



Generators of Bieberbach groups with 2-generated holonomy group

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Abstract

An n -dimensional Bieberbach group is the fundamental group of a closed flat n -dimensional manifold. K. Dekimpe and P. Penninckx conjectured that an n -dimensional Bieberbach group can be generated by n elements. In this paper, we show that the conjecture is true if the holonomy group is 2-generated (e.g. dihedral group, quaternion group or simple group) or the order of holonomy group is not divisible by 2 or 3. In order to prove this, we show that an n -dimensional Bieberbach group with cyclic holonomy group of order larger than two can be generated by $(n - 1)$ elements.

Keywords Crystallographic group · Bieberbach group · Generators · Cyclic group

1 Introduction

We first introduce the geometric definition of a crystallographic groups. A group Γ is said to be an n -dimensional crystallographic group if it is a discrete subgroup of $\mathbb{R}^n \rtimes O(n)$, which is the group of isometries of \mathbb{R}^n and it acts cocompactly on \mathbb{R}^n . By The First Bieberbach Theorem, [15, Theorem 2.1], $\Gamma \cap (\mathbb{R}^n \times I)$ is isomorphic to \mathbb{Z}^n and $\Gamma / \Gamma \cap (\mathbb{R}^n \times I)$ is a finite group called the *holonomy groups* of Γ . We say Γ is an n -dimensional Bieberbach group if it is an n -dimensional torsion-free crystallographic group. In this paper, we focus on the below conjecture.

Conjecture 1.1 [8, Dekimpe–Penninckx] Let Γ be an n -dimensional Bieberbach group. Then the minimum number of generators of Γ is less than or equal to n .

The conjecture was solved for some special cases. For example, the conjecture is true if the holonomy group is an odd prime p -group (see [1]), or the holonomy group is an elementary abelian p -group (see [8]). On the other hand, by a computer program namely CARAT, it has been checked that the conjecture is true if the Bieberbach group has dimension less than 7 (see [5]).

There is a connection between the number of generators of Bieberbach group and the number of generators of a finite group that can act freely on an n -torus (see [10]). Let G be

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a finite group. If G acts freely on an n -torus T^n , the quotient space T^n/G is a manifold M and we get a short exact sequence as below,

$$0 \longrightarrow \pi_1(T^n) \longrightarrow \pi_1(M) \longrightarrow G \longrightarrow 1$$

where $\pi_1(M)$ is an n -dimensional Bieberbach group. Hence if $\pi_1(M)$ can be generated by n elements, then the minimal number of generators of G should not be larger than n . For instance, we know that $(\mathbb{Z}/2\mathbb{Z})^{n+1}$ cannot act freely on T^n for $n \geq 1$. (see [8,10]).

Let G be a group and M be a $\mathbb{Z}G$ -module. Throughout this paper, we denote $d(G)$ to be the minimal number of generators of G and denote $rk_G(M)$ to be the minimal number of generators of M as a $\mathbb{Z}G$ -module. Our paper is divided into several sections. In Sect. 2, we give some basic definitions and some related properties of crystallographic groups. In Sect. 3, we discuss the number of generators of $\mathbb{Z}C_m$ -module, where C_m is a cyclic group of order m . In Sect. 4, we present our three main theorems. The below three theorems are our main results.

Theorem A *Let Γ be an n -dimensional crystallographic group with holonomy group isomorphic to $C_m = \langle g | g^m = 1 \rangle$ where $m \geq 3$.*

- (i) *If m is divisible by prime larger than 3, then $d(\Gamma) \leq n - 2$.*
- (ii) *If m is not divisible by prime larger than 3 and Γ is torsion-free, then $d(\Gamma) \leq n - 1$.*

The idea of the proof of Theorem A(i) is to consider $\Gamma \cap (\mathbb{R}^n \times I)$ as a $\mathbb{Z}C_p$ -module where p is prime larger than 3. We use the module structure to reduce the number of generators. For Theorem A(ii), we construct a surjective homomorphism from Γ to \mathbb{Z} . Then by studying how \mathbb{Z} acts on the kernel of the homomorphism, we can eliminate some redundant generators. By Theorem A, we get two corollaries. One shows that a general n -dimensional Bieberbach group can be generated by $2n$ elements. The other corollary shows an n -dimensional Bieberbach group with a simple group as holonomy group can be generated by $n - 1$ elements.

Theorem B *Let Γ be an n -dimensional crystallographic group with holonomy group isomorphic to a finite group G ,*

- (i) *If the order of G is not divisible by 2 or 3, then $d(\Gamma) \leq n$.*
- (ii) *If the order of G is odd and divisible by 3, then $d(\Gamma) \leq n + 1$.*

The idea of the proof of Theorem B is to apply results from [11] to get a relation between the number of generators of the finite group G and its Sylow p -subgroups. Then we apply results from [1] to prove Theorem B.

Theorem C *Let Γ be an n -dimensional Bieberbach group with 2-generated holonomy group. Then $d(\Gamma) \leq n$.*

The idea of the proof of Theorem C is to consider a Bieberbach subgroup with cyclic holonomy group. Then we apply Theorem A to get the desired bound for generators of the Bieberbach group Γ .

2 Background

In this section, we recall some properties of crystallographic group from [6,15]. Let Γ be an n -dimensional crystallographic group. By The First Bieberbach Theorem, [15, Theorem

2.1], $\Gamma \cap (\mathbb{R}^n \times I)$ is isomorphic to \mathbb{Z}^n and it is the maximal abelian subgroup with finite index, where I is the identity element in the orthogonal group. Therefore Γ can be expressed as the short exact sequence

$$0 \longrightarrow \mathbb{Z}^n \xrightarrow{\iota} \Gamma \xrightarrow{p} G \longrightarrow 1 \quad (1)$$

where G is a finite group, $\iota : \mathbb{Z}^n \hookrightarrow \Gamma$ is an inclusion map which maps e_i to (e_i, I) where e_1, \dots, e_n are the standard basis of \mathbb{Z}^n and $p : \Gamma \rightarrow G$ is a projection map which maps (a, A) to A . Given such a short exact sequence, it will induce a representation $\rho : G \rightarrow GL_n(\mathbb{Z})$ given by $\rho(g)x = \bar{g}\iota(x)\bar{g}^{-1}$, where $x \in \mathbb{Z}^n$ and \bar{g} is chosen arbitrarily such that $p(\bar{g}) = g$. In this case, we call the group G to be the *holonomy group* and the representation ρ to be the *holonomy representation* of Γ . It is well known that ρ is a faithful representation (see [15, Chapter 2]).

Now we are going to introduce the algebraic definition for crystallographic groups, which is equivalent to the geometric definition of crystallographic groups (see [15, Theorem 2.2]). We say Γ is an *n-dimensional crystallographic group* if it can be expressed as the below short exact sequence

$$0 \longrightarrow \mathbb{Z}^n \longrightarrow \Gamma \longrightarrow G \longrightarrow 1 \quad (2)$$

where G is finite group and the induce representation $\rho : G \rightarrow GL_n(\mathbb{Z})$ is a faithful representation. Given an n -dimensional crystallographic group Γ , with holonomy group G . Every element $\gamma \in \Gamma$ can be expressed as (a, g) where $a \in \mathbb{R}^n$ and $g \in G$. The operation in Γ is given by $(a_1, g_1)(a_2, g_2) = (a_1 + \rho(g_1)(a_2), g_1g_2)$, where $(a, g), (b, h) \in \Gamma$ and ρ is the holonomy representation. Notice that Γ will induce the holonomy representation $\rho : G \rightarrow GL_n(\mathbb{Z})$. Therefore we can consider $\Gamma \cap (\mathbb{R}^n \times I) \cong \mathbb{Z}^n$ as a $\mathbb{Z}G$ -module. We denote $rk_G(\mathbb{Z}^n)$ to be the minimal number of generators of \mathbb{Z}^n as a $\mathbb{Z}G$ -module. In particular, if G is a cyclic group with generator g and let $\rho : G \rightarrow GL_n(\mathbb{Z})$ where $g \mapsto M \in GL_n(\mathbb{Z})$ be its matrix holonomy representation. For convenience, we denote element $(a, g) \in \Gamma$ to be (a, M) and denote the $\mathbb{Z}G$ -module \mathbb{Z}^n to be \mathbb{Z}_M^n to specify that the G -action is given by the matrix M . We will denote I_n to be the identity matrix of dimension n and C_m to be a cyclic group of order m .

Remark 2.1 Let Γ be an n -dimensional crystallographic group with holonomy group G , where G is generated by m elements namely a_1, \dots, a_m . Then by sequence (1), we have the following two observations,

- (i) $d(\Gamma) \leq rk_G(\mathbb{Z}^n) + d(G)$.
- (ii) $\{\iota(e_1), \dots, \iota(e_n), \alpha_1, \dots, \alpha_m\}$ can be a generating set of Γ where e_1, \dots, e_n are the standard basis of \mathbb{Z}^n and α_i is chosen arbitrarily such that $p(\alpha_i) = a_i$ for all $i = 1, \dots, m$.

Definition 2.2 Let G be a group, M be a $\mathbb{Z}G$ -module and $\rho : G \rightarrow GL_m(\mathbb{Z})$ be the representation correspond to the $\mathbb{Z}G$ -module M .

- (i) N is a *submodule* of M if N is a subgroup of M which is closed under the action of ring elements.
- (ii) M is decomposable if M is the direct sum of submodules. M is indecomposable if M is not decomposable.
- (iii) M is \mathbb{Z} -*reducible* if there exists a matrix $N \in GL_m(\mathbb{Z})$ such that $N\rho(g)N^{-1} = \begin{pmatrix} * & * \\ 0 & * \end{pmatrix}$ for $g \in G$. M is \mathbb{Z} -*irreducible* if M is not \mathbb{Z} -reducible.

Now, we are going to give a short introduction to the properties of holonomy representation. Let M_1, \dots, M_k be square matrices with entries in \mathbb{Z} , we denote $\text{tri}(M_1, \dots, M_k)$ to be matrix of form as below,

$$\text{tri}(M_1, \dots, M_k) := \begin{pmatrix} M_1 & & * \\ & M_2 & \\ & & \ddots \\ 0 & & & M_k \end{pmatrix}$$

Let $\rho : C_m \rightarrow GL_n(\mathbb{Z})$ be a faithful representation. Since ρ is defined up to isomorphism, we are able to conjugate it by a suitable invertible matrix and assume $\rho(g) = \text{tri}(A_1, \dots, A_t)$ for some $t \in \mathbb{N}$ and A_1, \dots, A_t are square matrices such that $\mathbb{Z}_{A_1}^{\dim(A_1)}, \dots, \mathbb{Z}_{A_t}^{\dim(A_t)}$ are \mathbb{Z} -irreducible modules and $\sum_{j=1}^t \dim(A_j) = n$.

Remark 2.3 Let $M = \text{tri}(A_1, \dots, A_t)$ where A_1, \dots, A_t are square matrices. Denote the order of A_i to be a_i for $i = 1, \dots, t$ and m to be the order of M . Then the least common multiple of a_1, \dots, a_t equals to m . In particular, m is divisible by a_i for $i = 1, \dots, t$.

3 Generators of $\mathbb{Z}C_m$ -module

Let Γ be an n -dimensional crystallographic group with holonomy group isomorphic to C_m . We can consider $\Gamma \cap (\mathbb{R}^n \times I) \cong \mathbb{Z}^n$ as a $\mathbb{Z}C_m$ -module. Since we can restrict the C_m -action to be a C_k -action as long as m is divisible by k , we can also view \mathbb{Z}^n as a $\mathbb{Z}C_k$ -module. It is clear that $\text{rk}_{C_m}(\mathbb{Z}^n) \leq \text{rk}_{C_k}(\mathbb{Z}^n)$. The below lemma and proposition are on the number of generators of $\mathbb{Z}C_m$ -module.

Lemma 3.1 Let $\rho : C_p \rightarrow GL_n(\mathbb{Z})$ be a faithful representation and \mathbb{Z}^n be the correspondence $\mathbb{Z}C_p$ -module, where p is prime. Then $\text{rk}_{C_p}(\mathbb{Z}^n) \leq n - p + a$, where $a = 2$ if $p \leq 19$, otherwise $a = 3$.

Proof Let g be the generator of C_p . Assume $\rho(g) = \text{tri}(A_1, \dots, A_k)$ where $\mathbb{Z}_{A_1}^{\dim(A_1)}, \dots, \mathbb{Z}_{A_k}^{\dim(A_k)}$ are \mathbb{Z} -irreducible $\mathbb{Z}C_p$ -modules. By Remark 2.3, there exists $i \in \{1, \dots, k\}$ such that A_i has order p . By [7, Section 74] A_i has dimension $p - 1$ and the module $\mathbb{Z}_{A_i}^{\dim(A_i)}$ is isomorphic to an ideal in $\mathbb{Z}[\zeta]$ where ζ is a primitive p -root of unity. If $p \leq 19$, by [13, Section 29.1.3], the class number of $\mathbb{Z}[\zeta]$ is 1. Therefore the module $\mathbb{Z}_{A_i}^{\dim(A_i)}$ is a principle ideal and it is isomorphic to $\mathbb{Z}[\zeta]$. Hence $\text{rk}_{C_p}(\mathbb{Z}_{A_i}^{\dim(A_i)}) = 1$. Now assume $p > 19$. Since $\mathbb{Z}[\zeta]$ is a Dedekind domain. By [13, Section 7.1–2], every ideal in a Dedekind domain can be generated by two elements. Hence $\text{rk}_{C_p}(\mathbb{Z}_{A_i}^{\dim(A_i)}) \leq 2$. Therefore we have

$$\text{rk}_{C_p}(\mathbb{Z}^n) \leq n - \dim(A_i) + \text{rk}_{C_p}(\mathbb{Z}_{A_i}^{\dim(A_i)}) = n - p + 1 + \text{rk}_{C_p}(\mathbb{Z}_{A_i}^{\dim(A_i)}) \leq n - p + a$$

where $a = 2$ if $p \leq 19$, otherwise $a = 3$. \square

Proposition 3.2 Let $\rho : C_m \rightarrow GL_n(\mathbb{Z})$ be a faithful representation and \mathbb{Z}^n be the correspondence $\mathbb{Z}C_m$ -module of ρ , where $m \geq 3$.

- (i) If m is divisible by prime larger than 3, then $\text{rk}_{C_m}(\mathbb{Z}^n) \leq n - 3$.
- (ii) If m is not divisible by prime larger than 3, then $\text{rk}_{C_m}(\mathbb{Z}^n) \leq n - 1$.

Proof Let $m = p_1^{s_1} \cdots p_t^{s_t}$ be the prime decomposition of m and assume $p_1 < \cdots < p_t$. Let g be the generator of C_m .

(i): Consider $H = \langle g^{m/p_t} \rangle \cong C_{p_t}$, a subgroup of C_m . We can view \mathbb{Z}^n as a $\mathbb{Z}C_{p_t}$ -module where the C_{p_t} -action is given by $\rho|_H$. Since $\rho|_H$ is a faithful representation, by Lemma 3.1, we have $rk_{C_{p_t}}(\mathbb{Z}^n) \leq n - p_t + a$. Since m is divisible by prime larger than 3, we have $rk_{C_m}(\mathbb{Z}^n) \leq n - 3$.

(ii): We observe that m is either divisible by 3 or 4. If m is divisible by 3, we consider \mathbb{Z}^n as $\mathbb{Z}C_3$ -module. By Lemma 3.1, we have $rk_{C_3}(\mathbb{Z}^n) \leq n - 1$. Hence $rk_{C_m}(\mathbb{Z}^n) \leq n - 1$. Now we assume m is divisible by 4. Consider $H' = \langle g^{m/4} \rangle \cong C_4$, a subgroup of C_m . We can view \mathbb{Z}^n as a $\mathbb{Z}C_4$ -module by restricting the C_m -action to a C_4 -action, where the C_4 -action is given by $\rho|_{H'}$. We assume $\rho|_{H'}(g^{m/4}) = tri(M_1, \dots, M_k)$ where $\mathbb{Z}_{M_1}^{dim(M_1)}, \dots, \mathbb{Z}_{M_k}^{dim(M_k)}$ are \mathbb{Z} -irreducible $\mathbb{Z}C_4$ -modules. By Remark 2.3, there exists $i \in \{1, \dots, k\}$ such that M_i is a matrix of order 4. Let $\phi : C_4 \rightarrow GL_n(\mathbb{Z})$ be the corresponding representation of $\mathbb{Z}_{M_i}^{dim(M_i)}$. By [2, Section 5], there is only one faithful integral \mathbb{Z} -irreducible C_4 -representation up to equivalence. Hence we assume M_i is equivalent to $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. Let $y_1 = (1, 0) \in \mathbb{Z}^2$ and $y_2 = (0, 1) \in \mathbb{Z}^2$ be the standard basis of $\mathbb{Z}_{M_i}^2$. We have $\phi(g^{m/4})y_2 = y_1$. Hence $\mathbb{Z}_{M_i}^2$ can be generated by y_2 as a $\mathbb{Z}C_4$ -module. Since $rk_{C_4}(\mathbb{Z}_{M_i}^{m_i}) \leq m_i - 1$, we have $rk_{C_m}(\mathbb{Z}^n) \leq \sum_{z=1}^k rk_{C_4}(\mathbb{Z}_{M_z}^{m_z}) \leq n - 1$. \square

4 Main result

Theorem A Let Γ be an n -dimensional crystallographic group with holonomy group isomorphic to $C_m = \langle g | g^m = 1 \rangle$ where $m \geq 3$.

- (i) If m is divisible by prime larger than 3, then $d(\Gamma) \leq n - 2$.
- (ii) If m is not divisible by prime larger than 3 and Γ is torsion-free, then $d(\Gamma) \leq n - 1$.

Proof (i): By Remark 2.1, we have $d(\Gamma) \leq rk_{C_m}(\mathbb{Z}^n) + 1$. Since m is divisible by prime larger than 3, by Proposition 3.2, we have $rk_{C_m}(\mathbb{Z}^n) \leq n - 3$. Therefore we have $d(\Gamma) \leq n - 2$.
 (ii): By Remark 2.1, let $\Gamma = \langle \iota(e_1), \dots, \iota(e_n), \alpha \rangle$, where e_1, \dots, e_n are the standard basis of \mathbb{Z}^n and $p(\alpha) = g$. By [12, Proposition 1.4] and [15, Lemma 5.2], we have $b_1(\Gamma) = rk((\mathbb{Z}^n)^{C_m})$. It is well known that $b_1(\Gamma) \neq 0$ (see [15, Example 4.1]). Let $k = b_1(\Gamma) > 0$. Without loss of generality, every element of Γ can be expressed as $(a, tri(M, I_k))$ where $a \in \mathbb{R}^n$ and M is a square matrix of dimension $n - k$. In particular, let $\alpha = (x, tri(A, I_k))$ where $x = (x_1, \dots, x_n) \in \mathbb{R}^n$ and A is a square matrix of dimension $n - k$ which do not fix any non-trivial elements. In other words, $Au = u$ if and only if $u = 0$ for $u \in \mathbb{R}^{n-k}$. First we assume $x_{n-k+1} = \cdots = x_n = 0$. Let $v := (x_1, \dots, x_{n-k}) \in \mathbb{R}^{n-k}$. By simple calculations, we get $\alpha^m = ((\sum_{s=0}^{m-1} A^s v, 0, \dots, 0), I_n)$. Since $A(\sum_{s=0}^{m-1} A^s v) = \sum_{s=0}^{m-1} A^s v$, we have $\sum_{s=0}^{m-1} A^s v = 0$. There is a contradiction because $\alpha^m = (0, I_n)$. Therefore there exists $i \in \{n - k + 1, \dots, n\}$ such that $x_i = \frac{q}{z} \neq 0 \in \mathbb{Q}$. Define $f : \Gamma \rightarrow \mathbb{Z}$ where it maps $((y_1, \dots, y_n), tri(M, I_k)) \in \Gamma$ to $zy_i \in \mathbb{Z}$. Hence we have $f(\alpha) = q$, $f(\iota(e_i)) = z$ and $f(\iota(e_j)) = 0$ for all $j \neq i$. We claim that f is a surjective homomorphism. Let $\gamma_1 = ((m_1, \dots, m_n), tri(M_1, I_k)) \in \Gamma$ and $\gamma_2 = ((\bar{m}_1, \dots, \bar{m}_n), tri(M_2, I_k)) \in \Gamma$. By simple calculation, we get $\gamma_1 \gamma_2 = ((*, \dots, *, m_{n-k+1} + \bar{m}_{n-k+1}, \dots, m_n + \bar{m}_n), tri(M_1 M_2, I_k))$. Hence we have $f(\gamma_1) + f(\gamma_2) = f(\gamma_1 \gamma_2)$. Therefore f is a

homomorphism. Notice that q and z are coprime, there exists integers σ and τ such that $\sigma q + \tau z = 1$. Hence we have $f(\alpha^\sigma \iota(e_i)^\tau) = 1$. Therefore f is surjective. Observe that $\ker(f) = \langle \iota(e_1), \dots, \iota(\hat{e}_i), \dots, \iota(e_n) \rangle \cong \mathbb{Z}^{n-1}$. We have the below short exact sequence

$$0 \longrightarrow \ker(f) \cong \mathbb{Z}^{n-1} \longrightarrow \Gamma \xrightarrow{f} \mathbb{Z} \longrightarrow 0 \quad (3)$$

By [4, Chapter IV, Section 1], such short exact sequence will induce a representation $\rho : \mathbb{Z} \rightarrow GL_{n-1}(\mathbb{Z})$ given by $\rho(x)e_j = \bar{x}\iota(e_j)\bar{x}^{-1}$ where $f(\bar{x}) = x$ for all $j \neq i$. Consider the restriction $\bar{\rho} := \rho|_{q\mathbb{Z}} : q\mathbb{Z} \rightarrow GL_{n-1}(\mathbb{Z})$. We claim that $\ker(\bar{\rho}) = mq\mathbb{Z}$. Let $qx \in \ker(\bar{\rho})$ for any $x \in \mathbb{Z}$. We have $e_j = \bar{\rho}(qx)e_j = \alpha^x \iota(e_j) \alpha^{-x} = p(\alpha^x)e_j$ for all $j \neq i$. Hence $p(\alpha^x)$ needs to be an identity matrix. Therefore x is multiple of m or $x = 0$. Hence $\ker(\bar{\rho}) \subseteq mq\mathbb{Z}$. Since $p(\alpha^m)$ is an identity matrix, $\rho(mqx)(e_j) = \alpha^{mx} \iota(e_j) \alpha^{-mx} = p(\alpha^{mx})e_j = e_j$ for all $j \neq i$ and $x \in \mathbb{Z}$. Hence $mq\mathbb{Z} \subseteq \ker(\bar{\rho})$. Therefore we have $\ker(\bar{\rho}) = mq\mathbb{Z}$. Now we can obtain a faithful representation $\phi : a\mathbb{Z}/ma\mathbb{Z} \rightarrow GL_{n-1}(\mathbb{Z})$ given by $\phi(\bar{x}) = \bar{\rho}(x)$ where x is the representative of $\bar{x} \in a\mathbb{Z}/ma\mathbb{Z}$. Hence we can view \mathbb{Z}^{n-1} as a $\mathbb{Z}C_m$ -module with faithful C_m -representation. By Proposition 3.2, \mathbb{Z}^{n-1} can be generated by $n - 2$ elements. By (3), we have $d(\Gamma) \leq rk_{C_m}(\mathbb{Z}^{n-1}) + 1 \leq n - 1$. \square

The corollary below gives the general bound on the number of generators of general Bieberbach groups.

Corollary 4.1 *Let Γ be an n -dimensional Bieberbach group with holonomy group G . Then $d(\Gamma) \leq 2n$.*

Proof Let $|G| = p_1^{s_1} \cdots p_k^{s_k}$ be the prime decomposition of order of G . By [11, Theorem A], we have

$$d(G) \leq \max_{1 \leq i \leq k} d(P_i) + 1$$

where P_i is the Sylow p_i -subgroup of G for $i = 1, \dots, k$. We fix $j \in \{1, \dots, k\}$ such that $d(P_j) = \max_{1 \leq i \leq k} d(P_i)$. We first assume $p_j \geq 3$. We can consider $\Gamma \cap (\mathbb{R}^n \times I) \cong \mathbb{Z}^n$ as a $\mathbb{Z}P_j$ -module. By [1, Theorem A], we have $d(P_j) + rk_{P_j}(\mathbb{Z}^n) \leq n$. Hence we have $d(\Gamma) \leq d(G) + 1 + rk_{P_j}(\mathbb{Z}^n) \leq n + 1$. Now we assume $p_j = 2$. If G is a 2-group, then by [1, Theorem A], we have $d(\Gamma) \leq 2n$. If G is not a 2-group, then there exists $g \in G$ such that g has order $p \geq 3$. Hence we can consider \mathbb{Z}^n as a $\mathbb{Z}C_p$ -module. By Lemma 3.1, we have $rk_{C_p}(\mathbb{Z}^n) \leq n - 1$. By [1, Proposition 2.2], we have $d(P_j) \leq n$. Hence we have $d(\Gamma) \leq 2n$. \square

Corollary 4.2 *Let Γ be an n -dimensional Bieberbach group with holonomy group G , where G is a simple group but not C_2 . Then $d(\Gamma) \leq n - 1$.*

Proof By Remark 2.1, we have $d(\Gamma) \leq d(G) + rk_G(\mathbb{Z}^n)$. If G is a cyclic group of odd prime order, then by Theorem A, we have $d(\Gamma) \leq n - 1$. If G is not cyclic, by Burnside's Theorem, [9, Page 886], there exists a prime $p \geq 5$ such that the order of G is divisible by p . So we can view \mathbb{Z}^n as a $\mathbb{Z}C_p$ -module. By Lemma 3.1, we have $rk_{C_p}(\mathbb{Z}^n) \leq n - p + a \leq n - 3$, where $a = 2$ if $p \leq 19$, otherwise $a = 3$. By [3, Theorem B], we have $d(G) \leq 2$. Hence we have $d(\Gamma) \leq d(G) + rk_G(\mathbb{Z}^n) \leq 2 + rk_{C_p}(\mathbb{Z}^n) \leq n - 1$. \square

The rest of the paper will present the proof of Theorems B and C.

Theorem B *Let Γ be an n -dimensional crystallographic group with holonomy group isomorphic to a finite group G ,*

- (i) *If the order of G is not divisible by 2 or 3, then $d(\Gamma) \leq n$.*
- (ii) *If the order of G is odd and divisible by 3, then $d(\Gamma) \leq n + 1$.*

Proof Let $|G| = p_1^{s_1} \cdots p_k^{s_k}$ be the prime decomposition of the order of G , where $p_1 < \cdots < p_k$.

(i): First, we want to calculate the number of generators of the holonomy group G . By [1, Theorem A], we have

$$d(G) \leq \max_{1 \leq i \leq k} d(P_i) + 1$$

where P_i is the Sylow p_i -subgroup of G for $i = 1, \dots, k$. We fix $j \in \{1, \dots, k\}$ such that $d(P_j) = \max_{1 \leq i \leq k} d(P_i)$. Let $\rho : G \rightarrow GL_n(\mathbb{Z})$ be the holonomy representation for Γ . By definition, ρ is a faithful representation. Therefore P_i acts faithfully on \mathbb{Z}^n . By [1, Proposition 2.2], we have

$$d(G) \leq \frac{n - rk((\mathbb{Z}^n)^{P_j})}{p_j - 1} + 1$$

Now, we consider the lattice part. We can view $\Gamma \cap (\mathbb{R}^n \times I) \cong \mathbb{Z}^n$ as a $\mathbb{Z}P_j$ -module. By [1, Proposition 2.5], we have

$$rk_{P_j}(\mathbb{Z}^n) \leq \frac{(a - 1)(n - rk(\mathbb{Z}^n)^{P_j})}{p_j - 1} + rk(\mathbb{Z}^n)^{P_j}$$

where $a = 2$ if $p_j \geq 19$, otherwise $a = 3$. Therefore we have

$$\begin{aligned} d(\Gamma) &\leq d(G) + rk_{P_j}(\mathbb{Z}^n) \leq \frac{n - rk((\mathbb{Z}^n)^{P_j})}{p_j - 1} + 1 + \frac{(a - 1)(n - rk(\mathbb{Z}^n)^{P_j})}{p_j - 1} + rk(\mathbb{Z}^n)^{P_j} \\ &= \frac{a(n - rk(\mathbb{Z}^n)^{P_j})}{p_j - 1} + rk(\mathbb{Z}^n)^{P_j} + 1 \end{aligned}$$

We need to show

$$\frac{a(n - rk(\mathbb{Z}^n)^{P_j})}{p_j - 1} + rk(\mathbb{Z}^n)^{P_j} + 1 \leq n$$

We have

$$\begin{aligned} &\frac{a(n - rk(\mathbb{Z}^n)^{P_j})}{p_j - 1} + rk(\mathbb{Z}^n)^{P_j} + 1 \leq n \\ \iff &an - a \cdot rk(\mathbb{Z}^n)^{P_j} + (p_j - 1)rk(\mathbb{Z}^n)^{P_j} + p_j - 1 \leq n(p_j - 1) \\ \iff &(p_j - 1 - a)rk(\mathbb{Z}^n)^{P_j} \leq (p_j - 1 - a)n - (p_j - 1) \\ \iff &rk(\mathbb{Z}^n)^{P_j} \leq n - \frac{p_j - 1}{p_j - 1 - a} = n - 1 - \frac{a}{p_j - 1 - a} \end{aligned}$$

If $5 \leq p_j \leq 19$, we have $\frac{a}{p_j - 1 - a} = \frac{2}{p_j - 3} \leq 1$. If $p_j > 19$, we have $\frac{a}{p_j - 1 - a} = \frac{3}{p_j - 4} < 1$. Therefore we can conclude that if $rk(\mathbb{Z}^n)^{P_j} \leq n - 2$, then $d(\Gamma) \leq n$. By Cauchy's Theorem

[9, Page 93], P_j has an element $x \in P_j$ with order p_j . Let C_{p_j} be a cyclic subgroup of P_j generated by x . Consider $(\mathbb{Z}^n)^{C_{p_j}}$, where C_{p_j} acts faithfully on \mathbb{Z}^n via $\rho|_{C_{p_j}} : C_{p_j} \rightarrow GL_n(\mathbb{Z})$. By [7, Section 73], the degree of a faithful indecomposable C_{p_j} -representation is either $p_j - 1$ or p_j . If the degree is $p_j - 1$, then it has trivial fix point set. If the degree is p_j , then the fix point set is 1-dimensional. Observe that $rk(\mathbb{Z}^n)^{C_{p_j}}$ has maximum value when $\rho|_{C_{p_j}}$ is a direct sum of one faithful indecomposable sub-representation and all others are trivial sub-representations. Therefore $rk(\mathbb{Z}^n)^{C_{p_j}} \leq n - p_j + 1 \leq n - 4$. Hence we have $rk(\mathbb{Z}^n)^{P_j} \leq n - 4$. Therefore we can conclude $d(\Gamma) \leq n$.

(ii): By [1, Theorem A], we can assume G is not a p -group. By [11, Theorem A], we have

$$d(G) \leq \max_{1 \leq i \leq k} d(P_i) + 1$$

where P_i is the Sylow p_i -subgroup of G for $i = 1, \dots, k$. If $\max_{1 \leq i \leq k} d(P_i) \neq d(P_1)$, then by part (i), we have $d(\Gamma) \leq n$. Therefore we assume $\max_{1 \leq i \leq k} d(P_i) = d(P_1)$. We can consider the lattice part as a $\mathbb{Z}P_1$ -module. Since P_1 is a Sylow 3-subgroup, by [1, Theorem A], we have $d(P_1) + rk_{P_3}(\mathbb{Z}^n) \leq n$. Hence we can conclude that $d(\Gamma) \leq d(P_3) + rk_{P_3}(\mathbb{Z}^n) + 1 = n + 1$. \square

Theorem C *Let Γ be an n -dimensional Bieberbach group with 2-generated holonomy group. Then $d(\Gamma) \leq n$.*

Proof Let G be the holonomy group of Γ . Let x and y be the generators of G . They have order a and b respectively. If either $a = 1$ or $b = 1$, then G is a cyclic group. By [8, Theorem 5.7] and Theorem A, $d(\Gamma) \leq n$. Next, consider cases where $a \geq 3$ or $b \geq 3$. It is sufficient to consider only the case where $a \geq 3$. By Remark 2.1, let $\Gamma = \langle \iota(e_1), \dots, \iota(e_n), \alpha, \beta \rangle$, where e_1, \dots, e_n are the standard basis for \mathbb{Z}^n , $p(\alpha) = x$ and $p(\beta) = y$. Define $\Gamma' = \langle \iota(e_1), \dots, \iota(e_n), \alpha \rangle$. Notice that Γ' is an n -dimensional Bieberbach subgroup of Γ with holonomy group C_a . Since $a \geq 3$, by Theorem A, $d(\Gamma') \leq n - 1$. Hence we have $d(\Gamma) \leq n$. Finally, we assume $a = b = 2$. Consider element $xy \in G$. Since G is finite, xy has finite order. If xy is of order 1 (i.e. $xy = 1$), then $x = y$. So $G \cong C_2$. By [8, Theorem 5.7], $d(\Gamma) \leq n$. If xy is of order 2 (i.e. $xyxy = 1$), then $xy = yx$. Hence $G \cong C_2 \times C_2$. By [8, Theorem 5.7], we have $d(\Gamma) \leq n$. Lastly, we assume xy is of order k , where $k \geq 3$. We can rewrite the generating set of Γ to be $\{\iota(e_1), \dots, \iota(e_n), \alpha\beta, \beta\}$. Define $\Gamma'' = \langle \iota(e_1), \dots, \iota(e_n), \alpha\beta \rangle$, which is an n -dimensional Bieberbach subgroup of Γ with holonomy group isomorphic to C_k . By Theorem A, $d(\Gamma'') \leq n - 1$. Therefore $d(\Gamma) \leq n$. \square

By [1, 8] and the three main theorems in this paper, the Conjecture 1.1 is still open for certain cases of holonomy group where the minimal number of generators has at least three elements. For example, the case where the holonomy group is a 2-group or the order of holonomy group is even. By Corollary 4.1, the corresponding n -dimensional Bieberbach group can be generated by $2n$ elements. Another case is when the order of holonomy group is odd and divisible by 3. In this case, by Theorem B, the corresponding n -dimensional Bieberbach group can be generated by $n + 1$ elements. In order to prove this conjecture fully, we believe further study is needed in the key case where the holonomy group is a 2-group.

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