UNIVERSITY OF SOUTHAMPTON

UNIFORM DESSINS OF LOW GENUS

by

Robert Ian Syddall

Submitted for the degree of
Doctor of Philosophy

Faculty of Mathematical Studies

July 1997

UNIVERSITY OF SOUTHAMPTON

ABSTRACT

FACULTY OF MATHEMATICAL STUDIES

Doctor of Philosophy

UNIFORM DESSINS OF LOW GENUS

by Robert Ian Syddall

It is known that every dessin (map or hypermap) corresponds to a finite index subgroup of a triangle group and can be embedded naturally into some Riemann surface [**JS1**, **JS3**]. A dessin is uniform if its (hyper)vertices all have the same valency, its (hyper)edges all have the same valency, and its (hyper)faces all have the same valency; uniform dessins correspond to torsion-free subgroups of triangle groups. By the theorems of Belyĭ [**Bel**] and Wolfart [**Wo1**], a compact Riemann surface X is defined over the field of algebraic numbers $\overline{\mathbf{Q}}$ if and only if X carries a dessin (also see [**Gro**]). In this thesis we study the uniform dessins of genus $g \leq 3$ and investigate their connections with algebraic curves, Belyĭ's Theorem, and the absolute Galois group $\mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q})$.

An elliptic curve of modulus τ can be uniformized by a finite index subgroup of a Euclidean triangle group if and only if $\tau \in \mathbf{Q}(i)$ or $\tau \in \mathbf{Q}(\rho)$; these elliptic curves are said to have Euclidean Belyĭ uniformizations and naturally carry the uniform dessins of genus 1. Using results from number theory, it is proved that there are only five rational elliptic curves with Euclidean Belyĭ uniformizations. A classification of the genus 1 uniform maps is given which extends the notation for genus 1 regular maps found in [**CMo**]. Formulae are derived for the number of genus 1 uniform maps with a given number of vertices, and the reflexible maps are described. Belyĭ functions are computed in a number of cases, and arbitrarily large Galois orbits of genus 1 uniform dessins are constructed.

The existence of two uniform maps of genus $g > 1$ lying on conformally equivalent Riemann surfaces is considered. This leads naturally to the study of arithmetic Fuchsian groups [**Vi**] and motivates the definitions of arithmetic and non-arithmetic maps. General results are proved for non-arithmetic maps, and specific examples are given in the arithmetic case.

Contents

ACKNOWLEDGEMENTS

I would like to thank my supervisor Dr. David Singerman for his support and enthusiasm during the preparation of this thesis. I would also like to thank my other friends and colleagues for their many helpful comments and suggestions, especially Dr. Gareth Jones, Prof. Jurgen Wolfart, Dr. Manfred Streit, Dr. Colin Maclachlan and Prof. Marston Conder. I gratefully acknowledge the Engineering and Physical Sciences Research Council for their financial support.

I am particularly grateful to my family for their constant support and encouragement throughout my years of academic study, and to Holly for her patience and understanding.

Introduction

A map can be thought of as a two-cell decomposition of an orientable surface into vertices, edges and simply connected open regions called faces. Jones and Singerman [**JS1**] have developed a theory of maps on orientable surfaces in which every map $\mathcal{M}$ corresponds to a finite index subgroup Λ of a triangle group $\Gamma(m, 2, n)$. The map $\mathcal{M}$ can be embedded into the Riemann surface $X = \mathcal{U}/M$ (where $\mathcal{U}$ is the Riemann sphere Σ, the complex plane $\mathbf{C}$, or the upper half-plane $\mathbf{H}$) so that every automorphism of $\mathcal{M}$ extends naturally to an automorphism of its underlying Riemann surface X. More generally, a finite index inclusion $\Lambda \leq \Gamma(l_0, l_1, l_2)$ corresponds to a geometric object called a hypermap (for example see [**CoSi**]).

A Riemann surface X is compact if and only if it can be obtained as the normalization of some algebraic curve defined by an irreducible homogeneous polynomial $T(x, y, z) \in \mathbf{C}[x, y, z]$ (see [**Gri**] for precise details); X is said to be defined over a field $F \subseteq \mathbf{C}$ if we can choose $T(x, y, z) \in F[x, y, z]$. Belyĭ's Theorem [**Bel**] states that X is defined over the field of algebraic numbers $\overline{\mathbf{Q}}$ if and only if there exists a nonconstant meromorphic function $\beta : X \to \Sigma$ ramified over at most 3 points; β is called a Belyĭ function for X. As a corollary to Belyĭ's Theorem, Wolfart [**Wo1**, **Wo2**] has proved that X is defined over $\overline{\mathbf{Q}}$ if and only if X can be uniformized by a finite index subgroup of a cocompact triangle group. Hence the Riemann surfaces that carry finite maps and hypermaps are precisely those that are defined over $\overline{\mathbf{Q}}$.

The most familiar embeddings of maps into Riemann surfaces are the regular maps (those with the most symmetry). Examples include the Platonic solids embedded into the Riemann sphere, the regular maps of genus 2 [**Sh**] and the embedding of Klein's map into a Riemann surface of genus 3 [**Gre**]. However, regular maps are restrictive if one wishes to study the underlying Riemann surfaces; there are only two Riemann surfaces underlying the regular maps of genus 1 (the square torus and the hexagonal torus) and just three Riemann surfaces underlying the regular maps of genus 2. A larger family of Riemann surfaces can be obtained by relaxing the symmetry conditions and considering uniform maps. A map is *uniform* if all of its vertices have the same valency, all of its faces have the same valency, and it either has no free edges or all of its edges are free. Jones and Singerman [**JS1**] have proved that $\mathcal{M}$ is a uniform map if and only if $\mathcal{M}$ corresponds to a torsion-free subgroup of a triangle group.

The connections between Belyĭ functions, Galois theory and dessins d'enfants (we will think of these as maps or hypermaps) were first outlined by Grothendieck in his *Esquisse d'un programme* [**Gro**] (also see [**Sc**]). The absolute Galois group $\mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q})$ is the set of all field automorphisms of the algebraic numbers $\overline{\mathbf{Q}}$ fixing $\mathbf{Q}$. Grothendieck observed that $\mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q})$ acts naturally on Belyĭ functions and their associated Riemann surfaces via its action on their algebraic number coefficients, and that this induces an action of $\mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q})$ on the set of all dessins (maps and hypermaps); Grothendieck's idea was to use this geometric action as a tool for studying the structure of the absolute Galois group.

The study of plane trees (see [**SZ**]) has provided many examples illustrating the general theory of dessins d'enfants. To appreciate Grothendieck's ideas more fully, one would like to study dessins of higher genus; however there are few concrete examples for genus $g > 0$. In this thesis we will study the uniform dessins of genus $g \leq 3$, and develop techniques for calculating their associated algebraic curves and Belyĭ functions, and for constructing Galois orbits.

A map of genus 0 is uniform if and only if it is regular [**JS1**], and so the genus 0 uniform maps correspond to the five Platonic solids and several infinite families. The situation becomes much more interesting if we consider uniform maps of genus 1. Here the underlying Riemann surfaces correspond to elliptic curves, and we find many connections with number theory. Every elliptic curve corresponds to a point τ in the fundamental region of the modular group, and an elliptic curve can be uniformized by a finite index subgroup of a Euclidean triangle group if and only if its associated modulus τ lies in $\mathbf{Q}(i)$ or $\mathbf{Q}(\rho)$ (see Chapter 3); these elliptic curves are said to have *Euclidean Belyĭ uniformizations*, and by Belyĭ's Theorem are defined over the algebraic numbers. We prove that there are only five rational elliptic curves with Euclidean Belyĭ uniformizations.

The regular maps of genus 1 are classified in [**CMo**] by their Schläfli symbols $\{4,4\}_{p,q}$, $\{3,6\}_{p,q}$ and $\{6,3\}_{p,q}$. The study of elliptic curves with Euclidean Belyĭ uniformizations leads naturally to a classification of the uniform maps of genus 1;

if an elliptic curve E has a modulus $\tau \in \mathbf{Q}(i)$ then its corresponding uniform maps have type $(4,4)$ and are denoted by $\{\tau\}_{p+qi}$ where $p+qi \in \mathbf{Z}[i]$, while for $\tau \in \mathbf{Q}(\rho)$ the uniform maps have type $(6,3)$ and are denoted by $\{\tau\}_{p+q\rho}$ where $p+q\rho \in \mathbf{Z}[\rho]$. In particular, $\{i\}_{p+qi}$ and $\{\rho\}_{p+q\rho}$ correspond to the regular maps $\{4,4\}_{p,q}$ and $\{3,6\}_{p,q}$.

If E is an elliptic curve with modulus $\tau \in \mathbf{Q}(i)$ or $\tau \in \mathbf{Q}(\rho)$, then E carries a unique minimal map $\mathcal{M}_\tau$ such that every other uniform map lying on E has a strictly greater number of vertices, edges and faces (see §4.2). One can use the minimal maps to construct arbitrarily large Galois orbits of dessins, and the associated Belyĭ functions and elliptic curves are computed in a number of cases. In addition, we derive formulae for the number of genus 1 uniform maps with n vertices, and describe those that are reflexible.

Our study of uniform dessins extends naturally to higher genus. The regular maps and hypermaps of genus 2 are known (see [**CMo**] and [**BJ**]), and Threlfall [**Th**] has calculated the genus 2 uniform maps of type $(10,5)$. In §5.1 we enumerate all of the uniform maps and hypermaps of genus 2, and show that there are 978 uniform maps and 3133 uniform hypermaps (one can show that for every genus $g > 1$, there are only finitely many uniform dessins of genus g).

We have classified the uniform maps lying on a given elliptic curve of modulus $\tau \in \mathbf{Q}(i)$ or $\tau \in \mathbf{Q}(\rho)$, and one might ask if a similar classification exists for uniform maps of higher genus. Every uniform map corresponds to a conjugacy class of surface groups contained in a triangle group, and two surface groups define conformally equivalent Riemann surfaces if and only if they are conjugate in $\mathrm{PSL}_2(\mathbf{R})$. Hence the existence of two non-isomorphic uniform maps of type (m,n) lying on conformally equivalent Riemann surfaces is equivalent to finding two surface groups $M_1, M_2 \leq \Gamma(m,2,n)$ such that M_1 and M_2 are conjugate in $\mathrm{PSL}_2(\mathbf{R})$ but not conjugate in $\Gamma(m,2,n)$. This problem leads naturally to the study of arithmetic Fuchsian groups (see [**Vi**]), and motivates the definitions of arithmetic and non-arithmetic maps. One can easily distinguish the Riemann surfaces underlying non-arithmetic uniform maps (Theorem 5.21), while the situation for arithmetic maps is more complicated (but also more interesting). We prove some general results for non-arithmetic maps, and give examples in the arithmetic case.

Chapter 1

Riemann surfaces

We introduce the theory of Riemann surfaces and give some preliminary definitions and results. There are several classical ways of obtaining compact Riemann surfaces, for example using complex algebraic curves [**Gri**] or by means of cocompact groups [**JS2**]; a particularly interesting (and difficult) problem is how one might begin to unify these two different approaches. The theorems of Belyĭ [**Bel**] and Wolfart [**Wo1**] provide a good starting point:

A compact Riemann surface X can be obtained through the normalization of some algebraic curve defined over the field of algebraic numbers $\overline{\mathbf{Q}}$ if and only if X can be uniformized by a finite index subgroup of a cocompact triangle group.

In the following chapters we will give examples to illustrate this correspondence. For further details we refer the reader to [**JS2**], [**JS3**] and [**Gri**].

Abstract Riemann surfaces

We define a surface to be a Hausdorff topological space S which is locally homeomorphic to $\mathbf{C}$; that is each point $x \in S$ has an open neighbourhood $U_\alpha \subseteq S$ which is homeomorphic (by some mapping, ϕ_α say) to an open subset $V_\alpha \subseteq \mathbf{C}$. We call the homeomorphism $\phi_\alpha : U_\alpha \to V_\alpha$ a *chart* or *local coordinate mapping* on S. If $\{U_\alpha\}$ is a cover of S by open sets U_α, then

$$\Phi = \{\phi_\alpha : U_\alpha \to V_\alpha\}$$

is called an *atlas* of S. If $\phi_\alpha : U_\alpha \to V_\alpha$ and $\phi_\beta : U_\beta \to V_\beta$ are two charts in Φ with $U_\alpha \cap U_\beta \neq \emptyset$, then $\phi_\alpha(U_\alpha \cap U_\beta)$ and $\phi_\beta(U_\alpha \cap U_\beta)$ are open subsets of $\mathbf{C}$ and the homeomorphism

$$\phi_{\alpha\beta} = \phi_\alpha \circ \phi_{\beta^{-1}} : \phi_\beta(U_\alpha \cap U_\beta) \to \phi_\alpha(U_\alpha \cap U_\beta)$$

is called a *transition function* for Φ. An atlas Φ is said to be *holomorphic* if all of its transition functions $\phi_{\alpha\beta}$ are holomorphic.

Example 1.1. $\Phi = \{\mathrm{id}_{\mathbf{C}} : \mathbf{C} \to \mathbf{C}\}$ is a holomorphic atlas for the complex plane $\mathbf{C}$ consisting of just one chart and the identity map $\mathrm{id}_{\mathbf{C}}$. □

Example 1.2. The set Φ' consisting of all open sets $U \subseteq \mathbf{C}$ together with the identity mapping $\mathrm{id}_U : U \to \mathbf{C}$ is also a holomorphic atlas for $\mathbf{C}$. □

Two holomorphic atlases Φ and Ψ of a surface S are said to be *compatible* if the atlas $\Phi \cup \Psi$ is holomorphic (the atlases given in Examples 1.1 and 1.2 for the complex plane are compatible). Compatibility of atlases is an equivalence relation, and an equivalence class of holomorphic atlases is called a *complex structure* on S.

Definition 1.3. *A surface with a complex structure is called a Riemann surface.* □

Given two Riemann surfaces X and Y with holomorphic atlases Φ and Ψ, we can define holomorphic mappings between X and Y in terms of their charts.

Definition 1.4. *Let X and Y be Riemann surfaces with holomorphic atlases $\Phi = \{\phi_\alpha : U_\alpha \to V_\alpha\}$ and $\Psi = \{\psi_i : M_i \to N_i\}$. A holomorphic mapping $f : X \to Y$ with respect to the atlases Φ and Ψ is a family of continuous mappings*

$$f_\alpha : U_\alpha \to Y$$

such that

(i) $f_\alpha = f_\beta$ *on* $U_\alpha \cap U_\beta$ *for* $U_\alpha \cap U_\beta \neq \emptyset$;

(ii) $\psi_i \circ f_\alpha \circ {\phi_\alpha}^{-1}$ *is holomorphic on* $\phi_\alpha(f^{-1}(M_i) \cap U_\alpha)$ *whenever* $f^{-1}(M_i) \cap U_\alpha \neq \emptyset$. □

Examples 1.1 and 1.2 show that the complex plane $\mathbf{C}$ admits a complex structure and so forms a Riemann surface, which we denote by $\mathbf{C}$ (where there is no ambiguity we use the same notation for the Riemann surface and its underlying space). The one-point compactification of $\mathbf{C}$, $\mathbf{P}_1(\mathbf{C}) = \mathbf{C} \cup \{\infty\}$ also forms a Riemann surface [**JS2**], the Riemann sphere Σ. Holomorphic and meromorphic functions on a Riemann surface X (in the traditional sense of complex analysis) can then be thought of as being holomorphic mappings from X to $\mathbf{C}$ and Σ respectively in the sense of Riemann surfaces.

Let $f : X \to Y$ be a non-constant holomorphic mapping between compact connected Riemann surfaces (all surfaces considered in this thesis will be connected). Then there exists a positive integer n and a finite set of points $C(f) \subset Y$ such that $|f^{-1}(y)| = n$ for all $y \in Y \backslash C(f)$ and $1 \leq |f^{-1}(y)| < n$ for all $y \in C(f)$ (see [**Ac**]). The elements of $C(f)$ are called the *critical values* of f. The map $f : X \to Y$ is said to be an n-sheeted covering, *branched* or *ramified* above the points in $C(f)$ and *unramified* if $C(f) = \emptyset$. If one chooses suitable local coordinates at $x \in X$ and $f(x)$, then f 'looks like' $z \mapsto z^q$ for some positive integer q; the *order of branching* of f at x is equal to $q - 1$. If the order of branching at $x \in X$ is equal to zero, then x is said to be a *regular point* of f; otherwise x is a *critical point.*

Definition 1.5. *Two Riemann surfaces X and Y are conformally equivalent if there exists a holomorphic bijection $f : X \to Y$.* □

A holomorphic bijection $f : X \to X$ is called an *automorphism* of X, and the set $\mathrm{Aut}\,X$ of all automorphisms of X forms a group under composition. Every Riemann surface X is orientable ([**JS2**]), with the automorphisms of X preserving orientation. (Unless otherwise specified, all automorphisms will be orientation-preserving or conformal automorphisms; where necessary we will refer to orientation-reversing or anti-conformal automorphisms.)

Classical uniformization

A Riemann surface X is simply connected if its underlying topological space is simply connected. The Uniformization Theorem (for example see [**Sp**]) states that up to conformal equivalence there are just three simply connected Riemann surfaces: the Riemann sphere Σ, the complex plane $\mathbf{C}$, and the upper half-plane $\mathbf{H}$. The automorphism groups of these three Riemann surfaces are well-known:

(i) $\mathrm{Aut}\,\Sigma = \mathrm{PSL}_2(\mathbf{C}) = \{z \mapsto \frac{az+b}{cz+d} \mid a,b,c,d \in \mathbf{C}, ad - bc = 1\}$;

(ii) $\mathrm{Aut}\,\mathbf{C} = \{z \mapsto az + b \mid a, b \in \mathbf{C}, a \neq 0\}$; 1.6

(iii) $\mathrm{Aut}\,\mathbf{H} = \mathrm{PSL}_2(\mathbf{R}) = \{z \mapsto \frac{az+b}{cz+d} \mid a,b,c,d \in \mathbf{R}, ad - bc = 1\}$.

If Γ is a subgroup of $\mathrm{Aut}\,\mathcal{U}$ (where $\mathcal{U} = \Sigma, \mathbf{C}$ *or* $\mathbf{H}$) then Γ is said to act *discontinuously* on $\mathcal{U}$ if every $s \in \mathcal{U}$ has some neighbourhood V such that $V \cap \gamma(V) = \emptyset$ for all non-identity $\gamma \in \Gamma$; consequently such an action is free. Using ideas from the theory of covering spaces it can be shown that ([**JS2**]):

Theorem 1.7. *Every connected Riemann surface X is conformally equivalent to $\mathcal{U}/\Gamma$ ($\mathcal{U} = \Sigma, \mathbf{C}$ or $\mathbf{H}$) where Γ is a subgroup of $\mathrm{Aut}\,\mathcal{U}$ acting discontinuously on $\mathcal{U}$.*
□

Let $X = \mathcal{U}/\Gamma$ for some $\Gamma \leq \mathrm{Aut}\,\mathcal{U}$ acting discontinuously on $\mathcal{U}$. Then the group of conformal automorphisms of X, $\mathrm{Aut}\,X$, is isomorphic to $N(\Gamma)/\Gamma$ where $N(\Gamma)$ is the normalizer of Γ in $\mathrm{Aut}\,\mathcal{U}$. Furthermore, every group of automorphisms of X is isomorphic to K/Γ, for some group $K \leq \mathrm{Aut}\,\mathcal{U}$ containing Γ as a normal subgroup (see [**JS2**]).

An anti-conformal automorphism of the complex plane has the form $z \mapsto a\bar{z}+b$. The set of all conformal and anti-conformal automorphisms of $\mathbf{C}$ forms a group, denoted by $\overline{\mathrm{Aut}}\,\mathbf{C}$, which contains $\mathrm{Aut}\,\mathbf{C}$ with index 2. The groups $\overline{\mathrm{Aut}}\,\Sigma$ and $\overline{\mathrm{Aut}}\,\mathbf{H}$ are defined similarly: anti-conformal automorphisms of Σ have the form $z \mapsto \frac{a\bar{z}+b}{c\bar{z}+d}$ with $a, b, c, d \in \mathbf{C}$ and $ad - bc = 1$, while for $\mathbf{H}$ they have the form $z \mapsto \frac{a\bar{z}+b}{c\bar{z}+d}$ with $a, b, c, d \in \mathbf{R}$ and $ad - bc = -1$ (one can show that $\overline{\mathrm{Aut}}\,\mathbf{H} \cong \mathrm{PGL}_2(\mathbf{R})$). A Riemann surface $X = \mathcal{U}/\Gamma$ admits an anti-conformal automorphism if Γ is normalized by some element $n \in \overline{\mathrm{Aut}}\,\mathcal{U} \setminus \mathrm{Aut}\,\mathcal{U}$.

Groups with signature

Let g be a non-negative integer and $m_1, \ldots, m_r \in N \cup \{\infty\}$. Then a group with signature $(g; m_1, \ldots, m_r)$ is a group Γ with the presentation

$$\Gamma = < x_1, \ldots, x_r, a_1, b_1, \ldots, a_g, b_g \mid {x_1}^{m_1} = \ldots = {x_r}^{m_r} = \prod_{i=1}^{g} [a_i, b_i] \prod_{j=1}^{r} x_j = 1 >$$

where any relation of the form $x_i^{\infty} = 1$ is omitted; the m_i are called the periods of Γ. If none of the periods are equal to 1 or ∞, then two groups with signature $(g; m_1, \ldots, m_r)$ and $(g'; n_1, \ldots, n_s)$ are isomorphic if and only if $g = g'$ and $(m_1, \ldots, m_r)$ is a permutation of $(n_1, \ldots, n_s)$. If we define

$$\mu(\Gamma) = 2\pi \Big\{ (2g - 2) + \sum_{i=1}^{r} (1 - \frac{1}{m_i}) \Big\}$$

where $1/\infty = 0$, then it is known that an index k subgroup Γ_1 of Γ satisfies

$$\mu(\Gamma_1) = k\mu(\Gamma). \qquad 1.8$$

Singerman [**Sin1**] has proved the following. (Note: Only the case $\mu(\Gamma) > 0$ is considered in [**Sin1**], however the proof for $\mu(\Gamma) = 0$ follows similarly. For $\mu(\Gamma) < 0$ the groups involved are finite, and so each group can be checked directly.)

Theorem 1.9. *Let Γ be a group with signature $(g; m_1, \dots, m_r)$ where $\mu(\Gamma) \geq 0$. Then Γ contains a subgroup Γ_1 of index k with signature*

$$(g'; n_{11}, \dots, n_{1p_1}, \dots, n_{r1}, \dots, n_{rp_r})$$

if and only if there exists a finite permutation group G, transitive on k points, and an epimorphism $\theta : \Gamma \to G$ such that

(i) each permutation $\theta(x_j)$ has precisely p_j cycles of lengths less than m_j, these lengths being $\frac{m_j}{n_{j1}}, \dots, \frac{m_j}{n_{jp_j}}$;

(ii) $\mu(\Gamma_1) = k\mu(\Gamma)$.

If $m_j = \infty$, then in (i) the number of infinite periods of Γ_1 induced by m_j is equal to the number of cycles of $\theta(x_j)$. If stab(1) is the stabilizer of the point 1 in G, then $\Gamma_1 = \theta^{-1}(\text{stab}(1))$ is a subgroup of Γ with the required signature. □

A group Γ acts *properly discontinuously* on $\mathcal{U}$ if for each compact set $U \subset \mathcal{U}$,

$$\{\gamma \in \Gamma \mid \gamma U \cap U \neq \emptyset\}$$

is finite. Every group with signature acts as a properly discontinuous group of conformal isometries of one of the three simply-connected Riemann surfaces $\mathcal{U} = \Sigma, \mathbf{C}$ or $\mathbf{H}$ depending on whether $\mu(\Gamma) < 0$, $\mu(\Gamma) = 0$ or $\mu(\Gamma) > 0$. The quotient $\mathcal{U}/\Gamma$ will be a Riemann surface; compact if all of the periods m_i are finite, otherwise non-compact with one cusp or puncture for each infinite period.

Examples 1.10. (i) A group with signature $(g; -)$ is called a *surface group of genus g*. Every surface group is torsion-free and acts freely on $\mathcal{U}$. If Γ is a surface group, then the fundamental group of $\mathcal{U}/\Gamma$ is isomorphic to Γ.

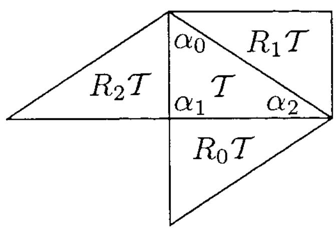


Figure 1.1

(ii) A group with signature $(0; l_0, l_1, l_2)$ is called a *triangle group* and will be denoted by $\Gamma(l_0, l_1, l_2)$. The extended triangle group $\Gamma^*(l_0, l_1, l_2)$ is the group generated by reflections in the sides of a triangle $\mathcal{T}$ with vertices $\alpha_0, \alpha_1, \alpha_2$ and corresponding angles $\pi/l_0, \pi/l_1, \pi/l_2$ (see Figure 1.1). The triangle will lie in Σ, $\mathbf{C}$ or $\mathbf{H}$ as $1/l_0 + 1/l_1 + 1/l_2 > 1, = 1$ or < 1. If R_i is a reflection in the side of $\mathcal{T}$ opposite to the vertex α_i $(i = 0, 1, 2)$ then $\Gamma^*(l_0, l_1, l_2)$ has the presentation

$$< R_0, R_1, R_2 \,|\, R_0^2 = R_1^2 = R_2^2 = (R_1R_2)^{l_0} = (R_2R_0)^{l_1} = (R_0R_1)^{l_2} = 1 >$$

(see [**Mag**]). The triangle group $\Gamma(l_0, l_1, l_2)$ is the index 2 subgroup of $\Gamma^*(l_0, l_1, l_2)$ consisting of the orientation-preserving transformations. Setting $x_0 = R_1R_2$, $x_1 = R_2R_0$ and $x_2 = R_0R_1$ so that x_i acts by rotating $\mathcal{T}$ anticlockwise through an angle $2\pi/l_i$ about α_i $(i = 0, 1, 2)$, we obtain the presentation

$$\Gamma(l_0, l_1, l_2) = < x_0, x_1, x_2 \,|\, x_0^{l_0} = x_1^{l_1} = x_2^{l_2} = x_0x_1x_2 = 1 > .$$

(iii) Using Theorem 1.9 we give some necessary conditions for the triangle group $\Gamma(l_0, l_1, l_2)$ to contain a surface group $(g; -)$ with index k. Since a surface group contains no periods, the epimorphism $\theta : \Gamma \to G$ must satisfy: $\theta(x_0)$ has precisely $\frac{k}{l_0}$ cycles of length l_0; $\theta(x_1)$ has precisely $\frac{k}{l_1}$ cycles of length l_1; and $\theta(x_2)$ has precisely $\frac{k}{l_2}$ cycles of length l_2. Hence l_0, l_1 and l_2 must all divide k, and further by Theorem 1.9(ii) g, k, l_0, l_1, l_2 must satisfy the Riemann-Hurwitz formula

$$2(g-1) = k\Big(1 - \frac{1}{l_0} - \frac{1}{l_1} - \frac{1}{l_2}\Big). \quad \square \qquad 1.11$$

A *Fuchsian group* is a discrete subgroup of $\mathrm{PSL}_2(\mathbf{R})$, the automorphism group of the upper half-plane $\mathbf{H}$, and it is known that a subgroup $\Gamma \leq \mathrm{PSL}_2(\mathbf{R})$ is a Fuchsian group if and only if Γ acts properly discontinuously on $\mathbf{H}$. A Fuchsian group Γ is of the *first kind* if its limit set $L(\Gamma) = \mathbf{R} \cup \{\infty\}$. Every finitely generated Fuchsian group of the first kind is a group with signature, where $\mu(\Gamma)$ is the hyperbolic measure of a fundamental region for Γ. For a proof of the following theorem see [**JS2**]:

Theorem 1.12. *Let Γ_1 and Γ_2 be Fuchsian surface groups. Then $\mathbf{H}/\Gamma_1$ and $\mathbf{H}/\Gamma_2$ are conformally equivalent if and only if Γ_1 and Γ_2 are conjugate in* $\mathrm{PSL}_2(\mathbf{R})$. $\square$

It is known (see [**JS2**, p.261]) that any cocompact Fuchsian group Γ satisfies $\mu(\Gamma) \geq \frac{\pi}{21}$. Hence if a cocompact Fuchsian group Γ contains an index k surface group Γ_1 with signature $(g; -)$, we have by 1.8 that

$$k = \frac{\mu(\Gamma_1)}{\mu(\Gamma)} = \frac{2\pi(2g-2)}{\mu(\Gamma)} \leq \frac{2\pi(2g-2)}{(\frac{\pi}{21})} = 84(g-1). \qquad 1.13$$

In particular, if the Fuchsian triangle group $\Gamma(l_0, l_1, l_2)$ (for $l_0, l_1, l_2 < \infty$) contains a surface group $(g; -)$ with index k, then by 1.13 we must have $k \leq 84(g-1)$.

Fundamental polygons and Schreier generators

Let Γ be a properly discontinuous group of isometries of $\mathcal{U}$. A *polygon* will be a closed, connected set with non-empty interior whose boundary is a union of geodesic segments (these will be called the sides of the polygon). A *fundamental polygon* $\mathcal{P}$ for Γ is a polygon (possibly with infinitely many sides) satisfying the following conditions:

F1. $\bigcup_{\gamma \in \Gamma} \gamma\mathcal{P} = \mathcal{U}$;

F2. $\dot{\mathcal{P}} \cap \gamma\dot{\mathcal{P}} = \emptyset$ for all $1 \neq \gamma \in \Gamma$ (where $\dot{\mathcal{P}}$ denotes the interior of $\mathcal{P}$);

F3. If b is a side of $\mathcal{P}$, then there exists a unique side $\bar{b}$ of $\mathcal{P}$ (possibly equal to b) and a unique element $\beta \in \Gamma$ such that $\beta\bar{b} = b$ and $\beta\mathcal{P}$ is a polygon adjacent to $\mathcal{P}$ along b (see Figure 1.2). We will say that β pairs $\bar{b}$ with b;

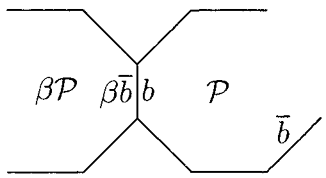


Figure 1.2

F4. $\mathcal{P}$ is locally finite. That is, any compact set $U \subset \mathcal{U}$ intersects only finitely many images of $\mathcal{P}$.

It is known (see [**Bea**]) that the side-pairing elements of $\mathcal{P}$ generate Γ, and so to a fundamental polygon $\mathcal{P}$ for Γ we have a generating set

$$\Phi = \{\beta_1, \beta_2, \ldots\}$$

consisting of elements which pair the sides of $\mathcal{P}$. Let Γ be a group with the generating set Φ, and suppose that Λ is a subgroup of Γ. A set T of right coset representatives for Λ in Γ which satisfies:

(i) $\Gamma = \dot{\bigcup}_{t \in T} \Lambda t$ (where $\dot{\cup}$ denotes the disjoint the union);

(ii) if $t_1 t_2 \ldots t_n$ is a word in Φ, then $t_1 \ldots t_n \in T \Rightarrow t_1 \ldots t_{n-1} \in T$,

is called a *(right) Schreier transversal* for Λ in Γ. For any $\gamma \in \Gamma$ there is a unique element $t \in T$ such that $\Lambda t = \Lambda\gamma$, and we let $\overline{\gamma} = t$. The set of elements

$$\{\, t\beta(\overline{t\beta})^{-1} \,|\, t \in T, \beta \in \Phi \,\}$$

generate Λ and are called the *Schreier generators* for Λ in Γ (see [**John**]). The following theorem of Hoare and Singerman [**HS**] proves that a Schreier transversal for Λ in Γ is exactly the set needed to construct a connected fundamental polygon for Λ. The result will be used extensively in Chapter 2.

Theorem 1.14. *Let Γ be a properly discontinuous group of isometries of $\mathcal{U}$ (where $\mathcal{U} = \Sigma$, $\mathbf{C}$, or $\mathbf{H}$) with a fundamental polygon $\mathcal{P}$. If Λ is a subgroup of Γ with a (right) Schreier transversal T, then*

$$\mathcal{P}' = \bigcup_{t \in T} t\mathcal{P}$$

is a fundamental polygon for Λ whose sides are paired by the Schreier generators corresponding to T.

Proof. Suppose that $t \in T$ has the form $t = s\beta$ where $\beta \in \Phi$ (these are taken to be words in Φ). Then $t\mathcal{P}$ and $s\mathcal{P}$ are adjacent along sb for some edge b of $\mathcal{P}$, and since T is a Schreier transversal we must have $s \in T$. Now by induction on the lengths of words in T, $\mathcal{P}'$ is connected. The boundary of $\mathcal{P}'$ will be the set of all edges sb of $s\mathcal{P}$ (b a side of $\mathcal{P}$ and $s \in T$) which have no $t\mathcal{P}$ ($t \in T$) adjacent to them; these sb are the sides of $\mathcal{P}'$. Hence $\mathcal{P}'$ is a connected polygon. We now show that $\mathcal{P}'$ is a fundamental polygon for Λ satisfying $F1$ to $F4$ above.

F1: This holds for $\mathcal{P}'$ since $\Lambda\mathcal{P}' = \Lambda T\mathcal{P} = \Gamma\mathcal{P} = \mathcal{U}$.

F2: Suppose that $x \in \mathcal{P}' \cap \lambda\mathcal{P}'$ for $1 \neq \lambda \in \Lambda$. Then $x \in s\mathcal{P} \cap \lambda t\mathcal{P}$ for some $s, t \in T$ with $s \neq \lambda t$, and because $\mathcal{P}$ is a fundamental polygon for Γ, we must have $x \in sb$ for some side b of $\mathcal{P}$. We now show that sb is on the boundary of $\mathcal{P}'$. For if not, then by our definition of the boundary for $\mathcal{P}'$, $\lambda t \in T$ and so by the Schreier property we have $\lambda \in T$. However $\lambda \in \Lambda$ so that $\Lambda\lambda = \Lambda 1$, and since $1 \in T$, we must have $\lambda = 1$, a contradiction. Therefore $\lambda t \notin T$ and $sb \ni x$ is on the boundary of $\mathcal{P}'$.

F3: Let $\beta \in \Phi$ pair $\bar{b}$ with b where $\bar{b}$ and b are sides of $\mathcal{P}$, and let $s \in T$. Then sb is a side of $\mathcal{P}'$ if and only if $s\beta \notin T$, and so if and only if there exists $t \in T$ with $\Lambda s\beta = \Lambda t$ and $t \neq s\beta$. This happens if and only if the Schreier generator $s\beta t^{-1} \neq 1$. In this case $t\bar{b}$ is a side of $\mathcal{P}'$ paired with sb by the (unique since $F3$ holds for $\mathcal{P}$) element $s\beta t^{-1}$.

F4: This holds for $\mathcal{P}'$ since it holds for $\mathcal{P}$. □

Algebraic curves and Belyĭ's theorem

If $T(x, y, z) \in \mathbf{C}[x, y, z]$ is an irreducible homogeneous polynomial with complex coefficients, then the algebraic curve

$$C_T = \{[x, y, z] \in P^2(\mathbf{C}) \,|\, T(x, y, z) = 0\}$$

can be normalized to obtain a compact Riemann surface X_T (we note that X_T is not necessarily isomorphic to C_T, see [**Ki**] or [**Ful**]). Conversely, given a compact Riemann surface X there exists an irreducible homogeneous polynomial $T(x, y, z) \in \mathbf{C}[x, y, z]$ such that X_T, the normalization of C_T, is isomorphic to X (see [**Gri**] for precise details). We say that a Riemann surface X is defined over a subfield $F \subseteq \mathbf{C}$ if $X \cong X_T$ for some polynomial $T(x, y, z) \in F[x, y, z]$.

A complex number $\alpha \in \mathbf{C}$ is an *algebraic number* if $f(\alpha) = 0$ for some non-zero polynomial $f(x) \in \mathbf{Z}[x]$, and α is said to be an *algebraic integer* if $f(x)$ is a monic polynomial. The set of all algebraic numbers forms a field, denoted $\overline{\mathbf{Q}}$. We are particularly interested in Riemann surfaces that are defined over finite extension fields of $\mathbf{Q}$.

Definition 1.15. *Let X be a compact Riemann surface. Then a non-constant holomorphic mapping $\beta : X \to \Sigma$ is called a Belyĭ function if $C(\beta) \subseteq \{0, 1, \infty\}$.* □

The group $\mathrm{Aut}\,\Sigma$ acts triply transitively on Σ, so if $f : X \to \Sigma$ is any non-constant holomorphic mapping with at most three critical values, then by composing f with an automorphism of Σ we can obtain a Belyĭ function. A compact Riemann surface with a Belyĭ function will be called a *Belyĭ surface*, and (X, β) will be called a *Belyĭ pair*. Belyĭ [**Bel**] (or see [**JS3**]) gave the following classification of compact Riemann surfaces defined over the algebraic numbers:

Theorem 1.16. *A compact Riemann surface X is defined over $\overline{\mathbf{Q}}$ if and only if there exists a Belyĭ function $\beta : X \to \Sigma$.* □

An alternative classification of compact Riemann surfaces defined over $\overline{\mathbf{Q}}$ was given by Wolfart [**Wo1**, **Wo2**]:

Theorem 1.17. *A compact Riemann surface X is defined over $\overline{\mathbf{Q}}$ if and only if $X \cong \mathcal{U}/\Lambda$ where Λ is a finite index subgroup of a cocompact triangle group and $\mathcal{U} = \Sigma, \mathbf{C}$ or $\mathbf{H}$.*

Proof. ($\Leftarrow$) Suppose that $X \cong \mathcal{U}/\Lambda$ where $\mathcal{U} = \Sigma, \mathbf{C}$ or $\mathbf{H}$ and Λ is a finite index subgroup of the triangle group $\Gamma = \Gamma(l_0, l_1, l_2)$ where $l_i < \infty$ for $i = 0, 1, 2$. Then $\Gamma(l_0, l_1, l_2)$ acts on a triangle with vertex angles $\pi/l_0, \pi/l_1, \pi/l_2$ as described in Examples 1.10(ii), and the projection $\pi : \mathcal{U} \to \mathcal{U}/\Gamma$ has at most three critical values $C(\pi)$ corresponding to the orbits of those vertices with $l_i > 1$. Since $\mathrm{Aut}\,\Sigma$ acts triply transitively on Σ, there is an isomorphism $\mathcal{U}/\Gamma \to \Sigma$ which sends the critical values $C(\pi)$ into $\{0, 1, \infty\}$ (one can use the Schwarz triangle functions to define this isomorphism, see [**Wo2**]). We note that the critical values of the projection $\mathcal{U}/\Lambda \to \mathcal{U}/\Gamma$ are contained in the set $C(\pi)$. Hence by composing the isomorphism $X \cong \mathcal{U}/\Lambda$ with the projection $\mathcal{U}/\Lambda \to \mathcal{U}/\Gamma$ and the above isomorphism $\mathcal{U}/\Gamma \cong \Sigma$ we obtain a Belyĭ function on X, so that X is defined over $\overline{\mathbf{Q}}$ by Theorem 1.16. ($\Rightarrow$) We refer the reader to [**Wo1**] for a proof of the converse. □

The connections between Belyĭ functions, algebraic curves and geometric objects called dessins d'enfants (otherwise called maps and hypermaps, see Chapter 2) were first outlined by Grothendieck in his *Esquisse d'un programme* [**Gro**] (or see [**Sc**]). In the following chapters we will produce examples illustrating Grothendieck's ideas, and their connections with the absolute Galois group $\mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q})$.

Chapter 2

Maps and hypermaps

We will use the term *dessin* to encompass two related theories: maps which are two-cell decompositions of orientable surfaces [**JS1**], and hypermaps [**JS3**, **CoSi**] which can be thought of as bipartite maps. Jones and Singerman [**JS1**, **JS3**] have shown that every finite dessin can be embedded naturally into some Riemann surface $X = \mathcal{U}/\Lambda$ where Λ is a finite index subgroup of a cocompact triangle group, and conversely that every finite index inclusion $\Lambda \leq \Gamma$ where Γ is a cocompact triangle group defines a dessin embedded into $\mathcal{U}/\Lambda$. Thus by the theorems of Belyĭ and Wolfart, the Riemann surfaces that carry finite dessins are precisely those that are defined over $\overline{\mathbf{Q}}$.

2.1. Topological and algebraic maps

For a detailed account of the material in this section, we refer the reader to [**JS1**]. A map is informally an embedding of a connected graph $\mathcal{G}$ into an orientable surface $\mathcal{S}$ such that the connected components of $\mathcal{S}\backslash\mathcal{G}$ (called the *faces* of the map) are simply connected. We note that $\mathcal{G}$ is allowed to contain loops and free-edges (i.e. edges for which only one end is incident with a vertex). If the map has an underlying surface $\mathcal{S}$ and an associated graph $\mathcal{G}$ with vertex set $\mathcal{V}$, then the map is represented by the triple $(\mathcal{G}, \mathcal{V}, \mathcal{S})$. Plane trees embedded into the complex plane $\mathbf{C}$ as studied by Shabat and Zvonkin [**SZ**] are examples of maps, as are the Platonic solids embedded into the Riemann sphere Σ.

Figure 2.1(a) shows a map with one vertex, one face and three edges (one of which is free) embedded into the torus. The torus is obtained by identifying opposite sides of the square.

(a) (b)

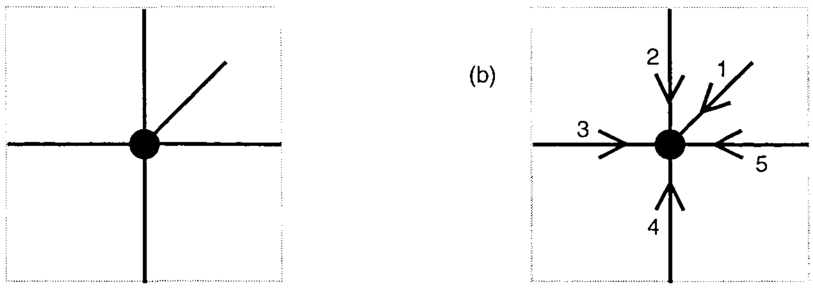


Figure 2.1: A map of genus 1

We now show how to associate an algebraic structure to a map $\mathcal{M}$. Whenever an edge intersects a vertex we put an arrow on the edge facing that vertex, as shown in Figure 2.2; every such vertex-edge pair is called a *dart* of $\mathcal{M}$. The genus 1 map of Figure 2.1(a) has five darts, and they are shown in Figure 2.1(b) labelled from 1 to 5.

Figure 2.2: A dart

Letting Ω denote the set of darts of $\mathcal{M}$, we define two permutations of Ω as follows: r_0 consists of cycles formed by going round each vertex in an anticlockwise direction, while r_1 is the permutation consisting of cycles which interchange the two darts on an edge or loop, and fix the dart on a free edge. The product $r_2 = r_1 r_0^{-1}$ consists of cycles which define anticlockwise rotations about each face of $\mathcal{M}$, where the composition is taken from left to right. We let $G = \mathrm{gp} < r_0, r_1 > \leq S^{\Omega}$ be the group generated by r_0 and r_1, and note that G is a transitive permutation group because the graph underlying $\mathcal{M}$ is connected; G will be called the *monodromy group* of $\mathcal{M}$. We define the *algebraic map* of $\mathcal{M}$ to be $Alg\,\mathcal{M} = (G, \Omega, r_0, r_1)$ and if r_0 and r_2 have orders m and n respectively, we say that $\mathcal{M}$ has *type* (m, n). We prove in Theorem 2.9 that from any algebraic map (G, Ω, r_0, r_1) where G is a transitive permutation group, one can recover a topological map $\mathcal{M}$ with $Alg\,\mathcal{M} = (G, \Omega, r_0, r_1)$. If $\mathcal{M}$ has type (m, n), then $\mathcal{M}$ is said to have *type dividing* (r, s) for all r, s such that $m|r$ and $n|s$.

The genus 1 map of Figure 2.1(b) has the set of darts $\Omega = \{1, 2, 3, 4, 5\}$ with $r_0 = (1\,2\,3\,4\,5)$, $r_1 = (1)(2\,4)(3\,5)$, and $r_2 = (1\,5\,2\,3\,4)$. The map has type $(5, 5)$.

Map subgroups

The triangle group $\Gamma(m,2,n)$ has a presentation of the form

$$\Gamma(m,2,n) = \mathrm{gp} < x_0, x_1 \mid x_0^m = x_1^2 = (x_1 x_0^{-1})^n = 1 > . \qquad 2.1$$

If $\mathcal{M}$ is a map of type (m,n) with $Alg\,\mathcal{M} = (G, \Omega, r_0, r_1)$, then there is an epimorphism $\theta : \Gamma(m,2,n) \to G$ given by $x_0 \mapsto r_0$ and $x_1 \mapsto r_1$. If $G_\alpha = \{g \in G \mid \alpha g = \alpha\}$ for any $\alpha \in \Omega$, then $M = \theta^{-1}(G_\alpha)$ is called the (*canonical*) *map subgroup* associated to $\mathcal{M}$. (All maps considered will have finite type (m,n), so that $\Gamma(m,2,n)$ is cocompact.) We can identify Ω with the set of right M-cosets in $\Gamma(m,2,n)$ by the bijection

$$Mh \longmapsto \alpha(h\theta) \qquad 2.2$$

where $h \in \Gamma(m,2,n)$. (The map 2.2 is well defined, since if $Mg = Mh$ then $gh^{-1} \in M$ and so $(gh^{-1})\theta \in G_\alpha$ which implies that $\alpha(g\theta) = \alpha(h\theta)$. By reversing this argument we see that 2.2 is injective, while $\alpha x \in \Omega$ is the image of My for any $y \in \theta^{-1}(x)$ so that 2.2 is surjective.) The permutation r_0 (respectively r_1) then corresponds to the action of x_0 (respectively x_1) on the right M-cosets in $\Gamma(m,2,n)$, and in this way any finite index subgroup $M \leq \Gamma(m,2,n)$ is the map subgroup for some algebraic map. An algebraic map may be defined more abstractly as a finite transitive permutation representation $\theta : \Gamma(m,2,n) \to G$.

Let $\mathcal{M}$ be a map with an associated map subgroup $M \leq \Gamma(m,2,n)$. From Theorem 1.14 we see that $X = \mathcal{U}/M$ (where $\mathcal{U} = \Sigma, \mathbf{C},$ or $\mathbf{H}$) defines a Riemann surface which can be constructed by 'gluing together' copies of a fundamental region for $\Gamma(m,2,n)$. It will be proved in this section that $\mathcal{M}$ can be embedded naturally into X, so that the edges of $\mathcal{M}$ are geodesics in X.

Map automorphisms

If $\mathcal{M}_i = (\mathcal{G}_i, \mathcal{V}_i, \mathcal{S}_i)$ $(i = 1,2)$ are topological maps, then a *morphism* $\phi : \mathcal{M}_1 \to \mathcal{M}_2$ is a (possibly branched) covering of surfaces $\phi : \mathcal{S}_1 \to \mathcal{S}_2$, preserving orientation, such that:

(i) $\phi^{-1}(\mathcal{G}_2) = \mathcal{G}_1$ and $\phi^{-1}(\mathcal{V}_2) = \mathcal{V}_1$;

(ii) all branch points have finite order.

We say that $\mathcal{M}_1$ *covers* $\mathcal{M}_2$ if there exists a morphism from $\mathcal{M}_1$ to $\mathcal{M}_2$. The following theorem concerning morphisms of maps was proved in [**JS1**]:

Theorem 2.3. *Let $\mathcal{M}_1$, $\mathcal{M}_2$ be maps of type (m, n). Then $\mathcal{M}_1$ covers $\mathcal{M}_2$ if and only if we can find map-subgroups $M_i \leq \Gamma(m, 2, n)$ for $\mathcal{M}_i$ $(i = 1, 2)$ with $M_1 \leq M_2$.* □

Two topological maps are isomorphic if there exists a morphism between them such that the covering of surfaces ϕ is a homeomorphism. Thus an automorphism of a topological map is a self-morphism induced by a homeomorphism of the underlying surface to itself. We can also define morphisms between algebraic maps: an *algebraic morphism* between $Alg\,\mathcal{M}_1 = (G_1, \Omega_1, r_0, r_1)$ and $Alg\,\mathcal{M}_2 = (G_2, \Omega_2, s_0, s_1)$ is a pair (δ, σ) of functions $\delta : \Omega_1 \to \Omega_2$, $\sigma : G_1 \to G_2$, where σ is a group homomorphism, $r_0\sigma = s_0$, $r_1\sigma = s_1$ and the diagram in Figure 2.3 commutes (the horizontal arrows in the diagram represent the actions of G_1 and G_2). Thus we require that $(\alpha g)\delta = (\alpha\delta)(g\sigma)$ for all $g \in G_1$, $\alpha \in \Omega_1$. Two algebraic maps are then isomorphic if there exists an algebraic morphism (δ, σ) between them where σ is a group isomorphism and δ is a bijection.

$$\begin{array}{ccc}
\Omega_1 \times G_1 & \longrightarrow & \Omega_1 \\
{\scriptstyle (\delta,\sigma)}\big\downarrow & & \big\downarrow{\scriptstyle \delta} \\
\Omega_2 \times G_2 & \longrightarrow & \Omega_2
\end{array}$$

Figure 2.3

The set of topological automorphisms of a map forms an infinite group since its edges can be continuously deformed, and each vertex can be perturbed in some small neighbourhood. We therefore follow Jones and Singerman by defining the automorphism group $\operatorname{Aut}\mathcal{M}$ of a map $\mathcal{M}$ to be the group of algebraic automorphisms of its associated algebraic map $Alg\,\mathcal{M}$.

Theorem 2.4. *Let $\mathcal{M}$ be a map with $Alg\,\mathcal{M} = (G, \Omega, r_0, r_1)$. Then*

$$\operatorname{Aut}\mathcal{M} \cong C_{S^\Omega}(G)$$

where $C_{S^\Omega}(G)$ is the centralizer of G in S^Ω.

Proof. The map is defined by the two permutations r_0 and r_1. Therefore $(\delta, \sigma) \in \operatorname{Aut}\mathcal{M}$ if and only if $r_0\sigma = r_0$, $r_1\sigma = r_1$, $r_0\delta = \delta r_0$ and $r_1\delta = \delta r_1$. Since r_0 and r_1

generate G, σ must be equal to the identity permutation and so

$$\mathrm{Aut}\,\mathcal{M} \cong \{\delta \in S^{\Omega} \mid \delta \text{ centralizes } G\}. \quad \square$$

It follows from Theorem 2.4 that $\mathrm{Aut}\,\mathcal{M}$ will be a finite group for any finite map $\mathcal{M}$. We will require the following results, which are proved in [**JS1**]:

Theorem 2.5. *Let M_1, M_2 be map subgroups of $\Gamma(m,2,n)$. Then they give rise to isomorphic maps if and only if they are conjugate in $\Gamma(m,2,n)$.* $\square$

Theorem 2.6. *If $\mathcal{M}$ has a map subgroup $M \leq \Gamma(m,2,n)$, then $Aut\,\mathcal{M} \cong N_{\Gamma}(M)/M$, where $N_{\Gamma}(M)$ is the normalizer of M in $\Gamma(m,2,n)$.* $\square$

Let $\mathcal{M}$ be a map of type (m,n) with $Alg\,\mathcal{M} = (G,\Omega,r_0,r_1)$, and the associated map subgroup $M = \theta^{-1}(G_\alpha) \leq \Gamma(m,2,n)$ where $\theta : \Gamma(m,2,n) \to G$ and G_α are defined as above. If we identify Ω with the set of right M-cosets in $\Gamma(m,2,n)$ as in 2.2, then the normalizer $N_{\Gamma}(M)$ acts by left-multiplication on the cosets; so $n \in N_{\Gamma}(M)$ acts by

$$Mg \longmapsto n^{-1}Mg = Mn^{-1}g.$$

This action commutes with the right action of $\Gamma(m,2,n)$, since $Mn^{-1}(gh) = M(n^{-1}g)h$ for all $n,g,h \in \Gamma$. Using Theorem 2.6 and the epimorphism $\theta : \Gamma(m,2,n) \to G$ we have

$$Aut\,\mathcal{M} \cong N_G(G_\alpha)/G_\alpha$$

where $N_G(G_\alpha)$ is the normalizer of G_α in G. We can then identify Ω with the set of right G_α-cosets in G, so that $n \in N_G(G_\alpha)$ acts by $G_\alpha g \longmapsto n^{-1}G_\alpha g = G_\alpha n^{-1} g$.

Quotient maps and universal maps

Let $\mathcal{A} = (G,\Omega,r_0,r_1)$ be an algebraic map. If T is a group of automorphisms of $\mathcal{A}$, then T induces an equivalence relation $\sim$ on Ω where $\alpha \sim \beta$ if $\alpha = \beta t$ for some $t \in T$. There is an action of G on the quotient set $\overline{\Omega} = \Omega/\sim$ given by $g : [\alpha] \mapsto [\alpha g]$, and if K is the kernel of this action, then $K \triangleleft G$ and $\overline{G} = G/K$ acts faithfully and transitively on $\overline{\Omega}$. Setting $\overline{r}_0 = Kr_0$ and $\overline{r}_1 = Kr_1$, we call $\overline{\mathcal{A}} = (\overline{G},\overline{\Omega},\overline{r}_0,\overline{r}_1)$ the quotient map of $\mathcal{A}$ by T.

Given the presentation for $\Gamma(m,2,n)$ in 2.1, the *universal algebraic map of type* (m,n) is defined to be

$$\hat{\mathcal{A}} = < \Gamma, |\Gamma|, x_0, x_1 >$$

where $\Gamma = \Gamma(m,2,n)$, $|\Gamma|$ is the set underlying $\Gamma(m,2,n)$, and $g \in \Gamma(m,2,n)$ acts on $|\Gamma|$ by right multiplication: $g : h \mapsto hg$ for all $h \in |\Gamma|$. Any subgroup $M \leq \Gamma(m,2,n)$ acts as a group of automorphisms of $\hat{A}$ by the left action $a : h \mapsto a^{-1}h$ for all $h \in |\Gamma|$, $a \in M$. We can therefore form the quotient algebraic map

$$\hat{A}/M = (\Gamma/M^\star, \Gamma/M, M^\star x_0, M^\star x_1)$$

where $\Gamma/M = \{Mg \,|\, g \in \Gamma\}$, $M^\star =$ the core of M in Γ, and $\Gamma/M^\star$ acts on Γ/M by $M^\star g : Mh \mapsto Mhg$ for all $g, h \in \Gamma$. It is proved in [**JS1**] that every algebraic map of type (m,n) can be obtained as a quotient of the universal algebraic map of type (m,n).

Maps on Riemann surfaces

The triangle group

$$\Gamma(m,2,n) = < x_0, x_1 | x_0^m = x_1^2 = (x_0x_1)^{-n} = 1 >$$

acts naturally on the $\frac{\pi}{m}, \frac{\pi}{2}, \frac{\pi}{n}$ triangle $\mathcal{T}$ of Figure 2.4(a) as described in Examples 1.10(ii); as usual $\mathcal{T}$ lies in $\mathcal{U} = \Sigma, \mathbf{C}$, or $\mathbf{H}$ as $\frac{1}{m} + \frac{1}{n} > \frac{1}{2}, = \frac{1}{2}$ or $< \frac{1}{2}$. If $\mathcal{P}$ is the fundamental polygon for Γ formed by reflecting $\mathcal{T}$ in the side with angles $\frac{\pi}{2}$ and $\frac{\pi}{m}$ as shown in Figure 2.4(b), then x_0 and x_1 are the side pairing transformations of $\mathcal{P}$. We draw a half-edge and a dart on $\mathcal{P}$ as shown in Figure 2.4(c). The set $\Omega = \{\gamma\mathcal{P} | \gamma \in \Gamma\}$ will tessellate $\mathcal{U}$, and the map formed by the set of half-edges will be called the *universal topological map* $\hat{\mathcal{M}}$ *of type* (m,n). The darts of $\hat{\mathcal{M}}$ will be identified with Ω in the obvious manner (they will be called the topological darts of $\hat{\mathcal{M}}$). Let $\mathcal{V}$ denote the vertex set of $\hat{\mathcal{M}}$ and $\mathcal{G}$ its underlying graph, so that $\hat{\mathcal{M}} = (\mathcal{G}, \mathcal{V}, \mathcal{U})$.

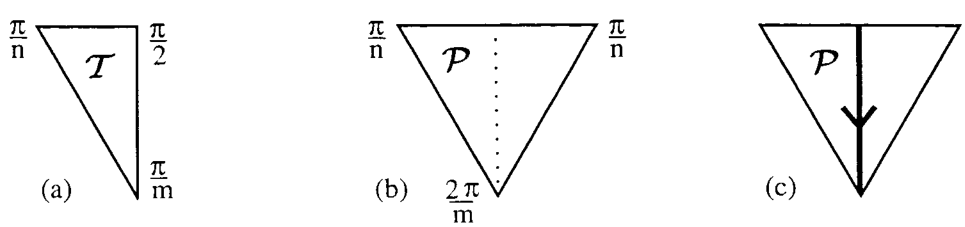


Figure 2.4

We will show that $\Gamma = \Gamma(m,2,n)$ acts in two different ways on $\hat{\mathcal{M}}$ through its action on the topological darts Ω; firstly as the monodromy group, and secondly as the automorphism group.

1. For the first action T_1 of $\Gamma(m,2,n)$ on $\hat{\mathcal{M}}$, we define

$$g : \gamma\mathcal{P} \mapsto \gamma g\mathcal{P}$$

for all $\gamma\mathcal{P} \in \Omega$ and $g \in \Gamma$. To check that T_1 is an action, we have for all $\gamma\mathcal{P} \in \Omega$ and $g, h \in \Gamma$

$$(gh) : \gamma\mathcal{P} \mapsto \gamma(gh)\mathcal{P}$$

which is equivalent to applying g

$$g : \gamma\mathcal{P} \mapsto \gamma g\mathcal{P} = (\gamma g)\mathcal{P}$$

and then h

$$h : (\gamma g)\mathcal{P} \mapsto (\gamma g)h\mathcal{P} = \gamma(gh)\mathcal{P}.$$

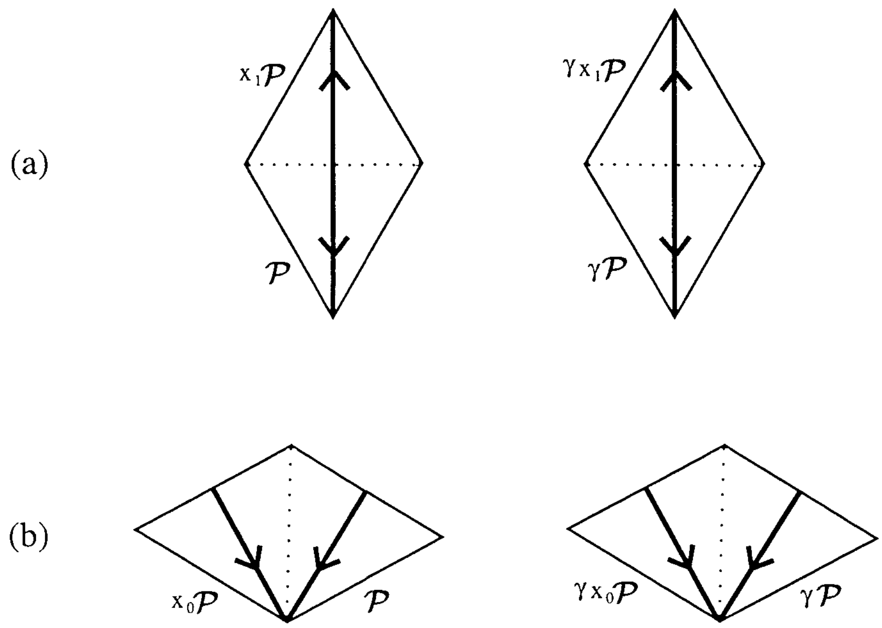


Figure 2.5

We now prove that $\Gamma(m,2,n)$ is acting as the monodromy group of $\hat{\mathcal{M}}$. For all $\gamma\mathcal{P} \in \Omega$, we have $x_1 : \gamma\mathcal{P} \mapsto \gamma x_1\mathcal{P}$ and $x_0 : \gamma\mathcal{P} \mapsto \gamma x_0\mathcal{P}$. As shown in Figure 2.5(a), the topological darts $\gamma\mathcal{P}$ and $\gamma x_1\mathcal{P}$ lie on the same edge, while from Figure 2.5(b) the topological dart $\gamma x_0\mathcal{P}$ corresponds to an anticlockwise rotation of the topological dart $\gamma\mathcal{P}$ about its vertex. Hence x_0 acts by rotating topological darts anticlockwise about vertices, and x_1 acts by interchanging the topological darts on an edge. Since $\Gamma(m,2,n) = < x_0, x_1 >$, this implies that $\Gamma(m,2,n)$ is isomorphic to the monodromy group of $\hat{\mathcal{M}}$. In this case we note that $\Gamma(m, 2, n)$ is not acting as a group of conformal automorphisms of the underlying surface $\mathcal{U}$.

The algebraic map associated to $\hat{\mathcal{M}}$ has the set of algebraic darts $|\Gamma(m, 2, n)|$ where the topological dart $\gamma\mathcal{P}$ corresponds to the algebraic dart γ for all $\gamma \in \Gamma$. The monodromy group of $Alg\,\hat{\mathcal{M}}$ is isomorphic to $\Gamma(m, 2, n)$ where $g \in \Gamma$ acts on the algebraic darts by $g : \gamma \mapsto \gamma g$ for all $\gamma \in |\Gamma|$; in particular x_0 induces anticlockwise rotations of the algebraic darts about each vertex and x_1 interchanges the algebraic darts on each edge. Hence the algebraic map $Alg\,\hat{\mathcal{M}}$ is isomorphic to the universal algebraic map $\hat{\mathcal{A}}$ defined above. We have therefore proved:

Theorem 2.7. *Let $\hat{\mathcal{M}}$ be the universal topological map of type (m, n). Then $Alg\,\hat{\mathcal{M}}$ is isomorphic to the universal algebraic map $\hat{\mathcal{A}}$ of type (m, n).* □

2. The second action T_2 of $\Gamma(m, 2, n)$ does induce conformal automorphisms of $\mathcal{U}$. We define

$$a : \gamma\mathcal{P} \mapsto a^{-1}\gamma\mathcal{P}$$

for all $\gamma\mathcal{P} \in \Omega$ and $a \in \Gamma$. Then $\Gamma(m, 2, n)$ acts transitively on Ω, and because

$$a^{-1}(\gamma g)\mathcal{P} = (a^{-1}\gamma)g\mathcal{P}$$

for all $a, g, \gamma \in \Gamma$, the two actions T_1 and T_2 commute. Therefore T_2 defines an action of $\Gamma(m, 2, n)$ as the automorphism group of $\hat{\mathcal{M}}$. Any subgroup $M \leq \Gamma(m, 2, n)$ will act as a group of automorphisms of $\hat{\mathcal{M}}$, and so we can form the quotient topological map $\hat{\mathcal{M}}/M = (\mathcal{G}/M, \mathcal{V}/M, \mathcal{U}/M)$ induced by the natural projection $p : \mathcal{U} \to \mathcal{U}/M$. Now M acts on the topological darts of $\hat{\mathcal{M}}$ (via the action T_2) and defines an equivalence relation on Ω corresponding to the orbits of this action; the equivalence classes have the form $[\gamma\mathcal{P}]_M$ for $\gamma \in \Gamma(m, 2, n)$, with $[\gamma_1\mathcal{P}]_M = [\gamma_2\mathcal{P}]_M$ if and only if $\gamma_1\gamma_2^{-1} \in M$.

Since the two actions T_1 and T_2 of $\Gamma(m, 2, n)$ commute, $\Gamma(m, 2, n)$ also acts on the set Ω/M by

$$g : [\gamma\mathcal{P}]_M \mapsto [\gamma g\mathcal{P}]_M$$

for all $g \in \Gamma$ and $\gamma\mathcal{P} \in \Omega$ (call this action T_3). An element $g \in \Gamma(m, 2, n)$ is in the kernel of this action if and only if $[\gamma g\mathcal{P}]_M = [\gamma\mathcal{P}]_M$ for all $\gamma \in \Gamma(m, 2, n)$; that is if and only if $\gamma g\gamma^{-1} \in M$ for all $\gamma \in \Gamma(m, 2, n)$, and so if and only if $g \in M^{\star}$, the core of M in $\Gamma(m, 2, n)$. If $\gamma\mathcal{P}, \gamma x_0\mathcal{P}, \ldots, \gamma x_0^{m-1}\mathcal{P}$ are the topological darts surrounding a vertex $v \in \mathcal{V}$ of $\hat{\mathcal{M}}$, then $[\gamma\mathcal{P}]_M, [\gamma x_0\mathcal{P}]_M, \ldots, [\gamma x_0^{r-1}\mathcal{P}]_M$ (where r is the smallest positive integer for which $\gamma x_0^r\gamma^{-1} \in M$) are the topological darts surrounding the vertex $p(v) \in \mathcal{V}/M$ of $\hat{\mathcal{M}}/M$. Similarly if $\gamma\mathcal{P}$ and $\gamma x_1\mathcal{P}$ are the topological darts on an edge of $\hat{\mathcal{M}}$, then $[\gamma\mathcal{P}]_M$ and $[\gamma x_1\mathcal{P}]_M$ (equal if $\gamma x_1\gamma^{-1} \in M$) are the darts on an edge of $\hat{\mathcal{M}}/M$. Hence the algebraic map corresponding to the quotient $\hat{\mathcal{M}}/M$ is isomorphic to the quotient algebraic map $\hat{\mathcal{A}}/M = (\Gamma/M^\star, \Gamma/M, M^\star x_0, M^\star x_1)$. Thus we have proved:

Theorem 2.8. *Let $\hat{\mathcal{M}}$ and $\hat{\mathcal{A}}$ be the universal topological and algebraic maps of type (m,n) respectively, with $M \leq \Gamma(m,2,n)$. Then $Alg\,(\hat{\mathcal{M}}/M) \cong \hat{\mathcal{A}}/M$.* □

We also note that the stabilizer of the topological dart $[\mathcal{P}]_M \in \Omega/M$ under the action T_3 of $\Gamma(m,2,n)$ is equal to M. Therefore for any $M \leq \Gamma(m,2,n)$, the topological map $\hat{\mathcal{M}}/M$ has a corresponding map subgroup $M \leq \Gamma(m,2,n)$.

Theorem 2.9. *If $\mathcal{A}$ is an algebraic map, then there is a topological map $\mathcal{M}$ such that $Alg\,\mathcal{M} \cong \mathcal{A}$.*

Proof. Let $\mathcal{A}$ have type (m,n). Then by the discussion of quotient algebraic maps we have $\mathcal{A} \cong \hat{\mathcal{A}}/M$, where $\hat{\mathcal{A}}$ is the universal algebraic map of type (m,n) and M is some subgroup of $\Gamma(m,2,n)$. If $\hat{\mathcal{M}}$ is the universal topological map of type (m,n), then by Theorem 2.8 we have $Alg\,(\hat{\mathcal{M}}/M) \cong \mathcal{A}$. Therefore the required map is $\mathcal{M} = \hat{\mathcal{M}}/M$. □

If $\mathcal{M}$ is a topological map of type (m,n) with a map subgroup $M \leq \Gamma(m,2,n)$, then the isomorphism $\mathcal{M} \cong \hat{\mathcal{M}}/M$ defines an embedding of $\mathcal{M}$ into the Riemann surface $X = \mathcal{U}/M$. In this way, every topological map can be embedded naturally into some Riemann surface.

Example 2.10. Let Λ_1 be a genus 2 subgroup of index 8 in $\Gamma(8,2,8)$ corresponding to the permutation representation

$$\begin{aligned} x_0 &\mapsto (1\,2\,3\,4\,5\,6\,7\,8) \\ x_1 &\mapsto (1\,5)(2\,6)(3\,7)(4\,8) \\ x_2 &\mapsto (1\,4\,7\,2\,5\,8\,3\,6) \end{aligned} \qquad 2.11$$

(see Theorem 1.9) where we take

$$\begin{aligned} \Gamma(8,2,8) &= \mathrm{gp} < x_0, x_1, x_2 \mid x_0^8 = x_1^2 = x_2^8 = x_0x_1x_2 = 1 > \\ &= \mathrm{gp} < x_0, x_1 \mid x_0^8 = x_1^2 = (x_0x_1)^{-8} = 1 > . \end{aligned} \qquad 2.12$$

Using the second presentation for $\Gamma(8,2,8)$ given in 2.12, we choose

$$\{1,\, x_0,\, x_0^2,\, x_0^3,\, x_0^4,\, x_0^5,\, x_0^6,\, x_0^7\} \qquad 2.13$$

to be a Schreier transversal for Λ_1 in $\Gamma(8,2,8)$ and obtain the corresponding Schreier generators

$$\{x_0^4x_1,\, x_0^5x_1x_0^7,\, x_0^6x_1x_0^6,\, x_0^7x_1x_0^5\}$$

(see Chapter 1). We have seen that the map $\mathcal{M}$ associated to the inclusion $\Lambda_1 \leq \Gamma(8,2,8)$ lies on the Riemann surface $X = \mathbf{H}/\Lambda_1$, and we construct $\mathcal{M}$ using the technique for constructing X given in Theorem 1.14. Take the hyperbolic triangle $\mathcal{T} = \alpha_0\alpha_1\alpha_2$ shown in Figure 2.6(a) with $\angle\alpha_2\alpha_0\alpha_1 = \frac{\pi}{8}$, $\angle\alpha_0\alpha_1\alpha_2 = \frac{\pi}{2}$, $\angle\alpha_1\alpha_2\alpha_0 = \frac{\pi}{8}$ and the let $\Gamma(8,2,8)$ act upon $\mathcal{T}$ as follows: x_0 is a $\frac{2\pi}{8}$ anticlockwise rotation about α_0, x_1 is a π rotation about α_1 and x_2 is a $\frac{2\pi}{8}$ anticlockwise rotation about α_2.

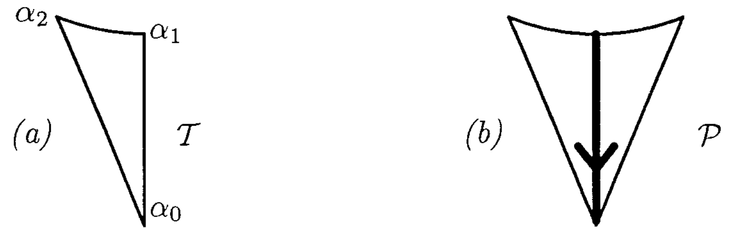


Figure 2.6

By reflecting $\mathcal{T}$ in the side $\alpha_0\alpha_1$ we obtain the fundamental region $\mathcal{P}$ for $\Gamma(8,2,8)$ shown in Figure 2.6(b); the bold line and dart drawn on $\mathcal{P}$ will form one half-edge of the map. We now obtain X (and hence the map $\mathcal{M}$) by 'gluing together' 8 copies of $\mathcal{P}$ as specified by the Schreier transversal. Thus we take the regions

$$\{\mathcal{P}, x_0\mathcal{P}, x_0^2\mathcal{P}, x_0^3\mathcal{P}, x_0^4\mathcal{P}, x_0^5\mathcal{P}, x_0^6\mathcal{P}, x_0^7\mathcal{P}\} \qquad 2.14$$

which, as shown in Figure 2.7, fit together to form a regular hyperbolic octagon. The Schreier generators pair the sides of the octagon as follows: $x_0^4x_1$ pairs the side A of $\mathcal{P}$ with the side E of $x_0^4\mathcal{P}$, $x_0^5x_1x_0^7$ pairs B with F, $x_0^6x_1x_0^6$ pairs C with G and $x_0^7x_1x_0^5$ pairs D with H. Hence X is represented as a regular hyperbolic octagon with opposite sides identified. The resulting map on X (formed by the bold lines) is of type $(8,8)$ and has one face, one vertex and four edges. The map has genus 2.

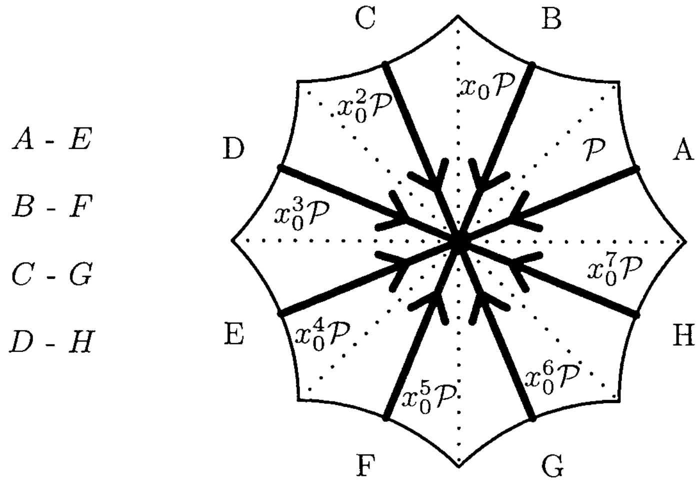


Figure 2.7

The eight darts of $\mathcal{M}$ can be associated with the eight cosets of M in $\Gamma(8,2,8)$ and labelled $1, 2, \ldots, 8$ according to the correspondence $i \mapsto Mx_0^{i-1}$ for $i = 1, \ldots, 8$. From Figure 2.8 we see that the defining permutations of $\mathcal{M}$ are $r_0 = (1\,2\,3\,4\,5\,6\,7\,8)$, $r_1 = (1\,5)(2\,6)(3\,7)(4\,8)$ and $r_2 = (1\,4\,7\,2\,5\,8\,3\,6)$, with $r_1 = r_0^4$. We therefore have $G = \mathrm{gp} < r_0, r_1 >= \mathrm{gp} < r_0 > \cong C_8$, the cyclic group of order 8. The only element in G that stabilizes the dart $\alpha = 1$ is the identity element e, and so $G_1 = \{e\}$ with $N_G(G_1) = G$. Hence the automorphism group of the map is given by

$$Aut\,\mathcal{M} \cong N_G(G_1)/G_1 \cong G \cong C_8.$$

These automorphisms can be realized as rotations about the centre of the octagon in Figure 2.8 through integer multiples of $\frac{\pi}{4}$.

If A is a group of automorphisms of $\mathcal{M}$ corresponding to a rotation of the octagon through an angle π, then A can be represented by its action on the eight darts as $A = \mathrm{gp} < (1\,5)(2\,6)(3\,7)(4\,8) >$. Letting $\overline{\mathcal{M}} = \mathcal{M}/A$ be the quotient map of $\mathcal{M}$ by A, $\overline{\mathcal{M}}$ has the set of darts $\{\overline{1}, \overline{2}, \overline{3}, \overline{4}\}$ where $\overline{1} = [1]$, $\overline{2} = [2]$, $\overline{3} = [3]$ and $\overline{4} = [4]$ with the defining permutations $\overline{r}_0 = (\overline{1}\ \overline{4}\ \overline{3}\ \overline{2})$, $\overline{r}_1 = (\overline{1})(\overline{2})(\overline{3})(\overline{4})$ and $r_2 = (\overline{1}\ \overline{2}\ \overline{3}\ \overline{4})$. Then $\overline{\mathcal{M}}$ is the genus 0 star map $\mathcal{S}_4$ shown in Figure 2.9. In general, the star map $\mathcal{S}_n$ is the genus 0 map with one vertex, one face and n free edges. The map $\mathcal{M}$ is a 2-sheeted cover of a map on the sphere; such a map is said to be *hyperelliptic* (a Riemann surface is hyperelliptic if it is a 2-sheeted cover of the Riemann sphere). We note that not all maps on hyperelliptic surfaces are themselves hyperelliptic [**Sin4**]. □

A - E

B - F

C - G

D - H

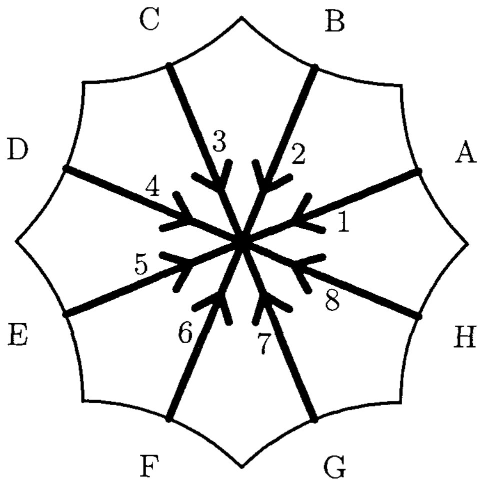


Figure 2.8

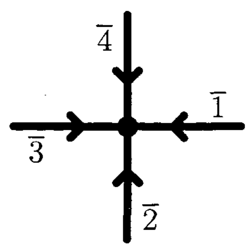


Figure 2.9

Example 2.15. We take a second genus 2 subgroup $\Lambda_2 \leq \Gamma(8,2,8)$ corresponding to the permutation representation

$$\begin{aligned} x_0 &\longmapsto (1\,2\,3\,4\,5\,6\,7\,8) \\ x_1 &\longmapsto (1\,7)(2\,4)(3\,5)(6\,8) \\ x_2 &\longmapsto (1\,6\,7\,8\,5\,2\,3\,4) \end{aligned} \qquad 2.16$$

and using the same Schreier transversal 2.13 of Example 2.10, we obtain the Schreier generators

$$\{x_0^3 x_1 x_0^7,\ x_0^6 x_1,\ x_0^7 x_1 x_0^3,\ x_0^4 x_1 x_0^6\}$$

for Λ_2 in $\Gamma(8,2,8)$. The map corresponding to the inclusion $\Lambda_2 \leq \Gamma(8,2,8)$ can now be constructed by gluing together the same eight copies of the fundamental region for $\Gamma(8,2,8)$ given in 2.14. The resulting hyperbolic octagon, shown in Figure 2.10, has the following side pairings: the Schreier generator $x_2^6 x_1$ pairs A with G, $x_2^3 x_1 x_2^7$ pairs B with D, $x_2^4 x_1 x_2^6$ pairs C with E and $x_2^7 x_1 x_2^3$ pairs F with H. The map we obtain is of genus 2 and type $(8,8)$, with one vertex, one face and four edges.

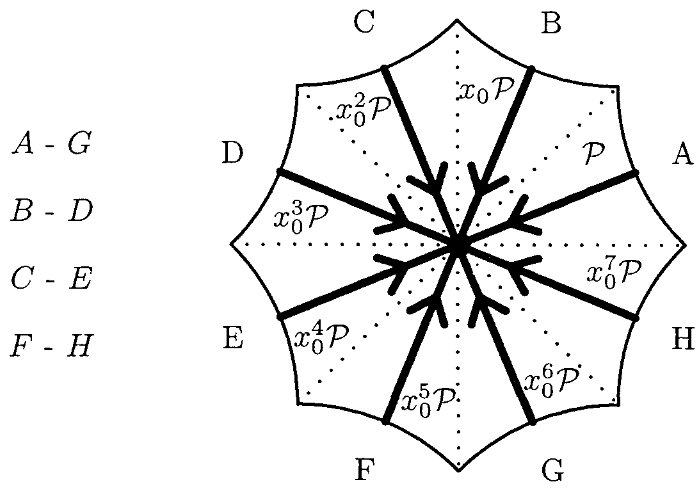


Figure 2.10

As in the previous example we associate the darts of $\mathcal{M}$ with the eight cosets of M in $\Gamma(8,2,8)$ by the correspondence $i \mapsto Mx_0^{i-1}$. The map is shown in Figure 2.11 with the defining permutations $r_0 = (1\,2\,3\,4\,5\,6\,7\,8)$, $r_1 = (1\,7)(2\,4)(3\,5)(6\,8)$ and $r_2 = (1\,6\,7\,8\,5\,2\,3\,4)$ where $G = \text{gp} < r_0, r_1 >$ has order 32 and contains eleven elements of order 2, four elements of order 4 and sixteen elements of order 8. The group G then has the form

$$(C_8 \rtimes C_2) \rtimes C_2$$

(see [**TW**]) where $G_1 \rtimes G_2$ is the semidirect product of G_1 with G_2, with G_1 as a normal subgroup. The stabilizer in G of the dart $\alpha = 1$ is given by

$$G_1 = < (2\,6)(4\,7), (3\,5)(4\,7) > \cong C_2 \times C_2$$

with the normalizer

$$N_G(G_1) = < (3\,5)(4\,7), (2\,6)(4\,7), (1\,8)(4\,7) > \cong C_2 \times C_2 \times C_2.$$

The automorphism group of the map is therefore

$$\text{Aut}\,\mathcal{M} \cong N_G(G_1)/G \cong C_2$$

and this can realized as a rotation about the centre of the octagon in Figure 2.11 through an angle π. Representing the automorphism group as a permutation of the darts of $\mathcal{M}$ gives $\text{Aut}\,\mathcal{M} = < (1\,5)(2\,6)(3\,7)(4\,8) >$. The map $\overline{\mathcal{M}} = \mathcal{M}/\text{Aut}\,\mathcal{M}$

then has the set of darts $\{\bar{1},\bar{2},\bar{3},\bar{4}\}$ where $\bar{i} = [i]$ for $i = 1,2,3,4$ and the defining permutations $\bar{r}_0 = (\bar{1}\ \bar{2}\ \bar{3}\ \bar{4})$, $\bar{r}_1 = (\bar{1}\ \bar{3})(\bar{2}\ \bar{4})$ and $\bar{r}_2 = (\bar{1}\ \bar{3}\ \bar{2}\ \bar{4})$. The resulting map has genus 1 and is shown in Figure 2.12. Thus $\mathcal{M}$ is a 2-sheeted cover of a map on the torus; such a map is said to be *elliptic-hyperelliptic.* □

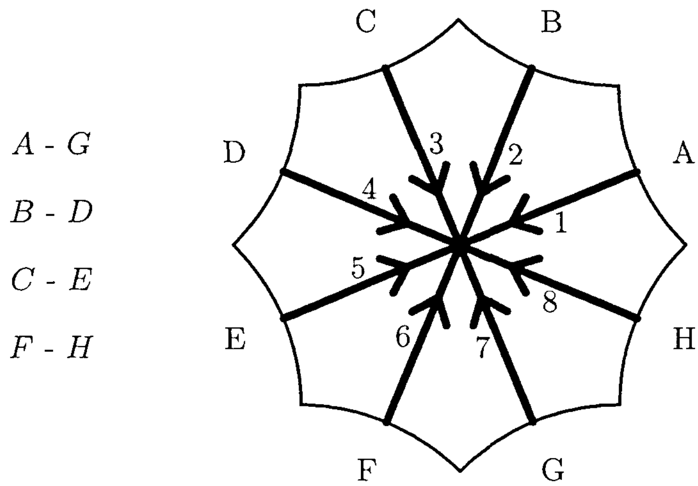


Figure 2.11

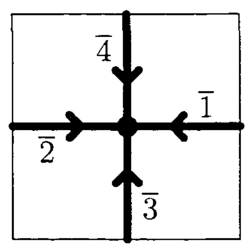


Figure 2.12

Uniform and regular maps

Definition 2.17. *A map $\mathcal{M}$ is uniform if all of its vertices have the same valency, all of its faces have the same valency, and it either has no free edges, or all of its edges are free.* □

The only uniform map with a free edge is the genus 0 star map $\mathcal{S}_n$ which has one vertex, one face and n free edges. The embeddings of the Platonic solids into the sphere are uniform maps of genus 0, while the maps in Examples 2.10 and 2.15 are uniform of type $(8,8)$. A stronger condition on a map is for it to be regular:

Definition 2.18. *A map $\mathcal{M}$ is regular if Aut $\mathcal{M}$ acts transitively on the darts of $\mathcal{M}$.* □

The automorphism group of the map $\mathcal{M}$ in Example 2.10 acts transitively upon the set of darts of $\mathcal{M}$, and so the map is regular. The genus 0 embeddings of the Platonic solids are also regular maps; indeed it is known that a genus 0 map is uniform if and only if it is regular (see [**JS1**]). More generally, every regular map is uniform (since if $Aut\,\mathcal{M}$ acts transitively on the darts, it must act transitively upon vertices and faces), however not all uniform maps are regular. For example, the uniform map in Example 2.15 is not regular because $Aut\,\mathcal{M}$ divides Ω into four orbits. The following classification of uniform and regular maps in terms of their canonical map subgroups is given in [**JS1**]:

Theorem 2.19. *A finite non-star map is uniform if and only if its canonical map-subgroup is torsion free.* □

Theorem 2.20. *Let $\mathcal{M}$ be a map with canonical map-subgroup $M \leq \Gamma(m, 2, n)$ and associated algebraic map (G, Ω, r_0, r_1). The following are equivalent:*

(i) $\mathcal{M}$ is regular;

(ii) $M \triangleleft \Gamma(m, 2, n)$;

(iii) (G, Ω) is a regular permutation group (i.e. $G_\alpha = \{e\}$ for all $\alpha \in \Omega$). □

Using Theorem 1.9 one can verify that the permutation representations in Examples 2.10 and 2.15 define torsion-free map subgroups $\Lambda_1, \Lambda_2 \leq \Gamma(8, 2, 8)$, so that by Theorem 2.19 the associated maps are uniform of type $(8, 8)$. Also note that in Example 2.10, the stabilizer in G of any dart of the map is trivial, so that by Theorem 2.20(iii) the map is regular.

Regular covers of maps

Given a subgroup $M \leq \Gamma(m, 2, n)$ the *core* of M in Γ, denoted $M^\star$, is the intersection of all conjugates of M in $\Gamma(m, 2, n)$. The core $M^\star$ is the largest subgroup of M that is normal in $\Gamma(m, 2, n)$, and $M^\star$ contains every other subgroup of M with this property.

Theorem 2.21. [**JS1**] *Every finite map $\mathcal{M}$ of type (m, n) can be finitely covered by a regular map of type (m, n).*

Proof. If $M \leq \Gamma(m, 2, n)$ is a canonical map-subgroup for $\mathcal{M}$, then M has finite index in $\Gamma(m, 2, n)$ and hence so does its core $M^\star$. Therefore the map $\mathcal{M}^\star$ with map-subgroup $M^\star$ is finite and since $M^\star \leq M$, $\mathcal{M}^\star$ covers $\mathcal{M}$ by Theorem 2.3. Since $M^\star \triangleleft \Gamma(m, 2, n)$, $\mathcal{M}^\star$ is a regular map of type dividing (m, n), and because $\mathcal{M}^\star$ covers $\mathcal{M}$, $\mathcal{M}^\star$ must have type (m, n). □

The map $\mathcal{M}^\star$ defined in the proof of Theorem 2.21 is the smallest regular map that covers $\mathcal{M}$. If $|M : M^\star| = k$ then we say that $\mathcal{M}$ has a minimal regular cover of index k. The index of the minimal regular cover of a map gives some measure of its regularity; the smaller the index, the more regular the map. For examples of regular covers, we refer the reader to §4.4 where the minimal regular cover is calculated for a family of toroidal maps.

2.2. Hypermaps

A hypermap is a generalization of a map in which an edge is allowed to intersect any finite number of vertices. We begin by giving an algebraic definition of a hypermap, and then describe three ways in which this can be interpreted topologically.

Algebraic and topological hypermaps

An algebraic hypermap $\mathcal{H}$ consists of a set of objects Ω called the *bits* of $\mathcal{H}$, together with two permutations r_0 and r_1 on Ω such that $G = \text{gp} < r_0, r_1 >$ is a transitive permutation group. The cycles in r_0 and r_1 correspond to *hypervertices* and *hyperedges* respectively, while the cycles of $r_2 = (r_0 r_1)^{-1}$ correspond to *hyperfaces*. The hypermap is represented by the quadruple (G, Ω, r_0, r_1), and if r_i has order l_i (for $i = 0, 1, 2$), then $\mathcal{H}$ is said to have *type* (l_0, l_1, l_2). The *degree* of $\mathcal{H}$ is equal to $|\Omega|$.

If $\mathcal{H} = (G, \Omega, r_0, r_1)$ has type (l_0, l_1, l_2), then there is a natural epimorphism $\theta : \Gamma(l_0, l_1, l_2) \to G$ given by $x_0 \mapsto r_0$, $x_1 \mapsto r_1$ where

$$\Gamma(l_0, l_1, l_2) = \text{gp} < x_0, x_1 \,|\, x_0^{l_0} = x_1^{l_1} = (x_0 x_1)^{-l_2} = 1 > .$$

If $G_\alpha = \{g \in G \,|\, \alpha g = \alpha\}$ for any $\alpha \in \Omega$, then $H = \theta^{-1}(G_\alpha)$ is called the *canonical hypermap subgroup* associated to $\mathcal{H}$. We can think of $\mathcal{H}$ as being the finite transitive permutation representation $\theta : \Gamma(l_0, l_1, l_2) \to G$.

Example 2.22. We let $\Omega = \{1, 2, 3, 4, 5, 6, 7, 8\}$ and take

$$\begin{aligned} r_0 &= (1\,2\,4\,8)(3\,6\,5)(7) \\ r_1 &= (1)(2\,3)(4\,5\,6\,7)(8) \\ r_2 &= (1\,8\,4\,7\,3)(2\,5)(6) \end{aligned} \qquad 2.23$$

so that the group $G = gp < r_0, r_1 >$ acts transitively on Ω. Then $\mathcal{H} = (G, \Omega, r_0, r_1)$ is an algebraic hypermap of type $(12, 4, 10)$, whose hypermap subgroup is obtained by considering the natural epimorphism $\theta : \Gamma(12, 4, 10) \to G$. □

The following topological interpretations of a hypermap $\mathcal{H}$ on an orientable surface $\mathcal{S}$ are due to Cori [**Cor**], Walsh [**Wa**] and James [**Ja**]:

Definition 2.24. The Cori representation. Hypervertices and hyperedges are represented by closed polygons, called 0-faces and 1-faces respectively. Hypervertices are mutually disjoint, as are the hyperedges, with the hypervertices intersecting the hyperedges at a finite number of points, corresponding to the *bits*. The orientation of $\mathcal{S}$ induces a cyclic ordering (in an anticlockwise direction) of the bits around each 0-face and 1-face, giving the permutations r_0 and r_1 respectively. Let $V \subset \mathcal{S}$ and $E \subset \mathcal{S}$ be the sets of hypervertices and hyperedges respectively. Then the components of $\mathcal{S} \backslash (V \cup E)$ are called the hyperfaces (2-faces), each one homeomorphic to an open disc and inducing the permutation $r_2 = (r_0 r_1)^{-1}$. Hypervertices, hyperedges and hyperfaces will be represented in black, grey and white respectively. Figure 2.13 shows the Cori representation $C(\mathcal{H})$ of the hypermap $\mathcal{H}$ in Example 2.22; the hypermap is drawn on a surface of genus 0. □

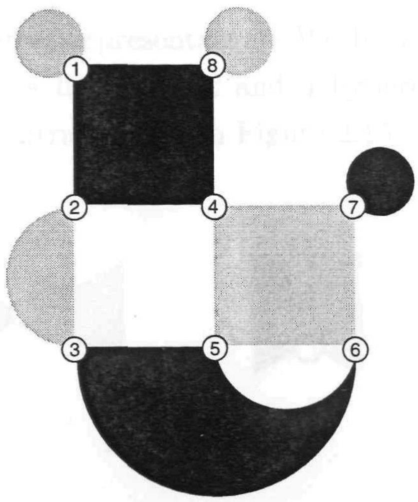


Figure 2.13: The Cori representation $C(\mathcal{H})$ of the hypermap $\mathcal{H}$

Definition 2.25. The Walsh representation. Starting with the Cori representation, hypervertices are replaced by black vertices (0-vertices) and hyperedges are replaced by white vertices (1-vertices). A 0-vertex and a 1-vertex are joined by an edge if and only if the associated hypervertex and hyperedge intersect at a bit, so that the

Walsh representation $W(\mathcal{H})$ is a bipartite map. The edges of $W(\mathcal{H})$ correspond to bits, with r_i giving an ordering of the edges around each i-vertex ($i = 0, 1$), and the permutation r_2 giving an ordering of the edges around each face. Figure 2.14 shows the Walsh representation of the hypermap in Example 2.22. □

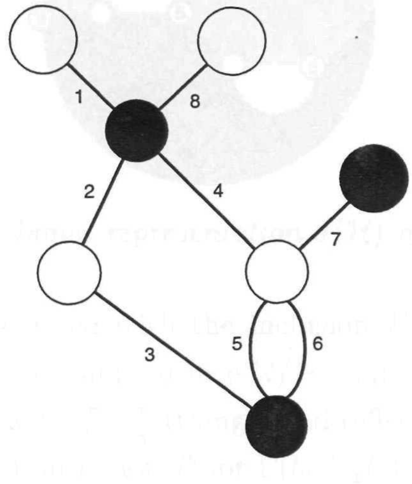


Figure 2.14: The Walsh representation $W(\mathcal{H})$ of the hypermap $\mathcal{H}$

Definition 2.26. The James representation. We begin once again with the Cori representation. Whenever a hypervertex and a hyperedge intersect, we 'squash' them together to form an extra edge as in Figure 2.15.

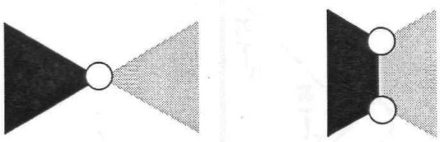


Figure 2.15

The James representation $J(\mathcal{H})$ is then a trivalent map with 0-faces, 1-faces and 2-faces corresponding to hypervertices, hyperedges and hyperfaces respectively. The bits correspond to those vertices in $J(\mathcal{H})$ about which the ordering of the faces in an anticlockwise direction is (012). Figure 2.16 shows the James representation of the hypermap in Example 2.22. □

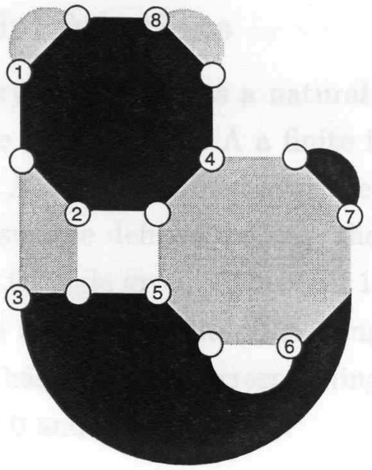


Figure 2.16: The James representation $J(\mathcal{H})$ of the hypermap $\mathcal{H}$

The hypermap $\mathcal{H}$ associated with the inclusion $H \leq \Gamma(l_0, l_1, l_2)$ can be embedded naturally into the Riemann surface $\mathcal{U}/H$. The Walsh hypermap $W(\mathcal{H})$ is embedded as follows: take a $\frac{\pi}{l_0}, \frac{\pi}{l_1}, \frac{\pi}{l_2}$ triangle and reflect it in the edge with angles $\frac{\pi}{l_0}, \frac{\pi}{l_1}$ to form the fundamental region $\mathcal{P}$ for $\Gamma(l_0, l_1, l_2)$ shown in Figure 2.17. The bold line drawn onto $\mathcal{P}$ will form one edge of the Walsh hypermap, and the black (resp. white) circle represents a hypervertex (resp. hyperedge). If $\{m_1, m_2, \ldots, m_k\}$ is a Schreier transversal for H in $\Gamma(l_0, l_1, l_2)$, then we obtain the hypermap $W(\mathcal{H})$ by gluing together the regions $m_1\mathcal{P}, \ldots, m_k\mathcal{P}$ and identifying sides according to the Schreier generators.

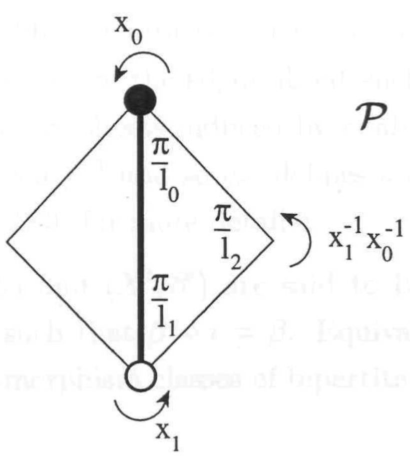


Figure 2.17

Other concepts such as hypermap coverings, isomorphisms, automorphisms and the definitions of uniform and regular hypermaps all follow by analogy with maps, and so we omit the details (for more information see [**JS3**], [**CMa**] and [**CoSi**]).

2.3. Dessins and Belyĭ functions

We have seen that every dessin admits a natural embedding into a Riemann surface X, where X has the form $\mathcal{U}/\Lambda$ for Λ a finite index subgroup of a triangle group; by Belyĭ's Theorem, X will be defined over the algebraic numbers $\overline{\mathbf{Q}}$. Conversely, if X is a Riemann surface defined over $\overline{\mathbf{Q}}$, then X can be uniformized by a finite index subgroup of a triangle group (Theorem 1.17) and so naturally carries a dessin. Let $\mathcal{B}_1$ denote the trivial bipartite map lying on the Riemann sphere Σ; as shown in Figure 2.18 $\mathcal{B}_1$ has one edge corresponding to the interval $[0,1]$ with a black and a white vertex at 0 and 1 respectively.

Figure 2.18. The trivial bipartite map $\mathcal{B}_1$

The Belyĭ pair (X,β) defines an n-sheeted branched cover $\beta : X \to \Sigma$ with critical values $C(\beta) \subseteq \{0,1,\infty\}$. Hence β will lift $\mathcal{B}_1$ to a (connected) bipartite map $\mathcal{B} = \beta^{-1}(\mathcal{B}_1)$ lying on X; the black (respectively white) vertices of $\mathcal{B}$ correspond to the points $\beta^{-1}(0)$ (respectively $\beta^{-1}(1)$), and the face centres of $\mathcal{B}$ will lie above $x = \infty$. The bipartite map $\mathcal{B}$ will have n edges, which we label from 1 to n, with each edge lying on a unique branch. Analytic continuation in a positive sense about $x = 0$ and $x = 1$ on the Riemann sphere will induce permutations of the n sheets which we denote by g_0 and g_1. The group $G = < g_0, g_1 >$ is called the monodromy group of the cover. Since β lifts the positive orientation of Σ to X, the permutations g_0 and g_1 define cyclic orderings of the edges about each black and white vertex of $\mathcal{B}$. If g_∞ is the permutation of sheets induced by analytic continuation about the point $x = \infty$, then $g_\infty = (g_0 g_1)^{-1}$ and so g_∞ defines a cyclic ordering of the edges about each face of $\mathcal{B}$ (see **[JS3]** for more details).

The Belyĭ pairs (X,β) and (X',β') are said to be equivalent if there is an isomorphism $i : X \to X'$ such that $\beta' \circ i = \beta$. Equivalence classes of Belyĭ pairs correspond precisely to isomorphism classes of bipartite maps (see **[Sc]** or **[JSt]** for example):

Lemma 2.27. *Let (X,β) and (X',β') be Belyĭ pairs, with associated bipartite maps $\mathcal{B}$ and $\mathcal{B}'$, and associated monodromy groups $G = < g_0, g_1 >$ and $G' = < g'_0, g'_1 >$ in S_N. Then the following are equivalent:*

(i) The Belyĭ pairs (X,β) and (X',β') are equivalent;

(ii) The bipartite maps $\mathcal{B}$, $\mathcal{B}'$ are isomorphic by a colour-preserving isomorphism;

(iii) The pairs (g_0, g_1) and (g'_0, g'_1) are conjugate in S_N, that is there exists some $s \in S_N$ with $sg_is^{-1} = g'_i$ for $i = 0, 1$. □

The Walsh representation of a hypermap defines a natural correspondence between hypermaps and bipartite maps. Thus a Belyĭ pair (X, β) corresponds to a hypermap $\mathcal{H}$ if we replace the black and white vertices of the bipartite map $\mathcal{B} = \beta^{-1}(\mathcal{B}_1)$ with hypervertices and hyperedges. The monodromy permutations g_0, g_1, g_∞ then define cyclic orderings of the bits about the hypervertices, hyperedges and hyperfaces respectively of $\mathcal{H}$.

A Belyĭ function β is said to be *pre-clean* if the points $\beta^{-1}(1)$ all have order of branching less than or equal to 1 (equivalently if the monodromy permutation g_1 satisfies $g_1^2 = 1$), and *clean* if they all have order of branching exactly equal to 1 (that is, if g_1 is a product of disjoint transpositions). The pair (X, β) is said to be (pre-)clean if β is (pre-)clean. Every pre-clean Belyĭ pair (X, β) corresponds to a map $\mathcal{M}$ if we replace the white vertices of the bipartite map $\mathcal{B} = \beta^{-1}(\mathcal{B}_1)$ with edge centres of $\mathcal{M}$, as shown in Figure 2.19. The monodromy permutation g_0 will then define a cyclic ordering of the darts around each vertex of $\mathcal{M}$, and g_1 (which is a disjoint product of transpositions and 1-cycles) will define the edges and free-edges of $\mathcal{M}$. Using this construction, clean Belyĭ pairs correspond to maps without free edges, while maps associated to pre-clean Belyĭ pairs may have free edges (these correspond to points in the set $\beta^{-1}(1)$ whose order of branching is equal to 0). If β is any Belyĭ function, then $\beta_W = 4\beta(1-\beta)$ is a clean Belyĭ function. If $\beta^{-1}(\mathcal{B}_1)$ defines a hypermap $\mathcal{H}$, then the map associated to $\beta_W^{-1}(\mathcal{B}_1)$ is the Walsh double of $\mathcal{H}$ (see Definition 2.43 and [**JS3**]).

Figure 2.19

The absolute Galois group $\mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q})$ is the set of all automorphisms of the algebraic numbers $\overline{\mathbf{Q}}$ that fix $\mathbf{Q}$. Now $\mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q})$ acts on Belyĭ pairs (and hence on equivalence classes of Belyĭ pairs) via its action on their algebraic number coefficients; so $\sigma \in \mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q})$ will send (X, β) to (X^σ, β^σ). Grothendieck [**Gro**] observed that the action of $\mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q})$ on equivalence classes of Belyĭ pairs induces a faithful action of $\mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q})$ on isomorphism classes of dessins; indeed $\mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q})$ is known to act faithfully on the set of genus 1 dessins, and even on the set of plane trees [**Sc**]. Jones and Streit [**JSt**] have determined the following invariants for Galois orbits of dessins:

Theorem 2.28. *The following properties of a dessin remain invariant under the action of Gal($\overline{\mathbf{Q}}/\mathbf{Q}$): genus; numbers of (hyper)vertices, (hyper)edges and (hyper)faces; valency partitions of (hyper)vertices, (hyper)edges and (hyper)faces; monodromy group and automorphism group (up to isomorphism).* □

However, it is still possible for two dessins in the same Galois orbit to be non-isomorphic. In §4.6 we give some examples of non-trivial Galois orbits.

2.4. Truncations and stellations

Jones [**Jon**] shows that an inclusion of triangle groups $\Gamma(l,m,n) \leq \Gamma(l',m',n')$ gives rise to a functor from the category of (hyper)maps of type (l,m,n) to those of type (l',m',n'). If the finite index inclusion $H \leq \Gamma(l,m,n)$ corresponds to a (hyper)map $\mathcal{H}$, then a second (hyper)map $\mathcal{H}'$ can be obtained from the inclusion $H \leq \Gamma(l',m',n')$; we would like to determine geometric operations which allow us to pass from $\mathcal{H}$ to $\mathcal{H}'$. In this section we define truncations and stellations of maps (see [**Sh**] and [**Coxe2**]) and show how they relate to some triangle group inclusions given in [**Sin2**]. Further examples of functors between maps are given in §5.3.

Definition 2.29. *If $\mathcal{M}$ is a map, then the dual map $D(\mathcal{M})$ is defined as follows: every q-valent vertex of $\mathcal{M}$ is replaced by a q-valent face of $D(\mathcal{M})$ and every q-valent face of $\mathcal{M}$ is replaced by a q-valent vertex of $D(\mathcal{M})$. The vertices of $D(\mathcal{M})$ are joined so that the edge centres of $D(\mathcal{M})$ coincide with the edge centres of $\mathcal{M}$.* □

Figure 2.20: A genus 1 map $\mathcal{M}$ and its dual $D(\mathcal{M})$

If we use the convention that a free edge has an 'edge centre' at its tip, then Definition 2.29 applies to maps with free edges. If $\mathcal{M}$ is a map of type (m,n) then its dual $D(\mathcal{M})$ will have type (n,m). The genus 1 map $\mathcal{M}$ in Figure 2.20(a) has type $(4, 4)$, as does its dual $D(\mathcal{M})$ shown in Figure 2.20(b); as usual the maps are obtained by identifying opposite sides of the squares.

Definition 2.30. *The type 1 truncation* $T_1(\mathcal{M})$ *of a map* $\mathcal{M}$ *is defined as follows: the vertices of* $T_1(\mathcal{M})$ *correspond to the edge centres of* $\mathcal{M}$*, and the vertices are joined so that a* q*-valent vertex of* $\mathcal{M}$ *is replaced by a* q*-valent face of* $T_1(\mathcal{M})$*, or equivalently so that a* q*-valent face of* $\mathcal{M}$ *is replaced by a* q*-valent face of* $T_1(\mathcal{M})$*.*

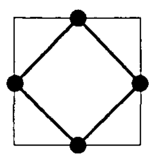


Figure 2.21: The type 1 truncations $T_1(\mathcal{M}) \cong T_1 D(\mathcal{M})$

The type 1 truncations $T_1(\mathcal{M})$ and $T_1 D(\mathcal{M})$ of the genus 1 map and its dual considered above are isomorphic to the map in Figure 2.21. This is a general phenomenon: $T_1(\mathcal{M})$ is defined in terms of the vertices, edge centres and face centres of $\mathcal{M}$. There is a one-to-one correspondence between the edge centres of $\mathcal{M}$ and the edge centres of $D(\mathcal{M})$, with q-valent vertices of $\mathcal{M}$ corresponding to q-valent faces of $D(\mathcal{M})$ and q-valent faces of $\mathcal{M}$ corresponding to q-valent vertices of $D(\mathcal{M})$. It follows that the resulting type 1 truncations are isomorphic:

$$T_1(\mathcal{M}) \cong T_1 D(\mathcal{M}). \qquad 2.31$$

Lemma 2.32. *Two maps* $\mathcal{M}_1$ *and* $\mathcal{M}_2$ *have isomorphic type 1 truncations if and only if* $\mathcal{M}_1$ *and* $\mathcal{M}_2$ *are isomorphic or dual.*

Proof. Isomorphic maps will have isomorphic type 1 truncations, and we have shown that $T_1(\mathcal{M}) \cong T_1 D(\mathcal{M})$ for any map $\mathcal{M}$. Now let $\mathcal{M}_1$ and $\mathcal{M}_2$ be maps with $T_1(\mathcal{M}_1) \cong T_1(\mathcal{M}_2)$. We assign a labelling L_1 to the faces of $T_1(\mathcal{M}_1)$ as follows: a face of $T_1(\mathcal{M}_1)$ is labelled v if it corresponds to a vertex of $\mathcal{M}_1$, and f if it corresponds to a face of $\mathcal{M}_1$ (Figure 2.22 shows the labelling for the genus 1 map $T_1(\mathcal{M})$).

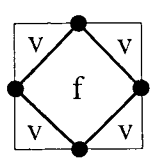


Figure 2.22: The face labelling for $T_1(\mathcal{M})$

This labelling has the property that no two adjacent faces of $T_1(\mathcal{M}_1)$ have the same label, and we note that L_1 uniquely determines the map $\mathcal{M}_1$; to recover $\mathcal{M}_1$ place a vertex in the centre of every v-face of $T_1(\mathcal{M}_1)$ and join two vertices whenever their associated v-faces share a common vertex in $T_1(\mathcal{M}_1)$. We define a similar labelling L_2 for the faces of $T_1(\mathcal{M}_2)$ corresponding to the map $\mathcal{M}_2$. The isomorphism between the maps $T_1(\mathcal{M}_1)$ and $T_1(\mathcal{M}_2)$ restricts to an isomorphism between their faces, and hence the labelling L_2 induces a labelling $\overline{L}_2$ on the faces of $T_1(\mathcal{M}_1)$ corresponding to $\mathcal{M}_2$.

The labellings L_1 and $\overline{L}_2$ define bipartite structures on the faces of $T_1(\mathcal{M})$ (i.e. faces are labelled either v or f and no two adjacent faces have the same label) and so L_1 and $\overline{L}_2$ are either the same or have v-faces and f-faces interchanged. In the former case L_1 and L_2 define isomorphic maps and $\mathcal{M}_1 \cong \mathcal{M}_2$, while in the latter case the vertices and face centres of $\mathcal{M}_1$ and $\mathcal{M}_2$ are interchanged so that $\mathcal{M}_1 \cong D(\mathcal{M}_2)$. □

Definition 2.33. *The type 1 stellation $S_1(\mathcal{M})$ of a map $\mathcal{M}$ is defined as follows: the vertices of $S_1(\mathcal{M})$ correspond to the vertices and face centres of $\mathcal{M}$, and the vertices are joined so that every vertex corresponding to a face centre of $\mathcal{M}$ is joined to the vertices surrounding that face (see Figure 2.23).* □

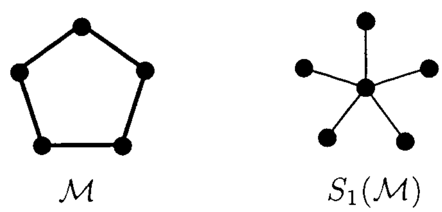


Figure 2.23

The type 1 stellation of the genus 1 map considered above is shown in Figure 2.24, and we note that $S_1(\mathcal{M})$ is dual to the truncated map $T_1(\mathcal{M})$ of Figure 2.21. It can be deduced from the definitions that for any map $\mathcal{M}$, the type 1 truncation and type 1 stellation of $\mathcal{M}$ are dual maps

$$S_1(\mathcal{M}) \cong DT_1(\mathcal{M})$$

and furthermore, since

$$S_1(\mathcal{M}) \cong DT_1(\mathcal{M}) \cong DT_1D(\mathcal{M}) \cong S_1D(\mathcal{M})$$

any map and its dual will have isomorphic type 1 stellations.

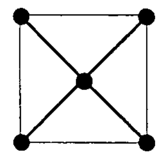


Figure 2.24: The type 1 stellation $S_1(\mathcal{M})$

From Singerman's list of triangle group inclusions [**Sin2**] we have

$$\Gamma(s,t,s) \leq_2 \Gamma(2t,2,s)$$

where s and t are chosen so that $\Gamma(s,t,s)$ is a Euclidean or hyperbolic triangle group (only hyperbolic triangle groups are considered in [**Sin2**], but some of the inclusions hold also for Euclidean groups). Let us fix the presentations

$$\begin{aligned} \Gamma(2t,2,s) &= \ < x_0, x_1 \,|\, x_0^{2t} = x_1^2 = (x_0x_1)^{-s} = 1 > \\ \Gamma(s,t,s) &= \ < y_0, y_1 \,|\, y_0^{s} = y_1^t = (y_0y_1)^{-s} = 1 > \end{aligned} \qquad 2.34$$

and define a homomorphism $\theta : \Gamma(2t,2,s) \to C_2$ by

$$\begin{aligned} x_0 &\longmapsto (12) \\ x_1 &\longmapsto (12) \\ (x_0x_1)^{-1} &\longmapsto (1)(2) \end{aligned}$$

so that $\theta^{-1}(\text{stab}(1))$ is isomorphic to $\Gamma(s,t,s)$ by Theorem 1.9 (note that we have chosen a particular inclusion $\Gamma(s,t,s) \leq \Gamma(2t,2,s)$). We choose the Schreier transversal $\{1, x_0\}$ for $\Gamma(s,t,s)$ in $\Gamma(2t,2,s)$ and obtain the corresponding Schreier generators $\{x_0x_1,\ x_0^2\}$.

If $\mathcal{M}$ is a map of type (m,n) then its type 1 truncation $T_1(\mathcal{M})$ will be a map of type $(4,s)$ where $s = \text{l.c.m.}(m,n)$. Now $\mathcal{M}$ has type dividing (s,s), and so we can find a map subgroup $M \leq \Gamma(s,2,s)$ for $\mathcal{M}$. Since $\Gamma(s,2,s) \leq_2 \Gamma(4,2,s)$, the inclusion $M \leq \Gamma(4,2,s)$ will define a second map $\mathcal{M}'$ of type $(4,s)$; we will prove that $\mathcal{M}'$ is the type 1 truncation of $\mathcal{M}$. The presentations given in 2.34 will be used with $t = 2$.

Lemma 2.35. *Let the finite index inclusion $M \leq \Gamma(s,2,s)$ correspond to a map $\mathcal{M}$. Then $\Gamma(s,2,s) \leq \Gamma(4,2,s)$ as defined above, and the inclusion $M \leq \Gamma(4,2,s)$ corresponds to the type 1 truncation $T_1(\mathcal{M})$.*

Proof. We construct $\mathcal{M}$ using the method given in Example 2.10: take the fundamental region $\mathcal{P}$ for $\Gamma(s, 2, s)$ shown in Figure 2.25 where the bold line represents one free-edge of the map $\mathcal{M}$. If $\{m_1, \ldots, m_k\}$ is a Schreier transversal for M in $\Gamma(s, 2, s)$, then $\mathcal{M}$ is constructed by gluing together the regions $\{m_1\mathcal{P}, \ldots, m_k\mathcal{P}\}$ and identifying sides according to the Schreier generators.

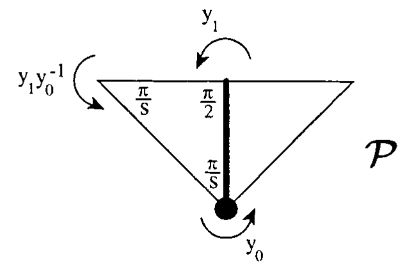


Figure 2.25

To construct the map $\mathcal{M}'$ corresponding to the inclusion $M \leq \Gamma(4, 2, s)$ we use the fundamental region $\mathcal{P}'$ for $\Gamma(4, 2, s)$ shown in Figure 2.26(a) where one free-edge is marked in bold. Since $\Gamma(s, 2, s) \leq_2 \Gamma(4, 2, s)$ we can glue together two copies of $\mathcal{P}'$ to form a fundamental region for $\Gamma(s, 2, s)$: taking the Schreier transversal $\{1, x_0\}$ for $\Gamma(s, 2, s)$ in $\Gamma(4, 2, s)$ given above, we glue together $\{\mathcal{P}', x_0\mathcal{P}'\}$ to obtain the fundamental region $\mathcal{Q}$ for $\Gamma(s, 2, s)$ shown in Figure 2.26(b). By Theorem 1.14, the Schreier generators pair the sides of $\mathcal{Q}$ and so we can identify them with the generators y_0, y_1 of $\Gamma(s, 2, s)$: $y_0 = x_1x_0^{-1}$, $y_1 = x_0^2$, $(y_0y_1)^{-1} = x_0^{-1}x_1$.

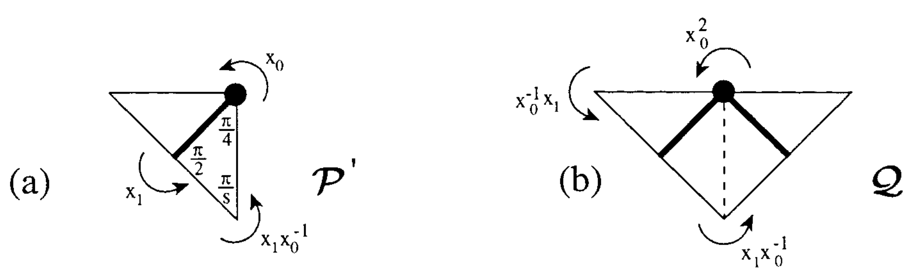


Figure 2.26

Therefore, if they are chosen carefully we can regard $\mathcal{P}$ and $\mathcal{Q}$ as being the same region with different edges marked in bold (of course in the Euclidean case $\mathcal{P}$ and $\mathcal{Q}$ must have the same area). Thus to construct $\mathcal{M}'$, we replace every fundamental region $\mathcal{P}$ used to construct $\mathcal{M}$ with the fundamental region $\mathcal{Q}$. The vertices of $\mathcal{M}'$ correspond to the edge centres of $\mathcal{M}$, and the vertices of $\mathcal{M}'$ are joined so that the

q-valent vertices of $\mathcal{M}$ are replaced by q-valent faces of $\mathcal{M}'$. Hence $\mathcal{M}' \cong T_1(\mathcal{M})$ by Definition 2.30. □

In the proof of Lemma 2.35 it was shown that the generators y_0, y_1 of $\Gamma(s,2,s)$ can be expressed in terms of the generators x_0, x_1 of $\Gamma(4,2,s)$:

$$\begin{aligned} y_0 &= x_1 x_0^{-1} \\ y_1 &= x_0^2 \\ (y_0 y_1)^{-1} &= x_0^{-1} x_1 \ . \end{aligned}$$

Conjugation of $\Gamma(s,2,s)$ by the element $x_1 \in \Gamma(4,2,s)$ will induce an outer automorphism of $\Gamma(s,2,s)$ interchanging its two conjugacy classes of elements of order s; let the image of $M \leq \Gamma(s,2,s)$ under this automorphism be $M^{x_1} \leq \Gamma(s,2,s)$.

Lemma 2.36. *Let $\mathcal{M}$ be the map corresponding to the finite index inclusion $M \leq \Gamma(s,2,s)$. Then the inclusion $M^{x_1} \leq \Gamma(s,2,s)$ corresponds to the dual map $D(\mathcal{M})$. In particular, $\mathcal{M}$ is self dual if and only if M and M^{x_1} are conjugate in $\Gamma(s,2,s)$.*

Proof. Let $[M]_{\Gamma(s,2,s)}$ represent the conjugacy class of M in $\Gamma(s,2,s)$. Then since $\Gamma(s,2,s) \leq_2 \Gamma(4,2,s)$ with $\Gamma(4,2,s) = \Gamma(s,2,s) \cup \Gamma(s,2,s)x_1$, we have $[M]_{\Gamma(4,2,s)} = [M]_{\Gamma(s,2,s)} \cup [M^{x_1}]_{\Gamma(s,2,s)}$. There are two cases to consider:

(i) If M and M^{x_1} are not conjugate in $\Gamma(s,2,s)$, then they correspond to non-isomorphic maps, say $\mathcal{M}_1$ and $\mathcal{M}_2$. Since M and M^{x_1} are conjugate in $\Gamma(4,2,s)$, $\mathcal{M}_1$ and $\mathcal{M}_2$ have isomorphic type 1 truncations by Lemma 2.35. Non-isomorphic maps with the same type 1 truncations must be dual by Lemma 2.32.

(ii) If M and M^{x_1} are conjugate in $\Gamma(s,2,s)$, then

$$[M]_{\Gamma(4,2,s)} = [M]_{\Gamma(s,2,s)}. \tag{2.37}$$

Let $\overline{M} \leq \Gamma(s,2,s)$ be the map subgroup corresponding to the dual map $D(\mathcal{M})$. Then M and $\overline{M}$ are conjugate in $\Gamma(4,2,s)$ because $\mathcal{M}$ and $D(\mathcal{M})$ have isomorphic type 1 truncations (Lemmas 2.32 and 2.35). Therefore M and $\overline{M}$ are conjugate in $\Gamma(s,2,s)$ by 2.37, and so $\mathcal{M}$ and $D(\mathcal{M})$ are isomorphic maps. □

If M is a finite index subgroup of the triangle group Γ_1, then there exists a Belyĭ function $\beta_1 : \mathcal{U}/M \to \mathcal{U}/\Gamma_1 \cong \Sigma$ such that $\beta_1^{-1}(\mathcal{B}_1)$ is isomorphic to the dessin corresponding to the inclusion $M \leq \Gamma_1$. Let Γ_2 be a second triangle group with $\Gamma_1 \leq \Gamma_2$. Then the projection $\mathcal{U}/\Gamma_1 \rightarrow \mathcal{U}/\Gamma_2$ can be thought of as a Belyĭ function β_2 from the sphere to itself, and the composition $\beta_3 = \beta_2 \circ \beta_1 : \mathcal{U}/M \rightarrow \mathcal{U}/\Gamma_2 \cong \Sigma$ is a Belyĭ function with $\beta_3^{-1}(\mathcal{B}_1)$ isomorphic to the dessin corresponding to the inclusion $M \leq \Gamma_2$ (for details see [**Jon**]).

The trivial map corresponding to $\Gamma(s, 2, s)$ is shown in Figure 2.27(a), and the map corresponding to the inclusion $\Gamma(s, 2, s) \leq \Gamma(4, 2, s)$ is shown in Figure 2.27(b); note that the sides of each fundamental region must be paired appropriately. The map in Figure 2.27(b) has the corresponding Belyĭ function

$$\beta_{T_1} : x \longmapsto \frac{-(x-1)^2}{4x}$$

with two branch points of order 1 at $x = 1$ and $x = -1$ with $\beta_{T_1}(1) = 0$ and $\beta_{T_1}(-1) = 1$. The only other points sent into $\{0, 1, \infty\}$ are $\beta_{T_1}(0) = \infty$ and $\beta_{T_1}(\infty) = \infty$. Hence if $\beta : X \rightarrow \Sigma$ is a Belyĭ function for a map $\mathcal{M}$, the composition $\beta_{T_1} \circ \beta : X \rightarrow \Sigma$ will be a Belyĭ function for the type 1 truncation $T_1(\mathcal{M})$.

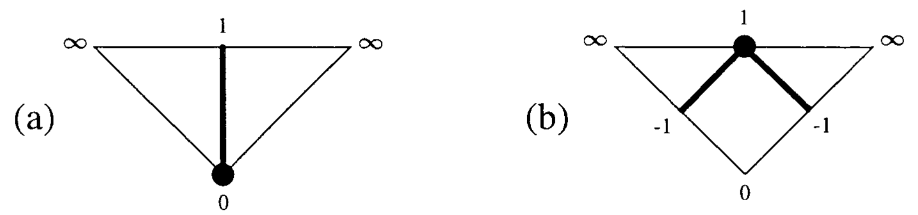


Figure 2.27

We now consider a second type of truncation in which q-valent vertices are replaced by q-gons, but the q-gons do not intersect one another.

Definition 2.38. *The type 2 truncation $T_2(\mathcal{M})$ of a map $\mathcal{M}$ is defined as follows: every edge of $\mathcal{M}$ corresponds to two vertices of $T_2(\mathcal{M})$, and the vertices are joined so that a q-valent vertex of $\mathcal{M}$ is replaced by a q-valent face of $T_2(\mathcal{M})$, as shown in Figure 2.28.* □

Figure 2.28: A map $\mathcal{M}$ and its type 2 truncation $T_2(\mathcal{M})$

Definition 2.39. *The type 2 stellation $S_2(\mathcal{M})$ of a map $\mathcal{M}$ is defined as follows: the vertices of $S_2(\mathcal{M})$ correspond to the vertices and face centres of $\mathcal{M}$, and the vertices are joined so that every vertex corresponding to a face F of $\mathcal{M}$ is joined to the vertices surrounding F, and to the vertices corresponding to the faces adjacent with F (see Figure 2.29).*

Figure 2.29: A map $\mathcal{M}$ and its type 2 stellation $S_2(\mathcal{M})$

Figure 2.30 shows the type 2 truncation and type 2 stellation of the genus 1 map considered above. We note that the two maps of Figure 2.30 are dual; in fact for any map $\mathcal{M}$ we have $T_2(\mathcal{M}) \cong DS_2(\mathcal{M})$.

$T_2(\mathcal{M})$ $S_2(\mathcal{M})$

Figure 2.30

Another inclusion from [**Sin2**] is

$$\Gamma(2s, 2, s) \leq_3 \Gamma(3, 2, 2s)$$

where we choose $s \geq 3$ so that $\Gamma(2s, 2, s)$ is either a Euclidean or hyperbolic triangle group. If we fix the presentations

$$\begin{aligned} \Gamma(3,2,2s) &= \ < x_0, x_1 \,|\, x_0^3 = x_1^2 = (x_0x_1)^{-2s} = 1 > \\ \Gamma(2s,2,s) &= \ < y_0, y_1 \,|\, y_0^{2s} = y_1^2 = (y_0y_1)^{-s} = 1 > \end{aligned} \qquad 2.40$$

and define a homomorphism $\theta : \Gamma(3, 2, 2s) \rightarrow C_3$ by

$$\begin{aligned} x_0 &\longmapsto (1\,2\,3) \\ x_1 &\longmapsto (1)(2\,3) \\ (x_0x_1)^{-1} &\longmapsto (1\,3)(2) \end{aligned}$$

then $\theta^{-1}(\mathrm{Stab}(1))$ is isomorphic to $\Gamma(2s,2,s)$ by Theorem 1.9 (note that we have chosen a particular inclusion $\Gamma(2s,2,s) \leq \Gamma(3,2,2s)$). We take the Schreier transversal $\{1, x_0, x_0^2\}$ for $\Gamma(2s,2,s)$ in $\Gamma(3,2,2s)$, and obtain the Schreier generators $\{x_0x_1x_0, x_1\}$.

If $\mathcal{M}$ is a map of type (m,n) with $s = \text{l.c.m.}(m,n)$, then $\mathcal{M}$ has type dividing $(2s,s)$ and so will correspond to a map subgroup $M \leq \Gamma(2s,2,s) \leq \Gamma(3,2,2s)$. We will prove that the map corresponding to the inclusion $M \leq \Gamma(3,2,2s)$ is the type 2 truncation of $\mathcal{M}$.

Lemma 2.41. *Let the finite index inclusion $M \leq \Gamma(2s,2,s)$ correspond to a map $\mathcal{M}$. Then $\Gamma(2s,2,s) \leq \Gamma(3,2,2s)$ as defined above, and the inclusion $M \leq \Gamma(3,2,2s)$ corresponds to the type 2 truncation $T_2(\mathcal{M})$.*

Proof. The proof is similar to that of Lemma 2.35. To construct the map $\mathcal{M}$ corresponding to the inclusion $M \leq \Gamma(2s,2,s)$, we take the fundamental region $\mathcal{P}$ for $\Gamma(2s,2,s)$ shown in Figure 2.31 where the bold line represents one free-edge of the map $\mathcal{M}$. If $\{m_1, \ldots, m_k\}$ is a Schreier transversal for M in $\Gamma(2s,2,s)$, then we construct $\mathcal{M}$ by gluing together the regions $\{m_1\mathcal{P} \ldots, m_k\mathcal{P}\}$ and identifying sides according to the Schreier generators.

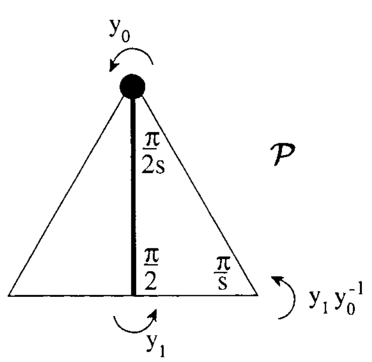


Figure 2.31

To construct the map $\mathcal{M}'$ corresponding to the inclusion $M \leq \Gamma(3,2,2s)$ we take the fundamental region $\mathcal{P}'$ for $\Gamma(3,2,2s)$ shown in Figure 2.32(a) where one free-edge is marked in bold. Since $\Gamma(2s,2,s) \leq_3 \Gamma(3,2,2s)$, we can glue together three copies of $\mathcal{P}'$ to form a fundamental region for $\Gamma(2s,2,s)$: taking the Schreier transversal $\{1, x_0, x_0^2\}$ for $\Gamma(2s,2,s)$ in $\Gamma(3,2,2s)$ given above, we glue together the regions $\{\mathcal{P}', x_0\mathcal{P}', x_0^2\mathcal{P}'\}$ to form the fundamental region $\mathcal{Q}$ for $\Gamma(2s,2,s)$ shown in Figure 2.32(b). By Theorem 1.14, the Schreier generators pair the sides of $\mathcal{Q}$ and so we can identify them with the generators y_0, y_1 of $\Gamma(2s, 2, s)$: $y_0 = x_0x_1x_0$, $y_1 = x_1$, $(y_0y_1)^{-1} = (x_0x_1)^{-2}$.

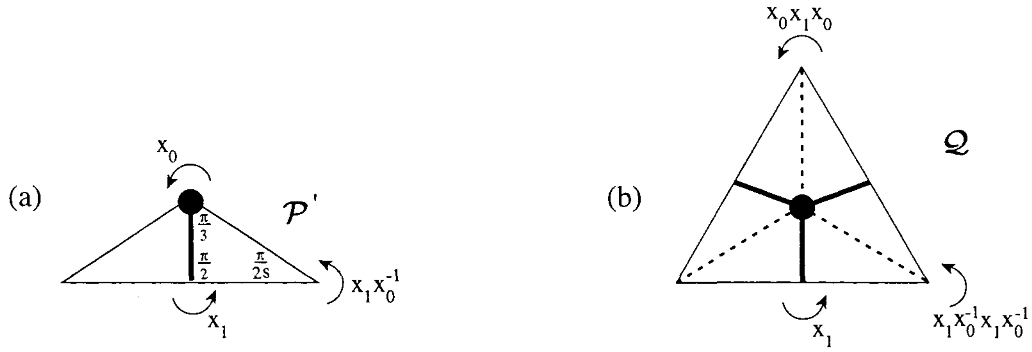


Figure 2.32

Therefore, if they are chosen carefully we can regard $\mathcal{P}$ and $\mathcal{Q}$ as being the same region with different highlighted edges. To construct the map $\mathcal{M}'$ we replace every fundamental region $\mathcal{P}$ used to form $\mathcal{M}$ with the fundamental region $\mathcal{Q}$. Consequently, every edge of $\mathcal{M}$ is replaced by two vertices of $\mathcal{M}'$, and the vertices of $\mathcal{M}'$ are joined so that every q-valent vertex of $\mathcal{M}$ is replaced by a q-valent face. Thus $\mathcal{M}' \cong T_2(\mathcal{M})$ by Definition 2.38. □

As shown in Lemma 2.41 we can express the generators y_0, y_1 of $\Gamma(2s, 2, s)$ in terms of the generators x_0, x_1 for $\Gamma(3, 2, 2s)$:

$$\begin{aligned} y_0 &= x_0x_1x_0 \\ y_1 &= x_1 \\ (y_0y_1)^{-1} &= (x_0x_1)^{-2}. \end{aligned} \qquad 2.42$$

The trivial map corresponding to $\Gamma(2s, 2, s)$ is shown in Figure 2.33(a), and the map corresponding to the inclusion $\Gamma(2s, 2, s) \leq \Gamma(3, 2, 2s)$ is shown in Figure 2.33(b) (the sides of each region must be paired appropriately). The map in Figure 2.33(b) has the corresponding Belyĭ function

$$\beta_{T_2} : x \longmapsto \frac{(4x-1)^3}{27x}$$

with two branch points: one of order 2 at $x = \frac{1}{4}$ and one of order 1 at $x = -\frac{1}{8}$ with $\beta_{T_2}(\frac{1}{4}) = 0$ and $\beta_{T_2}(-\frac{1}{8}) = 1$. The only other points sent into $\{0, 1, \infty\}$ are $\beta_{T_2}(0) = \infty = \beta_{T_2}(\infty)$ and $\beta_{T_2}(1) = 1$. Hence if $\beta : X \to \Sigma$ is a Belyĭ function for a map $\mathcal{M}$, the composition $\beta_{T_2} \circ \beta : X \to \Sigma$ is a Belyĭ function for the type 2 truncation $T_2(\mathcal{M})$.

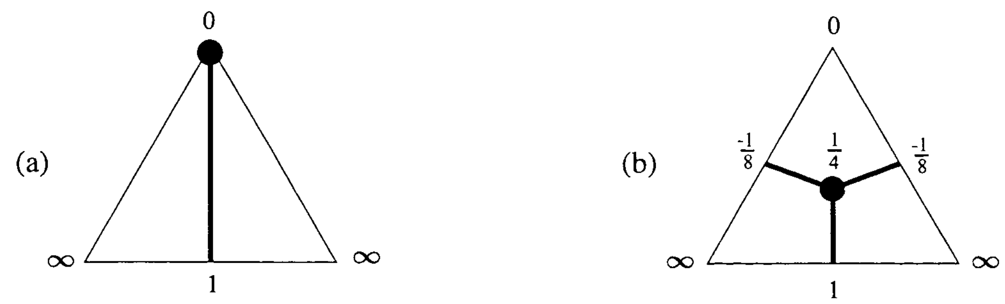


Figure 2.33

We recall that every Walsh map $W(\mathcal{H})$ naturally corresponds to a bipartite map (for example see [**Wa**] and §2.2).

Definition 2.43. *The Walsh double of $W(\mathcal{H})$ is the map formed by forgetting the bipartite structure on the vertices of $W(\mathcal{H})$.* □

Since any map has at most one (unique up to a choice of colouring) bipartite structure associated with its vertices, it is clear that (up to isomorphism) any map is the Walsh double of at most two hypermaps. If two hypermaps $\mathcal{H}_1$ and $\mathcal{H}_2$ have isomorphic Walsh doubles, then one can be obtained from the other by Machì's hypermap operation $\mathcal{H}_1 = \mathcal{H}_2^{(01)}$ which interchanges their hypervertices and hyperedges (see [**Mach**]). If a hypermap $\mathcal{H}$ has type (l_0, l_1, l_2), then $\mathcal{H}$ has type dividing (s, s, t) where $s = \text{l.c.m.}(l_0, l_1)$ and $t = l_2$. Consequently there exists a hypermap subgroup $H \leq \Gamma(s, s, t)$ corresponding to $W(\mathcal{H})$.

We consider again the inclusion

$$\Gamma(s, s, t) \leq_2 \Gamma(s, 2, 2t) \qquad 2.44$$

where s and t are chosen so that $\Gamma(s, s, t)$ is a Euclidean or hyperbolic triangle group. We fix the presentations

$$\Gamma(s, 2, 2t) = \; < x_0, x_1 \,|\, x_0^s = x_1^2 = (x_0 x_1)^{-2t} = 1 >$$

$$\Gamma(s, s, t) = \; < y_0, y_1 \,|\, y_0^s = y_1^s = (y_0 y_1)^{-t} = 1 >$$

and define a homomorphism $\theta : \Gamma(s, 2, 2t) \to C_2$ by

$$x_0 \longmapsto (1)(2)$$

$$x_1 \longmapsto (1\,2)$$

$$(x_0 x_1)^{-1} \longmapsto (1\,2)$$

so that $\theta^{-1}(\text{Stab}(1))$ is isomorphic to $\Gamma(s,s,t)$. We choose the Schreier transversal $\{1, x_1\}$ for $\Gamma(s,s,t)$ in $\Gamma(s,2,2t)$, and obtain the corresponding Schreier generators $\{x_0, x_1x_0x_1\}$.

Lemma 2.45. *Let the finite index inclusion $H \leq \Gamma(s,s,t)$ correspond to a hypermap $\mathcal{H}$. Then $H \leq \Gamma(s,s,t) \leq \Gamma(s,2,2t)$ as defined above, and the inclusion $H \leq \Gamma(s,2,2t)$ corresponds to the Walsh double of $\mathcal{H}$.*

Proof. The proof is similar to that of Lemma 2.35 and Lemma 2.41 and so we describe only the important step of defining a fundamental region for $\Gamma(s,s,t)$ in terms of a fundamental region for $\Gamma(s,2,2t)$. The hypermap $\mathcal{H}$ is built from the fundamental region $\mathcal{P}$ of $\Gamma(s,s,t)$ shown in Figure 2.34; the black and white circles correspond to hypervertices and hyperedges respectively.

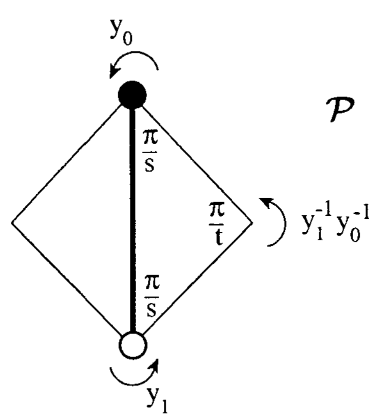


Figure 2.34

If $\mathcal{M}'$ is the map corresponding to the inclusion $M \leq \Gamma(s,2,2t)$, then we start with the fundamental region $\mathcal{P}'$ for $\Gamma(s,2,2t)$ shown in Figure 2.35(a). Since $\Gamma(s,s,t) \leq_2 \Gamma(s,2,2t)$, we can use the Schreier transversal $\{1, x_1\}$ given above to glue together the regions $\{\mathcal{P}', x_1\mathcal{P}'\}$ and form the fundamental region $\mathcal{Q}$ for $\Gamma(s,s,t)$ shown in Figure 2.35(b); the bold line will form one edge of the map $\mathcal{M}'$.

The Schreier generators $\{x_0, x_1x_0x_1\}$ will pair the sides of $\mathcal{Q}$, and so we can identify them with the generators y_0, y_1 of $\Gamma(s,s,t)$: $y_0 = x_0$, $y_1 = x_1x_0x_1$, $(y_0y_1)^{-1} = (x_0x_1)^{-2}$. Thus to construct the map $\mathcal{M}'$ we replace every fundamental region $\mathcal{P}$ used to form $\mathcal{H}$ with the fundamental region $\mathcal{Q}$. The map $\mathcal{M}'$ is formed by disregarding the labelling of hyperedges and hypervertices on the Walsh hypermap $\mathcal{H}$, and so $\mathcal{M}'$ is the Walsh double of $\mathcal{H}$. □

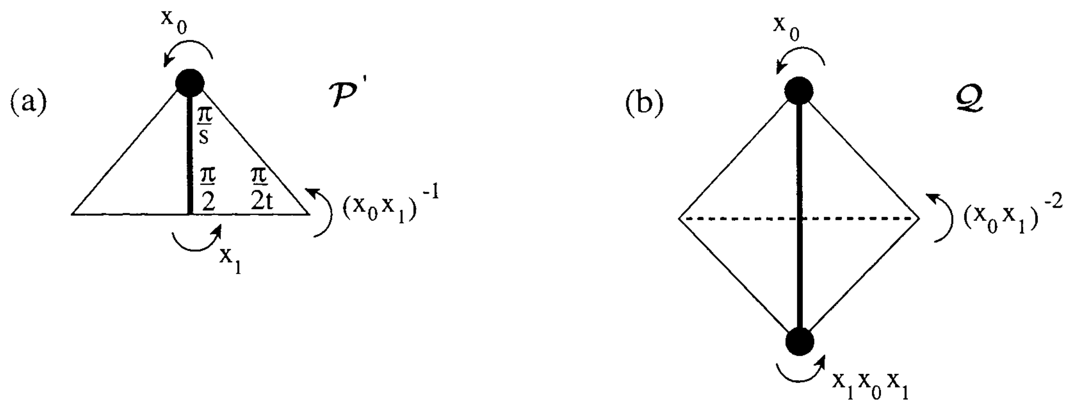


Figure 2.35

In the proof of Lemma 2.45, it was shown that we can express the generators y_0, y_1 of $\Gamma(s,s,t)$ in terms of the generators x_0, x_1 of $\Gamma(s,2,2t)$:

$$\begin{aligned} y_0 &= x_0 \\ y_1 &= x_1x_0x_1 \\ (y_0y_1)^{-1} &= (x_0x_1)^{-2} \end{aligned} \qquad 2.46$$

and hence that conjugation of $\Gamma(s,s,t)$ by the element $x_1 \in \Gamma(s,2,2t)$ induces an outer automorphism of $\Gamma(s,s,t)$ interchanging its two conjugacy classes of elements of order s. We let the image of $H \leq \Gamma(s,s,t)$ under this automorphism be $H^{x_1} \leq \Gamma(s,s,t)$.

Lemma 2.47. *Let the inclusion $H \leq \Gamma(s,s,t)$ correspond to a hypermap $\mathcal{H}$. Then the inclusion $H^{x_1} \leq \Gamma(s,s,t)$ corresponds to the hypermap $\mathcal{H}^{(01)}$ obtained from $\mathcal{H}$ by interchanging its hypervertices and hyperedges. In particular, $\mathcal{H} \cong \mathcal{H}^{(01)}$ if and only if H and H^{x_1} are conjugate in $\Gamma(s,s,t)$.*

Proof. We have observed that two hypermaps $\mathcal{H}_1$ and $\mathcal{H}_2$ have isomorphic Walsh doubles if and only if either $\mathcal{H}_1 \cong \mathcal{H}_2$ or $\mathcal{H}_1 \cong \mathcal{H}_2^{(01)}$. The proof is now similar to that of Lemma 2.36. □

If $\beta : X \to \Sigma$ is a Belyĭ function for a hypermap $\mathcal{H}$, then the composition of β with the Belyĭ function

$$\beta_W : x \mapsto 4x(1-x)$$

will be a Belyĭ function $\beta_W \circ \beta : X \to \Sigma$ for the Walsh double $W(\mathcal{H})$ (see §2.3).

Chapter 3

Elliptic curves and Belyĭ's Theorem

In this chapter we restrict our attention to genus 1 Riemann surfaces uniformized by finite index subgroups of Euclidean triangle groups. Such Riemann surfaces are said to have *Euclidean Belyĭ uniformizations*, and by Belyĭ's Theorem their associated elliptic curves are defined over the field of algebraic numbers $\overline{\mathbf{Q}}$. Using quadratic forms and a powerful result from algebraic number theory, we produce an algorithm to find (for a fixed positive integer k) the moduli of all elliptic curves with Euclidean Belyĭ uniformizations that are defined over extension fields of degree k over $\mathbf{Q}$. In particular, using the computational work of Berwick [**Ber**], we determine all elliptic curves with Euclidean Belyĭ uniformizations that are defined over the rational numbers $\mathbf{Q}$, and quadratic and cubic extensions of $\mathbf{Q}$.

3.1. Elliptic curves

Every compact Riemann surface of genus 1 has the form $\mathbf{C}/\Lambda$ where $\Lambda \leq \operatorname{Aut}\mathbf{C}$ is some group acting discontinuously on the complex plane [**JS2**]. As shown in 1.6,

$$\operatorname{Aut}\mathbf{C} = \{z \mapsto az + b \,|\, a, b \in \mathbf{C}, a \neq 0\}$$

where $z \mapsto az + b$ has a fixed point $z = \frac{b}{1-a}$ for $a \neq 1$. Hence Λ is a group of translations of the form $z \mapsto z + b$. It is shown further in [**JS2**] that Λ must be a *lattice*, that is a group generated by two independent translations $z \mapsto z + w_1$ and $z \mapsto z + w_1$ with $w_1, w_2 \in \mathbf{C}$ and $\frac{w_2}{w_1} \notin \mathbf{R}$. If we let $\mu\Lambda = \{\mu\omega \,|\, \omega \in \Lambda\}$, then two lattices Λ_1 and Λ_2 are *similar* if $\Lambda_1 = \mu\Lambda_2$ for some $\mu \in \mathbf{C}^\star$. Similarity defines an equivalence relation on the set of all lattices. The following theorem is well-known (for example see [**Ki**]):

Theorem 3.1. *$\mathbf{C}/\Lambda_1$ and $\mathbf{C}/\Lambda_2$ represent conformally equivalent Riemann surfaces if and only if Λ_1 and Λ_2 are similar lattices.* □

Given w_1, $w_2 \in \mathbf{C}$ with $\frac{w_2}{w_1} \notin \mathbf{R}$, let $\Lambda(w_1, w_2)$ denote the lattice generated by w_1 and w_2; that is $\Lambda(w_1, w_2) = \{mw_1 + nw_2 \,|\, m, n \in \mathbf{Z}\}$. If $\Lambda(w_1, w_2) = \Lambda(w_1', w_2')$ then $\{w_1, w_2\}$ and $\{w_1', w_2'\}$ are said to generate the same lattice.

Theorem 3.2. *$\Lambda(w_1, w_2) = \Lambda(w_1', w_2')$ if and only if $w_2' = aw_2 + bw_1$ and $w_1' = cw_2 + dw_1$ for $a, b, c, d \in \mathbf{Z}$ and $ad - bc = \pm 1$.* □

The *modulus* of a lattice $\Lambda(w_1, w_2)$ is defined to be the ratio $\tau = \frac{w_2}{w_1}$, where w_1 and w_2 are ordered so that $Im(\tau) > 0$. Every lattice determines a set of moduli corresponding to its different generating pairs, and since $\mu\Lambda(w_1, w_2) = \Lambda(\mu w_1, \mu w_2)$ with $\frac{w_2}{w_1} = \frac{\mu w_2}{\mu w_1}$, similar lattices determine the same moduli. It follows from Theorems 3.1 and 3.2 that $\Lambda(w_1, w_2)$ and $\Lambda(w_1', w_2')$ are similar lattices if and only if

$$\tau' = \frac{w_2'}{w_1'} = \frac{\mu(aw_2 + bw_1)}{\mu(cw_2 + dw_1)} = \frac{a\tau + b}{c\tau + d} \qquad 3.3$$

where $a, b, c, d \in \mathbf{Z}$ and $ad - bc = \pm 1$. Transformations of the form $z \mapsto \frac{az+b}{cz+d}$ with $a, b, c, d \in \mathbf{Z}$ and $ad - bc = -1$ act on $\mathbf{C}$ by interchanging the upper and lower half-planes. Since τ and τ' both lie in the upper half-plane $\mathbf{H}$, 3.3 must satisfy $ad - bc = 1$. The set of all transformations of the form $z \mapsto \frac{az+b}{cz+d}$ with $a, b, c, d \in \mathbf{Z}$ and $ad - bc = 1$ forms a group, called the *modular group*, denoted $\mathrm{PSL}_2(\mathbf{Z})$. Thus we have proved:

Theorem 3.4. *If $\Lambda = \Lambda(w_1, w_2)$ and $\Lambda' = \Lambda(w_1', w_2')$ are lattices in $\mathbf{C}$ with moduli $\tau = \frac{w_2}{w_1}$ and $\tau' = \frac{w_2'}{w_1'}$ lying in $\mathbf{H}$, then the following are equivalent:*

(i) $\mathbf{C}/\Lambda$, $\mathbf{C}/\Lambda'$ are conformally equivalent;

(ii) The lattices Λ, Λ' are similar;

(iii) $\tau' = T(\tau)$ for some $T \in \mathrm{PSL}_2(\mathbf{Z})$. □

Theorem 3.4 shows that there is a one-to-one correspondence between conformal equivalence classes of genus 1 Riemann surfaces and orbits of the modular group in the upper half-plane. The set

$$\mathcal{F} = \{z \in \mathbf{H} \,|\, |z| \geq 1, |Re(z)| \leq \frac{1}{2}\} \qquad 3.5$$

shown in Figure 3.1 is a fundamental region for the modular group acting on $\mathbf{H}$ with side pairing transformations $z \mapsto z+1$, $z \mapsto -\frac{1}{z}$; the corresponding Riemann surface $\mathbf{H}/\mathrm{PSL}_2(\mathbf{Z})$ is conformally equivalent to the complex plane. Under side pairings, each point in $\mathcal{F}$ represents a distinct conformal equivalence class of genus 1 Riemann surfaces. For convenience, we will say that each point in $\mathcal{F}$ defines a distinct Riemann surface, the technicality of side pairings being understood.

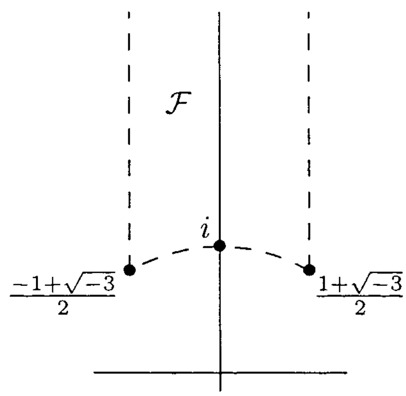


Figure 3.1

Definition 3.6. *An elliptic curve defined over a field F is an algebraic curve of genus 1 with coefficients in F, which contains at least one point with coefficients in F (the base point).* □

Unless otherwise specified, we will take F to be a subfield of the field of complex numbers $\mathbf{C}$. A classical result (see [**Kn**]) states that any elliptic curve defined over F is birationally equivalent to one in *Weierstrass normal form*

$$y^2 = 4x^3 - g_2x - g_3 \qquad 3.7$$

where $g_2, g_3 \in F$ and ${g_2}^3 - 27{g_3}^2 \neq 0$ (corresponding to the right hand side of 3.7 having distinct roots). By the above discussion, a Riemann surface defined by 3.7 has the form $\mathbf{C}/\Lambda$ for some lattice Λ lying in $\mathbf{C}$. The lattice Λ and g_2, g_3 are related by the following absolutely convergent series (see [**JS2**]):

$$g_2(\Lambda) = 60\sum_{\omega\in\Lambda}{}' \omega^{-4}\ , \quad g_3(\Lambda) = 140\sum_{\omega\in\Lambda}{}' \omega^{-6} \qquad 3.8$$

where $\sum'$ denotes the sum over all the non-zero lattice points. Furthermore, given any equation of the form 3.7 with ${g_2}^3 - 27{g_3}^2 \neq 0$, there exists a unique lattice Λ satisfying 3.8.

If we think of g_2, g_3 as being functions of the lattice Λ, then the following expression

$$j(\Lambda) = 1728\frac{g_2(\Lambda)^3}{g_2(\Lambda)^3 - 27g_3(\Lambda)^2} \tag{3.9}$$

called the j-function is also a function of Λ. It is easy to check that $j(\Lambda) = j(\mu\Lambda)$ for $\mu \in \mathbf{C}\backslash\{0\}$, and hence that $j(\Lambda)$ is an invariant of the similarity class of Λ. We have observed the one-to-one correspondence between similarity classes of lattices and orbits of the modular group acting on $\mathbf{H}$, so for $\tau \in \mathbf{H}$ we set $j(\tau) = j(\Lambda(1,\tau))$. The j-function then defines a $\mathrm{PSL}_2(\mathbf{Z})$-automorphic function on $\mathbf{H}$:

$$j(\tau) = 1728\frac{g_2(\tau)^3}{g_2(\tau)^3 - 27g_3(\tau)^2} \tag{3.10}$$

where $j(\tau) = j(T(\tau))$ for all $\tau \in \mathbf{H}$, $T \in \mathrm{PSL}_2(\mathbf{Z})$. The j-function 3.10 has the following important property (see for example [**Cox**]):

Theorem 3.11. *Let Λ, Λ' be lattices with moduli τ, τ' respectively. Then $\mathbf{C}/\Lambda$ and $\mathbf{C}/\Lambda'$ are conformally equivalent if and only if $j(\tau) = j(\tau')$.* □

As an example we will calculate the j-invariants and elliptic curve equations corresponding to the Riemann surfaces $\mathbf{C}/\Lambda(1,i)$ and $\mathbf{C}/\Lambda(1,\rho)$. We first observe that the square lattice $\Lambda = \Lambda(1,i)$ is invariant under multiplication by i, so that $\Lambda(1,i) = i\Lambda(1,i)$. Letting $\Lambda = \Lambda(1,i)$, by the absolute convergence of g_3 we have

$$\begin{aligned} g_3 &= 140\sum_{\omega\in\Lambda}{}' \omega^{-6} \\ &= 140\sum_{\omega\in\Lambda}{}' (i\omega)^{-6} \\ &= -140\sum_{\omega\in\Lambda}{}' \omega^{-6} = -g_3 \end{aligned}$$

and so $g_3 = -g_3 = 0$. Since any lattice Λ satisfies $g_2^3 - 27g_3^2 \neq 0$, it follows that $g_2 \neq 0$ and hence that

$$j(\Lambda(1,i)) = 1728\frac{g_2^3}{g_2^3 - 27g_3^2} = 1728\frac{g_2^3}{g_2^3} = 1728.$$

Therefore, the elliptic curve corresponding to $\mathbf{C}/\Lambda(1,i)$ has the form

$$y^2 = 4x^3 - g_2x \tag{3.12}$$

where g_2 is any non-zero complex number. All elliptic curves of the form 3.12 correspond to conformally equivalent Riemann surfaces, since for any $g_2 \in \mathbf{C}^\star$ the resulting elliptic curve has a j-invariant equal to 1728.

If $\rho = \frac{-1+\sqrt{-3}}{2}$ then the lattice $\Lambda = \Lambda(1,\rho)$ satisfies $\Lambda(1,\rho) = \rho\Lambda(1,\rho)$. By the absolute convergence of g_2 we have

$$\begin{aligned} g_2 &= 60 \sum_{\omega\in\Lambda}{}' \omega^{-4} \\ &= 60 \sum_{\omega\in\Lambda}{}' (\rho\omega)^{-4} \\ &= 60\rho^2 \sum_{\omega\in\Lambda}{}' \omega^{-4} = g_2\rho^2 \end{aligned}$$

so that $g_2 = g_2\rho^2 = 0$ and $g_3 \neq 0$ with

$$j(\rho) = \frac{g_2^3}{g_2^3 - 27g_3^2} = 0.$$

Hence for $g_3 \in \mathbf{C}^\star$ the elliptic curve

$$y^2 = 4x^3 - g_3$$

is conformally equivalent to $\mathbf{C}/\Lambda(1,\rho)$. The elliptic curves corresponding to the moduli $\tau = i$ and $\tau = \rho$ are defined over the rational number field $\mathbf{Q}$, as are their j-invariants $j(i)$ and $j(\rho)$.

Theorem 3.13. *An elliptic curve E is defined over a field F if and only if $j(E) \in F$, where $j(E)$ is the j-invariant of E.*

Proof. If E is defined over F then it is birationally equivalent to a curve E' in Weierstrass normal form with $g_2, g_3 \in F$, so $j(E) = j(E') \in F$. Conversely if $j(E) \in F$, we find $g_2, g_3 \in F$ satisfying 3.10 as follows:

(i) if $j(E) = 0$ set $g_2 = 0,\ g_3 = 1$;

(ii) if $j(E) = 1728$ set $g_2 = 1,\ g_3 = 0$;

(iii) if $j(E) \neq 0, 1728$ set $g_2 = g_3 = \frac{27j(E)}{j(E)-1728}$. □

The group law on the cubic

It is convenient to transform the standard Weierstrass model $\overline{E} : \overline{y}^2 = 4\overline{x}^3 - g_2\overline{x} - g_3$ to one of the form

$$E : y^2 = x^3 + ax + b \qquad 3.14$$

using the birational transformation $\overline{E} \to E$, $(\overline{x}, \overline{y}) \mapsto (x, y)$ where $x = \overline{x}$ and $y = \frac{\overline{y}}{2}$. The non-singular elliptic curve E defines a compact Riemann surface if we complete E by adding a point at infinity. Formally, this is done by working in homogeneous coordinates $[X, Y, Z]$ and setting $x = \frac{X}{Z}$, $y = \frac{Y}{Z}$ to obtain the projective curve

$$Y^2Z = X^3 + aXZ^2 + bZ^3.$$

A point $(x, y) \in E$ has the homogeneous coordinates $[x, y, 1]$, with the point at infinity corresponding to $\mathcal{O} = [0, 1, 0]$. With a slight abuse of notation, we let E denote the set of points of the elliptic curve $y^2 = x^3 + ax + b$ and the point at infinity $\mathcal{O}$.

Let $P_1 = (x_1, y_1)$ and $P_2 = (x_2, y_2)$ be two points on the elliptic curve $E - \{\mathcal{O}\}$. We define an addition law $P_1 + P_2$ for the points of E as follows:

(i) If $x_1 = x_2$ and $y_1 = -y_2$, then $P_1 + P_2 = \mathcal{O}$.

Otherwise

(ii) If $x_1 \neq x_2$ then let $\lambda = \frac{y_2 - y_1}{x_2 - x_1}$, $v = y_2 - \lambda x_2$ and $P_3 = P_1 + P_2$ where

$$x(P_3) = \lambda^2 - x_1 - x_2$$
$$y(P_3) = -\lambda x_3 - v$$

and $(x(P_3), y(P_3))$ denotes the (x, y)-coordinate of P_3.

(iii) If $x_1 = x_2$ and $y_1 = y_2$ then we use the duplication formula $P + P = [2]P$ where

$$x([2]P) = \frac{x^4 - 2ax^2 - 8bx + a^2}{4x^3 + 4ax + 4b}$$

and $y([2]P)$ can be calculated by substituting $x = x([2]P)$ into 3.14.

For any $P, Q \in E$ the addition law satisfies $P + \mathcal{O} = P = \mathcal{O} + P$, $(P + Q) + R = P + (Q + R)$ and $P + Q = Q + P$. If $P = (x, y) \in E$, then $-P = (x, -y) \in E$ satisfies $P + (-P) = \mathcal{O}$. Hence the points of E together with the law of addition defined above form an abelian group with identity element $\mathcal{O}$. (For all of this, see [**Sil**].)

Definition 3.15. *Let E_1 and E_2 be elliptic curves. Then an isogeny between E_1 and E_2 is a mapping*

$$\phi : E_1 \to E_2$$

induced by rational functions with $\phi(\mathcal{O}_1) = \mathcal{O}_2$ where $\mathcal{O}_1$ and $\mathcal{O}_2$ are the points at infinity of E_1 and E_2 respectively. □

Let $\phi : E_1 \to E_2$ be an isogeny of elliptic curves. It can be shown that ϕ is either the zero isogeny with $\phi(E_1) = \{\mathcal{O}_2\}$, or else ϕ is surjective and $\phi(E_1) = E_2$; E_1 and E_2 are said to be *isogenous* if $\phi(E_1) \neq \{\mathcal{O}_2\}$. An isogeny $\phi : E_1 \to E_2$ also induces a homomorphism between the underlying group structures of E_1 and E_2, so that

$$\phi(P + Q) = \phi(P) + \phi(Q) \tag{3.16}$$

for all $P, Q \in E_1$. As a consequence ([**Sil**]) it can be shown that a non-zero isogeny ϕ satisfies $|\phi^{-1}(U)| = |\phi^{-1}(V)|$ for all $U, V \in E_2$, and hence that every non-zero isogeny is unramified in the sense of Chapter 1 (we note that the term unramified has a more specific meaning in [**Sil**] with reference to the function fields of E_1 and E_2).

Proposition 3.17. [**Sil**, p.78] *Let E be an elliptic curve and Φ a finite subgroup of E. Then there is a unique elliptic curve E' and an isogeny $\phi : E \to E'$ such that $\ker \phi = \Phi$.* □

We will refer to E' as the quotient curve E/Φ, and as observed above, the isogeny $\phi : E \to E/\Phi$ is unramified. Vélu [**Ve**] has determined formulae for the isogeny ϕ and the quotient curve E/Φ in terms of the coefficients of E and the (x, y)-coordinates of the points in Φ. If $E : y^2 = x^3 + ax + b$ is an elliptic curve containing a finite group of points Φ, then for $P \in E - \Phi$ the isogeny $\phi : E \to E/\Phi$ is given by $\phi : P \mapsto (X, Y)$ where

$$\begin{aligned} X &= x(P) + \sum_{Q \in \Phi - \{\mathcal{O}\}} [x(P+Q) - x(Q)] \\ Y &= y(P) + \sum_{Q \in \Phi - \{\mathcal{O}\}} [y(P+Q) - y(Q)] \end{aligned} \tag{3.18}$$

and $\phi(P) = \mathcal{O}$ for $P \in \Phi$. The quotient curve E/Φ will have the the equation

$$Y^2 = X^3 + AX + B$$

where

$$\begin{aligned} A &= a - 5 \sum_{Q \in \Phi - \{\mathcal{O}\}} (3x(Q)^2 + a) \\ B &= b - 7 \sum_{Q \in F_2} (7x(Q)^3 + 5ax(Q) + 4b) \\ &\quad - 7 \sum_{Q \in \Phi - F_2 - \{\mathcal{O}\}} (5x(Q)^3 + 3ax(Q) + 2b) \end{aligned} \qquad 3.19$$

and F_2 denotes the set of points of $\Phi - \{\mathcal{O}\}$ with order 2. Extensive use will be made of isogenies induced by subgroups of orders 2 and 3, and so we determine the explicit equations of the isogenies and quotient curves in these cases.

Examples 3.20. (i) Let $E : y^2 = x^3 + ax + b$ be an elliptic curve. Then $(r, s) \in E$ is a point of order two if and only if $s = 0$, that is if and only if r is a root of the cubic polynomial $x^3 + ax + b$ [**ST**, p.40]. If $(r, 0) \in E$ has order 2, then setting $\Phi = < \mathcal{O}, (r, 0) >$ and using 3.18 we obtain the isogeny $\phi : E \to E/\Phi$, $\phi : (x, y) \mapsto (X, Y)$ where

$$(X, Y) = \left(x + \frac{3r^2 + a}{x - r},\ y + \frac{y(3r^2 + a)}{(x - r)^2} \right)$$

for all $(x, y) \in E - \Phi$ and $\phi(x, y) = \mathcal{O}$ otherwise. By 3.19 the corresponding quotient curve is $E/\Phi : Y^2 = X^3 + AX + B$ where

$$\begin{aligned} A &= -15r^2 - 4a \\ B &= b - 7(7r^3 + 5ar + 4b) \end{aligned}$$

(see [**Ve**] for more details).

(ii) A point $(r, s) \in E : y^2 = x^3 + ax + b$ has order 3 if and only if r is a root of the quartic polynomial $3x^4 + 6ax^2 + 12bx - a^2$ [**ST**, p.40]. If $(r, s) \in E$ is a point of order 3 generating the group $\Phi = < \mathcal{O}, (r, s), (r, -s) >$, then we obtain the isogeny $\phi : E \to E/\Phi$, $\phi : (x, y) \mapsto (X, Y)$ where

$$\begin{aligned} X &= x + \frac{6r^2 + 2a}{x - r} + \frac{4(r^3 + ar + b)}{(x - r)^2} \\ Y &= y - \frac{2y(3r^2 + a)}{(x - r)^2} - \frac{8y(r^3 + ar + b)}{(x - r)^3} \end{aligned}$$

for all $(x, y) \in E - \Phi$ and $\phi(x, y) = \mathcal{O}$ otherwise. The quotient curve has the equation $E/\Phi : Y^2 = X^3 + AX + B$ where

$$\begin{aligned} A &= -30r^2 - 9a \\ B &= b - 7(10r^3 + 6ar + 4b). \quad \square \end{aligned}$$

The Weierstrass pe-function

Let $\Lambda(\omega_1, \omega_2)$ be a lattice. The Weierstrass pe-function $\wp : \mathbf{C} \to \Sigma$ given by

$$\wp(z) = \frac{1}{z^2} + \sum_{\omega \in \Lambda}{}' \left(\frac{1}{(z-\omega)^2} - \frac{1}{\omega^2} \right)$$

is a meromorphic function satisfying $\wp(z + \omega) = \wp(z)$ for all $z \in \mathbf{C}$ and $\omega \in \Lambda$. Hence $\wp(z)$ is an elliptic function with respect to Λ, sometimes denoted $\wp(z, \Lambda)$. The poles of $\wp(z)$ and its derivative $\wp'(z)$ correspond to the lattice points Λ and have orders two and three respectively. The functions $\wp(z)$ and $\wp'(z)$ satisfy the relation

$$\wp'(z, \Lambda)^2 = 4\wp(z, \Lambda)^3 - g_2(\Lambda)\wp(z, \Lambda) - g_3(\Lambda)$$

and so for every complex number $z \in \mathbf{C} - \Lambda$ we obtain a point $\big(\wp(z), \wp'(z)\big)$ on the elliptic curve $\overline{E} : \overline{y}^2 = 4\overline{x}^3 - g_2\overline{x} - g_3$, with all $z \in \Lambda$ corresponding to the point at infinity $\mathcal{O}$. Conversely, given any $(\overline{x}, \overline{y}) \in \overline{E}$ there exists a complex number $z \in \mathbf{C}$ with $(\overline{x}, \overline{y}) = \big(\wp(z), \wp'(z)\big)$. The Riemann surfaces associated to $\overline{E}$ and $\mathbf{C}/\Lambda$ are isomorphic by the map $f : \mathbf{C}/\Lambda \to \overline{E}$ given by

$$f([z]) = \begin{cases} \big(\wp(z), \wp'(z)\big) & z \in \mathbf{C} - \Lambda \\ \mathcal{O} & z \in \Lambda \end{cases}$$

where $[z]$ denotes an equivalence class of points in $\mathbf{C}$ with $[z] = [u]$ if and only if $z - u \in \Lambda$. If we use the convention that $\big(\wp(\omega), \wp'(\omega)\big) = \mathcal{O}$ for all $\omega \in \Lambda$, then since the isomorphism satisfies $f([z] + [u]) = f([z]) + f([u])$, we have

$$\big(\wp(z+u), \wp'(z+u)\big) = \big(\wp(z), \wp'(z)\big) + \big(\wp(u), \wp'(u)\big) \qquad 3.21$$

for all $z, u \in \mathbf{C}$ where $z + u$ is the usual addition of complex numbers, and $\big(\wp(z), \wp'(z)\big) + \big(\wp(u), \wp'(u)\big)$ is the addition of points on an elliptic curve (see [**Sil**, p.158]).

Let $\mathcal{P}$ be a fundamental parallelogram for Λ with sides w_1 and w_2 as shown in Figure 3.2, and set $\frac{\omega_1+\omega_2}{2} = \frac{\omega_3}{2}$ and $e_i = \wp(\frac{\omega_i}{2})$ for $i = 1, 2, 3$. Then since $\frac{w_i}{2} + \frac{w_i}{2} = \omega_i \in \Lambda$, we use 3.21 to deduce that f maps the points $\{[0], [\frac{\omega_1}{2}], [\frac{\omega_2}{2}], [\frac{\omega_3}{2}]\}$ to the four points of order dividing 2 in $\overline{E}$ with the coordinates $\{\mathcal{O}, (e_1, 0), (e_2, 0), (e_3, 0)\}$. It is known ([**JS2**]) that the e_i are all distinct and correspond to the roots of the cubic polynomial $4\overline{x}^3 - g_2\overline{x} - g_3$. Hence the map $\pi : \mathbf{C}/\Lambda \to \Sigma$ given by $[z] \mapsto \wp(z)$ is a 2-sheeted cover of the Riemann sphere with branch points at $z = [0]$ and $z = [\frac{\omega_i}{2}]$ for $i = 1, 2, 3$.

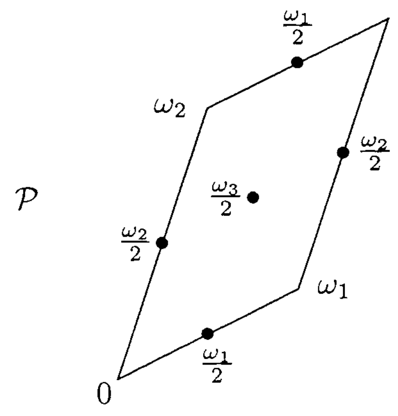


Figure 3.2

3.2. Euclidean Belyĭ uniformizations of elliptic curves

One can parameterize compact Riemann surfaces of genus 1 by the set of moduli $\tau \in \mathcal{F}$, the fundamental region for the modular group (see §3.1). For each $\tau \in \mathcal{F}$ the corresponding Riemann surface X_τ has the form $X_\tau = \mathbf{C}/\Lambda(1, \tau)$, and we have seen that X_τ may also be defined by the algebraic curve $y^2 = 4x^3 - g_2x - g_3$ where g_2, g_3 are functions of Λ. Belyĭ's Theorem tells us that an algebraic curve corresponding to X_τ can be defined over the field of algebraic numbers $\overline{\mathbf{Q}}$ if and only if X_τ can be uniformized by a finite index subgroup of a triangle group.

Definition 3.22. *A Riemann surface X of genus 1 has a Euclidean Belyĭ uniformization if $X = \mathbf{C}/\Lambda$, where Λ is a finite index subgroup of a Euclidean triangle group.* □

A complex number $\tau \in \mathbf{C}$ will be called a *modulus* if τ lies in $\mathbf{H}$, the upper half-plane. We will determine those moduli τ for which X_τ admits a Euclidean Belyĭ uniformization; i.e. X_τ can be uniformized by a finite index subgroup of a Euclidean triangle group. We note that by the Riemann-Hurwitz formula, a genus 1 subgroup of a Euclidean triangle group is torsion-free (and hence a surface group), while a genus 1 subgroup of a hyperbolic triangle group necessarily contains torsion. All Euclidean triangle groups are conjugate to $\Gamma(4, 2, 4)$, $\Gamma(3, 3, 3)$ or $\Gamma(6, 2, 3)$ in the isometry group of $\mathbf{C}$, and since $\Gamma(3, 3, 3) \leq \Gamma(6, 2, 3)$, Λ is a subgroup of a Euclidean triangle group if and only if Λ is a subgroup of $\Gamma(4, 2, 4)$ or $\Gamma(6, 2, 3)$.

Lemma 3.23. *A lattice Λ is contained in a triangle group $\Gamma(4,2,4)$ if and only if Λ is similar to $\Lambda(1,\tau)$ for some modulus $\tau \in \mathbf{Q}(i)$; a lattice Λ is contained in a triangle group $\Gamma(6,2,3)$ if and only if Λ is similar to $\Lambda(1,\tau)$ for some modulus $\tau \in \mathbf{Q}(\rho)$ where $\rho = \frac{-1+\sqrt{-3}}{2}$.*

Proof. (see [**JS1**,§7]) We can represent $\Gamma(4,2,4)$ as the group

$$< x_0, x_1 \,|\, x_0^4 = x_1^2 = (x_0 x_1)^{-4} = 1 >$$

where x_0 and x_1 are the transformations

$$x_0 : z \mapsto iz$$

$$x_1 : z \mapsto -z + 1$$

and $(x_0x_1)^{-1} : z \mapsto iz + 1$. So $\Gamma(4,2,4)$ consists of all transformations of the form $z \mapsto az + b$ where $a = \pm 1, \pm i$ and $b \in \mathbf{Z}[i]$. It is easy to see that if $a = -1$, i or $-i$ then $z \mapsto az + b$ has a fixed point in $\mathbf{C}$. If $\Lambda \leq \Gamma(4,2,4)$ is torsion-free, then Λ must be contained in the set of torsion-free elements $T = \{z \mapsto z + b \,|\, b \in \mathbf{Z}[i]\}$, where T is a normal subgroup of index 4 in $\Gamma(4,2,4)$. T is isomorphic to the Gaussian integer lattice under the obvious mapping, and so all torsion-free subgroups of $\Gamma(4,2,4)$ correspond to subgroups of $\Lambda(1,i)$; in particular, finite index torsion-free subgroups of $\Lambda(1,i)$ correspond to sublattices of $\Lambda(1,i)$. Every sublattice of $\Lambda(1,i)$ has the form $\Lambda(a+bi, c+di)$ with $a,b,c,d \in \mathbf{Z}$, which is similar to the lattice $\Lambda(1,\tau)$ for $\tau = \frac{c+di}{a+bi} \in \mathbf{Q}(i)$. Conversely, given a modulus $\tau \in \mathbf{Q}(i)$, we can write $\tau = \frac{c+di}{a+bi}$ for suitable $a,b,c,d \in \mathbf{Z}$. Thus $\Lambda(1,i)$ contains a lattice similar to $\Lambda(1,\tau)$, which by reversing the above argument corresponds to a torsion-free subgroup of $\Gamma(4,2,4)$.

The proof for $\Gamma(6,2,3)$ is similar: we represent $\Gamma(6,2,3)$ as the group

$$< x_0, x_1 \,|\, x_0^6 = x_1^2 = (x_0 x_1)^{-3} = 1 >$$

with $x_0 : z \mapsto -\rho^2 z$, $x_1 : z \mapsto -z+1$ and $(x_0x_1)^{-1} : z \mapsto \rho z + 1$. Thus $\Gamma(6,2,3)$ consists of the set of all transformations of the form $z \mapsto az + b$, where $a = \pm 1, \pm\rho, \pm\rho^2$ and $b \in \mathbf{Z}[\rho]$. It can be shown that $\Gamma(6,2,3)$ contains a normal subgroup of index 6 isomorphic to $\Lambda(1,\rho)$, and that every torsion-free subgroup of $\Gamma(6,2,3)$ corresponds to a sublattice of $\Lambda(1,\rho)$. The proof then proceeds as before. □

Thus we have proved:

Corollary 3.24. *The genus 1 Riemann surface X_τ admits a Euclidean Belyĭ uniformization if and only if τ is a modulus with $\tau \in \mathbf{Q}(i)$ or $\tau \in \mathbf{Q}(\rho)$.* □

It was shown in §3.1 that the Riemann surface X_i corresponds to the elliptic curve $y^2 = 4x^3 - x$, and that X_ρ corresponds to the curve $y^2 = 4x^3 - 1$. Now $i \in \mathbf{Q}(i)$ and $\rho \in \mathbf{Q}(\rho)$, so by Corollary 3.24 X_i and X_ρ have Euclidean Belyĭ uniformizations. In the representation of $\Gamma(4,2,4)$ as $\{z \mapsto az + b \,|\, a = \pm 1, \pm i,\ b \in \mathbf{Z}[i]\}$, we have $X_i \cong \mathbf{C}/T$ where T is the index 4 normal subgroup of $\Gamma(4,2,4)$ generated by $z \mapsto z+1$ and $z \mapsto z+i$. Similarly, $X_\rho \cong \mathbf{C}/S$ where S is the index 6 normal subgroup of $\Gamma(6,2,3)$ generated by $z \mapsto z+1$ and $z \mapsto z+\rho$.

The Riemann surfaces X_i and X_ρ admit Euclidean Belyĭ uniformizations, and so by Belyĭ's Theorem are defined over $\overline{\mathbf{Q}}$; indeed we have seen that they are defined over $\mathbf{Q}$. To obtain more general results on the fields of definition of elliptic curves with Euclidean Belyĭ uniformizations, we require some results from the theory of quadratic forms.

3.3. On number theory

For a more detailed study of the material in this section, we refer the reader to Davenport's treatment of quadratic forms [**Da**], and Chapter 7.2 of [**Coh**].

Quadratic forms

A *binary quadratic form* $Q(x,y)$ is a second degree homogeneous polynomial in two variables with integer coefficients:

$$Q(x,y) = px^2 + qxy + ry^2, \quad p,q,r \in \mathbf{Z} \qquad 3.25$$

where $Q(x,y)$ is *primitive* if p,q,r are coprime integers. Every quadratic form represents a set of integers

$$\{Q(\alpha,\beta)\,|\,\alpha,\beta \in \mathbf{Z}\} \qquad 3.26$$

and it is known that the quadratic forms $Q(x,y), Q'(x,y)$ represent the same sets of integers if and only if

$$Q(ax+by, cx+dy) = Q'(x,y) \qquad 3.27$$

with $a,b,c,d \in \mathbf{Z}, ad - bc = \pm 1$. A substitution with $ad - bc = 1$ is said to be *unimodular*.

Definition 3.28. *Two quadratic forms are properly equivalent if they are related by a unimodular substitution of the form 3.27.* □

Proper equivalence can be shown to be an equivalence relation on the set of all quadratic forms. Consider the quadratic form

$$Q(x,y) = 26x^2 + 102xy + 101y^2$$

under the unimodular transformation $Q'(x,y) = Q(2x-y, y-x)$. Then

$$\begin{aligned} Q'(x,y) &= 26(2x-y)^2 + 102(2x-y)(y-x) + 101(y-x)^2 \\ &= x^2 + 25y^2 \end{aligned}$$

and so $26x^2 + 102xy + 101y^2$ and $x^2 + 25y^2$ are properly equivalent.

Definition 3.29. *The discriminant of a quadratic form $px^2 + qxy + ry^2$ is $d = q^2 - 4pr$.* □

Both $Q(x,y) = 26x^2+102xy+101y^2$ and $Q'(x,y) = x^2+25y^2$ have discriminant $d = -100$. It is shown in [**Cox**] that any two properly equivalent forms have the same discriminant, although the converse is not true. For example the quadratic form

$$H(x,y) = 2x^2 + 2xy + 13y^2$$

has discriminant $d = -100$ but is not properly equivalent to $Q(x,y)$ or $Q'(x,y)$.

Since $d = q^2 - 4pr \equiv q^2 \bmod 4$, we must have either $d \equiv 0 \bmod 4$ or $d \equiv 1 \bmod 4$. Given $d \equiv 0, 1 \bmod 4$ we can find at least one quadratic form of discriminant d:

$$\begin{aligned} &\text{(i) } x^2 - \frac{d}{4}y^2 && \text{if } d \equiv 0 \bmod 4; \\ &\text{(ii) } x^2 + xy - \frac{d-1}{4}y^2 && \text{if } d \equiv 1 \bmod 4. \end{aligned} \qquad 3.30$$

The quadratic forms given in 3.30 are called the *principal forms* of discriminant d. The non-zero integers 3.26 represented by a quadratic form of negative discriminant are either all positive or all negative; the quadratic form then being either *positive or negative definite*. Quadratic forms with positive discriminant represent both positive and negative integers (so are called *indefinite*), while forms with discriminant equal to zero are just squares of linear forms. From now on we consider only primitive, positive definite, binary quadratic forms; that is forms of type 3.25 with $(p,q,r) = 1$, $q^2 - 4pr < 0$, and $p > 0$.

Definition 3.31. *A primitive positive definite form* $px^2 + qxy + ry^2$ *is said to be reduced if* $|q| \leq p \leq r$, *and* $q \geq 0$ *if either* $|q| = p$ *or* $p = r$. □

The quadratic forms $Q(x, y) = x^2 + 25y^2$ and $H(x, y) = 2x^2 + 2xy + 13y^2$ are reduced, with $Q(x, y)$ being the principal form of discriminant $d = -100$.

Theorem 3.32. *Every primitive positive definite form is properly equivalent to a unique reduced form.* □

Thus, every equivalence class of primitive positive definite forms has a canonical representative given by the unique reduced form for that class. Let $d < 0$ be the discriminant of a reduced form, and set $D = -d$ so that $D = 4pr - q^2$. From Definition 3.31 we deduce that $q^2 \leq p^2 \leq pr$, and hence that $D = 4pr - q^2 \geq 3pr$. There are only finitely many integers p, r satisfying $D \geq 3pr \geq 0$, and for each of these at most two possibilities for q. We conclude that there are only finitely many reduced forms having a given negative discriminant d.

Definition 3.33. *Let* $d < 0$ *be fixed. The class number* $h(d)$ *of* d *is the number of proper equivalence classes of primitive positive definite binary quadratic forms of discriminant* d. □

By Theorem 3.32, $h(d)$ is equivalently the number of reduced forms of discriminant d, and the discussion before Definition 3.33 shows that $h(d)$ is finite for $d < 0$. We have seen that $Q(x, y) = x^2 + 25y^2$ and $H(x, y) = 2x^2 + 2xy + 13y^2$ are two inequivalent reduced forms of discriminant $d = -100$. It is shown in [**Coh**, p.29] that $Q(x, y)$ and $H(x, y)$ are the only reduced forms of discriminant $d = -100$, and hence that the class number $h(-100) = 2$.

Quadratic imaginary numbers

Definition 3.34. $\tau \in \mathbf{H}$ *is a quadratic imaginary number if and only if* $p\tau^2 + q\tau + r = 0$ *for some* $p, q, r \in \mathbf{Z}$, *with* $q^2 - 4pr < 0$. □

Lemma 3.35. *There is a one-to-one correspondence between quadratic imaginary numbers lying in* $\mathbf{H}$ *and primitive positive definite quadratic forms.*

Proof. From Definition 3.34 every quadratic imaginary in $\mathbf{H}$ determines a quadratic polynomial, which is unique up to multiplication by a constant. We choose the unique polynomial pX^2+qX+r with $p,q,r \in \mathbf{Z}$, $q^2-4pr < 0$, $(p,q,r) = 1$ and $p > 0$ (conversely such a polynomial determines a unique quadratic imaginary in $\mathbf{H}$). The set of all such polynomials is in one-to-one correspondence with the set of primitive positive definite quadratic forms $px^2 + qxy + ry^2$. □

Cox [**Cox**] proves further that the quadratic imaginary numbers form an invariant subset of $\mathbf{H}$ under the action of the modular group on $\mathbf{H}$, and that two quadratic imaginary numbers lie in the same orbit under this action if and only if their associated quadratic forms are properly equivalent. The quadratic forms $Q(x,y) = 26x^2+102xy+101y^2$ and $Q'(x,y) = x^2+25y^2$ have associated quadratic imaginary numbers $\tau = \frac{-51+5i}{26}$ and $\tau' = 5i$ respectively. We have seen that $Q(x,y)$ and $Q'(x,y)$ are properly equivalent under the unimodular transformation $Q'(x,y) = Q(2x-y, y-x)$, and so τ and τ' lie in the same modular group orbit with $\tau' = \frac{2\tau-1}{1-\tau}$. Cox proves the following connection between reduced forms and the modular fundamental region:

Lemma 3.36. *A quadratic imaginary τ lies in the fundamental region $\mathcal{F}$ of the modular group if and only if its associated quadratic form is reduced.*

Proof. Let $px^2 + qxy + ry^2$ be a reduced quadratic form satisfying $|q| \leq p \leq r$, with its associated quadratic imaginary $\tau = \frac{-q+\sqrt{4pr-q^2}i}{2p} \in \mathbf{H}$. Then $|Re(\tau)| = |\frac{-q}{2p}| = \frac{|q|}{2p} \leq \frac{1}{2}$ and $|\tau| = \frac{pr}{p^2} = \frac{r}{p} \geq 1$, so $\tau \in \mathcal{F}$. Conversely, if $\tau \in \mathcal{F}$ is a quadratic imaginary satisfying $pX^2 + qX + r = 0$ with p,q,r in reduced form, then a similar argument implies that $|q| \leq p \leq r$. □

The quadratic forms $Q(x,y) = x^2 + 25y^2$ and $H(x,y) = 2x^2 + 2xy + 13y^2$ are reduced, and their associated quadratic imaginary numbers $5i$ and $\frac{-1+5i}{2}$ both lie in the modular fundamental region $\mathcal{F}$.

If $\tau \in \mathbf{H}$ is a quadratic imaginary number, we define the discriminant of τ be the discriminant of its associated quadratic form. We recall that an algebraic integer is a complex number which is the root of a monic polynomial with rational integer coefficients. We will require the following important theorem (for example see [**Coh**, p.377]):

Theorem 3.37. *Let $\tau \in \mathbf{H}$ be a quadratic imaginary number of discriminant $d < 0$. Then $j(\tau)$ is an algebraic integer of degree $h(d)$. Furthermore, the minimal* *polynomial over* $\mathbf{Z}$ *satisfied by* $j(\tau)$ *is given by*

$$\prod (X - j(\alpha)) = 0$$

where α *runs over the quadratic imaginary numbers associated to the reduced forms of discriminant* d. □

Example 3.38. We have seen that there are only two proper equivalence classes of quadratic forms of discriminant -100, so that $h(-100) = 2$. These are represented by the reduced forms

$$x^2 + 25y^2 \quad \text{and} \quad 2x^2 + 2xy + 13y^2$$

which have the associated quadratic imaginary numbers $\tau_1 = 5i$ and $\tau_2 = \frac{-1+5i}{2}$ respectively. By Theorem 3.37, the j-invariants $j(5i)$ and $j(\frac{-1+5i}{2})$ are algebraic integers of degree $h(-100) = 2$ and are conjugate in some quadratic extension field of $\mathbf{Q}$.

Consider the Riemann surfaces $X_{\tau_1} = \mathbf{C}/\Lambda(1, 5i)$ and $X_{\tau_2} = \mathbf{C}/\Lambda(1, \frac{-1+5i}{2})$. Then by Theorem 3.13, X_{τ_1} can be represented by an elliptic curve E_{τ_1} defined over the field $\mathbf{Q}(j(\tau_1))$. Similarly X_{τ_2} corresponds to an elliptic curve E_{τ_2} defined over the field $\mathbf{Q}(j(\tau_2))$. Since $j(\tau_1)$ and $j(\tau_2)$ are conjugate in some quadratic field, E_{τ_1} and E_{τ_2} are Galois-conjugate elliptic curves defined over a quadratic extension field of $\mathbf{Q}$. □

Quadratic residues

If $\gcd(d, m) = 1$, then d is called a *quadratic residue modulo* m if the congruence

$$x^2 \equiv d \bmod m$$

has a solution. If there is no solution, then d is called a *quadratic nonresidue modulo* m. Since $0^2 \equiv 0 \bmod 3$, $1^2 \equiv 1 \bmod 3$, and $2^2 \equiv 1 \bmod 3$ it follows that 1 is a quadratic residue mod 3, while 2 is a quadratic nonresidue mod 3. If p is an odd prime, then the *Legendre symbol* is defined to be

$$\left(\frac{d}{p}\right) = \begin{cases} 1 & \text{if } d \text{ is a quadratic residue mod } p \\ -1 & \text{if } d \text{ is a quadratic nonresidue mod } p \\ 0 & \text{if } p \mid d \end{cases}$$

so that $\left(\frac{1}{3}\right) = 1$ and $\left(\frac{2}{3}\right) = -1$. The following elementary properties of the Legendre symbol are proved in [**NZM** p.132]:

Theorem 3.39. If p is an odd prime then

(i) $\left(\frac{d}{p}\right) \equiv d^{\left(\frac{p-1}{2}\right)} \bmod p$;

(ii) $\left(\frac{d}{p}\right)\left(\frac{b}{p}\right) = \left(\frac{db}{p}\right)$;

(iii) $\left(\frac{-1}{p}\right) = (-1)^{\left(\frac{p-1}{2}\right)}$;

(iv) $d \equiv b \bmod p$ implies that $\left(\frac{d}{p}\right) = \left(\frac{b}{p}\right)$. □

The Legendre symbol $\left(\frac{d}{p}\right)$ requires p (the second argument) to be an odd prime. Jacobi introduced an extension to the Legendre symbol which allows the second argument to be any odd positive integer P. Writing $P = p_1 p_2 \dots p_s$ where the p_i are odd primes (not necessarily distinct) the Jacobi symbol is defined to be

$$\left(\frac{d}{P}\right) = \prod_{i=1}^{s} \left(\frac{d}{p_i}\right)$$

where $\left(\frac{d}{p_i}\right)$ is the Legendre symbol. A further extension to the Legendre symbol was given by Kronecker. This allows the second argument to be any positive integer, although the first argument is restricted to $d \equiv 0, 1 \bmod 4$. The Kronecker symbol is defined as follows:

Definition 3.40. [**Ro**, p.65] Let $d \equiv 0$ or $1 \bmod 4$ with d a non-square. The *Kronecker symbol* $\left(\frac{d}{n}\right)$ is defined for $n > 0$ by

(i) $\left(\frac{d}{n}\right) = 0$ if $\gcd(d, n) > 1$;

(ii) $\left(\frac{d}{1}\right) = 1$;

(iii) if d is odd $\left(\frac{d}{2}\right) = \left(\frac{2}{|d|}\right)$, a Jacobi symbol;

(iv) if $n = \prod_{i=1}^{s} p_i$ then $\left(\frac{d}{n}\right) = \prod_{i=1}^{s} \left(\frac{d}{p_i}\right)$, a product of Legendre symbols and possibly the symbol $\left(\frac{d}{2}\right)$. □

Hence we have the Kronecker symbols

$$\left(\frac{-4}{2}\right) = 0\,, \quad \left(\frac{-3}{2}\right) = \left(\frac{2}{3}\right) = -1. \qquad 3.41$$

Note that the Legendre, Jacobi and Kronecker symbols agree with one another on the intersections of their domains. There is therefore no ambiguity in the notation $\left(\frac{d}{n}\right)$, since it is obvious from the context which symbol is to be used.

3.4. Fields of definition of Euclidean Belyĭ uniformizations

It was proved in §3.2 that an elliptic curve E_τ with modulus τ admits a Euclidean Belyĭ uniformization if and only if τ is a quadratic imaginary number with $\tau \in \mathbf{Q}(i)$ or $\tau \in \mathbf{Q}(\rho)$. By Theorem 3.13, E_τ is defined over a field F if and only if $j(E_\tau) = j(\tau) \in F$. The results on class numbers and quadratic forms detailed in §3.3 give more information about F; $j(\tau)$ is an algebraic integer of degree $h(d)$ where d is the discriminant of τ, so that F is a field extension of degree $h(d)$ over $\mathbf{Q}$. The discriminants of quadratic imaginary numbers lying in $\mathbf{Q}(i)$ and $\mathbf{Q}(\rho)$ have a particularly simple form.

Lemma 3.42. *Let τ be a quadratic imaginary number. Then τ has discriminant $d = -4m^2$ (for some integer m) if and only if $\tau \in \mathbf{Q}(i)$, while τ has discriminant $d = -3m^2$ (for some integer m) if and only if $\tau \in \mathbf{Q}(\rho)$.*

Proof. Let $\tau \in \mathbf{Q}(i)$ have the associated quadratic form $px^2 + qxy + ry^2$. Then $\tau = \frac{-q+\sqrt{-d}i}{2p}$, where $d = q^2 - 4pr < 0$ is the discriminant of τ. Since $\tau \in \mathbf{Q}(i)$, $\sqrt{-d}$ must be an integer, and so $-d$ must be a square. In particular, $-d = 4pr - q^2 \equiv 0, 3 \bmod 4$ must be a square modulo 4. Hence $-d \equiv 0 \bmod 4$, and $d = -4m^2$ for some integer m. Conversely if $d = -4m^2$, it is obvious that $\tau \in \mathbf{Q}(i)$.

Similarly, if $\tau \in \mathbf{Q}(\rho)$ with the associated quadratic form $px^2 + qxy + ry^2$, then $\tau = \frac{-q+\sqrt{-d}i}{2p} \in \mathbf{Q}(\rho)$ and so we must have $\sqrt{-d} = \sqrt{3}m$ for some integer m. Consequently, $d = -3m^2$. □

Let us consider the moduli $\tau = i$ and $\tau = \rho$. The quadratic form associated to $\tau = i$ is $x^2 + y^2$, and so the corresponding discriminant is $d = -4$. For $\tau = \rho$ the quadratic form is $x^2 + xy + y^2$, for which the discriminant is $d = -3$.

Euclidean Belyĭ uniformizations of rational elliptic curves

An elliptic curve E_τ admits a Euclidean Belyĭ uniformization if and only if its modulus has the form $\tau \in \mathbf{Q}(i)$ or $\tau \in \mathbf{Q}(\rho)$, and E_τ is defined over $\mathbf{Q}$ if and only if its j-invariant $j(\tau)$ is rational. Hence by Theorem 3.37, a rational elliptic curve admits a Euclidean Belyĭ uniformization if and only if the discriminant of its modulus has class number 1. The following theorem, first proved by Heegner [**He**] in 1952, and later by Baker [**Ba**] and Stark [**St**] determines all negative discriminants with class number 1:

Theorem 3.43. *(Heegner-Baker-Stark) Let $d < 0$. Then $h(d) = 1$ if and only if*

$$d = -3, -4, -7, -8, -11, -12, -16, -19, -27, -28, -43, -67, -163. \quad \square$$

By Lemma 3.42, the moduli corresponding to elliptic curves with Euclidean Belyĭ uniformizations all have discriminants of the form $-3m^2$ or $-4m^2$, and so we require only 5 discriminants from Theorem 3.43:

$$d = -3, -4, -12, -16, -27. \tag{3.44}$$

In general, given a discriminant d having class number $h(d) = k$, one will obtain k distinct points in the modular region $\mathcal{F}$ corresponding to the k reduced forms of discriminant d. The 5 discriminants of 3.44 have class number 1, so each corresponds to a unique reduced form and hence to one modulus in $\mathcal{F}$. For example $d = -16$ corresponds to the reduced form

$$x^2 + 4y^2$$

with associated modulus $\tau = 2i$. The j-invariants of the moduli with class number 1 are well-known (see [**Cox**] for example) and we display them in Table 1 together with an equation for each elliptic curve E_τ.

By Belyĭ's Theorem, every rational elliptic curve has the form $\mathcal{U}/\Gamma$ where Γ is a finite index subgroup of a triangle group. From Table 1 we see that for only 5 rational elliptic curves will Γ be a subgroup of a Euclidean triangle group. For every other rational elliptic curve, Γ must be a genus 1 subgroup of a hyperbolic triangle group, and so must contain elements of finite order (i.e. Γ must have torsion). As a consequence we have the following result:

Theorem 3.45. *There are 5 rational elliptic curves that admit a Euclidean Belyĭ uniformization. Their j-invariants are $0, 1728, 54000, 287496$, or -12288000. Every other rational elliptic curve is (by Belyĭ's Theorem) of the form $\mathbf{H}/\Gamma$ where Γ is a subgroup of a hyperbolic triangle group.* $\square$

Elliptic curves defined over quadratic and cubic extensions of Q

The results of Theorem 3.45 were obtained by finding all of the solutions to

$$h(-3m^2) = 1 \quad \text{or} \quad h(-4m^2) = 1.$$

To find elliptic curves that are defined over extension fields of degree k over $\mathbf{Q}$, we need to find those values of d for which $h(d) = k$. Although this problem has been solved in principle by Goldfeld, Gross and Zagier ([**Coh**, p.229]), the explicit calculations have only been carried out for the cases $k \leq 4$. However, we require discriminants of the form $d = -3m^2$ or $-4m^2$, and in these cases formulae exist for the class number (see for example [**Cox**, p.148]):

$$h(-3m^2) = \begin{cases} 1 & m = 1 \\ \frac{m}{3}\prod_{p|m}\left(1 - \left(\frac{-3}{p}\right)\frac{1}{p}\right) & m > 1 \end{cases}$$

3.46

$$h(-4m^2) = \begin{cases} 1 & m = 1 \\ \frac{m}{2}\prod_{p|m}\left(1 - \left(\frac{-4}{p}\right)\frac{1}{p}\right) & m > 1 \end{cases}$$

where p ranges over all prime divisors of m. If p is an odd prime then $\left(\frac{v}{p}\right)$ ($v = -3, -4$) is the Legendre symbol, while if $p = 2$ the Kronecker symbol is used and $\left(\frac{-4}{2}\right) = 0$, $\left(\frac{-3}{2}\right) = -1$ by 3.41. From 3.46 one can deduce that $h(-3m^2)$ is either even or a power of three, while $h(-4m^2)$ is even for all $m > 2$. In particular, this means that there are no values of m for which $h(-4m^2) = 3$.

In §3.2 we stated a result from [**Cox**] that there are only two reduced forms of discriminant $d = -100$. By Theorem 3.32, this is equivalent to $h(-100) = 2$. Since $-100 = -4.5^2$ we have by the formula in 3.46 that

$$h(-4.5^2) = \frac{5}{2}\left(1 - \left(\frac{-4}{5}\right)\frac{1}{5}\right) = 2$$

which confirms the result. As a further example we take $d = -900 = -4(3.5)^2$, which using the formula from 3.46 gives $h(-4(3.5)^2) = 8$. The class number formulae 3.46 are not multiplicative functions of m; for example $h(-4.3^2) = 2 = h(-4.5^2)$ and so $h(-4.3^2)h(-4.5^2) = 4 \neq h(-4(3.5)^2)$. In order to make computation of the class number easier, we define the following two functions which are multiplicative in m

$$\overline{h}(-3m^2) = \begin{cases} 1 & m = 1 \\ m\prod_{p|m}\left(1 - \left(\frac{-3}{p}\right)\frac{1}{p}\right) & m > 1 \end{cases} \qquad 3.47(a)$$

$$\overline{h}(-4m^2) = \begin{cases} 1 & m = 1 \\ m\prod_{p|m}\left(1 - \left(\frac{-4}{p}\right)\frac{1}{p}\right) & m > 1 \end{cases} \qquad 3.47(b)$$

where $\overline{h}(-3.1^2) = h(-3.1^2) = 1 = \overline{h}(-4.1^2) = h(-4.1^2)$ and for $m > 1$, $\overline{h}(-3m^2) = 3h(-3m^2)$ and $\overline{h}(-4m^2) = 2h(-4m^2)$. If we calculate $\overline{h}(-900)$, then

$$\begin{aligned} \overline{h}(-4(3.5)^2) &= \overline{h}(-4.3^2)\,\overline{h}(-4.5^2) \\ &= 3\Big(1 + \frac{1}{3}\Big)5\Big(1 - \frac{1}{5}\Big) \\ &= 16 \end{aligned}$$

and since $\overline{h}(-900) = 2h(-900)$, we deduce that $h(-900) = 8$.

The functions $\overline{h}(-4m^2)$ and $\overline{h}(-3m^2)$ prove to be useful in determining all discriminants $d = -3m^2$ or $d = -4m^2$ with a given class number. For suppose that $h(-3m^2) = k$. Then either $m = 1$ in which case $h(-3.1^2) = 1 = k$, or else $m > 1$ and $h(-3m^2) = \frac{1}{3}\overline{h}(-3m^2)$. Hence solving $h(-3m^2) = k$ is equivalent to solving $\overline{h}(-3m^2) = 3k$ with the additional solution $m = 1$ when $k = 1$. Similarly, if $h(-4m^2) = k$, then either $m = 1$ and $h(-4.1^2) = 1 = k$, or $m > 1$ and $h(-4m^2) = \frac{1}{2}\overline{h}(-4m^2)$. Thus solving $h(-4m^2) = k$ is equivalent to solving $\overline{h}(-4m^2) = 2k$ with the additional solution $m = 1$ when $k = 1$.

Example 3.48. We have seen that $h(-900) = 8$. Using the multiplicative functions 3.47 we determine all discriminants of the form $d = -4m^2$ for which $h(d) = 8$. By the above discussion, this corresponds to finding all solutions to $\overline{h}(-4m^2) = 16$. Since $\overline{h}(-4m^2)$ is a multiplicative function of m, it is sufficient to determine all prime powers p^k for which $\overline{h}(-4(p^k)^2)$ is a divisor of 16. There are three cases to consider:

(1) If $p \equiv 1 \bmod 4$ then substituting $m = p^k$ into 3.47 gives

$$\overline{h}(-4(p^k)^2) = p^{k-1}(p-1) = 2, 4, 8 \text{ or } 16.$$

If $k > 1$ then $p = 2 \not\equiv 1 \bmod 4$. Hence we must have $k = 1$ and so $p = 5$ or $p = 17$ with $\overline{h}(-4.5^2) = 4$ and $\overline{h}(-4.17^2) = 16$.

(2) If $p \equiv 3 \bmod 4$ then substituting $m = p^k$ into 3.47 gives

$$\overline{h}(-4(p^k)^2) = p^{k-1}(p+1) = 2, 4, 8 \text{ or } 16.$$

As above we must have $k = 1$, and so either $p = 3$ or $p = 7$ with $\overline{h}(-4.3^2) = 4$ and $\overline{h}(-4.7^2) = 8$.

(3) Finally, if $m = 2^k$ we have

$$\overline{h}(-4(2^k)^2) = 2^k = 2, 4, 8 \text{ or } 16$$

which gives $\overline{h}(-4(2)^2) = 2$, $\overline{h}(-4(2^2)^2) = 4$, $\overline{h}(-4(2^3)^2) = 8$, and $\overline{h}(-4(2^4)^2) = 16$. If $\overline{h}(-4m^2) = 16$ then the divisors of m must be contained in the set

$$\{1, 2, 3, 4, 5, 7, 8, 16, 17\}$$

and it is easy to deduce that the only values of m for which $\overline{h}(-4m^2) = 16$ are $m = 12, 14, 15, 16, 17$ and 20. Hence there are six discriminants of the form $d = -4m^2$ for which $h(d) = 8$; $d = -576, -784, -900, -1024, -1156$, and -1600.

Using a similar method, one can show that there is only one discriminant of the form $d = -3m^2$ for which $h(d) = 8$; $d = -768$. □

By generalizing Example 3.48 one can determine all the discriminants $d = -3m^2$ or $d = -4m^2$ for which $h(d) = k$, where k is a positive integer. In particular when $k = 1$ we obtain the five discriminants shown in Table 1.

The only solutions to $h(-4m^2) = 2$ are $m = 3, 4, 5$ with corresponding discriminants $d = -36, -64, -100$, and the only solutions to $h(-3m^2) = 2$ are $m = 4, 5, 7$ with $d = -48, -75, -147$. These six discriminants have class-number two, and so each one corresponds to two reduced forms and hence two moduli lying in the modular fundamental region $\mathcal{F}$. The j-invariants of these moduli are algebraic integers of degree two by Theorem 3.37, and have been calculated by Berwick [**Ber**] (see Table 2). The twelve moduli of Table 2 correspond precisely to those elliptic curves that are defined over quadratic extensions of $\mathbf{Q}$ and admit Euclidean Belyĭ uniformizations. Using the j-invariants from Table 2 and Theorem 3.13 we give the equations of the corresponding elliptic curves in Table 3. As a consequence we have the following result:

Theorem 3.49. *There are 12 elliptic curves defined over* $\mathbf{Q}(\sqrt{m})$, *(m a square-free integer) that admit a Euclidean Belyĭ uniformization. Their j-invariants are listed in Table 2 and the corresponding elliptic curve equations are given in Table 3. Every other elliptic curve defined over a quadratic extension of* $\mathbf{Q}$ *is of the form*

$\mathbf{H}/\Gamma$, *where* Γ *is a subgroup of a hyperbolic triangle group. In particular, this is the case for elliptic curves defined over* $\mathbf{Q}(\sqrt{m})$ *for* $m \neq 2, 3, 5, 21$. □

We have seen that there are five discriminants of the form $-3m^2$ or $-4m^2$ which have class number one. The following lemma proves that for any other odd integer $k > 1$, there are at most two such discriminants having class number k.

Lemma 3.50. *Let* $k > 1$ *be an odd integer. Then there are no solutions to* $h(-4m^2) = k$ *and if* k *is not a power of three, there are no solutions to* $h(-3m^2) = k$. *If* $k = 3^r$ *for* $r > 0$, *then* $h(-3m^2) = k$ *has precisely two solutions,* $m = 3^{r+1}$ *and* $m = 2.3^r$.

Proof. By the discussion before Example 3.48, solving $h(-4m^2) = k > 1$ is equivalent to solving $\overline{h}(-4m^2) = 2k$. Since $\overline{h}(-4(2^a)^2) = 2^a$ by 3.47, there must be some odd prime power $p^b|m$ with $b > 0$ and $\overline{h}(-4(p^b)^2)|2k$. If $p \equiv 1 \bmod 4$ then $\overline{h}(-4(p^b)^2) = p^{b-1}(p-1) \equiv 0 \bmod 4$, while if $p \equiv 3 \bmod 4$ then $\overline{h}(-4(p^b)^2) = p^{b-1}(p+1) \equiv 0 \bmod 4$, both giving a contradiction since $2k \equiv 2 \bmod 4$ by assumption. Hence there are no solutions to $\overline{h}(-4m^2) = 2k$.

Similarly, solving $h(-3m^2) = k > 1$ is equivalent to solving $\overline{h}(-3m^2) = 3k$. Consider the prime power $p^a|m$ with $a > 0$ and $\overline{h}(-3(p^a)^2)|3k$. If $p \equiv 1 \bmod 3$ then $\overline{h}(-3(p^a)^2) = p^{a-1}(p-1) \equiv 0 \bmod 2$ because p is odd, contradicting the assumption that k is odd. If $p \equiv 2 \bmod 3$ then $\overline{h}(-3(p^a)^2) = p^{a-1}(p+1)$. Either $p^{a-1}(p+1) \equiv 0 \bmod 2$ for p odd, a contradiction, or else if $p = 2$ then $p^{a-1}(p+1) = 3.2^{a-1}$, and so $a = 1$. Finally if $p = 3$, then $\overline{h}(-3(3^a)^2) = 3^a$. Hence the only possibilities are $m = 3^{r+1}$ or $m = 2.3^r$ with $r > 0$ and $\overline{h}(-3m^2) = 3^{r+1}$, a power of three. □

Applying Lemma 3.50 to the case $k = 3$, we see that there are no solutions to $h(-4m^2) = 3$, while there are exactly two solutions to $h(-3m^2) = 3$: $m = 2.3$ and $m = 3^2$ with corresponding discriminants $d = -108$ and $d = -243$. These discriminants have class number 3, and so each one corresponds to three moduli lying in the modular fundamental region whose j-invariants are algebraic integers of degree 3. These j-invariants have also been calculated by Berwick [**Ber**, pp. 62-63] and we list them in Table 4. The equations for the three elliptic curves whose moduli have discriminant $d = -108$ are given in Table 5.

Theorem 3.51. *There are 6 elliptic curves defined over cubic extensions* $\mathbf{Q}(\theta)$ *of* $\mathbf{Q}$ *that admit Euclidean Belyĭ uniformizations. These are defined over* $\mathbf{Q}(\theta)$ *where*

$\theta^3 = 2$ *or* $\theta^3 = 3$, *and correspond to the six moduli in Table 4. Every other elliptic curve defined over a cubic extension of* $\mathbf{Q}$ *is of the form* $\mathbf{H}/\Gamma$, *where* Γ *is a subgroup of a hyperbolic triangle group.* □

In general, the total number of elliptic curves defined over extension fields of degree k over $\mathbf{Q}$ that admit Euclidean Belyĭ uniformizations is given by $kf(k)$, where $f(k)$ is the number of solutions to $h(-3m^2) = k$ or $h(-4m^2) = k$. Using Lemma 3.50 we can see that if $k > 1$ is odd and not a power of three then $f(k) = 0$, while if $k = 3^r$ for $r > 1$ then $f(k) = 2$. This means that there are no elliptic curves with Euclidean Belyĭ uniformizations defined over extension fields of degree k over $\mathbf{Q}$ where k is odd and not a power of three, while if $k = 3^r$ for $r > 0$, then there are exactly $2k$ elliptic curves defined over extension fields of degree $k = 3^r$ over $\mathbf{Q}$.

In Example 3.48 it was shown that there are seven solutions to $h(-3m^2) = 8$ or $h(-4m^2) = 8$, and so $f(8) = 7$. Therefore, there are 56 elliptic curves defined over extension fields of degree 8 over $\mathbf{Q}$ that admit Euclidean Belyĭ uniformizations. One can calculate the moduli of these elliptic curves by finding all of the reduced forms whose discriminants have class number 8, but although the j-invariants associated to these moduli are algebraic integers by Theorem 3.37, the precise values are unknown.

Chapter 4

Uniform dessins of genus 1

The regular maps of genus 1 were first described by Brahana [**Br1**] (also see [**Coxe1**]) and the regular hypermaps of genus 1 have been classified by Corn and Singerman [**CoSi**]. The elliptic curves with Euclidean Belyĭ uniformizations studied in Chapter 3 are precisely those that carry genus 1 uniform dessins. This correspondence leads naturally to a classification of the genus 1 uniform maps in terms of the moduli $\tau \in \mathcal{F}$ of their associated elliptic curves, and to the definition of a minimal map. We show that all reflexible genus 1 uniform maps occur as truncations and stellations of certain minimal maps, and that the minimal regular cover of a genus 1 uniform map is related to the discriminant of τ.

Cangül and Singerman [**CaSi**] have given formulae for the number of genus 1 regular maps with n vertices, and in §4.5 we extend these results to uniform maps. We conclude by constructing some Galois orbits of genus 1 uniform maps and calculate their associated Belyĭ functions.

4.1. Uniform maps and hypermaps of genus 1

The regular maps of genus 1 are classified in [**CMo**]. They have either square, triangular or hexagonal faces and have type $(4,4), (6,3)$ or $(3,6)$ respectively. Coxeter and Moser give the following method for constructing regular maps of type $(4,4)$ on the torus:

Choose any non-zero Gaussian integer $a + bi \in \mathbf{Z}[i]$ and form the sublattice $\Lambda(a + bi, -b + ai) \leq \Lambda(1, i)$. The projection of $\Lambda(1, i)$ to $\mathbf{C}/\Lambda(a + bi, -b + ai)$ induced by the natural projection of $\mathbf{C}$ to $\mathbf{C}/\Lambda(a+bi, -b+ai)$ gives a regular map of type $(4,4)$ on the torus, which is denoted $\{4,4\}_{a,b}$. (There is a difference in notation between

[CMo] and [JS1], but we use the established Coxeter notation for regular maps of genus 1.)

Example 4.1. Figure 4.1 shows the regular map $\{4,4\}_{3,-1}$ obtained by choosing the Gaussian integer $3-i$. □

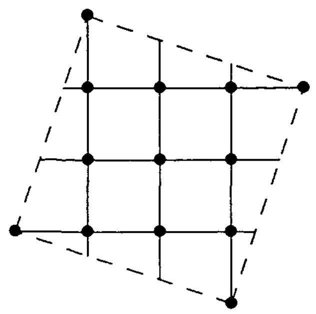


Figure 4.1

Similarly, by choosing some $0 \neq a + b\rho \in \mathbf{Z}[\rho]$ (where $\rho = \frac{-1+\sqrt{-3}}{2}$) and considering the sublattice $\Lambda(a+b\rho, a\rho+b\rho^2) \leq \Lambda(1,\rho)$, one obtains a regular map of type $(6,3)$ which is denoted $\{3,6\}_{a,b}$. The regular maps of type $(3,6)$ occur as the duals of regular maps of type $(6,3)$; the dual of $\{3,6\}_{a,b}$ being denoted $\{6,3\}_{a,b}$. It is shown in [**Coxe1**] that all regular maps of genus 1 can be obtained using these constructions.

Example 4.2. Figure 4.2(a) shows the regular map $\{3,6\}_{2,2}$ obtained by choosing $2+2\rho$, and Figure 4.2(b) gives its dual, the regular map $\{6,3\}_{2,2}$. □

Figure 4.2

Regular hypermaps of genus 1 were considered in [**CoSi**], and we now extend these results and those of Coxeter and Moser to classify the uniform maps and hypermaps of genus 1. Recall that a map of genus $g \geq 1$ is uniform if all of its vertices have the same valency, all of its faces have the same valency, and it contains no free edges; while a hypermap is said to be uniform if its hypervertices all have the same valency, its hyperedges all have the same valency, and its hyperfaces all have the same valency. By Theorem 2.19, uniform maps and hypermaps of genus 1 correspond to torsion-free subgroups of $\Gamma(4,2,4)$ giving maps of type $(4,4)$, torsion-free subgroups of $\Gamma(6,2,3) \cong \Gamma(3,2,6)$ giving maps of type $(6,3)$ and $(3,6)$, or torsion-free subgroups of $\Gamma(3,3,3)$ giving hypermaps of type $(3,3,3)$.

Using Lemma 3.23 we see that torsion-free subgroups of $\Gamma(4,2,4)$ correspond to sublattices of the Gaussian integer lattice $\Lambda(1,i)$. Since $\Lambda(1,i)$ represents the universal topological map associated to $\Gamma(4,2,4)$ (see §2.1), it follows that the map corresponding to the sublattice $\Lambda' \leq \Lambda(1,i)$ is given by the natural projection of $\Lambda(1,i)$ to $\mathbf{C}/\Lambda'$. These maps are uniform of type $(4,4)$.

Example 4.3. Let us consider the torsion-free subgroup of $\Gamma(4,2,4)$ which corresponds to the lattice $\Lambda(3,2+2i)$. The uniform map which results from projecting $\Lambda(1,i)$ to $\Lambda(3,2+2i)$ is shown in Figure 4.3. □

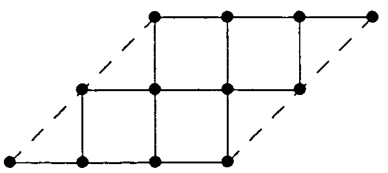


Figure 4.3

Similarly, a torsion-free subgroup of $\Gamma(6,2,3)$ can be represented by a sublattice $\Lambda' \leq \Lambda(1,\rho)$, the corresponding map of type $(6,3)$ being the natural projection of $\Lambda(1,\rho)$ to $\mathbf{C}/\Lambda'$. As with regular maps, every genus 1 uniform map of type $(3,6)$ corresponds to the dual of some uniform map of type $(6,3)$. We therefore identify two genus 1 uniform maps with each torsion-free subgroup of $\Gamma(6,2,3)$: one of type $(6,3)$ as constructed above, and its dual of type $(3,6)$.

Example 4.4. We take the torsion-free subgroup of $\Gamma(6,2,3)$ represented by the lattice $\Lambda(3+\rho, 2+4\rho)$, and display its associated uniform maps of type $(6,3)$ and $(3,6)$ in Figures 4.4(a) and 4.4(b) respectively. As noted above, the map in Figure 4.4(a) is dual to the map in Figure 4.4(b). □

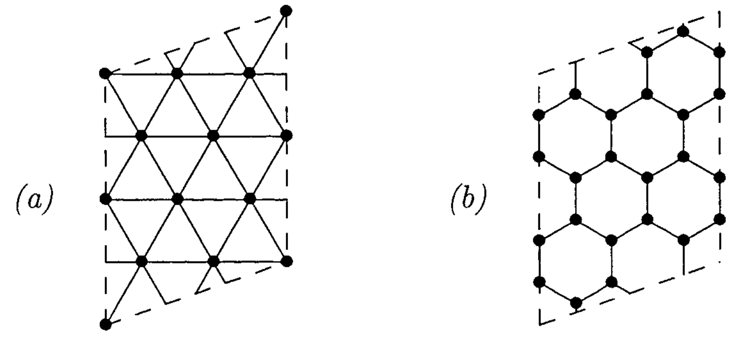


Figure 4.4

Since $\Gamma(3,3,3) \leq_2 \Gamma(3,2,6)$ by 2.44, every torsion-free subgroup of $\Gamma(3,3,3)$ is a torsion-free subgroup of $\Gamma(3,2,6)$, and so by Lemma 2.45 every genus 1 uniform hypermap of type $(3,3,3)$ has a corresponding Walsh double of type $(3,6)$. The following example illustrates this correspondence:

Example 4.5. Figure 4.5 shows two uniform Walsh hypermaps $\mathcal{W}_1$ and $\mathcal{W}_2$ where $\mathcal{W}_1 \cong \mathcal{W}^{(01)}$ (i.e. one may be obtained from the other by interchanging hypervertices and hyperedges). The Walsh doubles of $\mathcal{W}_1$ and $\mathcal{W}_2$ are both isomorphic to the map of Figure 4.4(b). Conversely, there is a bipartite structure on the vertices of the map in Figure 4.4(b) which is unique up to the choice of colouring; by interchanging the black and white vertices we obtain the Walsh hypermaps $\mathcal{W}_1$ or $\mathcal{W}_2$. □

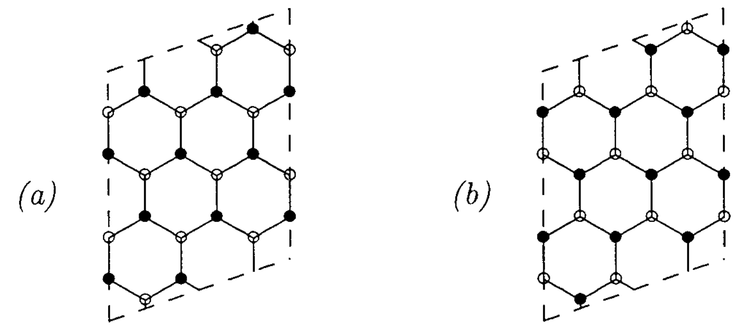


Figure 4.5

It will be proved that every genus 1 uniform map of type $(3,6)$ admits a bipartite structure on its vertices and furthermore that the two Walsh hypermaps obtained by interchanging the hypervertices and hyperedges are isomorphic; in particular this will mean that $\mathcal{W}_1$ and $\mathcal{W}_2$ in Figure 4.5 are isomorphic hypermaps. This will enable us to show that there is a one-to-one correspondence between: the set of genus 1 uniform hypermaps of type $(3,3,3)$, the set of genus 1 uniform maps of type $(3,6)$, and the set of genus 1 uniform maps of type $(6,3)$. We will need the following two lemmas.

Lemma 4.6. *If $M \leq \Gamma(3,2,6)$ is torsion-free, then $M \leq \Gamma(3,3,3)$.*

Proof. We can represent $\Gamma(3,2,6) = < x_0, x_1 \mid x_0^3 = x_1^2 = (x_0x_1)^{-6} = 1 >$ as the group of transformations generated by $x_0 : z \mapsto \rho z - \rho$, $x_1 : z \mapsto -z+1$ as in the proof of Lemma 3.23. We then think of $\Gamma(3,2,6)$ as being the set of all transformations of the form $z \mapsto az+b$ where $a = \pm 1, \pm\rho, \pm\rho^2$ and $b \in \mathbf{Z}[\rho]$. By Lemma 3.23 all torsion-free subgroups of $\Gamma(3,2,6)$ are subgroups of $\Lambda(1,\rho)$. Let $y_0 = x_0 : z \mapsto \rho z - \rho$ and $y_1 : x_1x_0x_1 : z \mapsto \rho z + 1$, so that gp $< y_0, y_1 >$ is an index 2 subgroup of $\Gamma(3,2,6)$ isomorphic to $\Gamma(3,3,3)$ by 2.46. Hence we can represent $\Gamma(3,3,3)$ as the set of elements $z \mapsto cz+d$ where $c = 1, \rho, \rho^2$ and $d \in \mathbf{Z}[\rho]$; in particular we note that $\Lambda(1,\rho) \leq \Gamma(3,3,3)$. Any torsion-free subgroup $M \leq \Gamma(3,2,6)$ is a subgroup of $\Lambda(1,\rho)$, and so $M \leq \Lambda(1,\rho) \leq \Gamma(3,3,3)$. □

Lemma 4.7. *Let $M_1, M_2 \leq \Gamma(3,2,6)$ be torsion-free. Then M_1, M_2 are conjugate in $\Gamma(3,2,6)$ if and only if they are conjugate in $\Gamma(3,3,3)$.*

Proof. If M_1 and M_2 are conjugate in $\Gamma(3,3,3)$ then they are certainly conjugate in $\Gamma(3,2,6)$. Conversely if M_1 and M_2 are conjugate in $\Gamma(3,2,6)$ then $M_1 = hM_2h^{-1}$ for some $h \in \Gamma(3,2,6)$. From the proof of Lemma 4.6 we can write $\Gamma(3,2,6) = \Gamma(3,3,3) \cup \Gamma(3,3,3)x_1$, so it suffices to show that if $M_1 = x_1M_2x_1^{-1}$, then M_1 and M_2 are conjugate in $\Gamma(3,3,3)$. Now M_2 is torsion-free, and so it has elements of the form $\phi : z \mapsto z+d$ for suitable $d \in \mathbf{Z}[\rho]$. Conjugating by x_1 gives

$$x_1\phi x_1^{-1} : z \mapsto z - d$$

so every element $\phi \in M_2$ is sent to its inverse $\phi^{-1} \in M_2$. Therefore $M_1 = x_1M_2x_1^{-1} = M_2$ and being equal, M_1 and M_2 are conjugate in $\Gamma(3,3,3)$. □

Thus by Lemmas 4.6 and 4.7 there is a bijection between conjugacy classes of torsion-free subgroups of $\Gamma(3,3,3)$ and conjugacy classes of torsion-free subgroups of $\Gamma(3,2,6)$, which induces by Theorem 2.5 a one-to-one correspondence between isomorphism classes of genus 1 uniform hypermaps of type $(3,3,3)$, genus 1 uniform maps of type $(6,3)$ and genus 1 uniform maps of type $(3,6)$. This correspondence can be defined as follows:

Theorem 4.8. *There is a one-to-one correspondence between the following three sets of elements: isomorphism classes of genus 1 uniform hypermaps of type* $(3, 3, 3)$, *isomorphism classes of genus 1 uniform maps of type* $(6, 3)$, *and isomorphism classes of genus 1 uniform maps of type* $(3, 6)$.

(i) The correspondence between genus 1 uniform maps of type $(3, 6)$ and $(6, 3)$ is defined by the operation of duality.

(ii) A genus 1 uniform hypermap $\mathcal{H}$ of type $(3, 3, 3)$ corresponds to a unique map of type $(3, 6)$ by taking the Walsh double associated to $\mathcal{H}$.

(iii) A genus 1 uniform map $\mathcal{M}$ of type $(3, 6)$ corresponds to a unique hypermap of type $(3, 3, 3)$ by defining a bipartite structure on the vertices of $\mathcal{M}$.

Proof. (i) This follows by considering the map subgroup inclusions $M \leq \Gamma(3, 2, 6) \cong \Gamma(6, 2, 3)$. (ii) and (iii). If H is the hypermap subgroup corresponding to $\mathcal{H}$, then $\Gamma(3, 3, 3) \leq \Gamma(3, 2, 6)$ and by Theorem 2.45 the inclusion $H \leq \Gamma(3, 2, 6)$ corresponds to the Walsh double of $\mathcal{H}$. Conversely if $M \leq \Gamma(3, 2, 6)$ is a map subgroup for a genus 1 uniform map $\mathcal{M}$ of type $(3, 6)$, then $M \leq \Gamma(3, 3, 3)$ by Lemma 4.6 and the inclusion $M \leq \Gamma(3, 3, 3)$ corresponds to a uniform hypermap $\mathcal{H}$ of type $(3, 3, 3)$ where $\mathcal{M}$ is the Walsh double of $\mathcal{H}$. This means that $\mathcal{M}$ admits a bipartite structure on its vertices, and that by a choice of hypervertices and hyperedges we obtain $\mathcal{H}$. Uniqueness follows from Lemma 2.47 and Lemma 4.7. □

In particular, this means that every uniform hypermap of type $(3, 3, 3)$ can be obtained by choosing a bipartite structure on the vertices of some uniform map of type $(3, 6)$. We recall that a uniform map or hypermap defined by a torsion-free subgroup $M \leq \Gamma(l_0, l_1, l_2)$ is regular if and only if M is a normal subgroup of $\Gamma(l_0, l_1, l_2)$. As a corollary to Lemma 4.7 we have:

Corollary 4.9. *Let* $M \leq \Gamma(3, 2, 6)$ *be torsion-free. Then* M *is normal in* $\Gamma(3, 2, 6)$ *if and only if* M *is normal in* $\Gamma(3, 3, 3)$.

Proof. By Lemma 4.7 there is a bijection between the conjugates of M in $\Gamma(3, 2, 6)$ and the conjugates of M in $\Gamma(3, 3, 3)$. The result follows from the fact that M is normal in Γ if and only if M has no other conjugates in Γ. □

Therefore, the correspondence between uniform maps and hypermaps defined in Theorem 4.8 restricts to a one-to-one correspondence between genus 1 regular hypermaps of type $(3,3,3)$, genus 1 regular maps of type $(6,3)$ and genus 1 regular maps of type $(3,6)$; this result is proved in [**CoSi**].

4.2. Classification of uniform maps of genus 1

It was shown in §4.1 that genus 1 uniform maps of type $(4,4)$ or $(6,3)$ correspond to sublattices of $\Lambda(1,i)$ or $\Lambda(1,\rho)$, and that by using simple geometric operations one can obtain any of the genus 1 uniform maps of type $(3,6)$ or the genus 1 uniform hypermaps of type $(3,3,3)$. For brevity, we restrict our attention in this section to uniform maps of type $(4,4)$ and $(6,3)$ and derive a classification of them in terms of their underlying lattices.

Uniform maps of type $(4,4)$

By Theorem 2.5, the maps associated to subgroups M_1 and M_2 of $\Gamma(4,2,4)$ are isomorphic if and only if M_1 and M_2 are conjugate in $\Gamma(4,2,4)$. It is therefore important for us to determine when two sublattices of $\Lambda(1,i)$ correspond to conjugate torsion-free subgroups of $\Gamma(4,2,4)$.

Lemma 4.10. *Let Λ_1 and Λ_2 be sublattices of $\Lambda(1,i)$. Then Λ_1 and Λ_2 correspond to conjugate torsion-free subgroups of $\Gamma(4,2,4)$ if and only if $\Lambda_1 = u\Lambda_2$ where u is a unit in $\mathbf{Z}[i]$.*

Proof. Using the representation of $\Gamma(4,2,4)$ given in Lemma 3.23, a lattice $\Lambda \leq \Lambda(1,i)$ corresponds to a torsion-free subgroup of $\Gamma(4,2,4)$ consisting of elements of the form $\phi : z \mapsto z + d$ where $d \in \mathbf{Z}[i]$. A general element of $\Gamma(4,2,4)$ has the form $g : z \mapsto az + b$ with $a = \pm 1$ or $\pm i$ and $b \in \mathbf{Z}[i]$. Since

$$g \circ \phi \circ g^{-1} : z \mapsto z + ad,$$

conjugation by g corresponds to multiplying the lattice Λ by a, where $a = \pm 1$ or $\pm i$. Hence the lattices $\Lambda, i\Lambda, -\Lambda, -i\Lambda$ form a complete set of conjugates in $\Gamma(4,2,4)$. We observe that $\Lambda = -\Lambda$ for any lattice Λ. □

We note that the uniform map obtained from a lattice $\Lambda \leq \Lambda(1,i)$ is independent of any specified basis for Λ. For if $\{\omega_1, \omega_2\}$ and $\{\omega'_1, \omega'_2\}$ are two bases for Λ, then $\Lambda(\omega_1, \omega_2) = \Lambda = \Lambda(\omega'_1, \omega'_2)$ and the maps associated to $\Lambda(\omega_1, \omega_2)$ and $\Lambda(\omega'_1, \omega'_2)$ are isomorphic since they have the same map subgroups. The following corollary will be of use:

Corollary 4.11. *Let $\Lambda(\omega_1, \omega_2) \leq \Lambda(1, i)$ be a lattice with non-zero elements $p + qi, r + si \in \mathbf{Z}[i]$. Then the lattices $(p + qi)\Lambda(\omega_1, \omega_2)$ and $(r + si)\Lambda(\omega_1, \omega_2)$ are conjugate in $\Gamma(4, 2, 4)$ if and only if $p + qi$ and $r + si$ are associates in $\mathbf{Z}[i]$.*

Proof. If $p + qi = u(r + si)$ for some unit $u \in \mathbf{Z}[i]$ then $(p + qi)\Lambda(\omega_1, \omega_2) = u(r + si)\Lambda(\omega_1, \omega_2)$ and the lattices are conjugate by Lemma 4.10. Conversely if they are conjugate in $\Gamma(4, 2, 4)$ then $(p + qi)\Lambda(\omega_1, \omega_2) = u(r + si)\Lambda(\omega_1, \omega_2)$ for some unit in $\mathbf{Z}[i]$. Hence there exist integers a, b, c, d such that

$$(p + qi)\omega_1 = u(r + si)(a\omega_1 + b\omega_2)$$
$$(p + qi)\omega_2 = u(r + si)(c\omega_1 + d\omega_2)$$

and choosing $v \in \mathbf{Z}[i]$ so that $\omega_1' = \frac{\omega_1}{v}$ and $\omega_2' = \frac{\omega_2}{v}$ are coprime gives

$$(p + qi)\omega_1' = u(r + si)(a\omega_1' + b\omega_2')$$
$$(p + qi)\omega_2' = u(r + si)(c\omega_1' + d\omega_2').$$

Since ω_1' and ω_2' are coprime, this implies that $(r + si)|(p + qi)$. Similarly one can show that $(p + qi)|(r + si)$ and hence that $p + qi$ and $r + si$ are associates in $\mathbf{Z}[i]$. □

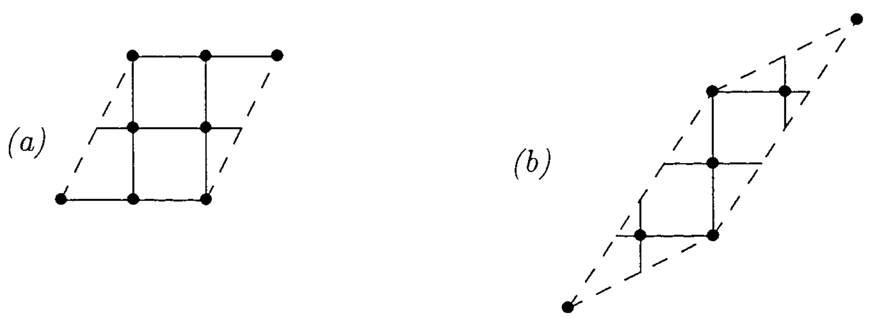


Figure 4.6

Example 4.12. Figures 4.6(a) and 4.6(b) show the uniform maps associated to the lattices $\Lambda(2, 1 + 2i)$ and $\Lambda(2 + i, 2 + 3i)$ respectively. The unimodular transformation

$$\begin{pmatrix} 1 & -2 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 1 + 2i \\ 2 \end{pmatrix} = \begin{pmatrix} -3 + 2i \\ -1 + 2i \end{pmatrix}$$

gives the following change of basis: $\Lambda(2, 1 + 2i) = \Lambda(-1 + 2i, -3 + 2i) = i\Lambda(2 + i, 2 + 3i)$. Therefore, by Lemma 4.10 the lattices $\Lambda(2, 1 + 2i)$ and $\Lambda(2 + i, 2 + 3i)$ are conjugate in $\Gamma(4, 2, 4)$ and so by Theorem 2.5 the maps in Figure 4.6 are isomorphic.

□

The regular maps of type $(4,4)$ correspond to normal subgroups of $\Gamma(4,2,4)$. Since a subgroup $K \leq H$ is normal if and only if K has no other conjugates in H, we have as a corollary to Lemma 4.10:

Corollary 4.13. *A sublattice $\Lambda \leq \Lambda(1,i)$ corresponds to a normal torsion-free subgroup of $\Gamma(4,2,4)$ if and only if $\Lambda = i\Lambda$.* □

Since any lattice satisfying the conditions of Corollary 4.13 corresponds to an ideal in the ring $\mathbf{Z}[i]$, a lattice corresponding to a normal subgroup of $\Gamma(4,2,4)$ will be called an *ideal lattice* of $\Lambda(1,i)$. Now $\mathbf{Z}[i]$ is a principal ideal domain [**Alle**, p.113], and so every ideal lattice of $\Lambda(1,i)$ has the form $\Lambda(a+bi, ai-b)$ for some $a+bi \in \mathbf{Z}[i]$. The ideal lattice $\Lambda(a+bi, ai-b) \leq \Lambda(1,i)$ corresponds to the regular map $\{4,4\}_{a,b}$ in Coxeter and Moser's notation (see §4.1). We now extend Coxeter and Moser's notation to give a description of all genus 1 uniform maps of type $(4,4)$.

We recall from §3.1 that if Λ is a lattice, then we can choose a basis $\{\omega_1, \omega_2\}$ with $\Lambda = \Lambda(\omega_1, \omega_2)$ such that the modulus $\tau = \frac{\omega_2}{\omega_1}$ of Λ lies in $\mathcal{F}$, the fundamental region for the modular group acting on the upper half-plane. Furthermore, the set of moduli $\tau \in \mathcal{F}$ parameterizes the set of all genus 1 compact Riemann surfaces. Therefore, given any sublattice $\Lambda \leq \Lambda(1,i)$, we can choose a basis $\{a+bi, c+di\}$ with $\Lambda = \Lambda(a+bi, c+di)$ such that the modulus $\tau = \frac{c+di}{a+bi}$ lies in $\mathcal{F}$, and the uniform map associated with Λ lies on the Riemann surface $\mathbf{C}/\Lambda$ of modulus τ. For example, the two (isomorphic) maps associated to the lattices $\Lambda(2, 1+2i)$ and $\Lambda(2+i, 2+3i)$ in Figure 4.6 lie on the Riemann surface of modulus $\frac{1+2i}{2}$. By specifying a modulus $\tau \in \mathbf{Q}(i)$ however, there are an infinite number of sublattices of $\Lambda(1,i)$ with modulus τ, each one associated to a uniform map.

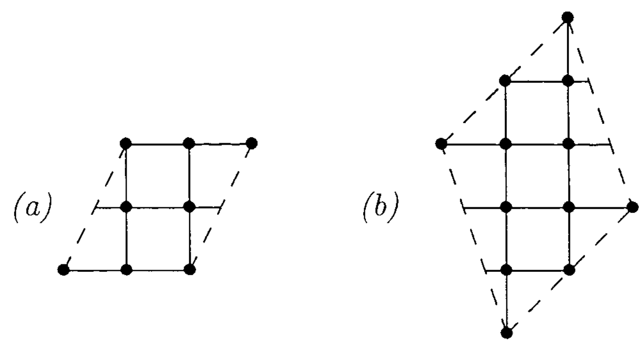


Figure 4.7

Example 4.14. The similar lattices $\Lambda(2, 1+2i)$ and $(1+i)\Lambda(2, 1+2i)$ both have the same modulus $\tau = \frac{1+2i}{2}$. Their associated uniform maps, shown in Figure 4.7, both lie on the same Riemann surface (of modulus $\frac{1+2i}{2}$) but are clearly non-isomorphic because they have different numbers of faces. Indeed, by considering the similar lattices given by $(p+qi)\Lambda(2, 1+2i)$ with $0 \neq p+qi \in \mathbf{Z}[i]$, one obtains an infinite family of uniform maps lying on the Riemann surface of modulus $\frac{1+2i}{2}$.

By Corollary 4.11, two lattices $(p+qi)\Lambda(2, 1+2i)$ and $(p'+q'i)\Lambda(2, 1+2i)$ will correspond to isomorphic uniform maps if and only if $p+qi$ and $p'+q'i$ are associates in $\mathbf{Z}[i]$. Therefore the maps corresponding to the lattices $(3+i)\Lambda(2, 1+2i)$ and $(-1+3i)\Lambda(2, 1+2i)$ are isomorphic, while the maps corresponding to $(3+i)\Lambda(2, 1+2i)$ and $(3-i)\Lambda(2, 1+2i)$ are not; we shall see that these last two maps are mirror images of each other. □

The lattice $\Lambda(a+bi, c+di)$ has a fundamental parallelogram of area

$$n = |ad - bc|$$

giving a subgroup of index n in $\Lambda(1, i)$. The map associated to $\Lambda(a+bi, c+di)$ will then have n faces, since each face is a square of unit area; $\frac{4n}{2} = 2n$ edges, since each face is bounded by 4 edges and 2 faces meet each edge; and $\frac{2(2n)}{4} = n$ vertices, since each edge is incident with 2 vertices and 4 edges meet at each vertex. If Λ is an index n sublattice of $\Lambda(1, i)$, then the uniform map $\mathcal{M}$ associated to Λ is also said to have *index n*. (It is important not to confuse the index of $\mathcal{M}$ with the index of its map subgroup. We recall that $\Lambda \leq \Gamma(4, 2, 4)$ is the *map subgroup* for $\mathcal{M}$ (see §2.1), and because $\Lambda \leq_n \Lambda(1, i) \leq_4 \Gamma(4, 2, 4)$, Λ has index $4n$ in $\Gamma(4, 2, 4)$). We have proved:

Lemma 4.15. *Let $\mathcal{M}$ be an index n uniform map of type $(4, 4)$. Then $\mathcal{M}$ has n faces, n vertices and $2n$ edges.* □

For example, the lattice $\Lambda(2, 1+2i)$ has index $n = 4$ in $\Lambda(1, i)$, and its corresponding map shown in Figure 4.7(a) has 4 faces, 4 vertices and 8 edges. For $0 \neq p+qi \in \mathbf{Z}[i]$, the lattice

$$(p+qi)\Lambda(a+bi, c+di) = \Lambda(pa - qb + i(pb+qa),\ pc - qd + i(pc+qd))$$

has index 4.16

$$\big|(pa-qb)(pc+qd) - (pb+qa)(pc-qd)\big| = (p^2+q^2)\big|(ad-bc)\big| = (p^2+q^2)n$$

in $\Lambda(1,i)$. Hence the lattice $(1+i)\Lambda(2,1+2i)$ has index $4(1^2+1^2)=8$ in $\Lambda(1,i)$; the corresponding uniform map of Figure 4.7(b) has 8 faces, 8 vertices and 16 edges.

Let us return to the family of lattices considered in Example 4.14. Since $1+2i$ and 2 are coprime Gaussian integers, every sublattice of $\Lambda(1,i)$ with modulus $\frac{1+2i}{2}$ has the form $(p+qi)\Lambda(2,1+2i)$ where $p+qi \in \mathbf{Z}[i]$. The map associated to $\Lambda(2,1+2i)$ has index 4, while the map associated to $(p+qi)\Lambda(2,1+2i)$ has index $4(p^2+q^2) \geq 4$ with equality if and only if $p+qi$ is a unit in $\mathbf{Z}[i]$, and so if and only if the uniform maps associated to $\Lambda(2,1+2i)$ and $(p+qi)\Lambda(2,1+2i)$ are isomorphic. Hence the map associated to the lattice $\Lambda(2,1+2i)$ is *minimal* in the sense that every other uniform map of type $(4,4)$ lying on the Riemann surface of modulus $\frac{1+2i}{2}$ has an index strictly greater than 4. We can define a minimal map for every genus 1 Riemann surface of modulus $\tau \in \mathbf{Q}(i)$:

Definition 4.17. *Given a modulus $\tau \in \mathbf{Q}(i)$, the minimal map $\mathcal{M}_\tau$ represented by τ corresponds to the lattice $\Lambda(a+bi, c+di)$ where $\tau = \frac{c+di}{a+bi}$ and $a+bi$, $c+di$ are coprime in the ring $\mathbf{Z}[i]$.* □

We note that $\mathcal{M}_\tau$ is unique, since although we can write $\tau = \frac{u(c+di)}{u(a+bi)}$ for any unit $u \in \mathbf{Z}[i]$, $\Lambda(u(a+bi), u(c+di)) = u\Lambda(a+bi, c+di)$ and so the associated maps are isomorphic by Lemma 4.10. Our classification now uses the fact that every genus 1 uniform map of type $(4,4)$ lies on a unique Riemann surface, and hence can be identified with a unique modulus $\tau \in \mathbf{Q}(i)$ lying in the modular fundamental region. Furthermore, τ can be expressed uniquely (up to multiplication by units) as $\tau = \frac{c+di}{a+bi}$ where $a+bi$ and $c+di$ are coprime Gaussian integers.

Notation 4.18. *Let $\mathcal{M}$ be a genus 1 uniform map of type $(4,4)$ associated to a sublattice $\Lambda \leq \Lambda(1,i)$. By choosing a suitable basis we can write $\Lambda = (p+qi)\Lambda(a+bi, c+di)$ where $a+bi$ and $c+di$ are coprime Gaussian integers and the modulus $\tau = \frac{c+di}{a+bi}$ lies in the modular fundamental region $\mathcal{F}$. Then $\mathcal{M}$ will be the map $\{\tau\}_{p+qi}$. Conversely, for a modulus $\tau \in \mathbf{Q}(i)$ lying in the modular fundamental region, $\{\tau\}_{p+qi}$ represents the uniform map corresponding to the lattice $(p+qi)\Lambda(a+bi, c+di)$ where $\tau = \frac{c+di}{a+bi}$ and $a+bi$, $c+di$ are coprime Gaussian integers.* □

The uniform map $\mathcal{M}$ of Figure 4.7(a) corresponds to the lattice $\Lambda(2,1+2i)$. Since 2 and $1+2i$ are coprime in $\mathbf{Z}[i]$ and $\frac{1+2i}{2} \in \mathcal{F}$, $\mathcal{M}$ is the map $\left\{\frac{1+2i}{2}\right\}_1$. Similarly, the map in Figure 4.7(b) corresponding to the lattice $(1+i)\Lambda(2,1+2i)$

has the form $\left\{\frac{1+2i}{2}\right\}_{1+i}$. The results in this section are brought together in the following lemma:

Lemma 4.19. *Let $\tau, \tau' \in \mathbf{Q}(i)$ be moduli lying in the modular fundamental region.*

(i) Every uniform map of type $(4,4)$ lying on the genus 1 Riemann surface of modulus τ has the form $\{\tau\}_{p+qi}$ for some $p+qi \in \mathbf{Z}[i]$;

(ii) The uniform maps $\{\tau\}_{p+qi}$ and $\{\tau'\}_{p'+q'i}$ are isomorphic if and only if $\tau = \tau'$ and $p+qi, p'+q'i$ are associates in the ring $\mathbf{Z}[i]$;

(iii) The minimal map $\mathcal{M}_\tau$ corresponds to the uniform map $\{\tau\}_1$;

(iv) The uniform map $\{\tau\}_{p+qi}$ is regular if and only if $\tau = i$. Hence $\{i\}_{p+qi}$ corresponds to Coxeter and Moser's $\{4,4\}_{p,q}$;

(v) If $\tau = \frac{c+di}{a+bi}$ where $a+bi, c+di$ are coprime Gaussian integers and $n = |ad-bc|$, then $\{\tau\}_{p+qi}$ is a uniform map with $(p^2+q^2)n$ faces, $(p^2+q^2)n$ vertices and $2(p^2+q^2)n$ edges.

Proof. (i) If a uniform map $\mathcal{M}$ lies on the Riemann surface of modulus τ, then it has an associated lattice $\Lambda \leq \Lambda(1,i)$ with a basis $\{\omega_1, \omega_2\}$ where $\omega_1, \omega_2 \in \mathbf{Z}[i]$, $\Lambda = \Lambda(\omega_1, \omega_2)$ and $\tau = \frac{\omega_2}{\omega_1} \in \mathcal{F}$. Writing $\omega_1 = (p+qi)(a+bi)$, $\omega_2 = (p+qi)(c+di)$ where $a+bi$ and $c+di$ are coprime Gaussian integers, $\mathcal{M}$ has the form $\{\tau\}_{p+qi}$.

(ii) We observe that every genus 1 uniform map lies on a unique Riemann surface of modulus $\tau \in \mathcal{F}$. By the $1-1$ correspondence between genus 1 Riemann surfaces and points in $\mathcal{F}$, if $\{\tau\}_{p+qi}$ and $\{\tau'\}_{p'+q'i}$ are isomorphic we must have $\tau = \tau'$. The lattices corresponding to $\{\tau\}_{p+qi}$ and $\{\tau\}_{p'+q'i}$ can be written as $(p+qi)\Lambda(a+bi, c+di), (p'+q'i)\Lambda(a+bi, c+di)$ where $\tau = \frac{c+di}{a+bi}$ and $a+bi, c+di$ are coprime Gaussian integers, and so by Corollary 4.11 and Theorem 2.5 the maps are isomorphic if and only if $p+qi, p'+q'i$ are associates in $\mathbf{Z}[i]$. The converse follows easily from Corollary 4.11.

(iii) This follows immediately from Definition 4.17.

(iv) Suppose that $\tau = i$. Then the lattice associated to $\{i\}_{p+qi}$ has the form $(p+qi)\Lambda(1,i)$, and since $i(p+qi)\Lambda(1,i) = (p+qi)\Lambda(i,-1) = (p+qi)\Lambda(1,i)$, the map is regular by Corollary 4.13 and Theorem 2.20. Conversely, if $\{\tau\}_{p+qi}$ is a regular map, then the associated lattice Λ corresponds to an ideal in $\mathbf{Z}[i]$ (see the remarks following Corollary 4.13), and so has the form $\Lambda = (p+qi)\Lambda(1,i)$ for some Gaussian integer $p+qi$. Therefore $\tau = \frac{i}{1} = i$.

(v) The map $\{\tau\}_{p+qi}$ has an associated lattice $(p+qi)\Lambda(a+bi, c+di)$, and hence has index $(p^2+q^2)n$ by 4.16. We now apply Lemma 4.15. □

Truncations of uniform maps

If $\mathcal{M}$ is a uniform map of type $(4,4)$, then its dual $D(\mathcal{M})$ and its type 1 truncation $T_1(\mathcal{M})$ will also be uniform maps of type $(4,4)$.

Lemma 4.20. *Every genus 1 uniform map of type $(4,4)$ is self-dual. If $\mathcal{M} = \{\frac{c+di}{a+bi}\}_{p+qi}$ is a uniform map, then its type 1 truncation $T_1(\mathcal{M})$ corresponds to the map $\{\frac{c+di}{a+bi}\}_{(1+i)(p+qi)}$.*

Proof. We let $\Gamma_1 = \Gamma(4,2,4) = < x_0, x_1 \mid x_0^4 = x_1^2 = (x_0x_1)^{-4} = 1 >$ where $x_0 : z \mapsto iz$, $x_1 : z \mapsto -z+1$. So Γ_1 consists of all the elements of the form $z \mapsto az+b$ where $a = \pm 1, \pm i$ and $b \in \mathbf{Z}[i]$. From Lemma 2.35 we see that Γ_1 contains an index 2 subgroup $\Gamma_2 \cong \Gamma(4,2,4)$ with $\Gamma_2 = < y_0, y_1 \mid y_0^4 = y_1^2 = (y_0y_1)^{-4} = 1 >$ where $y_0 = x_1x_0^{-1} : z \mapsto iz+1$, $y_1 = x_0^2 : z \mapsto -z$. Then Γ_2 consists of all elements of the form $z \mapsto \pm z + d$ and $z \mapsto \pm iz + d + 1$ where $d \in (1+i)\mathbf{Z}[i]$. Hence torsion-free subgroups of Γ_2 correspond to sublattices of $\Lambda(R, Ri)$ where $R = 1+i$. If $\mathcal{M}$ is the map $\{\frac{c+di}{a+bi}\}_{p+qi}$, then $\mathcal{M}$ has a map subgroup $M \leq \Gamma_2$ corresponding to the sublattice generated by $z \mapsto z + (p+qi)(aR+bRi)$ and $z \mapsto z+(p+qi)(cR+dRi)$. By Lemma 2.35 the type 1 truncation of $\mathcal{M}$ corresponds to the inclusion $M \leq \Gamma_1$, where we think of M as being the lattice generated by $z \mapsto z+(p+qi)(1+i)(a+bi)$ and $z \mapsto z+(p+qi)(1+i)(c+di)$. Hence $T_1(\mathcal{M})$ is the map $\{\frac{c+di}{a+bi}\}_{(1+i)(p+qi)}$.

Let $\mathcal{M}$ be a map with the map subgroup $M \leq \Gamma_2 \cong \Gamma(4,2,4)$. Then by Lemma 2.36 the dual of $\mathcal{M}$ corresponds to the map subgroup M^{x_1}. If $\mathcal{M}$ is uniform, then from the above $M \leq \Lambda(R, Ri)$ and so will contain elements of the form $\phi : z \mapsto z + eR + fRi$ for $e, f \in \mathbf{Z}$. We have $x_1\phi x_1^{-1} : z \mapsto z - eR - fRi$ and since any lattice contains its inverse elements, $M = M^{x_1}$. Therefore the map subgroups are equal and so $\mathcal{M}$ is isomorphic to its dual. □

Consider the uniform map $\{2i\}_1$ shown in Figure 4.8(a) with the corresponding lattice $\Lambda(1, 2i)$. By Lemma 4.20, $\{2i\}_1$ is self-dual and has the type 1 truncation $\{2i\}_{1+i}$ shown in Figure 4.8(b) corresponding to the lattice $(1+i)\Lambda(1, 2i)$

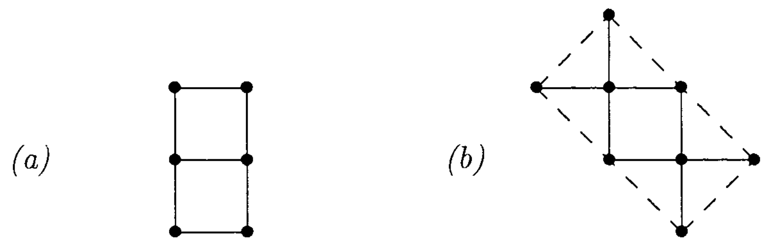


Figure 4.8

Uniform maps of type (6, 3)

Genus 1 uniform maps of type (6, 3) correspond to torsion-free subgroups of $\Gamma(6,2,3)$, and it was shown in §3.2 that these correspond precisely to sublattices of $\Lambda(1,\rho)$, where $\rho = \frac{-1+\sqrt{3}i}{2}$. Our classification follows analogously to the classification of uniform maps of type (4, 4) given above; the proofs required are almost identical, and so we only outline the details.

The ring $\mathbf{Z}[\rho]$ is a principal ideal domain with the group of units given by $\mathbf{Z}[\rho]^\star = \{\pm 1, \pm\rho, \pm\rho^2\}$ (see **[Alle]**). We now determine when two sublattices of $\Lambda(1,\rho)$ correspond to conjugate torsion-free subgroups of $\Gamma(6,2,3)$.

Lemma 4.21. *Let Λ_1 and Λ_2 be sublattices of $\Lambda(1,\rho)$. Then Λ_1 and Λ_2 correspond to conjugate torsion-free subgroups of $\Gamma(6,2,3)$ if and only if $\Lambda_1 = u\Lambda_2$ where u is a unit in $\mathbf{Z}[\rho]$. The sublattice $\Lambda \leq \Lambda(1,\rho)$ corresponds to a normal torsion-free subgroup of $\Gamma(6,2,3)$ if and only if $\Lambda = \rho\Lambda$.*

Proof. The proof follows similarly to that of Lemma 4.10: Using the representation of $\Gamma(6,2,3)$ given in Lemma 3.23 it can be shown that there are at most two other lattices conjugate to $\Lambda_1 \leq \Lambda(1,\rho)$ in $\Gamma(6,2,3)$, namely $\rho\Lambda_1$ and $\rho^2\Lambda_1$. Hence Λ_1 corresponds to a normal subgroup of $\Gamma(6,2,3)$ if and only if $\Lambda_1 = \rho\Lambda_1 = \rho^2\Lambda_1$. Otherwise, it can be shown that Λ_1, $\rho\Lambda_1$ and $\rho^2\Lambda_1$ are all distinct lattices. □

Every conjugacy class of torsion-free subgroups in $\Gamma(6,2,3)$ contains either one element (for normal subgroups) or exactly three elements (for non-normal subgroups). A lattice corresponding to a normal subgroup of $\Gamma(6,2,3)$ will be called an *ideal lattice* of $\Lambda(1,\rho)$ (since it corresponds to an ideal in $\mathbf{Z}[\rho]$). Every ideal lattice of $\Lambda(1,\rho)$ has the form $\Lambda(p+q\rho, p\rho+q\rho^2)$ for some $p+q\rho \in \mathbf{Z}[\rho]$ and corresponds to the regular map $\{3,6\}_{p,q}$ in the notation of Coxeter and Moser (see §4.1).

Corollary 4.22. *Let $\Lambda(\omega_1, \omega_2) \leq \Lambda(1, \rho)$ be a lattice with non-zero elements $p + q\rho, r + s\rho \in \mathbf{Z}[\rho]$. Then the lattices $(p + q\rho)\Lambda(\omega_1, \omega_2)$ and $(r + s\rho)\Lambda(\omega_1, \omega_2)$ are conjugate in $\Gamma(6, 2, 3)$ if and only if $p + q\rho$ and $r + s\rho$ are associates in $\mathbf{Z}[\rho]$.*

Proof. As for Corollary 4.11. □

The lattice $\Lambda(a + b\rho, c + d\rho)$ has a fundamental parallelogram of area

$$n = |ad - bc|$$

giving a subgroup of index n in $\Lambda(1, \rho)$. The map associated to $\Lambda(a + b\rho, c + d\rho)$ then has $2n$ faces, since two faces form a parallelogram having unit area; $\frac{3(2n)}{2} = 3n$ edges, since each face is bounded by 3 edges and 2 faces meet each edge; and $\frac{2(3n)}{6} = n$ vertices, since each edge is incident with 2 vertices and 6 edges meet at each vertex. If Λ is an index n sublattice of $\Lambda(1, \rho)$, then we will say that the uniform map of type $(6, 3)$ associated to Λ has index n. We have therefore proved:

Lemma 4.23. *Let $\mathcal{M}$ be an index n uniform map of type $(6, 3)$. Then $\mathcal{M}$ has n vertices, $3n$ edges and $2n$ faces.* □

If we now take $0 \neq p + q\rho \in \mathbf{Z}[\rho]$, then the lattice $(p + q\rho)\Lambda(a + b\rho, c + d\rho)$ will have index $(p^2 - pq + q^2)n$ in $\Lambda(1, \rho)$. Therefore, for every modulus $\tau \in \mathbf{Q}(\rho)$ there exists a *minimal map* $\mathcal{M}_\tau$ corresponding to the lattice $\Lambda(a + b\rho, c + d\rho)$ where $\tau = \frac{c+d\rho}{a+b\rho}$ and $a + b\rho, c + d\rho$ are coprime in $\mathbf{Z}[\rho]$. The minimal map $\mathcal{M}_\tau$ is the smallest uniform map of type $(6, 3)$ lying on the Riemann surface of modulus τ, and is unique since although we can multiply $a + b\rho$ and $c + d\rho$ by any unit $u \in \mathbf{Z}[\rho]$, the resulting lattices $u\Lambda(a + b\rho, c + d\rho)$ are conjugate in $\Gamma(6, 2, 3)$ by Lemma 4.21 and so the associated maps are isomorphic. We classify the genus 1 uniform maps of type $(6, 3)$ as follows:

Notation 4.24. *Let $\mathcal{M}$ be a genus 1 uniform map of type $(6, 3)$ associated to a sublattice Λ of $\Lambda(1, \rho)$. By choosing a suitable basis we can write $\Lambda = (p + q\rho)\Lambda(a + b\rho, c + d\rho)$ where $a + b\rho$ and $c + d\rho$ are coprime in the ring $\mathbf{Z}[\rho]$ and the modulus $\tau = \frac{c+d\rho}{a+b\rho}$ lies in the modular fundamental region. Then $\mathcal{M}$ will be the map $\{\tau\}_{p+q\rho}$. Given any modulus $\tau \in \mathbf{Q}(\rho)$ lying in the modular fundamental region, $\{\tau\}_{p+q\rho}$ represents the uniform map corresponding to the lattice $(p + q\rho)\Lambda(a + b\rho, c + d\rho)$ where $\tau = \frac{c+d\rho}{a+b\rho}$ and $a + b\rho, c + d\rho$ are coprime in $\mathbf{Z}[\rho]$.* □

Lemma 4.25. *Let $\tau, \tau' \in \mathbf{Q}(\rho)$ be moduli lying in the modular fundamental region.*

(i) Every uniform map of type $(6,3)$ lying on the Riemann surface of modulus τ has the form $\{\tau\}_{p+q\rho}$ for some $p+q\rho \in \mathbf{Z}[\rho]$;

(ii) The uniform maps $\{\tau\}_{p+q\rho}$ and $\{\tau'\}_{p'+q'\rho}$ are isomorphic if and only if $\tau = \tau'$ and $p+q\rho, p'+q'\rho$ are associates in the ring $\mathbf{Z}[\rho]$;

(iii) The minimal map $\mathcal{M}_\tau$ corresponds to the uniform map $\{\tau\}_1$;

(iv) The uniform map $\{\tau\}_{p+q\rho}$ is regular if and only if $\tau = \rho$. Hence $\{\rho\}_{p+q\rho}$ corresponds to Coxeter and Moser's $\{3,6\}_{p,q}$;

(v) If $\tau = \frac{c+d\rho}{a+b\rho}$ where $a+b\rho, c+d\rho$ are coprime in $\mathbf{Z}[\rho]$ and $n = |ad-bc|$, then $\{\tau\}_{p+q\rho}$ is a uniform map with $(p^2+q^2-pq)n$ vertices, $3(p^2+q^2-pq)n$ edges and $2(p^2+q^2-pq)n$ faces. □

Proof. (i) If a uniform map $\mathcal{M}$ lies on the Riemann surface of modulus τ, then it has an associated lattice $\Lambda \leq \Lambda(1,\rho)$ with a basis $\{\omega_1, \omega_2\}$ where $\omega_1, \omega_2 \in \mathbf{Z}[\rho]$, $\Lambda = \Lambda(\omega_1, \omega_2)$ and $\tau = \frac{\omega_2}{\omega_1} \in \mathcal{F}$. Writing $\omega_1 = (p+q\rho)(a+b\rho)$, $\omega_2 = (p+q\rho)(c+d\rho)$ where $a+b\rho$ and $c+d\rho$ are coprime in $\mathbf{Z}[\rho]$, $\mathcal{M}$ has the form $\{\tau\}_{p+q\rho}$.

(ii) Every genus 1 uniform map lies on a unique Riemann surface and so if $\{\tau\}_{p+q\rho}$ and $\{\tau'\}_{p'+q'\rho}$ are isomorphic we must have $\tau = \tau'$. The lattices corresponding to $\{\tau\}_{p+q\rho}$ and $\{\tau\}_{p'+q'\rho}$ can be written as $(p+q\rho)\Lambda(a+b\rho, c+d\rho), (p'+q'\rho)\Lambda(a+b\rho, c+d\rho)$ where $\tau = \frac{c+d\rho}{a+b\rho}$ and $a+b\rho, c+d\rho$ are coprime in $\mathbf{Z}[\rho]$, and so by Corollary 4.22 and Theorem 2.5 the maps are isomorphic if and only if $p+q\rho, p'+q'\rho$ are associates in $\mathbf{Z}[\rho]$. The converse follows easily from Corollary 4.22.

(iii) This follows from the definition of a minimal map given above.

(iv) Suppose that $\tau = \rho$. Then the lattice associated to $\{\rho\}_{p+q\rho}$ has the form $(p+q\rho)\Lambda(1,\rho)$. Using $\rho^2 = -1-\rho$, we find that $\rho(p+q\rho)\Lambda(1,\rho) = (p+q\rho)\Lambda(\rho,\rho^2) = (p+q\rho)\Lambda(1,\rho)$ and so the map is regular by Lemma 4.21 and Theorem 2.20. Conversely if $\{\tau\}_{p+q\rho}$ is a regular map, then as observed the corresponding lattice Λ has the form $\Lambda = \Lambda(p+q\rho, p\rho+q\rho^2) = (p+q\rho)\Lambda(1,\rho)$ for some $p+q\rho \in \mathbf{Z}[\rho]$. Therefore $\tau = \rho$.

(v) The map $\{\tau\}_{p+q\rho}$ has an associated lattice $(p+q\rho)\Lambda(a+b\rho, c+d\rho)$, and hence has index $(p^2-pq+q^2)n$. We now apply Lemma 4.23. □

We consider the uniform map in Figure 4.4(a) corresponding to the lattice $\Lambda(3+\rho, 2+4\rho)$. As $3+\rho$ and $2+4\rho$ are coprime in $\mathbf{Z}[\rho]$ and $\frac{2+4\rho}{3+\rho} = \frac{3+5\sqrt{3}i}{7} \in \mathcal{F}$, the map is denoted $\left\{\frac{2+4\rho}{3+\rho}\right\}_1$.

Stellations of uniform maps

If $\mathcal{M}$ is a uniform map of type $(6,3)$, then its type 2 truncation $T_2(\mathcal{M})$ will be a uniform map of type $(3,6)$. By taking the dual $DT_2(\mathcal{M})$ we obtain a map of type $(6,3)$ which, by the discussion following Definition 2.39, is isomorphic to the type 2 stellation $S_2(\mathcal{M})$.

Lemma 4.26. *Let $\left\{\frac{c+d\rho}{a+b\rho}\right\}_{p+q\rho}$ be a uniform map. Then its type 2 stellation $S_2(\mathcal{M})$ is the uniform map $\left\{\frac{c+d\rho}{a+b\rho}\right\}_{(2+\rho)(p+q\rho)}$.*

Proof. Letting $\Gamma_1 = \Gamma(3,2,6) = < x_0, x_1 \,|\, x_0^3 = x_1^2 = (x_0x_1)^{-6} = 1 >$ where $x_0 : z \mapsto \rho z - \rho$, $x_1 : z \mapsto -z+1$ we see that Γ_1 consists of all elements of the form $z \mapsto az+b$ where $a = \pm 1, \pm\rho, \pm\rho^2$ and $b \in \mathbf{Z}[\rho]$. By Lemma 2.41, Γ_1 contains an index 3 subgroup $\Gamma_2 = \Gamma(6,2,3)$ with $\Gamma_2 = < y_0, y_1 \,|\, y_0^6 = y_1^2 = (y_0y_1)^{-3} = 1 >$ and $y_0 = x_0x_1x_0 : z \mapsto -\rho^2 z - 1 - \rho$, $y_1 = x_1 : z \mapsto -z+1$. Hence Γ_2 consists of all elements of the form $z \mapsto cz+d$ and $z \mapsto -cz+d+1$ where $c = 1, \rho, \rho^2$ and $d \in (2+\rho)\mathbf{Z}[\rho]$. Torsion-free subgroups of Γ_2 then correspond to sublattices of $\Lambda(R, R\rho)$ where $R = 2+\rho$. If $\mathcal{M}$ is the map $\left\{\frac{c+d\rho}{a+b\rho}\right\}_{p+q\rho}$, then $\mathcal{M}$ has a map subgroup $M \leq \Gamma_2$ corresponding to the sublattice generated by $z \mapsto z + (p+q\rho)(aR+bR\rho)$ and $z \mapsto z + (p+q\rho)(cR+dR\rho)$. By Lemma 2.41 the type 2 truncation of $\mathcal{M}$ corresponds to the inclusion $M \leq \Gamma_1$; note that $T_2(\mathcal{M})$ has type $(3,6)$. Now $S_2(\mathcal{M})$ is the dual map of $T_2(\mathcal{M})$ and so $S_2(\mathcal{M})$ also corresponds to the inclusion $M \leq \Gamma_1$ by Theorem 4.8. Hence we think of M as being the lattice generated by $z \mapsto z + (p+q\rho)(2+\rho)(a+b\rho)$ and $z \mapsto z + (p+q\rho)(2+\rho)(c+d\rho)$, so that $S_2(\mathcal{M})$ is the map $\left\{\frac{c+d\rho}{a+b\rho}\right\}_{(2+\rho)(p+q\rho)}$. □

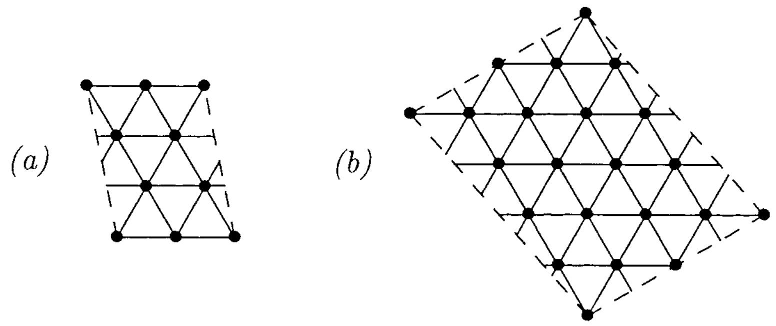


Figure 4.9

The uniform map $\{\frac{1+3\rho}{2}\}_1$ corresponding to the lattice $\Lambda(2, 1+3\rho)$ is shown in Figure 4.9(a). By Lemma 4.26 its type 2 stellation is the map $\{\frac{1+3\rho}{2}\}_{2+\rho}$ shown in Figure 4.9(b).

4.3. Reflexible uniform maps

A Riemann surface that admits an anti-conformal involution is said to be *symmetric*. It is known ([**Alli**]) that a genus 1 Riemann surface is symmetric if and only if its modulus τ lies on the boundary of the modular figure $\mathcal{F}$ or on the imaginary axis with $\text{Im}(\tau) \geq 1$ (or any point in the same orbit under the modular group). These moduli correspond precisely to the real values of $j(\tau)$.

A map is said to be *reflexible* if it admits an orientation-reversing involution and *chiral* otherwise. A *chiral pair* consists of two chiral maps, one being the reflected image of the other. The regular reflexible maps of genus 1 are classified in [**CMo**]: the regular map $\{i\}_{p+qi}$ is reflexible if and only if $pq(p-q) = 0$, while $\{\rho\}_{p+q\rho}$ is reflexible if and only if $pq(p-2q) = 0$ (this condition is equivalent to that given in [**CMo**] where the triangular lattice is generated by 1 and $-\rho^2$). We will extend these results to classify the reflexible uniform maps of genus 1. We recall that the extended triangle group $\Gamma^*(l_0, l_1, l_2)$ is the group generated by reflections in the sides of a triangle with angles $\frac{\pi}{l_i}$ for $i = 0, 1, 2$. The triangle group $\Gamma(l_0, l_1, l_2)$ is the index 2 subgroup of $\Gamma^*(l_0, l_1, l_2)$ consisting of all the conformal transformations.

If Λ is a lattice with the basis $\{\omega_1, \omega_2\}$, then $\overline{\Lambda}$ is the lattice generated by the complex conjugate basis $\{\overline{\omega}_1, \overline{\omega}_2\}$.

Theorem 4.27. *The lattice $\Lambda \leq \Lambda(1, i)$ is normalized by some anti-conformal element $h \in \Gamma^*(4,2,4) \setminus \Gamma(4,2,4)$ if and only if $\Lambda = u\overline{\Lambda}$ for some unit $u \in \mathbf{Z}[i]$.*

Proof. As in Lemma 3.23 we represent $\Gamma(4,2,4)$ as the set of transformations $\Psi = \{z \mapsto az + b \,|\, a = \pm 1, \pm i,\ b \in \mathbf{Z}[i]\}$. The extended triangle group $\Gamma^*(4,2,4)$ can be represented by the group of transformations generated by the elements of Ψ and the anti-conformal involution $z \mapsto \overline{z}$. Every lattice $\Lambda \leq \Lambda(1, i)$ consists of elements of the form $\phi : z \mapsto z + f$ for $f \in \mathbf{Z}[i]$. If Λ is normalized by some anti-conformal element $h : z \mapsto c\overline{z} + d$ of $\Gamma^*(4,2,4)$ then since

$$h\phi h^{-1} : z \mapsto z + c\overline{f}$$

we must have $\Lambda = u\overline{\Lambda}$ for some unit $u \in \mathbf{Z}[i]$. Conversely, if $\Lambda = u\overline{\Lambda}$ for a unit $u \in \mathbf{Z}[i]$, then Λ is normalized by the anti-conformal element $z \mapsto u\overline{z}$. □

Let $\Lambda_1 = \Lambda(1, 2i)$ and $\Lambda_2 = (1+i)\Lambda(1, 2i)$. Then $\Lambda_1 = \overline{\Lambda}_1$ and so Λ_1 is normalized by the element $z \mapsto \overline{z}$. We also have $\overline{\Lambda}_2 = (1-i)\Lambda(1, -2i) = (1-i)\Lambda(1, 2i) = i\Lambda_2$ and so Λ_2 is normalized by $z \mapsto i\,\overline{z}$. Before proving the main classification, we need the following simple lemma:

Lemma 4.28. *Let $\alpha \in \mathbf{Z}[i]$. Then α and $\overline{\alpha}$ are associates in $\mathbf{Z}[i]$ if and only if $\alpha = n, n(1+i)$ or any associate of these in $\mathbf{Z}[i]$ for some integer n.*

Proof. There are 4 cases to consider: $\alpha = \pm\overline{\alpha}$ and $\alpha = \pm i\,\overline{\alpha}$. By setting $\alpha = x + yi$ and equating real and imaginary parts, we deduce that $\alpha = n, in, n(1+i)$ or $n(i-1)$ for any integer n. Conversely, if $\alpha = n$ or $\alpha = n(1+i)$ (or some associate) then it is easy to check that α and $\overline{\alpha}$ are associates in $\mathbf{Z}[i]$. □

Theorem 4.29. *The uniform map $\{\frac{c+di}{a+bi}\}_{p+qi}$ is reflexible if and only if the following two conditions hold:*

(i) $\tau = \frac{c+di}{a+bi}$ lies on the boundary of the modular figure $\mathcal{F}$ or on the imaginary axis with $\mathrm{Im}(\tau) \geq 1$;

(ii) $p + qi = n, n(1+i)$ or any associate of these in $\mathbf{Z}[i]$ for some non-zero integer n.

Proof. We show first that the conditions are necessary. (i) As noted in §2.1, a map automorphism extends naturally to an automorphism of its underlying Riemann surface. Therefore a reflexible map must lie on a symmetric Riemann surface. By the discussion concerning symmetric Riemann surfaces of genus 1 at the beginning of this section, a reflexible map must have a modulus lying on the boundary of $\mathcal{F}$ or on the imaginary axis with $Im(\tau) \geq 1$. (ii) If $\{\frac{c+di}{a+bi}\}_{p+qi}$ is reflexible then the map must admit an anti-conformal automorphism, and so its associated lattice (map subgroup) must be normalized by some anti-conformal element in $\Gamma^{\star}(4,2,4)$. Hence by Theorem 4.27 its associated lattice must satisfy $(p+qi)\Lambda(a+bi, c+di) = u(p-qi)\Lambda(a-bi, c-di)$ for some unit $u \in \mathbf{Z}[i]$. Since $a+bi$ and $c+di$ are coprime (as are their conjugates), we deduce as in the proof of Corollary 4.11 that $p+qi$ and $p-qi$ are associates in $\mathbf{Z}[i]$. Therefore by Lemma 4.28 $p+qi = n, n(1+i)$ or some associate for a non-zero integer n.

The sufficiency of the conditions can be checked explicitly. If the minimal map $\{\frac{c+di}{a+bi}\}_1$ is reflexible, then it is easy to show that $\{\frac{c+di}{a+bi}\}_p$ and $\{\frac{c+di}{a+bi}\}_{p+pi}$ are reflexible for all $p \in \mathbf{Z}\backslash\{0\}$ by considering their underlying lattices and applying Theorem 4.27. We must therefore show that for all τ lying on the boundary of the modular fundamental region or on the imaginary axis, the minimal map $\mathcal{M}_\tau$ is reflexible. There are three cases to consider:

(a) Suppose τ lies on the imaginary axis. Then there exist coprime integers a, d with $\tau = \frac{di}{a}$ (note that a and di are also coprime Gaussian integers). The lattice associated to $\mathcal{M}_\tau$ is $\Lambda = \Lambda(a, di)$ which satisfies $\Lambda = \overline{\Lambda}$. Hence by Theorem 4.27 Λ is normalized by some anti-conformal element, and so $\mathcal{M}_\tau$ is reflexible.

(b) Suppose $Re(\tau) = \frac{1}{2}$. Then $\tau = \frac{c+di}{2c}$ where c, d are coprime integers. To find the minimal map associated with τ, we must express $\frac{c+di}{2c}$ in lowest terms. It can be shown that either $c + di$ and $2c$ are coprime Gaussian integers, or $c + di$ and $2c$ have exactly one non-unit divisor $1 + i$ (up to multiplication by a unit). In the former case the lattice associated to $\mathcal{M}_\tau$ is $\Lambda = \Lambda(2c, c + di)$ which satisfies $\Lambda = \overline{\Lambda}$. In the latter case, $2c = (1 + i)(1 - i)c$ and $c + di = (f + gi)(1 + i)$ for some Gaussian integer $f + gi$. Equating real parts gives $c = f - g$ so that the lattice associated to $\mathcal{M}_\tau$ is $\Lambda = \Lambda((1 - i)(f - g), f + gi)$. Then

$$\begin{aligned}\overline{\Lambda} &= \Lambda((1+i)(f-g), f-gi) \\ &= \Lambda(-(1+i)(f-g), -g+fi) \qquad \text{(change of basis)} \\ &= i\Lambda(-(1-i)(f-g), f+gi) \\ &= i\Lambda\end{aligned}$$

and $\mathcal{M}_\tau$ is reflexible.

(c) If τ lies on the unit circle, then there exist integers a, c, d with $\gcd(a, c, d) = 1$, $a^2 = c^2 + d^2$ and $\tau = \frac{c+di}{a}$. Since $a^2 = c^2 + d^2 = (c + di)(c - di)$, it can be shown that there exist coprime integers f and g such that $a = (f + gi)(f - gi)$ and $c + di = (f + gi)^2$. Since f and g are coprime integers, $f + gi$ and $f - gi$ must be coprime Gaussian integers, and so in reduced form we have $\frac{c+di}{a} = \frac{f+gi}{f-gi}$. The lattice associated to $\mathcal{M}_\tau$ clearly satisfies $\Lambda = \overline{\Lambda}$, and so $\mathcal{M}_\tau$ is reflexible. □

A similar characterization of reflexible genus 1 uniform maps of type $(6, 3)$ may be obtained by considering lattices of the extended triangle group $\Gamma^\star(6, 2, 3)$.

Theorem 4.30. *The lattice $\Lambda \leq \Lambda(1, \rho)$ is normalized by some anti-conformal element $h \in \Gamma^\star(6, 2, 3) \setminus \Gamma(6, 2, 3)$ if and only if $\Lambda = u\overline{\Lambda}$ for some unit $u \in \mathbf{Z}[\rho]$.*

Proof. As for Theorem 4.27 using the representation for $\Gamma(6, 2, 3)$ given in Lemma 3.23. □

As in Lemma 4.28 it is easy to show that $\alpha, \overline{\alpha} \in \mathbf{Z}[\rho]$ are associates in $\mathbf{Z}[\rho]$ if and only if $\alpha = n, n(2+\rho)$ or an associate of these in $\mathbf{Z}[\rho]$ for some integer n. The proof of the following theorem is similar to that of Theorem 4.29.

Theorem 4.31. *The uniform map* $\{\frac{c+d\rho}{a+b\rho}\}_{p+q\rho}$ *is reflexible if and only if the following two conditions hold:*

(i) $\tau = \frac{c+d\rho}{a+b\rho}$ *lies on the boundary of the modular figure* $\mathcal{F}$ *or on the imaginary axis with* $Im(\tau) \geq 1$.

(ii) $p + q\rho = n, n(2+\rho)$ or any associate of these in $\mathbf{Z}[\rho]$ for some non-zero integer n.

Proof. The necessity of the conditions follows as for the proof of Theorem 4.29. Since $\overline{2+\rho} = 1 - \rho = -\rho(2+\rho)$, it is also clear that if the minimal map $\{\frac{c+d\rho}{a+b\rho}\}_1$ is reflexible, then so are the maps $\{\frac{c+d\rho}{a+b\rho}\}_n$ and $\{\frac{c+d\rho}{a+b\rho}\}_{n(2+\rho)}$. Thus it remains to show that every minimal map $\mathcal{M}_\tau$ whose modulus lies on the boundary of the modular fundamental region or on the imaginary axis is reflexible.

(a) If τ lies on the imaginary axis, then $\tau = \frac{a(1+2\rho)}{c}$ where a and c are coprime integers. If $1+2\rho$ and c are coprime Gaussian integers, then $\Lambda = \Lambda(a(1+2\rho), c)$ is the lattice associated to $\mathcal{M}_\tau$ and since $\overline{\Lambda} = \Lambda(a(1+2\rho^2), c) = \Lambda(-a(1+2\rho), c) = \Lambda$, $\mathcal{M}_\tau$ will be reflexible by Theorem 4.30. Otherwise $c = -c'(1+2\rho)^2$ and $\mathcal{M}_\tau$ has the associated lattice $\Lambda(a, -c'(1+2\rho))$. Hence $\mathcal{M}_\tau$ is also reflexible.

(b) If $\mathrm{Re}(\tau) = \frac{1}{2}$ then we can write $\tau = \frac{a(1+2\rho)-c\rho}{c}$ where a and c are coprime integers. Then either $a(1+2\rho) - c\rho$ and c are coprime in $\mathbf{Z}[\rho]$ or else $c = -c'(1+2\rho)^2$. In both cases it can be shown that the lattice Λ associated to $\mathcal{M}_\tau$ satisfies $\overline{\Lambda} = \Lambda$, so that $\mathcal{M}_\tau$ is reflexible.

(c) If τ lies on the unit circle, then as in the proof of Theorem 4.29 it can be shown that the lattice associated to the minimal map $\mathcal{M}_\tau$ has the form $\Lambda = \Lambda(p+q\rho, p+q\rho^2)$. Since $\overline{\Lambda} = \Lambda$, $\mathcal{M}_\tau$ is reflexible. □

If $\mathcal{M}_1 = \{\frac{c+di}{a+bi}\}_{(p+qi)}$ and $\mathcal{M}_2 = \{\frac{c+d\rho}{a+b\rho}\}_{(p+q\rho)}$ are uniform maps, then for any non-zero integer $\alpha \in \mathbf{Z} - \{0\}$ the maps $\{\frac{c+di}{a+bi}\}_{\alpha(p+qi)}$ and $\{\frac{c+d\rho}{a+b\rho}\}_{\alpha(p+q\rho)}$ are said to be *enlargements* of $\mathcal{M}_1$ and $\mathcal{M}_2$ respectively. It follows from Theorems 4.29 and 4.31 that an enlargement of a reflexible uniform map is also reflexible. If $\{\frac{c+di}{a+bi}\}_1$ is reflexible, then its type 1 truncation $\{\frac{c+di}{a+bi}\}_{1+i}$ is reflexible by Theorem 4.29. Similarly, if $\{\frac{c+d\rho}{a+b\rho}\}_1$ is reflexible then its type 2 stellation $\{\frac{c+d\rho}{a+b\rho}\}_{2+\rho}$ is reflexible by Theorem 4.31.

By Theorems 4.29 and 4.31 all reflexible uniform maps of genus 1 occur as truncations and enlargements of reflexible minimal maps of type $(4,4)$, or stellations and enlargements of reflexible minimal maps of type $(6,3)$. We have therefore proved the following alternative characterization of reflexible genus 1 uniform maps:

Theorem 4.32. *Consider a modulus $\tau \in \mathbf{Q}(i)$ or $\mathbf{Q}(\rho)$. Then the minimal map $\mathcal{M}_\tau$ is reflexible if and only if τ lies on the boundary of the modular fundamental region or on the imaginary axis. Every reflexible genus 1 uniform map of type $(4,4)$ is an enlargement or type 1 truncation of a reflexible minimal map of type $(4,4)$. Every reflexible genus 1 uniform map of type $(6,3)$ is an enlargement or type 2 stellation of a reflexible minimal map of type $(6,3)$.* □

Figure 4.8 shows two maps, the minimal map $\mathcal{M}_{2i}$ and its type 1 truncation $\{2i\}_{1+i}$. The minimal map $\mathcal{M}_{2i}$ has the associated modulus $\tau = 2i$ and so is reflexible by Theorem 4.29; the type 1 truncation of $\mathcal{M}_{2i}$ is also reflexible by Theorem 4.32. Consider two maps of type $(6,3)$: the minimal map $\{1+3\rho\}_1$ and its type 2 stellation $\{1+3\rho\}_{2+\rho}$. Since $\mathrm{Re}(1+3\rho) = -\frac{1}{2}$, $\{1+3\rho\}_1$ is a reflexible minimal map, and hence its type 2 stellation is also reflexible.

4.4. Regular covers of uniform maps

It was shown in §2.1 that every finite map $\mathcal{M}$ of type (m,n) admits a finite cover by a regular map of type (m,n). If $M \leq \Gamma(m,2,n)$ is a canonical map subgroup for $\mathcal{M}$, then the map $\mathcal{M}^\star$ corresponding to the inclusion $M^\star \leq \Gamma(m,2,n)$ is a regular cover of $\mathcal{M}$, where $M^\star$ is the core of M in $\Gamma(m,2,n)$. Furthermore, $\mathcal{M}^\star$ is the minimal regular cover of $\mathcal{M}$ in the sense that every regular map that covers $\mathcal{M}$ also covers $\mathcal{M}^\star$.

A genus 1 uniform map of type $(4,4)$ has a map subgroup corresponding to a lattice Λ, where $\Lambda \leq \Lambda(1,i) \leq \Gamma(4,2,4)$. A lattice is normal in $\Gamma(m,2,n)$ if and only if it is an ideal lattice, that is $\Lambda = (p+qi)\Lambda(1,i)$ for $0 \neq p+qi \in \mathbf{Z}[i]$ (see the discussion following Corollary 4.13). If $\mathcal{M}$ is a genus 1 uniform map with the map subgroup $\Lambda \leq \Lambda(1,i)$, the regular covers of $\mathcal{M}$ correspond to the ideal lattices contained in Λ; in particular, the minimal regular cover will correspond to the maximal ideal lattice contained in Λ.

Example 4.33. We consider the lattice $\Lambda(a+bi, c+di)$ where $a+bi$ and $c+di$ are coprime Gaussian integers, and let $n = ad - bc$. Since

$$d(a+bi) - b(c+di) = ad - bc = n$$
$$\text{and} \quad -c(a+bi) + a(c+di) = (ad-bc)i = ni$$

$\Lambda(a+bi, c+di)$ contains the ideal lattice $\Lambda(n, ni)$. □

We will now prove that $\Lambda(n, ni)$ is the maximal ideal lattice contained in $\Lambda(a+bi, c+di)$.

Lemma 4.34. *Let $a+bi$, $c+di$ be coprime Gaussian integers with $ad-bc=n$. Then $\Lambda(n, ni)$ is the maximal ideal lattice contained in $\Lambda(a+bi, c+di)$.*

Proof. By Example 4.33, $\Lambda(a+bi, c+di)$ contains the ideal lattice $\Lambda(n, ni)$. If $\Lambda(p+qi, pi-q)$ is any ideal lattice contained in $\Lambda(a+bi, c+di)$, we will show that $\Lambda(p+qi, pi-q) \subseteq \Lambda(n, ni)$. Since $\Lambda(p+qi, pi-q) \subseteq \Lambda(a+bi, c+di)$, there exist integers $\alpha, \beta, \gamma, \delta$ such that

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \begin{pmatrix} a+bi \\ c+di \end{pmatrix} = \begin{pmatrix} p+qi \\ pi-q \end{pmatrix} \qquad 4.35$$

where we let $\Delta = \alpha\delta - \beta\gamma$. By calculating the index of the lattice $(p+qi)\Lambda(1,i)$ in $\Lambda(1,i)$, we deduce from 4.35 that

$$p^2 + q^2 = (ad-bc)(\alpha\delta - \beta\gamma) = n\Delta \qquad 4.36$$

from which it is easy to see that $\Delta = \alpha\delta - \beta\gamma \neq 0$ (since $p^2+q^2=0$ implies that $p = 0 = q$). Hence by taking inverses

$$\begin{pmatrix} a+bi \\ c+di \end{pmatrix} = \frac{1}{\Delta} \begin{pmatrix} \delta & -\beta \\ -\gamma & \alpha \end{pmatrix} \begin{pmatrix} p+qi \\ pi-q \end{pmatrix}$$

and so

$$\begin{aligned} \Delta(a+bi) &= (p+qi)(\delta - \beta i) \\ \Delta(c+di) &= (p+qi)(-\gamma + \alpha i). \end{aligned} \qquad 4.37$$

Since $a+bi$ and $c+di$ are coprime, $(p+qi)|\Delta$ and therefore

$$\Delta = (u+vi)(p+qi) \qquad 4.38$$

for some $u+vi \in \mathbf{Z}[i]$. Substituting 4.38 into 4.36 gives

$$p^2+q^2 = (p+qi)(p-qi) = n(p+qi)(u+vi)$$

and hence that

$$p - qi = n(u+vi), \quad p+qi = n(u-vi).$$

Therefore $\Lambda(p+qi, pi-q) = (u-vi)\Lambda(n, ni) \subseteq \Lambda(n, ni)$. □

Corollary 4.39. *Let $L = (r+si)\Lambda(a+bi, c+di)$ be a lattice where $a+bi, c+di$ are coprime Gaussian integers and $ad-bc = n$. Then $L' = (r+si)\Lambda(n, ni)$ is the maximal ideal lattice contained in L.*

Proof. Let $Q = (e+fi)\Lambda(1,i)$ be any ideal lattice in L; we will show that $Q \subseteq L'$. Since $Q \subseteq (r+si)\Lambda(a+bi, c+di)$, we can write $Q = (r+si)\Lambda(u+vi, ui-v)$ where $(r+si)(u+vi) = e+fi$ and $u+vi, ui-v \in \Lambda(a+bi, c+di)$. Now, $\Lambda(u+vi, ui-v)$ is an ideal lattice in $\Lambda(a+bi, c+di)$ and so $\Lambda(u+vi, ui-v) \subseteq \Lambda(n, ni)$ by Lemma 4.34. Hence $Q = (r+si)\Lambda(u+vi, ui-v) \subseteq (r+si)\Lambda(n, ni) = L'$. □

Similar results may be obtained for maximal ideal lattices contained in sublattices of $\Lambda(1,\rho)$; if $\mathbf{Z}[i]$ is replaced by $\mathbf{Z}[\rho]$ in Lemma 4.34 and Corollary 4.39, the proofs required are almost identical.

Lemma 4.40. *Let $L = (r+s\rho)\Lambda(a+b\rho, c+d\rho)$ be a lattice where $a+b\rho, c+d\rho$ are coprime in the ring $\mathbf{Z}[\rho]$ and $ad-bc = n$. Then $L' = (r+s\rho)\Lambda(n, n\rho)$ is the maximal ideal lattice contained in L.* □

Our results on maximal ideal lattices can be interpreted in terms of minimal regular covers of uniform maps.

Theorem 4.41. *Let $\mathcal{M}_1 = \left\{\frac{c+di}{a+bi}\right\}_{r+si}$ and $\mathcal{M}_2 = \left\{\frac{c+d\rho}{a+b\rho}\right\}_{r+s\rho}$ be uniform maps of genus 1 with $|ad-bc| = n$. Then $\{i\}_{n(r+si)}$ is the minimal regular cover of $\mathcal{M}_1$, and $\{\rho\}_{n(r+s\rho)}$ is the minimal regular cover of $\mathcal{M}_2$. In both cases the index of the minimal regular cover is equal to n.*

Proof. The uniform map $\mathcal{M}_1$ has a corresponding map subgroup $(r+si)\Lambda(a+bi, c+di) \leq \Lambda(1,i) \leq \Gamma(4,2,4)$ with $|ad-bc| = n$. Let $\mathcal{M}_1^\star$ be the minimal regular cover of $\mathcal{M}_1$ corresponding to the maximal ideal lattice contained in $(r+si)\Lambda(a+bi, c+di)$, which is $(r+si)\Lambda(n, ni)$ by Lemma 4.39 (we may assume that n is positive, since $\Lambda(n, ni) = \Lambda(-n, -ni)$). Therefore $\mathcal{M}_1^\star = \{i\}_{n(r+si)}$. Since $(r+si)\Lambda(n, ni)$ has index $n^2(r^2+s^2)$ in $\Lambda(1,i)$, and $(r+si)\Lambda(a+bi, c+di)$ has index $(r^2+s^2)|(ad-bc)| = (r^2+s^2)n$ in $\Lambda(1,i)$, the index of the minimal regular cover is $\frac{n^2(r^2+s^2)}{n(r^2+s^2)} = n$. Similarly, $\mathcal{M}_2$ has a map subgroup $(r+s\rho)\Lambda(a+b\rho, c+d\rho) \leq \Lambda(1,\rho) \leq \Gamma(6,2,3)$ which contains the maximal ideal lattice $(r+s\rho)\Lambda(n, n\rho)$ by Lemma 4.40. Hence the minimal regular cover of $\mathcal{M}_2$ is the map $\{\rho\}_{n(r+s\rho)}$, and the index of the cover is $\frac{n^2(r^2+s^2-rs)}{n(r^2+s^2-rs)} = n$. □

4.5. Enumerating uniform maps

Altshuler [**Alt**] has given bounds for the number of genus 1 uniform maps of type $(6,3)$ with n vertices. By Lemmas 4.15 and 4.23, the number of vertices of a genus 1 uniform map is uniquely determined by its index, and so Altshuler's problem generalizes to finding the number of uniform maps with a given index. The special case of enumerating genus 1 regular maps of index n was solved by Cangül and Singerman [**CaSi**] using the functions

$$\begin{aligned} R(n) &= \frac{1}{4}\Big|\{(p,q)\,|\,p^2+q^2=n \text{ and } p,q\in\mathbf{Z}\}\Big| \\ S(n) &= \frac{1}{6}\Big|\{(p,q)\,|\,p^2+q^2-pq=n \text{ and } p,q\in\mathbf{Z}\}\Big| \end{aligned} \qquad 4.42$$

where n is any positive integer. The multiplicative functions $R(n)$ and $S(n)$ may be evaluated as follows: $R(n)$ is equal to the number of positive divisors of n that are congruent to 1 mod 4 minus the number congruent to 3 mod 4; $S(n)$ is equal to the number of positive divisors of n that are congruent to 1 mod 3 minus the number congruent to 2 mod 3 (for proofs see [**Hu**, §12.4]).

Theorem 4.43. *The number of type* $(4,4)$ *genus 1 regular maps of index* n *is equal to* $R(n)$*, and the number of type* $(6,3)$ *genus 1 regular maps of index* n *is equal to* $S(n)$.

Proof. Every regular map of type $(4,4)$ satisfying the conditions of the theorem has the form $\{i\}_{p+qi}$ where $p^2+q^2=n$, and conversely every $p+qi$ for which $p^2+q^2=n$ determines such a map (see Lemma 4.19). If we identify the complex number $p+qi$ with the ordered pair (p,q), then the number of Gaussian integers $p+qi$ satisfying $p^2+q^2=n$ is equal to $4R(n)$. Note that every such $p+qi$ has four distinct associates in $\mathbf{Z}[i]$, $p+qi, -q+ip, -p-qi, q-ip$, which also satisfy $p^2+q^2=n$. By Lemma 4.19, two regular maps $\{i\}_{p+qi}$ and $\{i\}_{r+si}$ are isomorphic if and only if $p+qi$ and $r+si$ are associates in $\mathbf{Z}[i]$. Hence the number of non-isomorphic genus 1 regular maps of type $(4,4)$ and index n is equal to $R(n)$.

Similarly, every such regular map of type $(6,3)$ has the form $\{\rho\}_{p+q\rho}$ where $p+q\rho\in\mathbf{Z}[\rho]$ satisfies $p^2+q^2-pq=n$. The number of such elements is equal to $6S(n)$. The six associates of $p+q\rho$ correspond to isomorphic maps by Lemma 4.25, and every other $r+s\rho\in\mathbf{Z}[\rho]$ corresponds to a distinct regular map. Hence the number of genus 1 regular maps of type $(6,3)$ and index n is equal to $S(n)$. □

Before giving formulae for the number of genus 1 uniform maps of index n (Theorem 4.53), we discuss practical methods for constructing all uniform maps with a given index. The following two theorems relate the index of the minimal map $\mathcal{M}_\tau$ to the discriminant of its modulus τ. These results will be used in §4.6 to determine some Galois orbits of uniform maps.

Theorem 4.44. *Let a modulus $\tau \in \mathbf{Q}(i)$ have discriminant $d = -4n^2$ and the associated minimal map $\{\frac{c+di}{a+bi}\}_1$ where $\tau = \frac{c+di}{a+bi}$ and $a+bi, c+di$ are coprime in $\mathbf{Z}[i]$. Then $|ad-bc| = n$.*

Proof. It can be verified that τ satisfies the quadratic polynomial

$$(c^2+d^2)x^2 - 2(ac+bd)x + (a^2+b^2) = 0. \qquad 4.45$$

If the coefficients of 4.45 have greatest common divisor s (where s is a rational integer), then a direct calculation gives the discriminant of τ to be $\frac{-4}{s^2}(ad-bc)^2$. We now show that $s = 1$. If $s \neq 1$, then we can choose some Gaussian prime $p|s$ with $p|(c^2+d^2)$, $p|2(ac+bd)$ and $p|(a^2+b^2)$.

(1) Since $p|(a^2+b^2) = (a+bi)(a-bi)$, either $p|(a+bi)$ or $p|(a-bi)$. We suppose that $p|(a+bi)$.

(2) Also $p|(c^2+d^2) = (c+di)(c-di)$. If $p|(c+di)$ we contradict the assumption that $a+bi$ and $c+di$ are coprime. Hence $p|(c-di)$ which implies $\overline{p}|(c+di)$.

(3) Finally $p|2(ac+bd) = (a+bi)(c-di)+(a-bi)(c+di)$. Since $p|(a+bi)$, we conclude that either $p|(a-bi)$ or $p|(c+di)$. Now $p|(c+di)$ leads to a contradiction as in (2), and if $p|(a-bi)$ then $\overline{p}|(a+bi)$ which means $\overline{p}$ is a common factor of $a+bi$ and $c+di$, a contradiction.

A similar contradiction is reached if we suppose that $p|(a-bi)$ in (1). We therefore conclude that $s = 1$. □

Using a similar argument, we can also prove:

Theorem 4.46. *Let a modulus $\tau \in \mathbf{Q}(\rho)$ have discriminant $d = -3n^2$ and the associated minimal map $\{\frac{c+d\rho}{a+b\rho}\}_1$ where $\tau = \frac{c+d\rho}{a+b\rho}$ and $a+b\rho, c+d\rho$ are coprime in $\mathbf{Z}[\rho]$. Then $|ad-bc| = n$.* □

Hence the minimal map $\mathcal{M}_\tau$ lying on a surface of modulus $\tau \in \mathbf{Q}(i)$ or $\mathbf{Q}(\rho)$ will have index n, where τ has discriminant $-3n^2$ or $-4n^2$. The results of Theorem 4.41 on regular covers of uniform maps may be restated as follows:

Corollary 4.47. *Let $\mathcal{M}$ be a uniform map of genus 1 lying on a surface of modulus τ. Then $\tau \in \mathbf{Q}(i)$ or $\mathbf{Q}(\rho)$, and the index of the minimal regular cover of $\mathcal{M}$ is equal to n, where τ has discriminant $-4n^2$ or $-3n^2$.*

Proof. If $\mathcal{M}$ is a uniform map lying on a surface of modulus $\tau \in \mathbf{Q}(i)$, then $\mathcal{M}$ has the form $\{\frac{c+di}{a+bi}\}_{p+qi}$ by Lemma 4.19. By Theorem 4.41 the index of the minimal regular cover of $\mathcal{M}$ is $n = |ad - bc|$, and by Theorem 4.44, the discriminant of $\tau = \frac{c+di}{a+bi}$ is $-4n^2$. Similarly for $\tau \in \mathbf{Q}(\rho)$. □

Example 4.48. We consider the moduli $\tau_1 = 5i$ and $\tau_2 = \frac{-1+5i}{2}$. It was shown in Example 3.38 that both τ_1 and τ_2 have discriminant $d = -4.5^2$, and so their associated minimal maps have index 5 by Theorem 4.44. The minimal map $\mathcal{M}_{\tau_1} = \{5i\}_1$ is shown in Figure 4.10(a). In reduced form, $\frac{-1+5i}{2} = \frac{-3+2i}{1+i}$, and so $\mathcal{M}_{\tau_2} = \{\frac{-3+2i}{1+i}\}_1$; this map is shown in Figure 4.10(b). By Corollary 4.47, $\mathcal{M}_1$ and $\mathcal{M}_2$ have minimal regular covers of index 5; indeed by Theorem 4.41 both have the same minimal regular cover $\mathcal{M}^\star = \{i\}_5$.

In §3.4 we calculated that the class number $h(-100) = 2$, and hence that τ_1 and τ_2 are the only moduli in $\mathbf{Q}(i)$ lying in the modular fundamental region with discriminant -100. Any minimal map of type $(4,4)$ and index 5 must have an associated τ with discriminant -100 by Theorem 4.44, so that $\mathcal{M}_{\tau_1}$ and $\mathcal{M}_{\tau_2}$ are the only minimal maps of type $(4,4)$ with index 5. □

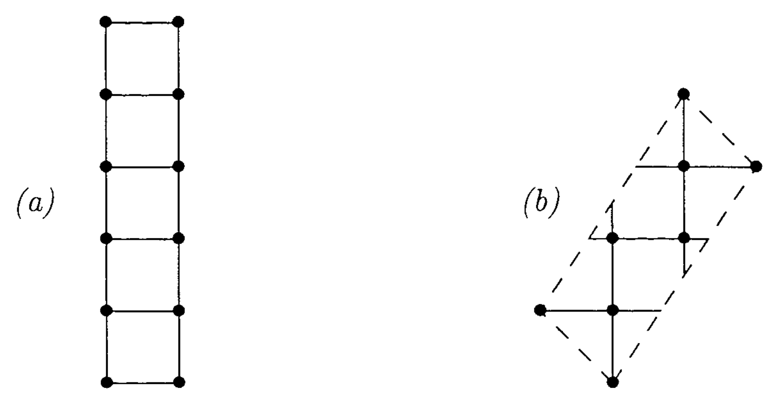


Figure 4.10

More generally, we might ask for the number of minimal maps of type $(4,4)$ and index n. By Theorem 4.44, this is equivalent to asking for the number of moduli $\tau \in \mathbf{Q}(i)$ lying in the modular fundamental region with discriminant $-4n^2$, and by

Definition 3.33 this is equal to the class number $h(-4n^2)$. Thus we have proved, with a similar argument for uniform maps of type $(6, 3)$:

Theorem 4.49. *Let $h(-3n^2)$ and $h(-4n^2)$ be the class number formulae of 3.46. Then the number of minimal maps of type $(4, 4)$ and index n is equal to $h(-4n^2)$; the number of minimal maps of type $(6, 3)$ and index n is equal to $h(-3n^2)$.* □

Example 4.50. We use Theorem 4.49 to calculate the number of uniform maps of type $(4, 4)$ and index 10. Every such uniform map has the form $\{\frac{a+bi}{c+di}\}_{p+qi}$ where

$$|ad - bc|(p^2 + q^2) = 10$$

(see 4.16) and $\{\frac{a+bi}{c+di}\}_1$ is a minimal map. There are four possibilities to consider:

(i) $|ad - bc| = 1$ and $p^2 + q^2 = 10$. Since $h(-4.1^2) = 1$, there is a unique minimal map $\{i\}_1$ with index 1 by Theorem 4.49. Also $R(10) = 2$, and so there are two possibilities for $p^2 + q^2 = 10$ up to multiplication by units in $\mathbf{Z}[i]$: $1 + 3i$ and $1 - 3i$. Since the maps $\{i\}_{r+si}$ and $\{i\}_{p+qi}$ are isomorphic if and only if $r + si$ and $p + qi$ are associates in $\mathbf{Z}[i]$ (see Lemma 4.19), this case contributes two maps of index 10: $\{i\}_{1+3i}$ and $\{i\}_{1-3i}$.

(ii) $|ad-bc| = 2$ and $p^2+q^2 = 5$. The class number $h(-4.2^2) = 1$ from 3.46, and so there is only one minimal map $\{2i\}_1$ with index 2 by Theorem 4.49. We have $R(5) = 2$, so that there are two possibilities for $p^2+q^2 = 5$ up to multiplication by units: $1 + 2i$ and $1 - 2i$. We therefore obtain two index 10 maps: $\{2i\}_{1+2i}$ and $\{2i\}_{1-2i}$.

(iii) $|ad - bc| = 5$ and $p^2 + q^2 = 2$. This case yields two maps of index 10: $\{5i\}_{1+i}$ and $\{\frac{-3+2i}{1+i}\}_{1+i}$.

(iv) $|ad - bc| = 10$ and $p^2 + q^2 = 1$. These will be minimal maps. Using the class-number formula in 3.46, $h(-4.10^2) = 4$, so there are four moduli in $\mathbf{Q}(i)$ lying in the modular fundamental region with discriminant -4.10^2. By finding these four moduli (as shown in §3.3) and writing them in reduced form, we obtain the maps $\{10i\}_1$, $\{\frac{5i}{2}\}_1$, $\{\frac{-2+3i}{2+2i}\}_1$ and $\{\frac{2+3i}{2-2i}\}_1$.

In total, there are are 10 non-isomorphic uniform maps of type $(4, 4)$ with index 10. □

By generalizing Example 4.50 we can express the number of genus 1 uniform maps of index n as Dirichlet products involving the class number formulae and the functions $R(n)$ and $S(n)$.

Theorem 4.51. *Let $M_{(4,4)}(n)$ and $M_{(6,3)}(n)$ denote the number of index n genus 1 uniform maps of type $(4,4)$ and $(6,3)$ respectively. Then*

$$M_{(4,4)}(n) = \sum_{t|n} h(-4t^2)R(\frac{n}{t})$$
$$M_{(6,3)}(n) = \sum_{t|n} h(-3t^2)S(\frac{n}{t})$$

where $h(-4t^2)$ and $h(-3t^2)$ are the class number formulae from 3.46.

Proof. A map of type $(4,4)$ satisfying the conditions of the theorem has the form $\{\frac{a+bi}{c+di}\}_{p+qi}$ where $\{\frac{a+bi}{c+di}\}_1$ is a minimal map with some index $t = |ad - bc|$ and $p^2 + q^2 = \frac{n}{t}$ (so that $t \mid n$). By Lemma 4.49, there are $h(-4t^2)$ distinct minimal maps with index t. For each of these minimal maps we want to choose all $p + qi$ such that $p^2 + q^2 = \frac{n}{t}$ and no two are associates in $\mathbf{Z}[i]$. The number of such choices is given by the function $R(\frac{n}{t})$, and so by starting with a minimal map of index t, we obtain $h(-4t^2)R(\frac{n}{t})$ uniform maps of index n. We now sum over all $t|n$.

The proof for maps of type $(6,3)$ is similar: every such map has the form $\{\frac{a+b\rho}{c+d\rho}\}_{p+q\rho}$ where $(p^2+q^2-pq)|ad-bc| = n$. For each $t = |ad-bc|$ where $t \mid n$ there are $h(-3t^2)$ minimal maps of type $(6,3)$ and index t. We must then determine all $p + q\rho$ for which $p^2 + q^2 - pq = \frac{n}{t}$ and no two are associates in the ring $\mathbf{Z}[\rho]$; the number of these is $S(\frac{n}{t})$. This gives $h(-3t^2)S(\frac{n}{t})$ uniform maps of index n, and we now sum over all $t \mid n$. □

We apply Theorem 4.51 to the case $n = 10$, considered in Example 4.50:

$$\begin{aligned} M_{(4,4)}(10) &= \sum_{t|10} h(-4t^2)R(\frac{10}{t}) \\ &= h(-4.1^2)R(10) + h(-4.2^2)R(5) + h(-4.5^2)R(2) + h(-4.10^2)R(1) \\ &= (1)(2) + (1)(2) + (2)(1) + (4)(1) \\ &= 10 \end{aligned}$$

which confirms the result. Although Theorem 4.51 provides a systematic way of building uniform maps of a given index from minimal maps, the resulting formulae fail to be multiplicative, and so are not practical for large values of n. We will see, however, that it is possible to express $M_{(4,4)}(n)$ and $M_{(6,3)}(n)$ as the sum of two multiplicative functions. The author would like to thank Gareth Jones for pointing out the following lemma, proved in [**Se** p.99]:

Lemma 4.52. *The lattice $\Lambda = \Lambda(\omega_1, \omega_2)$ contains $\sigma(n)$ sublattices of index n, where $\sigma(n)$ is equal to the sum of the positive divisors of n.*

Proof. Let $\Lambda(n)$ denote the set of index n sublattices of $\Lambda(\omega_1, \omega_2)$, and S_n the set of matrices $\left(\begin{smallmatrix} a & b \\ 0 & d \end{smallmatrix}\right)$ with $a, b, d \in \mathbf{Z}$, $ad = n$, $a \geq 1$ and $0 \leq b < d$. For $\sigma = \left(\begin{smallmatrix} a & b \\ 0 & d \end{smallmatrix}\right) \in S_n$ let Λ_σ be the lattice with basis

$$\omega_1' = a\omega_1 + b\omega_2, \ \omega_2' = d\omega_2,$$

then Λ_σ is an index n sublattice of $\Lambda(\omega_1, \omega_2)$. We will prove that the map $\sigma \mapsto \Lambda_\sigma$ is a bijection of S_n onto the set $\Lambda(n)$ of index n sublattices of Λ. Suppose that $\Lambda' \in \Lambda(n)$. We define

$$Y_1 = \Lambda/(\Lambda' + \mathbf{Z}\omega_2) \quad \text{and} \quad Y_2 = \mathbf{Z}\omega_2/(\Lambda' \cap \mathbf{Z}\omega_2)$$

where Y_1 and Y_2 are cyclic groups of orders (say) a and d respectively; Y_i is generated by the image of ω_i $(i = 1, 2)$. If we consider the exact sequence

$$0 \to Y_2 \xrightarrow{f} \Lambda/\Lambda' \xrightarrow{g} Y_1 \to 0$$

defined by $f : s\omega_2/(\Lambda' \cap \mathbf{Z}\omega_2) \mapsto s\omega_2/\Lambda'$ and $g : (r\omega_1 + s\omega_2)/\Lambda' \mapsto r\omega_1/(\Lambda' + \mathbf{Z}\omega_2)$ for integers r and s, then $Y_1 \cong (\Lambda/\Lambda')/f(Y_2)$ and so $n = ad$. Setting $\omega_2' = d\omega_2$, then $\omega_2' \in \Lambda'$. Also, there exists an element $\omega_1' \in \Lambda'$ such that $\omega_1' \equiv a\omega_1 \bmod \mathbf{Z}\omega_2$, and so we can write ω_1' in the form

$$\omega_1' = a\omega_1 + b\omega_2$$

for $b \in \mathbf{Z}$ uniquely determined modulo d. Furthermore, if we impose on b the inequality $0 \leq b < d$, then b, and so ω_1' are fixed. It is clear that $\omega_1' = a\omega_1 + b\omega_2$ and $\omega_2' = d\omega_2$ form a basis for Λ' with $ad = n$. Hence we have associated to every $\Lambda' \in \Lambda(n)$ a matrix $\sigma(\Lambda') \in S_n$. It can be checked that the maps $\sigma \mapsto \Lambda_\sigma$ and $\Lambda' \mapsto \sigma(\Lambda')$ are inverses to each other and so define a bijection between $\Lambda(n)$ and S_n.

Finally, a matrix $\sigma = \left(\begin{smallmatrix} a & b \\ 0 & d \end{smallmatrix}\right) \in S_n$ satisfies $ad = n$ and $0 \leq b < d$ where a, d are positive integers. So for every $d\,|n$ the choice of a is fixed. The possible values of b are then $0, 1, 2, \ldots d - 1$ from which it follows that $|S_n| = \sum_{d|n} d = \sigma(n)$. □

Theorem 4.53. *Let $M_{(4,4)}(n)$ and $M_{(6,3)}(n)$ be the number of index n, genus 1 uniform maps of type $(4,4)$ and $(6,3)$ respectively. Then*

$$M_{(4,4)}(n) = \frac{1}{2}(\sigma(n) + R(n))$$
$$M_{(6,3)}(n) = \frac{1}{3}(\sigma(n) + 2S(n))$$

where $R(n), S(n)$ are the functions from 4.42 and $\sigma(n)$ is equal to the sum of the positive divisors of n.

Proof. Genus 1 uniform maps of type $(4,4)$ and index n correspond to index n sublattices of $\Lambda(1,i)$. We define an equivalence relation on the set $\Lambda(n)$ of index n sublattices of $\Lambda(1,i)$ as follows: two lattices Λ_1, Λ_2 are equivalent if $\Lambda_1 = u\Lambda_2$ for a unit $u \in \mathbf{Z}[i]$. By Lemma 4.10, the maps corresponding to two lattices of $\Lambda(n)$ are isomorphic if and only if they are equivalent. In the proof of Lemma 4.10, it was shown that each equivalence class of lattice contains either 1 element (corresponding to a regular map) or 2 elements (corresponding to a non-regular map). The number of index n sublattices of $\Lambda(1,i)$ is equal to $\sigma(n)$ by Lemma 4.52 above, and the number of regular maps of index n is equal to $R(n)$ by Theorem 4.43. The number of distinct index n uniform maps is therefore equal to $\frac{1}{2}(\sigma(n) + R(n))$.

Similarly, a genus 1 uniform map of type $(6,3)$ and index n corresponds to an index n sublattice of $\Lambda(1,\rho)$. Define an equivalence relation on the set $\Lambda(n)$ of index n sublattices of $\Lambda(1,\rho)$ so that two lattices Λ_1, Λ_2 are equivalent if $\Lambda_1 = u\Lambda_2$ for a unit $u \in \mathbf{Z}[\rho]$. The maps associated to two lattices are isomorphic if and only if they are equivalent. Every equivalence class of lattice contains either 1 element (corresponding to a regular map) or 3 elements (corresponding to a non-regular map). Since $\Lambda(1,\rho)$ contains $\sigma(n)$ lattices of index n by Lemma 4.52, with $S(n)$ regular maps of index n, the number of index n uniform maps of type $(6,3)$ is equal to $\frac{1}{3}(\sigma(n) + 2S(n))$. □

The function $\sigma(n)$ is multiplicative, with

$$\sigma(p^k) = \frac{p^{k+1} - 1}{p - 1}$$

for any prime p and integer $k > 0$ (see [**NZM**, p.191]). We apply Theorem 4.53 to the case $n = 10$. Since $\sigma(10) = 1 + 2 + 5 + 10 = 18$ and $R(10) = 2$, the number of uniform maps of index 10 and type $(4,4)$ is given by

$$M_{(4,4)}(10) = \frac{1}{2}(18 + 2) = 10.$$

Table 6 gives the number of genus 1 uniform maps of index n for $1 \leq n \leq 20$; the results were obtained using Theorem 4.53.

The number of uniform maps of a given index has been expressed in two different ways: firstly as a Dirichlet product involving the class number formulae in Theorem 4.51, and secondly as a sum involving the function $\sigma(n)$ in Theorem 4.53. In §3.4 we introduced the multiplicative versions of the class number formulae, $\overline{h}(-4n^2)$ and $\overline{h}(-3n^2)$, where $\overline{h}(-4n^2) = 2h(-4n^2)$ and $\overline{h}(-3n^2) = 3h(-3n^2)$ for all $n > 1$. It can be shown that the formulae in Theorem 4.51 may be expressed as

$$M_{(4,4)}(n) = \frac{1}{2}\Big(\sum_{t|n} \overline{h}(-4t^2)R(\frac{n}{t}) \; + R(n)\Big)$$

$$M_{(6,3)}(n) = \frac{1}{3}\Big(\sum_{t|n} \overline{h}(-3t^2)S(\frac{n}{t}) \; + 2S(n)\Big)$$

where the two Dirichlet products, being composed of multiplicative functions, are themselves multiplicative. Furthermore, one can show (although this requires the consideration of a number of different cases) that

$$\sum_{t|n} \overline{h}(-4t^2)R(\frac{n}{t}) = \sigma(n) = \sum_{t|n} \overline{h}(-3t^2)S(\frac{n}{t})$$

from which we obtain the formulae given in Theorem 4.53.

4.6. Uniform maps and Belyĭ functions

Every genus 1 uniform map $\mathcal{M}$ is associated with a finite index subgroup of a Euclidean triangle group, and so can be embedded naturally into a Riemann surface of the form $X = \mathbf{C}/\Lambda$, where $\Lambda \leq \Gamma(4,2,4)$ or $\Lambda \leq \Gamma(6,2,3)$. By Theorem 1.17, X will be defined over the algebraic numbers $\overline{\mathbf{Q}}$, and so by Belyĭ's Theorem there exists a Belyĭ function $\beta : X \to \Sigma$ with critical values $C(\beta) \subseteq \{0, 1, \infty\}$; one can choose the Belyĭ function so that the Walsh double of $\beta^{-1}(\mathcal{B}_1)$ defines a map isomorphic to $\mathcal{M}$ (see §2.3).

In §3.4 we determined all elliptic curves with Euclidean Belyĭ uniformizations that are defined over the rational numbers $\mathbf{Q}$, and quadratic and cubic extensions of $\mathbf{Q}$. Table 7 lists the equations of these elliptic curves, together with their associated minimal maps. In this section we will construct Belyĭ functions for many of the minimal maps in Table 7.

Examples 4.54. (i) The minimal map $\mathcal{M}_\rho$ corresponds to the lattice $\Lambda(1,\rho)$ and lies on the elliptic curve $E_\rho : y^2 = 4x^3 - 1$. Figure 4.11 shows a fundamental parallelogram $\mathcal{P}$ for $\Lambda(1,\rho)$ with the vertices and edge centres of the minimal map represented by black and white vertices respectively.

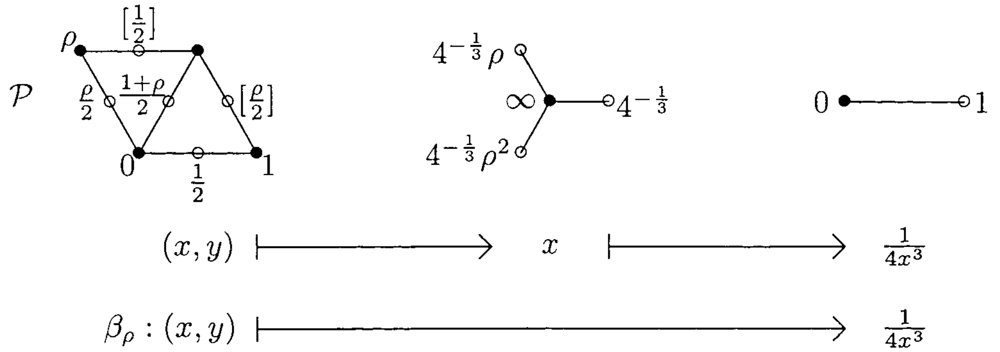


Figure 4.11

We can normalize $\Lambda(1,\rho)$ by a complex number $\alpha \in \mathbf{C}^\star$, so that the similar lattice $\Lambda(\alpha,\alpha\rho)$ satisfies $g_2(\Lambda(\alpha,\alpha\rho)) = 0$, $g_3(\Lambda(\alpha,\alpha\rho)) = -1$. The Weierstrass function $\wp(\alpha z, \alpha\Lambda(1,\rho)) = \alpha^{-2}\wp(z,\Lambda(1,\rho))$ then induces an isomorphism from $\mathbf{C}/\Lambda(1,\rho)$ to E_ρ [**Cox**, p.206] taking the vertex [0] to the point at infinity $\mathcal{O}$, and the edge centres $\{[\frac{1}{2}], [\frac{\rho}{2}], [\frac{1+\rho}{2}]\}$ to $\{(4^{-\frac{1}{3}}, 0), (4^{-\frac{1}{3}}\rho, 0), (4^{-\frac{1}{3}}\rho^2, 0)\}$. Hence the projection $E_\rho \to \Sigma$ defined by $(x, y) \mapsto x$ sends the vertex of $\mathcal{M}_\rho$ to infinity, and the three edge centres to the cube roots of $\frac{1}{4}$. By composing this projection with $x \mapsto \frac{1}{4x^3}$ (whose only critical values lie at $x = 0$ and $x = \infty$) we obtain the Belyĭ function $\beta_\rho : (x, y) \mapsto \frac{1}{4x^3}$. By construction, $\beta_\rho^{-1}(\mathcal{B}_1)$ is isomorphic to $\mathcal{M}_\rho$.

(ii) The minimal map $\mathcal{M}_i$ corresponds to the lattice $\Lambda(1,i)$ and lies on the elliptic curve $E_i : y^2 = 4x^3 - x$. The map $\mathcal{M}_i$ is drawn on the fundamental parallelogram $\mathcal{P}$ for $\Lambda(1,i)$ shown in Figure 4.12; the vertex corresponds to a black circle, the edge centres to white circles, and the face centre is marked with a dot.

For a suitable $\alpha \in \mathbf{C}^\star$, the Weierstrass function $\alpha^{-2}\wp(z,\Lambda(1,i))$ defines an isomorphism from $\mathbf{C}/\Lambda(1,i)$ to E_i mapping the vertex [0] to the point at infinity $\mathcal{O}$ and the edge and face centres $\{[\frac{1}{2}], [\frac{i}{2}], [\frac{1+i}{2}]\}$ to $\{(\frac{1}{2}, 0), (-\frac{1}{2}, 0), (0, 0)\}$. One can show that $\wp(\frac{1}{2}, \Lambda(1,i)) = -\wp(\frac{i}{2}, \Lambda(1,i))$, and hence that the two edge centres correspond to the points $(\frac{1}{2}, 0)$ and $(-\frac{1}{2}, 0)$ of E_i. Therefore, under the projection $(x, y) \mapsto x$, the vertex is sent to infinity, the face centre to $x = 0$, and the two edge centres to $x = \pm\frac{1}{2}$. Composing this with the map $x \mapsto \frac{1}{4x^2}$ gives the Belyĭ function $\beta_i : (x, y) \mapsto \frac{1}{4x^2}$. By construction, $\beta_i^{-1}(\mathcal{B}_1)$ defines a map isomorphic to $\mathcal{M}_i$. □

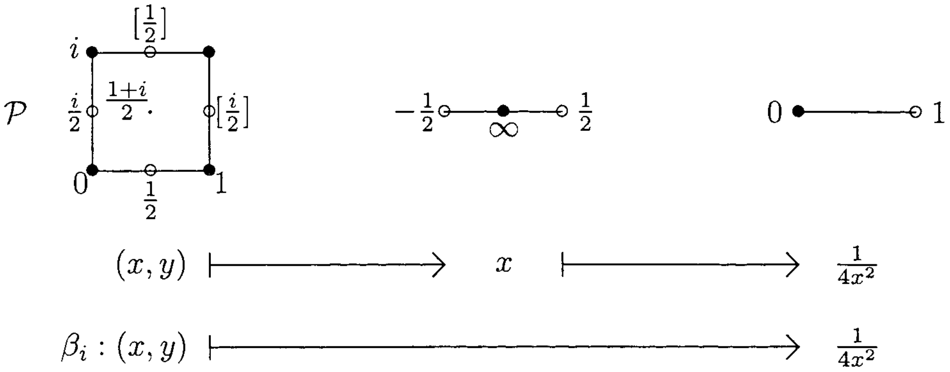


Figure 4.12

The action of the absolute Galois group $\mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q})$ upon Belyĭ pairs (or more precisely equivalence classes of Belyĭ pairs) induces an action of $\mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q})$ on the set of all dessins. The Belyĭ pairs (E_ρ, β_ρ) and (E_i, β_i) described in Examples 4.54 are defined over the rational number field $\mathbf{Q}$, and so they are fixed by every element of $\mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q})$. Therefore, the minimal maps $\mathcal{M}_\rho$ and $\mathcal{M}_i$ have orbits of length one under the action of $\mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q})$.

Lemma 4.55. *Let E be an elliptic curve with discriminant $-3m^2$ or $-4m^2$, and let β be a Belyĭ function for E where $\beta^{-1}(\mathcal{B}_1)$ is a minimal map lying on E. If $\sigma \in \mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q})$ is such that E and E^σ are conformally equivalent Riemann surfaces, then $\beta^{-1}(\mathcal{B}_1)$ and $(\beta^\sigma)^{-1}(\mathcal{B}_1)$ are isomorphic maps.*

Proof. We note that $\beta^{-1}(\mathcal{B}_1)$ is the unique minimal map lying on E, with every other uniform map on E having a strictly greater index. If E and E^σ are conformally equivalent, then by Theorem 2.28, $\beta^{-1}(\mathcal{B}_1)$ and $(\beta^\sigma)^{-1}(\mathcal{B}_1)$ are two uniform maps with the same index lying on E; therefore they must be isomorphic. □

Hence every Galois orbit of minimal maps corresponds to a Galois orbit of the underlying elliptic curves. This fact will allow us to construct arbitrarily large Galois orbits of minimal maps without explicitly constructing their Belyĭ functions. Firstly, we demonstrate the procedure with two examples from [**SiSy**].

Examples 4.56. (i) Consider the two elliptic curves listed in Table 2 that have discriminant -100. Their j-invariants, and hence the elliptic curves (see Table 7) are conjugate in the field $\mathbf{Q}(\sqrt{5})$. The elliptic curve E_1 with modulus $\tau_1 = 5i$ has the associated minimal map $\{5i\}_1$ of index 5 shown in Figure 4.10(a), and it follows from 4.16 that any other uniform map on E_1 has an index strictly greater than 5. Similarly, the elliptic curve E_2 with modulus $\tau = \frac{-1+5i}{2} = \frac{-3+2i}{1+i}$ has the minimal map $\{\frac{-3+2i}{1+i}\}_1$ of index 5 shown in Figure 4.10(b), and is the unique uniform map of index 5 lying on E_2.

The absolute Galois group $\mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q})$ will act on the elliptic curves (interchanging E_1 and E_2), and hence on their associated uniform maps. From Lemma 4.55 we see that if an element of $\mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q})$ fixes E_1 or E_2, then it must also fix the minimal maps lying on them. The minimal map $\{5i\}_1$ will be taken to a map on E_2 which, by Theorem 2.28, will be uniform of index 5. By the above discussion, $\{5i\}_1$ must be taken to $\{\frac{-3+2i}{1+i}\}_1$, and we conclude that the dessins of Figure 4.10 form an orbit under $\mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q})$.

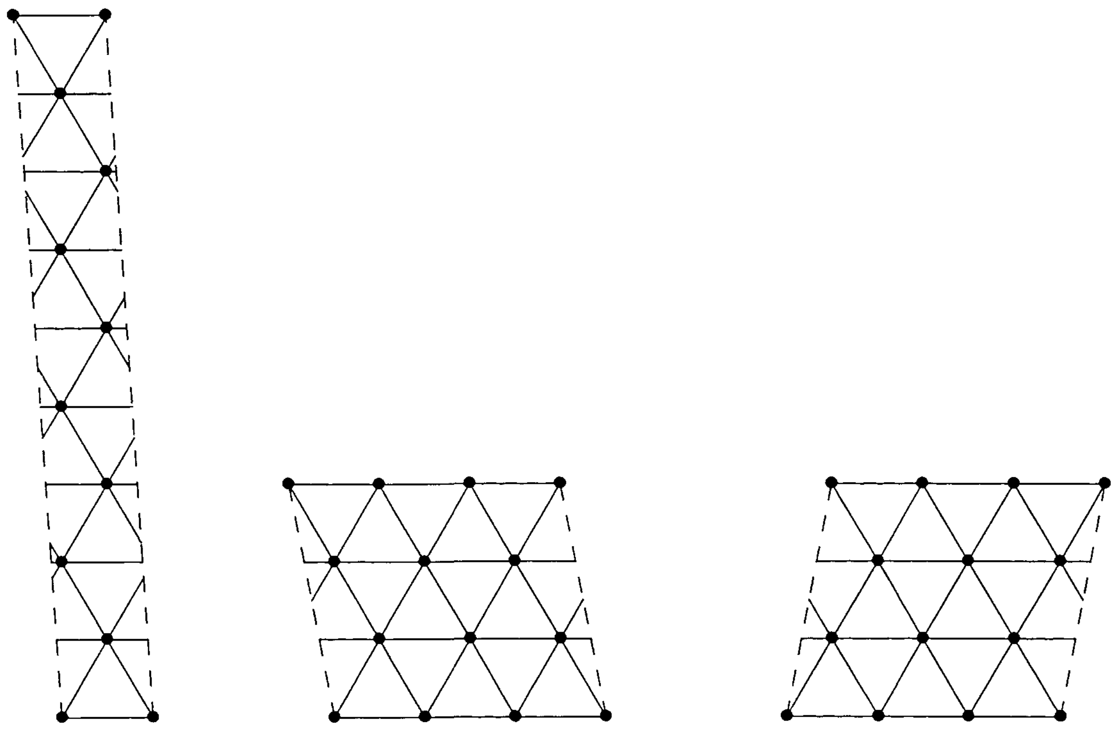


Figure 4.13. A Galois orbit, $d = -243$

(ii) The three elliptic curves E_1, E_2, E_3 whose associated moduli have discriminant -243 correspond to the minimal maps $\mathcal{M}_1 = \{4+9\rho\}_1$, $\mathcal{M}_2 = \{\frac{1+3\rho}{-3}\}_1$ and $\mathcal{M}_3 = \{\frac{2+3\rho}{-3}\}_1$ respectively. These minimal maps have index 9 and are illustrated in Figure 4.13; we note that any other uniform map lying on E_1, E_2, or E_3 will have an index strictly greater than 9. The three elliptic curves form an orbit under the action of $\mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q})$. Now, the minimal map $\mathcal{M}_1$ lying on E_1 will be taken to a map of index 9 lying on E_2, which by the above discussion must be the minimal map $\mathcal{M}_2$. Using a similar argument, we see that the minimal maps $\mathcal{M}_1, \mathcal{M}_2, \mathcal{M}_3$ form an orbit under $\mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q})$. □

These examples can be generalized as follows. Let E_τ denote the elliptic curve with modulus τ. If $\tau \in \mathbf{Q}(i)$, we can write $\tau = \frac{c+di}{a+bi}$ where $a+bi$ and $c+di$ are coprime Gaussian integers. On E_τ we can construct, as in Definition 4.17, the minimal map M_τ of type $(4,4)$ and index $n = |ad-bc|$. By Theorem 4.44 the discriminant of τ is $-4(ad-bc)^2 = -4n^2$, and every other uniform map lying on E_τ will have an index strictly greater than n.

We now fix a discriminant $d = -4n^2$. Let $\tau_1, \ldots, \tau_s$ be the $s = h(d)$ quadratic imaginary numbers lying in the modular region $\mathcal{F}$ and having discriminant d (we note that $\tau_k \in \mathbf{Q}(i)$ for $1 \leq k \leq s$). The j-values $j(\tau_1), \ldots, j(\tau_s)$, being the roots of the irreducible polynomial given in Theorem 3.37 form an orbit under $\mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q})$, and hence so do the elliptic curves $E_{\tau_1}, \ldots, E_{\tau_s}$. The elliptic curve E_{τ_k} $(1 \leq k \leq s)$ carries the unique minimal map $\mathcal{M}_{\tau_k}$ of index n as defined above, with every other uniform map on E_{τ_k} having an index strictly greater than n. The index of these maps is invariant under the action of $\mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q})$ by the result of Jones and Streit given in Theorem 2.28. Since by Lemma 4.55 an element of $\mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q})$ acts non-trivially on a minimal map if and only if it acts non-trivially on the underlying elliptic curve, the minimal maps $\mathcal{M}_{\tau_1}, \ldots, \mathcal{M}_{\tau_s}$ form an orbit under $\mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q})$. Conversely, if $\mathcal{M}_\tau$ and $\mathcal{M}_{\tau'}$ (for $\tau, \tau' \in \mathbf{Q}(i)$) are in the same Galois orbit then τ and τ' have the same index and hence the same discriminant. Thus we have proved

Theorem 4.57. *For $\tau, \tau' \in \mathbf{Q}(i)$ let $\mathcal{M}_\tau$, $\mathcal{M}_{\tau'}$ be two minimal maps. Then $\mathcal{M}_\tau$, $\mathcal{M}_{\tau'}$ are in the same orbit under Gal$(\overline{\mathbf{Q}}/\mathbf{Q})$ if and only if τ and τ' have the same discriminant.* □

Similarly, every $\tau \in \mathbf{Q}(\rho)$ corresponds to a unique minimal map $\mathcal{M}_\tau$ with index n, where τ has discriminant $-3n^2$ (see Theorem 4.46). By considering discriminants of the form $-3n^2$, one obtains a corresponding theorem about the orbits of minimal maps of type $(6,3)$ under $\mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q})$.

Theorem 4.58. *For $\tau, \tau' \in \mathbf{Q}(\rho)$, let $\mathcal{M}_\tau$, $\mathcal{M}_{\tau'}$ be two minimal maps. Then $\mathcal{M}_\tau$,*

$\mathcal{M}_{\tau'}$ are in the same orbit under Gal($\overline{\mathbf{Q}}/\mathbf{Q}$) if and only if τ and τ' have the same discriminant. □

Theorems 4.57 and 4.58 enable us to construct arbitrarily large Galois orbits of minimal maps. As an example, we will construct the Galois orbit of minimal maps of type $(4,4)$ and index 27; each map will have 27 vertices, 27 faces and 54 edges. By Theorem 4.44, all of these maps will have the form $\mathcal{M}_\tau$ where $\tau \in \mathbf{Q}(i)$ has discriminant $-4(27)^2 = -2916$. Using the class number formula in 3.46, we see that

$$h(-4(27)^2) = 18$$

and so there will be 18 such minimal maps in the Galois orbit. In order to find the minimal maps, we first find all of the values of $\tau \in \mathbf{Q}(i)$ lying in the modular fundamental region $\mathcal{F}$ with discriminant -2916, which by Lemma 3.36 is equivalent to finding all of the reduced quadratic forms

$$ax^2 + bxy + cy^2$$

with $b^2 - 4ac = -2916$, $\gcd(a,b,c) = 1$, $|b| \le a \le c$ and $b \ge 0$ if $|b| = a$ or $a = c$. The required values of τ are then the roots of the quadratic equations $aX^2 + bX + c$ with positive imaginary part. By expressing each of the τ values in reduced form, we obtain the 18 minimal maps

$$\{27i\}_1, \left\{\frac{-14+13i}{1+i}\right\}_1, \left\{\frac{-11+5i}{1+2i}\right\}_1, \left\{\frac{11+5i}{1-2i}\right\}_1, \left\{\frac{8+3i}{1-3i}\right\}_1, \left\{\frac{-8+3i}{1+3i}\right\}_1,$$

$$\left\{\frac{-1+9i}{3}\right\}_1, \left\{\frac{1+9i}{3}\right\}_1, \left\{\frac{-5+4i}{3+3i}\right\}_1, \left\{\frac{5+4i}{3-3i}\right\}_1, \left\{\frac{-7+3i}{2+3i}\right\}_1, \left\{\frac{7+3i}{2-3i}\right\}_1,$$

$$\left\{\frac{5+2i}{-1+5i}\right\}_1, \left\{\frac{5-2i}{1+5i}\right\}_1, \left\{\frac{-3+6i}{4+i}\right\}_1, \left\{\frac{3+6i}{4-i}\right\}_1, \left\{\frac{-5+3i}{4+3i}\right\}_1, \left\{\frac{5+3i}{4-3i}\right\}_1$$

which are shown in Figure 4.14. By Theorem 4.57 they form a complete Galois orbit. The author would like to thank Simon Cox for his help in producing the MATLAB code used to draw the diagrams of Figure 4.14.

Examples 4.59. (i) The minimal map $\mathcal{M}_{2+3\rho}$ lies on the elliptic curve $E_{2+3\rho}$: $y^2 = 4x^3 - 120x - 253$ and corresponds to the lattice $\Lambda(1, 2+3\rho)$ which, with a suitable change of basis, has the form $\Lambda(1, 3+3\rho)$. We will define an isogeny $\phi : E_{2+3\rho} \to E_\rho$ corresponding to the quotient of $E_{2+3\rho}$ by a group of order three,

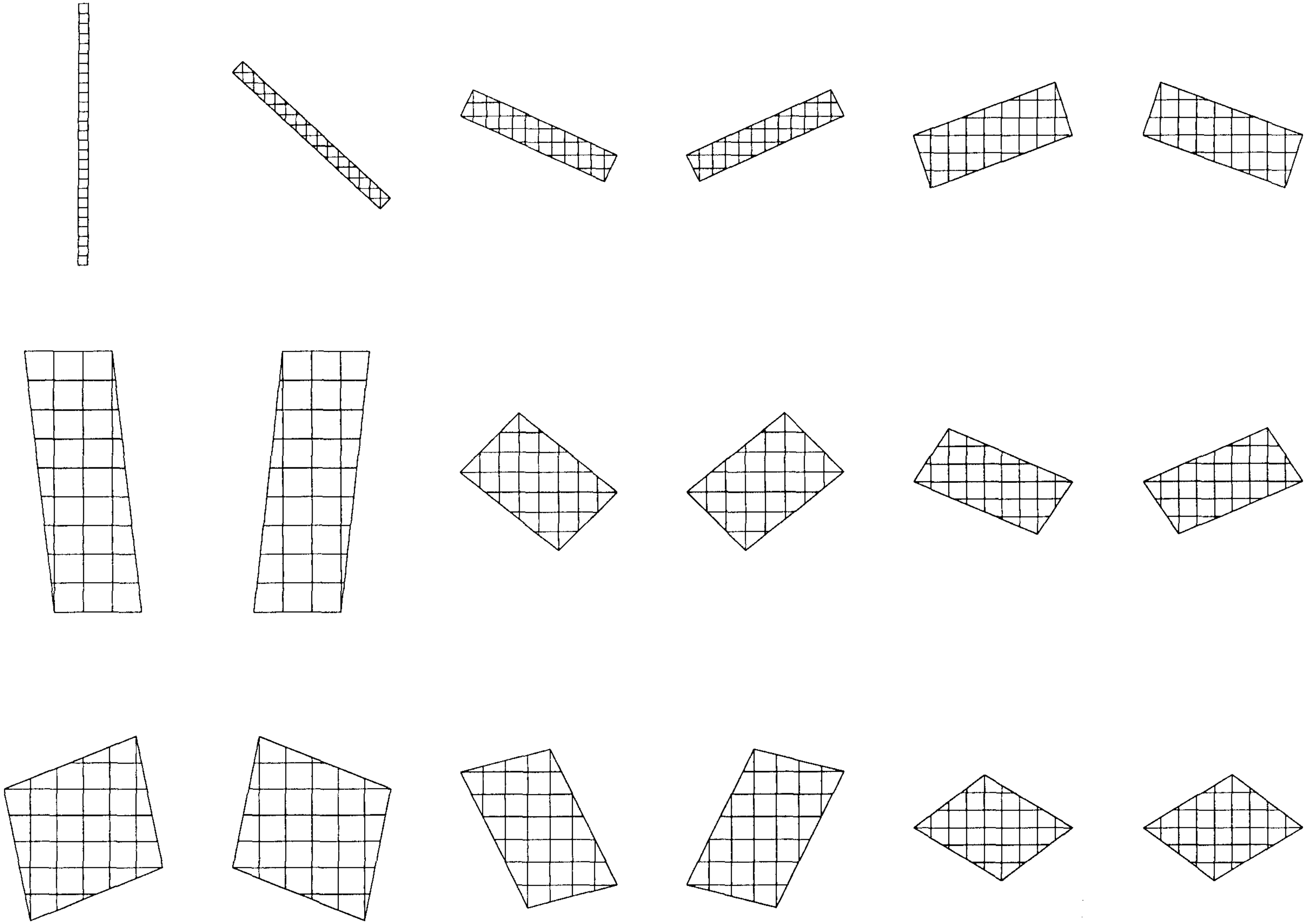


Figure 4.14. A Galois orbit, d = −2916

which as shown in Figure 4.15 will induce an unbranched cover of $\mathcal{M}_\rho$ by $\mathcal{M}_{2+3\rho}$. The composition $\beta_\rho \circ \phi : E_{2+3\rho} \longrightarrow \Sigma$ will then be a Belyĭ function for $\mathcal{M}_{2+3\rho}$.

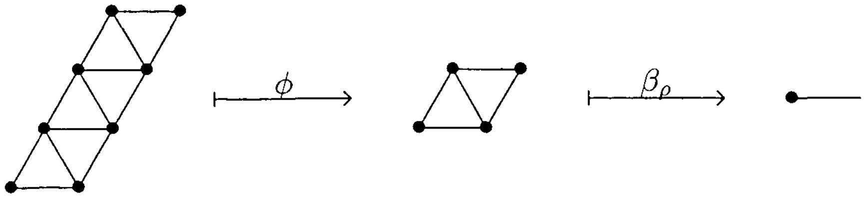


Figure 4.15

We transform $E_{2+3\rho}$ to the curve $\overline{E} : \overline{y}^2 = \overline{x}^3 - 30\overline{x} - \frac{253}{4}$ by the birational transformation $\phi_1 : (x, y) \mapsto (\overline{x}, \overline{y})$ where $\overline{x} = x$ and $\overline{y} = \frac{y}{2}$. As shown in Example 3.20(ii), (r, s) is a point of order 3 of $\overline{E}$ if and only if r is a root of the polynomial

$$3\overline{x}^4 - 180\overline{x}^2 - 759\overline{x} - 900. \qquad 4.60$$

Choosing an integer root $\overline{x} = -3$ of 4.60, we take the point $(-3, \frac{i}{2}) \in \overline{E}$ of order 3 with $[2](-3, \frac{i}{2}) = (-3, -\frac{i}{2})$ and $[3](-3, \frac{i}{2}) = \mathcal{O}$ and set

$$\Phi = < \mathcal{O}, (-3, \frac{i}{2}), (-3, -\frac{i}{2}) > .$$

The isogeny $\phi_2 : \overline{E} \longrightarrow \overline{E}/\Phi$ is given by $\phi_2(\mathcal{O}) = \phi_2(-3, \frac{i}{2}) = \phi_2(-3, -\frac{i}{2}) = \mathcal{O}$ and $\phi_2 : (\overline{x}, \overline{y}) \mapsto (X, Y)$ where

$$(X, Y) = \left(\frac{\overline{x}^3 + 6\overline{x}^2 + 3\overline{x} - 19}{(\overline{x} + 3)^2}, \frac{\overline{y}(\overline{x}^3 + 9\overline{x}^2 + 33\overline{x} + 47)}{(\overline{x} + 3)^3} \right)$$

otherwise. The quotient curve $\overline{E}/\Phi$ will have the equation

$$Y^2 = X^3 - \frac{729}{4}$$

which is isomorphic to the elliptic curve $E_\rho : \overline{Y}^2 = 4\overline{X}^3 - 1$ by the birational transformation $\phi_3 : (X, Y) \mapsto (\overline{X}, \overline{Y})$ where $\overline{X} = \frac{X}{9}$ and $\overline{Y} = \frac{2Y}{27}$. Hence the required isogeny $\phi : E_{2+3\rho} \longrightarrow E_\rho$ is given by the composition $\phi = \phi_3 \circ \phi_2 \circ \phi_1$ with $\phi(\mathcal{O}) = \phi(-3, i) = \phi(-3, -i) = \mathcal{O}$ and $\phi : (x, y) \mapsto (\overline{X}, \overline{Y})$ where

$$(\overline{X}, \overline{Y}) = \left(\frac{x^3 + 6x^2 + 3x - 19}{9(x + 3)^2}, \frac{y(x^3 + 9x^2 + 33x + 47}{27(x + 3)^3} \right)$$

otherwise. If we now take the composition $\beta_{2+3\rho} = \beta_\rho \circ \phi : E_{2+3\rho} \to \Sigma$ then

$$\beta_{2+3\rho} : (x,y) \longmapsto \frac{729(x+3)^6}{4(x^3+6x^2+3x-19)^3}$$

is the required Belyĭ function for the minimal map $\mathcal{M}_{2+3\rho}$.

(ii) The minimal map $\mathcal{M}_{3+6\rho}$ lies on the elliptic curve

$$E_{3+6\rho} : y^2 = sx^3 - 2805x - 1$$

where

$$s = 4(761257259 - 157058640\sqrt[3]{2} + 160025472\sqrt[3]{4})$$

and with a change of basis we take the corresponding lattice to be $\Lambda(1, 6+6\rho)$. As shown in Figure 4.16 we will construct an isogeny $\phi : E_{3+6\rho} \to E_{2+3\rho}$ corresponding to the quotient of $E_{3+6\rho}$ by a group of order two, and then compose this with the Belyĭ function $\beta_{2+3\rho}$ found above.

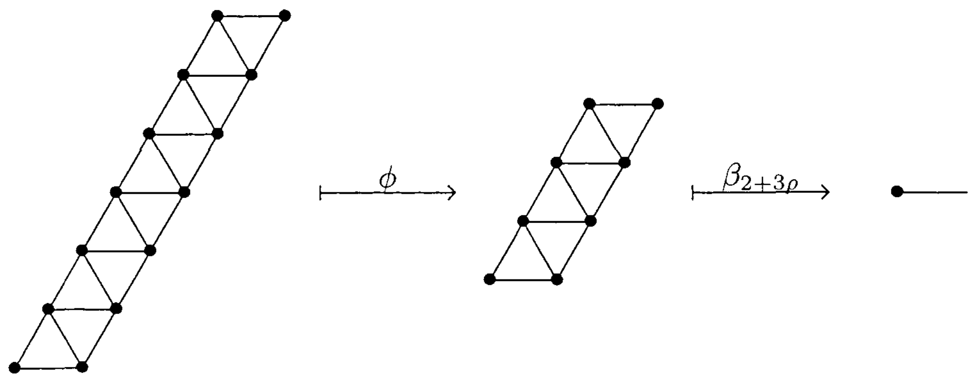


Figure 4.16

Using the birational transformation $\phi_1 : (x,y) \mapsto (\overline{x}, \overline{y})$ where $\overline{x} = x$ and $\overline{y} = \frac{y}{\sqrt{s}}$, we take the isomorphic curve

$$\overline{E} : \overline{y}^2 = \overline{x}^3 - \frac{2805}{s}\overline{x} - \frac{1}{s}.$$

It can be shown (for example using MAPLE) that

$$r = \frac{50111 - 30400\sqrt[3]{2} - 40448\sqrt[3]{4}}{97984073}$$

is a root of the cubic equation $\overline{x}^3 - \frac{2805}{s}\overline{x} - \frac{1}{s}$, so that $(r,0)$ is a point of order 2 of $\overline{E}$ by Example 3.20(i). Setting $\Phi = <\mathcal{O}, (r,0)>$, we obtain the isogeny $\phi_2 : \overline{E} \to \overline{E}/\Phi$ given by $\phi_2(\mathcal{O}) = \phi_2(r,0) = \mathcal{O}$, and $\phi_2 : (\overline{x}, \overline{y}) \mapsto (X, Y)$ where

$$X = \overline{x} - \frac{3(702421711735936\sqrt[3]{4} - 1261310690142071 + 116109236985392\sqrt[3]{2})}{25087450114628(97984073\overline{x} - 50111 + 30400\sqrt[3]{2} + 40448\sqrt[3]{4})}$$

$$Y = \overline{y} - \frac{3\overline{y}(702421711735936\sqrt[3]{4} - 1261310690142071 + 116109236985392\sqrt[3]{2})}{25087450114628(97984073\overline{x} - 50111 + 30400\sqrt[3]{2} + 40448\sqrt[3]{4})^2}$$

otherwise. The quotient curve is given by

$$Y^2 = X^3 + AX + B$$

where

$$\begin{aligned} A &= -15r^2 + 11220s^{-1} \\ B &= -s^{-1} - 7(7r^3 - 14025rs^{-1} - 4s^{-1}). \end{aligned}$$

The birational transformation $\phi_3 : (X, Y) \mapsto (\overline{X}, \overline{Y})$ where $\overline{X} = \frac{X}{j^2}$, $\overline{Y} = \frac{2Y}{j^3}$ and

$$j^2 = \frac{120B}{253A}$$

maps $\overline{E}/\Phi$ to the curve

$$E_{2+3\rho} : \overline{Y}^2 = 4\overline{X}^3 - 120\overline{X} - 253$$

so that the required isogeny from $E_{3+6\rho}$ to $E_{2+3\rho}$ is given by $\phi = \phi_3 \circ \phi_2 \circ \phi_1$. We now take the composition $\beta_{3+6\rho} = \beta_{2+3\rho} \circ \phi : E_{3+6\rho} \to \Sigma$ to obtain the required Belyĭ function $\beta_{3+6\rho}$ for the minimal map $\mathcal{M}_{3+6\rho}$; this Belyĭ function is listed in Table 8. (Where necessary, the Belyĭ functions in Table 8 are written as compositions of functions to preserve space. Note that the y-coordinate is merely denoted Y, since it is not required in the final Belyĭ function.)

By considering the non-trivial action of $\mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q})$ on $E_{3+6\rho}$ and the Belyĭ function for $\mathcal{M}_{3+6\rho}$, we obtain Belyĭ functions for the minimal maps $\mathcal{M}_{\frac{2+3\rho}{2}}$ and $\mathcal{M}_{\frac{1+3\rho}{2}}$; these are also listed in Table 8. □

In addition to the above examples, we have calculated Belyĭ functions for many of the minimal maps given in Table 7. These are listed in Table 8, and were obtained using the techniques developed in Examples 4.59, and the general isogenies determined in Examples 3.20.

Chapter 5

Higher genus and arithmetic groups

An enumeration of the regular maps of genus 2 was initiated by Erréra [**Er**] and completed by Threlfall [**Th**]; there are ten of them, and they are listed in Table 9 of [**CMo**]. The regular hypermaps of genus 2 have also been classified (see [**CoSi**] and [**BJ**]) and there are 42 of them. Threlfall [**Th**] also considers uniform maps (which are referred to as regelmässige Zellsysteme) and determines all of the genus 2 uniform maps of type $(10, 5)$. In §5.1 we complete Threlfall's classification by enumerating all uniform maps and hypermaps of genus 2.

One might ask if the classification of uniform maps on a given elliptic curve extends to higher genus uniform maps and their underlying Riemann surfaces. This problem leads to the study of arithmetic Fuchsian groups and motivates the definitions of arithmetic and non-arithmetic maps. We prove some general results for non-arithmetic uniform maps and give examples for the arithmetic case, including Klein's map on a Riemann surface of genus 3.

5.1. Uniform dessins of genus 2

By Theorem 2.19, every genus 2 uniform map of type (m, n) corresponds to a genus 2 surface group $\Lambda \leq \Gamma(m, 2, n)$ and by Theorem 2.5, two such surface groups determine isomorphic maps if and only if they are conjugate in $\Gamma(m, 2, n)$. Thus the number of genus 2 uniform maps of type (m, n) is equal to the number of conjugacy classes of genus 2 surface groups in $\Gamma(m, 2, n)$. Similarly, for uniform hypermaps of type (l_0, l_1, l_2) we must determine the number of conjugacy classes of genus 2 surface groups in $\Gamma(l_0, l_1, l_2)$. By Examples 1.10(iii), a necessary condition for the triangle group $\Gamma(l_0, l_1, l_2)$ to contain a genus 2 surface group with index k is that

$$2 = k\Big(1 - \frac{1}{l_0} - \frac{1}{l_1} - \frac{1}{l_2}\Big)$$

where l_0, l_1, l_2 all divide k. Furthermore, we have $1 \leq k \leq 84$ by 1.13. It is therefore an easy matter to determine all triples (l_0, l_1, l_2) with $l_0 \leq l_1 \leq l_2$ such that $\Gamma(l_0, l_1, l_2)$ *could* contain a genus 2 surface group. There are 22 possibilities, and these are listed in the first column of Table 9.

We use the MAGMA computer package to determine the number of conjugacy classes of genus 2 surface groups in each of the triangle groups of Table 9. By fixing a triangle group $\Gamma(l_0, l_1, l_2)$, the method is as follows:

(i) Enter the presentation of $\Gamma(l_0, l_1, l_2)$ into MAGMA and let k be the index of $(2; -)$ in $\Gamma(l_0, l_1, l_2)$:

$$\Gamma = < x_0, x_1 \,|\, x_0^{l_0} = x_1^{l_1} = (x_0 x_1)^{-l_2} = 1 >;$$

(ii) Use the *low index subgroup process* of MAGMA to select a representative from each conjugacy class of subgroups of index k in Γ;

(iii) For each such subgroup Λ, determine whether or not it is a surface group by testing the coset action of Γ on Λ and applying Theorem 1.9. Hence we require that the action of x_0 on the Λ-cosets is a product of l_0-cycles, the action of x_1 on the Λ-cosets is a product of l_1-cycles, and the action of x_2 on the Λ-cosets is a product of l_2-cycles.

The third column of Table 9 gives, for each triangle group Γ, the number of conjugacy classes of genus 2 surface groups contained in Γ. The data was produced using Program 1 (see Appendix II). The author would like to thank Prof. Marston Conder for his helpful suggestions with the MAGMA program.

From Table 9 we see that $\Gamma(8, 2, 8)$ contains four conjugacy classes of genus 2 surface groups, so that there are four genus 2 uniform maps of type $(8, 8)$. We also observe that $\Gamma(5, 2, 10)$ contains seven conjugacy classes of genus 2 surface groups, so there are seven genus 2 uniform maps of type $(5, 10)$, and seven of type $(10, 5)$; every uniform map of type $(5, 10)$ has a corresponding dual of type $(10, 5)$. In Table 10 we give for each map type (m, n) with $m \leq n$, the number of genus 2 uniform maps of that type. If there are p uniform maps of type (m, n), then for $m \neq n$ there are also p uniform maps of type (n, m); each map of type (m, n) being dual to one of type (n, m). It can be calculated from Table 10 that there are 978 genus 2 uniform maps.

Table 10 also lists the number of genus 2 uniform maps that are reflexible or hyperelliptic. A map corresponding to the inclusion $M \leq \Gamma(m, 2, n)$ is reflexible if and only if M is normalized by some anti-conformal element $n \in \Gamma^\star(m, 2, n) \backslash \Gamma(m, 2, n)$. Program 2 was used to determine the number of conjugacy classes of genus 2 surface groups contained in each extended triangle group $\Gamma^\star(m, 2, n)$. A genus 2 map corresponding to the inclusion $M \leq \Gamma(m, 2, n)$ is hyperelliptic if and only if there is an inclusion of the form $M \leq_2 (0; 2^{(6)}) \leq \Gamma(m, 2, n)$ [**Sin4**]. Every group $(0; 2^{(6)})$ contains a unique genus 2 surface group, and the hyperelliptic involution is unique for Riemann surfaces of genus $g > 1$ [**Ac**, p.44]. Thus there is a one-to-one correspondence between hyperelliptic genus 2 uniform maps of type (m, n) and conjugacy classes of $(0; 2^{(6)})$ contained in $\Gamma(m, 2, n)$; the number of these can be obtained by adapting Program 1.

Every map has an automorphism group (possibly trivial), and for each map type we list the order of the biggest automorphism group for a map of that type (see Program 3). This information is useful since it identifies the regular maps (see Theorem 2.20 for example). In Table 10 we let R denote a regular map, and we note that there is at most one genus 2 regular map for each type (also see [**KuNa**]).

Table 11 lists the number of genus 2 uniform hypermaps of type (l_0, l_1, l_2) for $2 < l_0 \leq l_1 \leq l_2$ and in addition the number of those that are reflexible and hyperelliptic. Letting R denote a regular hypermap, we see that there are three regular hypermaps of type $(5, 5, 5)$. For each hypermap type one can obtain further uniform hypermaps by applying Machì's hypermap operations, while every map of type (m, n) also corresponds to a hypermap of type $(m, 2, n)$. In total there are 3133 uniform hypermaps of genus 2.

Example 5.1. (see [**Sh**], [**SV**]) The genus 2 regular map $\mathcal{M}$ of type $(8, 8)$ shown in Figure 2.8 is a 2-sheeted cover of the star map $\mathcal{S}_4$ shown in Figure 2.9. Figure 5.1 shows $\mathcal{S}_4$ drawn on the Riemann sphere with its vertex at the origin, its face centre at infinity, and its four free edges at $x = \pm 1, \pm i$. Let X be the genus 2 Riemann surface associated to the algebraic curve

$$y^2 = x^5 - x.$$

Then the projection $\pi : X \to \Sigma,\ \ \pi : (x, y) \mapsto x$ is a 2-sheeted cover of the sphere, branched above $x = 0, \pm 1, \pm i, \infty$ so that the lift $\pi^{-1}(\mathcal{S}_4)$ defines a hyperelliptic uniform map of type $(8, 8)$ lying on X. From Table 10 we see that there is a unique genus 2 hyperelliptic uniform map of type $(8, 8)$, and so $\pi^{-1}(\mathcal{S}_4)$ must be isomorphic to $\mathcal{M}$. The composition of π with the map $x \mapsto x^4$ gives the Belyĭ function $\beta : X \to \Sigma,\ \ \beta : (x, y) \mapsto x^4$ for $\mathcal{M}$. □

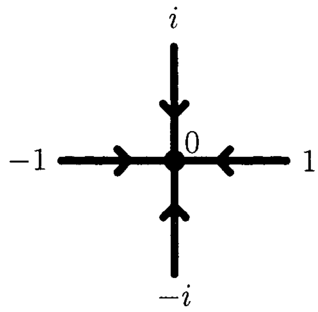


Figure 5.1

5.2. Quaternion algebras and arithmetic Fuchsian groups

We introduce the theory of quaternion algebras, and show how they can be used to construct certain Fuchsian groups. For more details, the reader is referred to [**Ka**], [**Pi**] and [**Vi**]. We begin with the definition of an algebra:

Definition 5.2. *An F-algebra A is a vector space over a field F which is also a ring with 1, such that $(\lambda a)b = \lambda(ab) = a(\lambda b)$ for all $a, b \in A$, $\lambda \in F$.* □

An F-algebra A is *associative* if $a(bc) = (ab)c$ for all $a, b, c \in A$, and a *division algebra* if for every $a \in A\backslash\{0\}$ there exists $a^{-1} \in A$ with $aa^{-1} = 1 = a^{-1}a$. A *central algebra* satisfies $Z(A) = F$, and a *simple algebra* contains no non-trivial 2-sided ideals. If F is a field then the set of two-by-two matrices with entries in F, $M_2(F)$, is an associative F-algebra. Note that $M_2(F)$ is not a division algebra.

Definition 5.3. *Let F be a field with $a, b \in F^\star$. A quaternion algebra over F is a 4-dimensional associative F-algebra A with basis $\{1, \mathbf{i}, \mathbf{j}, \mathbf{k}\}$ where 1 is the multiplicative identity of A, and the multiplication of basis elements given by*

$$\mathbf{i}^2 = a\,(= a.1),\ \mathbf{j}^2 = b\,(= b.1),\ \mathbf{ij} = -\mathbf{ji} = \mathbf{k}$$

is extended linearly to A. □

Every quaternion algebra is a 4-dimensional central simple algebra, and conversely every 4-dimensional central simple algebra is isomorphic to a quaternion algebra [**Pi**]. The quaternion algebra of Definition 5.3 will be denoted by the Hilbert symbol

$$\left(\frac{a, b}{F}\right)$$

and we note that by choosing different pairs of basis elements from $\{\mathbf{i}, \mathbf{j}, \mathbf{k}\}$, the Hilbert symbols

$$\left(\frac{a,b}{F}\right), \left(\frac{b,a}{F}\right), \left(\frac{a,-ab}{F}\right) \qquad 5.4$$

define isomorphic quaternion algebras. For any $\lambda, \mu \in F^\star$ the quaternion algebras

$$\left(\frac{a,b}{F}\right), \quad \left(\frac{\lambda^2 a, \mu^2 b}{F}\right) \qquad 5.5$$

are also isomorphic [**Pi**, p.19]. If $A = \left(\frac{a,b}{F}\right)$ is a quaternion algebra, then the map $f : A \to M_2(F(\sqrt{a}))$ given by

$$1 \longmapsto \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad \mathbf{i} \longmapsto \begin{pmatrix} \sqrt{a} & 0 \\ 0 & -\sqrt{a} \end{pmatrix}$$

$$\mathbf{j} \longmapsto \begin{pmatrix} 0 & 1 \\ b & 0 \end{pmatrix} \qquad \mathbf{k} \longmapsto \begin{pmatrix} 0 & \sqrt{a} \\ -b\sqrt{a} & 0 \end{pmatrix}$$

defines an isomorphism of A into $M_2(F(\sqrt{a}))$ where

$$f(x) = \begin{pmatrix} x_0 + x_1\sqrt{a} & x_2 + x_3\sqrt{a} \\ b(x_2 - x_3\sqrt{a}) & x_0 - x_1\sqrt{a} \end{pmatrix} \qquad 5.6$$

for $x = x_0 + x_1\mathbf{i} + x_2\mathbf{j} + x_3\mathbf{k} \in A$. If $a \in (F^\star)^2$, so that $F(\sqrt{a}) = F$, then f defines an isomorphism between A and $M_2(F)$ (to prove this one can show that $f(1), f(\mathbf{i}), f(\mathbf{j}), f(\mathbf{k})$ are linearly independent over F and hence generate a 4-dimensional F-module which, by a dimension count, must be equal to $M_2(F)$).

Any quaternion algebra $\left(\frac{a,b}{\mathbf{R}}\right)$ is isomorphic to one of

$$\left(\frac{-1,-1}{\mathbf{R}}\right), \left(\frac{-1,1}{\mathbf{R}}\right), \left(\frac{1,-1}{\mathbf{R}}\right), \left(\frac{1,1}{\mathbf{R}}\right)$$

by 5.5. Now $H = \left(\frac{-1,-1}{\mathbf{R}}\right)$ is a division algebra corresponding to Hamilton's quaternions. If we set $a = 1 = b$ then the map f in 5.6 defines an isomorphism between $\left(\frac{1,1}{\mathbf{R}}\right)$ and $M_2(\mathbf{R})$, which is not a division algebra. Finally, $\left(\frac{1,-1}{\mathbf{R}}\right)$ and $\left(\frac{-1,1}{\mathbf{R}}\right)$ are isomorphic to $M_2(\mathbf{R})$ by 5.4. Hence up to isomorphism there are only two distinct quaternion algebras defined over the real number field: Hamilton's quaternions H and the matrix algebra $M_2(\mathbf{R})$. The following theorem is a special case of Wedderburn's Structure Theorem (see [**Pi**]).

Theorem 5.7. *Any quaternion algebra over a field F is either a division algebra, or is isomorphic to $M_2(F)$.* □

If $A = \left(\frac{a,b}{F}\right)$ is a quaternion algebra with $x = x_0 + x_1\mathbf{i} + x_2\mathbf{j} + x_3\mathbf{k} \in A$, then the *conjugate* of x in A is defined to be $\overline{x} = x_0 - x_1\mathbf{i} - x_2\mathbf{j} - x_3\mathbf{k}$. We then define

$$\begin{aligned}\mathrm{Trd}(x) &= x + \overline{x} = 2x_0 \\ \mathrm{Nrd}(x) &= x\overline{x} = x_0^2 - ax_1^2 - bx_2^2 + abx_3^2\end{aligned}$$

where $\mathrm{Trd}(x)$ is the reduced trace and $\mathrm{Nrd}(x)$ the reduced norm of x. One can check that the reduced trace and reduced norm correspond to the trace and determinant respectively of the matrix representation for A given in 5.6.

Theorem 5.8. *A quaternion algebra A is a division algebra if and only if $Nrd(x) \neq 0$ for all $x \in A\backslash\{0\}$.*

Proof. If $\mathrm{Nrd}(x) \neq 0$ for all $x \in A\backslash\{0\}$, then since $\mathrm{Nrd}(x) = x\overline{x} \in F$ we have $x^{-1} = \overline{x}/\mathrm{Nrd}(x)$ and A is a division algebra. Conversely, if A is a division algebra, then every $x \in A\backslash\{0\}$ has an inverse $x^{-1} \in A$ for which $xx^{-1} = 1$. One can check that $\mathrm{Nrd}(xy) = \mathrm{Nrd}(x)\mathrm{Nrd}(y)$ for all $x, y \in A$, and so $\mathrm{Nrd}(x)\mathrm{Nrd}(x^{-1}) = \mathrm{Nrd}(1) = 1$ whence $\mathrm{Nrd}(x) \neq 0$. □

Now suppose that F is a number field with the ring of integers O_F, and that A is a quaternion algebra over F. Then $x \in A$ is an *integer element* of A if

$$\mathrm{Trd}(x) \in O_F \text{ and } \mathrm{Nrd}(x) \in O_F.$$

The integer elements of A do not necessarily form a ring, and so we study subsets of integer elements called orders.

Definition 5.9. *Let A be a quaternion algebra over a field F. An order O in A is a finitely generated ring of integer elements of A, containing O_F, such that $F.O = A$.*
□

If K is a field extension of F, then the tensor product of $A = \left(\frac{a,b}{F}\right)$ with K over F gives

$$\left(\frac{a,b}{F}\right) \otimes_F K \cong \left(\frac{a,b}{K}\right)$$

which will be denoted A_K. Thus A can be embedded naturally into A_K. If we assume further that F is a totally real number field of degree $n = |F : \mathbf{Q}|$, then there are n distinct Galois embeddings

$$\phi_i : F \to \mathbf{R} \qquad 1 \leq i \leq n \qquad 5.10$$

where we take ϕ_1 to be the identity. For each embedding ϕ_i we define A_{ϕ_i} to be the tensor product

$$\left(\frac{a,b}{F}\right) \otimes_{\phi_i} \mathbf{R} \cong \left(\frac{\phi_i(a), \phi_i(b)}{\mathbf{R}}\right) \qquad 1 \leq i \leq n$$

and say that A is *ramified* at ϕ_i if A_{ϕ_i} is isomorphic to Hamilton's quaternions H, and *unramified* at ϕ_i if A_{ϕ_i} is isomorphic to $M_2(\mathbf{R})$.

Let F be a totally real number field with n distinct embeddings $\phi_i : F \to \mathbf{R}$ as defined above. If A is a quaternion algebra over F, unramified at ϕ_1 but ramified at all other ϕ_i $(1 < i \leq n)$ then $A_{\phi_1} \cong M_2(\mathbf{R})$; we let $\rho : A \to M_2(\mathbf{R})$ be the restriction of this isomorphism to A, and note that ρ is uniquely determined up to $\mathrm{GL}_2(\mathbf{R})$-conjugation [**Ta1**]. If O is an order in A and

$$O^1 = \{x \in O \mid \mathrm{Nrd}(x) = 1\}$$

is the group of units in O, then $\rho(O^1) = \Gamma(A, O)$ is a subgroup of $\mathrm{SL}_2(\mathbf{R})$. The following theorem is a special case of [**Vi**, p.104, Theorem 1.1].

Theorem 5.11. *The group $\Gamma(A, O)$ is a discrete subgroup of $SL_2(\mathbf{R})$ with finite covolume, cocompact if A is a division algebra.* □

If we take the natural map $P : \mathrm{SL}_2(\mathbf{R}) \to \mathrm{PSL}_2(\mathbf{R})$, then $P\Gamma(A, O)$ is a discrete subgroup of $\mathrm{PSL}_2(\mathbf{R})$ with finite covolume and hence is a Fuchsian group of the first kind ($P\Gamma(A, O)$ is sometimes written Γ^1_O, see [**Bor**]). A Fuchsian group is said to be *derived from a quaternion algebra* if it is a finite index subgroup of some $P\Gamma(A, O)$. As an example we let $A = M_2(\mathbf{Q})$, the matrix quaternion algebra over $\mathbf{Q}$. Then $O = M_2(\mathbf{Z})$ is a maximal order in $M_2(\mathbf{Q})$ [**Vi**, p.25] and $O^1 = \mathrm{SL}_2(\mathbf{Z})$. Hence $P\Gamma(A, O) = \mathrm{PSL}_2(\mathbf{Z})$, the modular group $(0; 2, 3, \infty)$. Two groups are said to be *commensurable* if their intersection has finite index in both of them.

Definition 5.12. *A Fuchsian group Γ is said to be arithmetic if it is commensurable with some $P\Gamma(A, O)$.* □

Since an arithmetic Fuchsian group Γ is commensurable with a Fuchsian group of the first kind, Γ must also be of the first kind. If we let $\mathrm{tr}\gamma$ denote the trace of $\gamma \in \mathrm{PSL}_2(\mathbf{R})$ (this is slightly ambiguous, see [**MR1**]) then Takeuchi [**Ta1**] gives the following characterization of arithmetic Fuchsian groups in terms of their trace fields:

Theorem 5.13. *Let Γ be a Fuchsian group of the first kind. Then Γ is an arithmetic Fuchsian group if and only if*

(i) $k_1 = \mathbf{Q}(\mathrm{tr}\gamma \,|\, \gamma \in \Gamma)$ is an algebraic number field and the set of traces $tr\Gamma \subset O_{k_1}$, where O_{k_1} is the ring of integers in k_1, and

(ii) if $k_2 = \mathbf{Q}((\mathrm{tr}\gamma)^2 \,|\, \gamma \in \Gamma)$ then for every isomorphism $\phi : k_1 \to \mathbf{C}$ such that $\phi\big|_{k_2}$ is not the identity, $\phi(\mathrm{tr}\Gamma)$ is bounded in $\mathbf{C}$. □

If Γ is a Fuchsian group of the first kind, then let $\Gamma^{(2)}$ denote the subgroup of Γ generated by the set of elements $\{\gamma^2 \,|\, \gamma \in \Gamma\}$.

Theorem 5.14. [**Ta1**] *Let Γ be a Fuchsian group of the first kind. Then Γ is arithmetic if and only if $\Gamma^{(2)}$ is derived from a quaternion algebra.* □

If Γ is an arithmetic Fuchsian group, then $\Gamma^{(2)}$ is a finite index subgroup of some $P\Gamma(A, O)$ by Theorem 5.14. Takeuchi [**Ta1**] shows that the vector space spanned by $\Gamma^{(2)}$ over the field $k_2' = \mathbf{Q}(\mathrm{tr}\gamma \,|\, \gamma \in \Gamma^{(2)})$,

$$A(\Gamma^{(2)}) = \left\{ \sum a_i \gamma_i \,\middle|\, a_i \in k_2',\ \gamma_i \in \Gamma^{(2)} \right\}$$

is a quaternion algebra isomorphic to A. Therefore, the quaternion algebra associated to an arithmetic Fuchsian group Γ is uniquely determined up to isomorphism by Γ.

Theorem 5.15. [**Ta3**] *Let Γ_1 and Γ_2 be arithmetic Fuchsian groups. Then Γ_1 is commensurable with a $PGL_2(\mathbf{R})$-conjugate of Γ_2 if and only if their associated quaternion algebras are isomorphic.* □

For a Fuchsian group Γ, the *commensurator* of Γ is defined to be

$$\mathrm{Comm}(\Gamma) = \{t \in \mathrm{PGL}_2(\mathbf{R}) \,\big|\, \Gamma \text{ and } t\Gamma t^{-1} \text{ are commensurable}\}.$$

Note that $\Gamma \leq N_{\mathrm{PGL}_2(\mathbf{R})}(\Gamma) \leq \mathrm{Comm}(\Gamma)$ for all such Γ. We also note that if Γ_1 and Γ_2 are commensurable, then $\mathrm{Comm}(\Gamma_1) = \mathrm{Comm}(\Gamma_2)$. A result of Margulis (see [**Zi**]) gives an alternative characterization of arithmetic Fuchsian groups in terms of their commensurators:

Theorem 5.16. *Let Γ be a Fuchsian group of the first kind. Then Γ is a finite index subgroup of $Comm(\Gamma)$ if and only if Γ is non-arithmetic.* □

If $(m, 2, n)$ represents the triangle group $\Gamma(m, 2, n)$, then the following is a complete list of cocompact arithmetic Fuchsian triangle groups with one elliptic period equal to 2 (see [**Ta2**]):

$$\begin{gathered}(3,2,7),(3,2,8),(3,2,9),(3,2,10),(3,2,11),(3,2,12),(3,2,14),(3,2,16),\\(3,2,18),(3,2,24),(3,2,30),(4,2,5),(4,2,6),(4,2,7),(4,2,8),(4,2,10),\\(4,2,12),(4,2,18),(5,2,5),(5,2,6),(5,2,8),(5,2,10),(5,2,20),(5,2,30),\\(6,2,6),(6,2,8),(6,2,12),(7,2,7),(7,2,14),(8,2,8),(8,2,16),(9,2,18),\\(10,2,10),(12,2,12),(12,2,24),(15,2,30),(18,2,18).\end{gathered} \quad 5.17$$

We now suppose that Γ is a non-arithmetic cocompact Fuchsian triangle group with one elliptic period equal to 2, say $\Gamma = \Gamma(m, 2, n)$. We define $\mathrm{Comm}^+(\Gamma) = \mathrm{Comm}(\Gamma) \cap \mathrm{PSL}_2(\mathbf{R})$. Now Γ has finite index in $\mathrm{Comm}(\Gamma)$ by Theorem 5.16 and so the inclusion $\Gamma \leq \mathrm{Comm}^+(\Gamma)$ must also have finite index; as a consequence $\mathrm{Comm}^+(\Gamma)$ will also be a Fuchsian triangle group (see [**Sin2**]). Following [**Sin2**] we define a Fuchsian group to be maximal if there does not exist another Fuchsian group containing it with finite index. If we assume (without loss of generality) that $m \geq n$, then from [**Sin2**] $\Gamma(m, 2, n)$ will satisfy exactly one of the following:

(i) $\Gamma = \Gamma(m, 2, n)$ is maximal and so $\mathrm{Comm}^+(\Gamma) = \Gamma(m, 2, n)$;

(ii) $m = n$ with $\Gamma = \Gamma(n, 2, n) \lhd_2 \Gamma(4, 2, n)$. Since $\Gamma(4, 2, n)$ is maximal, $\mathrm{Comm}^+(\Gamma) = \Gamma(4, 2, n)$; 5.18

(iii) $m = 2n$ with $\Gamma = \Gamma(2n, 2, n) <_3 \Gamma(3, 2, 2n)$. Since $\Gamma(3, 2, 2n)$ is maximal, $\mathrm{Comm}^+(\Gamma) = \Gamma(3, 2, 2n)$.

We note that if Γ is a maximal Fuchsian triangle group, then the extended triangle group $\Gamma^\star$ will be maximal in $\mathrm{PGL}_2(\mathbf{R})$. If Γ is an arithmetic Fuchsian group, then $\mathrm{Comm}(\Gamma)$ is dense in $\mathrm{PGL}_2(\mathbf{R})$ and so $\mathrm{Comm}^+(\Gamma)$ is dense in $\mathrm{PSL}_2(\mathbf{R})$.

5.3. Arithmetic and non-arithmetic maps

A map $\mathcal{M}$ of type (m, n) will have a canonical map subgroup $M \leq \Gamma(m, 2, n)$ and as shown in §2.1 can be embedded naturally into the Riemann surface $X = \mathcal{U}/M$. In this way every map $\mathcal{M}$ is associated to a unique (up to conformal equivalence) Riemann surface $X = X(\mathcal{M})$. We therefore have a well defined function

$$R : \mathcal{M} \longmapsto X(\mathcal{M}) \quad 5.19$$

from isomorphism classes of maps to conformal equivalence classes of Riemann surfaces; note that by Belyĭ's Theorem all of the Riemann surfaces $X(\mathcal{M})$ will be defined over the algebraic numbers $\overline{\mathbf{Q}}$, and so R is not surjective. It was proved in Chapter 4 that every genus 1 Riemann surface with a modulus $\tau \in \mathbf{Q}(i)$ or $\tau \in \mathbf{Q}(\rho)$ carries a unique minimal map $\mathcal{M}_\tau$, and that two minimal maps are isomorphic if and only if their underlying Riemann surfaces are conformally equivalent. Hence R is injective on the set of genus 1 minimal maps of type $(4,4)$ or $(6,3)$.

A map of type (m,n) where $\frac{1}{m}+\frac{1}{n}<\frac{1}{2}$ corresponds to a finite index inclusion $M \leq \Gamma(m,2,n)$ of a Fuchsian triangle group. In this section we will see that the dichotomy between arithmetic and non-arithmetic Fuchsian triangle groups extends in some way to the maps they represent.

Definition 5.20. *Let $\mathcal{M}$ be a map of type (m,n) where $\frac{1}{m}+\frac{1}{n}<\frac{1}{2}$. Then $\mathcal{M}$ will be called an arithmetic map if $\Gamma(m,2,n)$ is one of the arithmetic triangle groups listed in 5.17; otherwise $\mathcal{M}$ will be called a non-arithmetic map. We will say that $\mathcal{M}$ has maximal type (m,n) if $\Gamma(m,2,n)$ is a maximal Fuchsian group in the sense of* [**Sin2**]. □

By the classification of non-arithmetic Fuchsian triangle groups in 5.18 above, a non-arithmetic map $\mathcal{M}$ of type (m,n) will satisfy exactly one of the following:

(i) $\Gamma(m,2,n)$ is maximal, so $\mathcal{M}$ is a non-arithmetic map of maximal type (m,n);

(ii) $m=n$ so that $\Gamma(n,2,n) \leq \Gamma(4,2,n)$ and by Lemma 2.35 the type 1 truncation $T_1(\mathcal{M})$ is a non-arithmetic map of maximal type $(4,n)$;

(iii) $m=2n$ so that $\Gamma(2n,2,n) \leq \Gamma(3,2,2n)$ and by Lemma 2.41 the type 2 truncation $T_2(\mathcal{M})$ is a non-arithmetic map of maximal type $(3,2n)$;

(iv) $n=2m$ so that the dual $D(\mathcal{M})$ is a map of type $(2m,m)$ and by (iii) above $T_2D(\mathcal{M})$ is a non-arithmetic map of maximal type $(3,2m)$.

Hence any non-arithmetic map easily extends to a non-arithmetic map of maximal type using the operations of duality and truncation defined in §2.4. Let $\mathcal{M}$ be a non-arithmetic map of maximal type (m,n) with a dual $D(\mathcal{M})$ of type (n,m). Since $m \neq n$, $\mathcal{M}$ and $D(\mathcal{M})$ are non-isomorphic maps lying on conformally equivalent Riemann surfaces (see Lemma 2.36). This is the only way in which two non-arithmetic uniform maps of maximal type can lie on conformally equivalent Riemann surfaces:

Theorem 5.21. *Let $\mathcal{M}_1$ and $\mathcal{M}_2$ be non-arithmetic uniform maps of maximal* *type. Then $X(\mathcal{M}_1) \cong X(\mathcal{M}_2)$ if and only if $\mathcal{M}_1$ and $\mathcal{M}_2$ are isomorphic or dual.*

Proof. It is clear that $X(\mathcal{M}_1) \cong X(\mathcal{M}_2)$ for $\mathcal{M}_1 \cong \mathcal{M}_2$ or $\mathcal{M}_1 \cong D(\mathcal{M}_2)$. If $m \neq n$, then every finite index inclusion $M \leq \Gamma(m,2,n)$ defines two non-isomorphic maps: one of type (m,n) and its dual of type (n,m). We note that for $t \in \mathrm{PSL}_2(\mathbf{R})$, the inclusions $M \leq \Gamma(m,2,n)$ and $tMt^{-1} \leq t\Gamma(m,2,n)t^{-1}$ define isomorphic maps. Suppose that $\mathcal{M}_1$ and $\mathcal{M}_2$ are non-arithmetic uniform maps of maximal type for which $X(\mathcal{M}_1) \cong X(\mathcal{M}_2)$. Then $\mathcal{M}_1, D(\mathcal{M}_1)$ and $\mathcal{M}_2, D(\mathcal{M}_2)$ correspond to (finite index) surface group inclusions $M_1 \leq \Gamma_1$ and $M_2 \leq \Gamma_2$ respectively where Γ_1 and Γ_2 are maximal non-arithmetic Fuchsian triangle groups. If $\mathbf{H}/M_1$ and $\mathbf{H}/M_2$ are conformally equivalent, then by Theorem 1.12 there exists some $t \in \mathrm{PSL}_2(\mathbf{R})$ for which $M_1 = tM_2t^{-1}$, and so $M_1 \leq \Gamma_1 \cap t\Gamma_2t^{-1}$. Now Γ_1 and $t\Gamma_2t^{-1}$ are maximal, non-arithmetic and commensurable so that by Theorem 5.16

$$\Gamma_1 = \mathrm{Comm}^+(\Gamma_1) = \mathrm{Comm}^+(t\Gamma_2t^{-1}) = t\Gamma_2t^{-1}$$

and hence $\Gamma_1 = t\Gamma_2t^{-1}$. Thus the maps $\mathcal{M}_2$ and $D(\mathcal{M}_2)$ associated to the inclusion $M_2 \leq \Gamma_2$ are isomorphic to the maps of the inclusion $M_1 = tM_2t^{-1} \leq t\Gamma_2t^{-1} = \Gamma_1$ which are just $\mathcal{M}_1$ and $D(\mathcal{M}_1)$. □

In particular, the function R defined in 5.19 is injective on the set of non-arithmetic uniform maps of maximal type (m,n) where $m < n$. The absolute Galois group $\mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q})$ acts on dessins, taking uniform maps to uniform maps. If $\mathcal{M}$ is a non-arithmetic uniform map of maximal type (m,n), then by Theorem 2.28 every map in the Galois orbit of $\mathcal{M}$ must also be a non-arithmetic uniform map of maximal type (m,n).

Corollary 5.22. *Let $\mathcal{M}$ be a non-arithmetic uniform map of maximal type. Then there is a one-to-one correspondence between isomorphism classes of maps in the Galois orbit of $\mathcal{M}$ and conformal equivalence classes of Riemann surfaces in the Galois orbit of $X(\mathcal{M})$.*

Proof. We restate the corollary: If $\mathcal{M}_1$ and $\mathcal{M}_2$ are non-arithmetic uniform maps of maximal type in the same Galois orbit, then $X(\mathcal{M}_1) \cong X(\mathcal{M}_2)$ if and only if $\mathcal{M}_1 \cong \mathcal{M}_2$. Clearly if $\mathcal{M}_1 \cong \mathcal{M}_2$ then $X(\mathcal{M}_1) \cong X(\mathcal{M}_2)$. By Theorem 5.21 if $X(\mathcal{M}_1) \cong X(\mathcal{M}_2)$ then either $\mathcal{M}_1 \cong \mathcal{M}_2$ or $\mathcal{M}_1 \cong D(\mathcal{M}_2)$. Being in the same Galois orbit, $\mathcal{M}_1$ and $\mathcal{M}_2$ must have the same type, say (m,n). Since $m \neq n$ for non-arithmetic maps of maximal type, $D(\mathcal{M}_2)$ has type $(n,m) \neq (m,n)$. Thus $\mathcal{M}_1$

and $D(\mathcal{M}_2)$ have different map types and so cannot lie in the same Galois orbit. We deduce that $\mathcal{M}_1 \cong \mathcal{M}_2$. □

Hence an element of $\mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q})$ acts non-trivially on a non-arithmetic uniform map $\mathcal{M}$ of maximal type if and only if it acts non-trivially on $X(\mathcal{M})$. A non-arithmetic uniform map of maximal type also has the property that it completely determines the automorphisms of its underlying Riemann surface.

Theorem 5.23. *Let $\mathcal{M}$ be a non-arithmetic uniform map of maximal type. Then every conformal (resp. anti-conformal) automorphism of $X(\mathcal{M})$ can be realized by a conformal (resp. anti-conformal) automorphism of $\mathcal{M}$.*

Proof. Let automorphism mean conformal or anti-conformal automorphism. We recall that every automorphism of $\mathcal{M}$ extends naturally to an automorphism of $X(\mathcal{M})$. The non-arithmetic uniform map $\mathcal{M}$ of maximal type corresponds to a (finite index) surface group inclusion $M \leq \Gamma$ where Γ is a maximal non-arithmetic Fuchsian triangle group. If $\Gamma^\star$ is the extended triangle group with $\Gamma \leq_2 \Gamma^\star \leq \mathrm{PGL}_2(\mathbf{R})$, then since Γ is maximal in $\mathrm{PSL}_2(\mathbf{R})$, $\Gamma^\star$ is maximal in $\mathrm{PGL}_2(\mathbf{R})$. Now $\overline{\mathrm{Aut}}\,\mathcal{M} \cong N_{\Gamma^\star}(M)/M$ and $\overline{\mathrm{Aut}}\,X(\mathcal{M}) \cong N_{\mathrm{PGL}_2(\mathbf{R})}(M)/M$ where of course $N_{\mathrm{PGL}_2(\mathbf{R})}(M) \leq \mathrm{Comm}(\Gamma)$. Note that Γ is non-arithmetic and so the inclusions $M \leq \Gamma \leq \Gamma^\star \leq \mathrm{Comm}(\Gamma)$ all have finite index. Since $\Gamma^\star$ is maximal, $\Gamma^\star = \mathrm{Comm}(\Gamma)$. Hence $N_{\mathrm{PGL}_2(\mathbf{R})}(M) \leq \Gamma^\star$ and we deduce that $\overline{\mathrm{Aut}}\,\mathcal{M} \cong \overline{\mathrm{Aut}}\,X(\mathcal{M})$. □

Singerman [**Sin4**] has proved that a regular map $\mathcal{M}$ is hyperelliptic if and only if $X(\mathcal{M})$ is hyperelliptic. Using Theorem 5.23 we extend this to non-arithmetic uniform maps of maximal type, and deduce a similar result for reflexible maps.

Corollary 5.24. *Let $\mathcal{M}$ be a non-arithmetic uniform map of maximal type. Then*

(i) $\mathcal{M}$ is hyperelliptic if and only if $X(\mathcal{M})$ is hyperelliptic;

(ii) $\mathcal{M}$ is reflexible if and only if $X(\mathcal{M})$ is reflexible.

Proof. This follows as an immediate corollary to Theorem 5.23. □

All of the triangle groups listed in Table 9 are arithmetic, and so all of the uniform maps of genus 2 are arithmetic. However, non-arithmetic uniform maps do exist; for example there are 335 non-arithmetic uniform maps of type $(6, 9)$ and genus 3. Indeed, since there are only finitely many arithmetic triangle groups, a uniform map of genus $g \geq 3$ will 'usually' be non-arithmetic.

We have proved that two non-arithmetic uniform maps of maximal type lie on conformally equivalent Riemann surfaces if and only if they are isomorphic or dual, and that a non-arithmetic uniform map $\mathcal{M}$ of maximal type completely determines the automorphisms of $X(\mathcal{M})$. In general, these results do not hold for arithmetic uniform maps; for example all Riemann surfaces of genus 2 are hyperelliptic [**FK**], but not all uniform maps of genus 2 are hyperelliptic (see Table 10). We conclude this chapter with some examples of arithmetic uniform maps; extensive use will be made of Theorem 1.14 and the techniques developed in the proof of Lemma 2.35.

Example 5.25. The arithmetic triangle groups $\Gamma(3,2,18)$ and $\Gamma(3,2,9)$ are commensurable (see the tables given in [**MR2**] and [**Ta3**]); we will use this fact to construct two arithmetic uniform maps that lie on conformally equivalent Riemann surfaces. By Theorem 1.9, $\Gamma(3,2,9)$ contains an index 4 subgroup isomorphic to $\Gamma(3,3,9)$ and $\Gamma(3,2,18)$ contains an index 2 subgroup isomorphic to $\Gamma(3,3,9)$. Fuchsian triangle groups are uniquely determined up to $\mathrm{PSL}_2(\mathbf{R})$-conjugation, and so we may assume that $\Gamma(3,2,9)$ and $\Gamma(3,2,18)$ contain the same $\Gamma(3,3,9)$ triangle group. We take the presentation

$$\Gamma(3,3,9) = < y_1, y_2 \,|\, y_1^3 = y_2^9 = (y_1 y_2)^{-3} = 1 >$$

where $y_0 = (y_1 y_2)^{-1}$. If ϕ_1 and ϕ_2 are permutation representations of $\Gamma(3,3,9)$ given by

$$\begin{array}{llll} \phi_1(y_1) = & (1\,3\,5)(2\,6\,7)(7\,8\,9) & \phi_2(y_1) = & (1\,2\,4)(3\,6\,8)(5\,9\,7) \\ \phi_1(y_2) = & (1\,7\,3\,4\,5\,8\,2\,9\,6) & \phi_2(y_2) = & (1\,9\,3\,4\,7\,8\,2\,5\,6) \\ \phi_1(y_0) = & (1\,2\,4)(3\,6\,8)(5\,9\,7) & \phi_2(y_0) = & (1\,3\,5)(2\,6\,7)(7\,8\,9) \end{array}$$

then by an application of Theorem 1.9, $H_1 = \phi_1^{-1}(\mathrm{stab}(1))$ and $H_2 = \phi_2^{-1}(\mathrm{stab}(1))$ are genus 2 surface groups contained in $\Gamma(3,3,9)$. Taking $\{1, y_2, y_2^2, \ldots, y_2^8\}$ to be

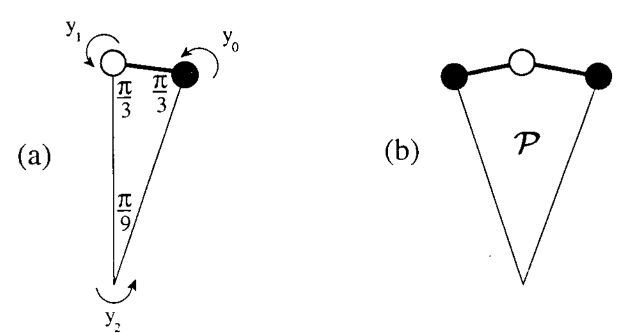
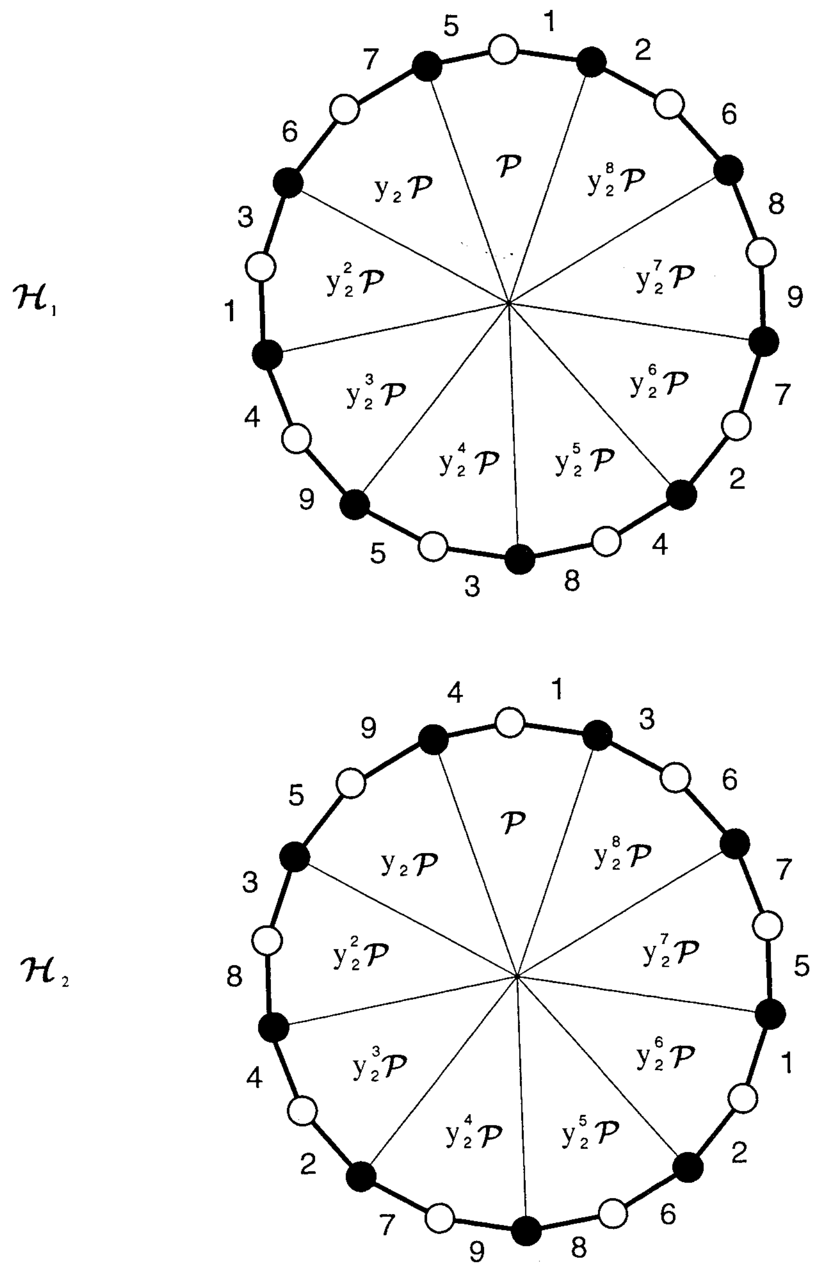


Figure 5.2



$\mathcal{H}_1$
$\mathcal{P}$
$y_2\mathcal{P}$
$y_2^2\mathcal{P}$
$y_2^3\mathcal{P}$
$y_2^4\mathcal{P}$
$y_2^5\mathcal{P}$
$y_2^6\mathcal{P}$
$y_2^7\mathcal{P}$
$y_2^8\mathcal{P}$
$\mathcal{H}_2$
$\mathcal{P}$
$y_2\mathcal{P}$
$y_2^2\mathcal{P}$
$y_2^3\mathcal{P}$
$y_2^4\mathcal{P}$
$y_2^5\mathcal{P}$
$y_2^6\mathcal{P}$
$y_2^7\mathcal{P}$
$y_2^8\mathcal{P}$

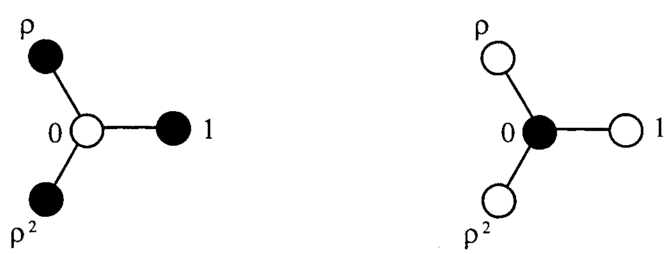
Figure 5.3

a Schreier transversal for H_1 and H_2 in $\Gamma(3,3,9)$, we obtain the corresponding Schreier generators

$$S_1 = \{y_1y_2^7,\ y_2^6y_1y_2,\ y_2^2y_1y_2^5,\ y_2^3y_1y_2^4,\ y_2^4y_1,\ y_2^8y_1y_2^8,\ y_2y_1y_2^3,\ y_2^5y_1y_2^2,\ y_2^7y_1y_2^6\}$$
$$S_2 = \{y_1y_2^3,\ y_2^6y_1y_2^6,\ y_2^2y_1y_2,\ y_2^3y_1,\ y_2^7y_1y_2^8,\ y_2^8y_1y_2^4,\ y_2^4y_1y_2^2,\ y_2^5y_1y_2^7,\ y_2y_1y_2^5\}$$

for H_1 and H_2 respectively. The inclusions $H_1, H_2 \leq \Gamma(3,3,9)$ will correspond to uniform hypermaps $\mathcal{H}_1, \mathcal{H}_2$ of type $(3,3,9)$ lying on the Riemann surfaces $\mathbf{H}/H_1$ and $\mathbf{H}/H_2$. If we take the $\frac{\pi}{3}, \frac{\pi}{3}, \frac{\pi}{9}$ triangle of Figure 5.2(a) and reflect in one side, we obtain the fundamental region $\mathcal{P}$ for $\Gamma(3,3,9)$ shown in Figure 5.2(b); the black and white circles represent hypervertices and hyperedges respectively. To form each hypermap we glue together the regions $\{\mathcal{P}, y_2\mathcal{P}, \ldots, y_2^8\mathcal{P}\}$ corresponding to the Schreier transversal and identify sides according to the Schreier generators. The resulting hypermaps are shown in Figure 5.3; for $1 \leq i \leq 9$ the ith Schreier generator of S_1 will pair the two sides of the 18-gon for $\mathcal{H}_1$ labelled i. Similarly for S_2 and $\mathcal{H}_2$.

Now $\mathcal{H}_1$ and $\mathcal{H}_2$ are non-isomorphic (one can check that their defining permutations are not simultaneously conjugate in S_9, see [**CMa**]), but they are related by Machì's hypermap operation $\mathcal{H}_1 = \mathcal{H}_2^{(01)}$ so that one is obtained from the other by interchanging hypervertices and hyperedges. By Lemmas 2.45 and 2.47, the Walsh doubles of $\mathcal{H}_1$ and $\mathcal{H}_2$ are isomorphic and so H_1 and H_2 will be conjugate inside $\Gamma(3,2,18)$. Therefore $\mathbf{H}/H_1$ and $\mathbf{H}/H_2$ are conformally equivalent Riemann surfaces (one can also see that the side identifications of each 18-gon are the same up to a rotation).



$\mathcal{H}_1/\mathrm{Aut}\,\mathcal{H}_1$ $\qquad$ $\mathcal{H}_2/\mathrm{Aut}\,\mathcal{H}_2$

Figure 5.4

It can be verified (for example by the generalization of Theorem 2.4 to hypermaps) that $\mathcal{H}_1$ and $\mathcal{H}_2$ both have automorphism groups of order 3, corresponding to a $\frac{2\pi}{3}$ rotation about the centre of each 18-gon. The quotient hypermaps $\mathcal{H}_1/\operatorname{Aut}\mathcal{H}_1$ and $\mathcal{H}_2/\operatorname{Aut}\mathcal{H}_2$ have genus zero, and are shown in Figure 5.4 with their 'free' hypervertices and hyperedges drawn at the cube roots of unity $1, \rho, \rho^2$.

Let X be the Riemann surface corresponding to the algebraic curve

$$y^3 = (x-1)^2(x-\rho)(x-\rho^2)$$

with the projection $\pi : X \to \Sigma$, $\pi : (x,y) \mapsto x$. Then $\pi : X \to \Sigma$ is a 3-sheeted cover of the sphere with four branch points of order 2 lying above $1, \rho, \rho^2, \infty$ and so by the Riemann-Hurwitz formula [**JS2**, p.196] X has genus $g = 2$. The genus 0 hypermap $\mathcal{H}_2/\operatorname{Aut}\mathcal{H}_2$ is lifted via π to a hypermap $\mathcal{H} = \pi^{-1}(\mathcal{H}_2/\operatorname{Aut}\mathcal{H}_2)$ lying on X; $\mathcal{H}$ has three hypervertices of valency 3 at $(0,1), (0,\rho), (0,\rho^2)$, three hyperedges of valency 3 at the critical points $(1,0), (\rho,0), (\rho^2,0)$ and one hyperface of valency 9 with centre the point at infinity. Thus $\mathcal{H}$ is a uniform hypermap of type $(3,3,9)$ lying on X.

We note that X has an automorphism of order 3 corresponding to the birational transformation $(x,y) \mapsto (x, \rho y)$ which induces an automorphism of the lifted hypermap $\mathcal{H}$. This automorphism fixes each of the hyperedges and cyclically permutes the hypervertices of $\mathcal{H}$. The quotient of $\mathcal{H}$ by the corresponding group of automorphisms of order three is the genus 0 hypermap $\mathcal{H}_2/\operatorname{Aut}\mathcal{H}_2$.

Using Table 11 we see that there are only four genus 2 uniform hypermaps of type $(3,3,9)$: the two discussed in this example with automorphism groups of order 3, and two with trivial automorphism groups (this can be checked by implementing Program 3 of Appendix II). Since the hypermaps $\mathcal{H}_1/\operatorname{Aut}\mathcal{H}_1$ and $\mathcal{H}_2/\operatorname{Aut}\mathcal{H}_2$ are not isomorphic, we deduce that $\pi^{-1}(\mathcal{H}_2/\operatorname{Aut}\mathcal{H}_2) \cong \mathcal{H}_2$.

Hence the composition of $\pi : X \to \Sigma$ with the map $x \mapsto x^3$ defines a Belyĭ function

$$\beta_{\mathcal{H}_2} : (x,y) \mapsto x^3$$

from X to Σ with $\beta_{\mathcal{H}_2}^{-1}(\mathcal{B}_1)$ isomorphic to $\mathcal{H}_2$. Now $\mathcal{H}_1$ can be obtained from $\mathcal{H}_2$ by interchanging hypervertices and hyperedges, so composing $\beta_{\mathcal{H}_2}$ with the map $x \mapsto 1-x$ gives the Belyĭ function

$$\beta_{\mathcal{H}_1} : (x,y) \mapsto 1 - x^3$$

from X to Σ for $\mathcal{H}_1$. If β is a Belyĭ function for a hypermap $\mathcal{H}$, then $4\beta(1-\beta)$ is a Belyĭ function for the Walsh double $W(\mathcal{H})$ (see §2.4). Therefore $\beta_W : (x,y) \mapsto 4x^3(1-x^3)$ is a Belyĭ function for the isomorphic Walsh doubles of $\mathcal{H}_1$ and $\mathcal{H}_2$.

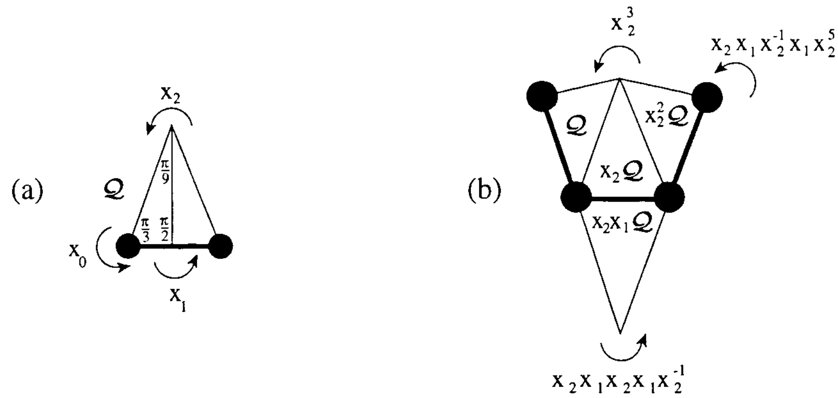


Figure 5.5

Since $\Gamma(3,3,9) \leq \Gamma(3,2,9)$, the subgroups H_1 and H_2 correspond to maps of type $(3,9)$ via the inclusions $H_1, H_2 \leq \Gamma(3,2,9)$. We will construct these maps using the techniques developed in §2.4. Using the presentation

$$\Gamma(3,2,9) = < x_1, x_2 \,|\, x_1^2 = x_2^9 = (x_1x_2)^{-3} = 1 >$$

where $x_0 = (x_1x_2)^{-1}$, Figure 5.5(a) shows a fundamental region $\mathcal{Q}$ for $\Gamma(3,2,9)$ formed by reflecting a $\frac{\pi}{3}, \frac{\pi}{2}, \frac{\pi}{9}$ triangle in one of its sides; one edge of the map is represented in bold. If we take the permutation representation ϕ_3 of $\Gamma(3,2,9)$ given by

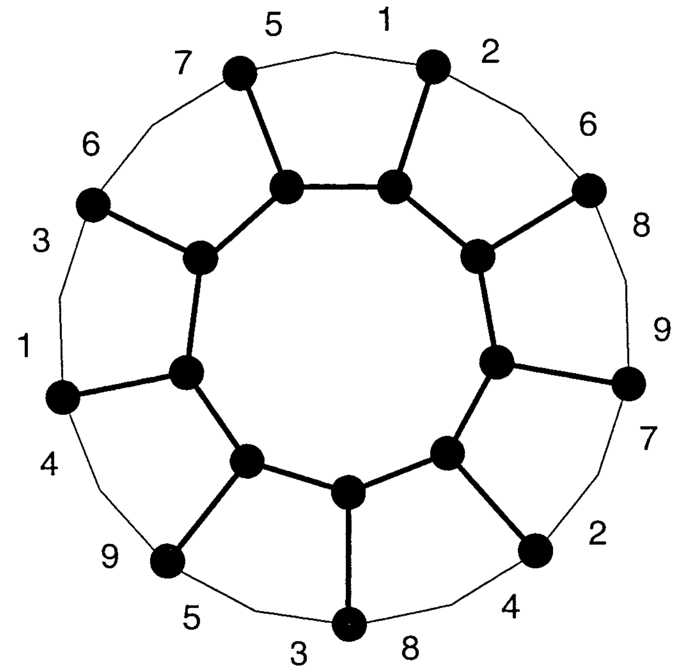
$$\phi_3(x_1) = (1\,3)(2\,4)$$
$$\phi_3(x_2) = (1\,2\,3)(4)$$
$$\phi_3(x_0) = (1)(2\,3\,4)$$

then $\phi_3^{-1}(\text{stab}(1))$ is isomorphic to $\Gamma(3,3,9)$ by Theorem 1.9; by our comments above we take this $\Gamma(3,3,9)$ to be contained in the intersection of $\Gamma(3,2,9)$ and $\Gamma(3,2,18)$. Taking the Schreier transversal for $\Gamma(3,3,9)$ in $\Gamma(3,2,9)$ to be $\{1, x_2, x_2^2, x_2x_1\}$, we obtain the Schreier generators $\{x_2^3, x_2x_1x_2x_1x_2^{-1}\}$. A fundamental region for $\Gamma(3,3,9)$ formed by gluing together the regions

$$\{\mathcal{Q}, x_2\mathcal{Q}, x_2^2\mathcal{Q}, x_2x_1\mathcal{Q}\}$$

is shown in Figure 5.5(b). The Schreier generators pair the sides of this region, and so we may identify them with the generators y_1, y_2 of $\Gamma(3,3,9)$ defined above: $y_1 = x_2^3$, $y_2 = x_2x_1x_2x_1x_2^{-1}$ and $y_0 = (y_1y_2)^{-1} = x_2x_1x_2^{-1}x_1x_2^5$. Hence the maps $\mathcal{M}_1$ and $\mathcal{M}_2$ corresponding to the inclusions $H_1, H_2 \leq \Gamma(3,2,9)$ can be obtained by replacing every fundamental region for $\Gamma(3,3,9)$ in the hypermap pictures of Figure 5.3 with the fundamental region for $\Gamma(3,3,9)$ shown in Figure 5.5(b). The resulting maps are displayed in Figure 5.6.

$\mathcal{M}_1$



$\mathcal{M}_2$

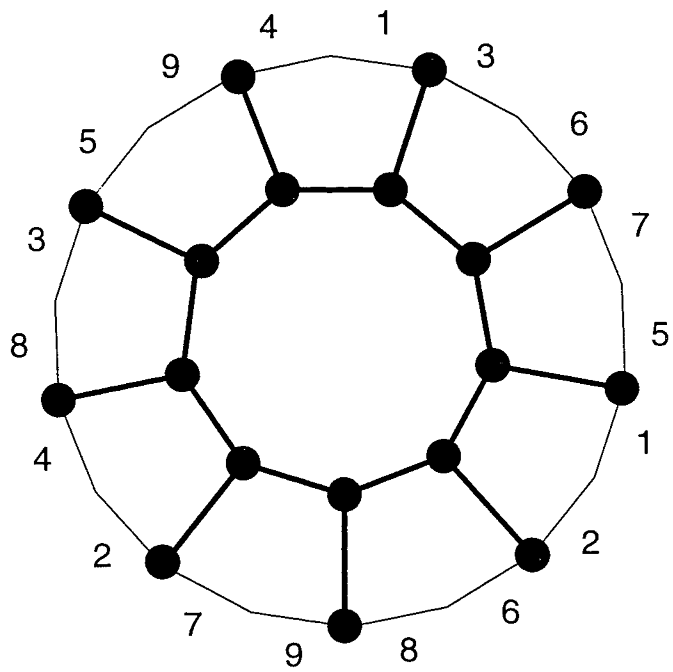


Figure 5.6

Now $\mathcal{M}_1$ and $\mathcal{M}_2$ are non-isomorphic because they have different automorphism groups: $\text{Aut}\,\mathcal{M}_1 \cong C_3$ the cyclic group of order 3, while $\text{Aut}\,\mathcal{M}_2 \cong D_6$ the dihedral group of order 12. Therefore we have constructed two non-isomorphic uniform maps which lie on the conformally equivalent Riemann surfaces $\mathbf{H}/H_1$ and $\mathbf{H}/H_2$. In particular we note that $\mathcal{M}_1$ and $\mathcal{M}_2$ are arithmetic uniform maps.

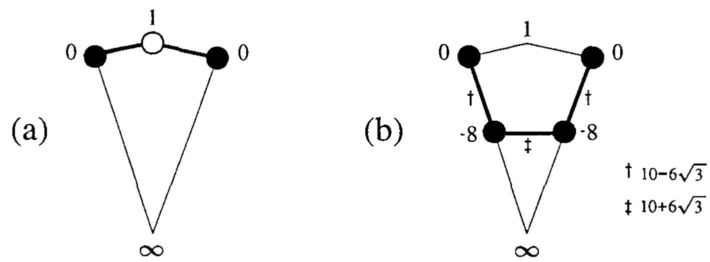


Figure 5.7

The trivial hypermap corresponding to $\Gamma(3,3,9)$ is shown in Figure 5.7(a), and the map corresponding to the inclusion $\Gamma(3,3,9) \leq \Gamma(3,2,9)$ is shown in Figure 5.7(b); note that the sides of each fundamental region must be paired appropriately. The map in Figure 5.7(b) has the corresponding Belyĭ function

$$\beta : x \mapsto \frac{-x(8+x)^3}{64(1-x)^3}$$

with four critical points: of order two at $x = -8$ and $x = 1$, and order one at $x = 10 \pm 6\sqrt{3}$ with $\beta(-8) = 0$, $\beta(10 \pm 6\sqrt{3}) = 1$ and $\beta(1) = \infty$. The only other points sent by β into $\{0, 1, \infty\}$ are $\beta(0) = 0$ and $\beta(\infty) = \infty$. Belyĭ functions for the maps $\mathcal{M}_1$ and $\mathcal{M}_2$ of Figure 5.6 are given by the compositions $\beta \circ \beta_{\mathcal{H}_1}$ and $\beta \circ \beta_{\mathcal{H}_2}$:

$$\beta_{\mathcal{M}_1} = \beta \circ \beta_{\mathcal{H}_1} : (x, y) \longmapsto \frac{(x^3-1)(9-x^3)^3}{64x^9}$$

$$\beta_{\mathcal{M}_2} = \beta \circ \beta_{\mathcal{H}_2} : (x, y) \longmapsto \frac{-x^3(8+x^3)^3}{64(1-x^3)^3}$$

where $\beta_{\mathcal{M}_1}, \beta_{\mathcal{M}_2} : X \to \Sigma$ (see §2.4 or [**Jon**] for more details). □

Example 5.26. An application of Theorem 1.9 shows that $\Gamma(3,2,7)$ and $\Gamma(3,2,14)$ both contain the triangle group $\Gamma(3,3,7)$; Fuchsian triangle groups are uniquely determined up to $\mathrm{PSL}_2(\mathbf{R})$-conjugation, and so we will assume that they contain the same $\Gamma(3,3,7)$. We take the presentation

$$\Gamma(3,2,7) =< x_1, x_2 \,|\, x_1^2 = x_2^7 = (x_1x_2)^{-3} = 1 >$$

where $x_0 = (x_1x_2)^{-1}$. If we define a permutation representation ϕ_1 of $\Gamma(3,2,7)$ by

$$\phi_1(x_1) = (1\,2)(3\,8)(4\,5)(6\,7)$$
$$\phi_1(x_2) = (1)(2\,3\,4\,5\,6\,7\,8)$$
$$\phi_1(x_0) = (1\,2\,3)(4\,8\,6)(5)(7)$$

then $\phi_1^{-1}(\mathrm{Stab}(1))$ is isomorphic to $\Gamma(3,3,7)$ by Theorem 1.9. We take the Schreier transversal

$$\{1, x_1, x_1x_2, x_1x_2^2, x_1x_2^3, x_1x_2^4, x_1x_2^5, x_1x_2^6\}$$

for $\Gamma(3,3,7)$ in $\Gamma(3,2,7)$ and obtain the corresponding Schreier generators

$$\{x_1x_2^2x_1x_2^4x_1, x_1x_2^4x_1x_2^2x_1\}$$

where $x_1x_2^2x_1x_2^4x_1$ and $x_1x_2^4x_1x_2^2x_1$ have order three and their product is equal to x_2^{-1}, which has order seven. Figure 5.8(a) shows a fundamental region $\mathcal{T}$ for $\Gamma(3,2,7)$; the bold line represents one edge of the map. By gluing together the regions $\{\mathcal{T}, x_1\mathcal{T}, \ldots, x_1x_2^6\mathcal{T}\}$ corresponding to the Schreier transversal we obtain a fundamental region $\mathcal{P}$ for $\Gamma(3,3,7)$ whose sides are paired by the Schreier generators, as shown in Figure 5.8(b).

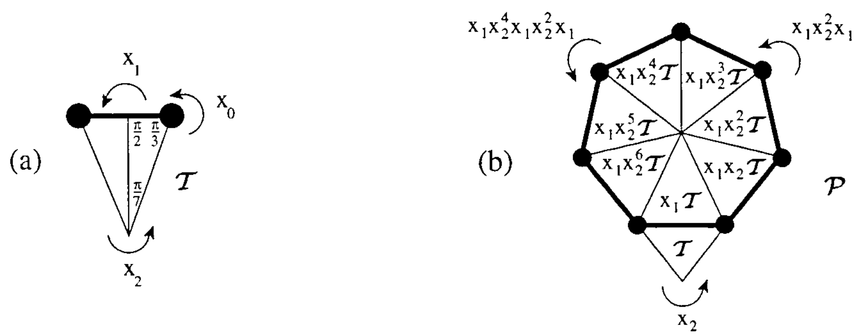


Figure 5.8

We take the following presentation for $\Gamma(3,3,7)$

$$< y_0, y_1, y_2 \,|\, y_0^3 = y_1^3 = y_2^7 = y_0y_1y_2 = 1 >$$

where $\Gamma(3,3,7)$ acts on the $\frac{\pi}{3}, \frac{\pi}{3}, \frac{\pi}{7}$ triangle of Figure 5.9(a) as described in Chapter 1; the black and white circles represent hypervertices and hyperedges respectively. Now $\mathcal{P}$ is a fundamental region for $\Gamma(3,3,7)$ and contains the $\frac{\pi}{3}, \frac{\pi}{3}, \frac{\pi}{7}$ triangle as shown in Figure 5.9(b). Hence we can identify the generators y_0, y_1, y_2 of $\Gamma(3,3,7)$ with the Schreier generators

$$y_0 = x_1x_2^2x_1x_2^4x_1, \quad y_1 = x_1x_2^4x_1x_2^2x_1, \quad y_2 = x_2$$

found above. Therefore if $\mathcal{H}$ is a hypermap corresponding to the inclusion $H \leq \Gamma(3,3,7)$, the map $\mathcal{M}$ corresponding to the inclusion $H \leq \Gamma(3,2,7)$ is formed by replacing every fundamental region for $\mathcal{H}$ shown in Figure 5.9(b) with the fundamental region of Figure 5.8(b).

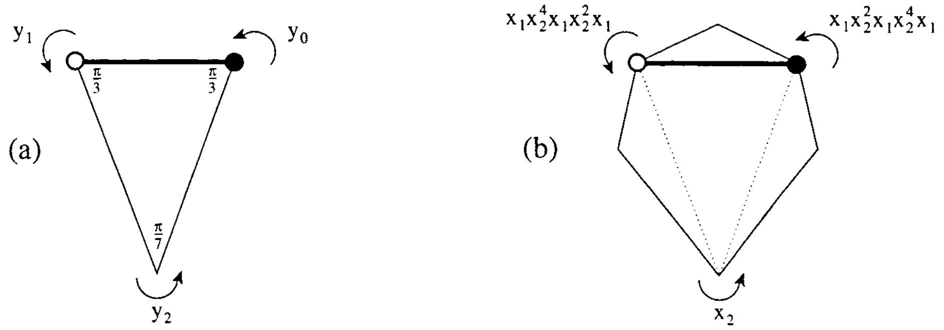


Figure 5.9

Singerman's embedding of the Fano plane as a hypermap $\mathcal{H}$ into Klein's Riemann surface of genus 3 [**Sin3**] is shown in Figure 5.10 (the Fano plane is the underlying hypergraph of $\mathcal{H}$). Now $\mathcal{H}$ corresponds to a permutation representation ϕ_2 of $\Gamma(3,3,7)$ where

$$\phi_2(y_0) = (1,11,17)(2,8,21)(3,12,18)(4,9,15)(5,13,19)(6,10,16)(7,14,20)$$
$$\phi_2(y_1) = (1,15,8)(2,19,12)(3,16,9)(4,20,13)(5,17,10)(6,21,14)(7,1,11)$$
$$\phi_2(y_2) = (1,2,3,4,5,6,7)(8,9,10,11,12,13,14)(15,15,16,17,18,19,20,21)$$

and $H = \phi_2^{-1}(\text{Stab}(1))$ is the hypermap subgroup for $\mathcal{H}$. Now H is a surface group of genus 3 by Theorem 1.9, and $G = \text{gp} < \phi_2(y_0), \phi_2(y_1) >$ is a group of order 21 (in fact $G \cong \frac{1}{2}\text{Aff}(1,7)$, the unique subgroup of index 2 in $\text{Aff}(1,7)$). Therefore H is a normal subgroup of $\Gamma(3,3,7)$ and $\mathcal{H}$ is a regular hypermap by Theorem 2.20.

If we let $\Gamma(3,2,14) =< w_1, w_2 \mid w_1^2 = w_2^{14} = (w_1w_2)^{-3} = 1 >$ then as shown in §2.4 we can assume that $H \leq \Gamma(3,3,7) \leq \Gamma(3,2,14)$. By Lemma 2.47 the conjugate

$H^{w_1} = w_1 H w_1^{-1}$ also lies in $\Gamma(3,3,7)$, and the inclusion $H^{w_1} \leq \Gamma(3,3,7)$ defines the hypermap $\mathcal{H}^{(01)}$ formed by interchanging the hypervertices and hyperedges of $\mathcal{H}$. Now $\mathcal{H}^{(01)}$ is a regular hypermap and has an underlying hypergraph isomorphic to the Fano plane, but $\mathcal{H}$ and $\mathcal{H}^{(01)}$ are not isomorphic hypermaps (to see this one can use MAGMA to construct the subgroups H and H^{w_1} explicitly and check that they are not conjugate in $\Gamma(3,3,7)$). Since H and H^{w_1} are conjugate in $\Gamma(3,2,14)$, the genus 3 Riemann surfaces $\mathbf{H}/H$ and $\mathbf{H}/H^{w_1}$ underlying $\mathcal{H}$ and $\mathcal{H}^{(01)}$ are conformally equivalent.

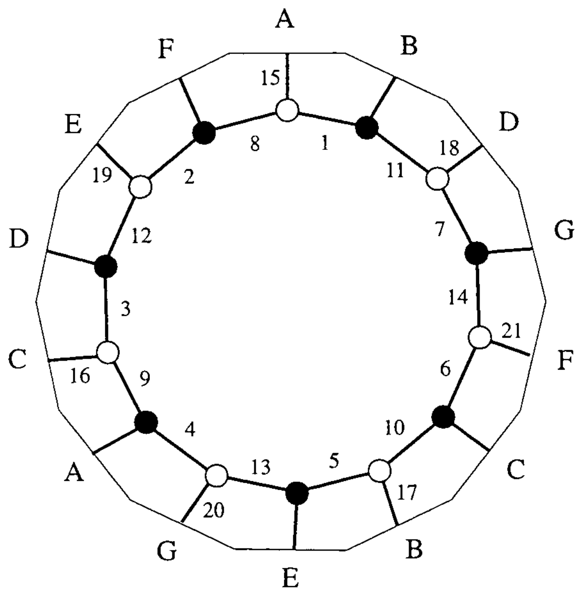


Figure 5.10

The inclusions $H \leq \Gamma(3,2,7)$ and $H^{w_1} \leq \Gamma(3,2,7)$ define the maps $\mathcal{M}_1$ and $\mathcal{M}_2$ respectively which are shown in Figure 5.11 (they are constructed using the procedure described above). Using MAGMA one can check that H is an index 168 normal subgroup of $\Gamma(3,2,7)$. By the Hurwitz bound **[JS2]** a Riemann surface of genus 3 has at most 168 automorphisms, and so $\mathrm{Aut}\,(\mathbf{H}/H) \cong \Gamma(3,2,7)/H$ is a Hurwitz group of order 168; this uniquely determines $\mathbf{H}/H$ as Klein's Riemann surface of genus 3 with $\mathrm{Aut}\,(\mathbf{H}/H) \cong \mathrm{PSL}_2(7)$ (see [**Macb**] and [**Gre**]). Hence $\mathrm{Aut}\,\mathcal{M}_1 \cong \mathrm{PSL}_2(7)$ and $\mathcal{M}_1$ is a regular map. By contrast, the normalizer of H^{w_1} in $\Gamma(3,2,7)$ is equal to $\Gamma(3,3,7)$ and so $\mathrm{Aut}\,\mathcal{M}_2 \cong \frac{1}{2}\mathrm{Aff}(1,7)$. The maps $\mathcal{M}_1$ and $\mathcal{M}_2$ are non-isomorphic because they have non-isomorphic automorphism groups.

We have therefore constructed two arithmetic uniform maps of type $(3,7)$ which can both be embedded into Klein's Riemann surface of genus 3. □

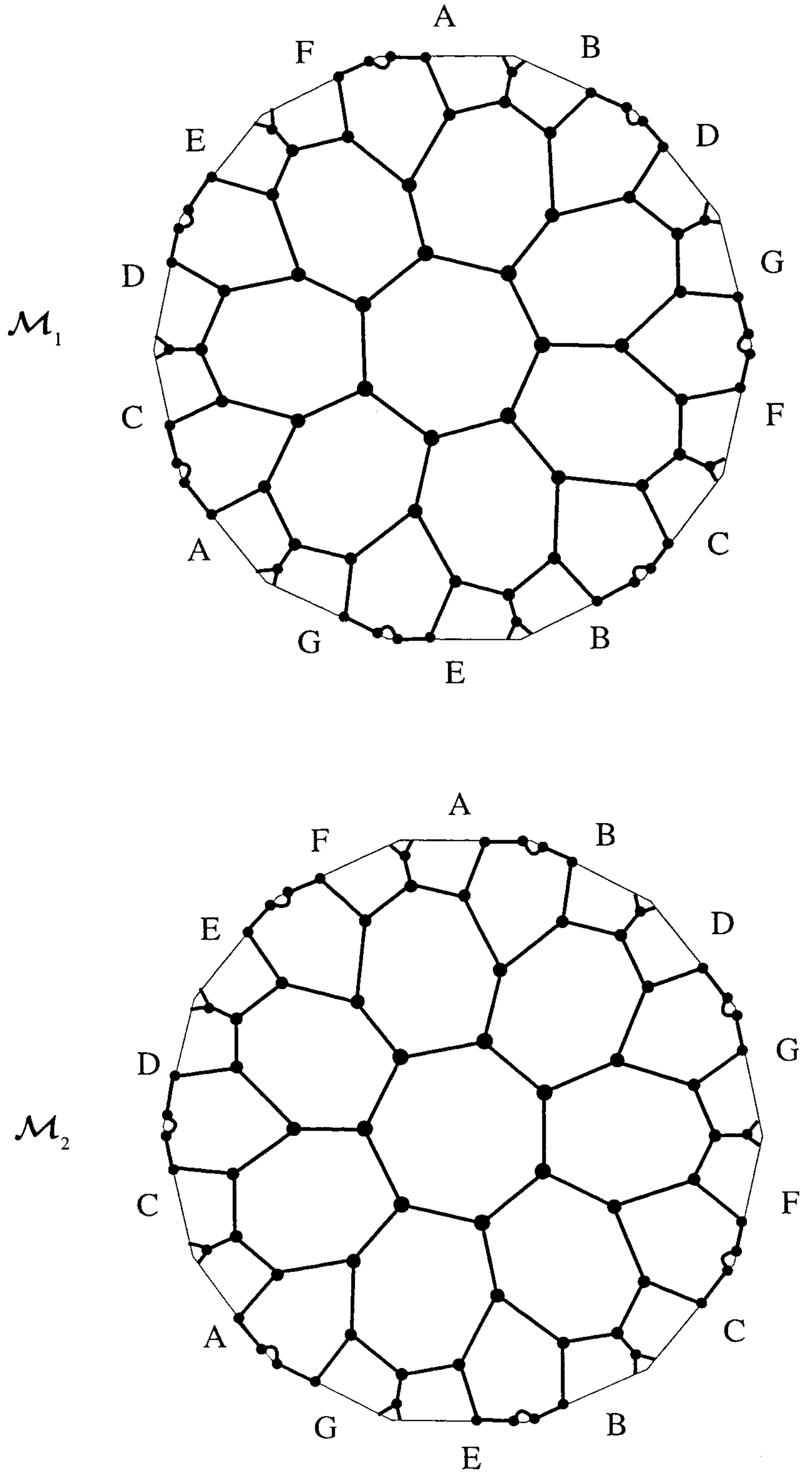


Figure 5.11

Appendix I

Tables

Table 1. The five rational elliptic curves with Euclidean Belyĭ uniformizations (§3.4)

d	τ	$j(\tau)$	Elliptic curve E_τ
-3	ρ	0	$y^2 = 4x^3 - 1$
-4	i	1728	$y^2 = 4x^3 - x$
-12	$1 + 2\rho$	54000	$y^2 = 4x^3 - 15x - 11$
-16	$2i$	287496	$y^2 = 4x^3 - 11x - 7$
-27	$2 + 3\rho$	-12288000	$y^2 = 4x^3 - 120x - 253$

Table 2. The six discriminants of the form $-3m^2$ or $-4m^2$ with class number 2 (§3.4)

d	τ_1	τ_2	$j(\tau_1)$, $j(\tau_2)$ *
-36	$3i$	$\frac{-1+3i}{2}$	$76771008 \pm 44330496\sqrt{3}$
-48	$2 + 4\rho$	$\frac{2+4\rho}{3}$	$1417905000 \pm 818626500\sqrt{3}$
-64	$4i$	$\frac{-1+2i}{2}$	$41113158120 \pm 29071392966\sqrt{2}$
-75	$\frac{1+5\rho}{3}$	$2 + 5\rho$	$-327201914880 \pm 146329141248\sqrt{5}$
-100	$5i$	$\frac{-1+5i}{2}$	$22015749613248 \pm 9845745509376\sqrt{5}$
-147	$\frac{2+7\rho}{3}$	$3 + 7\rho$	$-17424252776448000 \pm 3802283679744000\sqrt{21}$

* $j(\tau_1)$ with positive sign

Table 3. The twelve elliptic curves that are defined over quadratic extensions of $\mathbf{Q}$ *and admit Euclidean Belyĭ uniformizations (§3.4)*

τ	*Elliptic curve* E_τ
$3i$	$y^2 = 1568(44372 - 1767\sqrt{3})x^3 - 759x - 1$
$\frac{-1+3i}{2}$	$y^2 = 1568(44372 + 1767\sqrt{3})x^3 - 759x - 1$
$2+4\rho$	$y^2 = 12(937215 + 323408\sqrt{3})x^3 - 495x - 1$
$\frac{2+4\rho}{3}$	$y^2 = 12(937215 - 323408\sqrt{3})x^3 - 495x - 1$
$4i$	$y^2 = 4(83987 - 53808\sqrt{2})x^3 - 1081x - 77$
$\frac{-1+2i}{2}$	$y^2 = 4(83987 + 53808\sqrt{2})x^3 - 1081x - 77$
$2+5\rho$	$y^2 = 100(3777190 - 165393\sqrt{5})x^3 - 1320x - 1$
$\frac{1+5\rho}{3}$	$y^2 = 100(3777190 + 165393\sqrt{5})x^3 - 1320x - 1$
$5i$	$y^2 = 16(4622595160 - 269169\sqrt{5})x^3 - 46079x - 14$
$\frac{-1+5i}{2}$	$y^2 = 16(4622595160 + 269169\sqrt{5})x^3 - 46079x - 14$
$3+7\rho$	$y^2 = 588(164146563 - 94714\sqrt{21})x^3 - 42840x - 11$
$\frac{2+7\rho}{3}$	$y^2 = 588(164146563 + 94714\sqrt{21})x^3 - 42840x - 11$

Table 4. The two discriminants of the form $-3m^2$ *with class number 3 (§3.4)*

d	τ	$j(\tau)$	*Field*
	$3+6\rho$	$\alpha_1 + \alpha_2 2^{\frac{1}{3}} + \alpha_3 2^{\frac{2}{3}}$	$\mathbf{Q}(2^{\frac{1}{3}})$
-108	$\frac{2+3\rho}{2}$	$\alpha_1 + \alpha_2 2^{\frac{1}{3}}\rho + \alpha_3 2^{\frac{2}{3}}\rho^2$	$\mathbf{Q}(2^{\frac{1}{3}}\rho)$
	$\frac{1+3\rho}{2}$	$\alpha_1 + \alpha_2 2^{\frac{1}{3}}\rho^2 + \alpha_3 2^{\frac{2}{3}}\rho$	$\mathbf{Q}(2^{\frac{1}{3}}\rho^2)$
	$4+9\rho$	$\alpha_4 + \alpha_5 3^{\frac{1}{3}} + \alpha_6 3^{\frac{2}{3}}$	$\mathbf{Q}(3^{\frac{1}{3}})$
-243	$\frac{6+9\rho}{7}$	$\alpha_4 + \alpha_5 3^{\frac{1}{3}}\rho + \alpha_6 3^{\frac{2}{3}}\rho^2$	$\mathbf{Q}(3^{\frac{1}{3}}\rho)$
	$\frac{3+9\rho}{7}$	$\alpha_4 + \alpha_5 3^{\frac{1}{3}}\rho^2 + \alpha_6 3^{\frac{2}{3}}\rho$	$\mathbf{Q}(3^{\frac{1}{3}}\rho^2)$

$\alpha_1 = 50337742902000$, $\alpha_2 = 39953093016000$, $\alpha_3 = 31710790944000$,
$\alpha_4 = -618587635244888064000$, $\alpha_5 = -428904711070941184000$,
$\alpha_6 = -297385917043138560000$.

Table 5. Three elliptic curves that are defined over cubic extensions of $\mathbf{Q}$ *and admit Euclidean Belyĭ uniformizations (§3.4)*

τ	*Elliptic curve* E_τ
$3+6\rho$	$y^2 = 4(761257259 - 157058640\sqrt[3]{2} + 160025472\sqrt[3]{4})x^3 - 2805x - 1$
$\frac{2+3\rho}{2}$	$y^2 = 4(761257259 - 157058640\sqrt[3]{2}\rho + 160025472\sqrt[3]{4}\rho^2)x^3 - 2805x - 1$
$\frac{1+3\rho}{2}$	$y^2 = 4(761257259 - 157058640\sqrt[3]{2}\rho^2 + 160025472\sqrt[3]{4}\rho)x^3 - 2805x - 1$

Table 6. The number of genus 1 uniform maps of index $1 \le n \le 20$ *(§4.5)*

n	$\sigma(n)$	$R(n)$	$S(n)$	$M_{(4,4)}(n)$	$M_{(6,3)}(n)$
1	1	1	1	1	1
2	3	1	0	2	1
3	4	0	1	2	2
4	7	1	1	4	3
5	6	2	0	4	2
6	12	0	0	6	4
7	8	0	2	4	4
8	15	1	0	8	5
9	13	1	1	7	5
10	18	2	0	10	6
11	12	0	0	6	4
12	28	0	1	14	10
13	14	2	2	8	6
14	24	0	0	12	8
15	24	0	0	12	8
16	31	1	1	16	11
17	18	2	0	10	6
18	39	1	0	20	13
19	20	0	2	10	8
20	42	2	0	22	14

Table 7. Some minimal maps and their associated elliptic curves (§4.6)

$\mathcal{M}_\tau$	Elliptic curve equation E_τ
$\mathcal{M}_\rho$	$y^2 = 4x^3 - 1$
$\mathcal{M}_i$	$y^2 = 4x^3 - x$
$\mathcal{M}_{1+2\rho}$	$y^2 = 4x^3 - 15x - 11$
$\mathcal{M}_{2i}$	$y^2 = 4x^3 - 11x - 7$
$\mathcal{M}_{2+3\rho}$	$y^2 = 4x^3 - 120x - 253$
$\mathcal{M}_{3i}$	$y^2 = 1568(44372 - 1767\sqrt{3})x^3 - 759x - 1$
$\mathcal{M}_{\frac{-1+3i}{2}}$	$y^2 = 1568(44372 + 1767\sqrt{3})x^3 - 759x - 1$
$\mathcal{M}_{2+4\rho}$	$y^2 = 12(937215 + 323408\sqrt{3})x^3 - 495x - 1$
$\mathcal{M}_{\frac{2+4\rho}{3}}$	$y^2 = 12(937215 - 323408\sqrt{3})x^3 - 495x - 1$
$\mathcal{M}_{4i}$	$y^2 = 4(83987 - 53808\sqrt{2})x^3 - 1081x - 77$
$\mathcal{M}_{\frac{-1+2i}{2}}$	$y^2 = 4(83987 + 53808\sqrt{2})x^3 - 1081x - 77$
$\mathcal{M}_{\frac{1+5\rho}{3}}$	$y^2 = 100(3777190 + 165393\sqrt{5})x^3 - 1320x - 1$
$\mathcal{M}_{2+5\rho}$	$y^2 = 100(3777190 - 165393\sqrt{5})x^3 - 1320x - 1$
$\mathcal{M}_{5i}$	$y^2 = 16(4622595160 - 269169\sqrt{5})x^3 - 46079x - 14$
$\mathcal{M}_{\frac{-1+5i}{2}}$	$y^2 = 16(4622595160 + 269169\sqrt{5})x^3 - 46079x - 14$
$\mathcal{M}_{\frac{2+7\rho}{3}}$	$y^2 = 588(164146563 + 94714\sqrt{21})x^3 - 42840x - 11$
$\mathcal{M}_{3+7\rho}$	$y^2 = 588(164146563 - 94714\sqrt{21})x^3 - 42840x - 11$
$\mathcal{M}_{3+6\rho}$	$y^2 = 4(761257259 - 157058640\sqrt[3]{2} + 160025472\sqrt[3]{4})x^3 - 2805x - 1$
$\mathcal{M}_{\frac{2+3\rho}{2}}$	$y^2 = 4(761257259 - 157058640\rho\sqrt[3]{2} + 160025472\rho^2\sqrt[3]{4})x^3 - 2805x - 1$
$\mathcal{M}_{\frac{1+3\rho}{2}}$	$y^2 = 4(761257259 - 157058640\rho^2\sqrt[3]{2} + 160025472\rho\sqrt[3]{4})x^3 - 2805x - 1$

Table 8. Minimal maps and their associated Belyĭ functions (§4.6)

$\mathcal{M}_\tau$	Belyĭ function $\beta : E_\tau \to \Sigma$
$\mathcal{M}_\rho$	$\beta_1 : (x,y) \longmapsto \frac{1}{4x^3}$
$\mathcal{M}_i$	$\beta_2 : (x,y) \longmapsto \frac{1}{4x^2}$
$\mathcal{M}_{1+2\rho}$	$\beta_3 : (x,y) \longmapsto \frac{512(x+1)^3}{(2x+3)^3(2x-1)^3}$
$\mathcal{M}_{2i}$	$\beta_4 : (x,y) \longmapsto \frac{64(x+1)^2}{(2x+1)^4}$
$\mathcal{M}_{2+3\rho}$	$\beta_5 : (x,y) \longmapsto \frac{729(x+3)^6}{4(x^3+6x^2+3x-19)^3}$
$\mathcal{M}_{3i}$	$\beta_6 : (x,y) \longmapsto \frac{1882056627(237+45\sqrt{3}+155848x)^4}{32(3264+3361\sqrt{3})(19481x-17+32\sqrt{3})^2(155848x+305-83\sqrt{3})^4}$
$\mathcal{M}_{\frac{-1+3i}{2}}$	$\beta_7 : (x,y) \longmapsto \frac{1882056627(237-45\sqrt{3}+155848x)^4}{32(3264-3361\sqrt{3})(19481x-17-32\sqrt{3})^2(155848x+305+83\sqrt{3})^4}$
$\mathcal{M}_{2+4\rho}$	$\beta_8 : (x,y) \longmapsto \beta_3\left(\frac{3(5926\sqrt{3}-10245)}{176(35673x+219-64\sqrt{3})} + \frac{(603+502\sqrt{3})x}{4}, Y\right)$

(Table 8 continued)

$\mathcal{M}_\tau$	Belyĭ function $\beta : E_\tau \to \Sigma$
$\mathcal{M}_{\frac{2+4\rho}{3}}$	$\beta_9 : (x,y) \longmapsto \beta_3\left(\frac{3(-5926\sqrt{3}-10245)}{176(35673x+219+64\sqrt{3})} + \frac{(603-502\sqrt{3})x}{4}, Y\right)$
$\mathcal{M}_{4i}$	$\beta_{10} : (x,y) \longmapsto \beta_4\left(\frac{609-430\sqrt{2}}{16(1081x+25+64\sqrt{2})} + \frac{(33+2\sqrt{2})x}{4}, Y\right)$
$\mathcal{M}_{\frac{-1+2i}{2}}$	$\beta_{11} : (x,y) \longmapsto \beta_4\left(\frac{609+430\sqrt{2}}{16(1081x+25-64\sqrt{2})} + \frac{(33-2\sqrt{2})x}{4}, Y\right)$
$\mathcal{M}_{3+6\rho}$	$\beta_{12} : (x,y) \longmapsto \beta_5\left(\frac{-3(82248497\sqrt[3]{2}+139575268\sqrt[3]{4}-325185920)}{4048(97984073x+40448\sqrt[3]{4}-50111+30400\sqrt[3]{2})} + \frac{(11776+4141\sqrt[3]{2}+4196\sqrt[3]{4})x}{4}, Y\right)$
$\mathcal{M}_{\frac{2+3\rho}{2}}$	$\beta_{13} : (x,y) \longmapsto \beta_5\left(\frac{-3(82248497\rho\sqrt[3]{2}+139575268\rho^2\sqrt[3]{4}-325185920)}{4048(97984073x+40448\rho^2\sqrt[3]{4}-50111+30400\rho\sqrt[3]{2})} + \frac{(11776+4141\rho\sqrt[3]{2}+4196\rho^2\sqrt[3]{4})x}{4}, Y\right)$
$\mathcal{M}_{\frac{1+3\rho}{2}}$	$\beta_{14} : (x,y) \longmapsto \beta_5\left(\frac{-3(82248497\rho^2\sqrt[3]{2}+139575268\rho\sqrt[3]{4}-325185920)}{4048(97984073x+40448\rho\sqrt[3]{4}-50111+30400\rho^2\sqrt[3]{2})} + \frac{(11776+4141\rho^2\sqrt[3]{2}+4196\rho\sqrt[3]{4})x}{4}, Y\right)$

Table 9. Triangle groups containing genus 2 surface groups (§5.1)

Triangle Group	Index of $(2; -)$	$\#\,(2; -)$ up to conjugacy
$\Gamma(5,5,5)$	5	4
$\Gamma(3,6,6)$	6	4
$\Gamma(8,2,8)$	8	4
$\Gamma(4,4,4)$	8	6
$\Gamma(3,3,9)$	9	4
$\Gamma(5,2,10)$	10	7
$\Gamma(4,2,12)$	12	6
$\Gamma(6,2,6)$	12	13
$\Gamma(3,3,6)$	12	8
$\Gamma(3,4,4)$	12	10
$\Gamma(3,3,5)$	15	9
$\Gamma(4,2,8)$	16	19
$\Gamma(3,2,18)$	18	9
$\Gamma(5,2,5)$	20	21
$\Gamma(3,2,12)$	24	25
$\Gamma(4,2,6)$	24	40
$\Gamma(3,3,4)$	24	28
$\Gamma(3,2,10)$	30	20
$\Gamma(3,2,9)$	36	37
$\Gamma(4,2,5)$	40	75
$\Gamma(3,2,8)$	48	77
$\Gamma(3,2,7)$	84	155

Table 10. The uniform maps of genus 2 (§5.1)

Map type	# Uniform maps	# Reflexible uniform	# Hyper-elliptic	Order of largest aut. group	
(8, 8)	4	4	1	8	R
(5, 10)	7	5	1	10	R
(4, 12)	6	6	2	4	
(6, 6)	13	11	4	12	R
(4, 8)	19	17	10	16	R
(3, 18)	9	7	1	3	
(5, 5)	21	7	2	4	
(3, 12)	25	15	7	8	
(4, 6)	40	26	33	24	R
(3, 10)	20	10	7	10	
(3, 9)	37	11	13	12	
(4, 5)	75	27	65	8	
(3, 8)	77	37	51	48	R
(3, 7)	155	25	113	12	

Table 11. The uniform hypermaps of genus 2 (§5.1)

Hypermap type	# Uniform hypermaps	# Reflexible uniform	# Hyper-elliptic	Order of largest aut. group	
(5, 5, 5)	4	4	0	5	RRR
(3, 6, 6)	4	4	0	6	R
(4, 4, 4)	6	6	1	8	R
(3, 3, 9)	4	4	0	3	
(3, 3, 6)	8	6	0	3	
(3, 4, 4)	10	6	1	12	R
(3, 3, 5)	9	5	0	5	
(3, 3, 4)	28	14	1	24	R

Appendix II

Computer Programs

Program 1. (§5.1) This program runs on MAGMA and determines up to conjugacy all genus 2 surface groups that lie inside the triangle group $\Gamma(l, m, n)$ with index k. The surface groups are returned in the file 'surgps1'.

```
A<r,s> := Group<r,s | r^l = s^m = (r*s)^n = 1>;
surgps1:= [];
P := LowIndexProcess(A, <k,k>);
while not IsEmpty(P) do
    R := ExtractGroup(P);
    f,q,t := CosetAction(A,R);
    if CycleStructure(f(r)) eq [<l, k/l>]
        and CycleStructure(f(s)) eq [<m, k/m>]
        and CycleStructure(f(r)*f(s)) eq [<n, k/n>] then
        Append(~ surgps1, R);
    end if;
    NextSubgroup(~ P);
end while;
```

Program 2. (§5.1) Following the execution of Program 1, Program 2 will determine the genus 2 surface groups contained in $\Gamma(l, m, n)$ up to conjugacy in the extended triangle group $\Gamma^\star(l, m, n)$. The surface groups are returned in the file 'surgps2'.

```
B<x,y,z> := Group<x,y,z | x^2 = y^2 = z^2 = (x*y)^l = (y*z)^m = (x*z)^n = 1>;
C := sub<B | (x*y), (y*z)>;
```

```
h := hom < A-> C|r-> C.1, s-> C.2 >;
surgps2:= [];
for i in [1..#surgps1] do
    if IsEmpty(surgps2) then
        Append(~ surgps2, surgps1[i]);
    else p := 0;
        for j in [1..#surgps2] do
            if IsConjugate(B, h(surgps1[i]), h(surgps2[j])) then
                p := 1;
                break;
            end if;
        end for;
        if p eq 0 then
            Append(~ surgps2, surgps1[i]);
        end if;
    end if;
end for;
```

Program 3. (§5.1) Following the execution of Program 2, Program 3 will determine the automorphism groups of the uniform maps corresponding to the surface groups in the file 'surgps2'. The automorphism groups and their orders are returned in the files 'autgps' and 'autgpsorder' respectively.

```
autgps:= [];
autgpsorder:= [];
perm:=Sym(k);
for i in [1..#surgps2] do
    f, q, t := CosetAction(A, surgps2[i]);
    cent:= Centralizer(perm, q);
    Append(~ autgps, cent);
    Include(~ autgpsorder, Order(cent));
end for;
```

References

Ac R. D. M. Accola, *Topics in the theory of Riemann surfaces*, Lecture Notes in Mathematics 1595 (Springer-Verlag, New York, 1991).

Alle R. B. J. T. Allenby, *Rings, fields and groups, an introduction to abstract algebra*, Second edition (Edward Arnold, London, 1991).

Alli N. Alling, *Real elliptic curves* (North-Holland, Amsterdam, 1981).

Alt A. Altshuler, 'Construction and enumeration of regular maps on the torus', *Discr. Math.* 4 (1973) 201 – 217.

Ba A. Baker, 'Linear forms in the logarithms of algebraic numbers I', *Mathematika* 13 (1966) 204 – 216.

Bea A. F. Beardon, *The geometry of discrete groups*, Graduate Texts in Mathematics 91 (Springer-Verlag, New York, 1983).

Bel G. V. Belyĭ, 'On Galois extensions of a maximal cyclotomic field', *Izv. Akad. Nauk SSSR* 43 (1979) 269 – 276 (Russian); *Math USSR-Izv.* 14 (1980) 247 – 256 (English translation).

Ber W. E. H. Berwick, 'Modular invariants expressible in terms of quadratic and cubic irrationalities', *Proc. London Math. Soc.* 28 (1927) 53 – 69.

Bor A. Borel, 'Commensurability classes and volumes of hyperbolic 3-manifolds', *Ann. Sc. Norm. Sup. Pisa* 8 (1981) 1 – 33.

Br1 H. R. Brahana, 'Regular maps on an anchor ring', *Amer. J. Math.* 48 (1926) 225 – 240.

Br2 H. R. Brahana, 'Regular maps and their groups', *Amer. J. Math.* 49 (1927) 268 – 284.

BJ A. J. Breda d'Azevedo and G. A. Jones, 'Rotary hypermaps of genus 2', Preprint, Aveiro (1995).

CaSi İ. N. Cangül and D. Singerman, 'Normal subgroups of Hecke groups and regular maps', *Math. Proc. Camb. Phil. Soc.*, to appear.

Coh H. Cohen, *A course in computational algebraic number theory*, Second printing (Springer-Verlag, New York, 1995).

Cor R. Cori, 'Un code pour les graphes planaires et ses applications', *Astérisque*

27 (1975).

CMa R. Cori and A. Machì, 'Maps, hypermaps and their automorphisms: a survey I, II, III,' *Exposition. Math.* 10 (1992) 403 – 427, 429 – 447, 449 – 467.

CoSi D. Corn and D. Singerman, 'Regular hypermaps', *European J. Combin.* 9 (1988) 337 – 351.

Cox D. A. Cox, *Primes of the form* x^2+ny^2 (John Wiley & Sons, New York, 1989).

Coxe1 H. S. M. Coxeter, 'Configurations and maps', *Rep. Math. Colloq.* (2) 8 (1948) 18 – 38.

Coxe2 H. S. M. Coxeter, *Regular Polytopes*, Second edition, Macmillan Mathematics Paperbacks (Macmillan, New York, 1963).

CMo H. S. M. Coxeter and W. O. J. Moser, *Generators and relations for discrete groups* (Springer-Verlag, Berlin, 1965).

Da H. Davenport, *The higher arithmetic* (Hutchinson and Co. Ltd., London, 1970).

Er A. Erréra, 'Sur les polyèdres réguliers de l'Analysis Situs', *Acad. Roy. Belg. Cl. Sci. Mem. Coll.* in 8° (2) 7 (1922) 1 – 17.

FK H. M. Farkas and I. Kra, *Riemann surfaces*, Graduate Texts in Mathematics 71 (Springer-Verlag, New York, 1980).

Ful W. Fulton, *Algebraic Topology, a First Course*, Graduate Texts in Mathematics (Springer-Verlag, New York, 1995).

Gre J. Grey, 'From the history of a simple group', *Math. Intelligencer* 4 (1982) 59 – 67.

Gri P. A. Griffiths, *Introduction to algebraic curves*, American Mathematical Society 76 (Providence, Rhode Island, 1985).

Gro A. Grothendieck, *Esquisse d'un programme*, Preprint, Montpellier (1984).

He K. Heegner, 'Diophantische analysis und modulfunktionen', *Math. Zeit.* 56 (1952) 227 – 253.

HS A. H. M. Hoare and D. Singerman, 'The orientability of subgroups of plane groups', *Groups-St Andrews, 1981*, London Math. Soc. Lecture Note Ser. 71 221 – 227 (ed. C. M. Campbell and E. F. Robertson, Cambridge University Press, 1982).

Hu L. K. Hua, *Introduction to number theory* (Springer-Verlag, New York, 1982).

Ja L. D. James, 'Operations on hypermaps, and outer automorphisms', *European J. Combin.* 9 (1988) 551 – 560.

John D. L. Johnson, *Presentations of Groups*, London Math. Soc. Student Texts 15 (Cambridge Univ. Press, 1990).

Jon G. A. Jones, 'Inclusions between triangle groups, and functors between dessins', Preprint, Southampton (1997).

JS1 G. A. Jones and D. Singerman, 'Theory of maps on orientable surfaces', *Proc. London Math. Soc.* (3) 37 (1978) 273 – 307.

JS2 G. A. Jones and D. Singerman, *Complex functions, an algebraic and geometric viewpoint* (Cambridge University Press, Cambridge, 1987).

JS3 G. A. Jones and D. Singerman, 'Belyĭ functions, hypermaps and Galois groups', *Bull. London Math. Soc.* 28 (1996) 561 – 590.

JSt G. A. Jones and M. Streit, 'Galois groups, monodromy groups and cartographic groups', in *Geometric Galois actions: 2. The inverse Galois problem, moduli spaces and mapping class groups* (ed. L. Schneps) London Math. Soc. Lecture Note Series, to appear.

Ka S. Katok, *Fuchsian groups*, Chicago Lectures in Mathematics (The University of Chicago Press, Chicago, 1992).

Ki F. Kirwan, *Complex algebraic curves*, London Math. Soc. Student Texts 23 (Cambridge University Press, 1993).

Kn A. W. Knapp, *Elliptic curves*, Math. Notes 40, Princeton Univ. Press (Princeton, NJ, 1992).

KuNa T. Kuusalo and M. Näätänen, 'Geometric uniformization in genus 2', *Ann. Acad. Sci. Fenn. Ser. A I Math.* 20 (1995) 401 – 418.

Macb A. M. Macbeath, 'Generators of the linear fractional groups', *Proc. Sympos. Pure Math.* 12 (Amer. Math. Soc. 1967) 14 – 32.

Mach A. Machì, 'On the complexity of a hypermap', Discr. Math., 42 (1982) 221–226.

MR1 C. Maclachlan and G. Rosenberger, 'Two-generator arithmetic Fuchsian groups', *Math. Proc. Camb. Phil. Soc.* 93 (1983) 383 – 391.

MR2 C. Maclachlan and G. Rosenberger, 'Commensurability classes of two-generator

Fuchsian groups', London Math. Soc. Lecture Notes 173 (ed. W. J. Harvey with C. Maclachlan, Cambridge University Press, 1992)

Mag W. Magnus, *Noneuclidean tesselations and their groups* (Academic Press, New York, 1974).

NZM I. Niven, H. S. Zuckerman and H. L. Montgomery, *An introduction to the theory of numbers*, Fifth Edition (John Wiley & Sons, New York, 1991).

Pi R. S. Pierce, *Associative algebras*, GTM 88 (Springer-Verlag, New York, 1980).

Ro H. E. Rose, *A course in number theory*, Oxford Science Publications (Oxford Univ. Press, 1988).

Sc L. Schneps (ed.), *The Grothendieck theory of dessins d'enfants*, London Math. Soc. Lecture Note Ser. 200 (Cambridge University Press, 1994).

Se J-P. Serre, *A course in arithmetic*, Graduate Texts in Mathematics 7 (Springer-Verlag, New York, 1973).

SV G. B. Shabat and V. A. Voevodsky, 'Drawing curves over number fields', p. 199 – 227 in "*Grothendieck Festschrift III*" (ed. P. Cartier et *al.*) Progress in Math. 88 (Birkhäuser, Basel, 1990).

SZ G. B. Shabat and A. Zvonkin, 'Plane trees and algebraic numbers', *Jerusalem Combinatorics '93*, Amer. Math. Soc. Contemporary Math. Ser. 178, (ed. H. Barcelo and G. Kalai) 233 – 275.

Sh F.A. Sherk, 'The theory of regular maps', Ph.D. Thesis, University of Toronto (1957).

Sil J. H. Silverman, *The arithmetic of elliptic curves*, Graduate Texts in Mathematics 106 (Springer-Verlag, New York, 1986).

ST J. H. Silverman and J. Tate, *Rational points on elliptic curves*, Undergraduate Texts in Mathematics (Springer-Verlag, New York, 1992).

Sin1 D. Singerman, 'Subgroups of Fuchsian groups and finite permutation groups', *Bull. London Math. Soc.* 2 (1970) 319 – 323.

Sin2 D. Singerman, 'Finitely maximal Fuchsian groups', *J. London Math. Soc.* 2 (6) (1972) 29 – 38.

Sin3 D. Singerman, 'Klein's Riemann surface of genus 3 and regular imbeddings of finite projective planes', *Bull. London Math. Soc.* 18 (1986) 364 – 370.

Sin4 D. Singerman, 'Hyperelliptic maps and surfaces', *Math. Slovaca* 47 (1997) No. 1, 93 – 97.

SiSy D. Singerman and R. I. Syddall, 'Belyĭ uniformization of elliptic curves', *Bull. London Math. Soc.* 29 (1997) 443 – 451.

Sp G. Springer, *Introduction to Riemann surfaces* (Addison-Wesley, Reading, Massachusetts, 1957).

St H. M. Stark, 'A complete determination of the complex quadratic fields of class number one', *Michigan Math. J.* 14 (1967) 1 – 27.

Ta1 K. Takeuchi, 'A characterization of arithmetic Fuchsian groups', *J. Math. Soc. Japan* 27 (1975) 600 – 612.

Ta2 K. Takeuchi, 'Arithmetic triangle groups', *J. Math. Soc. Japan* 29 (1977) 91 – 106.

Ta3 K. Takeuchi, 'Commensurability classes of arithmetic triangle groups', J. Fac. Sci. Univ. Tokyo Sect. IA 24 (1977) 201 – 222.

TW A. D. Thomas and G. V. Wood, *Group Tables*, Shiva Math. Ser. (Shiva Publishing Ltd., Orpington, Kent, 1980).

Th W. Threlfall, 'Gruppenbilder', *Abh. sächs. Akad. Wiss. Math.-phys. K1* 41 (1932) 1 – 59.

Ve J. Vélu, 'Isogénies entre courbes elliptiques, *C. R. Acad. Sc. Paris*, t.273 (26 juillet 1971), Série A.

Vi M-F. Vignéras, *Arithmétique des algèbres de quaternions*, Lecture Notes in Math. 800 (Springer-Verlag, New York, 1980).

Wa T. R. S. Walsh, 'Hypermaps versus bipartite maps', *J. Comb. Theory Ser. B* 18 (1975) 155 – 163.

Wo1 J. Wolfart, 'Mirror-invariant triangulations of Riemann surfaces, triangle groups and Grothendieck dessins: variations on a theme of Belyĭ', Preprint, Frankfurt (1992).

Wo2 J. Wolfart, 'Two more proofs for the 'obvious' part of Belyĭ's theorem', Preprint, Frankfurt (1995).

Zi R. J. Zimmer, *Ergodic theory and semi-simple groups*, Monographs in Mathematics (Birkhäuser, Boston, 1984).