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An eigenvalue approximation for parameter–dependent undamped gyroscopic systems

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Abstract. Parameter-dependent eigenvalue problem occurs in a host of engineering contexts. Structural design under dynamic loading is concerned with the evaluation of eigenvalues for a large number of structures, each evaluation corresponding to a combination of design parameters. Exact calculation of the natural frequencies of all the models considered is computationally expensive. Here structural problems that possess gyroscopy, typically encountered in the analysis of rotating elastic structures, are considered. Approximate but inexpensive calculations are sought for this class of problems. In the present work, an algorithm for approximating the natural frequencies of undamped gyroscopic systems is presented. Numerical examples show excellent accuracy, while affording significant computational economy compared to exact calculations. The computational gain is found to be relatively more notable when the size of the considered problem is large.

1. Introduction

Eigenvalue problems in terms of a skew-symmetric matrix arise in the dynamics of different kinds of rotating structures like spinning disks, hard drive, bend saw, helicopter blades, vacuum cleaner, washing machine, etc. They all possess some common mathematical features that are generic to this class of problems. The governing equation of motion involves two real and symmetric matrices associated with the inertia and stiffness of the structure whereas gyroscopy, arising from the linear term within the kinetic energy, leads to a skew-symmetric matrix. In a design scenario, several design alternatives need to be analysed. Solving several eigenvalue problems leads to a large number of evaluation, which is computationally expensive. Here we propose approximation for a parameter-dependent eigenvalue problem, accurately and computationally economically.

The equation of free motion for undamped gyroscopic system is expressed

$$\mathbf{M}\ddot{\mathbf{q}}(t) + \mathbf{G}\dot{\mathbf{q}}(t) + \mathbf{K}\mathbf{q}(t) = \mathbf{0}, \quad (1)$$

where \mathbf{K} and \mathbf{M} are real symmetric stiffness and mass matrices of size $n \times n$ respectively, \mathbf{G} is $n \times n$ skew-symmetric gyroscopic matrix. Seeking a synchronous solution for (1) leads to a λ -matrix problem. Instead, (1) can be cast in the so-called state-space of double the size of the configuration space. Free vibration problem associated with the first order state-space equation is then given by the following generalised eigenvalue problem

$$\mathbf{G}^* \mathbf{u} = -\lambda \mathbf{M}^* \mathbf{u}, \quad (2)$$



where $\mathbf{G}^* = \begin{bmatrix} \mathbf{G} & \mathbf{K} \\ -\mathbf{K} & \mathbf{0} \end{bmatrix}$ is $2n \times 2n$ skew-symmetric matrix, $\mathbf{M}^* = \begin{bmatrix} \mathbf{M} & \mathbf{0} \\ \mathbf{0} & \mathbf{K} \end{bmatrix}$ is a positive-definite symmetric matrix [1], \mathbf{u} is an eigenvector and λ the associated eigenvalue. The geometry of engineering structures depends on a set of design parameters, such as thickness, length, height, material properties, etc. For such a practical design situation (2) can be expressed as a parameter-dependent eigenvalue problem

$$\mathbf{G}^*(p)\mathbf{u}(p) = -\lambda(p)\mathbf{M}^*(p)\mathbf{u}(p), \quad p_0 \leq p \leq p_f, \quad (3)$$

where p is a design parameter.

Several approaches for approximation of interval eigenvalue problems in terms of symmetric [2–5] and asymmetric matrices [6–8] have been suggested. These of the methods approximately predict the interval range for eigenvalues. By contrast, the proposed method provides an approximation for each value economically and with good accuracy. Previous work done by us [9–11] proved to be promising for symmetric and asymmetric parameter-dependent eigenvalue problems. The algorithms presented there were based on interpolated eigenvector. They showed excellent accuracy and computational efficiency in numerical examples. This idea is taken forward here and an extension of the same to skew-symmetric matrices is presented. The interpolated modes method is now extended to an approximation of a skew-symmetric parameter-dependent eigenvalue problem.

The eigenvalue problem associated with gyroscopic systems has several mathematical features that are different from that for non-gyroscopic problems [12, 13]. Meirovitch [14] introduced a method of solving an eigenvalue problem for gyroscopic system by transforming it to a standard form in term of one real symmetric and one skew-symmetric matrices. The solution of such eigenvalue problems contain complex eigenvalues and eigenvectors [1, 15]. Modal analysis for the response of gyroscopic systems [16–18]. Zheng et al. [19] introduced the method of how to solve a generalised eigenvalue problem of damped gyroscopic systems without finding the left eigenvector. However, the approximation of the parameter-dependent eigenvalue problem for rotating structures does not appear to have been considered before.

2. A mode interpolation algorithm for eigenvalue approximation in terms of a skew-symmetric matrix

Consider the generalised eigenvalue problem of the form (3), where \mathbf{G}^* is a skew-symmetric matrix. The main idea of the proposed method to solve two eigenvalue problems exactly at the ends of a parameter interval. Eigenvalues for the rest of the points inside the interval are then calculated economically but approximately. Before an interpolation takes place, the following eigenvalue problems need be solved exactly

$$\mathbf{G}_0^*\mathbf{u}_0 = -\lambda\mathbf{M}_0^*\mathbf{u}_0, \quad \mathbf{G}_f^*\mathbf{u}_f = -\lambda\mathbf{M}_f^*\mathbf{u}_f, \quad (4)$$

where $p_0 \leq p \leq p_f$, $\mathbf{G}_0^* = \mathbf{G}^*(p = p_0)$, $\mathbf{M}_0^* = \mathbf{M}^*(p = p_0)$, $\mathbf{G}_f^* = \mathbf{G}^*(p = p_f)$ and $\mathbf{M}_f^* = \mathbf{M}^*(p = p_f)$.

The eigenvalues of an eigenproblem in terms of a skew-symmetric matrix are purely imaginary. Complex numbers do not appear in an order. Therefore, the eigenvalues need to be sorted accordingly to some criteria. Eigenvectors need to be arranged correspondingly.

The approximate eigenvalues are calculated by Rayleigh quotient for a skew-symmetric matrix pencil

$$\hat{\lambda}(p) = -\frac{\hat{\mathbf{u}}^H(p)\mathbf{G}^*(p)\hat{\mathbf{u}}(p)}{\hat{\mathbf{u}}^H(p)\mathbf{M}^*(p)\hat{\mathbf{u}}(p)}, \quad p_0 \leq p \leq p_f, \quad (5)$$

with *exact* matrices at each value of p , i.e. using $\mathbf{G}^*(p)$ and $\mathbf{M}^*(p)$ respectively and a trial vector being taken as the interpolated vector given by

$$\hat{\mathbf{u}}(p) = \frac{(p_f - p)\mathbf{u}_0 + (p - p_0)\mathbf{u}_f}{p_f - p_0}, \quad p_0 \leq p \leq p_f, \quad (6)$$

which is a linear interpolation between two exact eigenvectors from initial and final eigenvalue problems (4).

3. A numerical example

A computer code for approximating the eigenvalues for a generalised parameter-dependent eigenvalue problem (3), in terms of a skew-symmetric and a symmetric matrices, was developed in the MATLAB environment [20].

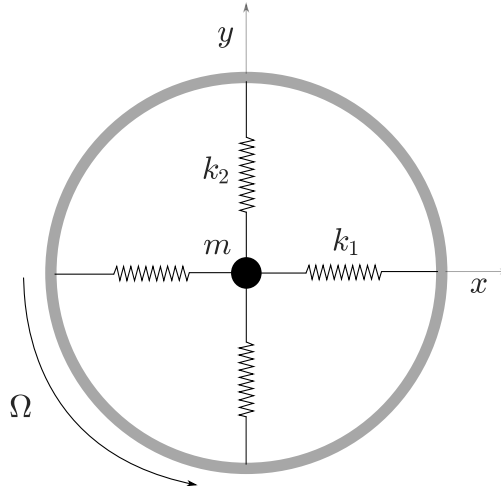


Figure 1. Rotating vibrating structure.

Consider a toy problem that possess the mathematical and physical features of rotating flexible structures as shown in Figure 1. The structure spins with different angular velocity Ω , k_1 and k_2 are the values of total stiffness of the springs along x and y directions respectively. The angular velocity Ω is considered as a parameter p in a parameter-dependent eigenvalue problem (3) and varies from 0.1 to 0.3, i.e. $\Omega_0 = 0.1$ and $\Omega_f = 0.3$. Here, matrices $\mathbf{G}^*(\Omega)$ and $\mathbf{M}^*(\Omega)$ of order $2n \times 2n$ are given by

$$\mathbf{G}^*(\Omega) = \begin{bmatrix} \mathbf{G}(\Omega) & \mathbf{K}(\Omega) \\ -\mathbf{K}(\Omega) & \mathbf{0} \end{bmatrix}, \quad \mathbf{M}^*(\Omega) = \begin{bmatrix} \mathbf{M} & \mathbf{0} \\ \mathbf{0} & \mathbf{K}(\Omega) \end{bmatrix}, \quad (7)$$

where $n \times n$ gyroscopic, mass and stiffness matrices are given by

$$\begin{aligned} \mathbf{G}(\Omega) &= \Omega \begin{bmatrix} 0 & -2m \\ 2m & 0 \end{bmatrix}, & \mathbf{M} &= \begin{bmatrix} m & 0 \\ 0 & m \end{bmatrix}, \\ \mathbf{K}(\Omega) &= \begin{bmatrix} k_1 & 0 \\ 0 & k_2 \end{bmatrix} - \Omega^2 \begin{bmatrix} -m & 0 \\ 0 & -m \end{bmatrix} = \mathbf{K} - \Omega^2 \mathbf{M}. \end{aligned} \quad (8)$$

A number of subdivisions of parametric range equals to 10 taken as the number of designs to be assessed. For this numerical example, the values of the model parameters are taken as

$k_1 = 1$, $k_2 = 2$ and $m = 1$. All the values are non-dimensional and hence only their numeric values are quoted here.

It is of our interest to compare results obtained by proposed method with the results obtained by a Rayleigh quotient approximation based on fixed reference modes. While Rayleigh quotient approximation is well known, here we provide trial vectors *that are parameter-dependent*, and not fixed, which turn out to be rich in the direction of the actual parameter-dependent eigenvector. Moreover, this calculation of the proposed trial vectors is extremely economical. The performance of the proposed approximation with respect to a fixed mode Rayleigh quotient

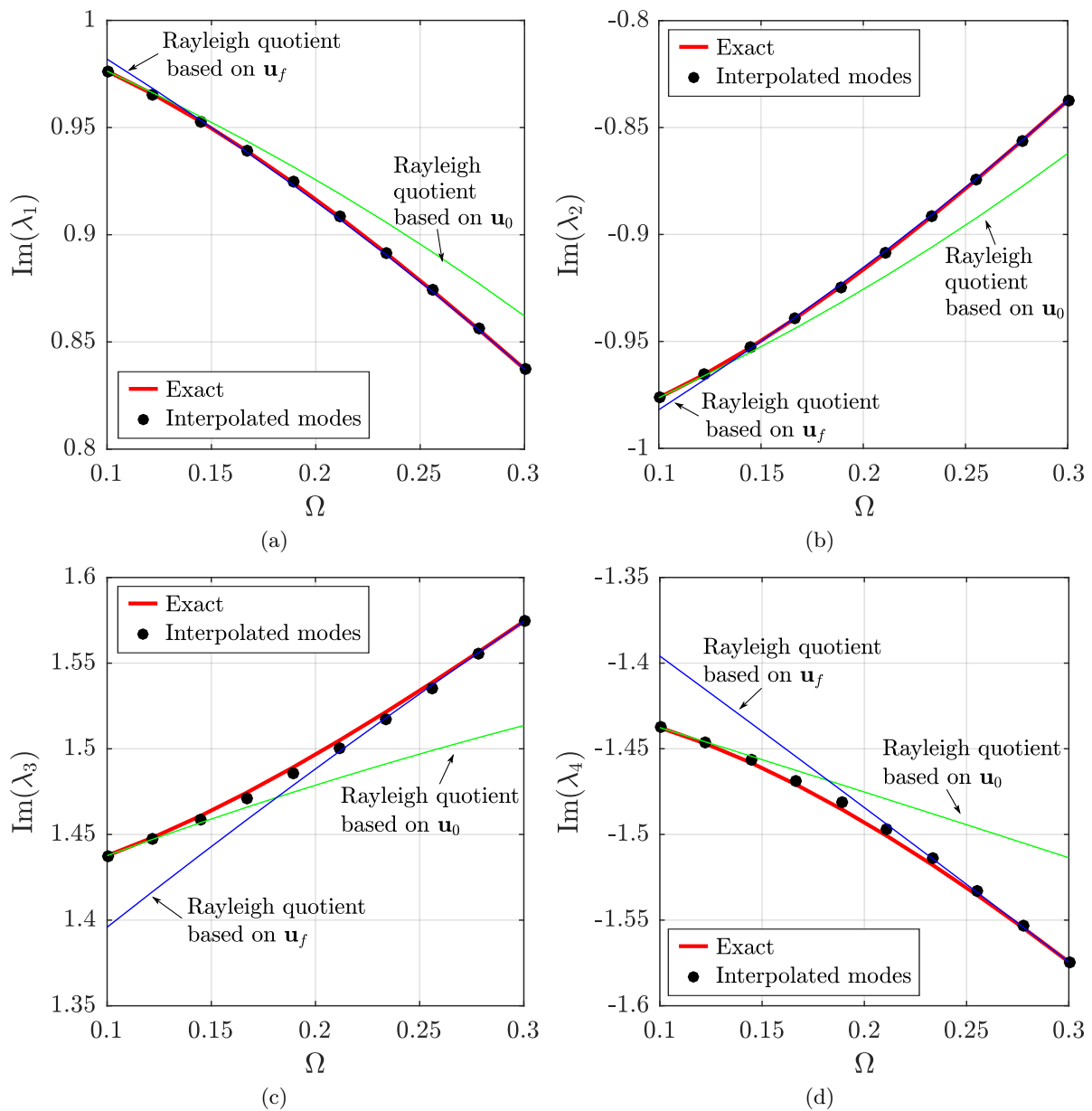


Figure 2. Imaginary parts of the first four eigenvalues as functions of parameter Ω of a parameter-dependent eigenvalue problem computed exactly (thick red line), by the interpolated modes approach (black dots) and reference fixed mode based Rayleigh quotient (thin lines as labelled).

approximation is compared next. As parameter p of the system changes, the eigenvalues along all the parametric range, approximated by Rayleigh quotient based on reference trial vectors at the two ends of parametric range, are given by

$$\tilde{\lambda}(\Omega) \approx -\frac{\mathbf{u}_0^H \mathbf{G}^*(\Omega) \mathbf{u}_0}{\mathbf{u}_0^H \mathbf{M}^*(\Omega) \mathbf{u}_0}, \quad \tilde{\tilde{\lambda}}(\Omega) \approx -\frac{\mathbf{u}_f^H \mathbf{G}^*(\Omega) \mathbf{u}_f}{\mathbf{u}_f^H \mathbf{M}^*(\Omega) \mathbf{u}_f}, \quad (9)$$

where \mathbf{u}_0 and \mathbf{u}_f are the exact eigenvectors from eigenproblems (4) respectively, $\Omega_0 \leq \Omega \leq \Omega_f$.

The approximations can be made for a certain number of modes that are required. In this example, all eigenvalues are approximated as the size of the problem is small. The eigenvalues are calculated in several ways: (i) exactly, (ii) by the interpolated modes method presented here, (iii) by Rayleigh quotient approximation based on fixed reference modes. Exact calculations require solving eigenvalue problems exactly at each parametric value along all the interval. Two eigenproblems, i.e. those expressed in (4), were solved first. The eigenvalues from these problems were sorted in ascending order by the absolute values of imaginary parts of the eigenvalues. The corresponding eigenvectors were normalised and sorted according to their eigenvalues. After this, the interpolated vector (6) was calculated and used in Rayleigh quotient (5) for eigenvalue approximation. Equations (9) were applied for approximation by Rayleigh quotient based on fixed references.

The imaginary parts of eigenvalues as functions of parameter are shown in Figure 2. Figures for real parts are omitted as they are equal to zero, i.e. eigenvalues are purely imaginary. The obtained eigenvalues appear as 2 conjugate pairs, where imaginary parts are equal but have opposite signs. The first conjugate pair is shown in Figures 2(a) – 2(b), the second is presented in Figures 2(c) – 2(d). Results obtained using the interpolated modes method marked by black dots are in excellent agreement with those calculated exactly as marked by thick red lines (Figures 2(a) – 2(d)).

The Rayleigh quotient approximation based on exact eigenvectors \mathbf{u}_0 and \mathbf{u}_f from fixed modes, which is presented in Figure 2 by thin lines as labelled, shows good accuracy closer to the taken references in the beginning and the end of the parameter range respectively. However, the accuracy deteriorates away from those reference points. The use of interpolated modes improves the accuracy impressively with respect to fixed trial vector based calculations.

The maximum percentage error of the interpolated modes method in this numerical example based on obtained results, shown in Figure 2, reaches 0.3 %. The errors in approximation while using reference fixed mode Rayleigh quotient are 3.9 % which is 13 times greater than that obtained from interpolated modes method. Therefore, the eigenvalues approximated by the interpolated modes method are more accurate than obtained by references fixed mode based Rayleigh quotient.

The proposed method is not only accurate but also computationally efficient. It is definitely cheaper to calculate only two eigenproblems from ends of parameter interval exactly than solve all problems inside parameter range. The approach presented here is particular of interest for large-scale problems, as it is computationally expensive to calculate them.

4. Conclusions

A method for approximating the natural frequencies of undamped gyroscopic systems is presented. While considering several design alternatives of a structure, one needs to solve a parameter-dependent eigenvalue problem. Solving several large-scale problems is computationally expensive. The proposed method, using only two exact eigenvectors from the ends of parametric interval and Rayleigh quotient, approximates the eigenvalues throughout parameter range economically.

The interpolated modes method was compared with Rayleigh quotient approximation, based on a fixed reference mode, to demonstrate the efficiency of the presented method. The

approximate methods were applied to a rotating vibrating structure. The results obtained demonstrate excellent accuracy in eigenvalue approximation by the interpolated modes method, which was more accurate compared to the Rayleigh quotient approximation using fixed modes.

The computational gain will be very significant if the proposed method is applied to the large-scale problems, which is in future implementation.

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