AN EIGENVECTOR INTERPOLATION BASED APPROXIMATION FOR PARAMETER-DEPENDENT ASYMMETRIC EIGENVALUE PROBLEMS

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ABSTRACT

Design exploration in the structural dynamics contexts involves assessment of a large family of designs, which frequently leads to a parameter-dependent eigenvalue problem. The present work is inspired by a class of structural dynamic problems in the presence of gyroscopy, control, or aeroelasticity that naturally give rise to asymmetric coefficient matrices in their governing equations of motion. Here, we restrict our attention to the case of a single parameter-dependent eigenvalue problem $\mathbf{A}(p)\mathbf{x}(p) = \lambda(p)\mathbf{x}(p)$ that are also encountered in the state-space formulation of general dynamical systems. These design alternatives vary significantly in their geometrical and physical characteristics. A need to solve a large number of similar eigenvalue problems thus arises. In design scenarios, it is computationally inefficient to carry out reanalysis for each nominally similar structural design. Here, an algorithm for approximating the natural frequencies for a range of parameter-dependent designs is presented. A method based on the interpolation of eigenvectors over the parameter interval, in order to calculate eigenvalues economically, is introduced. Numerical simulations are carried out for an asymmetric system to demonstrate how the proposed technique works. Approximate results compare well with the exact ones while providing significant computational economy. The computational saving is found to be increasingly more significant as the size of the problem increases. Finally, the computational complexity of the proposed algorithm is assessed.

NOMENCLATURE

λ	eigenvalue
$\mathbf{x}$	eigenvector
$\mathbf{x}_0$	eigenvector from the initial reference
$\mathbf{x}_f$	eigenvector from the final reference
$\breve{\mathbf{x}}$	interpolated vector
p	parameter

1. INTRODUCTION

Eigenvalue problems are found in a range of application from the vibration of structures to Google's Page Ranking algorithm. Dynamical systems, such as those containing gyroscopy, damping, or involving aeroelasticity lead to an asymmetric standard eigenvalue problem

$$\mathbf{A}\mathbf{x} = \lambda\mathbf{x}, \qquad \mathbf{A} \neq \mathbf{A}^T, \tag{1}$$

where $\mathbf{A} = \begin{bmatrix} -\mathbf{M}^{-1}(\mathbf{G}+\mathbf{C}) & -\mathbf{M}^{-1}(\mathbf{K}+\mathbf{H}) \\ \mathbf{I} & 0 \end{bmatrix}$, λ is an eigenvalue which corresponds to an eigenvector $\mathbf{x}$. Equation (1) represents the equation of motion

$$\mathbf{M}\ddot{\mathbf{x}} + (\mathbf{G}+\mathbf{C})\dot{\mathbf{x}} + (\mathbf{K}+\mathbf{H})\mathbf{x} = \mathbf{0}, \tag{2}$$

where $\mathbf{M}$ is the mass matrix; $\mathbf{G}$ is the gyroscopic matrix; $\mathbf{C}$ is the damping matric; $\mathbf{K}$ is the stiffness matrix; $\mathbf{H}$ is the circulatory matrix and $\mathbf{I}$ is an identity matrix [1]. In problems involving gyroscopy, for example, the matrix associated with the time-derivative term is skew-symmetric.

While creating a finite element model, the designers have to go through the evaluation of the dynamics of several designs alternatives. Finding the natural frequencies of each design leads to a large number of evaluations, which is computationally demanding. Therefore, at early stages of design search, a cheap approximate calculation is valuable. In this paper, we consider a general case of the form (1) where matrix $\mathbf{A} = \mathbf{A}(p)$ depends on a design parameter. An algorithm for approximating the eigenvalues of such a parameter-dependent problem is presented. The method is based on an eigenvector interpolation from eigenvectors evaluated only at the initial and final designs.

Several approaches [2, 3, 4] have been suggested for predicting the bounds for the eigenvalues and eigenvectors of a symmetric interval matrix.

Behnke [5] presented a method for finding eigenvalues bounds of a real symmetric parameter-dependent eigenvalue problem using the Temple quotient. However, the author specified that the method is suitable for matrices of small size up to 40×40.

Rohn [6] considered two special cases of symmetric and skew-symmetric matrices. They defined the bounds for real and imaginary parts of eigenvalues using min and max values, central matrix, radius matrix. Hladik [7] presented a computationally inexpensive formula for calculating the bounds on eigenvalues of real and complex parameter-dependent matrices in the same time improving previously mentioned results [6] by presenting low computational cost and tighter bounds for eigenvalues.

An asymmetric eigenvalue problem is a harder problem than the symmetric one, especially for large matrices. Several works attempt to solve this problem in more efficient way. Goldhirsch et al. [8] proposed an economical scheme, which includes filtering, construction and analysis of vectors, to obtain leading eigenvalues and eigenvectors for large asymmetric matrices. Saad [9] presented a technique based on Chebyshev polynomials to compute a few eigenvalues with the largest or the smallest real parts of asymmetric eigenvalue problems. Bai [10] presented comparative review of several algorithms and suggested a new technique to solve an asymmetric eigenvalue problem.

Most works approximate the bounds of eigenvalues or solve a single asymmetric eigenvalue problem in efficient way. By contrast, the aim of the research presented here is to develop a computationally economic approximation of eigenvalues for several design problems that depends on a parameter.

Previously there were some works done for the problem in terms of symmetric eigenvalue problems [11, 12]. Here we extend the same analysis for the case of asymmetric matrices that arises in a class of structural dynamic problems. Asymmetry of the coefficient matrices necessitates the use of a different set of orthogonality relations, a new definition of the Rayleigh quotient, and the consideration to an adjoint eigenvalue problem associated with the transpose of the original.

2. THE MODE INTERPOLATION ALGORITHM FOR EIGENVALUE APPROXIMATION

The algorithm presented here is a generalisation of an interpolated modes method which makes use of the Rayleigh quotient using a trial vector that is rich in its components along the exact eigenvector. Consider first a parameter-dependent eigenvalue problem in terms of a symmetric matrix $\mathbf{A}$

$$\mathbf{A}(p)\mathbf{x}(p) = \lambda(p)\mathbf{x}(p), \qquad (3)$$

where $p_0 \leq p \leq p_f$, $\mathbf{x}$ is an eigenvector that corresponds to an eigenvalue λ. A mode-interpolation based approximation was proposed in [11, 12], which is presented in this section before a generalisation of the same is taken up in section 3. The interpolation is based on two exactly calculated eigenvectors from the initial eigenvalue problem

$$\mathbf{A}_0\mathbf{x}_0 = \lambda\mathbf{x}_0, \qquad (4)$$

where $\mathbf{A}_0 = \mathbf{A}(p = p_0)$ and from the final eigenproblem

$$\mathbf{A}_f\mathbf{x}_f = \lambda\mathbf{x}_f, \qquad (5)$$

where $\mathbf{A}_f = \mathbf{A}(p = p_f)$. The approximate eigenvalues are calculated using the Rayleigh quotient

$$\breve{\lambda}(p) = \frac{\breve{\mathbf{x}}^T(p)\mathbf{A}(p)\breve{\mathbf{x}}(p)}{\breve{\mathbf{x}}^T(p)\breve{\mathbf{x}}(p)} \qquad (6)$$

with exact matrix $\mathbf{A}$ at each parametric point and the interpolated vector

$$\breve{\mathbf{x}}(p) = \frac{(p_f - p)\mathbf{x}_0 + \mathrm{sgn}(\mathbf{x}_0^T\mathbf{x}_f)(p - p_0)\mathbf{x}_f}{p_f - p_0}. \qquad (7)$$

Signum function of a dot product of eigenvectors $\mathbf{x}_0$ and $\mathbf{x}_f$ in (7) is used to ensure that the sense of the eigenvectors chosen at the ends of parametric range is such that the angle between them is acute, which is to ensure that interpolation is not erroneously carried out between two vectors of approximately opposite directions. So, equation (7) asserts that when the product of eigenvectors $\mathbf{x}_0^T\mathbf{x}_f > 0$, a positive sign is used in the interpolation. Otherwise when, $\mathbf{x}_0^T\mathbf{x}_f < 0$ the sign in (7) is changed to a negative.

In the previous work [12], the algorithm was tested on several numerical examples which showed excellent accuracy and computational efficiency. The interpolated modes method is now extended to an approximation of an *asymmetric* parameter-dependent eigenvalue problem which is taken up next.

3. ASYMMETRIC EIGENVALUE PROBLEM APPROXIMATION

Consider a standard parameter-dependent eigenvalue problem (3) in terms of asymmetric matrix $\mathbf{A} \neq \mathbf{A}^T$. Several modifications to the interpolated modes method for symmetric matrices [11, 12] need to be made now. They are dictated by new orthogonality relations for asymmetric matrices. In case of a single real symmetric matrix, eigenvectors are orthogonal. For asymmetric case, left and right eigenvectors of a matrix $\mathbf{A}$ corresponding to distinct eigenvalues are orthogonal [1]. In other words, eigenvectors of the matrix and its transposed matrix are biorthogonal. Therefore, an eigenvalue problem associated with transposed matrix $\mathbf{A}^T$ needs to be solved to find a left eigenvector. So, for the present case, four eigenproblems: (4), (5) and

$$\mathbf{A}_0^T\mathbf{y}_0 = \lambda\mathbf{y}_0 \tag{8}$$

$$\mathbf{A}_f^T\mathbf{y}_f = \lambda\mathbf{y}_f \tag{9}$$

should be solved before the interpolation can take place. Their eigenpairs are complex and do not appear in an order. It is impossible to determine which complex value is larger or smaller, as complex numbers consist of real and imaginary parts. Hence, they cannot be sorted in ascending or descending orders. Therefore, eigenvalues need to be sorted by some other criteria. Here, the eigenvalues are sorted in ascending order by their real parts. As a conjugate pair has equal real parts, it is important to sort the numbers within a pair in the same order too. For example, a complex number with a positive imaginary part is chosen first followed by one with a negative imaginary part. The corresponding eigenvectors are normalised and sorted by order of the corresponding eigenvalues.

A new definition of the Rayleigh quotient valid for asymmetric eigenvalue problems [1] needs to be involved

$$\breve{\lambda}(p) = \frac{\breve{\mathbf{y}}^T(p)\mathbf{A}(p)\breve{\mathbf{x}}(p)}{\breve{\mathbf{y}}^T(p)\breve{\mathbf{x}}(p)}, \tag{10}$$

where $\breve{\mathbf{y}}$ and $\breve{\mathbf{x}}$ are the left and the right eigenvectors given by

$$\breve{\mathbf{x}}(p) = \frac{(p_f - p)\mathbf{x}_0 + (p - p_0)\mathbf{x}_f}{p_f - p_0} \tag{11}$$

and

$$\breve{\mathbf{y}}(p) = \frac{(p_f - p)\mathbf{y}_0 + (p - p_0)\mathbf{y}_f}{p_f - p_0} \tag{12}$$

respectively. These two interpolations make use of exactly calculated eigenvectors $\mathbf{x}_0$, $\mathbf{x}_f$, $\mathbf{y}_0$ and $\mathbf{y}_f$ from using equations (4), (5), (8) and (9) respectively.

The interpolated modes method in [11] was inspired by the stationarity of the Rayleigh quotient. Equation (10) makes use of a Rayleigh quotient based approximation that utilises (i) actual parameter-dependent matrix $\mathbf{A}(p)$ at p, and (ii) interpolated left and right eigenvectors $\breve{\mathbf{y}}(p)$ and $\breve{\mathbf{x}}(p)$. We could make another Rayleigh quotient based approximation using exact $\mathbf{A}(p)$ at the parameter value p, but trial vectors fixed at one or the other end of the parameter interval. It would be interesting to compare how approximations based on the proposed interpolated modes compare with those employing eigenvectors fixed at one end.

If a parameter p of the system changes, the eigenvalues along all parametric range could be approximated by Rayleigh quotient based on reference trial vectors at the initial

$$\tilde{\lambda}(p) = \frac{\mathbf{y}_0^T\mathbf{A}(p)\mathbf{x}_0}{\mathbf{y}_0^T\mathbf{x}_0} \tag{13}$$

and final states of parametric range

$$\tilde{\tilde{\lambda}}(p) = \frac{\mathbf{y}_f^T\mathbf{A}(p)\mathbf{x}_f}{\mathbf{y}_f^T\mathbf{x}_f} \tag{14}$$

using the exact eigenvectors from eigenvalue problems (4), (8), (5) and (9) respectively.

4. A NUMERICAL EXAMPLE FOR SYSTEMS INVOLVING ASYMMETRIC MATRICES

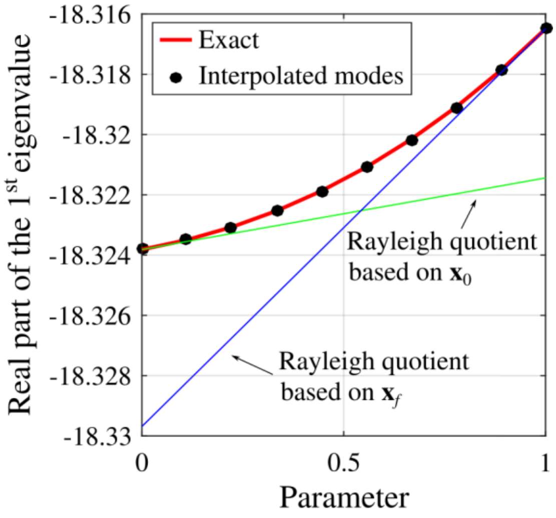
A computer program for approximating the eigenvalues for a standard asymmetric parameter-dependent eigenvalue problem is developed in the MATLAB environment [13]. Consider a standard parameter-dependent eigenvalue problem (3) involving an asymmetric matrix $\mathbf{A} \neq \mathbf{A}^T$ where p is in a range $0 \leq p \leq 1$ with 10 numbers of subdivisions over the parameter interval. Two arbitrary asymmetric matrices of size 1000×1000 were generated and assigned as initial $\mathbf{A}_0 = \mathbf{A}(p = 0)$ and final $\mathbf{A}_f = \mathbf{A}(p = 1)$ matrices. As matrices are randomly generated a parametric problem needs to be created to connect those. For simplicity of illustration, the entries of $\mathbf{A}(p)$ are taken to be linearly varying with the parameter p within the interval, so that

$$\mathbf{A}(p) = \mathbf{A}_0 + p(\mathbf{A}_f - \mathbf{A}_0). \quad (15)$$

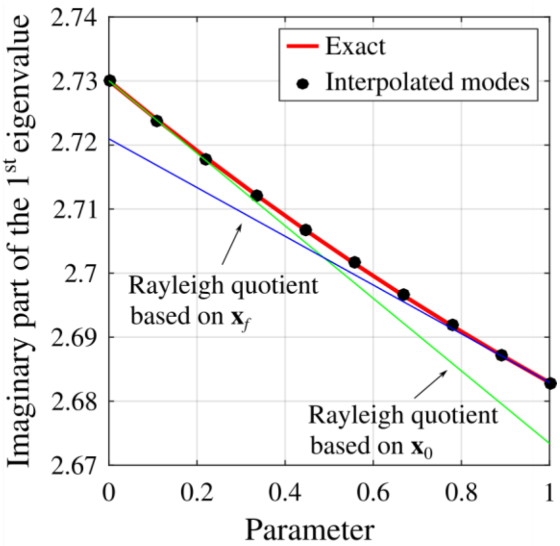
The numerical calculations were carried out within a MATLAB implementation which makes use of the method presented in the previous section. The first 10 eigenvalues are calculated (i) exactly, (ii) by the proposed interpolated modes method, and (iii) reference fixed mode based Rayleigh quotient. For demonstration, only the first two eigenvalues are presented in Figure 1 and 2.

As it was stated before, the eigenvalues of asymmetric eigenvalue problem are complex. So, approximations for their real and imaginary parts are presented separately. In this example, the first two eigenvalues are a conjugate pair shown in Figure 1 and 2. The figure for the real part of the second eigenvalues is omitted as a conjugate pair have identical real parts, shown in Figure 1(a), and opposite sign imaginary parts, shown in Figure 1(b) and 2.

The computed values of the interpolated modes method marked by black dots in Figure 1 and 2 are in excellent agreement with the exact eigenvalues presented by thick red line. The Rayleigh quotient approximation based on eigenvectors $\mathbf{x}_0$ and $\mathbf{x}_\mathrm{f}$ fixed at the ends $p = 0$ and $p = 1$ are marked by thin green and blue lines respectively Figure 1 and 2 as labelled.



(a)



(b)

Figure 1: Real (a) and imaginary (b) parts of the second eigenvalue λ_1 as a function of parameter p of an eigenvalue problem with respect to an asymmetric arbitrary matrix of size 1000×1000 computed exactly (thick red line), by the interpolated modes method (black dots) and reference fixed mode Rayleigh quotient (thin green and blue lines as labelled).

Approximations based on equations (13) and (14) show good eigenvalue prediction close to the taken reference. However, their accuracy deteriorates further away from the reference point. As opposed to this, the approximation based on the proposed method (10) matches extremely well throughout the whole parameter range $0 \leq p \leq 1$. The numerical results based on the interpolated modes are so good that the exact values (red lines) are practically indistinguishable from the approximate ones (black dots).

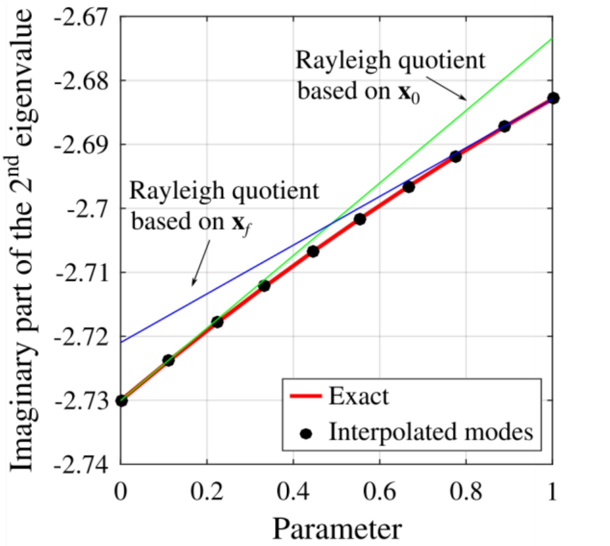


Figure 2: Imaginary part of the second eigenvalue λ_2 as a function of parameter p of an eigenvalue problem with respect to an asymmetric arbitrary matrix of size 1000×1000 computed exactly (thick red line), by the interpolated modes method (black dots) and reference fixed mode Rayleigh quotient (thin green and blue lines as labelled).

The percentage error of the first 10 modes between exact and approximated eigenvalues is calculated to assess the accuracy of the presented method and to compare the errors involve with those using the fixed reference based Rayleigh quotient approximation. The maximal error along parameter range for each real and imaginary parts are presented in Figure 3(a) and 3(b) respectively.

The error while using interpolated modes method, marked with black dots, is very close to zero with the largest being of $2.9 \times 10^{-4}\%$ and 0.03% in real and imaginary parts respectively. Such small error indicate excellent accuracy achieved in calculations while using the interpolated modes method. The accuracy of the fixed mode Rayleigh quotient approximation is worse than the proposed method, as the errors are significantly larger, which is shown in Figure 3.

Another advantage of the approximation presented here is the computational efficiency. In this example, the run time is 9.62 s to solve 10 eigenvalue problems of size 1000×1000 exactly within MATLAB environment. The interpolated modes approximation requires 5.78 s to approximate all the eigenvalues, which is in 1.7 times less than exact. Moreover, the actual approximation, Rayleigh quotient (10) and interpolated vectors (11), (12) took 1.98 s, as the rest of this time was spent on solving initial (4), (5) and final (8), (9) eigenproblems exactly. Calculating the exact eigenvectors form both ends of a parametric range is the most expensive procedure in the proposed algorithm but it is still more efficient than calculating each problem (10 in this case) exactly. This becomes even more significant when the number of such evaluations is even greater since the computational time would scale linearly with the number of such intermediate calculations.

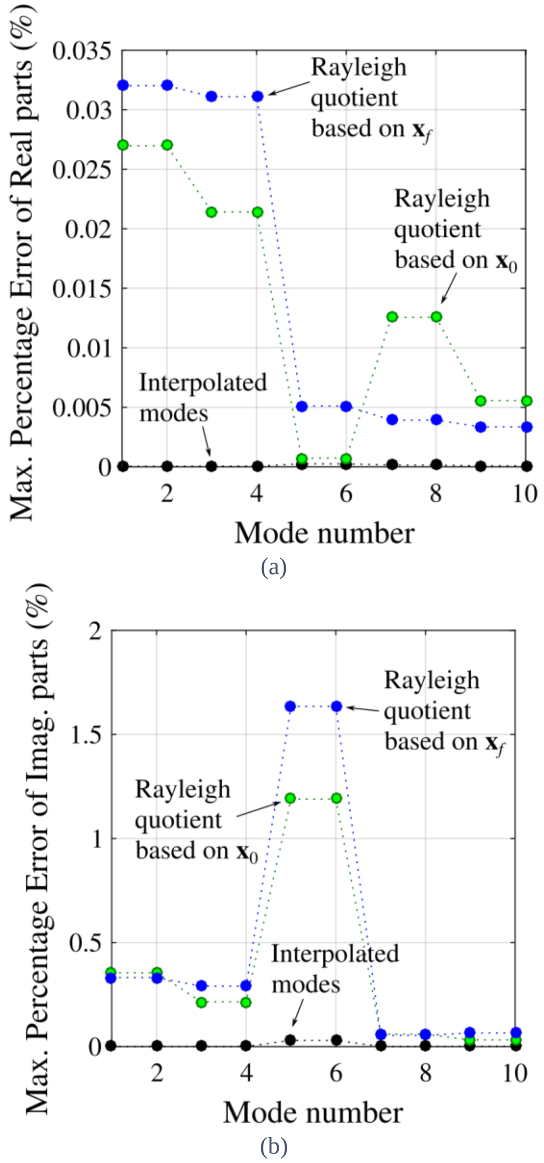


Figure 3: The maximum percentage error for the first 10 real (a) and imaginary (b) parts of eigenvalues over a parameter range calculated between exact and eigenvalues approximated by the interpolated modes method (black dots) and reference fixed mode Rayleigh quotient (blue and green dots as labelled).

In most of practical problems, considering only the first few modes is enough. In such cases, the algorithm required 4.22 s to approximate the first 10 eigenvalues, which is in 2.3 times less than exact calculations. This significant computational gain could even increase depending on the number of approximated values required, the size of the eigenvalue problem and the number of designs. Therefore, an asymmetric parameter-dependent eigenvalue problem can be approximately solved by the interpolated modes method which is computationally significantly inexpensive.

5. CONCLUSIONS

In this paper, the method for approximating the eigenvalues for systems depending on a design parameter and involving asymmetric matrices is proposed. The approach is based on a Rayleigh quotient approximation using the interpolated vectors that include the exact eigenvectors from eigenvalue problems at the ends of parametric range. The proposed method was applied on a numerical example of size 1000×1000 and 10 numbers of subdivisions over the parameter interval. The computed results show excellent accuracy of the approximated eigenvalues by the interpolated modes method and significant computational economy. Finally, the interpolated modes method was compared to the Rayleigh quotient fixed mode approximation that showed worse accuracy than the proposed method. The maximum percentage error over the parameter range were calculated and the method proposed here performed very favourably.

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