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A MODE SHAPE BASED APPROXIMATION FOR PARAMETER-DEPENDENT EIGENVALUE PROBLEMS ARISING IN STRUCTURAL VIBRATION

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Eigenvalue problems that depend on a parameter are frequently encountered in the structural design under dynamic loading. A designer inevitably has to consider several design alternatives, often in an automated way, via running computer programmes looped over a large number of design choices. Significant variations in geometrical and physical properties from one design to another are inevitable. A need to perform a large number of similar operations, to calculate natural frequencies for each case using finite element approach, thus arises. In design scenarios, it is computationally inefficient to carry out reanalysis for each of the design alternatives and approximations are frequently sought. In the present work, approximations for the frequency of vibration are proposed based on the interpolation of modes. The method is applied to practical problems arising in structural vibration. Approximate results compare well with the exact ones while providing considerable computational economy. The computational gain is found to be relatively more significant as the size of the problem increases. The computational complexity of the proposed algorithm is assessed.

Keywords: eigenvalue problem, reanalysis, numerical approximation

1. Introduction

We often come across eigenvalue problems associated with a matrix of large size when calculations have to be repeated for values of entries in a matrix that depend on a parameter. Such problems frequently appear while solving systems of differential equations. The physical contexts could range from computing frequency response of circuits to response of buildings due to earthquakes and computing energy levels of a quantum mechanical problem. An eigenvalue problem that depends on a parameter is also encountered in the structural design under dynamic loading. A special type of analysis — that of free vibration — leads to generalised eigenvalue problems. A designer often has to consider several design alternatives that are similar in many respects except that certain geometric parameters need changing. Then the appropriate calculation for every model is carried out. The need to perform a large number of similar operations to calculate natural frequencies for each case using finite element approach thus arises. The aim of the present research is to develop a computationally economical approximation of eigenvalues that depend on a parameter for a practical engineering problems.

The first general results for a real parameter–dependent eigenvalue problem were presented by Deif [1] and Rohn et al. [2]. At first, Deif [1] used eigenvalue inequalities and programming theorems to derive the formula for the bounds of the eigenvalues prediction. Consider a real symmetric parameter–dependent standard eigenvalue problem

$$\mathbf{A}(p)\mathbf{x}(p) = \lambda(p)\mathbf{x}(p), \quad p_0 \leq p \leq p_f, \tag{1}$$

where λ is an eigenvalue, $\mathbf{x}$ is an eigenvector and p is a parameter which describes a system. The mean value and maximum width of a real symmetric matrix $\mathbf{A}(p)$ are given by

$$\mathbf{A}_c = \frac{\mathbf{A}(p = p_f) + \mathbf{A}(p = p_0)}{2}, \qquad \mathbf{A}_\Delta = \frac{\mathbf{A}(p = p_f) - \mathbf{A}(p = p_0)}{2}. \tag{2}$$

Deif [1] derived the formula for predicting the interval range for eigenvalues. The upper and lower bounds of eigenvalues were obtained as

$$\lambda_i(p) = [\lambda(\mathbf{A}_c - \mathbf{S}\mathbf{A}_\Delta\mathbf{S}),\ \lambda(\mathbf{A}_c + \mathbf{S}A_\Delta\mathbf{S})], \quad i = 1, \dots n, \tag{3}$$

where $\mathbf{S}$ is a signature matrix which is defined in terms of positive and negative entries of the diagonalised matrix $s_{ij} = \text{sgn}(x_1^i), \dots, \text{sgn}(x_n^i)$, $i = 1, \dots, n$ of each component of eigenvector $\mathbf{x}_i$. Then, Deif and Rohn presented a method [2] for computing the set of real eigenvalues of a parameter–dependent matrix in respect to eigenvectors of a given sign pattern. Behnke [3] presented another algorithm for finding the bounds of eigenvalues. He suggested to find minimal and maximal number of Temple quotient on each step. Qiu et al. [4], Sofi et al. [5] considered a parameter–dependent generalised eigenvalue problem

$$\mathbf{K}(p)\mathbf{u}(p) = \lambda(p)\mathbf{M}(p)\mathbf{u}(p), \quad p_0 \le p \le p_f, \tag{4}$$

where $\mathbf{K}(p)$ and $\mathbf{M}(p)$ are stiffness and mass matrices respectively, λ is an eigenvalue, $\mathbf{u}$ is an eigenvector and p is a parameter which describes a system. In order to find the bounds of eigenvalues of a parameter–dependent generalised eigenvalue problem Qui et al. [4] and Sofi et al. [5] solved the following two eigenproblems

$$\mathbf{K}_0\underline{\mathbf{u}} = \underline{\lambda}\mathbf{M}_f\underline{\mathbf{u}}, \qquad \mathbf{K}_f\overline{\mathbf{u}} = \overline{\lambda}\mathbf{M}_0\overline{\mathbf{u}}, \tag{5}$$

where $\mathbf{K}_0 = \mathbf{K}(p = p_0)$, $\mathbf{M}_0 = \mathbf{M}(p = p_0)$ and $\mathbf{K}_f = \mathbf{K}(p = p_f)$, $\mathbf{M}_f = \mathbf{M}(p = p_f)$, $\underline{\lambda}$ and $\overline{\lambda}$ are the estimated lower and upper bounds of the eigenvalue, $\underline{\mathbf{u}}$ and $\overline{\mathbf{u}}$ are eigenvectors associated with the above problem having no particular significance. A huge contribution to this research area was made by Hladik et al. [6–10]. Based on some of their works, they developed several filtering algorithms for finding the bounds for eigenvalues [9, 10]. Their algorithm filters the interval from above and below making it tighter. All research articles of this topic are focused on calculation a range for an eigenvalue. Here we present an approach for approximately predicting eigenvalues of an eigenproblem economically.

2. Approximations using mode shape interpolation

Consider a real symmetric positive–definite parameter–dependent generalised eigenvalue problem (4). The essence of the interpolated modes method [11] is to use two eigensolutions that are calculated exactly, followed by relatively inexpensive calculations at a large number of intermediate steps in order to approximately and economically obtain eigenvalues. The two eigenvalue problems which correspond to initial and final values of the parameter p of the system need to be exactly solved first. At the initial point of the interval i.e. at $p = p_0$, and the final point $p = p_f$, we have respectively

$$\mathbf{K}_0\mathbf{u}_0 = \lambda_0\mathbf{M}_0\mathbf{u}_0 \quad \text{and} \quad \mathbf{K}_f\mathbf{u}_f = \lambda_f\mathbf{M}_f\mathbf{u}_f. \tag{6}$$

The approximate eigenvalues are obtained by using Rayleigh quotient approximation which employs the *exact* stiffness and mass matrices at p, i.e. $\mathbf{K}(p)$ and $\mathbf{M}(p)$ respectively, but interpolated mode $\mathbf{u}^*$ [11] as the approximate eigenvector such that

$$\lambda^*(p) = \frac{\mathbf{u}^{*T}\mathbf{K}(p)\mathbf{u}^*}{\mathbf{u}^{*T}\mathbf{M}(p)\mathbf{u}^*}, \quad p_0 \le p \le p_f, \tag{7}$$

where $\mathbf{u}^*$ is an interpolated vector taken as the approximate eigenvector at p and is given by

$$\mathbf{u}^* = \frac{(p_f - p)\mathbf{u}_0 + (p - p_0)\mathbf{u}_f}{p_f - p_0}, \quad p_0 \leq p \leq p_f. \tag{8}$$

Numerical implementation of this algorithm carried out by us demonstrated that this approach does not always give a good approximation for some eigenvalues. Here we have identified this problem and successfully resolved it. Since an eigenvector remains an eigenvector when multiplied by any arbitrary scalar, one has no control on its magnitude and sense. To ensure that the sense of the reference eigenvectors chosen at the ends of the interval is such that the angle between them is acute, Eq. (8) needs to be modified accounting for arbitrariness in the sense of the computed exact eigenvectors $\mathbf{u}_0$ and $\mathbf{u}_f$. Before approximating the eigenvalues, the sense of eigenvectors $\mathbf{u}_0$ and $\mathbf{u}_f$ is investigated by calculating the cosine between them

$$\cos(\alpha) = \frac{(\mathbf{u}_0^T \mathbf{u}_f)}{\|\mathbf{u}_0\| \, \|\mathbf{u}_f\|}. \tag{9}$$

The calculation of the denominator in (9) is not necessary and only dot product is calculated to determine the sign of the $\cos(\alpha)$ to ensure that acute angle between $\mathbf{u}_0$ and $\mathbf{u}_f$ needs to be taken before interpolation can take place.

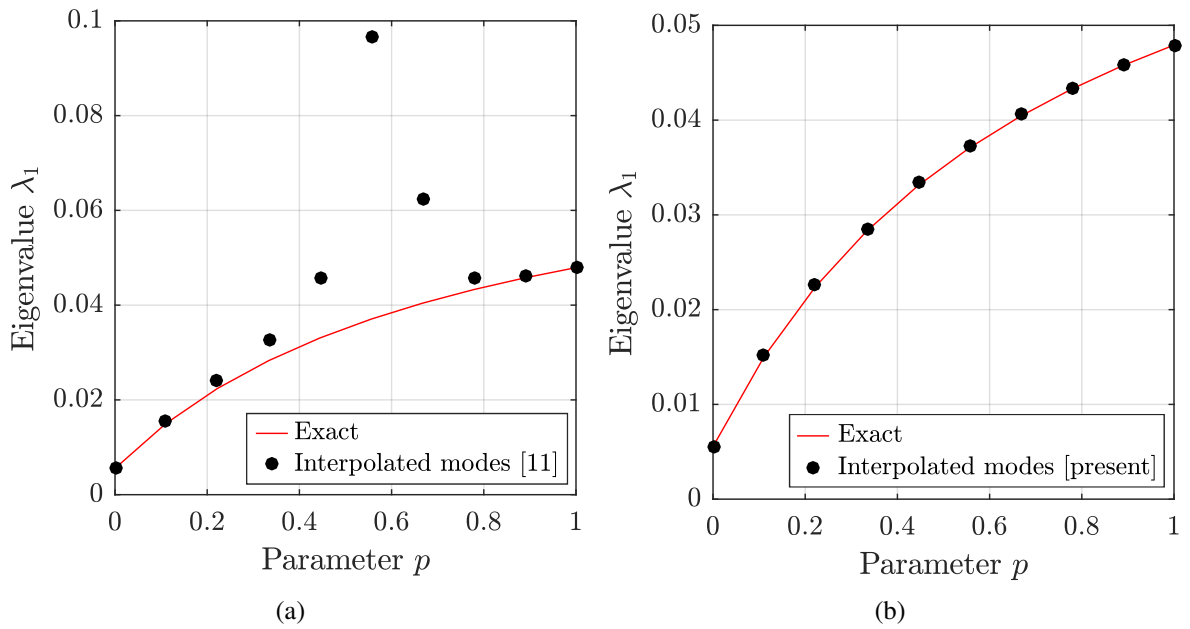

Figure 1: The first eigenvalue as a function of parameter p of a generalised eigenvalue problem with random symmetric parameter–dependent matrices of size 200×200 computed exactly (red line) and by the interpolated modes approach (black dots).

Therefore, the interpolated vector formula (8) is now modified as

$$\mathbf{u}^* = \frac{(p_f - p)\mathbf{u}_0 \pm (p - p_0)\mathbf{u}_f}{p_f - p_0}, \quad p_0 \leq p \leq p_f, \tag{10}$$

where the sign depends on the directions of eigenvectors $\mathbf{u}_0$ and $\mathbf{u}_f$. When the product of eigenvectors $\mathbf{u}_0^T \mathbf{u}_f > 0$, the positive sign in (10) should be used. Otherwise when, $\mathbf{u}_0^T \mathbf{u}_f < 0$ the negative sign in (10) needs to be used. Considering (10) and in conjunction with consideration to the correct sign, results in Fig. 1(a) are transformed to those Fig. 1(b), which shows significantly improved accuracy.

It should be noted that the method presented here is inspired by the Rayleigh quotient. The interpolated mode contains components of two eigenvectors calculated exactly at the ends of the

interval. Rayleigh quotient based approximation presented here is expected to be more accurate than one that uses trial vectors at one end of the interval or the other. Moreover, the additional computational expense in calculating the interpolated mode is minimal because it involves linearly combining two vectors in prescribed proportions.

If the parameter p of the system change, the eigenvalues along all parametric range are approximated by Rayleigh quotients based on reference trial vectors at the two ends of parameter range

$$\tilde{\lambda}(p) \approx \frac{\mathbf{u}_0^T \mathbf{K}(p) \mathbf{u}_0}{\mathbf{u}_0^T \mathbf{M}(p) \mathbf{u}_0}, \qquad \tilde{\tilde{\lambda}}(p) \approx \frac{\mathbf{u}_f^T \mathbf{K}(p) \mathbf{u}_f}{\mathbf{u}_f^T \mathbf{M}(p) \mathbf{u}_f}, \quad p_0 \leq p \leq p_f, \tag{11}$$

using the exact eigenvectors from both eigenproblems (6). The predicted eigenvalues of these two approximations are good only when they are at the ends of the interval. It means that each formula gives a satisfying result closer to the reference mode, but progressively becomes inaccurate for increased perturbation. The method presented here is applied to two problems to demonstrate its efficiency next.

3. Approximations for a parameter dependent generalised eigenvalue problem

The approximation proposed above is tested on algebraic problem first to test the efficiency of the scheme. Consider a real symmetric parameter–dependent positive–definite eigenvalue problem in terms of two matrices $\mathbf{K}(p)$ and $\mathbf{M}(p)$. Four random real symmetric matrices $\mathbf{K}_0$, $\mathbf{K}_f$, $\mathbf{M}_0$, $\mathbf{M}_f$ of size 2000×2000 were generated for the values of a parameter within the bounds of the parametric range. We assume that entries of a matrix vary linearly with the parameter p such that

$$\mathbf{K}(p) = \mathbf{K}_0 + p\frac{\mathbf{K}_f - \mathbf{K}_0}{n-1} \quad \text{and} \quad \mathbf{M}(p) = \mathbf{M}_0 + p\frac{\mathbf{M}_f - \mathbf{M}_0}{n-1}, \quad 0 \leq p \leq 1. \tag{12}$$

Here, $n = 10$ is the number of subdivisions of a parametric range. All 2000 eigenvalues are approximated by the interpolated modes approach (7) using (10) and compared with exact and Rayleigh quotient based results that make use of modes at the ends of the interval where exact calculations are carried out (11). The first and fifth eigenvalues are shown in Fig. 2 for illustration.

The exact results are marked with red thick line in Fig. 2. Interpolated modes method in black dots gives better approximation than any of Rayleigh quotient based approximations (thin lines in Fig. 2 as labelled) that do not use interpolated modes but those at either ends of the interval. The comparison illustrates the strength of making use of interpolated modes in Rayleigh quotient approximations.

The algorithm presented for predicting all eigenvalues for parameter–dependent generalised eigenvalue problem was run 10 times for each eigenproblem of different size. This is to study the trend of how the computational time depends on the size of the problem. Based on results obtained, the standard deviation and mean of computational time spent on solving the problems exactly and approximately are presented in Fig. 3.

The method presented here shows a significant computational economy especially when the problem involves large matrices. The computational time of exact approach is illustrated by a steep red line in Fig. 3. The difference between the time spent on exact and approximate calculations increases with the size of the input matrices in a generalised eigenvalue problem. The time spent on exact calculations is always greater than the time spent on approximating the eigenvalues which is shown by black dashed line in Fig. 3. The higher slope further demonstrates the advantage of the proposed approximation for large matrix problems. It clearly illustrates that using the interpolated modes method is quicker than solving eigenproblems exactly.

A measure of the scaling of computational time on the size of the problem is given by the complexity of the algorithm. In the interpolated modes method the eigenvalue problems are calculated

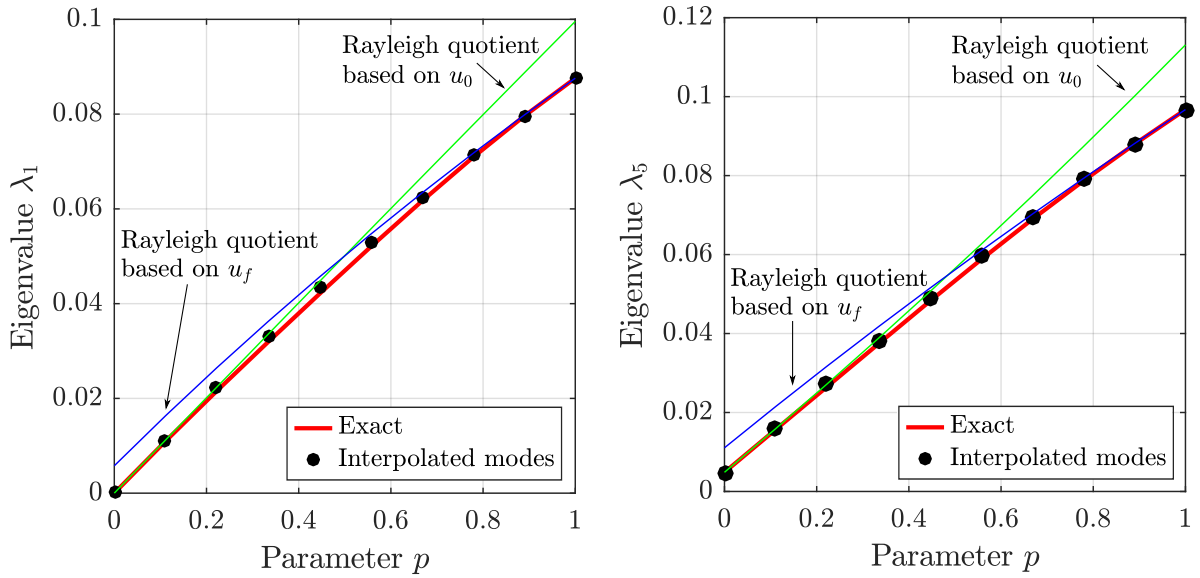

Figure 2: The dependence of the first and fifth eigenvalues upon the parameter p in generalised eigenvalue problem with two random symmetric positive–definite parameter–dependent matrices of size 2000×2000 computed exactly (red line), by an interpolated modes approach (black dots) and reference fixed mode based Rayleigh quotient.

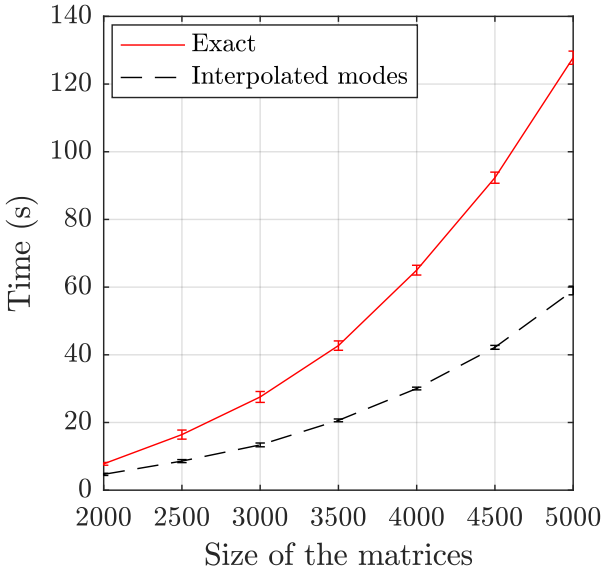

Figure 3: Average of 10 and standard deviation of computational time in a generalised eigenvalue problem considering the size of the randomly generated symmetric matrices, where with red line noted mean time for exact calculations of computing all eigenvalues and black dashed line represents mean time of predicting all eigenvalues by interpolated modes method.

exactly only at two end points of the interval and eigenvalues are predicted for other parameter steps. In contrast, where exact calculations are held in every parametric case.

We consider the ansatz that the computational time is proportional to the size of the matrix raised to some unknown power. To find this exponent and illustrate the difference in complexities of approaches, both axes in Fig. 3 are changed to logarithmic. Taking the coordinates of the red curve ends from, the slope is estimated as the exponent in the unknown power law scaling. This analysis showed that the complexity of the exact approach is nearly $O(n^{2.99})$, whereas the complexity of interpolated modes approach is $O(n^{2.71})$ highlighting the strength of the method for large scale problems. The

received results are close to the accepted computational complexity of order n^3 for an eigenvalue problem [12].

4. Ring vibration problem

The approximation presented in Section 2 and implemented on generalised eigenvalue problems is now applied to a practical natural vibration calculation problem. Rings are commonly found in many engineering applications. For example, many expandable biomedical implants mimic rings closely. A parametric CAD model is convenient to study in free vibration response of a family of rings with changing geometrical characteristics. A parametric finite element model of a ring is shown in

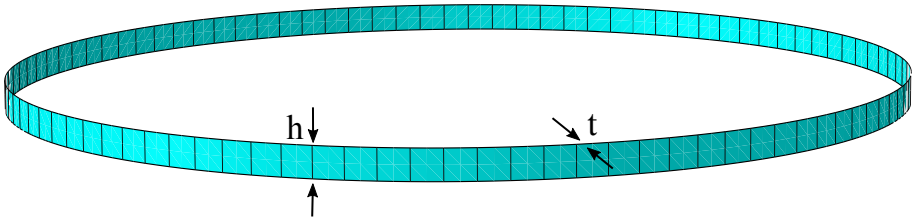

Figure 4: A finite element model of a ring using shell elements.

Fig. 4. The parameters and material properties of the model are presented in Table 1. The geometric parameters of our interest are the thickness t of the ring which plays the role of the parameter p within (4).

Table 1: Parameters of a ring model.

Parameter	Value
Diameter d	1.016 m
Height h	2.54×10^{-2} m
Thickness t	2.54×10^{-3} m$\leq t \leq 4.9 \times 10^{-3}$ m
Number of designs	10
Material	Aluminium
Mass Density ρ	2.712×10^{3} kg/m^3
Young's modulus E	6.826×10^{10} Pa
Poisson's ratio ν	0.334
Element type	Shell S8R
Number of elements	100
Number of nodes	500
Degree of freedom per node	6
Total degree of freedom	3000
Boundary conditions	Free
Software	Abaqus 6.14

Abaqus software [13] is used to import [14] the stiffness and mass matrices from the model for processing in order to implement the interpolated modes method presented here. Shell elements (S8R, an 8–node doubly curved thick shell, reduced integration) are selected.

The exact results (thick red line in Fig. 5) match closely with the approximated results obtained using the interpolated modes approach (black dots, Fig. 5). In the same figure, approximations based on Rayleigh quotient that employ fixed fixed modes at the ends of the interval are presented. The approach of interpolated modes shows a clear superiority.

In this example, stiffness matrix of a ring model with a height h and a thickness t is $\mathbf{K} \sim ht^3$ and mass matrix $\mathbf{M} \sim ht$ for in–plane bending modes. Hence, natural frequency of the ring depends on

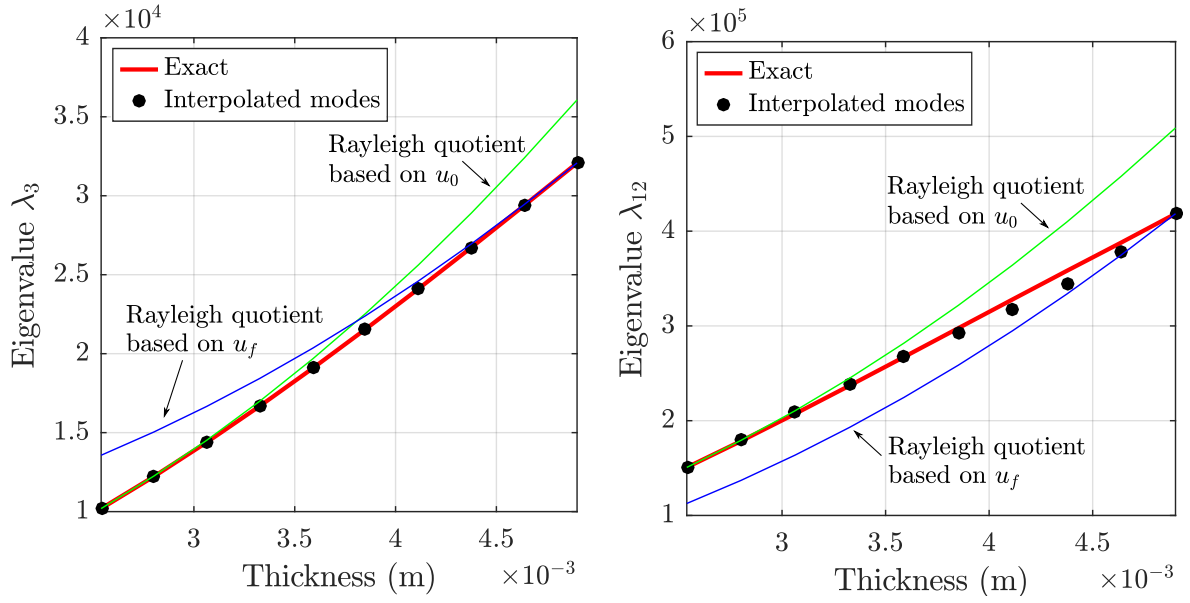

Figure 5: The dependence of the third and the twelfth eigenvalues upon the thickness for the free vibration problem of a ring by exact approach (red line), interpolated modes approach (black dots) and reference fixed mode based Rayleigh quotient.

its thickness t

$$\omega \sim \sqrt{\frac{ht^3}{ht}} \sim t. \tag{13}$$

This $\sim t$ dence is consistent with the calculations shown in Fig. 5.

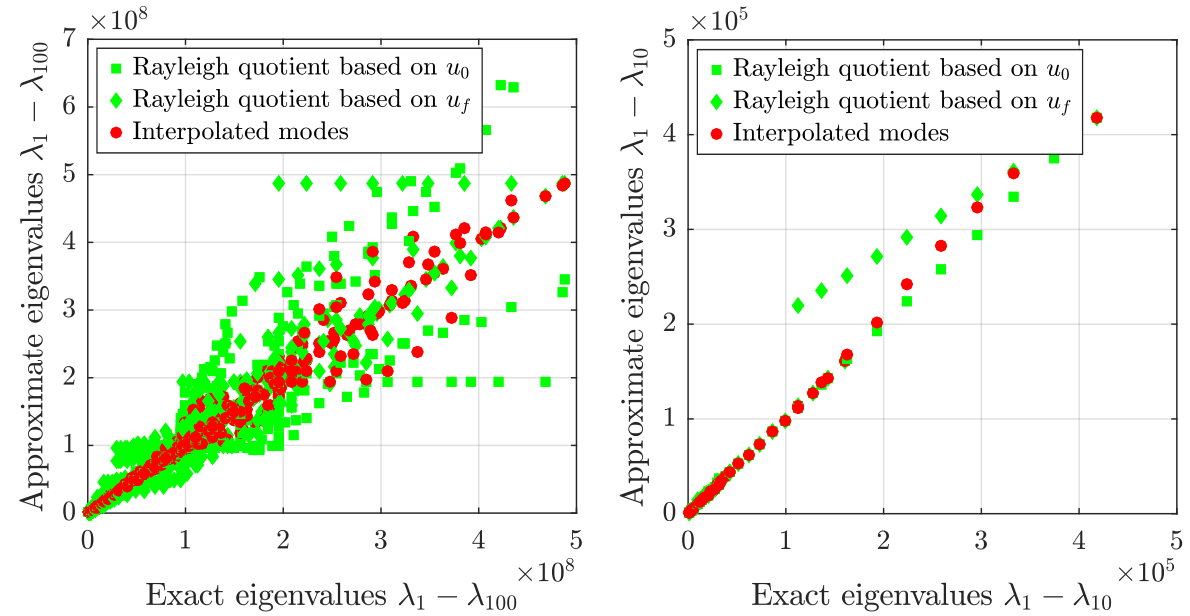

Figure 6: Exact eigenvalues vs approximate out of vibrating ring problem, where red dots represent eigenvalues calculated by interpolated modes method, green squares and diamonds represent reference fixed mode based Rayleigh quotient.

The exact answers are plotted against two different approximations in Fig. 6. The x coordinate of a point on a graph is an "exact" eigenvalue and y coordinate is approximate eigenvalue calculated by interpolated modes method and fixed mode based Rayleigh quotient. So, the closer the points are to

the diagonal of the graph, the more accurate is an approximation of a method. The first 100 as well as first 10 eigenvalues are presented in Fig. 6. It can be seen that the interpolated modes method gives better accuracy of eigenvalues than Rayleigh quotient based on $\mathbf{u}_0$ or $\mathbf{u}_f$ as almost all red dots which correspond to the interpolated modes method in Fig. 6 are aligned to the diagonal more closely. Also, the most time in the algorithm was spent on exact calculations 31.95 s, whereas to approximate all the eigenvalues took 13.64 s. So, it took two times quicker to predict the eigenvalues of the vibrating ring problem than to calculate them 10 times exactly.

5. Conclusions

An algorithm for approximating the eigenvalues from real symmetric positive–definite parameter–dependent generalised eigenvalue problem was presented. The formula was modified and now works for cases with different angles between eigenvectors. The algorithm was tested on real symmetric positive–definite randomly generated matrices first. Further the effectiveness of the proposed method was demonstrated for the ring vibration problem. The approximate method presented here shows excellent accuracy and with significantly less computational time to approximate the eigenvalues. Finally, we illustrated that the interpolated modes method gives better accuracy than Rayleigh quotient based modes calculated at fixed values of the parameter.

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