with $A = \{1, 2, 3\}$ and $B = \{1, 2\}$ (both with basepoint 1), where f is defined by $f(1) = 1$ and $f(2) = f(3) = 2$. This is exact at A since $h(\{1\}) = f^{-1}(1) = \{1\}$; however, the map f is not injective.

Example 2.10. For any $m \in \mathbb{R}$, let $\lfloor m \rfloor$ be the integer part of m. Suppose $A = B = \mathbb{Z}$ and $g : A \to B$ is defined by

$$g(a) = \begin{cases} \lfloor \sqrt{a} \rfloor & \text{if } a \geq 0, \\ -\lfloor \sqrt{-a} \rfloor & \text{if } a < 0. \end{cases}$$

Then the sequence of sets $1 \to A \xrightarrow{g} B \to 1$ is exact and g is surjective, but it is not injective: the set $g^{-1}(b)$ for $b \neq 0$ has cardinality $2|b| + 1$.

On the other hand, if in addition, the sets A, B and C in (2.1.4) are groups with basepoints the identity elements, and f and g are homomorphisms, then the sequence of sets is exact if and only if it is a exact sequence of groups. In particular, g induces an isomorphism between $g(B)$ and $B/f(A)$.

If Y is path connected, then the set $[X, Y]$ has a unique homotopy class containing all the constant maps. This class will be used as a basepoint of $[X, Y]$ if one is needed.

Definition 2.11. Let X be a topological space. If $n \geq 1$, the *n-th homotopy group* of X is

$$\pi_n(X) = [S^n, X]_*.$$

A map $f : X \to Y$ induces a homomorphism $f^* : \pi_n(X) \to \pi_n(Y)$ for all $n \geq 1$, and if $f \simeq g$ then $f^* = g^* : \pi_n(X) \to \pi_n(Y)$ for all $n \geq 0$. A topological space Y is *n-connected* if $\pi_n(Y) = 0$ for all $i \leq n$. A 1-connected topological space is called simply connected.

Definition 2.12. A map $f : X \to Y$ is a *homotopy equivalence* if there exists a map $g : Y \to X$ such that $f \circ g \simeq \mathbb{1}_Y$ and $g \circ f \simeq \mathbb{1}_X$. If there exists a homotopy equivalence $f : X \to Y$, the spaces X and Y are said to be *homotopy equivalent.*

A map $f : X \to Y$ is a *weak homotopy equivalence* if $f^* : \pi_n(X) \to \pi_n(Y)$ is an isomorphism for all n. Notice that a homotopy equivalence is also a weak homotopy equivalence.

2.2 H-spaces and co-H-spaces

Definition 2.13. A pointed topolological space X is an *H-space* if there is a map $\mu : X \times X \to X$ such that if $* : X \to X$ is the constant map to the basepoint, then the diagrams

$$\begin{array}{ccc} X & \xrightarrow{(\mathbb{1},*)} & X \times X \\ & \searrow^{\mathbb{1}_X} & \downarrow \mu \\ & & X \end{array} \quad \text{and} \quad \begin{array}{ccc} X & \xrightarrow{(*,\mathbb{1})} & X \times X \\ & \searrow^{\mathbb{1}_X} & \downarrow \mu \\ & & X \end{array}$$

commute up to homotopy. We say that μ is a *multiplication.* An H-space is *homotopy associative* if the following diagram commutes up to homotopy

$$\begin{array}{ccc} X \times X \times X & \xrightarrow{\mu \times \mathbb{1}} & X \times X \\ \downarrow \mathbb{1} \times \mu & & \downarrow \mu \\ X \times X & \xrightarrow{\quad \mu \quad} & X. \end{array}$$

A map $\iota : X \to X$ is a *homotopy inverse* of X if the diagrams

$$\begin{array}{ccc} X & \xrightarrow{(\iota,\mathbb{1})} & X \times X \\ & \searrow^{*} & \downarrow \mu \\ & & X \end{array} \quad \text{and} \quad \begin{array}{ccc} X & \xrightarrow{(\mathbb{1},\iota)} & X \times X \\ & \searrow^{*} & \downarrow \mu \\ & & X \end{array}$$

homotopy commute. An *H-group* is a homotopy associative H-space X with a homotopy inverse.

Example 2.14. A *topological group* G is a set G together with a group structure and a topology on G such that the function $(g, h) \mapsto gh^{-1}$ is a map $G \times G \to G$. All topological groups are H-groups.

Example 2.15. The unit spheres in $\mathbb{C}$ and $\mathbb{H}$, S^1 and S^3, are topological groups and therefore H-groups. The unit sphere S^7 in the division algebra $\mathbb{O}$ is an H-space that is not homotopy associative.

Definition 2.16. For a space X, the *loop space* ΩX is defined by

$$\Omega X = \{\alpha : I \to X \mid \alpha(0) = * = \alpha(1)\}.$$

Notice that $\Omega X = \mathrm{Map}^*(S^1, X)$. The map $\mu : \Omega X \times \Omega X \to \Omega X$, given by

$$\mu(\alpha, \alpha')(t) = \begin{cases} \alpha(2t) & \text{if } 0 \leq t \leq \frac{1}{2}, \\ \alpha'(2t-1) & \text{if } \frac{1}{2} \leq t \leq 1, \end{cases}$$

defines a multiplication in ΩX.

Indeed, the spaces

$$\Omega^n X = \mathrm{Map}^*(S^n, X)$$

are H-groups.

Given two H-spaces X and X' with multiplications μ and μ', respectively, a map $h : X \to X'$ is called an *H-map* or an *H-homomorphism* if the following diagram homotopy commutes

$$\begin{array}{ccc} X \times X & \xrightarrow{\mu} & X \\ \downarrow{\scriptstyle h\times h} & & \downarrow{\scriptstyle h} \\ X' \times X' & \xrightarrow{\mu'} & X'. \end{array}$$

If Y is an H-space with multiplication m, and X is any space, then we can endow $\mathrm{Map}^*(X, Y)$ with a multiplication defined as follows. Let $f, g \in \mathrm{Map}^*(X, Y)$ and define $f + g \in \mathrm{Map}^*(X, Y)$ as the composite

$$f + g : X \xrightarrow{\Delta} X \times X \xrightarrow{f\times g} Y \times Y \xrightarrow{m} Y.$$

This can be used to show the following result.

Theorem 2.17. *Let Y be an H-group and X be any pointed topological space. Then $[X, Y]_*$ has a group structure.* □

Definition 2.18. A pointed topological space Y with basepoint y_0 is a *co-H-space* if there is map $\sigma : Y \to Y \vee Y$, called a *comultiplication*, such that the diagrams

$$\begin{array}{ccc} X & \xrightarrow{\sigma} & X \vee X \\ & {\scriptstyle \mathbb{1}_X}\searrow & \downarrow{\scriptstyle p_1} \\ & & X \end{array} \quad \text{and} \quad \begin{array}{ccc} X & \xrightarrow{\sigma} & X \vee X \\ & {\scriptstyle \mathbb{1}_X}\searrow & \downarrow{\scriptstyle p_2} \\ & & X \end{array}$$

commute up to homotopy, where $p_1 = \{\mathbb{1}_X, *\}, p_2 = \{*, \mathbb{1}_X\} : X \vee X \to X$ are the projection maps to the first and the second factors, respectively. A co-H-space is *homotopy associative* if the following diagram commutes up to homotopy

$$\begin{array}{ccc} X & \xrightarrow{\sigma} & X \vee X \\ \downarrow{\scriptstyle \sigma} & & \downarrow{\scriptstyle \sigma\vee\mathbb{1}} \\ X \vee X & \xrightarrow{\mathbb{1}\vee\sigma} & X \vee X \vee X. \end{array}$$

A map $j : X \to X$ is a *homotopy inverse* of X if the diagrams

$$\begin{array}{ccc} X & \xrightarrow{\sigma} & X \vee X \\ & {\scriptstyle *}\searrow & \downarrow{\scriptstyle \{\mathbb{1},j\}} \\ & & X \end{array} \quad \text{and} \quad \begin{array}{ccc} X & \xrightarrow{\sigma} & X \vee X \\ & {\scriptstyle *}\searrow & \downarrow{\scriptstyle \{j,\mathbb{1}\}} \\ & & X \end{array}$$

homotopy commute. A *co-H-group* is a homotopy associative co-H-space X with a homotopy inverse.

Definition 2.19. Let X any pointed space and let $\Sigma X = X \wedge S^1$ be the *reduced suspension* of X. Then the map $\sigma : \Sigma X \to \Sigma X \vee \Sigma X$ defined by

$$\sigma(x \wedge s) = \begin{cases} (x \wedge 2s, *) & \text{if } 0 \leq s \leq \frac{1}{2}, \\ (*, x \wedge (2s-1)) & \text{if } \frac{1}{2} \leq s \leq 1, \end{cases}$$

defines a comultiplication in ΣX. In particular, by taking $X = \{x_0, x_1\}$, this gives a comultiplication on S^1.

Indeed, the spaces

$$\Sigma^n X = X \wedge S^n$$

are co-H-groups.

Given two co-H-spaces X and X' with comultiplications σ and σ', respectively, a map $g : X \to X'$ is called a *co-H-map* if the following diagram homotopy commutes

$$\begin{array}{ccc} X & \xrightarrow{g} & X \\ \sigma\downarrow & & \downarrow\sigma' \\ X \vee X & \xrightarrow{g \vee g} & X' \vee X'. \end{array}$$

If X is a co-H-space and Y is any space then the mapping space $\mathrm{Map}^*(X, Y)$ has a binary operation. Given $f, g \in \mathrm{Map}^*(X, Y)$ we define $f + g = \nabla \circ (f \vee g) \circ \sigma = \{f, g\} \circ \sigma$,

$$f + g : X \xrightarrow{\sigma} X \vee X \xrightarrow{f \vee g} Y \vee Y \xrightarrow{\nabla} Y.$$

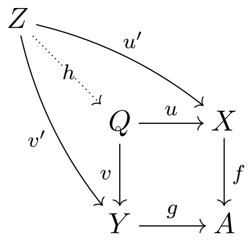
The following theorem states that the co-H-structure of X induces a group structure in homotopy sets.

Theorem 2.20. *Let X be a co-H-group and Y be any pointed topological space. Then $[X, Y]_*$ has a group structure.* □

2.3 Fibrations and cofibrations

Definition 2.21. Let $f : X \to A$ and $g : Y \to A$ be maps. The *pullback* of f and g is a space Q along with two maps $u : Q \to X$ and $v : Q \to Y$ such that $f \circ u = g \circ v$. Furthermore, the pullback satisfies the following universal property. If $u' : Z \to X$ and $v' : Z \to Y$ are maps such that $f \circ u' = g \circ v'$, then there exists a unique map $h : Z \to Q$

such that $u' = u \circ h$ and $v' = v \circ h$, i.e.

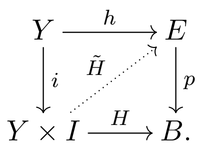


commutes. The rectangular diagram above is called a *pullback square*. Explicitly, we may take Q to be the space

$$Q = \{(x, y) \in X \times Y \mid f(x) = g(y)\},$$

and the maps u and v to be restrictions to Q of the projections $q_1 : X \times Y \to X$ and $q_2 : X \times Y \to Y$, respectively.

Definition 2.22. The map $p : E \to B$ has the *homotopy lifting property* with respect to a space Y if, given a homotopy $H : Y \times I \to B$ and a map $h : Y \to E$ such that $p \circ h(y) = H(y, 0)$, there exists a homotopy $\tilde{H} : Y \times I \to E$ with $\tilde{H}(y, 0) = h(y)$ and $p \circ \tilde{H} = H$. Diagrammatically, given any two maps h and H making the square in the following diagram commute, there exists a map $\tilde{H}$ making the whole of this diagram commute:



If the map $p : E \to B$ has the homotopy lifting property with respect to Y for all Y then we say that p is a *fibration*. We call B the *base space*, E the *total space* and the inverse image of a point $b \in B$, $F = p^{-1}(\{b\})$, is called the *fibre* over b. The sequence

$$F \xrightarrow{i} E \xrightarrow{p} B,$$

where i is the inclusion, is called a *fibration sequence*.

Example 2.23. If F and B are any spaces then the sequence

$$F \hookrightarrow B \times F \xrightarrow{p_1} B,$$

where p_1 is the projection onto the first factor, is a fibration sequence. The map p_1 is called the *trivial fibration*.

Example 2.24. For any space Y, the *path space* of Y, denoted PY, is the subspace of $\mathrm{Map}(I, Y)$ (with the compact open topology) defined by

$$PY = \mathrm{Map}^*(I, Y) = \{l \in \mathrm{Map}(I, Y) \mid l(1) = y_0\},$$

where y_0 is the basepoint of Y. The map $p : PY \to Y$ defined by $p(l) = l(0)$ is called the *path space fibration*, and it defines the following fibration sequence

$$\Omega Y \to PY \xrightarrow{p} Y.$$

Proposition 2.25. *Let $F \xrightarrow{i} E \xrightarrow{p} B$ be a fibration sequence, and let Y be any space. Then the sequence of sets*

$$[Y, F]_* \xrightarrow{i_*} [Y, E]_* \xrightarrow{p_*} [Y, B]_*$$

is exact. □

Let $f : X \to Y$ be a map. The *mapping path space* of f or the *homotopy fibre* of f, denoted F_f, is the space defined by the pullback square

$$\begin{array}{ccc} F_f & \xrightarrow{q_2} & PY \\ {\scriptstyle q_1}\downarrow & & \downarrow{\scriptstyle p} \\ X & \xrightarrow{f} & Y \end{array}$$

where p is the path space fibration.

Notice that if $X \xrightarrow{f} Y$ is a fibration with fibre F, then $F \simeq F_f$.

Proposition 2.26. *Let*

$$\begin{array}{ccc} Q & \xrightarrow{u} & X \\ {\scriptstyle v}\downarrow & & \downarrow{\scriptstyle f} \\ Z & \xrightarrow{g} & Y \end{array}$$

be a pullback square. If f is a fibration, then so is v. In this case u induces a homeomorphism $\tilde{u} : F_v \to F_f$ of fibres. □

Notice that by Proposition 2.26, there is a fibration sequence

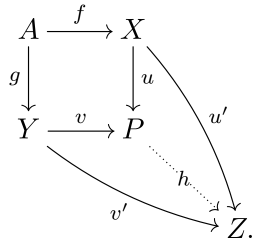
$$\Omega Y \to F_f \to X,$$

called the *principal fibration induced by f*.

Theorem 2.27. *Let Z be a topological space. For any pointed map $f : X \to Y$ the following sequence is a long exact sequence of sets ($i \geq 0$), groups ($i \geq 1$), and abelian groups ($i \geq 2$):*

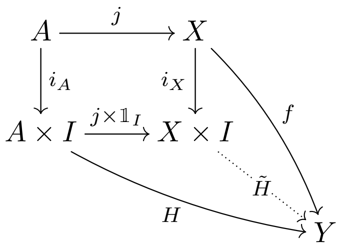
$$\cdots \to [Z, \Omega^i F_f]_* \to [Z, \Omega^i X]_* \to [Z, \Omega^i Y]_* \to$$
$$\cdots \to [Z, \Omega Y]_* \to [Z, F_f]_* \to [Z, X]_* \to [Z, Y]_*.$$ □

Let $f : A \to X$ and $g : A \to Y$ be maps. The *pushout* of f and g is a space P along with maps $u : X \to P$ and $v : Y \to P$ such that $u \circ f = v \circ g$. Moreover, the pushout satisfies the following universal property. If $u' : X \to Z$ and $v' : Y \to Z$ are maps such that $u' \circ f = v' \circ g$, then there exists a unique map $h : P \to Z$ such that $u' = h \circ u$ and $v' = h \circ v$:



The rectangular diagram above is called a *pushout square*.

An inclusion $j : A \to X$ has the *homotopy extension property* with respect to a space Y if for every map $f : X \to Y$ and every homotopy $H : A \times I \to Y$ such that $H(a, 0) = f \circ j(a)$, there exists a homotopy $\tilde{H} : X \times I \to Y$ such that the following diagram commutes



where i_A and i_X are inclusions. The map j is called a *cofibration* if it has homotopy extension property with respect to any space Y. If j is a cofibration then $C_j = X/j(A)$ is called the *cofibre* of j, and the sequence

$$A \xrightarrow{j} X \xrightarrow{q} C_j$$

is called a *cofibration sequence*, where q is the projection onto the quotient space.

Example 2.28. Let X be a space and $X \times I$ the cylinder over X. The *cone* on X, denoted CX, is the space $X \wedge I$ obtained by identifying $X \times \{1\} \cup \{*\} \times I$ in $X \times I$ to a single point. The map $j : X \to CX$ given by $j(x) = (x, 0)$ defines the following cofibration sequence

$$X \xrightarrow{j} CX \to \Sigma X.$$

Example 2.29. Given any spaces X and Y, the inclusions $i_1 : X \to X \vee Y$ and $i_2 : Y \to X \vee Y$ define the cofibration sequences

$$X \xrightarrow{i_1} X \vee Y \to Y,$$

$$Y \xrightarrow{i_2} X \vee Y \to X.$$

Proposition 2.30. *Let*

$$\begin{array}{ccc} A & \xrightarrow{f} & X \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle j} \\ Y & \xrightarrow{i} & P \end{array}$$

be a pushout square. If f is a cofibration, then so is i. In this case, j induces a homeomorphism $\tilde{j} : C_f \to C_i$ of cofibres. □

Proposition 2.31. *Let $A \xrightarrow{j} X \xrightarrow{q} C_j$ be a cofibration sequence, and let Y be any space. Then the sequence of sets*

$$[C_j, Y]_* \xrightarrow{q^*} [X, Y]_* \xrightarrow{j^*} [A, Y]_*$$

is exact. □

Given a map $f : X \to Y$, the *mapping cone* of f or the *homotopy cofibre* of f is the space C_f defined by the following pushout square

$$\begin{array}{ccc} X & \xrightarrow{i} & CX \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle j} \\ Y & \xrightarrow{q} & C_f. \end{array}$$

We also write $C_f = Y \cup_f CX$.

Notice that if $X \xrightarrow{f} Y$ is a cofibration with cofibre C, then $C \simeq C_f$.

By Proposition 2.30, the sequence

$$Y \xrightarrow{q} C_f \xrightarrow{\delta} \Sigma X$$

is a cofibration sequence, called the *principal cofibration induced by f*, and the map δ is called the *connecting map*.

The next theorem states that the mapping cone construction generates a long exact sequence of groups and sets.

Theorem 2.32. *For any pointed map $f : X \to Y$ and space Z, the following sequence is a long exact sequence of sets ($i \geq 0$), groups ($i \geq 1$), and abelian groups ($i \geq 2$):*

$$\begin{aligned} \cdots \to [\Sigma^i C_f, Z]_* \to [\Sigma^i Y, Z]_* \to [\Sigma^i X, Z]_* \to \\ \cdots \to [\Sigma X, Z]_* \to [C_f, Z]_* \to [Y, Z]_* \to [X, Z]_*. \end{aligned}$$ □

Proposition 2.33. *If $i : X \to Y$ is a cofibration and Z is any space, then the induced map*

$$i_* : \mathrm{Map}(Y, Z) \to \mathrm{Map}(X, Z)$$

is a fibration. □

Definition 2.34. For topological spaces X and Y, the *evaluation map*,

$$ev : \mathrm{Map}(X, Y) \times X \to Y,$$

is defined as $ev(f, x) = f(x)$. Applying Proposition 2.33 to the case when $X = *$ is a point and Z is a based space we obtain the fibration sequence

$$\mathrm{Map}^*(Y, Z) \hookrightarrow \mathrm{Map}(Y, Z) \xrightarrow{ev} Z$$

called the *evaluation fibration.*

2.4 Homotopy actions and coactions

Given a space X and an H-space W with multiplication m, a (right homotopy) *action* of W on X is a map $\phi : X \times W \to X$ such that the following diagrams homotopy commute

$$\begin{array}{ccc} X & \xrightarrow{j} & X \times W \\ & \searrow^{=} & \downarrow{\scriptstyle\phi} \\ & & X \end{array} \quad \text{and} \quad \begin{array}{ccc} X \times W \times W & \xrightarrow{\phi \times \mathbb{1}} & X \times W \\ \downarrow{\scriptstyle \mathbb{1} \times m} & & \downarrow{\scriptstyle\phi} \\ X \times W & \xrightarrow{\phi} & X, \end{array}$$

where j is the inclusion. We say that W *acts on* X *by* ϕ.

Example 2.35. If X is a homotopy-associative H-space, then X acts on itself by multiplication.

Example 2.36. Let $f : X \to Y$ be a map. Consider the principal fibration sequence

$$\Omega Y \xrightarrow{j} F_f \xrightarrow{p_1} X,$$

where j is the inclusion and p_1 is the projection onto the first factor. Recall that the loop space ΩX is an H-space. Let $m : \Omega X \times \Omega X \to \Omega X$ be the multiplication in ΩX. Notice that m can be extended to a map $m' : PY \times \Omega Y \to PY$ by

$$m'(\alpha, \alpha')(t) = \begin{cases} \alpha(2t) & \text{if } 0 \leq t \leq \frac{1}{2}, \\ \alpha'(2t - 1) & \text{if } \frac{1}{2} \leq t \leq 1. \end{cases}$$

The map $\phi : F_f \times \Omega Y \to F_f$ defined by $((x, \omega), \nu) \mapsto (x, m'(\omega, \nu))$, for $(x, \omega) \in F_f$ and $\nu \in \Omega Y$, is an action of ΩY on F_f.

Proposition 2.37. *Let* $\Omega Y \xrightarrow{\partial} F \xrightarrow{p} X$ *be a principal fibration and suppose there is a map* $s : X \to F$ *such that* $p \circ s \simeq \mathbb{1}_X$. *Then there is a weak homotopy equivalence*

$$F \simeq \Omega Y \times X.$$

Proof. Let $\phi : F \times \Omega Y \to F$ be the homotopy action of ΩY on F. Consider the homotopy commutative diagram

$$\begin{array}{ccccc} * \times \Omega Y & \xrightarrow{* \times \mathbb{1}} & \Omega Y \times \Omega Y & \xrightarrow{m} & \Omega Y \\ \downarrow & & \downarrow{\scriptstyle \partial \times \mathbb{1}} & & \downarrow{\scriptstyle \partial} \\ X \times \Omega Y & \xrightarrow{s \times \mathbb{1}} & F \times \Omega Y & \xrightarrow{\phi} & F \\ \downarrow & & \downarrow{\scriptstyle p \times *} & & \downarrow{\scriptstyle p} \\ X \times * & \xrightarrow{\simeq} & X \times * & \xrightarrow{\simeq} & X \end{array} \tag{2.4.1}$$

where m is the homotopy multiplication on ΩY and the columns are fibration sequences. Notice that the first and the third rows are homotopy equivalences. Applying the functor $\pi_*(-)$ to the diagram and the five lemma we get that the composite of the middle row induces isomorphisms in homotopy groups. This implies that the middle row is a weak homotopy equivalence, as asserted. □

Homotopy actions induce actions of a group of homotopy classes of maps on a homotopy set as follows. Let W be an H-group and $\phi : X \times W \to X$ be a homotopy action of W on a space X. If Z is any space and $f : Z \to X$ and $\alpha : Z \to W$ are any maps, then define f^α to be the composite

$$Z \xrightarrow{\Delta} Z \times Z \xrightarrow{f \times \alpha} X \times W \xrightarrow{\phi} X.$$

Thus the function $\theta : [Z, X]_* \times [Z, W]_* \to [Z, X]_*$ defined by $\theta(f, \alpha) = f^\alpha \in [Z, X]_*$ is an action of the group $[Z, W]_*$ on $[Z, X]_*$.

We can use the action of groups on homotopy sets to improve exactness of the end terms of the exact sequence that appears in the statement of Theorem 2.27. Let $f : X \to Y$ be a map and let

$$\cdots \to [Z, \Omega X]_* \xrightarrow{(\Omega f)^*} [Z, \Omega Y]_* \xrightarrow{\partial^*} [Z, F_f]_* \xrightarrow{v^*} [Z, X]_* \xrightarrow{f^*} [Z, Y]_*$$

be the exact sequence of homotopy sets induced by f. The proof of the next theorem can be found in [Ark11].

Theorem 2.38.

(1) Let $\rho, \sigma \in [Z, F_f]_*$. Then $v^*(\rho) = v^*(\sigma)$ if and only if there exists $\gamma \in [Z, \Omega Y]_*$ such that $\sigma = \rho^\gamma$.

(2) Let $\gamma, \delta \in [Z, \Omega Y]_*$. Then $\partial^*(\gamma) = \partial^*(\delta)$ if and only if there exists $\epsilon \in [Z, \Omega X]_*$ such that $\gamma = (\Omega f)^*(\epsilon) + \delta$. □

Given a space X and a co-H-space W with comultiplication σ, a (right) *coaction* of W on X is a map $\psi : X \to X \vee W$ such that the diagrams

$$\begin{array}{ccc} X & \xrightarrow{\psi} & X \vee W \\ & \searrow^{\mathbb{1}_X} & \downarrow p_1 \\ & & X \end{array} \quad \text{and} \quad \begin{array}{ccc} X & \xrightarrow{\sigma} & X \vee W \\ \psi \downarrow & & \downarrow \mathbb{1}\vee\sigma \\ X \vee W & \xrightarrow{\psi\vee\mathbb{1}} & X \vee W \vee W, \end{array}$$

where p_1 is the projection onto the first factor, homotopy commute.

Example 2.39. We can define a coaction for a principal cofibration as follows. Let $f : X \to Y$ be a map and consider the principal cofibration sequence

$$Y \xrightarrow{j} C_f \xrightarrow{q} \Sigma X,$$

where j is the inclusion and q is the projection. Recall that $C_f = CX \cup_f Y$ is the mapping cone of f. The map $\psi_0 : C_f \to C_f \vee \Sigma X$ defined by $\psi(y) = (y, *)$ for $y \in Y$ and

$$\psi_0(x,t) = \begin{cases} ((x,2t),*) & \text{if } 0 \le t \le \frac{1}{2}, \\ (*,(x,2t-1)) & \text{if } \frac{1}{2} \le t \le 1 \end{cases}$$

for $x \in X$ defines a coaction of ΣX on C_f.

A homotopy coaction induces an action of a group of homotopy classes of maps on a homotopy set. Let Q be a co-H-group with comultiplication c, and $\psi : X \to X \vee Q$ be a coaction of Q on X. If Z is any space and $g : X \to Z$ and $\beta : Q \to Z$ are any maps, then define $g^\beta : X \to Z$ as the composite

$$X \xrightarrow{\psi} X \vee Q \xrightarrow{g\vee\beta} Z \vee Z \xrightarrow{\nabla} Z.$$

An action of $\theta : [X,Z]_* \times [Q,Z]_* \to [X,Z]_*$ is defined by $\theta(g,\beta) = g^\beta \in [X,Z]_*$.

Using the action θ we can give a refinement of the last terms in Theorem 2.32 as follows.

Let $f : X \to Y$ be a map and let

$$\cdots \to [\Sigma Y, Z]_* \xrightarrow{(\Sigma f)^*} [\Sigma X, Z]_* \xrightarrow{\delta^*} [C_f, Z]_* \xrightarrow{q^*} [Y,Z]_* \xrightarrow{f^*} [X,Z]_*$$

be the exact sequence of homotopy sets induced by f. The proof of the following theorem can be found in [Ark11].

Theorem 2.40.

(1) Let $\rho, \sigma \in [C_f, Z]_*$. Then $q^*(\rho) = q^*(\sigma)$ if and only if there exists $\gamma \in [\Sigma X, Z]_*$ such that $\sigma = \rho^\gamma$.

(2) *Let* $\gamma, \xi \in [\Sigma X, Z]_*$. *Then* $\delta^*(\gamma) = \delta^*(\xi)$ *if and only if there exists* $\epsilon \in [\Sigma Y, Z]_*$ *such that* $\gamma = (\Sigma f)^*(\epsilon) + \xi$. □

2.5 Samelson and Whitehead products

We define two maps that will be crucial in the following chapters. Let G be an H-group. Given maps $f \in [X, G]_*$ and $g \in [Y, G]_*$, we define the map $c(f, g) : X \times Y \to G$ as the composite

$$X \times Y \xrightarrow{f \times g} G \times G \xrightarrow{[-,-]} G,$$

where $[-, -]$ is the commutator map, which is defined pointwise as $[x, y] = xyx^{-1}y^{-1}$. The restriction of $[-, -]$ to $G \vee G$ is nullhomotopic. Therefore, there exists an extension $\overline{[-, -]} : G \wedge G \to G$, and therefore $[-, -]$ factors as

$$G \times G \to G \wedge G \xrightarrow{\overline{[-,-]}} G.$$

Hence the map $c(f, g)$ factors as $X \times Y \to X \wedge Y \xrightarrow{\langle f,g \rangle} G$, where $\langle f, g\rangle$ is the composite

$$\overline{[-, -]} \circ (f \wedge g) : X \wedge Y \to G \wedge G \to G.$$

The map $\langle f, g\rangle : X \wedge Y \to G$ is called the *Samelson product* of $f : X \to G$ and $g : Y \to G$. From the cofibration sequence

$$X \vee Y \to X \times Y \to X \wedge Y \xrightarrow{*} \Sigma X \vee \Sigma Y \tag{2.5.1}$$

we obtain an exact sequence

$$0 \to [X \wedge Y, G]_* \to [X \times Y, G]_* \to [X \vee Y, G]_*.$$

Therefore, by Theorem 2.40, the Samelson product $\langle f, g\rangle$ is defined uniquely up to homotopy.

Samelson products are natural with respect to maps $f_1 : X_1 \to X$, $g_1 : Y_1 \to Y$, and H-maps of H-spaces $\psi : G \to H$, that is

$$\langle \psi \circ f \circ f_1, \psi \circ g \circ g_1\rangle \simeq \psi \circ \langle f, g\rangle \circ (f_1 \wedge g_1).$$

Observe that if G is homotopy commutative then the commutator is nullhomotopic and then we have the following.

Proposition 2.41. *The Samelson product vanishes if* G *is homotopy commutative.* □

Given maps $\bar{f} : \Sigma X \to Z$, $\bar{g} : \Sigma Y \to Z$ with respective adjoints

$$f : X \xrightarrow{E} \Omega\Sigma X \xrightarrow{\Omega\bar{f}} \Omega Z, \qquad g : Y \xrightarrow{E} \Omega\Sigma Y \xrightarrow{\Omega\bar{g}} \Omega Z,$$

where E is the suspension map, the *Whitehead product* $[\bar{f}, \bar{g}]$ is defined to be the adjoint of the Samelson product $\langle f, g\rangle$:

$$[\bar{f}, \bar{g}] : \Sigma(X \wedge Y) \xrightarrow{\Sigma\langle f,g\rangle} \Sigma\Omega Z \xrightarrow{ev} Z,$$

where ev is the evaluation map. As with Samelson products, Whitehead products are natural with respect to maps. That is, given $f_1 : X_1 \to X$, $g_1 : Y_1 \to Y$ and $h : Z \to Z'$ we get

$$[h \circ \bar{f} \circ \Sigma f_1, h \circ \bar{g} \circ \Sigma g_1] \simeq h \circ [\bar{f}, \bar{g}] \circ \Sigma(f_1 \wedge g_1).$$

Notice that if Z is an H-space then ΩZ is homotopy commutative. Then by Proposition 2.41 we have the following.

Proposition 2.42. *If Z is an H-space then the Whitehead product $[\bar{f}, \bar{g}]$ is trivial.* □

Chapter 3

Topology of S^3-bundles over S^4

In this chapter we discuss the topology and homotopy theory of S^3-bundles over S^4. We start by defining the notions of CW-complexes, manifolds and fibre bundles. We introduce the definition of a Moore space and then we discuss briefly the theory of localisation of spaces. The classification of S^3-bundles over S^4 is stated in Proposition 3.44. Then we move towards the homotopy classification of the total spaces of the sphere bundles M. In Theorem 3.45 we present a homotopy classification of the spaces M as it was given in the work of James and Whitehead [JW54]. In Proposition 3.47, we present a result due to Sasao regarding the homotopy theory of CW-complexes with homology groups isomorphic to some of those of the spaces M [Sas65]. We finish this chapter by presenting a classification of total spaces of S^3-bundles over S^4 due to Crowley and Escher [CE03]. This result is stated in Theorem 3.48.

3.1 CW-complexes and Moore spaces

Throughout this thesis we will denote the n-th homology and cohomology groups of a space X with the coefficient ring R by $H_n(X;R)$ and $H^n(X;R)$, respectively, and we will use $\tilde{H}_n(X;R)$ and $\tilde{H}^n(X;R)$ for the reduced homology and cohomology groups. In case the coefficient ring is $R = \mathbb{Z}$, the ring R will be omitted from the notation. We will also write $H_n(X,A)$, $H^n(X,A)$, $\tilde{H}_n(X,A)$ or $\tilde{H}^n(X,A)$ for the (co)homology groups of a pair (X,A).

Definition 3.1. A *CW-complex* or a *cellular complex* is a topological space X constructed inductively as follows:

(1) Start with a discrete space X^0. The elements of X^0 are the 0-*cells*.

(2) Let D^n denote the n-disk and e^n denote the open n-disk. Let $A = \{e_i^n\}$ be a (possibly empty) collection of open n-disks indexed by $i \in I$. If $A = \emptyset$, set $X^n = X^{n-1}$.

Otherwise, form the *n-skeleton* X^n by attaching open n-disks e_i^n, called simply *n-cells*, to X^{n-1} via maps $\varphi_i : S^{n-1} \to X^{n-1}$. The maps φ_i are called *attaching maps*. We define X^n as the quotient space

$$X^n = X^{n-1} \amalg_i D_i^n / \sim$$

where $x \sim \varphi_i(x)$ for $x \in \partial D_i^n$ and ∂D_i^n is the boundary of the n-disk $D_i^n = D^n$. Thus as sets $X^n = X^{n-1} \coprod_i e_i^n$.

(3) This process can end at a finite stage, setting $X = X^n$ for some $n < \infty$, or can continue indefinitely, setting $X = \bigcup_n X^n$.

Each cell e_i^n in a CW-complex has a *characteristic map* $\Phi_i : D^n \to X$ which extends the attaching map ϕ_i and is a homeomorphism from the interior of D_i^n onto e_i^n.

Alternatively, we can define a CW-complex X as follows. Let the n-skeleton of X be defined by the pushout square

$$\begin{array}{ccc} \amalg S_i^{n-1} & \xrightarrow{\amalg j_i'} & \amalg D_i^n \\ {\scriptstyle\varphi}\downarrow & & \downarrow \\ X^{n-1} & \xrightarrow{j} & X^n \end{array}$$

where $\varphi|_{S_i^{n-1}} = \varphi_i$ and the maps j, j_i' are inclusions. Then X^n or any space homeomorphic to X^n is a CW-complex of dimension at most n.

Example 3.2. The sphere S^n can be given the structure of a CW-complex with one cell e^0 and one cell e^n,

$$\begin{array}{ccc} S^{n-1} & \longrightarrow & D^n \\ {\scriptstyle\varphi}\downarrow & & \downarrow \\ e^0 = * & \longrightarrow & S^n \end{array}$$

where φ is the constant map.

Example 3.3. Given $S^n \subset \mathbb{R}^{n+1} - \{0\}$, $n \geq 1$, the anitpodal map $\rho : S^n \to S^n$ is defined by $\rho(x) = -x$. The real projective n-space $\mathbb{R}P^n$ can be obtained as the identification space with the equivalence relation $x \sim \rho(x)$. We can inductively construct $\mathbb{R}P^{n+1}$ with $\mathbb{R}P^n$ as the n-skeleton

$$\begin{array}{ccc} S^n & \longrightarrow & D^{n+1} \\ {\scriptstyle\varphi_{n+1}}\downarrow & & \downarrow \\ \mathbb{R}P^n & \longrightarrow & \mathbb{R}P^{n+1} \end{array}$$

where the attaching map φ_{n+1} is an identification map. In this construction, $\mathbb{R}P^n$ has cells in dimensions $0, 1, 2, 3, \ldots, n$. In this case it is not difficult to see that $\mathbb{R}P^1$ is homeomorphic to S^1.

Example 3.4. Similarly, for $\mathbb{C}$ and $\mathbb{H}$ we can define the complex projective n-space $\mathbb{C}P^n$ and the quaternionic projective n-space $\mathbb{H}P^n$ and give them cellular structures. For instance, $\mathbb{C}P^2$ has a cellular structure $e^0 \cup e^2 \cup e^4$, and the last attaching map $\eta : S^3 \to S^2 \cong \mathbb{C}P^1$ is a generator of the group $\pi_3(S^2) \cong \mathbb{Z}$. The quaternionic projective 2-space $\mathbb{H}P^2$ can be given the cellular structure $e^0 \cup e^4 \cup e^8$. The non-trivial attaching map $\nu : S^7 \to S^4 \cong \mathbb{H}P^1$ is a generator of an infinite cyclic subgroup of $\pi_7(S^4) \cong \mathbb{Z} \oplus \mathbb{Z}_{12}$.

Theorem 3.5 and Theorem 3.7 are due to Whitehead [Whi49].

Theorem 3.5. *If X and Y are CW-complexes and $f : X \to Y$ is a map, then f is a weak homotopy equivalence if and only if f is a homotopy equivalence.* □

For the next theorem, we need the following definition:

Definition 3.6. Let $n \geq 1$ be an integer or $n = \infty$, and let $f : X \to Y$ be a map. We say f is an *n-equivalence* if the induced map $f^* : \pi_i(X) \to \pi_i(Y)$ is an isomorphism for all $i < n$ and an epimorphism for $i = n$. We say f is a *homological n-equivalence* if $f^* : H_i(X) \to H_i(Y)$ is an isomorphism for all $i < n$ and an epimorphism for $i = n$.

Theorem 3.7. *Let X and Y be path-connected CW-complexes, let $f : X \to Y$ be a map and let $n \in \mathbb{Z} \cup \{\infty\}$ with $n \geq 1$.*

(1) If f is an n-equivalence, then f is a homological n-equivalence.

(2) If f is a homological n-equivalence and X, Y are simply connected, then f is an n-equivalence. □

We use Theorem 3.7 to obtain the following.

Proposition 3.8. *Let $Y \xrightarrow{q} C \xrightarrow{\delta} \Sigma X$ be a principal cofibration where all spaces are simply connected CW-complexes. Suppose there exists a map $s : C \to Y$ such that $s \circ q \simeq \mathbb{1}_Y$. Then there is a homotopy equivalence*

$$C \simeq Y \vee \Sigma X.$$

□

Proof. Let $\psi : C \to C \vee \Sigma X$ be a homotopy coaction of ΣX on C. Consider the homotopy commutative diagram

$$\begin{array}{ccccc}
Y & \xrightarrow{\simeq} & Y \vee * & \xrightarrow{\simeq} & Y \\
\downarrow{\scriptstyle q} & & \downarrow{\scriptstyle q \vee *} & & \downarrow \\
C & \xrightarrow{\psi} & C \vee \Sigma X & \xrightarrow{s \vee \mathbb{1}} & Y \vee \Sigma X \\
\downarrow{\scriptstyle \delta} & & \downarrow{\scriptstyle \delta \vee \mathbb{1}} & & \downarrow \\
\Sigma X & \xrightarrow{\sigma} & \Sigma X \vee \Sigma X & \xrightarrow{p_2} & \Sigma X.
\end{array} \tag{3.1.1}$$

where σ is a comultiplication in ΣX, p_2 is the projection and the columns are cofibration sequences. The first and the third rows are homotopy equivalences. Applying the five lemma to the extended commutative diagram of homology groups induced by (3.1.1) we obtain that the middle row induces an isomorphism in homology. By Theorem 3.7, as all spaces are simply connected CW-complexes it follows that the composite in the middle row is a weak homotopy equivalence. And finally by Theorem 3.5 it is a homotopy equivalence. □

We now introduce Moore spaces.

Definition 3.9. Let X be a co-H-space. A map of degree k, $k : X \to X$, is defined as the composite $X \xrightarrow{\mu_k} \bigvee_{i=1}^{k} X \xrightarrow{\nabla} X$, where μ_k is a choice of k-fold comultiplication. The *n-dimensional Moore space* $P^n(k)$ is the homotopy cofibre of the degree k map on S^{n-1}; that is, $P^n(k)$ is obtained attaching an n-cell to an $(n-1)$-sphere by a map of degree k, so this space is defined by the following pushout square

$$\begin{array}{ccc} S^{n-1} & \longrightarrow & D^n \\ {\scriptstyle k}\downarrow & & \downarrow \\ S^{n-1} & \longrightarrow & P^n(k). \end{array}$$

The homology groups of the Moore space are given as follows:

$$\tilde{H}_i(P^n(k)) = \begin{cases} \mathbb{Z}_k & \text{if } i = n-1, \\ 0 & \text{if } i \neq n-1, \end{cases}$$

where $\mathbb{Z}_k$ denotes the cyclic group of order k.

Example 3.10. The n-sphere S^n is an n-dimensional Moore space, $P^n(1)$.

Example 3.11. The real projective space $\mathbb{R}P^2$ is homotopy equivalent to the Moore space $P^2(2)$.

Let $k = p_1^{r_1} p_2^{r_2} \cdots p_m^{r_m}$ with $\{p_i\}_{i=1}^m$ distinct primes and $\{r_i\}_{i=1}^m$ positive integers. Let $X = P^n(p_1^{r_1}) \vee \cdots \vee P^n(p_m^{r_m})$, with $n \geq 2$, then we have

$$\tilde{H}_{n-1}(X) = \tilde{H}_{n-1}(P^n(p_1^{r_1})) \oplus \cdots \oplus \tilde{H}_{n-1}(P^n(p_m^{r_m}) \cong \mathbb{Z}_{p_1^{r_1}} \oplus \cdots \oplus \mathbb{Z}_{p_m^{r_m}} \cong \mathbb{Z}_k.$$

Notice that $\tilde{H}_{n-1}(X)$ is the only non trivial reduced homology group of X. Now the only non-trivial reduced homology group of the Moore space $P^n(k)$ is $\tilde{H}_{n-1}(P^n(k)) \cong \mathbb{Z}_k$. Indeed there is a homotopy equivalence

$$P^n(k) \simeq P^n(p_1^{r_1}) \vee P^n(p_2^{r_2}) \vee \cdots \vee P^n(p_m^{r_m}).$$

For $n > 2$, the Moore space $P^n(r)$ of degree r is the suspension $\Sigma P^{n-1}(r)$. We use Moore spaces to define homotopy groups with coefficients. If X is a pointed topological space, then the *n-th homotopy group of X with coefficients in $\mathbb{Z}_r$* is

$$\pi_n(X;\mathbb{Z}_r) = [P^n(r), X]_*.$$

Hence for $n > 3$, $\pi_n(X;\mathbb{Z}_r)$ is an abelian group since $P^n(r)$ is a double suspension. If X is an H-space, the homotopy set $[P^n(r), X]_*$ is a group for $n > 1$ and an abelian group for $n > 2$.

Theorem 3.12. *For $n \geq 2$ there is a natural exact sequence*

$$0 \to \pi_n(X) \otimes \mathbb{Z}_k \to \pi_n(X;\mathbb{Z}_k) \to \mathrm{Tor}^{\mathbb{Z}}(\pi_{n-1}(X), \mathbb{Z}_k) \to 0. \qquad \square$$

3.2 Localisation of spaces

The material presented in this section is based mainly in [Nei10].

Definition 3.13. Let $\mathcal{P} \subset \mathbb{Z}$ be the set of all primes, and let

$$\mathcal{P} = S \bigcup T$$

be a partition. An abelian group A is called *S-local* if every element of A is uniquely divisible by all elements of T, that is, multiplication by q, $q : A \to A$ is an isomorphism for all $q \in T$.

Example 3.14. Let $\overline{T}$ be the multiplicative monoid generated by T. The subring of the rationals

$$\mathbb{Z}[T^{-1}] = \mathbb{Z}_{(S)} = \left\{ \frac{a}{q} \mid a \in \mathbb{Z}, q \in \overline{T} \right\}$$

is an S-local abelian group, or equivalently, a $\mathbb{Z}_{(S)}$-module.

Definition 3.15. A map of abelian groups $f : A \to A'$ is an *S-local equivalence* if $f^* : Hom(A', B) \to Hom(A, B)$ is a bijection for all S-local abelian groups B. An *S-localisation of A* is an S-local group $\overline{A}$ such that there is an S-local equivalence $l : A \to \overline{A}$.

For any abelian group A, an S-localisation exists and is given by

$$A[T^{-1}] = A_{(S)} = \left\{ \frac{a}{q} \mid a \in A, q \in \overline{T} \right\}.$$

Definition 3.16. A simply connected pointed space X is called an *S-local space* if the homotopy groups $\pi_k(X)$ are S-local for all $k \geq 1$. For simply connected spaces X the universal coefficient theorem (Theorem 3.12) shows that X is S-local if and only if the homotopy groups $\pi_k(X;\mathbb{Z}_q)$ are trivial for all $k \geq 2$ and for all $q \in T$.

Definition 3.17. A map $f : A \to B$ of simply connected spaces is an *S-equivalence* if the map of S-local homology $f_* : H_*(A; \mathbb{Z}_{(S)}) \to H_*(B; \mathbb{Z}_{(S)})$ is an isomorphism.

Localisation of a space is obtained as a consequence of inverting a specific map α of spaces. For simply connected spaces, the Farjoun-Bousfield localisation theory specialises to the localisation of spaces at a set of primes S. This includes the rationalisation $X \to X_{(0)} = X \otimes \mathbb{Q}$, localisation at a prime p, $X \to X_{(p)} = X \otimes \mathbb{Z}_{(p)}$, and localisation away from a prime $X \to X\left[\frac{1}{p}\right] = X \otimes \mathbb{Z}\left[\frac{1}{p}\right]$. This theory shows that S-localisation exists and is unique up to homotopy equivalence. Thus given a simply connected space X there is map $\lambda : X \to X_{(S)}$ such that:

(1) $X_{(S)}$ is S-local.

(2) $\lambda : X \to X_{(S)}$ is an S-equivalence.

(3) for all maps $f : X \to Y$ with Y and S-local space, there is up to homotopy a unique extension $\tilde{f} : X_{(S)} \to Y$ of f.

Theorem 3.18. *If $X \to Y$ is a map of simply connected spaces, the following are equivalent:*

(1) $H_*(X) \to H_*(Y)$ is an S-local equivalence

(2) $\pi_*(X) \to \pi_*(Y)$ is an S-local equivalence

(3) $X_{(S)} \to Y_{(S)}$ is a homotopy equivalence. □

The following result shows that the homotopy type of a space may be decomposed into those of its localisations. A proof of this result can be found in [MP11].

Let I be an indexing set and let T_i be a set of primes for each $i \in I$. Let $T = \bigcup_{i \in I} T_i$, and $S = \bigcap_{i \in I} T_i$. Additionally, we assume that $T_i \cap T_j = S$ for all $i \neq j$ and that $T_i \neq S$ for all $i \in I$. Let X be a pointed connected topological space such that $\pi_1(X)$ is abelian. Let

$$\phi : X \to X_{(S)},$$

$$\phi_i : X \to X_{(T_i)},$$

$$\psi_i : X_{(T_i)} \to X_{(S)}$$

be localisations of X such that $\psi_i \phi_i \simeq \phi$ for each $i \in I$. Let

$$\phi_S : \prod_{i \in I} X_{T_i} \to (\prod_{i \in I} X_{T_i})_{(S)}$$

denote an S-localisation.

Theorem 3.19. *Let X be a T-local space. The following diagram is a homotopy pullback square*

$$\begin{array}{ccc} X & \xrightarrow{(\phi_i)} & \prod_{i\in I} X_{(T_i)} \\ \downarrow{\scriptstyle \phi} & & \downarrow{\scriptstyle \phi_S} \\ X_{(S)} & \xrightarrow[((\phi_i)_{(S)})]{} & (\prod_{i\in I} X_{(T_i)})_{(S)}. \end{array}$$

□

We state now some results on the homotopy theory of Moore spaces as it is discussed in [CMN79], which use localisation techniques.

Proposition 3.20. *Let p be an odd prime. If $m > 3$ the maps $p^r : P^m(p^r) \to P^m(p^r)$ are all nullhomotopic.* □

Proposition 3.21. *Let $m, n \geq 2$ and suppose and p is an odd prime. Then there is a homotopy equivalence*

$$P^{n+m}(p^r) \vee P^{n+m-1}(p^r) \simeq P^n(p^r) \wedge P^m(p^r).$$ □

Proposition 3.22. *If p is an odd prime and $n \geq 1$, then there is a homotopy equivalence*

$$\Omega\left(\bigvee_{k=0}^{\infty} P^{4n+2kn+3}(p^r)\right) \times S^{2n+1}\{p^r\} \simeq \Omega P^{2n+2}(p^r),$$

where $S^n\{p^r\}$ denotes the homotopy fibre of the map $p^r : S^n \longrightarrow S^n$. □

3.3 Manifolds

In this section we give a brief introduction to the topology of manifolds. In addition to the references mentioned at the beginning of this chapter, we use [Ark11, Hat02].

A *manifold* of dimension n or an n-manifold is a Hausdorff space M in which each point has an open neighbourhood homeomorphic to $\mathbb{R}^n$. By a result of Milnor [Mil59] any compact manifold M is homotopy equivalent to a CW-complex.

The dimension of M is characterised by the fact that for $x \in M$, the homology group $H_i(M, M - \{x\})$ is nonzero only for $i = n$:

$$\begin{aligned} H_i(M, M-\{x\}) &\cong H_i(\mathbb{R}^n, \mathbb{R}^n - \{0\}) \\ &\cong \tilde{H}_{i-1}(\mathbb{R}^n - \{0\}) \\ &\cong \tilde{H}_{i-1}(S^{n-1}). \end{aligned}$$

An n-manifold with boundary is a Haussdorff space M in which each point has an open neighbourhood homeomorphic either to $\mathbb{R}^n$ or to the space

$$\mathbb{R}^n_+ = \{(x_1, \dots, x_n) \in \mathbb{R}^n \mid x_n \geq 0\}.$$

Given a point $x \in M$, if an open neighbourhood of x is homeomorphic to $\mathbb{R}^n_+$, then $H_n(M, M - \{x\}) = 0$, whereas if it is homeomorphic to $\mathbb{R}^n$, then $H_n(M, M - \{x\}) \cong \mathbb{Z}$. The subspace of points x with $H_n(M, M - \{x\}) = 0$ is called the *boundary of M*. A compact manifold without boundary is called *closed*.

A *local orientation* of a manifold M without boundary at a point x is a choice of generator μ_x of the infinite cyclic group $H_n(M, M - \{x\})$. An *orientation* of an n-dimensional manifold M is a function $x \mapsto \mu_x$ assigning to each $x \in M$ a local orientation $\mu_x \in H_n(M, M - x)$, satisfying the following local consistency condition: each $x \in M$ has a neighbourhood $\mathbb{R}^n \subset M$ containing an open ball B of finite radius about x such that all the local orientations μ_y at points $y \in B$ are the images of a generator μ_B of $H_n(M, M - B) \cong H_n(\mathbb{R}^n, \mathbb{R}^n - B)$ under the maps $H_n(M, M - B) \to H_n(M, M - \{y\})$. If an orientation exists for M, then M is called *orientable*.

A *fundamental class* for a closed orientable n-manifold M with coefficients in R is an element of $H_n(M; R)$ whose image in $H_n(M, M - \{x\}; R)$ is a generator for all x. Given a topological space X, let $C_n(X; R)$ and $C^n(X; R)$ be the group of singular n-chains and n-cochains of X, respectively, with coefficient ring R. Define an R-bilinear cap product $\frown: C_k(X; R) \times C^l(X; R) \to C_{k-l}(X; R)$ for $k \geq l$ by setting

$$\sigma \frown \varphi = \varphi(\sigma|_{[v_0, \dots, v_l]})\sigma|_{[v_l, \dots, v_k]}$$

for $\sigma : \Delta^k \to X$ and $\varphi \in C^l(X; R)$, where Δ^k is a k-simplex. There is an induced cap product

$$H_k(X; R) \times H^l(X; R) \xrightarrow{\frown} H_{k-l}(X; R),$$

which is R-linear in each variable.

Poincaré Duality is stated below. A proof of this theorem can be found in [Hat02].

Theorem 3.23 (Poincaré Duality)**.** *Let M be a closed and oriented n-manifold with fundamental class $[M] \in H_n(M; R)$. The map $\phi : H^k(M; R) \to H_{n-k}(M; R)$ defined by*

$$\phi(\alpha) = [M] \frown \alpha$$

is an isomorphism for all k. □

The rank of $H_k(M; \mathbb{Q})$, called the *k-th Betti number*, describes the number of k-cells in M. Using the Universal coefficient theorem for cohomology it can be shown that

$$H^k(X) \cong \mathbb{Z}^{\beta_k} \oplus T_{k-1}$$

where β_k is the k-th Betti number of X and T_{k-1} is the torsion part of $H_{k-1}(X)$. Thus $H_k(X) \cong H^k(X)$ up to torsion. The Poincaré duality theorem states that the k-th cohomology group of a closed, oriented n-dimensional manifold M is isomorphic to the $(n-k)$-th homology group of M.

Definition 3.24. Let M_1 and M_2 be oriented closed connected n-manifolds. Their *connected sum* $M_1 \sharp M_2$ is an oriented closed connected n-manifold defined by deleting the interiors of n-cells B_1 in M_1 and B_2 in M_2 and attaching the resulted punctured manifolds to each other by a homeomorphism $h : \partial B_1 \to \partial B_2$, so that

$$M_1 \sharp M_2 = (M_1 - \mathrm{Int} B_1) \cup_h (M_2 - \mathrm{Int} B_2). \tag{3.3.1}$$

3.4 Fibre bundles

In Example 2.14, we defined a topological group as a group G together with a topology on G such that the binary operation and the inverse functions are continuous respect to the topology. Now we define an important family of topological groups.

Definition 3.25. A *Lie group* G is a group that is a differentiable manifold such that the multiplication map

$$\mu : G \times G \to G, \quad (g, h) \mapsto gh$$

and the inversion map

$$\iota : G \to G, \quad g \mapsto g^{-1}$$

are differentiable. If the underlying manifold of a Lie group G is connected or compact then we say that G is connected or compact. Two Lie groups are *locally isomorphic* if there exists a homeomorphism between two neighborhoods of the identities compatible with the product. A Lie group is orientable as manifold and, indeed, an orientation at the identity can be translated to an arbitrary point by left translation. Any Lie group G is homeomorphic to $K \times \mathbb{R}^n$, with K a compact subgroup of G and $n = \dim G - \dim K$.

Example 3.26. Let $GL(n, \mathbb{R})$ be the set of real invertible $n \times n$ matrices with the group structure given by matrix multiplication and the topology given as follows. Using the isomorphism as vector spaces $GL(n) \cong \mathbb{R}^{n^2}$, we can think of $A \in GL(n, \mathbb{R})$ as an element of $\mathbb{R}^{n^2}$. In this way $GL(n, (R))$ can be regarded as an open subspace of $\mathbb{R}^{n^2}$ with the relative topology. Now as $\mathbb{R}^{n^2}$ is a differentiable manifold, so is any open subset. Thus $GL(n, \mathbb{R})$ is a differentiable manifold. The map $GL(n, \mathbb{R}) \times GL(n, \mathbb{R}) \to GL(n, \mathbb{R})$ given by $(A, B) \mapsto AB^{-1}$ is continuous since the entries of AB^{-1} are rational functions of A and B. Then $GL(n, \mathbb{R})$, called the *general linear group of real $n \times n$ matrices*, is a Lie group. Analogously, it is easy to check that the group $GL(n, \mathbb{C})$ of complex invertible $n \times n$ matrices is also a Lie group.

Example 3.27. Every subgroup H of a topological group G with the relative topology is also a topological group. The following are subgroups of $GL(n, \mathbb{R})$:

(1) The *special linear group* $SL(n, \mathbb{R}) = \{A \in GL(n, \mathbb{R}) \mid det(A) = 1\}$, where $det(A)$ is the determinant of A.

(2) The *orthogonal group* $O(n, \mathbb{R}) = \{A \in GL(n, \mathbb{R}) \mid AA^T = 1\}$, where A^T denotes the transpose of A.

(3) The *special orthogonal group* $SO(n, \mathbb{R}) = O(n, \mathbb{R}) \cap SL(n, \mathbb{R})$.

The following are subgroups of $GL(n, \mathbb{C})$:

(1) $SL(n, \mathbb{C}) = \{A \mid \det(A) = 1\}$.

(2) $O(n, \mathbb{C}) = \{A \mid AA^T = 1\}$.

(3) The *unitary group* $U(n) = \{A \mid \overline{A}A^T = 1\}$, where $\overline{A}$ denotes the complex conjugate of A.

(4) The *special unitary group* $SU(n) = U(n) \cap SL(n, \mathbb{C})$. In the case $n = 2$ it is also known that $S^3 \cong SU(2)$.

From now on we will omit $\mathbb{R}$ from our notation for the general linear groups of real matrices and their subgroups.

Definition 3.28. A *bundle* consists of a topological space E called the *total space*, a space B called the *base space*, a map

$$p : E \to B$$

of E onto B called the *projection*, and a space F called the *fibre*, such that for each element $b \in B$, the set $p^{-1}(b)$ is homeomorphic to F. Finally for each $b \in B$ there is a neighbourhood V of b and a homeomorphism $\phi : V \times F \to p^{-1}(V)$ such that the diagram

$$\begin{array}{ccc} V \times F & \xrightarrow{\phi} & p^{-1}(V) \\ & {}_{p_1}\searrow \quad \swarrow_{p} & \\ & V & \end{array}$$

commutes, where p_1 is a projection onto the first factor. Intuitively, we can think of a bundle as a union of fibres parametrized by B and glued together by the topology of E. If $p : E \to B$ is a projection with fibre F we say that

$$F \to E \xrightarrow{p} B$$

is a bundle.

Example 3.29. The map $p : F \times B \to B$ given by the projection onto the second factor is a bundle map and the bundle defined in this way is called *trivial*.

Example 3.30. Let $\mathbb{F} = \mathbb{R}, \mathbb{C}$ or $\mathbb{H}$ be a field and let $d \in \mathbb{Z}$ such that if $\mathbb{F} = \mathbb{R}, \mathbb{C}, \mathbb{H}$ then $d = 1, 2, 4$, respectively. Then the sphere $S^{d(n+1)-1}$ can be viewed as a subset of $\mathbb{F}^{n+1}$ for each $n \geq 0$, as a subset of all elements of norm 1. In particular, $S^{d-1} \subset \mathbb{F}$. Define an equivalence relation on $S^{d(n+1)-1} \subset \mathbb{F}^{n+1}$ by setting $x \sim y$ if $x = cy$ for some $c \in S^{d-1} \subset \mathbb{F}$. Then the equivalence classes are homeomorphic to S^{d-1}, and the projective plane $\mathbb{F}P^n$ is just the quotient space $S^{d(n+1)-1}/\sim$. Thus we have a bundle

$$S^{d-1} \longrightarrow S^{d(n+1)-1} \xrightarrow{\phi} \mathbb{F}P^n,$$

where maps ϕ, called *Hopf maps*, are just the projection maps $S^{d(n+1)-1} \to S^{d(n+1)-1}/\sim$.

A *cross section* of a bundle (E, p, B) is a map $s : B \to E$ such that $ps = \mathbb{1}_B$.

A bundle carries, as a part of its structure, a group G of transformations of the fibre F. We will include this information in the definition of coordinate bundle.

Definition 3.31. Let G be a topological group. A (*right*) *action* of G on a topological space Y is a map

$$\theta : Y \times G \to Y$$

such that, denoting $\theta(y, g)$ by yg, the following hold:

(1) For each $y \in Y$, $g, h \in G$, $y(gh) = (yg)h$;

(2) For each $y \in Y$, $y1 = y$, where 1 is the identity of G.

In this case we say that G *acts on* Y and Y is a (*right*) G-*space*. Notice that for a fixed $g \in G$, the map $y \mapsto yg$ is a homeomorphism of Y onto itself. Thus θ gives a homeomorphism from G into the group of homeomorphisms of Y.

Similarly, a space X is a *left* G-*space* if there exists a map $\theta : G \times Y \to Y$ such that $\theta(gh, y) = \theta(g, \theta(h, y))$ and $\theta(1, y) = y$ for $g, h \in G$ and $y \in Y$. Notice that for any left action $\theta : G \times Y \to Y$ we can define a right action $Y \times G \to Y$ by $(y, g) \mapsto \theta(g^{-1}, y)$. Hence in general these concepts are equivalent and it is usually enough to consider only right actions or only left actions.

Definition 3.32. We say that G acts

(1) *effectively* on Y if whenever $yg = y$ for all $y \in Y$, we have $g = 1$;

(2) *transitively* on Y if for any $y_1, y_2 \in Y$ there exists an element $g \in G$ such that $y_2 = y_1 g$;

(3) *freely* on Y if whenever $yg = y$ for some $y \in Y$, we have $g = 1$.

Definition 3.33. Let E, B, F be topological spaces, $p : E \to B$ a projection, G a topological group acting effectively on F, and $\{V_i\}_{i\in I}$ an open cover of B. Suppose that for each $i \in I$ we have a homeomorphism

$$\phi_i : V_i \times F \to p^{-1}(V_i)$$

such that the diagram

$$\begin{array}{ccc} V_i \times F & \xrightarrow{\phi_i} & p^{-1}(V_i) \\ & {\scriptstyle p_1}\searrow \quad \swarrow{\scriptstyle p} & \\ & V_i & \end{array}$$

commutes. For $b \in V_i$, define a homeomorphism $\phi_{i,b} : F \to p^{-1}(b)$ by $\phi_{i,b}(x) = \phi_i(b, x)$. Then the tuple $\xi = (E, p, B, F, G)$ is said to be a *coordinate bundle* if in addition the following two conditions are satisfied:

(1) for each pair $i, j \in I$ and each $b \in V_i \cap V_j$ the homeomorphism

$$\phi_{j,b}^{-1}\phi_{i,b} : F \to F$$

coincides with the map $f \mapsto fg$ for some $g = g_{ji}(b) \in G$;

(2) for each pair i, j in I, the function

$$g_{ji} : V_i \cap V_j \to G$$

is continuous.

The maps ϕ_i are called the *coordinate functions*, the maps g_{ji} are called the *coordinate transformations*, and G is called the *structure group*.

As in the definition of bundle, the spaces E, B and F are called the total space, the base space and the fibre, respectively. Two coordinate bundles ξ and ξ' are said to be *strictly equivalent* if they have the same total space, base space, projection, fibre, and group, and their coordinate functions $\{\phi_j\}$, $\{\phi'_k\}$ are such that

$$\bar{g}_{kj}(x) = {\phi'}_{k,x}^{-1}\phi_{j,x},$$

with $x \in V_j \cap V'_k$, coincides with the action of an element $g \in G$, and the map

$$\bar{g}_{k,j} : V_j \cap V'_k \to G$$

is continuous. This equivalence condition can be stated just by saying that the union of the two sets of coordinate functions is a set of coordinate functions of a bundle. We can check that this indeed defines an equivalence relation.

Definition 3.34. A *fibre bundle* $[\xi]$ is an equivalence class of coordinate bundles. Notice that we can regard a fibre bundle as a maximal coordinate bundle with respect to all possible coordinate functions.

In further discussions we will study fibre bundles through their representatives, coordinate bundles. From now on by a bundle we will mean a coordinate bundle.

The real orthogonal linear group $O(n+1)$ acts transitively on S^n, which is regarded as the unit sphere in $\mathbb{R}^{n+1}$.

Definition 3.35. A linear *n-sphere bundle* or an *S^n-bundle* is a bundle in which the fibre is an n-sphere and the structure group is the orthogonal group $O(n+1)$. An *orientable n-sphere bundle* is a bundle in which the fibre is an n-sphere and the group is the special orthogonal group $SO(n+1)$.

Definition 3.36. A bundle $\xi = (E, p, B, F, G)$ is called a *principal G-bundle* or a *principal bundle* if $F = G$ and G acts on F by left translations.

In the next chapter we reintroduce the definition of a principal G-bundle giving more information on the properties of the spaces and maps involved. For the purpose of this chapter we can keep Definition 3.36. Next, we give an example of a principal G-bundle which we will use in further discussions.

Example 3.37. There is a principal $\mathbb{Z}_2$-bundle

$$\mathbb{Z}_2 \to S^3 \xrightarrow{\rho} SO(3).$$

To see this define the map ρ as follows. Regarding S^3 as the group of quaternions of norm 1, the subset of S^3 such that the real part is zero is a 2-sphere S^2. Notice that this subset is the intersection of S^3 with the subspace of $\mathbb{H}$ orthogonal to 1. Let $\rho : S^3 \to SO(4)$ be the continuous homomorphism defined by

$$\rho(u)v = uvu^{-1}$$

where $u, v \in S^3$. We now show that $\rho(u) \in SO(3)$. Clearly $\rho(u)$ is a linear map, since $v \mapsto uvu^{-1}$ is linear in the $v_\alpha \in \mathbb{R}$, where $v = v_1 + iv_2 + jv_3 + kv_4$, and $\alpha \in \{1, 2, 3, 4\}$. The transformation is orthogonal as $|uvu^{-1}| = |u||v||u|^{-1} = |v|$. Since $u1u^{-1} = 1$, and S^2 is orthogonal to 1, it follows that $\rho(u)$ fixes S^2. Now since reals are the only quaternions that commute with i, j and k, it follows that the kernel of ρ is the set $\{1, -1\}$. The coset of this subgroup are the pairs, u and $-u$. Thus by Example 3.3 we obtain that $\rho(S^3)$ is homeomorphic to $\mathbb{R}P^3$. It is not hard to see that the one parameter subgroup of matrices leaving fixed the quaternions i, j and k are contained in $\rho(S^3)$, which implies that $SO(3) \subset \rho(S^3)$. Therefore ρ is a projection map and $\mathbb{Z}_2$ acts on S^3 with the antipodal map.

Example 3.38. Let us consider the standard action of $SO(n+1)$ on $\mathbb{R}^{n+1}$. We define the map $p : SO(n+1) \to S^n$ by

$$p(u) = ue_0,$$

where $u \in SO(n+1)$ and $e_0 = (1, 0, \cdots, 0)$. The homomorphism $h : SO(n) \to SO(n+1)$ defined by

$$v \mapsto \begin{pmatrix} 1 & 0 \\ 0 & v \end{pmatrix}$$

allows to identify $SO(n)$ with the subgroup of $SO(n+1)$ that fixes e_0. The multiplication in $SO(n+1)$ defines an action of $SO(n)$ on $SO(n+1)$, $(u, v) \mapsto uv$ for $u \in SO(n+1)$ and $v \in SO(n)$. Now let $x \in S^3$. If $u \in p^{-1}(x)$ then $ue_0 = x$ and $uve_0 = ue_0 = x$. The action of $SO(n)$ on $SO(n+1)$ is therefore fibre preserving and free. We can regard $SO(n+1)$ as the total space of a principal $SO(n)$-bundle over S^n,

$$SO(n) \to SO(n+1) \xrightarrow{p} S^n.$$

Theorem 3.39. *If G is the topological group of transformations of F, and $\{V_{ij}\}$, $\{g_{ij}\}$ are sets of coordinate transformations in B, then there exists a bundle ξ with base space B, fibre F, group G and the coordinate transformations $\{g_{ij}\}$.* □

Definition 3.40. Let ξ, ξ' be two bundles having the same fibre and the same structure group. A *bundle map* is a map $f : E \to E'$ with the following properties:

(1) f maps each fibre $p^{-1}(b) = F_b$ of E homeomorphically onto a fibre $F_{b'}$ of E' inducing a continuous map $u : B \to B'$ such that the diagram

$$\begin{array}{ccc} E & \xrightarrow{f} & E' \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle p'} \\ B & \xrightarrow{u} & B' \end{array}$$

commutes;

(2) if $b \in V_j \cap u^{-1}(V'_k)$, and $f_b : F_b \to F_{b'}$ is the map induced by f, then the map

$$\bar{g}_{kj}(b) = {\phi'}^{-1}_{k,b'} f_b \phi_{j,b} : F \to F$$

coincides with the operation of an element of G;

(3) the map $\bar{g}_{kj} : V_j \cap u^{-1}(V'_k) \to G$ is continuous.

It is not hard to verify that bundles and bundle maps form a category. Given two bundles ξ and ξ' having the same base space, fibre and group, a bundle map $u : E \to E'$ is called an *equivalence over* B if the diagram

$$\begin{array}{ccc} E & \xrightarrow{u} & E' \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle p'} \\ B & = & B \end{array}$$

commutes.

We can also replace the spaces B, F and G by homeomorphic spaces to reduce the number of equivalence classes.

Definition 3.41. Given an arbitrary bundle $\xi = (E, p, B, F, G)$, the *associated principal G-bundle* of ξ is the bundle given by Theorem 3.39, where $F = G$.

The next theorem relates the classification of bundles to that of their associated principal G-bundles. In Chapter 4 we will discuss the classification of principal G-bundles.

Theorem 3.42. *Two bundles $\xi = (E, p, B, F, G)$ and $\xi' = (E', p', B, F, G)$ are equivalent if and only if their associated principal G-bundles are equivalent.* □

The next proposition states that bundles have the homotopy lifting property.

Proposition 3.43. *If $p : E \to B$ is a bundle map with fibre F, then p is a fibration with fibre homeomorphic to F.* □

3.5 Classification of S^3-bundles over S^4

The total spaces M of S^3-bundles over S^4 have been of interest in both topology and geometry since the work of Milnor on exotic spheres. In [Mil56b] Milnor showed that there exist total spaces of S^3-bundles over S^4 that are homeomorphic to S^7 but not diffeomorphic to it. In 1974 Gromoll and Meyer [GM74] showed that one of these exotic spheres admits a metric with non-negative sectional curvature. Several decades later Grove and Ziller [GZ00] showed that all total spaces of S^3-bundles over S^4 admit a metric with non-negative sectional curvature. In 2003 Crowley and Escher gave a classification of these manifolds up to diffeomorphism, homeomorphism and homotopy equivalence [CE03].

The homotopy classification of these spaces started with the work of Steenrod on the classification of sphere bundles over spheres [Ste44, Ste51]. Following the work of Steenrod on the classification of k-sphere bundles over n-spheres we give a classification of S^3-bundles over S^4. Recall that an orientable n-sphere bundle is a bundle in which the fibre is an n-sphere and the group is the special orthogonal group $SO(n+1)$. Thus S^3-bundles over S^4 have the group $SO(4)$ as a structure group. The following result requires additional results related to the theory of principal G-bundle so that the proof will be presented in Chapter 4.

Proposition 3.44. *The equivalence classes of S^3-bundles over S^4 are in one-to-one correspondence with elements of $\pi_3(SO(4))$.*

We describe the generators of the group $\pi_3(SO(4)) \cong \mathbb{Z} \oplus \mathbb{Z}$, following [Ste44]. In our discussion we will consider actions on the left.

First, by Example 3.38 there is a principal $SO(3)$-bundle

$$SO(3) \to SO(4) \xrightarrow{p} S^3,$$

where $p(u) = u1$. Let $\sigma : S^3 \to SO(4)$ be the homomorphism defined by

$$\sigma(u)v = uv.$$

Since $p(r) = r(1)$, it follows that $p\sigma(u) = u$. Therefore $p\sigma = \mathbb{1}_{S^3}$, implying that the long exact sequence

$$\cdots \xrightarrow{\partial^*} \pi_n(SO(3)) \longrightarrow \pi_n(SO(4)) \xrightarrow{p^*} \pi_n(S^3) \longrightarrow \cdots$$

splits and $\pi_n(SO(4)) \cong \pi_n(SO(3)) \oplus \pi_n(S^3)$, $n \geq 0$. Now, recall the setup in Example 3.37, where the map $\rho : S^3 \to SO(3) \subset SO(4)$ given by $\rho(u)v = uvu^{-1}$defines a principal $\mathbb{Z}_2$-bundle

$$\mathbb{Z}_2 \to S^3 \xrightarrow{\rho} SO(3). \tag{3.5.1}$$

From (3.5.1) we obtain an exact sequence of homotopy groups

$$\pi_i(\mathbb{Z}_2) \longrightarrow \pi_i(S^3)) \xrightarrow{\rho^*} \pi_i(SO(3)) \longrightarrow \pi_{i-1}(\mathbb{Z}_2) \tag{3.5.2}$$

which shows that ρ induces isomorphisms $\pi_i(S^3) \cong \pi_i(SO(3))$ for $i \geq 2$. In particular $\pi_3(S^3) \cong \pi_3(SO(3)) \cong \mathbb{Z}$. Therefore $\pi_3(SO(4)) \cong \pi_3(SO(3)) \oplus \pi_3(S^3) \cong \mathbb{Z} \oplus \mathbb{Z}$, with generators ρ and σ.

By Proposition 3.44 we obtain a doubly indexed family of S^3-bundles over S^4, namely, $\xi_{l,m} = (M, \pi, S^4, S^3, SO(4))$. Writing $M = M_{l,m}$ for the total space of $\xi_{l,m}$ with the corresponding projection map $\pi : M \to S^4$ we have that the bundle

$$S^3 \to M_{l,m} \xrightarrow{\pi} S^4 \tag{3.5.3}$$

is the S^3-bundle over S^4 classified by $l\rho + m\sigma \in \pi_3(SO(4))$. These bundles are not all pairwise distinct since there is a non-trivial action of $\pi_0(O(4)) \cong \mathbb{Z}_2$ on $\pi_3(SO(4))$ (see

[Ste44]). Denoting α_0 the non-trivial element of $\pi_0(O(4))$ the action is given by

$$\alpha_0(\rho) = \rho$$

$$\alpha_0(\sigma) = \rho - \sigma$$

Changing orientation in the fibre bundles produces a equivalence between $\xi_{l,m}$ and $\xi_{l+m,-m}$. The bundles $\xi_{l,0}$ are reducible to the group $SO(3)$. Since $SO(3)$ fixes a 0-sphere on S^3, the fixed points provide subbundles which are cross-sections. Therefore, from the exact sequence induced by (3.5.3),

$$\cdots \to \pi_{i+1}(S^4) \xrightarrow{0} \pi_i(S^3) \to \pi_i(M_{l,0}) \to \pi_i(S^4) \xrightarrow{0} \cdots, \tag{3.5.4}$$

we obtain

$$\pi_i(M_{l,0}) \cong \pi_i(S^3) \times \pi_i(S^4) \tag{3.5.5}$$

for $i \geq 1$.

The structure group of the bundles $\xi_{0,m}$ can be reduced to the group $SU(2) \cong S^3$. Since we may regard the bundles $\xi_{0,m}$ as principal $SU(2)$-bundles, it follows that in the long exact sequence of homotopy groups induced by (3.5.3)

$$\cdots \to \pi_4(S^4) \xrightarrow{\delta^*} \pi_3(S^3) \to \pi_3(M_{l,0}) \to \pi_3(S^4) \to \cdots, \tag{3.5.6}$$

the connecting map δ^* sends a generator of $\pi_4(S^4)$ into m times a generator of $\pi_3(S^3)$. Since $\pi_3(S^4) \cong 0$, exactness of the homotopy sequence implies that $\pi_3(M_{0,m}) \cong \mathbb{Z}_m$. If $|m| \neq |m'|$, then the spaces $M_{0,m}$, $M_{0,m'}$ are not homeomorphic. From (3.5.5) we get $\pi_3(M_{l,0}) \cong \mathbb{Z}$. This shows that $M_{l,0}$ and $M_{0,m}$ are not homeomorphic if $m \neq 0$. More generally, according to Escher and Crowley [CE03] $H_3(M_{l,m}) \cong \mathbb{Z}_m$.

As the manifolds $M_{l,m}$ are simply connected we can give them the following minimal CW-structure

$$M_{l,m} = e^0 \cup e^3 \cup e^4 \cup e^7. \tag{3.5.7}$$

The 4-skeleton of $M_{l,m}$ is then the pushout

$$\begin{array}{ccc} S^3 & \longrightarrow & D^4 \\ {\scriptstyle m}\big\downarrow & & \big\downarrow \\ S^3 & \longrightarrow & P^4(m) \end{array}$$

where m is the degree m map. James and Whitehead classified the manifolds $M_{l,0}$ up to homotopy. Let $M_{l,0}, M_{l',0}$ be the total spaces of S^3-bundles over S^4 classified by the elements $(l,0),(l',0) \in \mathbb{Z} \oplus \mathbb{Z}$, respectively.

Theorem 3.45. *$M_{l,0}$ is homotopy equivalent to $M_{l',0}$ if and only if $l \equiv \pm l'$ mod 12.*

Proof. See [JW54] Theorem 1.6. □

In [Sas65] Sasao investigated the homotopy type of complexes K such that

$$H_i(K) = \begin{cases} \mathbb{Z}_k & \text{if } i = 3, \\ \mathbb{Z} & \text{if } i = 0, 7, \\ 0 & \text{if } i \neq 0, 3, 7. \end{cases}$$

For $m \geq 2$, the total spaces $M_{l,m}$ of S^3-bundles over S^4 belong to the family of complexes K. Since any complex K has the homotopy type of a complex L which is obtained by attaching a $(2n+1)$-cell to $P^4(m)$, it is sufficient to consider the homotopy type of L. In order to do this Sasao determined the homotopy groups $\pi_6(P^4(m))$.

If $f : X \to Y$ is a map and $f(A) \subseteq B$ for $A \subset X$ and $B \subseteq Y$, then we write $\bar{f} : (X, A) \to (Y, B)$ for the maps of pairs determined by f. Define $P(X; A, B)$ by

$$P(X; A, B) = \{\gamma \in \mathrm{Map}(I, X) \mid \gamma(0) \in A \text{ and } \gamma(1) \in B\}.$$

Thus if $B = \{*\}$, $P(X; A, \{*\})$ is the subspace of PX consisting of paths that begin in A and end in $\{*\}$, so that $P(X; A, \{*\})$ is just the homotopy fibre of the inclusion map $A \to X$.

Definition 3.46. For $A \subseteq X$ and an abelian group G, the *n-th relative homotopy group* of the pair (X, A) is

$$\pi_n(X, A) = \pi_{n-1}(P(X; A, \{*\}),$$

for $n \geq 1$.

A map $f : (X, A) \to (Y, B)$ of pairs induces a map $P(X; A, \{*\}) \to P(Y; B, \{*\})$, and hence a homomorphism

$$f_* : \pi_n(X, A) \to \pi_n(X, B)$$

for $n \geq 2$. Now the sequence of spaces

$$\Omega X \to P(X; A, \{*\}) \xrightarrow{q} A \tag{3.5.8}$$

is a fibration, where $q(\omega) = \omega(0)$ for $\omega \in P(X; A, \{*\})$. From (3.5.8) we obtain the following exact sequence

$$\cdots \to \pi_{n+1}(X, A) \to \pi_n(A) \to \pi_n(X) \to \pi_n(X, A) \to \pi_{n-1}(A) \to \cdots . \tag{3.5.9}$$

The sequence (3.5.9) is called the *exact homotopy sequence of the pair* (X, A).

Now consider the exact sequence generated by $(P^4(m), S^3)$:

$$\cdots \longrightarrow \pi_7(P^4(m), S^3) \xrightarrow{\partial} \pi_6(S^3) \xrightarrow{i_*} \pi_6(P^4(m)) \xrightarrow{j_*} \pi_6(P^4(m), S^3) \longrightarrow \cdots.$$

Thus we have

$$\pi_4(S^3) \xrightarrow{i_*} \pi_4(P^4(m)) \longrightarrow \pi_4(P^4(m), S^3) \xrightarrow{\partial} \pi_3(S^3) \xrightarrow{i_*} \pi_3(P^4(m))$$

and the maps i_* are surjective, as i is the inclusion of the bottom cell in $P^4(m)$. Therefore we have a short exact sequence

$$0 \longrightarrow \pi_4(P^4(m), S^3) \xrightarrow{\partial} \mathbb{Z} \xrightarrow{i_*} \mathbb{Z}_m \longrightarrow 0$$

Therefore $\pi_4(P^4(m), S^3) \cong \mathbb{Z}$. Let χ_4 denote the generator of $\pi_4(P^4(m), S^3)$ and let $[\iota_3, \chi_4]$ be the Whitehead product of the generator $\iota_3 \in \pi_3(S^3)$ and χ_4. By Lemma 2 in [Sas65], we have $\partial([\iota_3, \chi_4]) = 0$. Let σ be any element of $\pi_6(P^4(m))$ such that its image under j_* is $[\iota_3, \chi_4]$.

Now we state a result concerning the attaching maps of CW-complexes with the homotopy types of S^3-bundles over S^4. Let (a, b) denote the greatest common divisor of two integers a and b. Let ν be the generator of $\pi_6(S^3)$.

Proposition 3.47. *The following hold:*

(1) if m is odd, then

$$\pi_6(P^4(m)) \cong \mathbb{Z}_{(m,12)}\{i_*(\nu)\} \oplus \mathbb{Z}_m\{\sigma\};$$

(2) if m is even multiple of $(m, 12)$, then

$$\pi_6(P^4(m)) \cong \mathbb{Z}_{(m,12)}\{i_*(\nu)\} \oplus \mathbb{Z}_m\{\sigma\} \oplus \mathbb{Z}_2;$$

(3) if m is even and an odd multiple of $(m, 12)$, then

$$\pi_6(P^4(m)) \cong \mathbb{Z}_{(m,12)/2}\{\lambda_3\} \oplus \mathbb{Z}_{2m}\{\sigma\} \oplus \mathbb{Z}_2,$$

where $\lambda_3 = \frac{2m}{(m,12)}\sigma + i_*(\nu)$.

Proof. See the proof of Theorem in [Sas65]. □

The proof of Proposition 3.47 has many elaborate computations. In Chapter 4, we present a simpler computation for the odd primary part considering the localisation of spaces at an odd prime p. Crowley and Escher determined the homotopy classification of spaces $M_{l,m}$ with $m > 0$, which happens to coincide with the homeomorphism classification.

Theorem 3.48. *Let $m, m' > 0$.*

(1) The manifolds $M_{l',m'}$ and $M_{l,m}$ are orientation preserving homotopy equivalent if and only if $m = m'$ and $m' \equiv \alpha l \pmod{(m, 12)}$ where $\alpha^2 \equiv 1 \pmod{(m, 12)}$.

(2) Orientation reversing homotopy equivalences between any $M_{l',m}$ and $M_{l,m}$ can only exist when $m = 2^\varepsilon p_1^{i_1} \cdots p_k^{i_k}$, where p_i is a prime, $p_i \equiv 1 \pmod 4$, and $\varepsilon \in \{0, 1\}$. Furthermore, if n is of this form with $\varepsilon = 0$, then the single oriented homotopy type admits an orientation reversing self homotopy equivalence; if $\varepsilon = 1$, then $M_{l',m}$ is orientation reversing homotopy equivalent to $M_{l,m}$ if and only if $l' + l \not\equiv 0 \pmod 2$.

Proof. See the proof of Theorem 1.1 in [CE03]. □

Chapter 4

Homotopy theory of gauge groups

The topology and homotopy theory of principal G-bundles and their associated gauge groups are presented in this chapter. In the first section we summarise information on classical Lie groups. In Section 4.2 we describe the topology of principal G-bundles and give some well-known results on their classification up to bundle isomorphism. We introduce in Section 4.3 the definition of gauge groups and prove Theorem 4.16, which will be widely used throughout the next chapters. In Section 4.4 we focus on the homotopy theory of gauge groups of principal G-bundles over spaces of dimension $n \leq 4$ and we present some results. We also mention some results for the classifying spaces of gauge groups. In the last section we mention research related to gauge groups over high dimensional manifolds that has been done inside and outside of homotopy theory.

4.1 Classical Lie groups

This section is based on [Mim95]. To study the homotopy theory of Lie groups it suffices to consider compact Lie groups. Any abelian compact connected Lie group of dimension n is isomorphic to an n-torus. A *maximal torus* T of G is a subgroup which is a torus such that if $T \subset U \subset G$ and U is a torus then $T = U$. The *rank* of G is the dimension of a maximal torus of G.

A compact connected Lie group G is called *simple* if it is non-abelian and has no proper closed normal subgroups of dimension higher than 0. If the centre of G is finite we say that G is *semi-simple*. Compact connected Lie groups are locally isomorphic to direct products of tori and simple non-abelian Lie groups. Thus the classification of Lie groups, up to local isomorphism, reduces to that of simple Lie groups.

Theorem 4.1 (Classification of simple Lie groups)**.** *The connected compact simple Lie groups are exactly the following:*

	dimension	*linear group*	*universal cover*
$A_n(n \geq 1)$	$n(n+2)$	$SU(n+1)$	
$B_n(n \geq 2)$	$n(2n+1)$	$SO(2n+1)$	$Spin(2n+1)$
$C_n(n \geq 3)$	$n(2n+1)$	$Sp(n)$	
$D_n(n \geq 2)$	$n(2n-1)$	$SO(2n)$	$Spin(2n)$
G_2	*14*		
F_4	*52*		
E_6	*78*		
E_7	*133*		
E_8	*248*		.

In the classification theorem, the first four families of groups are called the *classical Lie groups* and the last five groups are called the *exceptional Lie groups*.

Let G be one of U, SU, O, SO, Sp, so that $G(n) = U(n), SU(n), O(n), SO(n), Sp(n)$. We can obtain an inclusion $G(n) \to G(n+1)$ by the map $A \mapsto A \oplus I_1$. The *infinite dimensional classical Lie group* $\mathbf{G}$ (or $G(\infty)$) is the group

$$\mathbf{G} = \bigcup_n G(n)$$

which is endowed with the weak topology.

From the fibrations $G(n+1)/G(n) = S^{d(n+1)-1}$ (where $d = 4$ if $G = Sp$, $d = 2$ if $G = U$ and $d = 1$ if $G = O$) we obtain

$$\begin{aligned}
\pi_k(\mathbf{Sp}) &= \pi_k(Sp(n)) \quad \text{for } n \geq (k-1)/4; \\
\pi_k(\mathbf{U}) &= \pi_k(U(n)) \quad \text{for } n \geq (k+1)/2; \\
\pi_k(\mathbf{O}) &= \pi_k(O(n)) \quad \text{for } n \geq k+2.
\end{aligned}$$

The homotopy groups of the classical Lie groups G can be obtained using the classification theorem and the homotopy exact sequences associated to each bundle where G is involved. Table A.1 in the Appendix contains some relevant information on the higher homotopy groups of simple Lie groups as it is presented in [Jam95].

4.2 Principal G-bundles

This section is based on [Hus66, Sel08, Ste51]. Let X be a right G-space. Two elements $x, x' \in X$ are *G-equivalent* if there exists an element $g \in G$ such that $xg = x'$. This is an equivalence relation, and the set of all xg for $g \in G$, denoted xG, is called the *orbit* of $x \in X$. Let X/G denote the set of all orbits xG, for $x \in X$. Thus X/G is a topological space with the quotient topology defined by the quotient map $p : X \to X/G$, $p(x) = xG$.

Definition 4.2. A *principal G-bundle*, denoted $\xi = \{P, p, B\}$ or $P \xrightarrow{p} B$, is a bundle with projection $p : P \to B$ and an action $P \times G \to P$, $(x, g) \mapsto xg$, such that the following hold:

(1) the map $P \times G \to P \times P$ given by

$$(x, g) \mapsto (x, xg) \qquad x \in P, \;\; g \in G$$

is a homeomorphism onto its image;

(2) $B = P/G$ and the projection p is the quotient map;

(3) for all $b \in B$ there exists an open neighborhood V together with a homeomorphism $\phi : V \times G \to p^{-1}(V)$ such that the diagram

$$\begin{array}{ccc} V \times G & \xrightarrow{\phi} & p^{-1}(V) \\ & {\scriptstyle p_1}\searrow \quad \swarrow{\scriptstyle p} & \\ & V & \end{array}$$

commutes, and for all $x \in p^{-1}(V)$ and $g \in G$, $\phi^{-1}(xg) = \phi^{-1}(x)g$, where the action on $V \times G$ is given by $(x, g)g' = (x, gg')$.

Notice that property (1) in Definition 4.2 implies that the action θ is free, and properties (1) and (2) imply that the fibre of the bundle is homeomorphic to G. The space P is the *total space*, B is the *base space* and G is the *structure group*. We say that the sequence

$$G \to P \xrightarrow{p} B$$

is a *principal G-bundle* over B. For simplicity we can also denote it by $P \to B$.

Definition 4.3. Let $\xi = \{E, p, B, F, G\}$ be a bundle over B, and let $f : B' \to B$ be a map. The *induced bundle of ξ under f*, denoted by $f^*(\xi) = \{f^*(E), f^*(p), B', F, G\}$, is defined by the pullback of f and p:

$$\begin{array}{ccc} f^*(E) & \longrightarrow & E \\ {\scriptstyle f^*(p)}\downarrow & & \downarrow{\scriptstyle p} \\ B' & \xrightarrow{f} & B. \end{array}$$

That is, $f^*(\xi)$ is a bundle that has B' as the base space, the pullback

$$f^*(E) = \{(b_1, x) \in B' \times E \mid f(b_1) = p(x)\}$$

as the total space, and the map $f^*(p)$ given by $(b_1, x) \mapsto b_1$ as the projection.

Proposition 4.4. *Let $\xi = \{P, p, B\}$ be a principal G-bundle. Given a map $f : B' \to B$, the induced bundle $f^*(\xi)$ is a principal G-bundle.*

Proof. See Proposition 4.1 in [Hus66]. □

Definition 4.5. Given two principal G-bundles $\xi = \{P, p, B\}$ and $\xi' = \{P', p', B\}$ over a space B, a *principal bundle map* or *principal bundle morphism* over B is a map $f : P \to P'$ such that

(1) the diagram

$$\begin{array}{ccc} P & \xrightarrow{f} & P' \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle p'} \\ B & = & B \end{array}$$

commutes, and

(2) f is *G-equivariant*, that is,

$$f(xg) = f(x)g$$

for all $g \in G$ and all $x \in P$.

Since the composition of a principal morphisms is a principal morphism we can speak of the category of principal G-bundles and principal bundle morphisms.

Theorem 4.6. *Every principal bundle map over B is an isomorphism.* □

A bundle isomorphism of principal G-bundles defines an equivalence relation. We denote by $Prin_G(B)$ the set of isomorphism classes of principal G-bundles over B.

An open cover $\{U_i\}_{i \in I}$ of a topological space is *numerable* if there exists a locally finite partition of unity $\{u_i\}_{i \in I}$ such that $\overline{u_i^{-1}((0,1])} \subset U_i$ for each $i \in I$.

Definition 4.7. A principal G-bundle $\xi = \{P, p, B\}$ is *numerable* if there is a numerable open cover $\{U_i\}_{i \in I}$ of B such that the bundle $\xi|_{U_i} = \{p^{-1}(U_i), p|_{p^{-1}(U_i)}, U_i\}$ is trivial for each $i \in I$. Notice that each principal G-bundle over a paracompact Hausdorff space is numerable.

Definition 4.8. A numerable principal G-bundle $\xi_0 = \{E_0, p_0, B_0\}$ is called a *universal G-bundle* if the following hold:

(1) For each numerable principal G-bundle ξ over B there exists a map $f : B \to B_0$ such that ξ and $f^*(\xi_0)$ are isomorphic over B.

(2) If f, $g : B \to B_0$ are two maps such that $f^*(\xi_0)$ and $g^*(\xi_0)$ are isomorphic over B, then f and g are homotopic.

In other words, a numerable principal G-bundle $\xi_0 = \{E_0, p_0, B_0\}$ is a universal G-bundle if, for any pointed space B with numerable covering, the pullback of p_0 and maps $f : B \to B_0$ induces a bijection

$$[B, B_0] \to Prin_G(B).$$

The next theorem gives a characterisation of universal G-bundles, and its proof can be found in [Ste51].

Theorem 4.9. *If $\xi = \{P, p, B\}$ is a principal G-bundle, then ξ is a universal bundle if and only if P is contractible.*

Now we consider the construction given by Milnor for a universal G-bundle. Let G be a topological group. Let EG be the infinite join

$$EG = G * G * G * \cdots = \varinjlim G^{*n}.$$

Explicity, as a set,

$$EG = \{(g_0, t_0, g_1, t_1, \ldots, g_n, t_n, \ldots) \in (G \times I)^\infty \mid \{i \mid t_i \neq 0\} \text{ is finite}, \textstyle\sum_i t_i = 1\}/\sim$$

where the equivalence relation $\sim$ is generated by the relations

$$(g_0, t_0, \ldots, g_{n-1}, t_{n-1}, g_n, 0, g_{n+1}, t_{n+1}, \ldots) \sim (g_0, t_0, \ldots, g_{n-1}, t_{n-1}, g'_n, 0, g_{n+1}, t_{n+1}, \ldots)$$

for all $g_n, g'_n \in G$. A G-action on EG is given by

$$(g_0, t_0, g_1, t_1, \ldots, g_n, t_n, \ldots) \cdot g = (g_0 g, t_0, g_1 g, t_1, \ldots, g_n g, t_n, \ldots).$$

Let $BG = EG/G$. We also write $E_n G = G^{*(n+1)}$ and $B_n G = E_n G/G$, referring to the inclusions $B_0 G \hookrightarrow B_1 G \hookrightarrow B_2 G \hookrightarrow \cdots \hookrightarrow B_n G \hookrightarrow \cdots$ as the *Milnor filtration* on BG. Observe that $G^{*n} = G * \cdots * G = \Sigma^{n-1} G \wedge \cdots \wedge G$. Therefore $EG = \varinjlim G^{*n}$ is ∞-connected, implying that it is contractible. The following theorem is due to Milnor [Mil56a].

Theorem 4.10. *For every topological group G, the quotient map*

$$EG \to BG$$

is a universal G-bundle. □

The bundle $G \to EG \to BG$ is called *the universal G-bundle of G*, and since homotopy classes of maps into BG classify numerable principal G-bundles, BG is called the *classifying space of the group G*.

Corollary 4.11. *Let X be a space with the homotopy type of a CW-complex. There is a one-to-one correspondence*

$$Prin_G(X) \leftrightarrow [X, BG]. \qquad \square$$

Proposition 4.12. *Let G be a topological group. Suppose that G has the homotopy type of a CW complex. Then G is homotopy equivalent to the loop space ΩBG.*

Proof. Given a fibration

$$F \to E \to B$$

with E contractible, there is a homotopy equivalence $F \simeq \Omega B$ (see Proposition 2.37). As EG is contractible, the universal bundle

$$G \to EG \to BG$$

gives a homotopy equivalence $\Omega BG \to G$. $\square$

Proof of Proposition 3.44. By Theorem 3.42 there is a bijection between S^3-bundles over S^4 and the associated principal $SO(4)$-bundles over S^4. Thus it suffices to compute the set of classes of principal $SO(4)$-bundles over S^4. By Corollary 4.11, there is a one-to-one correspondence

$$Prin_{SO(4)}(S^4) \leftrightarrow [S^4, BSO(4)].$$

As $SO(4)$ is connected, the unpointed and pointed homotopy sets coincide so that we have

$$[S^4, BSO(4)] = [S^4, BSO(4)]_* = \pi_4(BSO(4)).$$

Finally by Proposition 4.12 we obtain $\pi_4(BSO(4) \cong \pi_3(SO(4))$. $\square$

4.3 Gauge groups

Let $P \xrightarrow{p} B$ be a principal G-bundle over B. A bundle isomorphism of P over B to itself, $f : P \to P$, is called an *automorphism of* P. We can endow $\mathrm{Map}(P, P)$ with the compact-open topology (see Ch.1). For two right G-spaces X and Y, let $\mathrm{Map}^G(X, Y)$ be the subspace of maps $f \in \mathrm{Map}(X, Y)$ such that $f(xg) = f(x)g$ for all $g \in G$ and all $x \in X$.

Definition 4.13. Let $P \xrightarrow{p} B$ be a principal G-bundle over B. The *gauge group* of P, denoted $\mathcal{G}_P(B)$, is the group of automorphisms of P endowed with the induced topology from $\mathrm{Map}(P, P)$, with group multiplication given by composition. In particular, given a principal G-bundle $P \xrightarrow{p} B$, its gauge group $\mathcal{G}_P(B)$ is a subspace of $\mathrm{Map}^G(P, P)$.

For a topological group G we denote by $\mathrm{Ad}(G)$ the right G-space G with right action (called the *adjoint action*) given by

$$(x, s) \mapsto x^s = s^{-1}xs.$$

Proposition 4.14. *The gauge group $\mathcal{G}_P(B)$ can be identified with the mapping space*

$$\mathrm{Map}^G(P, \mathrm{Ad}(G)) = \{\phi : P \to G \mid \phi(xg) = g^{-1}\phi(x)g\}.$$

Proof. Let $u \in \mathcal{G}_P(B)$. Then for any $x \in P$ we have that $p(x) = pu(x)$, where p is the projection map. Thus $x, u(x)$ are G-equivalent and so to each element $u \in \mathcal{G}_P(B)$ we can assign the continuous function $\phi_u : P \to G$ defined by the relation

$$u(x) = x\phi_u(x). \tag{4.3.1}$$

As u satisfies $u(x)g = u(xg)$ we obtain

$$u(x)g = u(xg) = xg\phi_u(xg).$$

This together with (4.3.1) gives

$$x\phi_u(x)g = xg\phi_u(xg).$$

Since the action of G is free, the last equation implies $\phi_u(x)g = g\phi_u(xg)$ and therefore $\phi(xg) = g^{-1}\phi(x)g$, for all $x \in P$, $g \in G$. Moreover, the function which assigns to each automorphism $u \in \mathcal{G}_P(B)$ the function $\phi_u \in \mathrm{Map}^G(P, \mathrm{Ad}(G))$ is a continuous bijection. Indeed, if $\phi \in \mathrm{Map}^G(P, \mathrm{Ad}(G))$, then we define $u = u_\phi \in \mathcal{G}_P(B)$ by

$$u(x) = x\phi(x) \in \mathcal{G}_P(B)$$

and, since the action of G on P is free, the map $\phi \mapsto u_\phi$ is inverse to $u \mapsto \phi_u$. □

Proposition 4.15. *Let P be a principal G-bundle over B. If either P is trivial or G is abelian, then*

$$\mathcal{G}_P(B) \cong \mathrm{Map}(B, G).$$

Proof. By Proposition 4.14 we have $\mathcal{G}_P(B) = \mathrm{Map}^G(P, \mathrm{Ad}(G))$. Consider two cases:

(1) Suppose that $P = B \times G$ is trivial. Given $f \in \mathrm{Map}^G(B \times G, \mathrm{Ad}(G))$, we may define $f_B \in \mathrm{Map}(B, G)$ by $f_B(b) = f(b, 1)$; and conversely, given $f' \in \mathrm{Map}(B, G)$, we may define $f'_P \in \mathrm{Map}^G(B \times G, \mathrm{Ad}(G))$ by $f'_P(b, g) = f'(b)^g$. It is not difficult to check that the maps $f \mapsto f_B$ and $f' \mapsto f'_P$ are continuous maps between $\mathrm{Map}^G(B \times G, \mathrm{Ad}(G))$ and $\mathrm{Map}(B, G)$ and that they are inverse to each other.

(2) Suppose now that G is abelian. Then the action of G on $\mathrm{Ad}(G)$ is trivial and so $f(xg) = f(x)$ for all $f \in \mathrm{Map}^G(P, \mathrm{Ad}(G))$, $x \in P$ and $g \in G$. Hence all such f factor through some map $B \to G$, and conversely, for every $f' \in \mathrm{Map}(B, G)$ we have $f' \circ p \in \mathrm{Map}^G(P, \mathrm{Ad}(G))$, where $p : P \to B$ is the projection. Thus $f' \mapsto f' \circ p$ gives a homeomorphism $\mathrm{Map}(B, G) \to \mathrm{Map}^G(B \times G, \mathrm{Ad}(G))$. $\square$

Let $\xi = (EG, \pi, BG)$ be the universal bundle of G. We use the following notation. For a principal G-bundle $p : P \to B$ and a map $f : B \to BG$, let $\mathrm{Map}_f(B, BG)$ denote the subspace of $\mathrm{Map}(B, BG)$ consisting of maps $g : B \to BG$ such that $g^*(\xi)$ and $f^*(\xi)$ are isomorphic over B. In other words, the subscript f denotes the component of $\mathrm{Map}(B, BG)$ which contains the map $f : B \to BG$ that induces P. Let $\mathcal{G}_f(B) = \mathcal{G}_P(B)$.

The following theorem is crucial for this work. This result is proved in [Got72] and [AB83]. Throughout this work we will consider principal G-bundles over manifolds with G a Lie group, so that from now on by a principal G-bundle we will mean a numerable principal G-bundle with structure group G a Lie group.

Theorem 4.16. *Let $\xi = \{P, p, B\}$ be a principal G-bundle classified by $f : B \to BG$. There is a homotopy equivalence*

$$B\mathcal{G}_f(B) \simeq \mathrm{Map}_f(B, BG).$$

Proof. Let

$$G \to EG \to BG$$

be the universal bundle of G. Consider the mapping space $\mathrm{Map}^G(P, EG)$. The gauge group $\mathcal{G}_f(B)$ acts on this space by composition on the right. By Proposition 4.14 we have that given $u \in \mathcal{G}_f(B)$, there exist a unique $\phi_u \in \mathrm{Map}^G(P, \mathrm{Ad}(G))$ such that $u(x) = x\phi_u(x)$. Thus for $w \in \mathrm{Map}^G(P, EG)$ and $u \in \mathcal{G}_f(B)$, the action is given by

$$(wu)(x) = w(x)\phi_u(x)$$

and the action is free. We define a map $\tilde{p} : \mathrm{Map}^G(P, EG) \to \mathrm{Map}_f(B, BG)$ by assigning to each $w \in \mathrm{Map}^G(P, EG)$ the quotient map $h \in \mathrm{Map}_f(B, BG)$ on the base space of the bundles, as depicted in the following diagram

$$\begin{array}{ccc} P & \xrightarrow{w} & EG \\ \downarrow{\scriptstyle p} & & \downarrow{\scriptstyle p_0} \\ B & \xrightarrow{h} & BG. \end{array} \qquad (4.3.2)$$

Thus if $\tilde{p}(w') = \tilde{p}(w)$ then there is a $u \in \mathcal{G}_f(B)$ such that $w' = wu$. This defines a principal fibration

$$\mathcal{G}_f(B) \longrightarrow \mathrm{Map}^G(P, EG) \longrightarrow \mathrm{Map}_f(B, BG).$$

As EG is contractible so is the space $\mathrm{Map}^G(P, EG)$, which means that this is a universal bundle for $\mathcal{G}_f(B)$ and

$$B\mathcal{G}_f(B) \simeq \mathrm{Map}_f(B, BG)$$

as claimed. □

Definition 4.17. Let $P_f \xrightarrow{p} B$ be a principal G-bundle and P_{b_0} be the fibre at the base point b_0 of B. The pointed gauge group, denoted $\mathcal{G}^*_f(B)$, is the subgroup of $\mathcal{G}_f(B)$ which fixes the fibre at b_0, that is,

$$\mathcal{G}^*_f(B) = \{u \in \mathcal{G}_f(B) \mid u(x) = x \text{ for all } x \in P_{b_0}\}.$$

By Theorem 4.16 we have the following homotopy equivalence

$$B\mathcal{G}^*_f(B) \simeq \mathrm{Map}^*_f(B, BG).$$

The study of the topology of gauge groups and their classifying spaces is strongly motivated by applications in other areas in mathematics such as differential geometry and mathematical physics. For instance, Donaldson [Don86] used topological information of gauge groups of principal $SU(2)$-bundles over 4-manifolds to distinguish differentiable structures on homeomorphic manifolds. In mathematical physics, the description of the dynamical behaviour of elementary particles in a 4-dimensional space-time is still an open question. From a mathematical viewpoint, gauge theories correspond to the differential geometry and topology of fibre bundles (see for instance [CM94, Ati88]). Defining concrete applications of the homotopy theory of the gauge groups in other areas of mathematics go beyond the scope of this work. However it is worth mentioning the links to other areas for these are at the core of the motivations behind the present work.

4.4 Homotopy theory of gauge groups

For the next discussion we will assume that all spaces are pointed, compact, connected, and have the homotopy type of a CW-complex. Let $P_f \xrightarrow{p} B$ be a principal G-bundle over B with gauge group $\mathcal{G}_f(B)$. The subindex in P_f, $\mathcal{G}_f(B)$ and $B\mathcal{G}_f(B)$ denotes the classifying map $f \in [B, BG]$ of the principal G-bundle. It follows from Theorem 4.16 that there is a commutative diagram

$$\begin{array}{ccc} B\mathcal{G}^*_f(B) & \longrightarrow & B\mathcal{G}_f(B) \\ \downarrow \simeq & & \downarrow \simeq \\ \mathrm{Map}^*_f(B, BG) & \longrightarrow & \mathrm{Map}_f(B, BG). \end{array} \tag{4.4.1}$$

Recall that the evaluation map $ev : \mathrm{Map}(X,Y) \to Y$ is defined by $ev(f) = f(x_0)$, where x_0 is the basepoint of X, with fibre $\mathrm{Map}^*(X,Y)$. Consider the evaluation fibration

$$\mathrm{Map}^*(B,BG) \longrightarrow \mathrm{Map}(B,BG) \xrightarrow{ev} BG. \tag{4.4.2}$$

The restriction of the evaluation map to path components induces evaluation fibrations

$$\mathrm{Map}^*_f(B,BG) \longrightarrow \mathrm{Map}_f(B,BG) \xrightarrow{ev} BG. \tag{4.4.3}$$

Thus by (4.4.1), the gauge group $\mathcal{G}_f(B)$ and the classifying space $B\mathcal{G}_f(B)$ fit in the following fibration sequence

$$\mathcal{G}^*_f(B) \longrightarrow \mathcal{G}_f(B) \longrightarrow G \xrightarrow{\partial} B\mathcal{G}^*_f(B) \xrightarrow{i} B\mathcal{G}_f(B) \xrightarrow{ev} BG, \tag{4.4.4}$$

where i is the inclusion and ∂ is the connecting map. Thus the gauge group is the homotopy fibre of the connecting map ∂. Hence it is expected that the homotopy types of the gauge groups are determined by the properties of the connecting map.

We present some useful results on the homotopy types of pointed mapping spaces. There is a bijection between the set of homotopy classes of maps $[X,Y]_*$ from X to Y and the set of path components $\pi_0(\mathrm{Map}^*(X,Y))$ of $\mathrm{Map}^*(X,Y)$.

Proposition 4.18. *Let Y be any space.*

(1) If X is an H-group then $\mathrm{Map}^*_f(Y,X) \simeq \mathrm{Map}^*_0(Y,X)$ for all $[f] \in [Y,X]_*$.

(2) If X is a co-H-group then $\mathrm{Map}^*_f(X,Y) \simeq \mathrm{Map}^*_0(X,Y)$ for all $[f] \in [X,Y]_*$.

Proof. It suffices to prove only one part of the statement since we obtain similar results dualising the arguments. Thus we prove part b). Suppose X is a co-H-group. Then the comultiplication on X induces an H-group structure on $Z = \mathrm{Map}^*(X,Y)$, given by homotopy-associative multiplication $m : Z \times Z \to Z$ and homotopy inverse $i : Z \to Z$, say. Let $f \in Z$ be a map, and let Z_f and Z_0 be path components of Z containing the map f and the trivial map $* \in Z$, respectively. Consider maps $\Theta = m(f,-) : Z \to Z$, defined by the composite

$$\Theta(\alpha) : X \xrightarrow{\sigma} X \vee X \xrightarrow{f \vee \alpha} Y \vee Y \xrightarrow{\nabla} Y,$$

and $\Psi = m(i(f),-) : Z \to Z$, defined by the composite

$$\Psi(\alpha) : X \xrightarrow{\sigma} X \vee X \xrightarrow{i(f) \vee \alpha} Y \vee Y \xrightarrow{\nabla} Y.$$

Note that the maps Θ and Ψ satisfy $\Theta(Z_0) \subseteq Z_f$ and $\Psi(Z_f) \subseteq Z_0$. We aim to show that Θ and Ψ and homotopy inverses, thus inducing a homotopy equivalence $Z_0 \simeq Z_f$.

We will show that $\Psi \circ \Theta \simeq \mathbb{1}_Z$; the argument for $\Theta \circ \Psi$ is similar. Consider the homotopy commutative diagram

$$\begin{array}{ccccccc}
Z & \xrightarrow{(f,\mathbb{1}_Z)} & Z \times Z & \xrightarrow{(i,\mathbb{1}_Z)\times \mathbb{1}_Z} & Z \times Z \times Z & \xrightarrow{\mathbb{1}_Z \times m} & Z \times Z \\
\downarrow{\scriptstyle j_2} & & \downarrow{\scriptstyle * \times \mathbb{1}_Z} & & \downarrow{\scriptstyle m \times \mathbb{1}_Z} & & \downarrow{\scriptstyle m} \\
Z \times Z & = & Z \times Z & = & Z \times Z & \xrightarrow{m} & Z,
\end{array}$$

where $f : Z \to Z$ is the constant map at $f \in Z$ and $j_2 : Z \to Z \times Z$ is the inclusion into the second factor, given by $j_2(\alpha) = (*, \alpha)$. Here the square on the right homotopy commutes because of homotopy associativity of the map m, and the one on its left homotopy commutes because of properties of the homotopy inverse i. The composite of maps on the top and the right of the diagram is just $\Psi \circ \Theta = m(i(f), m(f, -))$, and the map $m \circ j_2$ along the left and the bottom of the diagram is homotopic to $\mathbb{1}_Z$. Thus $\Psi \circ \Theta \simeq \mathbb{1}_Z$, as required. □

Recall that given a cofibration sequence

$$A \xrightarrow{\varphi} X \xrightarrow{q} C_\varphi \xrightarrow{\delta} \Sigma A$$

there is a coaction $\psi_0 : C_\varphi \to C_\varphi \vee \Sigma A$ of ΣA onto the cofibre C_φ (see Example 2.39). This coaction defines an action on homotopy sets. We have already seen that closed manifolds have the homotopy type of a CW-complex. Let

$$S^{n-1} \xrightarrow{\varphi} M^{n-1} \xrightarrow{q} M \xrightarrow{\delta} S^n \tag{4.4.5}$$

be the cofibration sequence induced by the attaching map φ onto the $(n-1)$-skeleton M^{n-1}. Here $M \simeq C_\varphi$ is a CW-complex. Consider the exact sequence induced by the cofibration sequence (4.4.5),

$$[S^n, Y]_* \xrightarrow{\delta^*} [M, Y]_* \xrightarrow{q^*} [M^{n-1}, Y]_* \xrightarrow{\varphi^*} [S^{n-1}, Y]_*.$$

By Theorem 2.40, the group $\pi_n(Y)$ induces an action on $[M, Y]$, and the orbit of $f \in [M, Y]$ under this action is the preimage $(q^*)^{-1}(q^* f)$. We can use this action to obtain homotopy equivalences between the path components of $\mathrm{Map}^*(M, Y)$, which in turn gives equivalences between classifying spaces of the gauge groups and between gauge groups.

Theorem 4.19. *Let M be an n-dimensional closed manifold and Y a connected CW-complex. Given a map $f : M \to Y$ there is a homotopy equivalence*

$$\mathrm{Map}^*_f(M, Y) \simeq \mathrm{Map}^*_{f \cdot \alpha}(M, Y)$$

where $\alpha \in \pi_n(Y)$ and $f \cdot \alpha$ is the image of the action of α on f.

Proof. The proof is similar to that of Proposition 4.18. Given a map $f : M \to Y$ and γ a representative of the class $\alpha \in \pi_n(Y)$, we define $\Theta_\gamma : \mathrm{Map}^*_f(M, Y) \to \mathrm{Map}^*_{f\cdot\alpha}(M, Y)$ by the composite

$$\Theta_\gamma(g) : M \xrightarrow{\psi_0} S^n \vee M \xrightarrow{\gamma\vee g} Y \vee Y \xrightarrow{\nabla} Y$$

where ψ_0 is the coaction of S^n onto M. It is not hard to see that the map $\Theta_{\tilde{\gamma}}$, where $\tilde{\gamma}$ is a representative of $-\alpha \in \pi_n(Y)$, is a homotopy inverse of Θ_γ. □

Now if $Y = BG$, then from Theorem 4.19 we obtain results on the homotopy types of the classifying spaces of gauge groups of principal G-bundles over M.

Corollary 4.20. *Let $P_f \to M$ be a principal G-bundle over an n-dimensional closed manifold with classifying map $f \in [M, BG]$. Then*

$$B\mathcal{G}^*_f(M) \simeq B\mathcal{G}^*_{f\cdot\alpha}(M).$$

Moreover, if the map δ^* is a surjection then

$$B\mathcal{G}^*_f(M) \simeq B\mathcal{G}^*_0(M)$$

for any $f \in [M, BG]$. □

It is clear that statements above can be extended to any finite n-dimensional CW-complex with one n-cell. In the case of the unpointed classifying spaces of the gauge groups, $B\mathcal{G}_f(M) \simeq \mathrm{Map}_f(M, BG)$, the answer on the homotopy types might be different in general. Masbaum [Mas91] showed that when $M = S^4$ and $G = SU(2)$ the number of homotopy types of spaces $B\mathcal{G}_f(M)$ is infinite. Kono and Tsukuda [KT00] found similar results for the path components of $\mathrm{Map}(M, BSU(2))$ for M a simply connected 4-manifold.

In contrast to the results obtained on the number of homotopy types of unpointed classifying spaces of gauge groups, Crabb and Sutherland [CS00] proved that there are finitely many homotopy types of gauge groups even when the number of isomorphism classes of bundles is countable. If the number of isomorphism classes of principal G bundles over a closed simply connected manifold is finite, it is clear that the number of homotopy types of the gauge groups is also finite. It is expected that the homotopy types of gauge groups depend on the properties of the connecting map ∂ in the evaluation fibration

$$G \xrightarrow{\partial_f} \mathrm{Map}^*_f(\Sigma Y, BG) \longrightarrow \mathrm{Map}_f(B, BG). \qquad (4.4.6)$$

In [Lan73] Lang showed that if $B = \Sigma Y$, then the adjoint of the connecting map ∂ in the fibration (4.4.6) is a Whitehead product. Notice that if $B = \Sigma Y$ then the pointed exponential law and Proposition 4.12 imply that there is a homotopy equivalence

$$\mathrm{Map}^*(\Sigma Y, BG) \simeq \mathrm{Map}^*(Y, G).$$

We restate the result of Lang in terms of Samelson products.

Lemma 4.21. *The adjoint $G \wedge Y \to G$ of the composite*

$$G \xrightarrow{\partial_f} \mathrm{Map}^*_f(Y, G) \xrightarrow{\simeq} \mathrm{Map}^*_0(Y, G)$$

is homotopic to the Samelson product $\langle \mathbb{1}_G, f\rangle$. □

We present some results on the classification of gauge groups of principal G-bundles over S^4. We start by recalling the seminal work of Kono on the $SU(2)$-gauge groups over S^4 [Kon91], proved in a slightly different manner.

The principal $SU(2)$-bundles over S^4 are classified by the set $[S^4, BSU(2)]$. Since we can identify $SU(2)$ with S^3 we get $[S^4, BSU(2)] \cong \pi_4(BS^3) \cong \pi_3(S^3) \cong \mathbb{Z}$. Let $P_k \to S^4$ be the principal G-bundle classified by $k \in \mathbb{Z}$ and $\mathcal{G}_k$ its gauge group. We will use the notation

$$L_k = \mathrm{Map}^*_k(S^4, BSU(2))$$

for the component of $L = \mathrm{Map}^*(S^4, BSU(2))$ classified by $k \in \mathbb{Z}$. From the evaluation fibration we obtain a homotopy fibration sequence

$$\mathcal{G}_k \to SU(2) \xrightarrow{\partial_k} L_k \to B\mathcal{G}_k \to BSU(2), \tag{4.4.7}$$

where the gauge group $\mathcal{G}_k$ appears as the homotopy fibre of the map ∂_k. Notice that $\partial_k \in [SU(2), L_k] = [S^3, L_k\pi_3(L_k) \cong \pi_3(L_0)$, where the isomorphism is a consequence of the fact that $L_0 \simeq L_k$ (see Proposition 4.18).

Lemma 4.22. *Let X be an H-space such that $\pi_k(X)$ is finite for all $k \geq 0$. Then there exists a homotopy equivalence*

$$X \to \prod_p X_{(p)}.$$

Proof. As X is an H-space then $\pi_1(X)$ is abelian. Then for each prime p, we consider the localisation $\phi_{i_p} : X \to X_{(p)}$. By Theorem 3.19 there is a homotopy pullback square

$$\begin{array}{ccc} X & \longrightarrow & \prod_p X_{(p)} \\ \downarrow & & \downarrow \\ X_{(0)} & \longrightarrow & (\prod_p X_{(p)})_{(0)} \end{array} \tag{4.4.8}$$

By hypothesis $\pi_k(X)$ is finite for all $k \geq 0$. Thus $\pi_k(X_{(0)}) \cong \pi_k(X)_{(0)} = 0$, for all $k \geq 0$. Therefore $X_{(0)}$ and $(\prod_p X_{(p)})_{(0)}$ are contractible. Hence using this fact in (4.4.8) we obtain the homotopy equivalence as required. □

Given a map $h : X \to Y$, let F_h denote its homotopy fibre. Let $\epsilon : S^3 \to L$ be a map that generates $\pi_3(L) = \pi_3(\Omega^4 BS^3) \cong \pi_6(S^3) \cong \mathbb{Z}_{12}$.

Lemma 4.23. *$F_{n\epsilon} \simeq F_{m\epsilon}$ if and only if $(12, n) = (12, m)$.*

Proof. There is an isomorphism

$$\pi_k(L) \cong \pi_{k+3}(S^3)$$

for any $k \geq 1$. Therefore, as all homotopy groups of L are finite and L is an H-space, by Lemma 4.22 there is a homotopy equivalence

$$\theta : L \to \prod_p L_{(p)}. \tag{4.4.9}$$

As $\pi_3(L) \cong \mathbb{Z}_{12}$ we have

$$\begin{aligned}\pi_3(L) &\cong \pi_3\left(L_{(2)} \times L_{(3)} \times \prod_{p \notin \{2,3\}} L_{(p)}\right) \\ &\cong \pi_3(L_{(2)}) \oplus \pi_3(L_{(3)}) \oplus \pi_3\left(\prod_{p \notin \{2,3\}} L_{(p)}\right) \cong \mathbb{Z}_4 \oplus \mathbb{Z}_3.\end{aligned}$$

Thus the map θ induces an isomorphism

$$\theta^* : \pi_3(L) \to \mathbb{Z}_4 \oplus \mathbb{Z}_3$$

by sending ϵ to $(1, 1)$.

Let $i : L \to L$ and $i_{(p)} : L_{(p)} \to L_{(p)}$ (for $p = 2, 3$) be homotopy equivalences defined by the rule $x \mapsto x^{-1}$. Using the maps $i_{(p)}$ we define

$$\tilde{\mu}_2 = i_{(2)} \times \mathbb{1} : L_{(2)} \times \prod_{p \neq 2} L_{(p)} \to L_{(2)} \times \prod_{p \neq 2} L_{(p)}, \tag{4.4.10}$$

$$\tilde{\mu}_3 = i_{(3)} \times \mathbb{1} : L_{(3)} \times \prod_{p \neq 3} L_{(p)} \to L_{(3)} \times \prod_{p \neq 3} L_{(p)}. \tag{4.4.11}$$

These maps are homotopy equivalences and induce isomorphisms $i^*, \mu_p^* : \pi_3(L) \to \pi_3(L)$ (for $p = 2, 3$), where $\mu_p = \theta^{-1} \circ \tilde{\mu}_p \circ \theta$. Thus we obtain:

$$i^*(\epsilon) = -\epsilon = 11\epsilon,$$

$$\mu_2^*(\epsilon) = (\theta^*)^{-1}((-1,1)) = 7\epsilon,$$

$$\mu_3^*(\epsilon) = (\theta^*)^{-1}((1,-1)) = 5\epsilon.$$

We computed the images of the above isomorphism for $0 \le n < 12$. The results are shown in the next table. In particular, this shows that two elements $n\epsilon$ and $m\epsilon$ such that $(12,n) = (12,m)$ can be obtained from one another through some of the isomorphisms i^*, μ_2^* or μ_3^*.

$(12,n)$	n with $0 \le n < 12$	$n\epsilon$
1	1, 5, 7, 11	ϵ, $5\epsilon = \mu_3^*(\epsilon)$, $7\epsilon = \mu_2^*(\epsilon)$, $11\epsilon = i^*(\epsilon)$
2	2, 10	2ϵ, $10\epsilon = i^*(2\epsilon)$
3	3, 9	3ϵ, $9\epsilon = i^*(3\epsilon)$
4	4, 8	4ϵ, $8\epsilon = i^*(4\epsilon)$
6	6	6ϵ
12	0	0

Thus if $(12,m) = (12,n)$ then there are homotopy commutative diagrams:

$$\begin{array}{ccccc} F_{n\epsilon} & \longrightarrow & SU(2) & \xrightarrow{n\epsilon} & L \\ \downarrow{\scriptstyle\gamma} & & \| & & \downarrow{\scriptstyle h} \\ F_{m\epsilon} & \longrightarrow & SU(2) & \xrightarrow{m\epsilon} & L \end{array} \tag{4.4.12}$$

where h is one of homotopy equivalences i, μ_2, μ_3.

Finally, from the long exact sequence

$$\cdots \longrightarrow \pi_3(SU(2)) \xrightarrow{n\epsilon} \pi_3(L) \longrightarrow \pi_2(F_{n\epsilon}) \longrightarrow \pi_2(SU(2)),$$

if $F_{n\epsilon} \simeq F_{m\epsilon}$, then the orders of $n\epsilon$ and $m\epsilon$ coincide, that is, $12/(12,n) = 12/(12,m)$. □

Let $\mathcal{G}_k$ denote the gauge group of the principal $SU(2)$-bundle with classifying map $k \in \pi_3(SU(2))$.

Theorem 4.24. *$\mathcal{G}_k$ is homotopy equivalent to $\mathcal{G}_{k'}$ if and only if $(12,k) = (12,k')$.*

Proof. $\mathcal{G}_k$ is the homotopy fibre of the connecting map ∂_k in

$$SU(2) \xrightarrow{\partial_k} \Omega^3 SU(2) \to \mathrm{Map}_k(S^3, BSU(2)). \tag{4.4.13}$$

Identify $SU(2)$ with S^3 in (4.4.13). Thus by Lemma 4.21, the map ∂_k is homotopic to $\langle \iota, k\iota \rangle \in [S^3 \wedge S^3, S^3]$, where the map $\iota = \mathbb{1}_{S^3} : S^3 \to S^3$ is a generator of $\pi_3(S^3)$. The Samelson product is bilinear so that $\langle \iota, k\iota \rangle \simeq k \circ \langle \iota, \iota \rangle$. In [Sam54] it is shown that the

Samelson product $\langle \iota, \iota \rangle$ is a generator of $\pi_6(S^3) \cong \mathbb{Z}_{12}$. Thus $\epsilon = \langle \iota, \iota \rangle$ and the result follows from this fact and Lemma 4.23. □

In [Ter05] Terzić studied the rational homotopy types of gauge groups and classifying spaces of principal G-bundles over simply connected 4-manifolds, with G a compact simply connected semi-simple Lie group. We present some results of this work.

Let M be a simply connected 4-manifold and G be a compact simply connected simple Lie group. There exists a 7-equivalence $\theta : BG \to K(\mathbb{Z}, 4)$ (See Proposition 2.1 [Kaj06]). Thus for any space X of dimension $n \leq 6$ we have $[X, BG] = [X, K(\mathbb{Z}, 4)]$. In particular, if $X = M$ then $[M, BG] = H^4(M)$. As M is simply connected then it is orientable. Hence by Poincaré duality we have that $H^4(M) = \mathbb{Z}$. Therefore we have $Prin_G(M) = H^4(M) = \mathbb{Z}$. The element $k \in H^4(M) = \mathbb{Z}$ that classifies the bundle $P_k \to M$ is known as the Chern class of the bundle. Giving a cellular structure to M, the attaching map of the top cell induces the following exact sequence

$$S^3 \xrightarrow{\varphi} M^2 \xrightarrow{i} M \xrightarrow{\delta} S^4 \xrightarrow{\Sigma\varphi} \Sigma M^2. \tag{4.4.14}$$

From Table A.1 we have $\pi_4(BG) \cong \pi_3(G) \cong \mathbb{Z}$ for all G. From (4.4.14) there is an action of $\pi_4(BG)$ onto $[M, BG]$. The induced map δ^* is trivial and by Corollary 4.20 we have homotopy equivalences

$$B\mathcal{G}_k^*(M) \simeq B\mathcal{G}_0^*(M)$$

for all $k \in \mathbb{Z}$. This implies $\mathcal{G}_k^*(M) \simeq \mathcal{G}_0^*(M)$ for all $k \in \mathbb{Z}$. By Proposition 4.15 we have that $\mathcal{G}_0^*(M) \simeq \mathrm{Map}^*(M, G)$. Recall that there is a fibration sequence

$$\mathcal{G}_0^*(B) \longrightarrow \mathcal{G}_0(B) \longrightarrow G \tag{4.4.15}$$

From the exact sequence induced by (4.4.14) after applying the functor $[-, G]$, and the fibration in (4.4.15), Terzić obtained information on the rational homotopy groups $\pi_j(\mathcal{G}_0(M)) \otimes \mathbb{Q}$ of the gauge groups when G is a semisimple Lie group. For the following theorem we set $\pi_j(\mathcal{G}_0(M)) := \pi_j(\mathcal{G}_0(M)) \otimes \mathbb{Q}$.

Theorem 4.25. *Let* $\mathrm{rk}(G)$ *denote the rank of the group* G *and let* $b_n(M)$ *be the* n*-th Betti number of* M*.*

(1) The ranks of the rational homotopy groups of $\mathcal{G}_0^*(M)$ *are given by*

$$\mathrm{rk}(\pi_j(\mathcal{G}_0^*(M))) = b_2(M)\,\mathrm{rk}(\pi_{j+2}(G)) + \mathrm{rk}(\pi_{j+4}(G)), \qquad j \in \mathbb{N}.$$

(2) The ranks of the homotopy groups of the group $\mathcal{G}(M)$ *are given by*

$$\mathrm{rk}(\pi_j(\mathcal{G}_0(M))) = b_2(M)\,\mathrm{rk}(\pi_{j+2}(G)) + \mathrm{rk}(\pi_{j+4}(G)) + \mathrm{rk}(\pi_j(G)), \qquad j \in \mathbb{N}.$$

Proof. See Proposition 1 and Proposition 2 in [Ter05]. □

Studying the integral homotopy theory of path components and gauge groups is a complicated task in many cases. An intermediate step between the integral and the rational homotopy theories is studying the p-local homotopy theory of these mapping spaces one prime at a time. There is a general result proved by Theriault that can be used to get information on the p-local homotopy types of the gauge groups. Let Y be an H-space with a homotopy inverse, let $k : Y \to Y$ be the k-th power map, and let F_k be the homotopy fibre of the map $k \circ f$, where $f : X \to Y$ is a map of finite order m.

Lemma 4.26. *Let X be a space and Y be an H-space with a homotopy inverse. Suppose there is a map $X \xrightarrow{f} Y$ of finite order m. If $(m, k) = (m, k')$ then F_k and $F_{k'}$ are homotopy equivalent when localised rationally or at any prime.*

Proof. See [The10a, Lemma 3.1]. □

4.5 Gauge groups over high dimensional manifolds

The homotopy theory of gauge groups of principal G-bundles $P \xrightarrow{p} M$ when M is a low dimensional manifold has been widely studied in homotopy theory due to the connections to other areas in mathematics. In the last decade new formulations of gauge theories have been developed which include high dimensional manifolds with special geometric structures. In [DT98] Donaldson and Thomas exposed some ideas to construct gauge theories in higher dimensions. These ideas were formalised later on in [DS11] and particular attention was paid in the case where $\dim M \in \{6, 7, 8\}$. Literature on gauge theories for high dimensional manifolds has increased considerably in recent years.

The case when M is a 7-dimensional manifold has received attention in both differential geometry and mathematical physics alike, and currently we can find a good amount of literature for this particular case (see for instance [LL09, Wal13, SEW15]). The manifolds that present the required geometric properties are called G_2-manifolds. Examples of constructions of this kind of manifolds can be found in [CHNP15], where it is showed that there are G_2-manifolds M such that

$$M = S^3 \times_l S^4 \sharp^k (S^3 \times S^4),$$

where $S^3 \times_l S^4$ denotes one of the manifolds $M_{l,m}$ such that $l \neq 0 \pmod 1 2$ and $\sharp^k$ is the connected sum of k copies of the trivial sphere bundle $S^3 \times S^4$.

In homotopy theory, the study of gauge groups over high dimensional manifolds is almost unexplored. There are, however, some results when M is a high dimensional sphere and

G is either $SU(n)$ or $Sp(2)$. In the case of $G = SU(n)$, the principal G-bundles over S^n are classified by elements of $\pi_{n-1}(G)$. These results are stated below.

Theorem 4.27. *For $n \in \{7, 8, 9, 10, 15, 16, 17, 18, 23, 24, 25\}$, there is a unique homotopy type of the gauge groups of all the principal $SU(2)$-bundles over S^n, and it is the one of the trivial bundle, namely,*

$$\mathrm{Map}(S^n, S^3) \simeq \Omega_0^n S^3 \times S^3.$$

Proof. See [CS09] Proposition 2. □

Theorem 4.28. *Denote by ε a generator of $\pi_6(BSU(3))$ and by $\mathcal{G}_k(S^6)$ the gauge group of the principal $SU(3)$-bundle over S^6 classified by $k\varepsilon$. Then $\mathcal{G}_k \simeq \mathcal{G}_{k'}$ if and only if $(120, k) = (120, k')$.*

Proof. See [HK07] Theorem 1.1. □

Hamanaka, Kaji and Kono obtained a homotopy classification of the gauge groups of principal $Sp(2)$-bundles over S^8.

Theorem 4.29. *Denote by ϵ_7' a generator of $\pi_7(Sp(2)) \cong \mathbb{Z}$ and by $\mathcal{G}_k$ the gauge group of principal $Sp(2)$-bundle over S^8 classified by $k\epsilon_7'$ $(k \in \mathbb{Z})$. Then $\mathcal{G}_k \simeq \mathcal{G}_{k'}$) if and only if $(140, k) = (140, k')$, where (a, b) denotes the GCD of a and b.*

Proof. See [HKK08] Theorem 1. □

Chapter 5

Results for S^3-bundles over S^4

This chapter presents the main results on homotopy decomposition of gauge groups over principal G-bundles. We consider the space $\text{Map}^*(M, BG)$ for a simply connected simple compact Lie group G and a total space M of an S^3-bundle over S^4. In Section 5.1 we describe the set $Prin_G(M)$ in the case $\pi_6(G) \cong 0$ (Proposition 5.6). We give a homotopy decomposition of pointed and unpointed gauge groups of principal G-bundles over M with $\pi_6(G) \cong 0$, when M has torsion-free homology (Section 5.2, Theorems 5.9 and 5.10) and when M has torsion in homology (Section 5.3, Theorems 5.14 and 5.15). In Section 5.4 we describe homotopy equivalences between unpointed gauge groups over S^7 (Theorem 5.17). Throughout this chapter we assume that all spaces have the homotopy type of CW complexes with a non-degenerate basepoint and finitely many cells in each dimension.

5.1 Classification of principal G-bundles

Recall that, for given spaces X and Y, we denote by $[X, Y] = \pi_0(\text{Map}(X, Y))$ and $[X, Y]_*$ the sets of homotopy classes of unpointed and pointed maps from X to Y, respectively. Given a map $f : X \to Y$, we denote its homotopy class by the same letter f. The finite cyclic group of n elements is denoted $\mathbb{Z}_n$. The localisation of $\mathbb{Z}$ at a prime p is denoted $\mathbb{Z}_{(p)}$.

Let M be the total space of an S^3-bundle over S^4,

$$S^3 \to M \xrightarrow{\pi} S^3.$$

By Proposition 3.44, the spaces M are classified by elements $l\rho+m\sigma \in \pi_3(SO(4)) \cong \mathbb{Z}\times\mathbb{Z}$, where ρ, σ are generators of $\pi_3(SO(4))$ which form a basis. Let $M = M_{l,m}$ be the S^3-bundle over S^4 classified by $l\rho + m\sigma \in \pi_3(SO(4))$. We want to classify, up to bundle isomorphism, the principal G-bundles over manifolds $M_{l,m}$, with G a simply connected simple compact Lie group.

Given a compact topological space X and a topological group G, there is a one-to-one correspondence between $Prin_G(X)$ and $[X, BG]$, where BG is the classifying space of G (see Corollary 4.11). The evaluation fibration

$$\mathrm{Map}^*(M_{l,m}, BG) \to \mathrm{Map}(M_{l,m}, BG) \xrightarrow{ev} BG$$

induces an exact sequence of homotopy sets

$$\pi_1(BG) \xrightarrow{\partial} [M_{l,m}, BG]_* \to [M_{l,m}, BG] \xrightarrow{ev^*} \pi_0(BG).$$

The induced map ev^* is trivial as BG is connected, and the coset space of $\partial(\pi_1(BG))$ coincides with the orbit space of the action of $\pi_1(BG) \cong \pi_0(G)$ on $[M_{l,m}, BG]_*$. Since all groups G considered in this work are connected, this action is trivial, therefore we have $[M_{l,m}, BG]_* \cong [M_{l,m}, BG]$. Hence from now on we will drop the star symbol of $[M_{l,m}, BG]_*$.

In order to compute the sets $[M_{l,m}, BG]$ we make use of cofibration sequences where the spaces $M_{l,m}$ and $\Sigma M_{l,m}$ are involved. In Chapter 3 we have described the topology of the spaces $M_{l,m}$. We will recall some information on the structure of the spaces $M_{l,m}$ to obtain results on the suspensions $\Sigma M_{l,m}$.

There are homeomorphisms [Ste51] $M_{l,m} \cong M_{-l,-m}$ and $M_{l,m} \cong M_{l+m,-m}$ so that we will only consider the case $m \geq 0$. Since any space $M_{l,m}$ is a simply connected manifold, we can give a minimal cellular structure by $e^3 \cup_{\varphi'} e^4 \cup_\varphi e^7$, where φ' and φ are the attaching maps of the 4-cell and the 7-cells respectively. The 4-skeleton $M^4_{l,m}$ is given by the pushout

$$\begin{array}{ccc} S^3 & \longrightarrow & D^4 \\ {\scriptstyle\varphi'}\downarrow & & \downarrow \\ S^3 & \longrightarrow & S^3 \cup_{\varphi'} D^4 \cong M^4_{l,m} \end{array} \tag{5.1.1}$$

From the discussion of Section 3.5, the map φ' is a degree m map, where m is the index in $M_{l,m}$.

Lemma 5.1. *The map $\pi^* : H^4(S^4) \to H^4(M_{l,m})$ induced by the projection map is an isomorphism if $m = 0$, reduction mod m if $m > 0$ and, in particular, the constant map if $m = 1$.*

Proof. Consider the Serre spectral sequence of the sphere bundle

$$S^3 \longrightarrow M_{l,m} \xrightarrow{\pi} S^4$$

which converges to $H^*(M_{l,m})$, and let y_3 and x_4 be suitable generators of $H^3(S^3) \cong \mathbb{Z}$ and $H^4(S^4) \cong \mathbb{Z}$ respectively. Then the $E_2^{p,q}$ page in the spectral sequence has the following form

$$\begin{array}{c|cc} 3 & y_3 & y_3x_4 \\ & & \\ 0 & 1 & x_4 \\ \hline & 0 & 4 \end{array}$$

Thus we have that $E_2^{p,q} = E_4^{p,q} = H^p(S^3) \otimes H^q(S^4)$, and for dimensional reasons there is at most one non-trivial differential, namely $d_4(y_3) = mx_4$. This implies the result. □

If $m = 0$, then $M^4 \simeq S^3 \vee S^4$. Therefore

$$M_{l,0} \simeq (S^3 \vee S^4) \cup_\varphi e^7,$$

for $\varphi \in \pi_6(S^3 \vee S^4)$.

All the sphere bundles $M_{l,0} \xrightarrow{\pi} S^4$ admit cross sections. In this case the exact sequences of the fibre bundles show that the homotopy groups of the manifolds $M_{l,0}$ are isomorphic to those of the total space of the trivial bundle $S^3 \times S^4$. The homology groups of $M_{l,0}$ are also isomorphic to the ones of the trivial bundle:

$$H_i(M_{l,0}) \cong \begin{cases} \mathbb{Z} & \text{if } i = 0, 3, 4, 7, \\ 0 & \text{otherwise.} \end{cases}$$

In [JW54] James and Whitehead classified the manifolds $M_{l,0}$ up to homotopy equivalence. Theorem 3.45 states that $M_{l,0} \simeq M_{l',0}$ if and only if $l \equiv \pm l' \pmod{12}$.

If $m > 0$, then the 4-skeleton $M^4_{l,m}$ is the Moore space $P^4(m)$. In [CE03] Crowley and Escher classified the homotopy types of manifolds $M_{l,m}$ for $m > 0$. They showed that there is an orientation preserving homotopy equivalence $M_{l,m} \simeq M_{l',m'}$ if and only if $m = m'$ and $l' \equiv \alpha l \pmod{(m, 12)}$ where $\alpha^2 \equiv 1 \pmod{(m, 12)}$ (see Theorem 3.48). In the case $m = 1$ we have $P^4(1) \simeq *$, and from Theorem 3.48 we have

$$M_{l,1} \simeq S^7$$

for all $l \in \mathbb{Z}$. The homology groups of the manifolds $M_{l,m}$ for $m \geq 1$ are

$$H_i(M_{l,m}) = \begin{cases} \mathbb{Z}_m & \text{if } i = 3, \\ \mathbb{Z} & \text{if } i = 0, 7, \\ 0 & \text{otherwise.} \end{cases}$$

Thus given $M_{l,m}$ and $M_{l,m'}$, if $m \neq m'$ then $\pi_3(M_{l,m}) \not\cong \pi_3(M_{l,m'})$, and therefore these spaces are not homotopy equivalent. A minimal cellular structure for $M_{l,m}$ is given by

$$M_{l,m} \simeq P^4(m) \cup_\varphi e^7,$$

for some $\varphi \in \pi_6(P^4(m))$. As $\pi : M_{l,m} \to S^4$ is a fibre bundle where the base space is simply connected, we can use the Serre spectral sequence to obtain information on the properties of the projection maps. Now we prove a general statement regarding the suspension of the total spaces of S^{n-1}-bundles over S^n.

Lemma 5.2. *Let $\pi : X \to S^n$ be an S^{n-1}-bundle over S^n, $n \geq 3$, with a cross section. Then $X^n \simeq S^{n-1} \vee S^n$ and there is a homotopy equivalence*

$$\Sigma X \simeq \Sigma Y \vee S^{n+1},$$

where Y is the homotopy cofibre of the composite $S^{2n-2} \xrightarrow{\varphi} S^{n-1} \vee S^n \xrightarrow{p_1} S^{n-1}$. Here the map $\varphi : S^{2n-2} \to S^{n-1} \vee S^n$ is the attaching map of the top cell of X, and the map $p_1 : S^{n-1} \vee S^n \to S^{n-1}$ is the projection onto the first component.

Proof. The manifold X is homotopy equivalent to a CW-complex X^{CW} with the following cellular structure

$$X \simeq X^{CW} = e^{n-1} \cup e^n \cup_\varphi e^{2n-1}.$$

Where φ is the attaching map of the top cell. Set $X = X^{CW}$. Let X^n be the n-skeleton of X. There is a homotopy commutative diagram

$$\begin{array}{ccccc} S^{n-1} & \hookrightarrow & X^n & \xrightarrow{q} & S^n \\ \| & & \downarrow & & \| \\ S^{n-1} & \hookrightarrow & X & \xrightarrow{\pi} & S^n \end{array} \tag{5.1.2}$$

where the top row is the cofibration sequence induced by the inclusion of the bottom cell into the n-skeleton, the bottom row is the fibration sequence of the sphere bundle, and q is the quotient map. Since $X^n = X^{2n-2}$ by connectivity, the map q also has a right homotopy inverse, implying that there is a homotopy equivalence $X^n \simeq S^{n-1} \vee S^n$.

Now consider the cofibration sequence induced by $S^{2n-2} \xrightarrow{\varphi} S^n \vee S^{n-1}$:

$$S^{2n-2} \xrightarrow{\varphi} S^{n-1} \vee S^n \xrightarrow{i} X \xrightarrow{\rho} S^{2n-1} \xrightarrow{\Sigma\varphi} S^n \vee S^{n+1} \xrightarrow{\Sigma i} \Sigma X, \tag{5.1.3}$$

where i is the inclusion and ρ is the pinch map to the $(2n-1)$-cell. By the Hilton-Milnor Theorem [Hil55, Mil72] there is an isomorphism

$$\pi_{2n-2}(S^{n-1} \vee S^n) \cong \pi_{2n-2}(S^{2n-2}) \times \pi_{2n-2}(S^n) \times \pi_{2n-2}(S^{n-1}).$$

With this decomposition, the element $[\iota_{n-1}, \iota_n] \in \pi_{2n-2}(S^{n-1} \vee S^n)$, where $\iota_{n-1} \simeq \mathbb{1}_{S^{n-1}}$ and $\iota_n \simeq \mathbb{1}_{S^n}$, factors through a generator of $\pi_{2n-2}(S^{2n-2})$. For any $\alpha \in \pi_{2n-2}(S^{n-1})$ and $\beta \in \pi_{2n-2}(S^n)$, let $\underline{\alpha}$ and $\underline{\beta}$ be the elements of $\pi_{2n-2}(S^{n-1} \vee S^n)$ which are represented by the maps

$$\underline{\alpha} : S^{2n-2} \xrightarrow{\alpha} S^{n-1} \hookrightarrow S^{n-1} \vee S^n$$

and

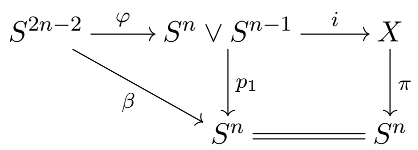
$$\underline{\beta} : S^{2n-2} \xrightarrow{\beta} S^n \hookrightarrow S^{n-1} \vee S^n.$$

In this way any $\varphi \in \pi_{2n-2}(S^{n-1} \vee S^n)$ can be expressed as

$$\varphi = t[\iota_{n-1}, \iota_n] + \underline{\alpha} + \underline{\beta} \tag{5.1.4}$$

for some $t \in \mathbb{Z}$.

Consider the diagram



The triangle homotopy commutes by definition of φ and $\underline{\beta}$, and the square homotopy commutes by the commutativity of right square in (5.1.2). Thus $\beta \simeq \pi \circ i \circ \varphi$, but $i \circ \varphi$ is nullhomotopic since i and φ are consecutive maps in a cofibration. Hence β is nullhomotopic and therefore so is $\underline{\beta}$. Hence (5.1.4) is reduced to

$$\varphi = t[\iota_{n-1}, \iota_n] + \underline{\alpha}.$$

After suspension we have $\Sigma\varphi = \Sigma\underline{\alpha}$ since $\Sigma[\iota_{n-1}, \iota_n] \simeq *$. Let Y be the homotopy cofibre of the map $\alpha : S^{2n-2} \to S^{n-1}$. Thus if $\Sigma\alpha \simeq *$ then $\Sigma\varphi \simeq *$. Therefore the map Σi in (5.1.3) has a left homotopy inverse, and $\Sigma X \simeq S^{2n} \vee S^n \vee S^{n+1}$. If instead $\Sigma\alpha$ is not nullhomotopic, then $\Sigma\varphi \not\simeq *$. Consider the following part of the homotopy cofibration sequence (5.1.3)

$$S^{2n-1} \xrightarrow{\Sigma\varphi} S^{n+1} \vee S^n \xrightarrow{\Sigma i} \Sigma X.$$

Thus $\Sigma\varphi = \Sigma\underline{\alpha} = j \circ \Sigma\alpha$, where $j : S^n \longrightarrow S^n \vee S^{n+1}$ is the inclusion into the wedge. Therefore $\Sigma X \simeq \Sigma Y \vee S^{n+1}$, where Y is defined by the cofibration sequence $S^{2n-2} \xrightarrow{\alpha} S^{n-1} \longrightarrow Y$ for $\alpha \in \pi_{2n-2}(S^{n-1})$. $\square$

Proposition 5.3. *Let $M_{l,m}$ be the total space of an S^3-bundle over S^4 classified by an element $l\rho + m\sigma \in \pi_3(SO(4)) \cong \mathbb{Z} \oplus \mathbb{Z}$. Suppose $M_{l,m}$ has cross sections. There is a homotopy equivalence*

$$\Sigma M_{l,m} \simeq \Sigma Y_l \vee S^5,$$

where Y_l is the homotopy cofibre of the composite $S^6 \xrightarrow{\varphi} S^3 \vee S^4 \xrightarrow{p_1} S^3$. Moreover $\Sigma M_{l,m} \simeq \Sigma M_{l',m}$ if and only if $l' \equiv \pm l \pmod{12}$. In particular, if $l \equiv 0 \pmod{12}$, there is a homotopy equivalence

$$\Sigma M_{l,m} \simeq S^8 \vee S^4 \vee S^5.$$

Proof. Let $M_{l,m}$ be an S^3-bundle with cross sections. This manifold satisfies the conditions of Lemma 5.2. Thus there is a homotopy equivalence

$$\Sigma M_{l,m} \simeq \Sigma Y_{l,m} \vee S^5, \tag{5.1.5}$$

where $Y_{l,m}$ is the homotopy cofibre of the composite $S^6 \xrightarrow{\varphi} S^3 \vee S^4 \xrightarrow{p_1} S^3$. Recall that all manifolds $M_{l,m}$ with cross sections satisfy $m = 0$. Thus the homotopy type of $\Sigma M_{l,m}$ and therefore, that of the space $Y_{l,m}$, only depends on the integer l. Set $Y_l := Y_{l,m}$. Let ν' be a generator of $\pi_6(S^3) \cong \mathbb{Z}_{12}$ [Tod63]. We can write the attaching map of the top cell as

$$\varphi = [\iota_3, \iota_4] + t_l \underline{\nu'},$$

where $\iota_i \simeq \mathbb{1}_{S^i}$, for $i = 3, 4$ and $t_l \underline{\nu'} \in \pi_6(S^3 \vee S^4)$ for some $t_l \in \mathbb{Z}_{12}$ such that the map sending l to t_l is a surjective homomorphism, namely, the J-homomorphism [JW54]. After suspension we have $\Sigma\varphi \simeq t_l \Sigma \underline{\nu'}$. The homotopy equivalence shows that the homotopy type of $\Sigma M_{l,m}$ only depends on ΣY_l, which is homotopy equivalent to a CW-complex obtained by attaching an 8-cell to S^4 via the map $t_l \Sigma\nu' \in \pi_7(S^4) = \mathbb{Z} \oplus \mathbb{Z}_{12}$. It is known that the element $\Sigma\nu'$ generates a subgroup of order 12 in $\pi_7(S^4)$ [Tod63]. Observe that two spaces ΣY_l, $\Sigma Y_{l'}$ are homotopy equivalent if and only if there is a homotopy equivalence $\theta : S^4 \to S^4$ such that

$$\theta^*(t_l \Sigma\nu') = t_{l'} \Sigma\nu',$$

where θ^* is the automorphism of $\pi_7(S^4)$ induced by θ. Thus we need to compute the set of classes of self equivalences of S^4. Since $[S^4, S^4] = \pi_4(S^4) \cong \mathbb{Z}$, there are two classes of self-equivalences, namely, $\pm\mathbb{1}_{S^4}$. Since, $t_l, t_{l'} \in \mathbb{Z}_{12}$, we have that $\Sigma Y_l \simeq \Sigma Y_{l'}$ if and only if $l' \equiv \pm l \pmod{12}$. In particular, when $M_{l,0}$ is the product bundle we have that $l = 0$ and $\Sigma\varphi \simeq *$. This implies that $\Sigma M_{l,m} \simeq S^8 \vee S^5 \vee S^4$ if and only if $l \equiv 0 \pmod{12}$.

□

In order to obtain results on the gauge groups over manifolds $M_{l,m}$ with torsion in homology we will require localisation at a prime $p \geq 5$. The cofibration $S^n \xrightarrow{m} S^n \to P^{n+1}(m)$ induces a fibration

$$\mathrm{Map}^*(P^{n+1}(m), BG) \to \mathrm{Map}^*(S^n, BG) \xrightarrow{m^*} \mathrm{Map}^*(S^n, BG),$$

where m^* is the m-th power map. Let $\Omega^n BG\{m\}$ denote the space $\mathrm{Map}^*(P^{n+1}(m), BG)$. Let $v_p(m)$ be the p-adic valuation of m at p.

Proposition 5.4. *Let $M_{l,m}$ be the total space of an S^3-bundle over S^4 with $m > 1$. Localised at $p \geq 5$ there exists a local homotopy equivalence*

$$\Sigma M_{l,m} \simeq P^5(p^r) \vee S^8,$$

where $r = v_p(m)$.

Proof. There exists a cofibration sequence

$$S^6 \xrightarrow{\varphi} P^4(m) \xrightarrow{i} M_{l,m} \xrightarrow{\rho} S^7 \xrightarrow{\Sigma\varphi} P^5(m) \xrightarrow{\Sigma i} \Sigma M_{l,m}, \tag{5.1.6}$$

where φ is the attaching map of the top cell, i is the inclusion and ρ is the pinch map. Now suppose that all spaces are localised at a prime $p \geq 5$ with $r = v_p(m)$. Consider the cofibration sequence

$$S^3 \xrightarrow{m} S^3 \xrightarrow{q} P^4(m). \tag{5.1.7}$$

We will split the argument into two cases: $r = 0$ and $r \geq 1$.

If $r = 0$ then the degree map m is invertible in $\mathbb{Z}_{(p)}$, so the map m is a homotopy equivalence in the cofibration sequence (5.1.7), and therefore $P^4(m) \simeq *$. From (5.1.6) we can see that the attaching map φ is nullhomotopic and therefore $M_{l,m} \simeq S^7$. Moreover, we can write $P^5(1) = P^5(p^0) \simeq *$. Hence there is a homotopy equivalence $\Sigma M_{l,m} \simeq P^5(p^r) \vee S^8$ for $r = 0$.

If $r \geq 1$ then the degree map m is not invertible. Localising at p we obtain

$$\pi_3(P^4(m)) \cong \mathbb{Z}_m \otimes \mathbb{Z}_{(p)} \cong \mathbb{Z}_{p^r}.$$

As $P^4(m)$ is 2-connected,

$$\pi_3(P^4(m)) \cong H_3(P^4(m); \mathbb{Z}_{(p)}) \cong \mathbb{Z}_{p^r},$$

and this is the only non-zero homology group of $P^4(m)$. Therefore $P^4(m) \simeq P^4(p^r)$. In [Sas65] Sasao computed the homotopy group $\pi_6(P^4(m))$. He showed that

$$\pi_6(P^4(m)) \cong \begin{cases} \mathbb{Z}_{(m,12)} \oplus \mathbb{Z}_m & \text{if } v_2(m) = 0, \\ \mathbb{Z}_{(m,12)/2} \oplus \mathbb{Z}_{2m} \oplus \mathbb{Z}_2 & \text{if } 1 \leq v_2(m) \leq 2, \\ \mathbb{Z}_{(m,12)} \oplus \mathbb{Z}_m \oplus \mathbb{Z}_2 & \text{if } v_2(m) \geq 3. \end{cases}$$

In all cases, localising at $p \geq 5$ we obtain

$$\pi_6(P^4(m)) \cong \mathbb{Z}_{p^r}.$$

We now give an alternative construction of a generator of $\pi_6(P^4(m))$ to that given by Sasao using localisation at a prime $p \geq 5$. Let $\bar{\sigma} \in \pi_6(P^4(m)) \cong \mathbb{Z}_{p^r}$ be a generator. We can write the attaching map of the top cell as $\varphi = t \cdot \bar{\sigma}$ with $t \in \mathbb{Z}_{p^r}$. Notice that if $\Sigma\varphi \simeq *$ then Σi has a left homotopy inverse, implying $\Sigma M_{l,m} \simeq P^4(p^r) \vee S^8$. We claim that the generator $\bar{\sigma}$ suspends trivially.

Let $\nu : P^4(p^r) \to P^4(p^r)$ be the identity map. Since ν is a suspension there is a Whitehead product $[\nu,\nu] : \Sigma P^3(p^r) \wedge P^3(p^r) \to P^4(p^r)$. By Proposition 3.21 there is a p-local homotopy equivalence

$$\Sigma P^3(p^r) \wedge P^3(p^r) \simeq P^7(p^r) \vee P^6(p^r).$$

This homotopy equivalence precomposed with the inclusion of $P^7(p^r)$ into the wedge determines a map $\widehat{[\nu,\nu]} : P^7(p^r) \to P^4(p^r)$. By Proposition 3.22 there is a p-local homotopy equivalence

$$\phi : S^3\{p^r\} \times \Omega\mathcal{A} \to \Omega P^4(p^r) \tag{5.1.8}$$

where $\mathcal{A} = \bigvee_{k=0}^{\infty} P^{4+2k+3}(p^r)$ and $S^n\{p^r\}$ denotes the homotopy fibre of the degree map $p^r : S^n \to S^n$. Using (5.1.8) we get

$$\pi_6(P^4(p^r)) \cong \pi_5(\Omega P^4(p^r)) \cong \pi_5(S^3\{p^r\}) \oplus \pi_5(\Omega\mathcal{A}).$$

Notice that there is a homotopy fibration given by

$$\Omega S^3 \longrightarrow S^3\{p^r\} \longrightarrow S^3.$$

As 2 and 3 are inverted we have $\pi_5(S^3) = 0$ and $\pi_5(\Omega S^3) \cong \pi_6(S^3) = 0$ and therefore we obtain $\pi_5(S^3\{p^r\}) = 0$. Now $\pi_5(\Omega\mathcal{A}) \cong \pi_6(\mathcal{A}) \cong \pi_6(P^7(p^r)) \cong \mathbb{Z}_{p^r}$, where the last two isomorphisms are given by the high connectivity of the factors in the wedge defining $\mathcal{A}$ and the Hurewicz isomorphism, respectively. Thus a generator $\bar{\sigma}$ of $\pi_6(P^4(p^r))$ is represented by the map

$$\bar{\sigma} : S^6 \hookrightarrow P^7(p^r) \xrightarrow{\widehat{[\nu,\nu]}} P^4(p^r).$$

Since $\widehat{[\nu,\nu]}$ factors through the Whitehead product $[\nu,\nu]$, which suspends trivially, we obtain $\Sigma\bar{\sigma} \simeq *$, as claimed. □

We now start computing the set of principal G-bundles over spaces $M_{l,m}$. In order to proceed with the classification of the principal G-bundles and the homotopy decomposition of the gauge groups, it will be necessary to study the cofibration sequence associated to the inclusion $i : S^3 \to M_{l,m}$

$$S^3 \xrightarrow{i} M_{l,m} \longrightarrow C_{l,m}(i) \xrightarrow{\delta} S^4, \tag{5.1.9}$$

where $C_{l,m}(i)$ is the homotopy cofibre of i, q is the pinch map and δ is the connecting map.

Lemma 5.5. *For $m \neq 1$, the projection $\pi : M_{l,m} \to S^4$ extends to a map $\tilde{\pi} : C_{l,m}(i) \to S^4$ such that the composite $S^4 \hookrightarrow C_{l,m}(i) \xrightarrow{\tilde{\pi}} S^4$ is a homotopy equivalence. Moreover, $C_{l,m}(i) \simeq S^4 \vee S^7$.*

Proof. Consider the following diagram

$$\begin{array}{ccccccc} S^3 & \xrightarrow{i} & M_{l,m} & \xrightarrow{q} & C_{l,m}(i) & \longrightarrow & S^4 \\ & & \downarrow \pi & \swarrow \tilde{\pi} & & & \\ & & S^4. & & & & \end{array} \tag{5.1.10}$$

Since $\pi_3(S^4) = 0$, there is an extension $\tilde{\pi} : C_{l,m}(i) \to S^4$. Notice that the cofibre $C_{l,m}(i)$ can be built as a CW-complex with one 7-cell attached to a 4-sphere. Thus $C_{l,m}(i)$ fits into the following cofibration sequence

$$S^6 \xrightarrow{\theta} S^4 \xrightarrow{g} C_{l,m}(i), \tag{5.1.11}$$

with g the inclusion and $\theta \in \pi_6(S^4) \cong \mathbb{Z}_2$.

Suppose first $m = 0$. Then the map $\pi : M_{l,0} \to S^4$ has a cross section $S^4 \to M_{l,0}$. The homotopy commutativity of (5.1.10) implies that the map $\tilde{\pi}$ also has a right homotopy inverse. Therefore the composite $\tilde{\pi} \circ g$ is a homotopy equivalence, as claimed.

Now suppose $m > 1$. The map $S^4 \xrightarrow{g} C_{l,m}(i)$ is the inclusion of the bottom cell and induces an isomorphism $g^* : H^4(C_{l,m}(i)) \xrightarrow{\cong} H^4(S^4) \cong \mathbb{Z}$. Consider the commutative diagram:

$$\begin{array}{ccc} H^4(M_{l,m}) & \xleftarrow{q^*} & H^4(C_{l,m}(i)). \\ \uparrow \pi^* & \nearrow \tilde{\pi}^* & \\ H^4(S^4) & & \end{array} \tag{5.1.12}$$

By Lemma 5.1, π^* is reduction mod m. From (5.1.12) we obtain the following composite

$$\pi^* : \mathbb{Z} \xrightarrow{\tilde{\pi}^*} \mathbb{Z} \xrightarrow{q^*} \mathbb{Z}_m,$$

which is reduction mod m. Thus $\tilde{\pi}^* = \pm 1 \pmod{m}$. Consider the homotopy commutative diagram

$$\begin{array}{ccccccc} S^3 & \xrightarrow{m} & S^3 & \longrightarrow & P^4(m) & \xrightarrow{q'} & S^4 \\ \downarrow \xi & & \| & & \downarrow i & & \downarrow \pi' \\ \Omega S^4 & \xrightarrow{\delta} & S^3 & \longrightarrow & M_{l,m} & \xrightarrow{\pi} & S^4 \end{array} \tag{5.1.13}$$

where the top row is a cofibration sequence and the bottom row is a fibration sequence. We can apply the cohomology functor to the bottom row fibration producing an exact sequence in low degrees. This shows that the connecting map δ induces multiplication by m. From the left square we obtain that ξ is a degree one map, as it is the inclusion of the bottom cell. By the Peterson-Stein formula the adjoint of the map ξ is homotopic to π'. Therefore π' is a homotopy equivalence.

Notice that in cohomology $q^* \circ \tilde{\pi}^* = (\pi' \circ q')^*$. Therefore $\tilde{\pi}^*$ is an isomorphism and the map $\tilde{\pi} \circ g$ is a homotopy equivalence as required.

Finally, as $\tilde{\pi} \circ g$ is a homotopy equivalence, the map θ in (5.1.11) is nullhomotopic, implying that $C_{l,m}(i) \simeq S^4 \vee S^7$. □

There is a one-to-one correspondence between $Prin_G(M_{l,m})$ and $[M_{l,m}, BG]$ (see Corollary 4.11). Let $m = 1$. From the homotopy classification of $M_{l,m}$ we know that all manifolds $M_{l,1}$ are homotopy equivalent to S^7. Therefore we get

$$[M_{l,1}, BG] \cong [S^7, BG] \cong \pi_7(BG) \cong \pi_6(G).$$

The following proposition is a classification of principal G-bundles over spaces $M_{l,m}$ with $m \neq 1$ and $\pi_6(G) = 0$, extending the result for the case $m = 1$.

Proposition 5.6. *Let G be a simply connected simple compact Lie group such that $\pi_6(G) \cong 0$. Then*

$$Prin_G(M_{l,m}) \cong \mathbb{Z}_m.$$

More precisely,

(1) if $m = 0$ then $Prin_G(M_{l,m}) \cong \mathbb{Z}$;

(2) if $m > 1$ then $Prin_G(M_{l,m}) \cong \mathbb{Z}_m$.

Moreover, the projection $M_{l,m} \xrightarrow{\pi} S^4$ induces a map

$$\pi^* : [S^4, BG] \to [M_{l,m}, BG]$$

which is a bijection if $m = 0$ and a surjection if $m > 1$.

Proof. By the classification theorem of principal G-bundles there is a one-to-one correspondence between $Prin_G(M_{l,m})$ and $[M_{l,m}, BG]$. Thus we compute the set $[M_{l,m}, BG]$.

Let $f : M_{l,m} \to BG$ be a map. From Table A.1 we have $\pi_3(BG) \cong \pi_2(G) \cong 0$ for all simply connected simple compact Lie groups G. Hence the composite $S^3 \hookrightarrow M_{l,m} \xrightarrow{f} BG$ is nullhomotopic. Using Lemma 5.5 to identify the homotopy cofibre of i as $S^4 \vee S^7$, there is a homotopy commutative diagram

$$\begin{array}{ccccccccc} S^3 & \xrightarrow{i} & M_{l,m} & \xrightarrow{q} & S^4 \vee S^7 & \xrightarrow{\delta} & S^4 & \xrightarrow{\Sigma i} & \Sigma M_{l,m} \\ & & \downarrow f & \swarrow \tilde{f} & & & & & \\ & & BG & & & & & & \end{array} \tag{5.1.14}$$

where the top row is a cofibration sequence and $\tilde{f} : S^4 \vee S^7 \to BG$ is an extension of f.

Let $M^4_{l,m}$ be the 4-skeleton of $M_{l,m}$, so that $M^4_{l,m} \simeq S^3 \vee S^4$ or $M^4_{l,m} \simeq P^4(m)$. In any case, $M^4_{l,m}$ is a co-H-space. From the exact sequence induced by the attaching map of the 4-cell,

$$S^3 \xrightarrow{m} S^3 \longrightarrow M^4_{l,m} \longrightarrow S^4, \tag{5.1.15}$$

we obtain an exact sequence of groups

$$[S^4, BG] \xrightarrow{m^*} [S^4, BG] \longrightarrow [M^4_{l,m}, BG] \longrightarrow 0 \tag{5.1.16}$$

where $\pi_4(BG) \cong \pi_3(G) \cong \mathbb{Z}$ (from Table A.1) and $m^* : \mathbb{Z} \to \mathbb{Z}$ is multiplication by m. There is a coaction $\psi = \psi_{l,m} : S^4 \to S^4 \vee S^4$ associated to the cofibration (5.1.15) which induces an action of homotopy sets,

$$\psi^* : [S^4, BG] \times [S^4, BG] \to [S^4, BG].$$

Exactness of (5.1.16) implies that $[M^4_{l,m}, BG] = \mathbb{Z}_m$, and by construction the orbits under the action ψ^* are equal to the cosets of the image of m^*. The map $S^3 \xrightarrow{i} M_{l,m}$ factors through the 4-skeleton $M^4_{l,m}$. Therefore we have a homotopy commutative diagram

$$\begin{array}{ccccccc} S^3 & \longrightarrow & M^4_{l,m} & \longrightarrow & S^4 & \xrightarrow{m} & S^4 \\ \| & & \downarrow & & \downarrow{\scriptstyle i_1} & & \| \\ S^3 & \xrightarrow{i} & M_{l,m} & \longrightarrow & S^4 \vee S^7 & \xrightarrow{\delta} & S^4 \end{array} \tag{5.1.17}$$

where $i_1 : S^4 \to S^4 \vee S^7$ is the inclusion of the first factor into the wedge. Let $\psi' : S^4 \vee S^7 \to S^4 \vee S^7 \vee S^4$ be the coaction of S^4 on $S^4 \vee S^7$. From (5.1.17) we obtain a homotopy commutative diagram as follows

$$\begin{array}{ccc} S^4 & \xrightarrow{\psi} & S^4 \vee S^4 \\ {\scriptstyle i_1}\downarrow & & \downarrow{\scriptstyle i_1 \vee \mathbb{1}} \\ S^4 \vee S^7 & \xrightarrow{\psi'} & S^4 \vee S^7 \vee S^4. \end{array} \tag{5.1.18}$$

Applying the functor $[-, BG]$ we obtain a commutative diagram of homotopy groups

$$\begin{array}{ccc} \pi_4(BG) \times \pi_7(BG) \times \pi_4(BG) & \xrightarrow{(\psi')^*} & \pi_4(BG) \times \pi_7(BG) \\ {\scriptstyle i_1^* \times \mathbb{1}}\downarrow & & \downarrow{\scriptstyle i_1^*} \\ \pi_4(BG) \times \pi_4(BG) & \xrightarrow{\psi^*} & \pi_4(BG). \end{array} \tag{5.1.19}$$

By hypothesis $\pi_7(BG) \cong \pi_6(G) = 0$, implying that the vertical arrows in (5.1.19) are isomorphisms. Therefore $(\psi')^* = \psi^*$.

From (5.1.14) we obtain an exact sequence of homotopy sets

$$\begin{array}{ccccccc} [S^4, BG] & \xrightarrow{\delta^*} & [S^4 \vee S^7, BG] & \xrightarrow{q^*} & [M_{l,m}, BG] & \longrightarrow & 0 \\ \| & & \| & & & & \\ \mathbb{Z} & & \mathbb{Z}. & & & & \end{array}$$

Since the homotopy set $[M_{l,m}, BG]$ might not be a group we will need to use the action $(\psi')^*$ to compute it. From Theorem 2.40 we have that $[M_{l,m}, BG]$ is the orbit space of the action $(\psi')^*$ on $\pi_4(BG) \times \pi_7(BG) \cong \pi_4(BG)$. Therefore, since $(\psi')^* = \psi^*$, $[M_{l,m}, BG] = \mathbb{Z}$ if $m = 0$, and $[M_{l,m}, BG] = \mathbb{Z}_m$ if $m > 1$.

Finally we analyse the induced map

$$\pi^* : [M_{l,m}, BG] \to [S^4, BG].$$

By Lemma 5.5, the projection $\pi : M_{l,m} \to S^4$ extends to a map $\tilde{\pi} : S^4 \vee S^7 \to S^4$ so that the restriction to S^4 is the degree 1 map. Since $\pi_6(G) = 0$,

$$[S^4 \vee S^7, BG] = [S^7, BG] \times [S^4, BG] \cong \pi_6(G) \times [S^4, BG] \cong [S^4, BG].$$

Consider the commutative diagram

$$\begin{array}{ccc} [M_{l,m}, BG] & \xleftarrow{q^*} & [S^4 \vee S^7, BG] \\ {\scriptstyle \pi^*}\uparrow & \nearrow {\scriptstyle \tilde{\pi}^*} & \\ [S^4, BG] & & \end{array} \tag{5.1.20}$$

where the induced map $\tilde{\pi}^*$ is an isomorphism. We have already shown that q^* is an isomorphism if $m = 0$ and a surjection if $m > 1$. Therefore $\pi^* : [M_{l,m}, BG] \to [S^4, BG]$ is an isomorphism if $m = 0$, and a surjection if $m > 1$. □

Let $\Sigma\nu'$ be the suspension of a generator of $\pi_6(S^3)$. Let Y_l be the homotopy cofibre of a map $t_l \Sigma\nu' \in \pi_7(S^4)$.

Lemma 5.7. *There is a homotopy commutative diagram*

$$\begin{array}{ccccccc} * & \longrightarrow & S^7 & = & S^7 & \longrightarrow & * \\ \downarrow & & \downarrow & & \downarrow{\scriptstyle \gamma} & & \downarrow \\ M_{l,m} & \longrightarrow & S^4 \vee S^7 & \xrightarrow{\delta} & S^4 & \longrightarrow & \Sigma M_{l,m} \\ \| & & \downarrow & & \downarrow & & \| \\ M_{l,m} & \xrightarrow{\pi} & S^4 & \xrightarrow{q} & C_{l,m}(\pi) & \longrightarrow & \Sigma M_{l,m} \end{array} \tag{5.1.21}$$

where each row and column is a cofibration sequence. Furthermore the map $S^4 \xrightarrow{q} C_{l,m}(\pi)$ is identified with the composite

$$S^4 \xrightarrow{*+\gamma} S^4 \hookrightarrow C_{l,m}(\pi)$$

and there are homotopy equivalences

$$C_{l,m}(\pi) \simeq \Sigma Y_l.$$

Proof. By Lemma 5.5, there is a homotopy commutative diagram

$$\begin{array}{ccccc} S^3 & \xrightarrow{i} & M_{l,m} & \xrightarrow{g} & S^4 \vee S^7 \\ & & \| & & \downarrow{\tilde{\pi}} \\ & & M_{l,m} & \xrightarrow{\pi} & S^4 \end{array} \tag{5.1.22}$$

where the top row i is the cofibration sequence induced by the inclusion of the bottom cell and $\tilde{\pi}$ is the homotopy extension of the projection map π. We can extend the diagram (5.1.22) to the right to generate the lower part in (5.1.21), where $C_{l,m}(\pi)$ is the cofibre of the map π. The inclusion $S^7 \hookrightarrow S^4 \vee S^7$ generates the upper part of the diagram where γ is homotopic to the restriction of δ to S^7. Therefore the whole diagram, where each column and row is a cofibration, homotopy commutes. From the exact sequence induced by the middle row

$$\begin{array}{ccccccc} H^4(S^4) & \xrightarrow{\delta^*} & H^4(S^4 \vee S^7) & \xrightarrow{q^*} & H^4(M_{l,m}) & \longrightarrow & 0 \\ \| & & \| & & \| & & \\ \mathbb{Z} & & \mathbb{Z} & & \mathbb{Z}_m & & \end{array} \tag{5.1.23}$$

we conclude that δ restricted to S^4 is the degree m map. Suppose $m = 0$. Using Proposition 5.3 we obtain a decomposition of the suspension

$$\Sigma M_{l,m} \simeq \Sigma Y_l \vee S^5, \tag{5.1.24}$$

where Y_l is the homotopy cofibre of a map $t_l \Sigma \nu' \in \pi_7(S^4)$ and t_l depends on l. Since π has a cross section, so does the map $\Sigma\pi$. Therefore, from the commutative diagram we have $\Sigma M_{l,m} \simeq S^5 \vee C_{l,m}(\pi)$. Comparing this equivalence with (5.1.24) we conclude that $\Sigma C_{l,m}(\pi) \simeq \Sigma Y_l$. □

5.2 Homotopy decompositions: torsion-free case

Let $\mathrm{Map}(X, Y)$ be the mapping space of continuous maps from X to Y, and denote by $\mathrm{Map}^*(X, Y)$ the space of pointed maps. Similarly let $\mathrm{Map}_f(X, Y)$ and $\mathrm{Map}^*_f(X, Y)$ be the path components of the corresponding mapping spaces containing the map f. Let $B\mathcal{G}_f(X,G)$ denote the classifying space of $\mathcal{G}_f(X,G)$. There are homotopy equivalences (see Theorem 4.16)

$$B\mathcal{G}^*_f(X) \simeq \mathrm{Map}_f(X, BG), \tag{5.2.1}$$

$$B\mathcal{G}_f(X) \simeq \mathrm{Map}^*_f(X, BG), \tag{5.2.2}$$

Recall that for G connected, $[X, BG] = \pi_0(\mathrm{Map}^*(X, BG)) = \pi_0(\mathrm{Map}(X, BG))$. By Proposition 5.6 if $\pi_g(G) \cong 0$ and $M_{l,m}$ with torsion-free homology and not homotopic to S^7, $\pi : M_{l,m} \to S^4$ induces a function $\pi^* : [S^4, BG] \to [M_{l,m}, BG] = \mathbb{Z}$. In order to obtain homotopy decomposition of the unpointed gauge groups, we have to study the cofibration and fibration sequences induced by the projection map. Let $M_{l,m}$ be a manifold such that $m = 0$. Notice that there is a one-to-one correspondence

$$\pi_0(\mathrm{Map}^*(S^4, BG)) \to [M_{l,m}, BG] \tag{5.2.3}$$

and the projection map induces homotopy fibrations

$$F^{l,0} \to \mathrm{Map}^*(S^4, BG) \xrightarrow{\pi^*} \mathrm{Map}^*(M_{l,m}, BG),$$

$$F^{l,0}_k \to \mathrm{Map}^*_k(S^4, BG) \xrightarrow{\pi^*_k} \mathrm{Map}^*_k(M_{l,m}, BG),$$

where π^*_k is the restriction to the k-th component and $F^{l,0}$ and $F^{l,0}_k$ are the corresponding homotopy fibres. Using the bottom row in the commutative diagram of Lemma 5.7 we obtain the following fibration sequence for the $k = 0$ case

$$\mathrm{Map}^*(\Sigma Y_l, BG) \to \mathrm{Map}^*_0(S^4, BG) \xrightarrow{\pi^*_0} \mathrm{Map}^*_0(M_{l,m}, BG),$$

where we can identify $\mathrm{Map}^*(\Sigma Y_l, BG) \simeq \mathrm{Map}^*(Y_l, G)$. Next we state a general result on the homotopy types of the spaces $F^{l,0}_k$.

Lemma 5.8. *Let $F^{l,0}_k(\pi)$ be the homotopy fibre of $\pi^*_k : \mathrm{Map}^*_k(S^4, BG) \to \mathrm{Map}^*_k(M_{l,m}, BG)$. There are homotopy equivalences*

$$F^{l,0}_k(\pi) \simeq \mathrm{Map}^*(Y_l, G), \ \textit{for all } k \in \mathbb{Z}.$$

In particular, if $l \equiv 0 \pmod{12}$, then $F^{l,0}_k(\pi) \simeq \Omega^3 G \times \Omega^7 G$.

Proof. The inclusion of the bottom cell into $M_{l,m}$ induces a fibration sequence

$$\mathrm{Map}^*(S^4 \vee S^7, BG) \xrightarrow{q^*} \mathrm{Map}^*(M_{l,m}, BG) \xrightarrow{i^*} \mathrm{Map}^*(S^3, BG) \tag{5.2.4}$$

Applying the functor $\mathrm{Map}^*(-, BG)$ to the diagram in Lemma 5.7, we can fit the fibration sequence (5.2.4) into a homotopy commutative diagram

$$\begin{array}{ccccc}
\mathrm{Map}^*(\Sigma Y_l, BG) & \longrightarrow & \mathrm{Map}^*(S^4, BG) & \xrightarrow{\gamma^*} & \mathrm{Map}^*(S^7, BG) \\
\downarrow & & \downarrow{\scriptstyle (m+\gamma)^*} & & \| \\
\mathrm{Map}^*(S^4, BG) & \longrightarrow & \mathrm{Map}^*(S^4 \vee S^7, BG) & \xrightarrow{p_2^*} & \mathrm{Map}^*(S^7, BG) \\
{\scriptstyle \pi^*}\downarrow & & {\scriptstyle q^*}\downarrow & & \downarrow \\
\mathrm{Map}^*(M_{l,m}, BG) & = & \mathrm{Map}^*(M_{l,m}, BG) & \longrightarrow & * \\
 & & {\scriptstyle i^*}\downarrow & & \\
 & & \mathrm{Map}^*(S^3, BG) & &
\end{array} \tag{5.2.5}$$

where rows and columns are fibrations. Notice that the map q^* induces bijection between path components. Identifying $\mathrm{Map}_k^*(S^4 \vee S^7, BG)$ with $\mathrm{Map}_k^*(S^4, BG) \times \mathrm{Map}^*(S^7, BG)$ we see that p_2^* is the projection to the second factor. Restricting the map q^* to the k-th component we obtain the following homotopy commutative diagram

$$\begin{array}{ccccc}
\mathrm{Map}^*(\Sigma Y_l, BG) & \longrightarrow & \Omega^4 BG & \xrightarrow{\gamma^*} & \Omega^7 BG \\
\downarrow & & \downarrow{\scriptstyle (\theta_k \circ m+\gamma)^*} & & \| \\
\Omega_k^4 BG & \xrightarrow{\tilde{\pi}} & \Omega_k^4 BG \times \Omega^7 BG & \xrightarrow{p_2^*} & \Omega^7 BG \\
{\scriptstyle \pi^*}\downarrow & & {\scriptstyle q^*}\downarrow & & \downarrow \\
\mathrm{Map}_k^*(M_{l,m}, BG) & = & \mathrm{Map}_k^*(M_{l,m}, BG) & \longrightarrow & *
\end{array} \tag{5.2.6}$$

where each row and middle and right columns (and therefore the left column) are fibrations. Here the map $\theta_k : \Omega_0^4 BG \to \Omega_k^4 BG$ is the homotopy equivalence defined by

$$\omega \mapsto \mu \circ (\omega \times k_0) \circ \Delta,$$

where μ is a homotopy multiplication in $\Omega^4 BG$ and k_0 is a choice of base point in $\Omega_k^4 BG$. Finally we have homotopy equivalences

$$F_k^{l,0}(\pi) \simeq \mathrm{Map}^*(\Sigma Y_l, BG) \simeq \mathrm{Map}^*(Y_l, \Omega BG) \simeq \mathrm{Map}^*(Y_l, G). \qquad \square$$

If Y is an H-group, or if X is a co-H-group, then all the path components of $\mathrm{Map}^*(X, Y)$ are homotopy equivalent (see Proposition 4.18). Therefore in our case, if $M_{l,m} \simeq S^7$, given $k, k' \in [M_{l,m}, BG]$, the path components $\mathrm{Map}_k^*(M_{l,m}, BG)$ and $\mathrm{Map}_{k'}^*(M_{l,m}, BG)$ are homotopy equivalent and, as a consequence, so are the pointed gauge groups. In the case when $M_{l,m}$ is not homotopy equivalent to S^7, it is not known if the path components of $\mathrm{Map}^*(M_{l,m}, BG)$ have the same homotopy type. We prove a result on the homotopy types of the pointed gauge groups over manifolds $M_{l,m}$ with torsion-free homology and $m \neq 1$. Let $P_k \to M_{l,m}$ be a principal G-bundle classified by $k \in \mathbb{Z}$ and let $\mathcal{G}_k^*(M_{l,m})$ be its pointed gauge group. Let $\nu : S^6 \to S^3$ be the generator of $\pi_6(S^3) \cong \mathbb{Z}_{12}$ as it is given in [Tod63]. For $l \in \mathbb{Z}$ with $l \not\equiv 0 \pmod{12}$, let Y_l be the homotopy cofibre of $\xi_l = l\nu$.

Theorem 5.9. *Let G be a simply connected simple compact Lie group with $\pi_6(G) \cong 0$. Let $M_{l,m}$ be the total space of an S^3-bundle over S^4 with torsion-free homology not homotopy equivalent to S^7. Then there are homotopy equivalences*

$$\mathcal{G}_k^*(M_{l,m}) \simeq \Omega^4 G \times \mathrm{Map}^*(Y_l, G), \text{ for all } k \in \mathbb{Z}.$$

In particular, if $l \equiv 0 \pmod{12}$ then there is a homotopy splitting.

$$\mathcal{G}_k^*(M_{l,0}) \simeq \Omega^4 G \times \Omega^3 G \times \Omega^7 G.$$

Proof. Let $M_{l,m}$ be a manifold with torsion-free homology, that is $m = 0$, and let $\mathcal{G}_k^*(M_{l,0})$ be the gauge group classified by $k \in \mathbb{Z}$. By Lemma 5.8 there is a fibration sequence

$$\mathrm{Map}^*(Y_l, G) \to \mathrm{Map}_k^*(S^4, BG) \xrightarrow{\pi_k^*} \mathrm{Map}_k^*(M_{l,m}, BG). \tag{5.2.7}$$

Extend the fibration sequence to the left. Consider the following part of the fibration

$$\Omega\mathrm{Map}_0^*(S^4, BG) \simeq \Omega\mathrm{Map}_k^*(S^4, BG) \xrightarrow{\Omega\pi_k^*} \Omega\mathrm{Map}_k^*(M_{l,m}, BG) \longrightarrow \mathrm{Map}^*(Y_l, G). \tag{5.2.8}$$

Since $m = 0$, $M_{l,m}$ has cross sections. Let $s : M_{l,m} \to S^4$ be a map such that the diagram

$$\begin{array}{ccc} S^4 & \xrightarrow{s} & M_{l,m} \\ & \searrow^{=} & \downarrow{\scriptstyle \pi} \\ & & S^4 \end{array} \tag{5.2.9}$$

commutes. Applying the functor $\mathrm{Map}^*(-, BG)$ to the diagram (5.2.9) we obtain the following homotopy commutative diagram

$$\begin{array}{ccc} \mathrm{Map}^*(S^4, BG) & \xleftarrow{s^*} & \mathrm{Map}^*(M_{l,m}, BG) \\ & \nwarrow^{=} & \uparrow{\scriptstyle \pi^*} \\ & & \mathrm{Map}^*(S^4, BG). \end{array} \tag{5.2.10}$$

Thus the map Ωs^* is a homotopy retraction of $\Omega\pi^*$, which implies that there is a homotopy splitting

$$\Omega\mathrm{Map}_k^*(M_{l,0}, BG) \simeq \Omega\mathrm{Map}_0^*(S^4, BG) \times \mathrm{Map}^*(Y_l, G).$$

We can identify $\Omega\mathrm{Map}_0^*(S^4, BG) \simeq \Omega^4 G$ and $\mathcal{G}_k^*(M_{l,m}) \simeq \Omega\mathrm{Map}_k^*(M_{l,m}, BG)$. Putting things together we obtain

$$\mathcal{G}_k^*(M_{l,0}) \simeq \Omega^4 G \times \mathrm{Map}^*(Y_l, G).$$

Finally, note that when $l \equiv 0 \pmod{12}$ we have $\Sigma Y_l \simeq S^4 \vee S^8$ and therefore we get

$$\mathcal{G}_k^*(M_{l,0}) \simeq \Omega^4 G \times \Omega^3 G \times \Omega^7 G. \qquad \square$$

We look at the evaluation map to obtain homotopy decomposition of the unpointed gauge groups. The restriction of evaluation map to the k-th component defines a fibration sequence

$$\Omega \mathrm{Map}_k(M_{l,m}, BG) \longrightarrow G \xrightarrow{\partial_k} \mathrm{Map}_k^*(M_{l,m}, BG) \longrightarrow \mathrm{Map}_k(M_{l,m}, BG) \xrightarrow{ev_k} BG \tag{5.2.11}$$

where ∂_k is the connecting map. Thus the gauge group $\mathcal{G}_k(M_{l,m}) \simeq \Omega \mathrm{Map}_k(M_{l,m}, BG)$ appears as the homotopy fibre of the connecting map ∂_k. Hence, it is expected that the properties of ∂_k determine the homotopy type of the gauge groups over the manifolds $M_{l,m}$.

By Proposition 5.6, if $m = 0$ the projection $M_{l,m} \xrightarrow{\pi} S^4$ induces an bijection between path components of $\mathrm{Map}^($$M_{l,m}, BG)$ and those of $\mathrm{Map}^*(S^4, BG)$. Therefore, the evaluation map induces a commutative diagram

$$\begin{array}{ccccccc} G & \xrightarrow{\phi_k} & \mathrm{Map}_k^*(S^4, BG) & \longrightarrow & \mathrm{Map}_k(S^4, BG) & \xrightarrow{ev_k} & BG \\ \| & & \downarrow \pi_k^* & & \downarrow \pi^* & & \| \\ G & \xrightarrow{\partial_k} & \mathrm{Map}_k^*(M_{l,m}, BG) & \longrightarrow & \mathrm{Map}_k(M_{l,m}, BG) & \xrightarrow{ev_k} & BG \end{array} \tag{5.2.12}$$

which defines the map ϕ_k. We write $\Omega \mathrm{Map}_k(S^4, BG) \simeq \mathcal{G}_k(S^4)$, where $\mathcal{G}_k(S^4)$ is the gauge group of the principal G-bundle classified by $k \in \pi_3(G) \cong \mathbb{Z}$.

Theorem 5.10. *Let G be a simply connected simple compact Lie group with trivial $\pi_6(G)$ and let $M_{l,m} \to S^4$ be a sphere bundle with cross section. Let $P_k \to M_{l,m}$ be a principal G-bundle classified by $k \in \mathbb{Z}$. There are homotopy decompositions*

$$\mathcal{G}_k(M_{l,0}) \simeq \mathcal{G}_k(S^4) \times \mathrm{Map}^*(Y_l, G).$$

Moreover, if $l \equiv 0 \pmod{12}$ there are homotopy equivalences

$$\mathcal{G}_k(M_{l,0}) \simeq \mathcal{G}_k(S^4) \times \Omega^3 G \times \Omega^7 G.$$

Proof. We argue along the lines of [The10b]. Consider the restriction of the map $\pi^* : \mathrm{Map}^*(S^4, BG) \to \mathrm{Map}^*(M_{l,m}, BG)$ to the k-th component. By Lemma 5.8 there is a fibration sequence

$$\Omega \mathrm{Map}_k(M_{l,m}, BG) \to \mathrm{Map}^*(Y_l, G) \to \mathrm{Map}_k^*(S^4, BG) \xrightarrow{\pi_k^*} \mathrm{Map}_k^*(M_{l,m}, BG). \tag{5.2.13}$$

We identify $\mathrm{Map}_k(M_{l,m}, BG) \simeq \mathcal{G}^*_k(M_{l,m})$. The left square in (5.2.12) along with 5.2.13 induce the following homotopy commutative diagram

$$\begin{array}{ccccc}
* & \longrightarrow & \mathcal{G}^*_k(M_{l,m}) & = & \mathcal{G}^*_k(M_{l,m}) \\
\downarrow & & \downarrow & & \downarrow{\scriptstyle \delta^*} \\
\mathcal{G}_k(S^4) & \longrightarrow & \mathcal{G}_k(M_{l,0}) & \xrightarrow{h} & F^{l,0}_k(\pi) \\
\| & & \downarrow & & \downarrow{\scriptstyle q^*_{l,0}} \\
\mathcal{G}_k(S^4) & \longrightarrow & G & \xrightarrow{\phi_k} & \mathrm{Map}^*_k(S^4, BG) \\
\downarrow & & \downarrow{\scriptstyle \partial_k} & & \downarrow{\scriptstyle \pi^*_k} \\
* & \longrightarrow & \mathrm{Map}^*_k(M_{l,0}, BG) & = & \mathrm{Map}^*_k(M_{l,0}, BG)
\end{array} \tag{5.2.14}$$

which defines the map h.

By Theorem 5.9 the map δ^* has a right homotopy inverse which implies that the map h also has a right homotopy inverse. The group structure on $\mathcal{G}_k(M_{l,0})$ allows to multiply to obtain a composite

$$\mathcal{G}_k(S^4) \times \mathrm{Map}^*(\Sigma Y_l, BG) \to \mathcal{G}_k(M_{l,0}) \times \mathcal{G}_k(M_{l,0}) \to \mathcal{G}_k(M_{l,0}),$$

which is a homotopy equivalence.

If $l \equiv 0 \pmod{12}$ then by Lemma 5.8 we have that $\Sigma Y_l \simeq S^4 \vee S^8$ and therefore $\mathrm{Map}^*(\Sigma Y_l, BG) \simeq \Omega^3 G \times \Omega^7 G$. □

Theorem 5.10 implies that the determination of the homotopy type of $\mathcal{G}_k(M_{l,0})$ is reduced to determining that of $\mathcal{G}_k(S^4)$. These gauge groups have been computed for different Lie groups. For example, from [The15, Theorem 1.1] we obtain the following corollary. Let (n_1, n_2) denote the greatest common divisor of n_1 and n_2, and notice that 120 is divisible by 16 positive integers.

Corollary 5.11. *Suppose M is either $S^3 \times S^4$ or any twisted product $S^3 \tilde{\times}_l S^4$. Let $P_k \to M$ and $P_{k'} \to M$ be principal $SU(5)$-bundles classified by $k, k' \in \mathbb{Z}$. Then $\mathcal{G}_k(M) \simeq \mathcal{G}_{k'}(M)$ if and only if $(120, k) = (120, k')$ when localised rationally or at any prime p. In particular, there are 16 distinct homotopy types of the groups $\mathcal{G}_k(M)$ for $k \in \mathbb{Z}$ when localised rationally or at any prime.*

5.3 Homotopy decompositions: non torsion-free case

Now we focus on the case of gauge groups of principal G-bundles over manifolds $M_{l,m}$ for $m > 1$, which have homology groups with torsion. We will require that all spaces are localised at a prime $p \geq 5$. The statements and proofs are similar to those obtained in the previous section. However, in this case we will obtain results for the loop space of the gauge group, $\Omega\mathcal{G}_k(M_{l,m})$.

Lemma 5.12. *There is a homotopy commutative diagram*

$$\begin{array}{ccccccccccc}
* & \longrightarrow & S^7 & = & S^7 & \longrightarrow & * & \longrightarrow & S^8 & = & S^8 \\
\downarrow & & \downarrow & & \downarrow{\scriptstyle \gamma} & & \downarrow & & \downarrow & & \downarrow{\scriptstyle \Sigma\gamma} \\
M_{l,m} & \xrightarrow{g} & S^4 \vee S^7 & \xrightarrow{\delta} & S^4 & \longrightarrow & \Sigma M_{l,m} & \longrightarrow & S^5 \vee S^8 & \xrightarrow{\Sigma\delta} & S^5 \\
\| & & \downarrow & & \downarrow & & \| & & \downarrow & & \downarrow \\
M_{l,m} & \xrightarrow{\pi} & S^4 & \xrightarrow{q} & C_{l,m}(\pi) & \longrightarrow & \Sigma M_{l,m} & \longrightarrow & S^5 & \xrightarrow{\Sigma q} & \Sigma C_{l,m}(\pi)
\end{array} \tag{5.3.1}$$

where each row and column is a cofibration sequence. Furthermore, after localisation at a prime $p \geq 5$, the map $S^5 \xrightarrow{\Sigma q} \Sigma C_{l,m}(\pi)$ is identified with the composite

$$S^5 \xrightarrow{m} S^5 \hookrightarrow \Sigma C_{l,m}(\pi)$$

and there are homotopy equivalences

$$\Sigma C_{l,m}(\pi) \simeq S^5 \vee S^9.$$

Proof. By Lemma 5.5, there is a homotopy commutative diagram

$$\begin{array}{ccccc}
S^3 & \xrightarrow{i} & M_{l,m} & \xrightarrow{g} & S^4 \vee S^7 \\
& & \| & & \downarrow{\scriptstyle \tilde{\pi}} \\
& & M_{l,m} & \xrightarrow{\pi} & S^4
\end{array} \tag{5.3.2}$$

where the top row is the cofibration sequence induced by the inclusion of the bottom cell and $\tilde{\pi}$ is the homotopy extension of the projection map π. Arguing along the lines of Lemma 5.7 we can extend (5.3.2) to the right to obtain a homotopy commutative diagram as shown in (5.3.1). Note that $\delta = \beta + \gamma$ where $\beta \in \pi_4(S^4)$ and $\gamma \in \pi_7(S^4) \cong \mathbb{Z} \oplus \mathbb{Z}_{12}$. Using the long exact sequence induced in homology by the middle row of (5.3.1), we can see that β is the degree m map. Thus we can identify the map Σq with the composite

$$S^5 \xrightarrow{m} S^5 \hookrightarrow \Sigma C_{l,m}(\pi).$$

The homotopy group $\pi_8(S^5)$ becomes trivial after localisation at a prime $p \geq 5$ [Tod63]. Since $\Sigma C_{l,m}(\pi)$ is the homotopy cofibre of the map $\Sigma\gamma \in \pi_8(S^5)$, after localisation at a prime $p \geq 5$ there are homotopy equivalences

$$\Sigma C_{l,m} \simeq S^5 \vee S^9.$$

□

Lemma 5.13. *After localisation at a prime $p \geq 5$, there is a fibration sequence*

$$\Omega^9 BG \times \Omega^5 BG \xrightarrow{*\times m^*} \Omega^5 BG \xrightarrow{\underline{\pi}_k^*} \Omega \mathrm{Map}_k^*(M_{l,m}, BG),$$

where m^* is the m-th power map, and the map $\underline{\pi}_k^*$ is identified with the composite

$$\Omega \mathrm{Map}_0^*(S^4, BG) \xrightarrow{\Omega\theta_k} \Omega \mathrm{Map}_k^*(S^4, BG) \xrightarrow{\Omega\pi_k^*} \Omega \mathrm{Map}_k^*(M_{l,m}, BG),$$

where $\theta_k : \Omega_0^4 BG \to \Omega_k^4 BG$ is a homotopy equivalence.

Proof. Applying the functor $\mathrm{Map}^*(-, BG)$ to the diagram in Lemma 5.3.1, we obtain a homotopy commutative diagram

$$\begin{array}{ccccc}
\mathrm{Map}^*(C_{l,m}(\pi), BG) & \longrightarrow & \mathrm{Map}^*(S^4, BG) & \xrightarrow{\gamma^*} & \mathrm{Map}^*(S^7, BG) \\
\downarrow{\scriptstyle q^*} & & \downarrow{\scriptstyle (m+\gamma)^*} & & \Vert \\
\mathrm{Map}^*(S^4, BG) & \longrightarrow & \mathrm{Map}^*(S^4 \vee S^7, BG) & \xrightarrow{p_2^*} & \mathrm{Map}^*(S^7, BG) \\
\downarrow{\scriptstyle \pi^*} & & \downarrow{\scriptstyle g^*} & & \downarrow \\
\mathrm{Map}^*(M_{l,m}, BG) & = & \mathrm{Map}^*(M_{l,m}, BG) & \longrightarrow & * \\
 & & \downarrow{\scriptstyle i^*} & & \\
 & & \mathrm{Map}^*(S^3, BG) & &
\end{array} \qquad (5.3.3)$$

where rows and columns are fibrations. We can identify $\mathrm{Map}^*(S^4 \vee S^7, BG)$ with $\mathrm{Map}^*(S^4, BG) \times \mathrm{Map}^*(S^7, BG)$ so that p_2^* is the projection of the second factor. The following diagram is obtained by restricting the map i^* to the k-th component and composing with the homotopy equivalence $\theta_k : \Omega^4 BG \to \Omega^4 BG$ given by $\omega \mapsto \mu \circ (\omega \times k_0) \circ \Delta$ for a fixed $k_0 \in \Omega_k^4 BG$:

$$\begin{array}{ccccc}
\mathrm{Map}^*(C_{l,m}(\pi), BG) & \longrightarrow & \Omega^4 BG & \xrightarrow{\gamma^*} & \Omega^7 BG \\
\downarrow & & \downarrow{\scriptstyle (\theta_k \circ m+\gamma)^*} & & \Vert \\
\coprod_{i\in\mathbb{Z}} \Omega^4_{im+k} BG & \xrightarrow{\tilde{\pi}} & \coprod_{i\in\mathbb{Z}} \Omega^4_{im+k} BG \times \Omega^7 BG & \xrightarrow{p_2^*} & \Omega^7 BG \\
\downarrow{\scriptstyle \pi_k^*} & & \downarrow{\scriptstyle g_k^*} & & \downarrow \\
\mathrm{Map}_k^*(M_{l,m}, BG) & = & \mathrm{Map}_k^*(M_{l,m}, BG) & \longrightarrow & *
\end{array} \qquad (5.3.4)$$

Here all rows and the middle and right columns, and hence the left column, are fibrations. Note that since the projection map $g^* : \mathrm{Map}^*(S^4, BG) \to \mathrm{Map}^*(M_{l,m}, BG)$ induces a surjection in path components, the homotopy fibre of i^* restricted to the k-th component is not path connected. Applying the functor $\Omega(-)$ to the previous diagram we obtain the following homotopy commutative diagram

$$\begin{array}{ccccc} \Omega\mathrm{Map}^*(C_{l,m}(\pi), BG) & \longrightarrow & \Omega^5 BG & \xrightarrow{\Omega\gamma^*} & \Omega^8 BG \\ \downarrow{\tilde{q}_k^*} & & \downarrow{\Omega(m+\gamma)^*} & & \| \\ \Omega^5 BG & \longrightarrow & \Omega^5 BG \times \Omega^8 BG & \xrightarrow{p_2^*} & \Omega^8 BG \\ \downarrow{\pi_k^*} & & \downarrow{\Omega g_k^*} & & \downarrow \\ \Omega\mathrm{Map}_k^*(M_{l,m}, BG) & = & \Omega\mathrm{Map}_k^*(M_{l,m}, BG) & \longrightarrow & * \end{array} \tag{5.3.5}$$

where we have identified $\Omega\Omega_k^4 BG$ with $\Omega\Omega_0^4 BG \simeq \Omega^5 BG$ for all k; note that localised away from $p = 2$ we have $\pi_4(G) = 0$ and so $\Omega^5 BG$ is connected. Note that the adjoint of $\Omega\gamma^*$ is homotopic to $(\Sigma\gamma)^*$. Localise at a prime $p \geq 5$. Taking adjoints and using Lemma 5.12 we obtain a string of homotopy equivalences

$$\Omega\mathrm{Map}^*(C_{l,m}(\pi), BG) \simeq \mathrm{Map}^*(\Sigma C_{l,m}(\pi), BG) \simeq \Omega^9 BG \times \Omega^5 BG,$$

and $\tilde{q}_k^* \simeq * \times m^*$, where m^* is the m-th power map. □

Now we give results on the homotopy decomposition of the pointed gauge groups. Recall that for any space X, the cofibration sequence $S^n \xrightarrow{k} S^n \to P^{n+1}(k)$ induces the following fibration sequence

$$\mathrm{Map}^*(P^{n+1}(k), BG) \to \Omega^n X \xrightarrow{k^*} \Omega^n X,$$

where k^* is the k-th power map. Let $\Omega^n G\{k\} := \mathrm{Map}^*(P^{n+1}(k), BG)$.

Theorem 5.14. *Let G be a simply connected simple compact Lie group with $\pi_6(G) \cong 0$. Let $P_k \to M_{l,m}$ be a principal G-bundle classified by $k \in [M_{l,m}, BG]$, and $m > 1$. Localising at a prime $p \geq 5$ there are p-local homotopy equivalences*

$$\mathcal{G}_0^*(M_{l,m}) \simeq \Omega^3 G\{p^r\} \times \Omega^7 G,$$

$$\Omega\mathcal{G}_k^*(M_{l,m}) \simeq \Omega^4 G\{p^r\} \times \Omega^8 G,$$

where $r = v_p(m)$ is the valuation of m at p.

Proof. Localise at a prime $p \geq 5$ and let $v_p(m) = r$. First suppose $k = 0$. Using Lemma 5.4, there is a homotopy equivalence

$$\Sigma M_{l,m} \simeq P^4(p^r) \vee S^8.$$

Thus we obtain a string of homotopy equivalences

$$\Omega\mathrm{Map}_0^*(M_{l,m}, BG) \simeq \mathrm{Map}^*(\Sigma M_{l,m}, BG) \simeq \mathrm{Map}^*(P^4(p^r) \vee S^8, BG).$$

Taking adjoints we obtain

$$\mathrm{Map}^*(P^4(p^r)\vee S^8, BG)\simeq \mathrm{Map}^*(P^3(p^r)\vee S^7, G)\simeq \mathrm{Map}^*(P^3(p^r),G)\times \mathrm{Map}^*(S^7,G).$$

Since $\mathcal{G}_0^*(M_{l,m})\simeq \Omega\mathrm{Map}_0^*(M_{l,m},BG)$, we get $\mathcal{G}_0^*(M_{l,m})\simeq \Omega^3G\{p^r\}\times\Omega^7G$.

Now suppose $k\neq 0$. By Lemma 5.13 there is a fibration sequence

$$\Omega^9BG\times\Omega^5BG\xrightarrow{*\times m^*}\Omega^5BG\xrightarrow{\underline{\pi}_k^*}\Omega\mathrm{Map}_k^*(M_{l,m},BG),$$

where m^* is the m-th power map, and the map $\underline{\pi}_k^*$ is identified with the composite

$$\Omega\mathrm{Map}_0^*(S^4,BG)\xrightarrow{\Omega\theta_k}\Omega\mathrm{Map}_k^*(S^4,BG)\xrightarrow{\Omega\pi_k^*}\Omega\mathrm{Map}_k^*(M_{l,m},BG),$$

where $\theta_k:\Omega_0^4BG\to\Omega_k^4BG$ is a homotopy equivalence. Note that the homotopy fibre of the map $*\times m^*$ is homotopy equivalent to $\Omega^2\mathrm{Map}_k^*(M_{l,m},BG)$, which can be identified with $\Omega\mathcal{G}_k^*(M_{l,m})$. Now identifying Ω_0^5BG with $\Omega_0^5BG\times *$ it is easy to see that there is a homotopy equivalence

$$\Omega\mathcal{G}_k^*(M_{l,m})\simeq\Omega^4G\{p^r\}\times\Omega^8G,$$

as required. □

Now by Proposition 5.6, $M_{l,m}\xrightarrow{\pi}S^4$ induces a surjection $[S^4,BG]\xrightarrow{\pi^*}[M_{l,m},BG]$ if $m>1$. Therefore, by the naturality of the evaluation map, we obtain a commutative diagram

$$\begin{array}{ccccccc}\Omega G & \xrightarrow{\Omega\phi_k} & \Omega\mathrm{Map}_k^*(S^4,BG) & \longrightarrow & \Omega\mathrm{Map}_k(S^4,BG) & \xrightarrow{ev_k} & G\\ \| & & \downarrow{\scriptstyle \pi^*} & & \downarrow{\scriptstyle\pi^*} & & \|\\ \Omega G & \xrightarrow{\Omega\partial_k} & \Omega\mathrm{Map}_k^*(M_{l,m},BG) & \longrightarrow & \Omega\mathrm{Map}_k(M_{l,m},BG) & \xrightarrow{ev_k} & G\end{array} \qquad (5.3.6)$$

which defines the map $\Omega\phi_k$. More precisely, if $m>1$ then $\mathrm{Map}(M_{l,m},BG)$ has m components and $\mathrm{Map}(S^4,BG)\xrightarrow{\pi^*}\mathrm{Map}(M_{l,m},BG)$ sends the k-th component of $\mathrm{Map}(S^4,BG)$ to the $\bar{k}$-th component of $\mathrm{Map}(M_{l,m},BG)$, where $\bar{k}$ is the mod m reduction of k.

Theorem 5.15. *Let $m>1$ be an integer and $p\geq 5$ be a prime. Let $P_k\to M_{l,m}$ be a principal G-bundle classified by $k\in\mathbb{Z}_m$, where G is a simply connected simple compact Lie group. There are p-local homotopy equivalences*

(1) $\mathcal{G}_0(M_{l,m})\simeq\Omega^7G\times G$ if $v_p(m)=0$;

(2) $\Omega\mathcal{G}_k(M_{l,m})\simeq\Omega^8G\times X_k$ if $v_p(m)\geq 1$, where X_k fits into a homotopy fibration

$$\Omega^4G\{m\}\to X_k\to\Omega G.$$

Moreover, if $v_p(m)=r\geq 1$ and $p^r|k$, then $X_k\simeq\Omega G\times\Omega^4G\{m\}$.

Proof. Let K be a CW-complex such that $M_{l,m} \simeq K$. Then K is obtained attaching a 7-cell to a Moore space $P^4(m)$. Localise all spaces at a prime $p \geq 5$, so that $\pi_6(G) \cong 0$ for any simply connected simple compact Lie group. Suppose that $v_p(m) = 0$. Then $P^4(m) \simeq *$ and therefore $M_{l,m} \simeq K \simeq S^7$. Thus in this case there is only one principal G-bundle over $M_{l,m}$ up to isomorphism, namely, the trivial bundle. Since the map ev_0 in (5.2.11) has a section and this is a principal fibration we obtain a p-local homotopy equivalence

$$\mathcal{G}_0(M_{l,m}) \simeq \Omega^7 G \times G.$$

Suppose now that $v_p(m) \geq 1$. By Theorem 5.14, there is a p-local homotopy equivalence

$$\Omega \mathcal{G}_k^*(M_{l,m}) \simeq \Omega^8 G \times \Omega^4 G\{m\}.$$

This implies that there is fibration sequence

$$\Omega^8 G \times \Omega^4 G\{m\} \xrightarrow{\delta^*} \Omega^8 G \times \Omega^4 G \xrightarrow{*\times m^*} \Omega^4 G \xrightarrow{\pi_k^*} \Omega \mathrm{Map}_k^*(M_{l,m}, BG) \simeq \mathcal{G}_k^*(M_{l,m}). \tag{5.3.7}$$

Therefore we have $\delta^* \simeq \mathbb{1} \times j$ where j is the inclusion map. The evaluation fibration along with (5.3.7) induce a homotopy commutative diagram

$$\begin{array}{ccccc}
 & & \Omega^9 BG \times \Omega^5 BG\{m\} & = & \Omega^9 BG \times \Omega^5 BG\{m\} \\
 & & \downarrow j' & & \downarrow \mathbb{1}\times j \\
\Omega \mathcal{G}_k(S^4) & \longrightarrow & \Omega \mathcal{G}_k(M_{l,m}) & \xrightarrow{h} & \Omega^9 BG \times \Omega^5 BG \\
\| & & \downarrow & & \downarrow *\times m^* \\
\Omega \mathcal{G}_k(S^4) & \longrightarrow & \Omega G & \xrightarrow{\Omega \phi_k} & \Omega^5 BG \\
 & & \downarrow \Omega \partial_k & & \downarrow \pi_k^* \\
 & & \Omega \mathrm{Map}_k^*(M_{l,m}, BG) & = & \Omega \mathrm{Map}_k^*(M_{l,m}, BG)
\end{array} \tag{5.3.8}$$

which defines the map h; here we identify $\Omega^5 BG \simeq \Omega \Omega_0^4 BG \simeq \Omega \Omega_k^4 BG$. Let $\bar{h}$ be the composite

$$\Omega \mathcal{G}_k(M_{l,m}) \xrightarrow{h} \Omega^9 BG \times \Omega^5 BG \xrightarrow{p_1} \Omega^9 BG,$$

where p_1 is the projection onto the first factor. The top square of (5.3.8) shows that $\bar{h}$ has a right homotopy inverse. Let X be the homotopy fibre of the map $\bar{h}$. Then there is a homotopy equivalence

$$\Omega \mathcal{G}_k(M_{l,m}) \simeq X \times \Omega^9 BG. \tag{5.3.9}$$

Finally from (5.3.8) and (5.3.9) there exists a homotopy pullback square

$$\begin{array}{ccccc} & & \Omega^4 G\{m\} & = & \Omega^4 G\{m\} \\ & & \downarrow & & \downarrow \\ \Omega\mathcal{G}_k(S^4) & \longrightarrow & X_k & \longrightarrow & \Omega^4 G \\ \| & & \downarrow & & \downarrow m^* \\ \Omega\mathcal{G}_k(S^4) & \longrightarrow & \Omega G & \xrightarrow{\Omega\phi_k} & \Omega^4 G. \end{array}$$

Let $r = \nu_p(m)$. Then, if $p^r|k$ then the map $\Omega\phi_k$ lifts through m^*. Therefore, by the properties of the pullback there is a map $\zeta : \Omega G \to X_k$ which is a homotopy section. Thus in this case we have a splitting $X_k \simeq \Omega G \times \Omega^4 G\{m\}$. □

5.4 Homotopy types of gauge groups over S^7

In this section we discuss the classification of the homotopy types of the gauge groups over manifolds $M_{l,1}$. As all manifolds $M_{l,1}$ are homotopy equivalent to S^7, the following results will be expressed in terms of S^7. Recall that $Prin_G(S^7)$ is in one-to-one correspondence with the set $[S^7, BG]$. In Table 5.1 we collect information on the sets $Prin_G(S^7) = [S^7, BG]$. Here G^* is any of the simply connected simple compact Lie groups not isomorphic to $SU(3)$, G_2 or $SU(2) \cong Sp(1)$.

TABLE 5.1

G	$SU(2)$	$SU(3)$	G_2	G^*
$[S^7, BG]$	$\mathbb{Z}_{12}$	$\mathbb{Z}_6$	$\mathbb{Z}_3$	0

Let $P_k \to S^7$ be a principal G-bundle classified by $k \in [S^7, BG]$. We have seen already that as S^7 is a co-H-space, $\text{Map}_k^*(S^7, BG) \simeq \text{Map}_0^*(S^7, BG)$, which implies that for any $k \in [S^7, BG]$ there exists a homotopy equivalence $\mathcal{G}_k^*(S^7) \simeq \mathcal{G}_0^*(S^7)$. In what follows we discuss the results on the homotopy classification of the unpointed gauge groups over S^7.

Consider the fibration sequence

$$\mathcal{G}_k(S^7) \longrightarrow G \xrightarrow{\partial_k} \text{Map}_k^*(S^7, BG) \longrightarrow \text{Map}_k(S^7, BG) \xrightarrow{ev} BG \qquad (5.4.1)$$

where ev is the evaluation map. Thus the connecting map ∂_k is an element of

$$[G, \text{Map}_k^*(S^7, BG)] \cong [G, \text{Map}_0^*(S^7, BG)] = [G, \Omega_0^6 G].$$

By Lemma 4.21 the adjoint of the connecting map, denoted ∂^k, is homotopic to the Samelson product $\langle k\gamma, \mathbb{1}_G\rangle$, where γ is a generator of $[S^7, BG] \cong \pi_6(G)$ and $\mathbb{1}_G$ is the identity map on G. It is clear that the order of ∂_k is bounded by both the number of principal G-bundles and the order of $[G, \Omega_0^6 G]$.

Lemma 5.16. *The set $[SU(3), \Omega_0^6 SU(3)]$ is isomorphic to*

$$\mathbb{Z}_4 \oplus \mathbb{Z}_2^2 \oplus \mathbb{Z}_3^2 \oplus \mathbb{Z}_7.$$

Proof. Taking the adjoint we get $[SU(3), \Omega^6 SU(3)] \cong [\Sigma^6 SU(3), SU(3)]$. After suspension there is a homotopy equivalence [Mim69]

$$\Sigma^6 SU(3) \simeq \Sigma^7 \mathbb{C}P^2 \vee S^{14}.$$

Let $p > 2$ be a prime. There exists a p-local homotopy equivalence

$$\Sigma^6 SU(3) \simeq_p S^9 \vee S^{11} \vee S^{14}$$

Thus localized at $p > 2$ we get

$$\begin{aligned} [\Sigma^6 SU(3), SU(3)]_{(p>2)} &= [S^9 \vee S^{11} \vee S^{14}, S^3 \times S^5] \\ &= \pi_9(S^3 \times S^5) \oplus \pi_{11}(S^3 \times S^5) \oplus \pi_{14}(S^3 \times S^5) \end{aligned}$$

Using the information on the homotopy groups of spheres [Tod63] we obtain

$$[\Sigma^6 SU(3), SU(3)]_{(p)} = \mathbb{Z}_3^2 \oplus \mathbb{Z}_7.$$

Now consider localization at $p = 2$. First by (5.4)

$$\begin{aligned} [\Sigma^6 SU(3), SU(3)]_{(2)} &= [\Sigma^7 \mathbb{C}P^2 \vee S^{14}, SU(3)] \\ &= [\Sigma^7 \mathbb{C}P^2, SU(3)] \oplus \pi_{14}(SU(3)) \\ &= [\Sigma^7 \mathbb{C}P^2, SU(3)] \oplus \mathbb{Z}_4 \oplus \mathbb{Z}_2 \end{aligned}$$

since $\pi_{14}(SU(3)) \cong \mathbb{Z}_4 \oplus \mathbb{Z}_2$ [MT63]. The next step is to determine the homotopy set $[\Sigma^7 \mathbb{C}P^2, SU(3)]$. Consider the cofibration sequence

$$S^{10} \xrightarrow{\eta_9} S^9 \xrightarrow{i} \Sigma^7 \mathbb{C}P^2 \xrightarrow{q} S^{11} \xrightarrow{\eta_{10}} S^{10} \tag{5.4.2}$$

where $\eta_n = \Sigma^{n-2}\eta : S^{n+1} \to S^n$ with $\eta : S^3 \to S^2$ the Hopf map. Applying the functor $[-, SU(3)]$ we get the following sequence of homotopy sets

$$\pi_{10}(SU(3)) \xrightarrow{\eta_{10}^*} \pi_{11}(SU(3)) \xrightarrow{q^*} [\Sigma^7 \mathbb{C}P^2, SU(3)] \xrightarrow{i^*} \pi_9(SU(3)). \tag{5.4.3}$$

As $\pi_9(SU(3)) = 0$, the group $[\Sigma^7 \mathbb{C}P^2, SU(3)]$ is isomorphic to the cokernel of η_{10}^*. It is known that $\pi_{10}(S^5) \cong \mathbb{Z}_2$, generated by $\nu_5 \eta_8^2$, and $\pi_{11}(S^5) \cong \mathbb{Z}_2$, generated by ν_5^2. In

[MT63] it is shown that the projection map $q : SU(3) \to S^5$ induces an isomorphism for π_{10} and an epimorphism for π_{11}. Thus there are maps $[\nu_5\eta_8^2] : S^{10} \to SU(3)$ and $[\nu_5^2] : S^{11} \to SU(3)$ such that the following diagrams homotopy commute

$$\begin{array}{ccc} S^{10} & & \\ {\scriptstyle [\nu_5\eta_8^2]}\downarrow & \searrow^{\nu_5\eta_8^2} & \\ SU(3) & \xrightarrow{q} & S^5, \end{array}$$

$$\begin{array}{ccc} S^{11} & & \\ {\scriptstyle [\nu_5^2]}\downarrow & \searrow^{\nu_5^2} & \\ SU(3) & \xrightarrow{q} & S^5. \end{array}$$

and the maps $[\nu_5\eta_8^2]$ and $[\nu_5^2]$ are the generators of $\pi_{10}(SU(3)) \cong \mathbb{Z}_2$ and $\pi_{11}(SU(3)) \cong \mathbb{Z}_4$ [MT63], respectively. Observe that the composite

$$S^{11} \xrightarrow{\eta_{10}} S^{10} \xrightarrow{[\nu_5\eta_8^2]} SU(3) \xrightarrow{q} S^5$$

is $\nu_5\eta_8^3$. The map $\nu_5\eta_8^3$ cannot be homotopic to ν_5^2 since stably $\nu_5\eta_8^3 \simeq *$ while $\nu_5^2 \not\simeq *$. Thus $q \circ [\nu_5\mu_8^2] \circ \eta_{10}$ is nullhomotopic and therefore $\eta_{10}^* = 0$. Thus the homomorphism $q^* : \pi_{11}(SU(3)) \to [\Sigma^7\mathbb{C}P^2, SU(3)]$ is an isomorphism and $[\Sigma^7\mathbb{C}P^2, SU(3)]_{(p=2)} = \mathbb{Z}_4 \oplus \mathbb{Z}_2^2$. Finally, putting things together we obtain $[\Sigma^6 SU(3), SU(3)] \cong \mathbb{Z}_4^2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_3^2 \oplus \mathbb{Z}_7$. □

We give a result on the number of homotopy types of gauge groups of principal G-bundles over S^7. In the case of the 2-primary component for $G = SU(3)$ this has not been resolved so that we will state the result for the odd primary part.

Theorem 5.17. *Let G be a simply connected simple compact Lie group and let $P_k \to S^7$ and $P_{k'} \to S^7$ be principal G-bundles. Then*

(1) for $G = SU(2) \cong Sp(1)$ or $G = G_2$, there is a homotopy equivalence $\mathcal{G}_k(S^7) \simeq \mathcal{G}_{k'}(S^7)$ when localised rationally or at any prime if and only if $(3, k) = (3, k')$;

(2) for $G = SU(3)$, there is a homotopy equivalence $\mathcal{G}_k(S^7) \simeq \mathcal{G}_{k'}(S^7)$ when localised rationally or at a prime $p \geq 3$ if and only if $(3, k) = (3, k')$;

(3) otherwise, the gauge group of the unique principal G-bundle decomposes as

$$\mathcal{G}_0(S^7) \simeq \Omega^7 G \times G.$$

Proof. Let $\partial^k \in [\Sigma^6 G, G]$ be the adjoint of the connecting map ∂_k in (5.4.1).

(1) Suppose first that $G = SU(2)$. We identify the Lie group $SU(2)$ with the unit quaternions S^3. Thus there is an isomorphism

$$[\Sigma^6 SU(2), SU(2)] \cong [\Sigma^6 \wedge S^3, S^3] \cong \pi_9(S^3).$$

According to [Tod63], $\pi_9(S^3) \cong \mathbb{Z}_3$, and so the order of ∂^k is at most 3. We also know that the map ∂^k is homotopic to the Samelson product $\langle k\gamma, \iota_3\rangle$, where γ is a generator of $\pi_6(SU(2))$ and $\iota_3 : S^3 \to S^3 \cong SU(2)$ is the identity map on S^3. From the fibration sequence

$$\mathrm{Map}_k^*(S^7, BG) \xrightarrow{i} \mathrm{Map}_k(S^7, BG) \xrightarrow{ev} BG$$

we obtain the following commutative diagram of groups

$$\begin{array}{ccccccc} \pi_3(S^3) & \longrightarrow & \pi_3(\Omega^7_0 BS^3) & \longrightarrow & \pi_3(\mathrm{Map}_k(S^7, BS^3)) & \longrightarrow & 0, \\ & \searrow^{\partial^k} & \downarrow \cong & \nearrow & & & \\ & & [S^6 \wedge S^3, S^3] & & & & \end{array} \qquad (5.4.4)$$

where $\partial^k(f) := \langle k\gamma, f\rangle$ for any $f \in \pi_3(S^3) \cong \mathbb{Z}$. Thus $\pi_3(\mathrm{Map}_k(S^7, BS^3)) \cong \mathrm{coker}\, \partial^k$. Linearity in the Samelson product implies that $\langle k\gamma, \iota_3\rangle \simeq k\langle \gamma, \iota_3\rangle$. Thus we only have to determine the order of $\langle \gamma, \iota_3\rangle$.

Notice that if $\mathcal{G}_k(S^7) \simeq \mathcal{G}_{k'}(S^7)$ then $\pi_n(\mathcal{G}_k(S^7)) \cong \pi_n(\mathcal{G}_{k'}(S^7))$ for all $n \geq 0$. In particular, from (5.4.4) we obtain

$$\pi_2(\mathcal{G}_k(S^7)) \cong \pi_3(\mathrm{Map}_k(S^7)) \cong \pi_3(\mathrm{Map}_{k'}(S^7)) \cong \pi_2(\mathcal{G}_{k'}(S^7)).$$

According to [Sam54], the Samelson product $\langle \iota_3, \iota_3\rangle \in \pi_6(S^3)$ has order 12, therefore this map is a generator of $\pi_6(S^3) \cong \mathbb{Z}_{12}$. Hence the adjoint of the map ∂^1 is homotopic to the iterated commutator map $\langle\langle \iota_3, \iota_3\rangle, \iota_3\rangle$. Observe that $\langle\langle \iota_3, \iota_3\rangle, \iota_3\rangle$ is an element of $[SU(2) \wedge SU(2) \wedge SU(2), SU(2)] = [S^3 \wedge S^3 \wedge S^3, S^3] \cong [S^9, S^3] \cong \pi_9(S^3) \cong \mathbb{Z}_3$, thus the order of $\langle\langle \iota_3, \iota_3\rangle, \iota_3\rangle$ is either 1 or 3. According to [KK10, Theorem 2], $SU(2)$ localised at $p = 3$ is not nilpotent of class 2, hence $\langle\langle \iota_3, \iota_3\rangle, \iota_3\rangle$ is non-trivial. Thus it must have order 3. Since this map is the adjoint of ∂^1, we have that the order of ∂^1 is 3. From (5.4.4) and information of the homotopy groups of spheres we obtain an exact sequence

$$\mathbb{Z} \xrightarrow{\partial^k} \mathbb{Z}_3 \xrightarrow{i^*} \pi_3(\mathrm{Map}_k(S^7, BS^3)) \longrightarrow 0.$$

Therefore $|\mathrm{coker}\, \partial^k| = (3, k)$. Thus if $\pi_3(\mathrm{Map}_k(S^7, BS^3)) \cong \pi_3(\mathrm{Map}_{k'}(S^7, BS^3))$ then $(3, k) = (3, k')$. From the previous discussion we see that if $\mathcal{G}_k(S^7) \simeq \mathcal{G}_{k'}(S^7)$ then $(3, k) = (3, k')$.

Now suppose instead that $G = G_2$. Let $\iota : S^3 \xrightarrow{\iota_3} S^3 \hookrightarrow G$ be a generator of $\pi_3(G)$. The map $\langle \iota, \iota \rangle$ represents a generator of $\pi_6(G_2)$ [Mim67]. Thus we have $\partial^1 \simeq \langle\langle \iota, \iota \rangle, \mathbb{1}_{G_2}\rangle : S^6 \wedge G_2 \longrightarrow G_2$. Consider the following composite

$$\theta : S^6 \wedge S^3 \xrightarrow{\mathbb{1}\wedge\iota} S^6 \wedge G_2 \xrightarrow{\langle\langle \iota,\iota\rangle,\mathbb{1}_{G_2}\rangle} G_2.$$

Thus $\theta = \langle\langle \iota, \iota \rangle, \iota\rangle$. We claim that θ is not nullhomotopic. Consider the inclusion $S^3 \overset{i}{\hookrightarrow} G_2$. According to [Jam95], there exists a homotopy fibration

$$S^3 \overset{i}{\hookrightarrow} G_2 \longrightarrow S^{11}$$

after localisation at 3. Thus the induced map $i^* : \pi_m(S^3) \to \pi_m(G_2)$ is an isomorphism for $m \leq 9$. Recall that, localising at $p = 3$, the map $\langle\langle \iota_3, \iota_3, \rangle, \iota_3\rangle \in \pi_9(S^3)$ is essential. Therefore the map θ has order 3. Thus θ generates $\pi_9(G_2) \cong \mathbb{Z}_3$. By definition, θ is the restriction of ∂^1 to $S^3 \subset G_2$. Thus the order of ∂^1 and hence of its adjoint ∂_1 is at least 3. Now, from Proposition 5.6 we know that $Prin_{G_2}(S^7) = \pi_6(G_2) \cong \mathbb{Z}_3$. Thus, as there are 3 isomorphism classes of principal G_2-bundles over S^7, the order of the map ∂_1 is at most 3. The upper and the lower bounds of the order of ∂_1 coincide and therefore the order of ∂_1 is 3. We also obtain an exact sequence as in (5.4.4) to show that if $\mathcal{G}_k(S^7) \simeq \mathcal{G}_{k'}(S^7)$ then $(3, k) = (3, k')$.

Finally a simple application of Lemma 4.26 shows that $\mathcal{G}_k(S^7) \simeq \mathcal{G}_{k'}(S^7)$ whenever $(3, k) = (3, k')$ for $G = SU(2) \cong Sp(1)$ and $G = G_2$.

(2) Suppose all spaces are localised at a prime $p \geq 3$. We can get an upper bound on the order of ∂^1 through Lemma 5.16. It was shown that $\partial^k \in [\Sigma^6 SU(3), SU(3)] \cong \mathbb{Z}_3^2 \oplus \mathbb{Z}_7$. Let β be the order of ∂^1. Then β divides $|\mathbb{Z}_3^2 \oplus \mathbb{Z}_7| = 63$. We also have that $\beta \leq |Prin_{SU(3)}(S^7)| = 6$. Therefore the order of ∂^1 localised at a prime $p \geq 3$ is at most 3.

Suppose all spaces are localise at $p = 3$. Then there is a p-local homotopy equivalence $SU(3) \simeq S^3 \times S^5$, and the composite $\iota : S^3 \xrightarrow{\iota_3} S^3 \hookrightarrow SU(3)$ is a generator of $\pi_3(SU(3))$. Let $\langle \iota, \iota \rangle$ be a generator of $\pi_6(S^3) \cong \pi_6(SU(3); 3) \cong \mathbb{Z}_3$. Consider the composite

$$S^9 \cong S^6 \wedge S^3 \xrightarrow{\mathbb{1}\wedge\iota} S^6 \wedge SU(3) \xrightarrow{\langle\langle \iota,\iota\rangle,\mathbb{1}_{SU(3)}\rangle} SU(3).$$

The element $\langle\langle \iota, \iota \rangle, \iota\rangle$ is non-trivial in $\pi_9(SU(3); 3) \cong \mathbb{Z}_3$. Therefore localised at $p = 3$ the map $\langle\langle \iota, \iota \rangle, \iota\rangle$ has order 3. Thus using an exact sequence as in (5.4.4), we see that if $\mathcal{G}_k(S^7) \simeq \mathcal{G}'_k(S^7)$ then $(3, k) = (3, k')$. Finally, applying Lemma 4.26 we complete the proof of (2).

(3) If $G \neq SU(2), SU(3), G_2$ or $Sp(1)$, then $\pi_7(BG) \cong 0$. Thus there is a single principal G-bundle over S^7 which must be the trivial bundle, implying that the map ∂^k is nullhomotopic. Therefore the principal fibration

$$\Omega \mathrm{Map}^*(S^7, BG) \to \mathcal{G}_0(S^7) \to G$$

splits and $\mathcal{G}_0(S^7) \simeq \Omega^7 G \times G$. □

Remark 5.18. We want to point out that part (i) of Theorem 5.17 contrasts with a results given in [CS09].

Proposition 5.19 ([Proposition 2 [CS09]). *For $n \in \{7, 8, 9, 10, 15, 16, 17, 18, 23, 24, 25\}$, there is a unique homotopy type of the gauge groups of all the principal $SU(2)$-bundles over S^n, and it is the one of the trivial bundle, namely,*

$$\mathrm{Map}(S^n, S^3) \simeq \Omega_0^n S^3 \times S^3.$$ □

According to Proposition 5.19, if $G = S^3$ all gauge groups over S^7 are homotopy equivalent. However, Theorem 5.17 shows that given two elements $k, k' \in [S^7, BS^3]$, it is not always true that $\mathcal{G}_k(S^7) \simeq \mathcal{G}_{k'}(S^7)$.

Chapter 6

Connected sums and other extensions

We extend our results on principal G-bundles over $M_{l,m}$ to some cases that were not covered in Chapter 5. We also give results on the homotopy theory of gauge groups of principal G-bundles over certain connected sums. In the first part of Chapter 6 we consider the homotopy theory of gauge groups over manifolds $M_{l,0}$ for the cases $G = SU(2), SU(3), G_2$. The main results of the first two sections of Chapter 6 are Proposition 6.3 and Theorem 6.6. In the last three sections of this chapter we will study the homotopy theory of some 2-connected 7-manifolds with torsion-free homology. We will provide homotopy decomposition for certain connected sums of manifolds $M_{l,0}$. The proofs are very similar to those presented in Chapter 5. The ultimate goal is to give homotopy decompositions of gauge groups over certain connected sums. We present some information on the classification of 2-connected 7-manifolds M. We then prove Proposition 6.11 and Theorem 6.12.

6.1 Principal G-bundles: $G = SU(2), SU(3), G_2$

Let $M_{l,m}$ be the sphere bundle classified by $(l, m) \in \mathbb{Z} \oplus \mathbb{Z}$. In this section we restrict to principal G-bundles over manifolds $M_{l,0}$ and $G = SU(2)$, $SU(3)$ or G_2. From the classification of S^3-bundles over S^4, all the projection maps $\pi : M_{l,0} \to S^4$ have cross sections, and we have the following homotopy equivalences

- $M_{l,0} \simeq M_{l',0}$ if and only if $l \equiv \pm l' \pmod{12}$;
- $M_{l,0} \simeq S^3 \times S^4$ if and only if $l \equiv 0 \pmod{12}$.

Let $o(G)$ denote the order of $\pi_6(G)$. In the study of gauge groups of principal G-bundles over $M_{l,0}$, the value of l for each $M_{l,0}$ becomes important when working with Lie groups G such that $o(G) \neq 1$. We write $S^3 \tilde{\times}_l S^4 = M_{l,0}$ for the cases where $l \not\equiv 0 \pmod{12}$.

Given a manifold $M_{l,0}$, there is a homotopy cofibration sequence

$$S^6 \xrightarrow{\varphi} S^3 \vee S^4 \xrightarrow{i} M_{l,0} \xrightarrow{q} S^7 \xrightarrow{\Sigma\varphi} S^4 \vee S^5, \tag{6.1.1}$$

where φ is the attaching map of the top cell, i is the inclusion of the 4-skeleton and q is the pinch map to the top cell. We can express the attaching map as $\varphi = [\iota_3, \iota_4] + t\underline{\alpha}$, where $[\iota_3, \iota_4]$ is the Whitehead product of the identity maps in S^3 and S^4, and $\underline{\alpha}$ is a generator of $\pi_6(S^3)$. We obtain an exact sequence

$$[S^4 \vee S^5, BG] \xrightarrow{(\Sigma\varphi)^*} [S^7, BG] \xrightarrow{q^*} [M_{l,0}, BG] \xrightarrow{i^*} [S^3 \vee S^4, BG] \xrightarrow{\varphi^*} [S^6, BG]. \tag{6.1.2}$$

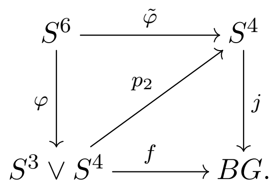
Lemma 6.1. *Let $M_{l,0}$ be an S^3-bundle over S^4 classified by an element $(l, 0) \in \mathbb{Z} \oplus \mathbb{Z}$, and let G be a simply connected simple compact Lie group with $o(G) \neq 1$. Then the map φ^* induced by the attaching map is trivial.*

Proof. From the homotopy groups of Lie groups (Table A.1), for $G = G_2$ we have $\pi_5(G_2) = 0$. There is an isomorphism $[S^6, BG_2] \cong \pi_5(G_2)$, and therefore $\varphi^* = 0$.

Now suppose that $G \neq G_2$, that is, $G = SU(n)$ for $n \in \{2, 3\}$. By connectivity of BG, any map $f : S^3 \vee S^4 \longrightarrow BSU(n)$ factors as the composite

$$S^3 \vee S^4 \xrightarrow{p_2} S^4 \hookrightarrow BSU(n)$$

where $p_2 : S^3 \vee S^4 \to S^4$ is the projection onto the second component. Thus there is a commutative diagram induced by the attaching map φ as follows



As $M_{l,0}$ is the total space of a sphere bundle over S^4, the map p_2 factors as the composite $p_2 : S^3 \vee S^4 \xrightarrow{i} M_{l,0} \to S^4$ where $i \circ \varphi$ is nullhomotopic, and therefore the composite $\tilde{\varphi} = p_2 \circ \varphi$ is nullhomotopic. Hence given $f \in [S^3 \vee S^4, BG]$ we have $\varphi^*(f) = f \circ \varphi = j \circ p_2 \circ \varphi = j \circ \tilde{\varphi}$. As $\tilde{\varphi}$ is nullhomotopic, so is $j \circ \tilde{\varphi}$, and therefore φ^* is the zero map. $\square$

Recall, (n, m) denotes the greatest common divisor of n and m.

Lemma 6.2. *The image of* $(\Sigma\varphi)^*$ *is equal to* $\mathbb{Z}_n$, *where* $n = \frac{o(G)}{(o(G),l)}$.

Proof. First notice that if $o(G) = 1$ then $(\Sigma\varphi)^* : \pi_3(G) \oplus \pi_5(G) \to \pi_6(G)$ is the zero map. Therefore $\mathrm{im}(\Sigma\varphi)^* \cong \mathbb{Z}_{\frac{1}{(1,l)}} \cong \mathbb{Z}_1 \cong 0$. Now suppose $o(G) > 1$.

If $l \equiv 0 \pmod{12}$ then we have that $M_{l,0} \simeq S^3 \times S^4$ with attaching map φ a Whitehead product. In this case, $\Sigma\varphi$ is nullhomotopic, implying that $(\Sigma\varphi)^*$ is the trivial map. Thus $\mathrm{im}(\Sigma\varphi)^* \cong \mathbb{Z}_{\frac{o(G)}{(o(G),l)}} \cong \mathbb{Z}_1 \cong 0$.

Now if $l \not\equiv 0 \pmod{12}$ then from the exact sequence (6.1.2) we have $\mathrm{im}(\Sigma\varphi)^* = \ker q^*$. A map $f : S^7 \to BG$ is in the kernel of q^* if and only if there is an extension $\tilde{f} : S^4 \vee S^5 \to BG$, such that the diagram

$$\begin{array}{ccccc} M_{l,0} & \xrightarrow{q} & S^7 & \xrightarrow{\Sigma\varphi} & S^4 \vee S^5 \\ & & \downarrow f & \swarrow \tilde{f} & \\ & & BG & & \end{array} \tag{6.1.3}$$

homotopy commutes.

As $\pi_6(S^3) \cong \mathbb{Z}_{12}$ is generated by the Samelson product $\langle \iota, \iota \rangle$, the generator $\tilde{\gamma}$ of $\pi_6(G)$ factors as $\tilde{\gamma} : S^6 \xrightarrow{\langle \iota, \iota \rangle} S^3 \hookrightarrow G$. The adjoint of $\tilde{\gamma}$,

$$\tilde{\gamma}^{\mathrm{Ad}} : S^7 \xrightarrow{\Sigma\langle \iota, \iota \rangle} S^4 \hookrightarrow BG,$$

is a generator of $\pi_7(BG)$.

For $G = SU(3)$, consider the exact sequence of homotopy groups induced by the fibre bundle $S^3 \to SU(3) \to S^5$,

$$\pi_6(S^3) \xrightarrow{i_1^*} \pi_6(SU(3)) \xrightarrow{p_1^*} \pi_6(S^5) \xrightarrow{\delta^*} \pi_5(S^3) \to \pi_5(SU(3)). \tag{6.1.4}$$

From Table A.1 we have $\pi_6(S^5) \cong \pi_5(S^3) \cong \mathbb{Z}_2$ and $\pi_5(SU(3)) \cong \mathbb{Z}$. This implies that the last map in (6.1.4) is the zero map. Therefore δ^* is surjective, and as δ^* is a map between two copies of $\mathbb{Z}_2$, it must be an isomorphism. In turn, p_1^* is the zero map. Thus any map $S^6 \to SU(3)$ factors as a composite

$$S^6 \to S^3 \xrightarrow{i_1} SU(3).$$

For $G = G_2$, note that localised at $p = 3$ there is an exact sequence [Mim95]

$$\pi_6(S^3) \xrightarrow{i_2^*} \pi_6(G_2) \xrightarrow{p_2^*} \pi_6(S^{11}).$$

Since $\pi_6(S^{11}) \cong 0$, the map i_2^* is surjective when localised at 3. Since $\pi_6(G_2) \cong \mathbb{Z}_3$ is invariant under localisation at 3, the map i_2^* is surjective integrally as well. Thus any map $S^6 \to G_2$ factors as a composite

$$S^6 \to S^3 \xrightarrow{i_2} G_2.$$

In the next table we collect this information on the generators of the non-trivial groups $\pi_6(G)$, that is, when $G = SU(2), SU(3)$ or G_2 (alternatively, see [Mim67]).

G	$\pi_6(G)$	generator
$SU(2) \cong S^3$	$\mathbb{Z}_{12}$	$\langle \iota, \iota \rangle$
$SU(3)$	$\mathbb{Z}_6$	$i_1 \circ \langle \iota, \iota \rangle$
G_2	$\mathbb{Z}_3$	$i_2 \circ \langle \iota, \iota \rangle$

Since $l \not\equiv 0 \pmod{12}$, we have that $p_1 \circ \Sigma\varphi \simeq l\Sigma\langle \iota, \iota \rangle$ (where $p_1 : S^4 \vee S^5 \to S^4$ is the pinch onto the first wedge summand), and this map is not nullhomotopic. Consider the diagram

$$\begin{CD} M_{l,0} @>h>> BG \\ @VqVV @AAiA \\ S^7 @>lk\Sigma\langle \iota,\iota \rangle>> S^4 \\ @V\Sigma\varphi VV @AAkA \\ S^4 \vee S^5 @>p_1>> S^4 \end{CD} \tag{6.1.5}$$

where k is the degree k map. The maps $i \circ lk\Sigma\langle \iota, \iota \rangle$ are therefore in the kernel of q^*. Thus the result follows from the exact sequence (6.1.2), the diagram and the table of the homotopy groups $\pi_6(G)$. □

Now we present a classification of principal G-bundles which generalises Proposition 5.6 (1) for the torsion-free case. The next result includes the groups G such that $o(G) > 1$: $SU(2)$, $SU(3)$ and G_2. We will therefore give a different proof to the one given for Proposition 5.6.

Proposition 6.3. *Let G be a simply connected simple compact Lie group. Then*

$$Prin_G(M_{l,0}) = \mathbb{Z} \times \mathbb{Z}_{(o(G),l)}.$$

Moreover, the projection $\pi : M_{l,0} \to S^4$ induces a bijection $\pi^* : \pi_4(BG) \to [M_{l,0}, BG]$ if $(l, o(G)) = 1$.

Proof. Recall the exact sequence given by (6.1.2),

$$[S^4 \vee S^5, BG] \xrightarrow{(\Sigma\varphi)^*} [S^7, BG] \xrightarrow{q^*} [M_{l,0}, BG] \xrightarrow{i^*} [S^3 \vee S^4, BG] \xrightarrow{\varphi^*} [S^6, BG]. \tag{6.1.6}$$

First, if $o(G) = 1$ then by Theorem 2.40, the map i^* is injective and by Lemma 6.1 it is surjective. Therefore we obtain a one-to-one correspondence between $Prin_G(M_{l,0})$ and $\pi_4(BG) \cong \mathbb{Z}$, recovering the conclusion of Proposition 5.6 in the case $m = 0$.

Now suppose $o(G) > 1$. Since $[M_{l,0}, BG]$ is not a group, more care is needed when discussing exactness: for instance, $q^* = 0$ in (6.1.6) would not necessarily imply that i^* is injective. For $j \in \mathbb{Z}$, let $\alpha_j \in \pi_4(BG)$ be the map corresponding to j under the isomorphism $\pi_4(BG) \cong \mathbb{Z}$. According to Theorem 3.2.1 in [Rut67], we can define maps $\Gamma(\alpha_j, \varphi) : [S^4 \vee S^5, BG] \to [S^7, BG]$ for each $j \in \mathbb{Z}$ such that

$$[M_{l,0}, BG] = \bigcup_{j \in \mathbb{Z}} \operatorname{coker} \Gamma(\alpha_j, \varphi).$$

Moreover, from Theorem 3.3.3 in [Rut67] we have that if φ^* is a homomorphism then

$$\Gamma(\alpha_j, \varphi) = (\Sigma\varphi)^*. \tag{6.1.7}$$

By Lemma 6.1, the map $\varphi^* : \pi_3(BG) \times \pi_4(BG) \to \pi_6(BG)$ is the zero map. Therefore, equality in (6.1.7) holds and we have

$$[M_{l,0}, BG] = \mathbb{Z} \times \operatorname{coker}(\Sigma\varphi)^*. \tag{6.1.8}$$

Using Lemma 6.2 we obtain

$$\operatorname{coker}(\Sigma\varphi)^* = \pi_6(G)/\operatorname{im}(\Sigma\varphi)^* \cong \mathbb{Z}_{(o(G),l)}.$$

Finally, suppose that $q^* = 0$: equivalently, $\operatorname{coker}(\Sigma\varphi)^* = 0$, that is, $(o(G), l) = 1$. Then $[M_{l,0}, BG] \cong \mathbb{Z}$ and $i^* : [M_{l,0}, BG] \to [S^4 \vee S^3, BG] \cong \pi_4(BG)$ is a bijection. Now the map $\pi : M_{l,0} \to S^4$ has a section so that the composite $S^4 \hookrightarrow M_{l,0} \xrightarrow{\pi} S^4$ is a homotopy equivalence. Therefore the composite

$$S^4 \hookrightarrow S^3 \vee S^4 \xrightarrow{i} M_{l,0} \xrightarrow{\pi} S^4 \tag{6.1.9}$$

is a homotopy equivalence. Thus applying the functor $[-, BG]$ to (6.1.9) shows that map

$$\pi^* : \pi_4(BG) \to [M_{l,0}, BG]$$

is a bijection. $\square$

6.2 Gauge groups: $G = SU(2), SU(3), G_2$

In this section we use a similar strategy to that of the previous chapter to give a homotopy decomposition of the gauge groups over certain manifolds $S^3 \tilde{\times}_l S^4$ when $o(G) \neq 1$.

Let $M_{l,0} = S^3 \tilde{\times}_l S^4$ be such that $(l, o(G)) = 1$. By Proposition 6.3, the projection map $\pi : M_{l,0} \to S^4$ induces a map

$$\pi^* : \mathrm{Map}^*(S^4, BG) \to \mathrm{Map}^*(M_{l,0}, BG)$$

that gives a bijection between path components of mapping spaces. Let

$$F_k \to \mathrm{Map}^*_k(S^4, BG) \xrightarrow{\pi^*_k} \mathrm{Map}^*_k(M_{l,0}, BG)$$

be the fibration sequence of the restriction of π^* to the k-th component. By an application of Lemma 5.2 to the bundle $M_{l,0} \xrightarrow{\pi} S^4$ we obtain a homotopy decomposition

$$\Sigma M_{l,0} \simeq \Sigma Y_l \vee S^5, \tag{6.2.1}$$

where the homotopy type of the space Y_l depends on the attaching map of the top cell of $M_{l,0}$. By a similar argument as the one given in the proof of Lemma 5.8 we obtain the following result.

Lemma 6.4. *Let $o(G) > 1$. Given a manifold $S^3 \tilde{\times}_l S^4$ such that $(o(G), l) = 1$, there are homotopy equivalences*

$$F_k^{l,0} \simeq \mathrm{Map}^*(Y_l, G), \quad k \in \mathbb{Z}. \qquad \square$$

Arguing as in the proof of Theorem 5.9 we obtain the following homotopy equivalence.

Theorem 6.5. *Let G be a simply connected simple compact Lie group such that $o(G) > 1$. Given a principal G-bundle $P_k \to S^3 \tilde{\times}_l S^4$, if $(l, o(G)) = 1$, then there is a homotopy equivalence*

$$\mathcal{G}^*_k(S^3 \tilde{\times}_l S^4) \simeq \Omega^4 G \times \mathrm{Map}^*(Y_l, G). \qquad \square$$

The next theorem is a result on the homotopy decomposition of gauge groups. Although the proof is similar to the one given for Theorem 5.10, we give a sketch of the proof.

Theorem 6.6. *Let $M_{l,0}$ such that $(l, o(G)) = 1$ and $o(G) > 1$. Then there is a homotopy equivalence*

$$\mathcal{G}_k(S^3 \tilde{\times}_l S^4) \simeq \mathcal{G}_k(S^4) \times \mathrm{Map}^*(Y_l, G), \qquad k \in \mathbb{Z}.$$

Proof. As in Theorem 5.10, we use the evaluation fibration along with Proposition 6.3 and Lemma 6.4 to obtain the following commutative diagram

$$\begin{array}{ccccc}
 & & \Omega\mathrm{Map}^*_k(S^3\tilde{\times}_l S^4, BG) & = & \Omega\mathrm{Map}^*_k(S^3\tilde{\times}_l S^4, BG) \\
 & & \downarrow & & \downarrow{\scriptstyle \delta^*} \\
\mathcal{G}_k(S^4) & \longrightarrow & \mathcal{G}_k(S^3\tilde{\times}_l S^4) & \xrightarrow{h} & \mathrm{Map}^*(Y_l, G) \\
\| & & \downarrow & & \downarrow{\scriptstyle g^*} \\
\mathcal{G}_k(S^4) & \longrightarrow & G & \xrightarrow{\phi_k} & \mathrm{Map}^*_k(S^4, BG) \\
 & & \downarrow{\scriptstyle \partial_k} & & \downarrow{\scriptstyle \pi^*_k} \\
 & & \mathrm{Map}^*_k(S^3\tilde{\times}_l S^4, BG) & = & \mathrm{Map}^*_k(S^3\tilde{\times}_l S^4, BG)
\end{array} \qquad (6.2.2)$$

which defines the map h. By hypothesis the map $\pi : S^3\tilde{\times}_l S^4 \to S^4$ has a cross section, so that the map g in the cofibration

$$S^3\tilde{\times}_l S^4 \xrightarrow{\pi} S^4 \xrightarrow{g} \Sigma Y_l$$

is nullhomotopic. This implies that the map $g^* : \mathrm{Map}^*_k(Y_l, G) \to \mathrm{Map}^*_k(S^4, BG)$ in (6.2.2) is also nullhomotopic. Therefore the map δ^* has a right homotopy inverse and so does h. The group structure on $\mathcal{G}_k(M_{l,0})$ allows to multiply to obtain a composite

$$\mathcal{G}_k(S^4) \times \mathrm{Map}^*(Y_l, G) \to \mathcal{G}_k(S^3\tilde{\times}_l S^4) \times \mathcal{G}_k(S^3\tilde{\times}_l S^4) \to \mathcal{G}_k(S^3\tilde{\times}_l S^4),$$

which is a homotopy equivalence. □

6.3 Classification of closed 2-connected 7-manifolds

In Chapter 4 we discussed some applications of the study of principal G-bundles over high dimensional manifolds, such as those of dimension 7. All the spaces $M_{l,0}$ are 2-connected 7-manifolds, so we wanted to extend the study of the homotopy theory of gauge groups of principal bundles over some other closed 2-connected 7-manifolds. We start by presenting some information on the classification of these manifolds.

Let M be a closed 2-connected 7-manifold. Then M is orientable, and by the Poincaré duality theorem, the non-trivial homology groups of M are given as follows

$$H_k(M) = \begin{cases} \mathbb{Z} & \text{if } k = 0, 7, \\ \mathbb{Z}^d & \text{if } k = 4, \\ \mathbb{Z}^d \oplus \bigoplus_{i=1}^t \mathbb{Z}_{r_i} & \text{if } k = 3, \end{cases} \qquad (6.3.1)$$

where $d, t, \{r_i\}_{i=1}^t$ are positive integers. Let M_1 and M_2 be closed oriented connected 7-manifolds. Recall that their connected sum $M_1 \sharp M_2$ is an oriented closed connected 7-manifold defined by deleting the interiors of 7-cells D_1 in M_1 and D_2 in M_2 and attaching the resulted punctured manifolds to each other by a homeomorphism $h : \partial D_1 \to \partial D_2$, so that

$$M_1 \sharp M_2 = (M_1 - \mathrm{Int} D_1) \cup_h (M_2 - \mathrm{Int} D_2). \tag{6.3.2}$$

Let $TH^4(M) \cong TH_3(M)$ denote the torsion subgroup of $H^4(M)$. There is a nonsingular symmetric bilinear map, called the *torsion linking form*

$$b : TH^4(M) \otimes TH^4(M) \to \mathbb{Q}/\mathbb{Z},$$

defined in [Wal67].

Consider the triple of invariants of M given by $(H^4(M), b, p_M)$, where $p_M \in 2H^4(M)$. From the definition of the invariants it follows that the invariants of the connected sum of two manifolds are the direct sum of the two sets of invariants. That is, if M_1 and M_2 have invariants $(H^4(M_1), b_1, p_{M_1})$ and $(H^4(M_2), b_2, p_{M_2})$ the connected sum $M_1 \sharp M_2$ has invariants $(H^4(M_1) \oplus H^4(M_2), b, (p_{M_1}, p_{M_2}))$ where

$$b((x_1, x_2), (y_1, y_2)) = b_1(x_1, y_1) + b_2(x_2, y_2)$$

for $x_1, y_1 \in TH^4(M_1),\ x_2, y_2 \in TH^4(M_2)$.

The proofs of Proposition 6.7, Theorem 6.8 and Theorem 6.9 are found in [Wil72].

Proposition 6.7. *The invariants $(H^4(M), b, p_M)$ for the manifolds M can assume the following values: the group $H^4(M)$ can be any finitely generated abelian group, the map $b : TH^4(M) \times TH^4(M) \to \mathbb{Q}/\mathbb{Z}$ can be any non-singular symmetric bilinear map, and $p_M \in H^4(M)$ can be any even element.* □

The linking form is called *irreducible* if it cannot be expressed as a proper sum of two maps, and we say that a manifold M with invariants $(H^4(M), b, p_M)$ is *indecomposable* if either $H^4(M)$ is finite and b is irreducible, or $H^4(M) \cong \mathbb{Z}$.

Theorem 6.8. *Any 2-connected 7-manifold M is a connected sum of indecomposable manifolds.* □

In Theorem 4 of [Wal63], Wall determines that if b is an irreducible map, then $TH^4(M)$ is one of 0, $\mathbb{Z}_{p^k}$ for a prime p, and $\mathbb{Z}_{2^k} \oplus \mathbb{Z}_{2^k}$. Hence for indecomposable manifolds M, the group $H^4(M)$ is isomorphic to 0, $\mathbb{Z}$, $\mathbb{Z}_{p^k}$ or $\mathbb{Z}_{2^k} \oplus \mathbb{Z}_{2^k}$.

Theorem 6.9. *For an indecomposable manifold M, let $G = H^4(M)$.*

(1) The indecomposable manifolds M with $G \cong \mathbb{Z}$ or $\mathbb{Z}_{p^k}$, for p an odd prime, are classified by the invariants (G, b, β).

(2) For the indecomposable manifolds M with $G \cong \mathbb{Z}_{2^k}$ or $\mathbb{Z}_{2^k} \oplus \mathbb{Z}_{2^k}$ there are two cases:

(a) If $|G| \leq 4$ then the manifolds with β not divisible by 4 are classified by the invariants (G, b, β); there are two distinct manifolds for each value of (G, b, β) if β divisible by 4.

(b) If $|G| > 4$ then the manifolds with β divisible by 4 are classified by the invariants (G, b, β); there are two distinct manifolds for each value of (G, b, β) if β is not divisible by 4. □

By [Hat02, Proposition 4C.1] we can give M a minimal cellular structure so that the 4-skeleton of M has the following homotopy type

$$M^4 = \bigvee_{i=1}^{d} (S_i^3 \vee S_i^4) \vee \bigvee_{i=1}^{t} P^4(p_i^{r_i}).$$

The cellular structure of M is given by

$$M = \left(\bigvee_{i=1}^{d} (S^3 \vee S^4) \vee \bigvee_{i=1}^{t} P^4(p_i^{r_i}) \right) \cup_\Phi e^7 \tag{6.3.3}$$

where $\Phi : S^6 \longrightarrow M^4$ is the attaching map of the top cell e^7. Thus by Theorem 6.9 any closed 2-connected 7-manifold M is a connected sum

$$M \cong M_1 \sharp M_2 \sharp \cdots \sharp M_d \tag{6.3.4}$$

where the manifold M_i, for $1 \leq i \leq d$, belongs to one of the following families of indecomposable 2-connected 7-manifolds:

(A) $M_A \simeq S^7$,

(B) $M_B \simeq (S^3 \vee S^4) \cup_\Phi e^7 \simeq S^3 \times_l S^4$,

(C) $M_C \simeq P^4(p^r) \cup_\Phi e^7$ for a prime p, and

(D) $M_D \simeq (P^4(2^r) \vee P^4(2^r)) \cup_\Phi e^7$.

Notice that the manifolds $M_{l,m}$ belong to the families (A), (B) and (C).

6.4 Principal G-bundles over connected sums

In this section we classify principal G-bundles over 2-connected 7-manifolds M which are connected sums of S^3-bundles over S^4 with torsion-free homology. Thus we can write

$$M \simeq M_{l_1} \sharp \cdots \sharp M_{l_d}$$

where $l_i = (l_i, 0) \in \mathbb{Z} \oplus \mathbb{Z}$, so that M_{l_i} is the S^3-bundle over S^4 with an attaching map $\varphi_i : S^6 \to M^4_{l_i}$. There is a homotopy cofibration sequence

$$S^6 \xrightarrow{\Phi} M^4 \xrightarrow{i} M \xrightarrow{q} S^7 \xrightarrow{\Sigma\Phi} \Sigma M^4 \xrightarrow{\Sigma i} \Sigma M \tag{6.4.1}$$

where $M^4 = \bigvee^d S_i^3 \vee \bigvee^d S_i^4$ is the 4-skeleton of M. According to [Ish81], the attaching map Φ can be expressed as

$$\Phi = \sum_{i=1}^{d} \varphi_i = \sum_{i=1}^{d} (\alpha_i + [\iota_4^i, \iota_3^i]) \tag{6.4.2}$$

where for each i, the maps $\iota_3^i \simeq \mathbb{1}_{S_i^3}$ and $\iota_4^i \simeq \mathbb{1}_{S_i^4}$ are generators of $\pi_3(S_i^3)$ and $\pi_4(S_i^4)$, respectively, and the map $\alpha_i : S^6 \to S_i^3$ is an element of $\pi_6(S_i^3)$.

Lemma 6.10. *Let M be a connected sum of sphere bundles M_{l_i}, for $1 \leq i \leq d$. Then the map*

$$\Phi^* : [M^4, BG] \to \pi_6(BG)$$

is trivial.

Proof. By connectivity we have $[M^4, BG] = [\bigvee^{d} S_i^3 \vee \bigvee^{d} S_i^4, BG] = [\bigvee^{d} S_i^4, BG]$. Therefore any map $f : M^4 \to BG$, factors as the composite

$$M^4 \xrightarrow{\text{pinch}} \bigvee^{d} S_i^4 \xrightarrow{j} BG.$$

for some $j \in [\bigvee^d S_i^4, BG]$. Thus there is a commutative diagram

$$\begin{array}{ccc} S^6 & \xrightarrow{\tilde{\Phi}} & \bigvee^{d} S_i^4 \\ {\scriptstyle\Phi}\downarrow & \nearrow{\scriptstyle \text{pinch}} & \downarrow{\scriptstyle j} \\ M^4 & \xrightarrow{f} & BG. \end{array}$$

where $\tilde{\Phi} = \text{pinch} \circ \Phi$. We have $\Phi = \sum_{i=1}^{d} \varphi_i$, and for each i, the composite

$$S^6 \xrightarrow{\varphi_i} S_i^3 \vee S_i^4 \xrightarrow{\text{pinch}} S_i^4$$

is nullhomotopic. Indeed, just notice that by (6.4.2) we have $\varphi_i = \alpha_i + [\iota_i^3, \iota_i^4]$ for some $\alpha_i \in \pi_6(S_i^3)$, and both $\text{pinch} \circ \alpha_i$ and $\text{pinch} \circ [\iota_i^3, \iota_i^4]$ are nullhomotopic. Therefore $\text{pinch} \circ \Phi$ is nullhomotopic, and given any $f \in [M^4, BG]$ we have $\Phi^*(f) = f \circ \Phi = *$. □

The manifold M has a cellular structure given by $(\bigvee_{i=1}^{d} S_i^3 \vee \bigvee_{i=1}^{d} S_i^4) \cup_\Phi e^7$. Define $\Pi : M \to \bigvee_{i=1}^{d} S_i^4$ as the composite

$$\Pi : M \xrightarrow{\text{pinch}} \bigvee^{d} M_{l_i} \xrightarrow{\bigvee^d \pi_i} \bigvee^{d} S_i^4,$$

where the maps $\pi_i : M_{l_i} \to S_i^4$ are projections.

Proposition 6.11. *If $\pi_6(G) \cong 0$ then $Prin_G(M) = \mathbb{Z}^d$. Moreover, the map*

$$\Pi^* : \bigoplus_{i=1}^{d} \pi_4(BG) \to [M, BG]$$

is a bijection.

Proof. The set $Prin_G(M)$ is in one-to-one correspondence with $[M, BG]$. Hence it suffices to compute $[M, BG]$. The cofibration sequence (6.4.1) induces an exact sequence of sets

$$[\Sigma M^4, BG] \xrightarrow{(\Sigma\Phi)^*} [S^7, BG] \xrightarrow{q^*} [M, BG] \xrightarrow{i^*} [M^4, BG] \xrightarrow{\Phi^*} [S^6, BG]. \qquad (6.4.3)$$

By Lemma 6.10 the map Φ^* is trivial, implying that i^* is onto. As $\pi_7(BG) \cong \pi_6(G) = 0$, the action of $\pi_6(G)$ on $[M, BG]$ is trivial and therefore the map i^* is injective. Hence the map i^* is a bijection and $Prin_G(M) = [M^4, BG] = [\bigvee^d S_i^4, BG] = \mathbb{Z}^d$.

Now consider the composite

$$\bigvee^{d} S_i^4 \hookrightarrow \bigvee^{d} S_i^3 \vee \bigvee^{d} S_i^4 \xrightarrow{\simeq} M^4 \xrightarrow{i} M \xrightarrow{\Pi} \bigvee^{d} S_i^4, \qquad (6.4.4)$$

and consider the diagram

$$\begin{array}{ccc} \bigvee^d S_i^4 & = & \bigvee^d S_i^4 \\ \big\downarrow & & \big\uparrow {\scriptstyle \bigvee^d \pi_i} \\ M^4 & \hookrightarrow & \bigvee^d M_{l_i} \\ \big\downarrow {\scriptstyle i} & & \big\uparrow {\scriptstyle \text{pinch}} \\ M & = & M. \end{array} \qquad (6.4.5)$$

The bottom square homotopy commutes because of the cellular structure of M, and the top square homotopy commutes since the composite

$$S_i^4 \hookrightarrow S_i^4 \vee S_i^3 \xrightarrow{\simeq} M_{l_i}^4 \hookrightarrow M_{l_i} \xrightarrow{\pi_i} S^4$$

is homotopic to the identity map.

In particular, commutativity of (6.4.5) implies that the composite (6.4.4) is a homotopy equivalence. Thus applying the functor $[-, BG]$ to the composite (6.4.4) we obtain a composite

$$\bigoplus_{i=1}^{d}\pi_4(BG)\xrightarrow{\Pi^*}[M,BG]\xrightarrow{i^*}[M^4,BG]\xrightarrow{\cong}\bigoplus_{i=1}^{d}[S_i^4\vee S_i^3,BG]\to\bigoplus_{i=1}^{d}\pi_4(BG)$$

that is an isomorphism, and since the last two maps are isomorphisms and i^* is a bijection, it follows that the map $\Pi^* : [\bigvee^d S^4, BG] \to [M, BG]$ is also a bijection. □

6.5 Gauge groups of principal G-bundles over connected sums

Let M be a manifold satisfying the condition of Proposition 6.11. Then there is a one-to-one correspondence between the sets $Prin_G(M)$ and $[M, BG] = \mathbb{Z}^d$, where $2 \leq d \in \mathbb{N}$. Thus every principal G-bundle over M is classified by an element $K = (k_1, \ldots, k_d) \in \mathbb{Z}^d$. Let P_K denote the prinicipal G-bundle classified by K. Let

$$\mathrm{Map}^*(\bigvee^d S_i^4, BG) \longrightarrow \mathrm{Map}(\bigvee^d S_i^4, BG) \xrightarrow{ev} BG. \tag{6.5.1}$$

be the evaluation fibration associated with the mapping space $\mathrm{Map}(\bigvee^d S_i^4, BG)$. The restriction of the evaluation map to the component $\mathrm{Map}_K(\bigvee^d S_i^4, BG)$, where $K \in \mathbb{Z}^d$, defines the fibration

$$G \xrightarrow{\phi_K} \mathrm{Map}_K^*(\bigvee^d S_i^4, BG) \longrightarrow \mathrm{Map}_K(\bigvee^d S_i^4, BG) \xrightarrow{ev_K} BG. \tag{6.5.2}$$

Denote by $\mathcal{G}_K(\bigvee^d S_i^4)$ the homotopy fibre of the connecting map ϕ_K of the fibration sequence (6.5.2).

Theorem 6.12. *Let $P_K \to M$ be a principal G-bundle over $M \simeq M_{l_1}\sharp\cdots\sharp M_{l_d}$ classified by $K = (k_1, \ldots, k_d) \in \mathbb{Z}^d$. Suppose G is a simply connected simple compact Lie group such that $\pi_6(G) = 0$. Then there exists a homotopy decomposition*

$$\mathcal{G}_K(M) \simeq \mathcal{G}_K(\bigvee^d S_i^4) \times \mathrm{Map}^*(C_\Pi, BG)$$

where C_Π is the homotopy cofibre of the map $\Pi : M \to \bigvee^d S^4$.

Proof. Let $\Phi : S^6 \to M^4$ be the attaching map of the top cell. The map Π defines a homotopy cofibration sequence

$$M \xrightarrow{\Pi} \bigvee_{i=1}^{d} S^4 \xrightarrow{g} C_\Pi \xrightarrow{\delta} \Sigma M \xrightarrow{\Sigma\Pi} \bigvee_{i=1}^{d} S^5 \tag{6.5.3}$$

with C_Π the homotopy cofibre of Π. Consider the evaluation fibration

$$\mathrm{Map}_K^*(M, BG) \to \mathrm{Map}_K(M, BG) \xrightarrow{ev_K} BG, \tag{6.5.4}$$

The evaluation map is natural, and by Proposition 6.3, the map $\Pi : M \to \bigvee_{i=1}^d S^4$ makes the homotopy fibration diagram

$$\begin{array}{ccccccc}
G & \xrightarrow{\phi_K} & \mathrm{Map}_K^*(\bigvee^d S^4, BG) & \longrightarrow & \mathrm{Map}_K(\bigvee^d S^4, BG) & \xrightarrow{ev_K} & BG \\
\| & & \downarrow{\Pi^*} & & \downarrow{\Pi^*} & & \| \\
G & \xrightarrow{\partial_K} & \mathrm{Map}_K^*(M, BG) & \longrightarrow & \mathrm{Map}_K(M, BG) & \xrightarrow{ev_K} & BG
\end{array}$$

commute. There is a homotopy commutative diagram

$$\begin{array}{ccccc}
 & & \mathrm{Map}_K^*(\Sigma M, BG) & = & \mathrm{Map}_K^*(\Sigma M, BG) \\
 & & \downarrow & & \downarrow{\delta^*} \\
\mathcal{G}_K(\bigvee^d S^4) & \longrightarrow & \mathcal{G}_K(M) & \xrightarrow{h} & \mathrm{Map}^*(C_\Pi, BG) \\
\| & & \downarrow & & \downarrow{g^*} \\
\mathcal{G}_K(\bigvee^d S^4) & \longrightarrow & G & \xrightarrow{\phi_K} & \mathrm{Map}_K^*(\bigvee^d S^4, BG) \\
 & & \downarrow{\Pi^*} & & \downarrow{\Pi^*} \\
 & & \mathrm{Map}_K^*(M, BG) & = & \mathrm{Map}_K^*(M, BG)
\end{array}$$

which defines the map h. The map g is nullhomotopic, hence the induced map g^* is also nullhomotopic. Therefore the map δ^* has a right homotopy inverse, which implies that h has a homotopy inverse. The group structure on $\mathcal{G}_k(M)$ allows us to multiply to obtain a homotopy equivalence

$$\mathcal{G}_k(\bigvee^d S^4) \times \mathrm{Map}^*(C_\Pi, BG) \to \mathcal{G}_k(M) \times \mathcal{G}_k(M) \to \mathcal{G}_k(M)$$

as required. □

Theorem 6.12 shows that the homotopy types of gauge groups over connected sums $\mathcal{G}_K(M)$ depend on the homotopy types of the mapping spaces

$$\mathcal{G}_K(\bigvee^d S_i^4) \simeq \Omega \mathrm{Map}_K(\bigvee^d S_i^4, BG).$$

We will try to obtain some information on the homotopy type of these mapping spaces. Let $K = (k_1, \ldots, k_d) \in \mathbb{Z}^d$. Consider the following fibration

$$\mathcal{G}_K\left(\bigvee_{i=1}^d S_i^4\right) \longrightarrow G \xrightarrow{\partial_K} \mathrm{Map}_K^*(\bigvee_{i=1}^d S_i^4, BG) \longrightarrow \mathrm{Map}_K(\bigvee_{i=1}^d S_i^4, BG) \xrightarrow{ev_K} BG \tag{6.5.5}$$

which by the pointed exponential law and Proposition 4.12 can be rewritten as

$$\mathcal{G}_K\left(\bigvee_{i=1}^d S_i^4\right) \longrightarrow G \xrightarrow{\partial_K} \mathrm{Map}_K^*(\bigvee_{i=1}^d S_i^3, G). \tag{6.5.6}$$

The space $\mathcal{G}_K(\bigvee^d S_i^4)$ is the homotopy fibre of ∂_K so that the properties of ∂_K determine the homotopy types of the fibre. Taking the adjoint of ∂_K we obtain

$$\begin{array}{ccc} G \wedge \left(\bigvee_{i=1}^d S_i^3\right) & \xrightarrow{\partial^K} & G \\ \Vert & & \Vert \\ \bigvee_{i=1}^d \left(G \wedge S_i^3\right) & \xrightarrow{\bigvee_{i=1}^d \partial^{k_i}} & G \end{array} \tag{6.5.7}$$

where

$$\partial^K \in \left[\bigvee_{i=1}^d \left(G \wedge S_i^3\right), G\right] \cong \bigoplus_{i=1}^d [G \wedge S_i^3, G] \tag{6.5.8}$$

and

$$\partial^{k_i} \simeq \langle \mathbb{1}_G, k_i\gamma_i\rangle \in [G \wedge S_i^3, G]; \tag{6.5.9}$$

here γ_i is a generator of $[S_i^3, G] \cong \mathbb{Z}$. Let $\mathcal{H}$ denote the set of distinct homotopy types of the spaces $\mathcal{G}_K(\bigvee^d S_i^4)$, for $K \in \mathbb{Z}^d$.

Proposition 6.13. *The order of $\mathcal{H}$ is bounded by*

$$|\mathcal{H}| \leq |\langle \mathbb{1}_G, \gamma\rangle|^d$$

where γ is a generator of $\pi^3(G)$. □

Table of homotopy groups

X	$\pi_1(X)$	$\pi_2(X)$	$\pi_3(X)$	$\pi_4(X)$	$\pi_5(X)$	$\pi_6(X)$	$\pi_7(X)$
$Sp(1)$	0	0	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_{12}$	$\mathbb{Z}_2$
$Sp(n), n \geq 2$	0	0	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	0	$\mathbb{Z}$
$SU(3)$	0	0	$\mathbb{Z}$	0	$\mathbb{Z}$	$\mathbb{Z}_6$	0
$SU(n), n \geq 4$	0	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$
$Spin(7)$	0	0	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$
$Spin(8)$	0	0	$\mathbb{Z}$	0	0	0	$\mathbb{Z} \oplus \mathbb{Z}$
$Spin(n), n \geq 9$	0	0	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$
$SO(3)$	$\mathbb{Z}_2$	0	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_{12}$	$\mathbb{Z}_2$
$SO(5)$	$\mathbb{Z}_2$	0	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	0	$\mathbb{Z}$
$SO(6)$	$\mathbb{Z}_2$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$
$SO(7)$	$\mathbb{Z}_2$	0	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$
$SO(8)$	$\mathbb{Z}_2$	0	$\mathbb{Z}$	0	0	0	$\mathbb{Z} \oplus \mathbb{Z}$
$SO(n), n \geq 9$	$\mathbb{Z}_2$	0	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$
G_2	0	0	$\mathbb{Z}$	0	0	$\mathbb{Z}_3$	0
F_4, E_6, E_7, E_8	0	0	$\mathbb{Z}$	0	0	0	0
S^2	0	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_{12}$	$\mathbb{Z}_2$
S^3	0	0	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_{12}$	$\mathbb{Z}_2$
S^4	0	0	0	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z} \oplus \mathbb{Z}_{12}$
S^5	0	0	0	0	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$

Table A.1: Homotopy groups of connected compact simple Lie groups [Jam95] and spheres S^n for $2 \leq n \leq 5$ [Tod63]. Notice that there are isomorphisms $Sp(1) \cong SU(2) \cong Spin(3)$, $Sp(2) \cong Spin(5)$ and $SU(4) \cong Spin(6)$

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