UNIVERSITY OF SOUTHAMPTON

FACULTY OF SOCIAL, HUMAN AND MATHEMATICAL SCIENCES

Mathematical Sciences

Relativistic fluid dynamics and electromagnetic media

by

Konstantinos Palapanidis

Thesis for the degree of Doctor of Philosophy

October 2018

UNIVERSITY OF SOUTHAMPTON

<u>ABSTRACT</u>

FACULTY OF SOCIAL, HUMAN AND MATHEMATICAL SCIENCES

Mathematical Sciences

<u>Thesis for the degree of Doctor of Philosophy</u>

RELATIVISTIC FLUID DYNAMICS AND ELECTROMAGNETIC MEDIA

by Konstantinos Palapanidis

In this thesis we describe fluid media with electromagnetic properties in the context of general relativity. Using the variational principle we derive the Einstein equations from the Einstein-Hilbert action, the Euler-Lagrange equations for a multicomponent fluid and the Maxwell equations. We provide a covariant description of linear electromagnetic media and we also discuss media with non linear electromagnetic properties. We also provide a formula that generalises the expression for the Lagrangian of linear media, to that of non linear media and we discuss a set of constraints for linear electromagnetic media in terms of the material derivative. We discuss a model for a multifluid with general electromagnetic properties. We also derive the limit for the single fluid ideal magnetohydrodynamics in general relativistic context. In the final part we look into the linear stability of specific systems using the geometric optics method along with the notion of "fast" and "slow" variables. Employing this method we reproduce a number of results in Newtonian context, building gradually to the derivation of the magnetorotational instability. Additionally, we discuss the vanishing magnetic field of this configuration. Subsequently, considering an unperturbed background spacetime we derive the characteristic equations describing the relativistic inertial waves, the relativistic Rayleigh shearing instability and the relativistic magnetorotational instability. Finally, by assuming a low velocity and flat metric limit of the relativistic equations we reproduce the Newtonian characteristic equations.

Contents

Declaration of Authorship

I, Konstantinos Palapanidis, declare that the thesis entitled *Relativistic fluid dynamics and electromagnetic media* and the work presented in the thesis are both my own, and have been generated by me as the result of my own original research. I confirm that:

- this work was done wholly or mainly while in candidature for a research degree at this University;
- where any part of this thesis has previously been submitted for a degree or any other qualification at this University or any other institution, this has been clearly stated;
- where I have consulted the published work of others, this is always clearly attributed;
- where I have quoted form the work of others, the source is always given. With the exception of such quotations, this thesis is entirely my own work;
- I have acknowledged all main sources of help;
- where the thesis is based on work done by myself jointly with others, I have made clear exactly what was done by others and what I have contributed myself;
- none of this work has been published before submission.

Signed: ....

Date: ....

Acknowledgements

I would like to thank my advisor Prof. Nils Andersson for supervising and helping me to accomplish this project and for motivating me during several occasions as well. His continuous guidance was of crucial importance throughout this project.
Also I would like to thank Prof. James Vickers, Prof. Ian Hawke, Prof. Abraham Harte and the external examiner, Prof. Éric Gourgoulhon, for their helpful comments and discussions as well. In addition, I would like to thank Dr. Kyriaki Dionysopoulou and Dr. Stuart Wells for the help provided during the first year of my studies.
I would also like to thank Stephanos Tyros, Georgios Katsianis, Dionysios Syrigos, Dimitra Matthaiou and Stelios Kotronis for all the discussions and enjoyable time we have had. Special thanks to Dionysios for insightful discussions regarding parts of this project. I am also very grateful to Apostolos Tovletoglou and Maria Georgiadou who were a great company and have always shown true hospitality, making me feel always welcome at their home here in Southampton. I would also like to thank my friends Zisis Karampaglis and Chrysa Kirmitsaki for the comfort and hospitality at their home in London.
I would like also to thank Despoina Pazouli for reading, commenting and discussing my work and for being there all the time supporting me. Her presence always softened the difficulties I encountered. Finally, I would like to express my gratitude to my family, my sister Maria, my mother Olympia, and especially my father Panagiotis, who has been the greatest teacher I have ever had and always listened to my ideas. Their support in all occasions has always been of crucial importance in carrying out this endeavour.

CHAPTER 1

Introduction

In this thesis we consider the theory of media with fluid and electromagnetic properties in the context of general relativity and examine the dynamical behaviour of specific systems using the method of geometric optics. The purpose of this work is to provide a description that is relevant for the modelling of specific aspects of astrophysical structures, such as neutron stars and their environments. We will firstly discuss the properties of neutron stars and then outline the plan of our study.

Since their discovery in 1968 by J. Bell [1], neutron stars have been studied in great depth and extent and today there are many models that describe them. Because they exhibit extreme properties, the various models usually need to combine more than one aspect of physics. The main goal of these theoretical models is to cover and explain the phenomena that we observe. Additionally, observations of neutron stars are continuously improved and refined and as a result there is a need for the production of more accurate theoretical models.

Neutron stars are stellar objects which are formed when main sequence stars with mass $M \gtrsim 10M_{\odot}$ collapse. They have a radius that lies within the range of $\sim 9-12$ km [2, 3], while their mass ranges within $\sim 1.2 - 2M_{\odot}$ [2] (and references therein), with the largest theoretically predicted mass $\sim 2-3M_{\odot}$ [4]. It is apparent that such masses and radii imply extremely high densities. In fact, neutron stars are known to be the most dense objects in the universe, having core densities of the order of 10^{15} g/cm^3, a value that exceeds the standard nuclear saturation density (2.8×10^{14} g/cm^3). Furthermore, their high compactness indicates that general relativistic effects are important and should be taken into to account.

Most, if not all, neutron stars rotate with periods, that vary depending on their age, in the range between milliseconds and seconds. The rotation period of neutron stars is almost constant, since they slow at a very small rate, and is measured with very high precision. The typical rate at which a neutron star slows its rotation, which is called spin down, is $\sim 10^{-13}$s/s [5, 6]. Nevertheless, there have been observed occasions where the star spins up for a brief period of time or, less frequently, it spins-down. These incidents are called "glitch" and "anti-glitch" respectively [7, 8, 9, 10, 11], and they are hypothesized to relate with the structure of the neutron star. More specifically they may occur due to the rapid re-organization of a solid crust, which is a part of the neutron star's structure (as we will discuss later). This phenomenon is called a starquake[1]. Alternatively, glitches indicate the existence of superconducting and superfluid properties of matter in the bulk of the neutron star [12, 13, 14].

The magnetic field of neutron stars is predicted (mainly by assuming that the star is a magnetic dipole that radiates energy and by using the period and the spin down of the neutron star) to range from 10^8 G to about 10^{15} G. The latter value is characteristic of a special kind of neutron stars, the magnetars. Furthermore, the magnetic field of a magnetar indicates that a precise description of electromagnetism requires the consideration of quantum electrodynamical corrections [15, 16, 17]. Empty space may appear as an electromagnetic medium with properties different from the classical vacuum [18, 19, 20].

Despite the extensive research related to neutron stars, there is still some ambiguity regarding their structure. However, it is widely accepted that there is stratification of the phases of matter, which depends on the distance from the center of the star. Following this assumption, there are four main regions in the interior of the star. These are, from the outermost to the innermost: the outer crust, the inner crust, the outer core and the inner core. Below we will briefly describe each of these regions in order to provide some reasoning for the theoretical model that we will develop.

The outer crust is thought to be solid having a depth between 300 m and 500 m and a mass of around 1% of the star's mass [21]. It is composed of iron (^{56}Fe) ions that form a body-centered cubic lattice and electrons [22], while the density

[1]When the re-organization takes place in the crust only, we use the term "crustquake" instead.

scales up to $\sim 10^{11}$ g/cm^3. Since the crust has crystalline structure it is anticipated that, in general, its matter possesses elastic and electromagnetic properties. The elasticity theory in the context of general relativity has been introduced by Carter in [23], while subsequent work has been done in [24, 25, 26]. The electromagnetic properties of matter in a general relativistic context were formally introduced in [27] as a continuation to the theory presented in [23]. It is worth noting that the two theories, i.e. that of elasticity and that of electromagnetism in matter, have some similarities in their mathematical formulation.

The inner crust lies beneath the outer crust and its density ranges between $\sim 10^{11}$ g/cm^3 and $\sim 10^{14}$ g/cm^3. Here the neutrons are partially free and may exist in superfluid state. Additionally, it is predicted that in the deepest layers of the inner crust there is a peculiar phase of matter, the “pasta” phase [22, 28]. In this region (where the “pasta” phase occurs) the nuclei acquire non-spherical shapes, resembling rods and slabs which justifies the name, “pasta”. Furthermore, in this peculiar phase the matter is theorized to exhibit liquid crystal behaviour [28]. Matter in this state possesses both solid and liquid properties (and hence the name). Note that liquid crystals are anisotropic in the sense that they behave as liquids towards some direction in space and as solids towards some other direction[2]. The crystal structure indicates that the matter in these regions may have, as in the outer crust, electromagnetic properties different than those of vacuum.

The outer core comprises the largest part of the star and has a range of the order of kilometres while the density is around $\sim 10^{14}$ g/cm^3. In this region, neutron star matter is composed of superfluid neutrons, which is the main component, type-II superconducting protons and electrons. Superconductivity and superfluidity are states of matter that normally (i.e. in laboratory experiments) occur at very low temperatures, around 0 K, which is somewhat contradictory given the very high temperatures of neutron stars ($T \sim 10^8$ K). However, the corresponding Fermi temperature of such dense matter is even higher, of the order of $T_F \sim 10^{12}$ K, and thus the existence of the superconducting and superfluid states is expected.

Finally, the innermost part of a neutron star's structure is the inner core which lies at the centre and is usually considered to be smaller than the outer core. Here the density is considered to be between $\sim 10^{14}$ g/cm^3 and $\sim 10^{15}$ g/cm^3. There are many hypotheses regarding the phase and composition of the matter, and even the

[2]For example, a matter type of liquid crystals which could exist in a neutron star is the Smectics A [28]. This kind of liquid crystal is structured in many separate layers. Each layer is composed of elongated molecules, arranged (with the long side) perpendicular to the layer. It possesses liquid properties along the layers, that is each layer behaves as a two-dimensional fluid. Also it possesses solid (elastic) properties in the direction perpendicular to the layers, which means that different layers cannot be mixed together [29].

nature of matter [21]. Various models predict different compositions for the inner core and use a variety of equations of state. This plethora of models explains the difference between the possible densities. In general it has been theorised that matter in this region exists in phases that exhibit non-linear electromagnetic properties. Such exotic states, as for example a ferromagnetic phase, are suggested to occur at densities a few times the nuclear saturation density [30, 31, 32, 33].
In many cases, we consider the existence of disks consisting of gas and dust in the proximity of neutron stars. These structures interact gravitationally and electromagnetically with the star and may also produce accretion. The process of accretion has been employed as an explanation for observed X-ray emission [34, 35]. Furthermore, the interaction of the disk with the star's magnetic field plays an important role, since it may affect the stability and therefore the evolution of the disk [36].

With this motivation in mind, this thesis is structured in the following way. In Chapter 2 we discuss the covariant geometrical framework of the relativistic fluid model. We introduce the basic concepts of relativistic spacetime along with the 1+3 split, which we mainly use. We also discuss Eulerian and Lagrangian variations and we introduce matter space. These mathematical tools provide the basis for the description of the multicomponent fluid of the following chapter. We also discuss Carter's material derivative [23], a generalisation of the Lie derivative that is later employed in the description of non-linear electromagnetic media.
In Chapter 3 we employ the variational principle to derive the Einstein equations from the Einstein-Hilbert action, the Euler-Lagrange equations of a multicomponent isotropic fluid and the electromagnetic field-equations. We also look into the description of electromagnetism in linear and non-linear media, and discuss specific cases. We provide a formula that generalises the Lagrangian expression of linear media to non-linear media and we discuss a set of constraints for linear electromagnetic media in terms of the material derivative. Subsequently, combining the fluid and electromagnetic theory we demonstrate a general model for a multicomponent fluid medium with general electromagnetic properties. Finally, we look into the single fluid ideal magnetohydrodynamics limit of the previously mentioned model, since astrophysical processes in neutron star environments are usually described by this limit.
In Chapter 4 we perform a linear perturbation analysis using plane waves in order to examine the dynamical properties of various systems. We employ the geometric optics method considering "fast" and "slow" quantities to calculate Newtonian and relativistic instabilities. We discuss the choice of the observer and the stability cri-

terion. The latter investigates whether the plane wave solution exhibits oscillation or exponential growth i.e. whether the linear perturbations are stable or unstable. More specifically real values of the angular frequency of the plane wave imply stability while complex values imply unstable behaviour. In order to gain some insight into the method we derive in Newtonian context the sound waves, the Alfvén waves, and the continuous limit of Taylor-Rayleigh and Kelvin-Helmholtz instabilities. We also consider configurations that model astrophysical disks and calculate the inertial waves, the Rayleigh shearing instability and the magnetorotational instability. For the latter, we also discuss the vanishing magnetic field limit. Considering an unperturbed background, we derive the characteristic equations describing the relativistic inertial waves, the relativistic Rayleigh shearing instability and the relativistic magnetorotational instability. By considering a low velocity and flat metric limit of the relativistic equations we reproduce the Newtonian characteristic equations.

Finally, in Chapter 5 we summarise the ideas of the thesis and discuss possible extensions of the work.

CHAPTER 2

The covariant description

This chapter introduces the mathematical framework and tools used in subsequent chapters. First we discuss spacetime concepts and the 1+3 decomposition, which provides an intuitive covariant description of tensorial quantities. Then we discuss Eulerian and Lagrangian variations which will be used in Chapter 3 to obtain the equations of motion in the systems under consideration and finally we introduce the concept of matter space and material derivative which are used for describing multifluid systems. Since the notions presented here are extensively discussed in literature, our discussion is brief and intended to serve as a basis for the subsequent parts.

2.1 Spacetime concepts

In general relativity, we consider the spacetime, a 4-dimensional manifold $\mathcal{M}$ which admits a metric tensor[1] g_{ab} with Lorentzian signature $(-,+,+,+)$. The metric tensor is a symmetric tensor i.e. $g_{ab} = g_{ba}$, which defines the invariant infinitesimal

[1] Note that Latin indices are abstract while Greek indices are concrete taking values $\mu = 0, 1, 2, 3$.

distance between two spacetime points through

$$\mathrm{d}s^2 = g_{ab}\,\mathrm{d}x^a\mathrm{d}x^b, \tag{2.1.1}$$

where x^a are spacetime coordinates[2]. The metric determinant, g, is non-singular i.e. $g \neq 0$, and hence the inverse of the metric is defined via the identity

$$g^{ab}\,g_{bc} = \delta^a{}_c, \tag{2.1.2}$$

where $\delta^a{}_c$ is the Kronecker delta is given by

$$\delta_a^{\ b} = \begin{cases} +1, & a = b \\ 0, & a \neq b \end{cases}, \tag{2.1.3}$$

and the components of this tensor are the same in all coordinate systems. It follows from the definition of the inverse that, g^{ab} is a second rank tensor. The metric and its inverse can be used to raise and lower indices of tensors. Assuming the vector V^a and the covector W_a, we can lower and raise the indices using the relations $V_a = g_{ab}V^b$ and $W^a = g^{ab}\,W_b$.

From the definition of the infinitesimal distance 2.1.1, we see that it can take positive, negative or zero values. Therefore, the vectors can be characterized as spacelike if $g_{ab}\,V^aV^b > 0$, as timelike if $g_{ab}V^aV^b < 0$, and as null if $g_{ab}V^aV^b = 0$. The integral curves of vectors are characterized as spacelike, timelike or null according to the vectors. Timelike curves can be parametrised by proper time τ which is related to the infinitesimal distance $\mathrm{d}s^2$ through[3]

$$\mathrm{d}\tau^2 = -\mathrm{d}s^2. \tag{2.1.4}$$

The tangent vector to an observer's timelike curve, i.e. the observer's worldline

$$u^a = \frac{\mathrm{d}x^a}{\mathrm{d}\tau}, \tag{2.1.5}$$

is the observer's 4-velocity. Using equations (2.1.1) and (2.1.5) it follows that $u^a u_a = -1$. The 4-velocity introduces a split of spacetime into space and time. Contracting a tensor index with the 4$-$velocity projects this tensor along the 4-velocity and

[2]Note that indices that appear twice (one upstairs and one downstairs) are contracted. This operation implies summation over the range of indices through $V_aW^a = V_\mu W^\mu = V_0W^0 + V_1W^1 + V_2W^2 + V_3W^3$

[3]We use geometrised units i.e. $c = G = 1$. This means that all quantities have units that are integer powers of length.

therefore we obtain a temporal component of the tensor with respect to the observer. We can also project tensors on the observer' s instantaneous rest space, which is orthogonal to the 4-velocity, using the projector tensor h_{ab}, a symmetric tensor defined through

$$h_{ab} = g_{ab} + u_a u_b. \tag{2.1.6}$$

Quantities which are obtained by contraction with h_{ab} are referred to as spatial. It follows from the definition above that $h_{ab}u^b = 0$, $h_a^{\ a} = 3$ and $h_{ab}h^b_{\ c} = h_{ac}$. The projection tensor also serves as the metric of the observer's local 3 dimensional co-moving frame. The decomposition of tensors using the 4-velocity and the projection tensor, are given for a vector V^a through

$$\begin{aligned} V^a &= \delta^a_b V^b \\ &= -u^a u_b V^b + h^a_{\ b} V^b \\ &= V^{\|} u^a + V^a_{\perp}, \end{aligned} \tag{2.1.7}$$

where $V^{\|} = -V^a u_a$ and $V^a_{\perp} = h^a_{\ b} V^b$. The formula is generalised for any tensor by multiplying all indices by the Kronecker delta and then using equation (2.1.6) to calculate the projected components.

We also introduce the Levi-Civita tensor ϵ_{abcd}, a totally antisymmetric tensor. In a coordinate system x^μ, the components of this tensor are defined through

$$\epsilon_{\mu\nu\sigma\rho} = \sqrt{-g}\,[\mu\nu\sigma\rho]\,, \tag{2.1.8}$$

where

$$[\mu\nu\sigma\rho] = \begin{cases} +1, & \mu\nu\sigma\rho \text{ is an even permutation of } 0123 \\ -1, & \mu\nu\sigma\rho \text{ is an odd permutation of } 0123 \\ 0, & \text{otherwise} \end{cases}, \tag{2.1.9}$$

is the totally antisymmetric symbol with $[0123] = 1$. The Levi-Civita tensor with all indices up can be defined in two ways. Either by using the formula

$$\epsilon_{abcd}\epsilon^{abcd} = 4!, \tag{2.1.10}$$

which implies that the components in the coordinate system x^μ are

$$\epsilon^{\mu\nu\sigma\rho} = \frac{1}{\sqrt{-g}}[\mu\nu\sigma\rho], \tag{2.1.11}$$

or by raising the indices with the metric tensor

$$\epsilon^{abcd} = g^{ae} g^{bf} g^{cg} g^{dh} \epsilon_{efgh}, \tag{2.1.12}$$

which implies that

$$\epsilon_{abcd} \epsilon^{abcd} = -4!, \tag{2.1.13}$$

or in general,

$$\epsilon_{abcd} \epsilon^{efgh} = -4! \delta_{[a}{}^{e} \delta_{b}{}^{f} \delta_{c}{}^{g} \delta_{d]}{}^{h}, \tag{2.1.14}$$

which means that the components of the Levi-Civita tensor in the coordinate system x^μ with indices upstairs is related to the totally antisymmetric symbol through

$$\epsilon^{\mu\nu\sigma\rho} = -\frac{1}{\sqrt{-g}} [\mu\nu\sigma\rho]. \tag{2.1.15}$$

The difference in sign between equations (2.1.11) and (2.1.15) arises due to the indefiniteness of the Lorentzian metric we use in general relativity. Here, as in most of the literature we use equation (2.1.15).
We also define the three dimensional Levi-Civita tensor by contracting the four dimensional Levi-Civita tensor with the observer's 4-velocity u^a. This leads to

$$\epsilon_{abc} = u^d \epsilon_{dabc}. \tag{2.1.16}$$

We also have $\epsilon_{abc} u^c = 0$ which follows from the total antisymmetry of ϵ_{abcd}. Additionally, the four dimensional Levi-Civita tensor is referred to as the volume form since it is related to the volume element of spacetime $\sqrt{-g}$.

A key notion of the geometric framework is the generalisation of the partial differentiation on a curved spacetime is the covariant derivative ∇_a. The covariant derivative of a tensor $T_{ab\ldots}{}^{cd\ldots}$ is defined through

$$\begin{aligned} \nabla_m T_{ab\ldots}{}^{cd\ldots} = \partial_m T_{ab\ldots}{}^{cd\ldots} - \Gamma^{e}{}_{am} T_{eb\ldots}{}^{cd\ldots} - \Gamma^{e}{}_{bm} T_{ae\ldots}{}^{cd\ldots} - \ldots \\ + \Gamma^{c}{}_{em} T_{ab\ldots}{}^{ed\ldots} + \Gamma^{d}{}_{em} T_{ab\ldots}{}^{ce\ldots} + \ldots, \end{aligned} \tag{2.1.17}$$

where $\Gamma^a{}_{bc}$ are the Christoffel symbols of the second kind,symmetric in the two lower indices, obtained through

$$\Gamma^{a}{}_{bc} = \frac{1}{2} g^{ad} \left(\partial_c g_{db} + \partial_b g_{dc} - \partial_d g_{bc} \right). \tag{2.1.18}$$

The definition above also implies that the metric tensor is constant with respect to the covariant derivative, that is

$$\nabla_a g_{bc} = 0. \tag{2.1.19}$$

Furthermore, by applying the decomposition to the covariant derivative of a tensor $\nabla_e T_{ab...}{}^{cd...}$ we get the time derivative (i.e. the total derivative[4] of the tensor along the worldline of the observer with respect to the proper time [38]) and the orthogonally projected covariant derivative of the tensor

$$\dot{T}_{ab...}{}^{cd...} = \frac{\mathrm{d}}{\mathrm{d}\tau} T_{ab...}{}^{cd...} = u^e \nabla_e T_{ab...}{}^{cd...}, \tag{2.1.20}$$

and

$$\bar{\nabla}_e T_{ab...}{}^{cd...} = h_e^{\ f} \nabla_f T_{ab...}{}^{cd...}, \tag{2.1.21}$$

respectively. Various authors [38, 39], use the completely projected version of the covariant derivative. That is, all free indices are contracted with the projection tensor. The resulting tensor is a purely spatial quantity. Here, in analogy to the definition (2.1.20) of the time derivative we contract with the projection tensor only along the operator index.

We also introduce the Lie derivative, a differential operator that generalises the notion of the directional derivative on manifolds that may not be flat [40]. The Lie derivative[5] of a tensor $T_{ab...}{}^{cd...}$ along the vector V^a, denoted as $\pounds_V T_{ab...}{}^{cd...}$ is given by

$$\begin{aligned} \pounds_V T_{ab...}{}^{cd...} = V^e \partial_e T_{ab...}{}^{cd...} \\ - T_{ab...}{}^{ed...} \partial_e V^a - T_{ab...}{}^{ce...} \partial_e V^a - \ldots \\ + T_{eb...}{}^{cd...} \partial_a V^e + T_{ae...}{}^{cd...} \partial_b V^e + \ldots . \end{aligned} \tag{2.1.22}$$

Since we are using Christoffel symbols that are symmetric in the lower indices the partial derivatives in the expression above may be substituted with the covariant derivative.

Tensor quantities obtain different components when expressed in different coordinate systems. Assuming a coordinate transformation $x'^\mu = x'^\mu (x^\nu)$ the components V^ρ

[4]The total derivative mentioned here is the directional derivative of vector calculus, see [37].

[5]Many authors show the Lie derivative as $(\pounds_V T)_{ab...}{}^{cd...}$ in order to indicate the fact that the indices refer to the operator as a whole rather than just to the differentiated tensor. This practise is used for the covariant derivative as well. In this work, although we do not follow this practise, we mean in both cases that the tensor rank refers to the whole tensor (i.e. the differential operator and the differentiated quantity)

of a vector in the coordinate system x^μ transform to the components V'^ρ through

$$V'^\rho = \frac{\partial x'^\rho}{\partial x^\nu} V^\nu, \tag{2.1.23}$$

while for the components of the covector W_ρ in the coordinate system x^μ transform to W'_ρ through

$$V'_\rho = \frac{\partial x^\nu}{\partial x'^\rho} V_\nu. \tag{2.1.24}$$

The transformation rules above hold for the components of any tensor. The upstairs indices transform through equation (2.1.23) while the downstairs indices transform according to equation (2.1.24). Extensive discussion on the concepts introduced in this section may be found in [41, 42, 43, 44, 45, 46].

2.2 Eulerian and Lagrangian variations

Here we will briefly present the notion of Eulerian and Lagrangian variations [47, 48]. As shown in the following sections variations are employed to obtain the equations of motion of the various physical systems we consider. Variations are also used to linearise the differential equations governing a system in order to perform a stability analysis.

Assuming a coordinate system x^μ and a scalar field $\phi(x^\mu)$ the value of the field at a point x_0^μ is $\phi(x_0^\mu)$. Let x_1^μ be a point infinitesimally close to x_0^μ such that

$$x_1^\mu = x_0^\mu + \delta x^\mu, \tag{2.2.1}$$

where δx^μ is the displacement between the two points. We calculate the value of the field at x_1^μ in two ways. The first way is to take a first order Taylor approximation around x_0^μ

$$\phi(x_1^\mu) = \phi(x_0^\mu) + \left.\frac{\partial \phi}{\partial x^\mu}\right|_{x^\mu = x_0^\mu} \delta x^\mu. \tag{2.2.2}$$

The second way is to assume that the value of the field at x_1^μ, i.e. ϕ' is related to the value at the initial point through

$$\phi'(x_1^\mu) = \phi(x_0^\mu) + \Delta\phi(x_0^\mu), \tag{2.2.3}$$

where $\Delta\phi$ denotes the difference between the quantities. Subtracting equation (2.2.2) from equation (2.2.3) we get

$$\phi'(x_1^\mu) - \phi(x_1^\mu) = \Delta\phi(x_0^\mu) - \left.\frac{\partial\phi}{\partial x^\mu}\right|_{x^\mu = x_0^\mu} \delta x^\mu. \tag{2.2.4}$$

Setting[6] $\phi'(x_1^\mu) - \phi(x_1^\mu) \equiv \delta\phi$ and noting that the second term of the right hand side of equation (2.2.4) is the Lie derivative of the field ϕ along the vector field ξ^a, with components given by $\xi^\mu \equiv \delta x^\mu$, the preceding equation becomes

$$\Delta\phi = \delta\phi + \pounds_\xi \phi. \tag{2.2.5}$$

In equation (2.2.5), $\delta\phi$ is the Eulerian variation, representing the change of the field at a specific point while $\Delta\phi$ is the Lagrangian variation which shows the total change of the field between two points infinitesimally close, i.e. the change due to the Eulerian variation plus the amount due to the Lie dragging of the field between the two points induced by equation (2.2.1).

Note also that the Eulerian variation commutes with the partial differentiation operator ∂_a,

$$\delta\left(\partial_a \phi\right) = \partial_a \left(\delta\phi\right). \tag{2.2.6}$$

Also, the chain rule of differentiation for composite functions holds for δ in the sense that

$$\delta\left(f\left(V^a\right)\right) = \frac{\partial f}{\partial V^a} \delta V^a, \tag{2.2.7}$$

where $f = f\left(V^a\right)$ is some function and V^a some vector. The chain rule holds for the Lagrangian variation and the Lie derivative, as well [40].

2.3 The matter space description

Since we are interested in describing dynamics of media in spacetime we will follow the matter space description which was initially introduced by Carter in [23, 27]. As discussed in section 2.1 at each point of an observer's worldline we can introduce a 3 dimensional rest space orthogonal to the 4-velocity at that point. With respect to

[6] Here we calculate $\delta\phi$ at x_1^μ while $\Delta\phi$ and the derivative are calculated at x_0^μ. Assuming that, to linear order $\phi'(x_1^\mu) - \phi(x_1^\mu) = \phi'(x_0^\mu) - \phi(x_0^\mu)$ we get that $\delta\phi(x_0^\mu) = \delta\phi(x_1^\mu)$.

an observer the energy-matter content exists in this 3D rest space. Therefore it is reasonable to introduce a 3D space, the "matter space", where the material media exist. As we discuss matter space and spacetime are related, yet matter space can be treated separately from spacetime. We assume a set of coordinates q^A on matter space with $A = 1, 2, 3$ (since matter space is 3 dimensional).
We assume that these coordinates are related to the spacetime coordinates x^a through a continuous map q such that

$$q : x^a \mapsto q^A(x^a). \tag{2.3.1}$$

This map is not invertible and this fact has the following implication: Consider a spacetime point which lies on a worldline of an observer. This point, according to the map above, is mapped to a point on matter space. However, the contrary does not hold. A point on matter space is not mapped to a specific point on spacetime, since the map (2.3.1) is not invertible, but is rather mapped to any point of the observer's worldline. An extensive discussion of the rigorous mathematical framework on matter space may be found in [23, 27]. We denote the quantities of matter space using capital letter indices, and these quantities are considered to be scalars with respect to spacetime. In this section, we assume for simplicity that the medium consists of a single component. In the general case of many components there are as many matter spaces and related matter space coordinate systems as the number of components. The transformation between spacetime and matter space tensors is performed by the Jacobian of the map (2.3.1), i.e. by the partial derivatives of the matter space coordinates with respect to spacetime coordinates, given by

$${e_a}^A = \frac{\partial q^A}{\partial x^a} = \partial_a q^A = \nabla_a q^A. \tag{2.3.2}$$

These transformation quantities[7] provide a projection of matter space tensors with indices downstairs $S_{AB\ldots}$ to the respective spatial tensors $S_{ab\ldots}$ in spacetime, through

$$S_{ab\ldots} = {e_a}^A {e_b}^B \ldots \; S_{AB\ldots}. \tag{2.3.3}$$

A fundamental quantity, for the description of material media in matter space, is the number density n_{ABC} which represents the density of particles (or continuum material in general) and in this work is considered to be conserved. The description where number density is not conserved is provided in [50, 51]. The number den-

[7] In continuum mechanics these quantities are called "two-point tensors" since they have indices in both manifolds [49].

sity is a 3-form[8] in matter space (that is, a totally antisymmetric tensor of rank 3 with indices downstairs) and is a function of matter space coordinates only. The corresponding spacetime quantity n_{abc} is given by

$$n_{abc} = n_{ABC}\, e_a{}^A e_b{}^B e_c{}^C, \tag{2.3.4}$$

which is a spacetime 3-form. We can show that this form is closed, that is

$$\nabla_{[d} n_{abc]} = 0, \tag{2.3.5}$$

by applying the covariant derivative to equation (2.3.4) and calculating the totally antisymmetric part

$$\begin{aligned} &\nabla_{[d} n_{abc]} = \nabla_{[d} \left(n_{ABC} e_a{}^A e_b{}^B e_{c]}{}^C \right) = \\ &\partial_{[d} \left(n_{ABC} \right) e_a{}^A e_b{}^B e_{c]}{}^C + n_{ABC} \nabla_{[d} \left(e_a{}^A e_b{}^B e_{c]}{}^C \right) = \\ &\frac{\partial n_{ABC}}{\partial q^D}\, e_{[d}{}^D e_a{}^A e_b{}^B e_{c]}{}^C + n_{ABC} \nabla_{[d} \left(e_a{}^A e_b{}^B e_{c]}{}^C \right) = 0, \end{aligned} \tag{2.3.6}$$

where for the first term in the second row we used the chain rule for derivatives after we substituted the covariant with partial derivative (this substitution is permitted since matter space quantities are scalars with respect to spacetime). This term vanishes because we have considered the antisymmetric part of four projection vectors, each possessing three components in matter space. In any case, at least two of the four capital letter indices will be the same (since, as we stated, the material indices obtain three different values) and thus the term vanishes. For the second term we work as follows: Since the expression is totally antisymmetric and the Christoffel symbols, as given in equation (2.1.18), are symmetric in the two lower indices the terms containing the Christoffel symbols vanish and so we can substitute the covariant derivative with the partial derivative. Using the definition of the projections given by equation (2.3.2) and since the partial derivatives commute it is trivial to show that this term vanishes as well[9].

Multiplying and contracting the number density form with the Levi-Civita tensor we get the number density current

$$n^a = \frac{1}{3!} \epsilon^{bcda}\, n_{bcd}, \tag{2.3.7}$$

[8] For an intuitive geometrical description of differential forms, see [41].

[9] A similar calculation is shown in Appendix (A.1.4) for the derivation of one of Maxwell equations. Although the antisymmetric quantity in that case is a 2-form (the Faraday tensor) the result is the same for the rank 3 totally antisymmetric quantity $e_{[a}{}^A e_b{}^B e_{c]}{}^C$ that appears in equation (2.3.6).

which describes the material flow in spacetime. The number density current is conserved in the sense that

$$\nabla_a n^a = 0, \tag{2.3.8}$$

which is derived after multiplying equation (2.3.5) by the Levi-Civita tensor and using equation (2.3.7). This implies that equations (2.3.5) and (2.3.8) are equivalent statements of the conservation condition for the medium.
The number density current can be decomposed as

$$n^a = nu^a, \qquad u_a u^a = -1, \tag{2.3.9}$$

where n is the scalar number density of the medium and u^a the 4-velocity of the medium. Multiplying this equation by ϵ_{abcd} and contracting the first index we obtain

$$n_{abc} = n\epsilon_{abc}, \tag{2.3.10}$$

where we used equations (2.3.7) and (A.1.1) for the left hand part and equation (2.1.16) for the right hand part. This equation has the following implication. If we consider the matter space 3-form n_{ABC}/n and use the transformation (2.3.3) we have

$$\frac{1}{n} n_{ABC} e_a{}^A e_b{}^B e_c{}^C = \frac{1}{n} n_{abc} = \epsilon_{abc}, \tag{2.3.11}$$

where we used for the last equality equation (2.3.10). This equation implies that there is a 3-form in matter space ϵ_{ABC} which transformed to spacetime, provides through equation (2.3.3) the spatial Levi-Civita symbol. This result is anticipated since in n-dimensions the n-forms differ only by a scalar quantity (see [42] for further discussion). We proceed with the calculation of the Lie derivative of the matter space coordinates with respect to the number density current. Using equations (2.3.7) and (2.3.4) we get

$$£_n q^A = n^a \partial_a q^A = \frac{1}{3!} n_{BCD}\, \epsilon^{abcd}\, e_a{}^A e_b{}^B e_c{}^C e_d{}^D = 0. \tag{2.3.12}$$

The last equality follows, as previously, from the fact that the expression is totally antisymmetric and matter space indices are 3 dimensional. There will be at least two projection tensors with same material index and thus the expression vanishes. Equation (2.3.12) shows that the material coordinates are dragged along the flow produced by the number density current. Using equation (2.3.9) we can show that the projection vectors are orthogonal to the 4-velocity

$$u^a e_a{}^A = 0. \tag{2.3.13}$$

Expressing the Lie derivative, given in equation (2.1.22), of the projection vectors $e_a{}^A$ along the 4-velocity with partial derivatives we get

$$\begin{aligned} \pounds_u e_a{}^A &= u^b \partial_b \partial_a q^A + \left(\partial_b q^A\right) \partial_a u^b \\ &= u^b \partial_a \partial_b q^A + \left(\partial_b q^A\right) \partial_a u^b = \\ &= \partial_a \left(u^b \partial_b q^A\right) = 0, \end{aligned} \tag{2.3.14}$$

which shows that the projection vectors are Lie dragged along the flow of u^a, as well. In the above, we used the commutating property of the partial derivative and equation (2.3.13). Using the projection vectors it is possible to transform a general spacetime tensor$S^{ab\dots}$ to matter space through

$$S^{AB\dots} = e_a{}^A e_b{}^B \dots S^{ab\dots}. \tag{2.3.15}$$

The transformed quantity $S^{AB\dots}$ though, does not contain any information of the temporal components of $S^{ab\dots}$. This is anticipated since according to equation (2.3.13) the 4-velocity is orthogonal to the transformation vectors. Applying equation (2.3.15) to the metric tensor g^{ab} (or to h^{ab} since only spatial components survive) we get

$$h^{AB} = e_a{}^A e_b{}^B g^{ab} = e_a{}^A e_b{}^B h^{ab}, \tag{2.3.16}$$

which is a symmetric second rank matter space tensor. The determinant of this tensor is given by

$$\det\left(h^{AB}\right) = \frac{1}{3!} \epsilon_{ABC} \epsilon_{DEF} h^{AD} h^{BE} h^{CF}, \tag{2.3.17}$$

and by substituting h^{AB} from equation (2.3.16) we get

$$\begin{aligned} \det\left(h^{AB}\right) &= \frac{1}{3!} \epsilon_{ABC} \epsilon_{DEF} e_a{}^A e_d{}^D g^{ad} e_b{}^B e_e{}^E g^{be} e_c{}^C e_f{}^F g^{cf} \\ &= \frac{1}{3!} \epsilon_{abc} \epsilon^{abc} = 1, \end{aligned} \tag{2.3.18}$$

where we used the transformation (2.3.3) for ϵ_{ABC} and equation (A.1.7). Since the determinant of h^{AB} is non-zero, this matter space tensor has an inverse h_{AB} such that

$$h_{AC}h^{CB} = \delta_A^{\ B}, \tag{2.3.19}$$

where $\delta_A^{\ B}$ is the matter space Kronecker delta. The tensor h^{AB} can be perceived as the 3D metric of matter space since it is defined by the transformation of the spacetime metric.

We may now introduce the vectors $e_a^{\ A}$ defined through

$$e_A^{\ a} = e_b^{\ B} h_{AB} g^{ab}. \tag{2.3.20}$$

These vectors are orthogonal to the $e_a^{\ A}$ covectors which can be seen by substituting equation (2.3.16) in equation (2.3.19)

$$\begin{aligned} h_{AC} e_a^{\ C} e_b^{\ B} g^{ab} &= \delta_A^{\ B} \\ e_A^{\ b} e_b^{\ B} &= \delta_A^{\ B}. \end{aligned} \tag{2.3.21}$$

We can show that the vectors $e_A^{\ a}$ are orthogonal to u_a since

$$u_a e_A^{\ a} = u_a e_b^{\ B} h_{AB} g^{ab} = u^b e_b^{\ B} h_{AB} = 0, \tag{2.3.22}$$

and additionally they introduce the following transformations between spacetime and matter space quantities

$$S_{AB\ldots} = e_A^{\ a} e_B^{\ b} \ldots\ S_{ab\ldots}, \tag{2.3.23}$$

and

$$S^{ab\ldots} = e_A^{\ a} e_B^{\ b} \ldots\ S^{AB\ldots}. \tag{2.3.24}$$

These transformations provide the freedom of transforming tensors between matter space and spacetime independently of the their type. It is possible to express the $e_a^{\ A}$ co-vectors with respect to the vectors $e_A^{\ a}$ multiplying both sides of the definition (2.3.20) by h^{AB} and g_{ab}

$$\begin{aligned} g_{ab} h^{AB} e_A^{\ a} &= e_c^{\ C} h_{AC} g^{ac} g_{ab} h^{AB} \\ e_b^{\ B} &= g_{ab} h^{AB} e_A^{\ a}. \end{aligned} \tag{2.3.25}$$

Using the transformation (2.3.23) for g_{ab} (or h_{ab}), provides the matter space metric with indices downstairs

$$h_{AB} = e_A^{\ a} e_B^{\ b}\ g_{ab} = e_A^{\ a} e_B^{\ b}\ h_{ab}, \tag{2.3.26}$$

which is directly verified if we multiply by contraction on both sides h^{BC} and use equations (2.3.21) and (2.3.25). In the same way, substituting the matter space metric h_{AB} on the right-hand-side of equation (2.3.3) provides h_{ab}. Note that substitution of either the projection tensor or the spacetime metric in equations (2.3.3) and (2.3.23) provides the same matter space tensor. Using equations (2.3.3) and (2.3.20) we can show that

$$e_a{}^B e_B^b = e_a{}^B e_c{}^C h_{BC} g^{bc} = h_{ac} g^{bc} = h_a{}^b. \tag{2.3.27}$$

From the result above, we see that contraction of tensors $e_a{}^A$ and e_B^b with respect to the matter space indices provides the projection tensor with mixed indices. In comparison, contracting the spacetime indices of the same quantities, as seen in equation (2.3.21) yields the matter space Kronecker delta.

We may now define the transformation of mixed tensors between matter space and spacetime using both covariant and contravariant projection vector quantities via the relations

$$S_{a\ldots}{}^{b\ldots} = e_a{}^A e_B{}^b \ldots S_{A\ldots}{}^{B\ldots}, \tag{2.3.28}$$

and

$$S_{A\ldots}{}^{B\ldots} = e_A{}^a e_b{}^B \ldots S_{a\ldots}{}^{b\ldots}. \tag{2.3.29}$$

Finally, using the former of the two transformations for $\delta_A{}^B$ yields

$$h_a{}^b = e_a{}^A e_B^b \delta_A{}^B = e_a{}^A e_A^b, \tag{2.3.30}$$

which shows that the matter space Kronecker delta is transformed to the spacetime projection tensor with mixed indices.

2.3.1 Variation of matter space quantities

In this section we demonstrate the Lagrangian and Eulerian variations (see section 2.2) of various quantities which are used for the derivation of the equations of motion for the medium. By definition, the Lagrangian variation of the matter space coordinates is zero

$$\Delta q^A = 0, \tag{2.3.31}$$

which, combined with equation (2.2.5) yields

$$\delta q^A = -\xi^a \nabla_a q^A, \tag{2.3.32}$$

The Lagrangian variation of the projection vectors ${e_a}^A$ vanishes as well

$$\begin{aligned} \Delta {e_a}^A &= \delta {e_a}^A + \pounds_\xi {e_a}^A = \delta {e_a}^A + \xi^b \partial_b {e_a}^A + {e_b}^A \partial_a \xi^b \\ &= \partial_a \left(\delta q^A\right) + \xi^b \partial_a \partial_b q^A + \left(\partial_b q^A\right)\left(\partial_a \xi^b\right) \\ &= \partial_a \left(\delta q^A\right) + \partial_a \left(\xi^b \partial_b q^A\right) = \partial_a \Delta q^A = 0, \end{aligned} \tag{2.3.33}$$

where we have used equations (2.2.5), (2.2.6) and (2.3.31). For a material tensor $S_{AB\ldots}$ that is a function of the matter space coordinates we then have

$$\Delta S_{AB\ldots} = \frac{\partial S_{AB\ldots}}{\partial q^C} \Delta q^C = 0. \tag{2.3.34}$$

Using equations (2.3.3) and (2.3.33) it is straightforward by direct substitution to show that the Lagrangian variation of the spacetime projection of $S_{AB\ldots}$ is zero as well

$$\Delta S_{ab\ldots} = 0. \tag{2.3.35}$$

Furthermore, if a spacetime tensor $S_{ab\ldots}$ has a vanishing Lie derivative along the material flow and is orthogonal to the 4-velocity in all indices then it is uniquely related to a "fixed" material tensor $S_{AB\ldots}$. The orthogonality to the 4-velocity ensures that the mapping from matter space to spacetime given by equations (2.3.3) and (2.3.23) will be one to one. Applying the Lie derivative on both sides of transformation (2.3.3) and using equation (2.3.14) we obtain

$$\begin{aligned} \pounds_u S_{ab\ldots} &= {e_a}^A {e_b}^B \ldots \; \pounds_u S_{AB\ldots} \\ &= {e_a}^A {e_b}^B \ldots \; u^c \nabla_c S_{AB\ldots}, \end{aligned} \tag{2.3.36}$$

where the second line follows from the fact that the material tensor $S_{AB\ldots}$ is a scalar with respect to spacetime. Therefore, the vanishing of the Lie derivative implies that the time derivative $u^c \nabla_c S_{AB\ldots}$ vanishes[10] and thus the tensor does not change along the flow of the 4-velocity. In that sense $S_{ab\ldots}$ is "fixed" or "materially constant". In other words, a materially constant tensor is not functionally dependent on proper time (i.e. does not change along the worldline of the observer) but depends only on

[10] The vanishing of the Lie derivative could also happen due the orthogonality of some of ${e_a}^A$ to the $u^c \nabla_c S_{AB\ldots}$ term. However, considering the general case that holds for any tensor, the time derivative has to vanish.

the material coordinates. The converse, i.e that a matter space tensor depending only on the matter space coordinates transforms to a Lie derived orthogonal to the flow spacetime tensor, can be shown by building equation (2.3.36) backwards. In this case the orthogonality of the spacetime tensor arises from the transformation (2.3.3). A similar statement for the general case of mixed material and spacetime tensors will be demonstrated in the next section where we will introduce a generalization of the Lie derivative.

Using equations (2.2.5) and (2.3.7), the Eulerian variation of the number density current is

$$\begin{aligned}\delta n^a &= -\pounds_\xi n^a + \Delta n^a = -\pounds_\xi n^a + \Delta\left(\frac{1}{3!}\epsilon^{abcd} n_{bcd}\right) \\ &= -\pounds_\xi n^a + \frac{1}{3!}\left(\Delta n_{bcd}\right)\epsilon^{abcd} + \frac{1}{3!}\left(\Delta\epsilon^{abcd}\right) n_{bcd} \\ &= n^b\nabla_b\xi^a - \xi^b\nabla_b n^a - \frac{1}{3!}\cdot\frac{1}{2} n_{bcd}\epsilon^{abcd} g^{st}\Delta g_{st} \\ &= n^b\nabla_b\xi^a - \xi^b\nabla_b n^a - n^a\left(\nabla_b\xi^b + \frac{1}{2}g^{st}\delta g_{st}\right),\end{aligned} \tag{2.3.37}$$

where we have used equations (A.1.9) and (A.1.13) from the Appendix. We see that the Eulerian variation of the number density current is related to the variation of the metric tensor as well as the infinitesimal displacement ξ^a. This result is useful for the derivation of the equations of motion for the medium [52].

2.4 Carter's material derivative

In this section, we provide the notion of materially constant tensors with indices upstairs. This serves as a generalisation of the Lie derived orthogonal spacetime tensors with indices downstairs of the previous section and was introduced by Brandon Carter in [23] and [27]. We start with the Lie derivative and then demonstrate the derivation of the formula for the material[11] derivative for a mixed spacetime tensor. Our initial assumption is that for a covariant, completely orthogonal tensor (i.e. $S_{ab\ldots c}u^a = S_{ab\ldots c}u^b = \ldots = S_{ab\ldots c}u^c = 0$), the material derivative, $\not{\pounds}$, is equal to the Lie derivative

$$\not{\pounds}_u S_{ab\ldots} = \pounds_u S_{ab\ldots}, \tag{2.4.1}$$

[11] Various authors use the term "convective derivative" instead of "material derivative".

which implies that for a scalar the material derivative is

$$\not{\mathcal{L}}_u \phi = u^a \partial_a \phi = u^a \nabla_a \phi. \tag{2.4.2}$$

Note that the material derivative is assumed to follow the Leibniz rule of differentiation. As we show, the material derivative provides information about the dynamical behaviour of quantities on matter space. The vanishing of the material derivative of a spacetime tensor orthogonal to the 4-velocity, means that the respective matter space tensor obtained through equation (2.3.29) (we remind the reader that the orthogonality ensures the mapping between matter space and spacetime is one to one) is constant along the 4-velocity and vice versa. Intuitively, the Kronecker delta, which is by definition a constant tensor, should be constant along the 4-velocity as well. This implies that the projection of the Kronecker delta from matter space to spacetime (i.e. the quantity ${h_a}^b$ see equation (2.3.30)) should be materially constant

$$\not{\mathcal{L}}_u {h_a}^b = 0. \tag{2.4.3}$$

We consider the equation above as a requirement for the material derivative and use it as an assumption in the following calculations. To calculate the material derivative of a mixed orthogonal tensor we start by working out the material derivative of the projection tensor h^{ab}. The material derivative of the covariant projection tensor h_{ab} is

$$\not{\mathcal{L}}_u h_{ab} = 2\nabla_{(a} u_{b)} + 2u_{(a}\dot{u}_{b)}, \tag{2.4.4}$$

and by using equations (2.4.3) and (2.4.4) we get

$$\begin{aligned} \not{\mathcal{L}}_u {h_a}^b = \not{\mathcal{L}}_u \left(h_{ac}h^{cb}\right) &= h_{ac}\not{\mathcal{L}}_u h^{cb} + h^{cb}\not{\mathcal{L}}_u h_{ac} \\ h^{da}h_{ac}\not{\mathcal{L}}_u h^{cb} &= -h^{da}h^{cb}\left(2\nabla_{(c}u_{a)} + 2u_{(a}\dot{u}_{c)}\right) \\ \not{\mathcal{L}}_u h^{db} &= -2\nabla^{(d}u^{b)} - 2u^{(d}\dot{u}^{b)}. \end{aligned} \tag{2.4.5}$$

The material derivative of an orthogonal mixed tensor is

$$\not{\mathcal{L}}_u {S_{a\ldots}}^{b\ldots} = \not{\mathcal{L}}_u \left(S_{ac\ldots}h^{cb}\ldots\right), \tag{2.4.6}$$

where we used the orthogonal projection tensor to raise the indices. Combining the above with equation (2.4.5) and applying Leibniz's rule, we find that

$$\not{\mathcal{L}}_u {S_{a\ldots}}^{b\ldots} = u^c\nabla_c {S_{a\ldots}}^{b\ldots} + {S_{c\ldots}}^{b\ldots}\nabla_a u^c + \ldots - {S_{a\ldots}}^{c\ldots}\left(\nabla_c u^b - u^b\dot{u}_c\right)\ldots. \tag{2.4.7}$$

Therefore, we conclude that the material derivative of an orthogonal mixed tensor is the Lie derivative for a mixed tensor minus the terms containing the 4-acceleration. So far we have calculated the material derivative for an orthogonal mixed tensor. Below we extend the definition so that it is applicable for any mixed spacetime tensor. To do this we need to employ the tranformation vectors defined in section 2.3. Using equations (2.3.14), (2.4.1), (2.4.3) and (2.3.30) we get

$$\begin{aligned} \not{\mathcal{L}}_u h_a{}^b &= \not{\mathcal{L}}_u \left(e_a{}^A e_A{}^b\right) \\ &= e_a{}^A \not{\mathcal{L}}_u e_A{}^b = 0, \end{aligned} \tag{2.4.8}$$

which implies that the transformation vector is materially constant

$$\not{\mathcal{L}}_u e_A{}^b = 0. \tag{2.4.9}$$

Additionally, substituting the definition (2.1.6) in equation (2.4.8) yields

$$\begin{aligned} \not{\mathcal{L}}_u h_a{}^b &= \not{\mathcal{L}}_u \delta_a{}^b - \not{\mathcal{L}}_u u_a u^b \\ &= -u_a \not{\mathcal{L}}_u u^b - u^b \not{\mathcal{L}}_u u_a = 0, \end{aligned} \tag{2.4.10}$$

and therefore the 4-velocity is materially constant. Using equation (2.4.7) to calculate equation (2.4.9) we get

$$\begin{aligned} & u^c \nabla_c e_A{}^a - e_A{}^c \nabla_c u^a - e_A{}^c \dot{u}_c u^a = 0 \\ & \dot{e}_A{}^a = e_A{}^c \nabla_c u^a + e_A{}^c \dot{u}_c u^a, \end{aligned} \tag{2.4.11}$$

while equation (2.1.22) used with covariant derivatives, implies that

$$\begin{aligned} & u^c \nabla_c e_a{}^A + e_c{}^A \nabla_a u^c = 0 \\ & \dot{e}_a{}^A = -e_c{}^A \nabla_a u^c. \end{aligned} \tag{2.4.12}$$

To obtain the material derivative formula for a general tensor (not necessarily orthogonal to the observer's 4-velocity) we first need to derive an equation similar to (2.3.29) for a general spacetime tensor. This is possible by considering the projection vectors (2.3.2) and additionally one more, along the 4-velocity. These four quantities[12] which are denoted with $\tilde{e}_a{}^{\hat{A}}$ (where $\hat{A} = 0, 1, 2, 3$) are defined through

$$\tilde{e}_a{}^{\hat{A}} = u_a, \qquad \hat{A} = 0, \tag{2.4.13}$$

[12]These four tensors are also called "tetrad" since they define a basis of 4 independent vectors at each point of spacetime [53, 54]. They are also sometimes refered to as "vierbein" or "vielbein" for the case of n independent vectors.

$$\tilde{e}_a{}^{\hat{A}} = e_a{}^A, \qquad \hat{A} = A = 1,2,3 \tag{2.4.14}$$

and

$$\tilde{e}_{\hat{A}}{}^a = u^a, \qquad \hat{A} = 0, \tag{2.4.15}$$

$$\tilde{e}_{\hat{A}}{}^a = e_A{}^a, \qquad \hat{A} = A = 1,2,3. \tag{2.4.16}$$

Since the 4-velocity is orthogonal to the transformation quantities $e_a{}^A$ and $e^a{}_A$, equations (2.4.13-2.4.16) satisfy a condition similar to the orthogonality condition given by equation (2.3.21) as follows

$$\tilde{e}_a{}^{\hat{A}} \tilde{e}_{\hat{B}}{}^a = \delta_{\hat{A}}{}^{\hat{B}}, \qquad \hat{A}, \hat{B} = 1,2,3 \tag{2.4.17}$$

$$\tilde{e}_a{}^{\hat{A}} \tilde{e}_{\hat{B}}{}^a = -\delta_{\hat{A}}{}^{\hat{B}}, \qquad \hat{A} = 0 \qquad \text{or} \qquad \hat{B} = 0. \tag{2.4.18}$$

The minus sign in equation (2.4.18) arises from the fact that the 4-velocity is a timelike vector. Using the previous definitions it is possible to rewrite equations (2.4.11) and (2.4.12) using $\tilde{e}_{\hat{A}}{}^a$ as

$$\dot{\tilde{e}}_{\hat{A}}{}^a = \tilde{e}_{\hat{A}}{}^c \nabla_c u^a + \tilde{e}_{\hat{A}}{}^c \dot{u}_c u^a, \tag{2.4.19}$$

and

$$\dot{\tilde{e}}_a{}^{\hat{A}} = -\tilde{e}_c{}^{\hat{A}} \nabla_a u^c - \tilde{e}_c{}^{\hat{A}} u^c \dot{u}_a. \tag{2.4.20}$$

It is obvious that for $\hat{A} = 0$ we get, in both cases, the 4-acceleration $\dot{u}_a$ while for $\hat{A} = 1,2,3$ we get equations (2.4.11, 2.4.12). Note that we have added the term $u^c \dot{u}_a$ in equation (2.4.20). This term does not alter equation (2.4.12) since it either provides an identity for the 4-acceleration, or it vanishes.
We are now in a position to calculate the material derivative for a general spacetime tensor $S_{a\ldots}{}^{b\ldots}$. We have

$$\begin{aligned} \not{\mathcal{L}}_u \left(S_{a\ldots}{}^{b\ldots} \tilde{e}_{\hat{A}}{}^a \tilde{e}_b{}^{\hat{B}} \ldots \right) &= u^c \nabla_c \left(S_{a\ldots}{}^{b\ldots} \tilde{e}_{\hat{A}}{}^a \tilde{e}_b{}^{\hat{B}} \ldots \right) \\ \not{\mathcal{L}}_u \left(S_{a\ldots}{}^{b\ldots} \right) \tilde{e}_{\hat{A}}{}^a \tilde{e}_b{}^{\hat{B}} \ldots &= \dot{S}_{a\ldots}{}^{b\ldots} \tilde{e}_{\hat{A}}{}^a \tilde{e}_b{}^{\hat{B}} \ldots + S_{a\ldots}{}^{b\ldots} \dot{\tilde{e}}_{\hat{A}}{}^a \tilde{e}_b{}^{\hat{B}} \ldots \\ &\quad + S_{a\ldots}{}^{b\ldots} \tilde{e}_{\hat{A}}{}^a \dot{\tilde{e}}_b{}^{\hat{B}} \ldots, \end{aligned} \tag{2.4.21}$$

where the first line equality follows from equation (2.4.2) since the quantity in the parentheses is a scalar from the spacetime point of view. Substituting equations (2.4.19) and (2.4.20) in the right-hand-side of the second line of the equation above we get

$$\begin{aligned}\not{\mathcal{L}}_u \left(S_{a\ldots}{}^{b\ldots}\right) \tilde{e}_{\hat{A}}{}^{a}\tilde{e}_{b}{}^{\hat{B}}\ldots &= \dot{S}_{a\ldots}{}^{b\ldots}\tilde{e}_{\hat{A}}{}^{a}\tilde{e}_{b}{}^{\hat{B}} + S_{a\ldots}{}^{b\ldots}\left(\tilde{e}_{\hat{A}}{}^{c}\nabla_c u^a + \tilde{e}_{\hat{A}}{}^{c}\dot{u}_c u^a\right)\tilde{e}_{b}{}^{\hat{B}}\ldots \\ &+ S_{a\ldots}{}^{b\ldots}\tilde{e}_{\hat{A}}{}^{a}\left(-\tilde{e}_{c}{}^{\hat{B}}\nabla_b u^c - \tilde{e}_{c}{}^{\hat{B}}u^c\dot{u}_b\right)\ldots \\ &= \left[\dot{S}_{a\ldots}{}^{b\ldots} + S_{c\ldots}{}^{b\ldots}\left(\nabla_a u^c + \dot{u}_a u^c\right)\ldots\right. \\ &\left. - S_{a\ldots}{}^{c\ldots}\left(\nabla_c u^b + u^b\dot{u}_c\right)\ldots\right]\tilde{e}_{\hat{A}}{}^{a}\tilde{e}_{b}{}^{\hat{B}}\ldots,\end{aligned} \tag{2.4.22}$$

where we have interchanged dummy indices a with c, and so finally we get

$$\not{\mathcal{L}}_u S_{a\ldots}{}^{b\ldots} = \dot{S}_{a\ldots}{}^{b\ldots} + S_{c\ldots}{}^{b\ldots}\left(\nabla_a u^c + \dot{u}_a u^c\right)\ldots - S_{a\ldots}{}^{c\ldots}\left(\nabla_c u^b + u^b\dot{u}_c\right)\ldots. \tag{2.4.23}$$

which is the formula for the material derivative of a general spacetime tensor.

The material derivative generalises the Lie derivative in terms of material constancy according to the following argument, which is similar to the argument of the previous section. If $S_{a\ldots}{}^{b\ldots}$ is an orthogonal tensor with vanishing material derivative then the respective matter space tensor is materially constant along the worldlines of u^a. This arises, in analogy to equation (2.3.36), from equation (2.4.21). In this case the projection vectors appearing in the equality are only those of matter space (i.e. $e_a{}^A$) and the quantity in the parentheses on the left hand side is the matter space tensor $S_{A\ldots}{}^{B\ldots}$. Vanishing of the material derivative of the spacetime tensor (i.e the term $\not{\mathcal{L}}_u \left(S_{a\ldots}{}^{b\ldots}\right) \tilde{e}_{\hat{A}}{}^{a}\tilde{e}_{b}{}^{\hat{B}}\ldots$ in the left hand side) implies that the time derivative of the respective matter space tensor (i.e. the term $u^c\nabla_c\left(S_{a\ldots}{}^{b\ldots}\tilde{e}_{\hat{A}}{}^{a}\tilde{e}_{b}{}^{\hat{B}}\ldots\right)$ in right hand side) is vanishing as well. Considering that the worldlines of u^a are parametrised by the proper time τ then, from the matter space point of view, the matter space tensor is independent of the proper time and thus it is materially constant (i.e. a function of the material coordinates q^A only).

CHAPTER 3

Variational principle

Assuming a Lagrangian density which is a function of some scalar, vector or tensor fields, the action of our system is given by the integral on all spacetime Ω of the Lagrangian density under consideration

$$I_{\text{tot}} = \int_{\Omega} \mathcal{L}_{\text{tot}} \sqrt{-g}\, \mathrm{d}^4 x, \tag{3.0.1}$$

where $\sqrt{-g}\, d^4x$ is the invariant volume element at each point on Ω and g is the determinant of the metric tensor. $\mathcal{L}_{\text{tot}}$ is usually decomposed in a sum of Lagrangians each related to some aspects of the system. In order to obtain the equations of motion for the fields we demand that the Eulerian variation of the action is zero, $\delta I_{\text{tot}} = 0$. It follows from equation (3.0.1) that

$$\int_{\Omega} \delta \left(\mathcal{L}_{\text{tot}} \sqrt{-g} \right) \mathrm{d}^4 x = 0. \tag{3.0.2}$$

The Euler-Lagrange equtions for a scalar field [55], say $\phi = \phi\left(x^a\right)$ described by the Lagrangian $\mathcal{L}_\phi = \mathcal{L}_\phi\left(\phi, \nabla_a\phi\right)$ follow from equation (3.0.2) through

$$\begin{aligned}
&\int_\Omega \left(\frac{\partial\mathcal{L}_\phi}{\partial\phi}\delta\phi + \frac{\partial\mathcal{L}_\phi}{\partial\nabla_a\phi}\delta\nabla_a\phi\right)\sqrt{-g}\mathrm{d}^4x = 0 \\
&\int_\Omega \left[\frac{\partial\mathcal{L}_\phi}{\partial\phi}\delta\phi - \left(\nabla_a\frac{\partial\mathcal{L}_\phi}{\partial\nabla_a\phi}\right)\delta\phi + \nabla_a\left(\frac{\partial\mathcal{L}_\phi}{\partial\nabla_a\phi}\delta\phi\right)\right]\sqrt{-g}\mathrm{d}^4x = 0,
\end{aligned} \tag{3.0.3}$$

where we used equation (2.2.6) and integrated by parts the second term in the first line. Also, we have assumed that the independent variables of the Lagrangian are the scalar field and its gradient $\nabla_a\phi$. The integral above can be separated into integrals, namely

$$\int_\Omega \left(\frac{\partial\mathcal{L}_\phi}{\partial\phi} - \nabla_a\frac{\partial\mathcal{L}_\phi}{\partial\nabla_a\phi}\right)\delta\phi\sqrt{-g}\mathrm{d}^4x \tag{3.0.4}$$

and

$$\int_\Omega \nabla_a\left(\frac{\partial\mathcal{L}_\phi}{\partial\nabla_a\phi}\delta\phi\right)\sqrt{-g}\mathrm{d}^4x. \tag{3.0.5}$$

The latter integral contains the divergence of the varied field. Using the divergence theorem [49] we transform this integral through

$$\int_\Omega \left(\nabla_a\delta V^a\right)\sqrt{-g}\,\mathrm{d}^4x = \int_{\partial\Omega}\left(\delta V^a\right)n_a\sqrt{|h|}\mathrm{d}^3x, \tag{3.0.6}$$

where $\delta V^a = \frac{\partial\mathcal{L}_\phi}{\partial\nabla_a\phi}\delta\phi$, $\partial\Omega$ denotes the 3-dimensional boundary of Ω, n_a is normal to $\partial\Omega$ boundary and $\sqrt{|h|}$ is the volume element of the boundary. According to the divergence theorem we may choose any 3-surfaces enclosing Ω. Assuming that $\partial\Omega$ extends to infinity and that the variations vanish at infinity, the above integral vanishes identically. This means, through equation (3.0.3), that the integral (3.0.4) has to vanish. Assuming that the variation of the action is independent of the variations of the fields (i.e. $\delta\phi$ in this case) the coefficient of the variation has to vanish and therefore we get

$$\frac{\partial\mathcal{L}_\phi}{\partial\phi} - \nabla_a\frac{\partial\mathcal{L}_\phi}{\partial\nabla_a\phi} = 0, \tag{3.0.7}$$

which is the Euler-Lagrange equation for the scalar field ϕ. This demonstrates the idea of the variational principle for the simplest case, i.e. that of a scalar field. Deriving equations for systems described by higher rank tensor fields, as in the following sections, is straightforward.

3.1 Einstein-Hilbert action

The Einstein-Hilbert action provides the equations of motion for a gravitational field without sources i.e. those of vacuum spacetime. Starting with the Einstein-Hilbert Lagrangian

$$\mathcal{L}_{\mathrm{EH}} = R, \tag{3.1.1}$$

where $R = R_{ab}g^{ab}$ is the Ricci scalar which is the trace of the Ricci tensor R_{ab}, the variation of the action is

$$\int_{\Omega} \delta \left(R\sqrt{-g}\right) \mathrm{d}^4 x = 0. \tag{3.1.2}$$

The integrated quantity is varied with respect to the metric tensor with indices upstairs, which yields

$$\delta \left(R\sqrt{-g}\right) = \sqrt{-g} \left(g^{ab}\, \delta R_{ab} + R_{ab}\, \delta g^{ab}\right) + R_{ab}\, g^{ab}\, \delta(\sqrt{-g}). \tag{3.1.3}$$

In order to calculate the expression above we have to calculate the variations of the Ricci tensor and the determinant of the metric.

3.1.1 Variation of the metric determinant

The quantity $\sqrt{-g}$ being a part of the volume element in equation (3.0.1), will appear in any Lagrangian under consideration. To derive the variation of $\sqrt{-g}$ we work as follows. The metric determinant g is given by

$$g = \sum_a g_{ab_i} \Delta^{ab_i}, \tag{3.1.4}$$

where in this equation the summation convention of repeating indices does not hold. b_i denotes a specific index (any of them), while Δ^{ab} is the cofactor matrix of the metric tensor given by

$$\Delta^{ab} = g\, g^{ab}. \tag{3.1.5}$$

Using equations (3.1.4) and (3.1.5) we get the intermediate result

$$\frac{\partial g}{\partial g_{ab}} = g\, g^{ab}, \tag{3.1.6}$$

and thus the derivative of $\sqrt{-g}$ with respect to the metric is

$$\frac{\partial\sqrt{-g}}{\partial g_{ab}} = \frac{1}{2}\sqrt{-g}g^{ab}. \tag{3.1.7}$$

It follows that the variation of $\sqrt{-g}$ with respect to g_{ab} is

$$\delta(\sqrt{-g}) = \frac{\partial\sqrt{-g}}{\partial g_{ab}}\delta g_{ab} = \frac{1}{2}\sqrt{-g}g^{ab}\delta g_{ab}. \tag{3.1.8}$$

Using (2.1.2) and that $\delta_a^{\ b}$ is constant the variation of the metric with indices downstairs is related to that with the indices upstairs through

$$\delta g_{ab} = -g_{ac}\, g_{bd}\, \delta g^{cd}. \tag{3.1.9}$$

It also follows that the partial derivative of the metric with indices upstairs with respect to the metric with indices downstairs is

$$\frac{\partial g^{ab}}{\partial g_{cd}} = -g^{a(c}g^{d)b}. \tag{3.1.10}$$

In the following section we proceed with the calculation of the variation of Christoffel symbols which is an intermediate step towards calculating the variation of the Riemann tensor and eventually of the Ricci tensor.

3.1.2 The variation of the Christoffel symbols

Varying the definition (2.1.18) of the Christoffel symbols yields

$$\begin{aligned}\delta\Gamma^a{}_{bc} =& \frac{1}{2}\delta g^{ad}\left(\partial_c g_{db} + \partial_b g_{dc} - \partial_d g_{bc}\right) \\ &+ \frac{1}{2}g^{ad}\left(\partial_c \delta g_{db} + \partial_b \delta g_{dc} - \partial_d \delta g_{bc}\right),\end{aligned} \tag{3.1.11}$$

and by using equations (2.1.18) and (3.1.9), after rearranging terms we get

$$\delta\Gamma^a{}_{bc} = \frac{1}{2}g^{ad}\left(\partial_c\delta g_{db} + \partial_b\delta g_{dc} - \partial_d\delta g_{bc} - 2\Gamma^f{}_{bc}\,\delta g_{df}\right). \tag{3.1.12}$$

Adding and subtracting $2\Gamma^f{}_{cd}\,\delta g_{fb}$ and $2\Gamma^f{}_{bd}\,\delta g_{fc}$ to the previous equation we obtain

$$\delta\Gamma^a{}_{bc} = \frac{1}{2}g^{ad}\left(\nabla_c\delta g_{bd} + \nabla_b\delta g_{cd} - \nabla_d\delta g_{bc}\right), \tag{3.1.13}$$

which demonstrates the fact that the variation of the Christoffel symbols is a tensor.

3.1.3 The variation of the Ricci tensor

In order to derive the variation of the Ricci tensor R_{ab} we start with the Riemann tensor ${R^a}_{bcd}$ defined by

$$ {R^a}_{bcd} = \partial_c {\Gamma^a}_{bd} - \partial_d {\Gamma^a}_{bc} + {\Gamma^e}_{bd} \, {\Gamma^a}_{ec} - {\Gamma^e}_{bc} \, {\Gamma^a}_{ed}. \tag{3.1.14} $$

Working in normal coordinates the Christoffel symbols vanish (${\Gamma^a}_{bc} = 0$) and the metric tensor is constant ($\partial_c g_{ab} = 0$) at an arbitrary point P_0. Then the Riemann tensor acquires the following simpler form

$$ {R^a}_{bcd} = \partial_c {\Gamma^a}_{bd} - \partial_d {\Gamma^a}_{bc}, \tag{3.1.15} $$

while the variation is (since δ commutes with the partial differentiation)

$$ \delta {R^a}_{bcd} = \partial_c \delta {\Gamma^a}_{bd} - \partial_d \delta {\Gamma^a}_{bc}. \tag{3.1.16} $$

Now since the variation of the Christoffel symbols is a tensor (see section 3.1.2) and the partial derivative is equivalent to the covariant derivative in normal coordinates, we obtain a tensor equation that holds in all coordinate systems

$$ \delta {R^a}_{bcd} = \nabla_c \delta {\Gamma^a}_{bd} - \nabla_d \delta {\Gamma^a}_{bc}. \tag{3.1.17} $$

Contracting a and d we get the variation of the Ricci tensor

$$ \delta R_{ab} = \nabla_c \delta {\Gamma^c}_{ab} - \nabla_b \delta {\Gamma^c}_{ac}, \tag{3.1.18} $$

which is known as the "Palatini equation". Substituting this result and equations (3.1.8), (3.1.9) in (3.1.3) we have

$$ \delta \left(R\sqrt{-g} \right) = -R^{ab} \, \delta g_{ab} + \frac{1}{2} R g^{ab} \, \delta g_{ab} + \nabla_c \left(g^{ab} \, \delta {\Gamma^c}_{ab} - g^{ac} \, \delta {\Gamma^b}_{ab} \right), \tag{3.1.19} $$

and thus equation (3.1.2) obtains the following form

$$\begin{aligned}&\int_\Omega \left(\frac{1}{2}Rg^{ab} - R^{ab}\right)\delta g_{ab}\sqrt{-g}\mathrm{d}^4x\\&+\int_\Omega \nabla_c\left(g^{ab}\,\delta\Gamma^c{}_{ab} - g^{ac}\,\delta\Gamma^b{}_{ab}\right)\sqrt{-g}\mathrm{d}^4x = 0.\end{aligned} \tag{3.1.20}$$

The second term of the previous equation, using the divergence theorem, vanishes. Finally, assuming that the integral is independent of the variation of the metric we obtain the Einstein equations for vacuum spacetime

$$G^{ab} \equiv R^{ab} - \frac{1}{2}Rg^{ab} = 0, \tag{3.1.21}$$

where G^{ab} is the Einstein tensor. Note also that the divergence of the Einstein tensor vanishes (i.e. $\nabla_a G^{ab} = 0$) due the contracted Bianchi identity [56].
In the following sections we consider systems that contain matter and energy and therefore we introduce additional terms to the Lagrangian. In those cases the matter-energy counterpart of the Einstein tensor is the Einstein-Hilbert stress-energy-momentum tensor[1], a symmetric second rank tensor denoted usually by T^{ab}. This tensor contains all the information about the matter-energy part of the system under consideration. In this case the Einstein equation becomes

$$G^{ab} = 8\pi T^{ab}. \tag{3.1.22}$$

From the vanishing of the divergence of the Einstein tensor we obtain

$$\nabla_a T^{ab} = 0, \tag{3.1.23}$$

which provides a set of conservation equations for the material part of the system.

3.2 Hydrodynamics of multifluids

Here, following Carter's formalism [57, 52], we will describe the dynamics of multifluid media. To do this we will employ the matter space description which was introduced in section 2.3, generalizing it for many components. We will derive the

[1]Various authors refer to this tensor as stress-energy or energy-momentum tensor. Here we use these names interchangeably. Note that in all cases we refer to the Einstein-Hilbert energy momentum tensor.

stress-energy-momentum tensor and the equations of motion for a multicomponent fluid focusing, for the sake of clarity, only on the hydrodynamic part of the medium. As in the single component case we will introduce the fundamental quantities describing the medium. To distinguish the various components we introduce additional indices denoted by the roman letters $\mathrm{x}, \mathrm{y}, \mathrm{z}$ that are evaluated in the range of the components. For example, in the case of a medium consisting of protons and neutrons we will have $\mathrm{x}, \mathrm{y}, \mathrm{z} = \mathrm{p}, \mathrm{n}$. The indices of the fluid's components indices (i.e. $\mathrm{x}, \mathrm{y}, \ldots$) are only labels for the fluid species and are shown upstairs or downstairs interchangeably. In general, other choices for the components of the fluid can be electrons, other elementary entities which collectively form fluids, or entropy. Furthermore, Einstein summation convention does not hold for the component indices and so any summation will be shown explicitly. Although entropy is intuitively more abstract than the other components (which consist of particles and therefore are pictured more clearly), it is considered, within this formulation, to be a component of the medium. Such a consideration is justified since at the lengthscales that hyrdodynamics apply, the entropy contained in a volume element in spacetime may be considered as a fluid, similarly to the other components. Furthermore the inclusion of entropy as a dynamical variable allows to describe systems that exhibit heat transfer [57].

Assuming that there are as many matter spaces as there are components, the number density[2] form for each component n^{x}_{abc} will be closed (see equation (2.3.5)). Following the calculation of section 2.3 the respective number density 4-current for each component n^a_{x} is given by

$$n^a_{\mathrm{x}} = \frac{1}{3!}\epsilon^{bcda} n^{\mathrm{x}}_{bcd}, \qquad \text{and} \qquad n^{\mathrm{x}}_{abc} = \epsilon_{dabc} n^d_{\mathrm{x}}, \tag{3.2.1}$$

and is related to the fluid's 4-velocity u^a_{x} through

$$n^a_{\mathrm{x}} = n_{\mathrm{x}} u^a_{\mathrm{x}}, \qquad \text{where} \qquad u^a_{\mathrm{x}} u^{\mathrm{x}}_a = -1. \tag{3.2.2}$$

The 4-velocity of the x component is related to that of an other observer[3] u^a frame through

$$u^a_{\mathrm{x}} = \gamma_{\mathrm{x}} \left(u^a + v^a_{\mathrm{x}}\right), \quad \text{and} \quad \gamma_{\mathrm{x}} = \left(1 - v^a_{\mathrm{x}} v^{\mathrm{x}}_a\right)^{-\frac{1}{2}}, \tag{3.2.3}$$

[2] Although entropy is not characterised by a "number", contrary to the other material components (where e.g. 'proton number' makes sense), we refer to n_{x} in all cases as 'number density'.

[3] This is an arbitrary observer introduced here to demonstrate the transformation between frames.

where $v^a_{\rm x}$ is the relative spatial velocity between frames u^a and $u^a_{\rm x}$ (with $u^a v^{\rm x}_a = 0$) and $\gamma_{\rm x}$ is the Lorentz factor. Following equation (2.3.8) the number density current is conserved for each component seperately

$$\nabla_a n^a_{\rm x} = 0. \tag{3.2.4}$$

The square of $n^a_{\rm x}$ is given by $n^2_{\rm x} = -n^a_{\rm x} n^{\rm x}_a$ where we introduced the minus sign to ensure that it is a positive quantity. In a similar manner, the contraction of number density currents between different components is given by $n^2_{\rm xy} = -n^a_{\rm x} n^{\rm y}_a$. The Eulerian variation of $n^a_{\rm x}$ is given by equation (2.3.37)

$$\delta n^a_{\rm x} = n^b_{\rm x} \nabla_b \xi^a_{\rm x} - \xi^b_{\rm x} \nabla_b n^a_{\rm x} - n^a_{\rm x} \left(\nabla_b \xi^b_{\rm x} + \frac{1}{2} g^{cd} \delta g_{cd} \right). \tag{3.2.5}$$

To describe the medium we assume a Lagrangian density $\mathcal{L}_F$ which is a function of $n^2_{\rm x}$ and $n^2_{\rm xy}$. This implies that the medium is locally isotropic as there are no preferred directions. The dependence on these invariants implies that we are looking exclusively into the hydrodynamic part of the medium. Calculating variations with respect to the number density currents we obtain the equations of motion for the fluids and with respect to the metric tensor we obtain the energy momentum tensor of the medium. The variation of $\mathcal{L}_{\rm F}\sqrt{-g}$ with respect the number density current and the metric yields

$$\delta(\sqrt{-g}\mathcal{L}_{\rm F}) = \sqrt{-g} \sum_{\rm x} \frac{\partial \mathcal{L}_{\rm F}}{\partial n^a_{\rm x}} \delta n^a_{\rm x} + \left(\mathcal{L}_{\rm F} \frac{\partial \sqrt{-g}}{\partial g_{ab}} + \sqrt{-g} \frac{\partial \mathcal{L}_{\rm F}}{\partial g_{ab}} \right) \delta g_{ab}, \tag{3.2.6}$$

and by considering that the Lagrangian is a function of $n^2_{\rm x}$ and $n^2_{\rm xy}$ the previous result obtains the following form

$$\begin{aligned} \delta(\sqrt{-g}\mathcal{L}_{\rm F}) =& \sqrt{-g} \sum_{\rm x} \left(\frac{\partial \mathcal{L}_{\rm F}}{\partial n^2_{\rm x}} \frac{\partial n^2_{\rm x}}{\partial n^a_{\rm x}} + \sum_{\rm x \neq y} \frac{\partial \mathcal{L}_{\rm F}}{\partial n^2_{\rm xy}} \frac{\partial n^2_{\rm xy}}{\partial n^a_{\rm x}} \right) \delta n^a_{\rm x} + \mathcal{L}_{\rm F} \frac{\partial \sqrt{-g}}{\partial g_{ab}} \delta g_{ab} \\ &+ \sqrt{-g} \sum_{x} \left(\frac{\partial \mathcal{L}_{\rm F}}{\partial n^2_{\rm x}} \frac{\partial n^2_{\rm x}}{\partial g_{ab}} + \frac{1}{2} \sum_{x \neq y} \frac{\partial \mathcal{L}_{\rm F}}{\partial n^2_{\rm xy}} \frac{\partial n^2_{\rm xy}}{\partial g_{ab}} \right) \delta g_{ab}. \end{aligned} \tag{3.2.7}$$

The reason behind the $\frac{1}{2}$ factor in the coefficient of the metric variation is that $n^2_{\rm xy} = n^2_{\rm yx}$ and each term should be considered once. Note that the $\frac{1}{2}$ factor is not introduced in the number density current variation, because $n^2_{\rm xy}$ should be taken into account for each fluid component separately. Since we want to present a general result valid for multiple equations of state, we have not yet determined the exact functional form of $\mathcal{L}_{\rm F}$. It is therefore useful to use the following quantities to denote the partial derivatives of $\mathcal{L}_{\rm F}$

$$\mathcal{B}^{\rm x} = -2\frac{\partial \mathcal{L}_{\rm F}}{\partial n_{\rm x}^2} \quad \text{and} \quad \mathcal{A}^{\rm xy} = -\frac{\partial \mathcal{L}_{\rm F}}{\partial n_{\rm xy}^2}. \tag{3.2.8}$$

Using these equations, the variation now reads

$$\begin{aligned}\delta(\sqrt{-g}\mathcal{L}_{\rm F}) = \sqrt{-g}\sum_{\rm x}\left(\mathcal{B}^{\rm x} n_a^{\rm x} + \sum_{\rm x\neq y}\mathcal{A}_{\rm xy} n_a^{\rm y}\right)\delta n_{\rm x}^a + \frac{1}{2}\sqrt{-g}\mathcal{L}_{\rm F} g^{ab}\delta g_{ab} \\ + \sqrt{-g}\sum_{\rm x}\left(\frac{1}{2}\mathcal{B}^{\rm x} n_{\rm x}^a n_{\rm x}^b + \frac{1}{2}\sum_{x\neq y}\mathcal{A}^{\rm xy} n_{\rm x}^a n_{\rm y}^b\right)\delta g_{ab}.\end{aligned} \tag{3.2.9}$$

The momentum of each fluid component $\mu_a^{\rm x}$ is defined through

$$\mu_a^{\rm x} = \frac{\partial \mathcal{L}}{\partial n_a^{\rm x}}, \tag{3.2.10}$$

where $\mathcal{L}$ denotes the Lagrangian of the system under consideration (in the present section we have $\mathcal{L} = \mathcal{L}_{\rm F}$). Therefore the momenta are given by equation (3.2.9) through

$$\mu_a^{\rm x} = \mathcal{B}^{\rm x} n_a^{\rm x} + \sum_{\rm x\neq y}\mathcal{A}_{\rm xy} n_a^{\rm y}. \tag{3.2.11}$$

As we can see in the above expression, the momentum for a specific component does not depend solely on the number density current of that component but there are additional terms that are related to the rest of the components since they are coupled through $A_{\rm xy}$. This relation between the momenta and the number density currents of other components is known as "entrainment" and shows that the momentum is not aligned with the respective number density current [58, 59, 57, 60, 61]. Using the momenta we are able to define a 3rd rank contravariant totally antisymmetric tensor as we did for $n_{\rm x}^a$

$$\mu_{\rm x}^{abc} = \epsilon^{dabc}\mu_d^{\rm x}. \tag{3.2.12}$$

Additionally, in the case one of the components is entropy then the related momentum $\mu_{\rm s}$ is the temperature [52]. Equation (3.2.9) can be written

$$\delta(\sqrt{-g}\mathcal{L}_{\rm F}) = \sqrt{-g}\sum_{\rm x}\mu_a^{\rm x}\delta n_{\rm x}^a + \frac{1}{2}\sqrt{-g}\left(\mathcal{L}_{\rm F} g^{ab} + \sum_{\rm x}\mu_{\rm x}^b n_{\rm x}^a\right)\delta g_{ab}, \tag{3.2.13}$$

and by using equation (3.2.5) to substitute $\delta n^a_{\rm x}$ we get

$$\begin{aligned}\delta(\sqrt{-g}\mathcal{L}_{\rm F}) = {} & \frac{1}{2}\sqrt{-g}\left(\Psi_{\rm F} g^{cd} + \sum_x \mu^c_{\rm x} n^d_{\rm x}\right)\delta g_{cd} \\ & + \sqrt{-g}\sum_{\rm x}\left(\mu^{\rm x}_a n^b_{\rm x}\nabla_b \xi^a_{\rm x} - \mu^{\rm x}_a \xi^b_{\rm x}\nabla_b n^a_{\rm x} - \mu^{\rm x}_a n^a_{\rm x}\nabla_b \xi^b_{\rm x}\right).\end{aligned} \tag{3.2.14}$$

In the above expression, $\Psi_{\rm F}$ is the generalized pressure of the multifluid, defined as

$$\Psi_{\rm F} = \mathcal{L}_{\rm F} - \sum_{\rm x}\mu^{\rm x}_a n^a_{\rm x}. \tag{3.2.15}$$

After some manipulation in the last term of equation (3.2.14) (see section A.1.3) the variation obtains the final form

$$\begin{aligned}\delta(\sqrt{-g}\mathcal{L}_{\rm F}) = {} & \frac{1}{2}\sqrt{-g}\left(\Psi_{\rm F} g^{cd} + \sum_x \mu^c_{\rm x} n^d_{\rm x}\right)\delta g_{cd} \\ & + \sqrt{-g}\sum_{\rm x}(-f^{\rm x}_a \xi^a_{\rm x}) + \sum_{\rm x}\nabla_b\left(\frac{1}{2}\mu^{bef}_{\rm x} n^{\rm x}_{efa}\xi^a_{\rm x}\right),\end{aligned} \tag{3.2.16}$$

where $f^{\rm x}_a$ is the force density for component {x} given by

$$f^{\rm x}_a = n^b_{\rm x}\omega^{\rm x}_{ba}, \tag{3.2.17}$$

and $\omega^{\rm x}_{ab}$ is the vorticity of component {x}, a second rank, covariant, totally antisymmetric tensor

$$\omega^{\rm x}_{ab} = 2\nabla_{[a}\mu^{\rm x}_{b]}. \tag{3.2.18}$$

Assuming that the variation of the Lagrangian is independent of $\xi^a_{\rm x}$, and noting that $\xi^a_{\rm x}$ are independent variables, the equations of motion for each species is given by

$$f^{\rm x}_a = 0. \tag{3.2.19}$$

The stress-energy-momentum tensor T^{ab} for a Lagrangian density $\mathcal{L}$ is obtained by varying the Lagrangian with respect to the metric and is given by

$$T^{ab}\delta g_{ab} = \frac{2}{\sqrt{-g}}\delta\left(\sqrt{-g}\mathcal{L}\right). \tag{3.2.20}$$

For the system under consideration the above expression appears in the first term in equation (3.2.16) and the respective tensor T_{F}^{ab} is

$$T_{\mathrm{F}}^{ab} = \Psi_{\mathrm{F}} g^{ab} + \sum_{\mathrm{x}} \mu_{\mathrm{x}}^{a} n_{\mathrm{x}}^{b}. \tag{3.2.21}$$

Finally, the last term of equation (3.2.16) is a divergence containing the infinitesimal displacements ξ_{x}^{a} and by transforming that part to a surface integral it will vanish, according to the divergence theorem (see eq. (3.0.6)).

3.2.1 The case of a two-constituent single fluid

A special case of the multifluids discussed in the previous section is a single fluid with two components. Since there are two constituents, the Lagrangian is a function of the two number densities n_{x}, where in this case $\mathrm{x} = \{\mathrm{i}, \mathrm{s}\}$ are assumed to be ions and entropy. The fact that this is a single fluid means it is characterised by a single 4-velocity which is tangent to the worldlines of the medium's particles. Therefore the number density currents are given by

$$n_{\mathrm{i}}^{a} = n_{\mathrm{i}} u_{\mathrm{i}}^{a}, \tag{3.2.22}$$

and

$$n_{\mathrm{s}}^{a} = n_{\mathrm{s}} u_{\mathrm{i}}^{a}, \tag{3.2.23}$$

where u_{i}^{a} is the fluid's 4-velocity (with $u_{\mathrm{i}}^{a} u_{a}^{\mathrm{i}} = -1$) and n_{i}, n_{s} are the ion number density and entropy number density, respectively. As in the previous section we have $n_{\mathrm{x}}^{2} = -n_{\mathrm{x}}^{a} n_{a}^{\mathrm{x}}$ and the conservation of number density currents, given by equation (3.2.4). Additionally, since this fluid is characterised by a single 4-velocity we have only one matter space. There is only one infinitesimal displacement ξ^{a}, and the variation on the number density current is given instead of equation (3.2.5) is given by

$$\delta n_{\mathrm{x}}^{a} = n_{\mathrm{x}}^{b} \nabla_{b} \xi^{a} - \xi^{b} \nabla_{b} n_{\mathrm{x}}^{a} - n_{\mathrm{x}}^{a} \left(\nabla_{b} \xi^{b} + \frac{1}{2} g^{cd} \delta g_{cd} \right). \tag{3.2.24}$$

In this case since the two number density currents are aligned (parallel to u^{a}) there is no entrainment. Thus, the momenta using equation (3.2.10) are given by

$$\mu_{a}^{\mathrm{x}} = \mathcal{B}^{\mathrm{x}} n_{a}^{\mathrm{x}} \tag{3.2.25}$$

where $\mathcal{B}^{\mathrm{x}}$ is given by equation (3.2.8). The generalised pressure Ψ_{F} is given by equation (3.2.15). The variation of the Lagrangian for the two-constituent fluid is given by

$$\begin{aligned}\delta(\sqrt{-g}\mathcal{L}_{\mathrm{F}}) = \frac{1}{2}\sqrt{-g}\left(\Psi_{\mathrm{F}} g^{cd} + \sum_{x}\mu^{c}_{\mathrm{x}} n^{d}_{\mathrm{x}}\right)\delta g_{cd} \\ + \sqrt{-g}\sum_{\mathrm{x}}(-f^{\mathrm{x}}_{a}\xi^{a}) + \sum_{\mathrm{x}}\nabla_{b}\left(\frac{1}{2}\mu^{bef}_{\mathrm{x}} n^{\mathrm{x}}_{efa}\xi^{a}\right).\end{aligned} \tag{3.2.26}$$

As in the previous section the last term vanishes, while the first term is the Einstein-Hilbert energy momentum tensor of the fluid given by equation (3.2.21). The second term, provides the equation of motion for the fluid. The f^{a}_{x} is related to vorticity through equation (3.2.17) and the vorticity in turn is given by equation (3.2.18). Here we have only one equation of motion, given by

$$\sum_{\mathrm{x}} f^{\mathrm{x}}_{a} = 0, \tag{3.2.27}$$

since, as we mentioned before, there is only one ξ^{a}.

It also worth mentioning the following aspect of the two-constituent single fluid. Combining the number density currents conservation given by equation (3.2.4) for n_{i} and n_{s} we get

$$-\nabla_{a} u^{a} = \frac{\dot{n}_{\mathrm{i}}}{n_{\mathrm{i}}} = \frac{\dot{n}_{\mathrm{s}}}{n_{\mathrm{s}}}. \tag{3.2.28}$$

It follows that

$$\begin{aligned}\frac{\dot{n}_{\mathrm{i}}}{n_{\mathrm{i}}} &= \frac{\dot{n}_{\mathrm{s}}}{n_{\mathrm{s}}} \\ u^{a}_{\mathrm{i}}\nabla_{a}(\log n_{\mathrm{i}}) &= u^{a}_{\mathrm{i}}\nabla_{a}(\log n_{\mathrm{s}}) \\ u^{a}_{\mathrm{i}}\nabla_{a}\left(\log\frac{n_{\mathrm{s}}}{n_{\mathrm{i}}}\right) &= 0 \\ u^{a}_{\mathrm{i}}\nabla_{a}\frac{n_{\mathrm{s}}}{n_{\mathrm{i}}} &= 0,\end{aligned} \tag{3.2.29}$$

which means that the ratio $\frac{n_{\mathrm{s}}}{n_{\mathrm{i}}}$ is conserved along the fluid worldlines. The number densities are physical quantities divided by volume, n_{i} is ion particle number per volume and n_{s} is entropy per volume. It follows, the previously mentioned ratio is entropy per particle number, which is also referred to as specific entropy[4] [48, 62].

[4]The definition here of specific entropy is different from the bibliography by a constant multiplicative factor, which is the fixed rest mass per baryon. Nevertheless, this difference does not alter equation (3.2.29).

The conservation of specific entropy Σ along u_{i}^a

$$u_{\mathrm{i}}^a \nabla_a \Sigma = 0, \tag{3.2.30}$$

is referred to as the "adiabatic" [62, 41] or sometimes "isentropic" condition [48]. In this thesis we will use the former.

3.3 Electromagnetism

In this section we will look into electromagnetism. We will first discuss linear media, describing the classical vacuum as the simplest case of a linear medium, the isotropic media, and media with spontaneous excitation. We will also investigate electromagnetism in non-linear media, by describing a general Lagrangian used in quantum electrodynamics as a simple isotropic model of a non-linear medium. We also we the material derivative to provide an expression for non-linear media that resembles the respective of the linear case. Subsequently, we discuss the coupling between electromagnetism and matter and how the electromagnetic and material properties of a medium relate.

3.3.1 Electromagnetism in linear media

The electromagnetic field is described by the Faraday tensor F_{ab} an antisymmetric second rank tensor which is related to the 4-vector potential A_a through[5]

$$F_{ab} = \nabla_a A_b - \nabla_b A_a. \tag{3.3.1}$$

The Faraday tensor may be defined with partial instead of covariant derivatives because the Christoffel symbols are symmetric in the lower indices and cancel. The Lagrangian of electromagnetism in linear media [63], is given by

$$\mathcal{L}_{\mathrm{EM}}^{\mathrm{lin}} = -\frac{1}{8}\chi^{abcd} F_{ab} F_{cd} + j_{\mathrm{F}}^a A_a, \tag{3.3.2}$$

[5]As we will show in section 3.3.6 the completely antisymmetric part of the covariant derivative of Faraday tensor is vanishing. From a geometric point of view this means that the Faraday tensor is a closed 2-form which implies that (at least) locally it is exact i.e. it may be written in terms of a non-unique 4-vector as in equation (3.3.1). See [42] for a discussion on closed and exact forms.

where χ^{abcd} is the constitutive tensor which contains the information about the electrical and magnetic properties of the medium, j_{F}^{a} is the "free 4-current". This Lagrangian is linear in the sense that the constitutive tensor is not a function of the electromagnetic field (i.e. the Faraday tensor or the 4-vector potential). Additionaly it is described by a sum of two terms, the first is source-free which means that it is valid in the case where electromagnetic currents are absent while the second takes into account currents. The currents are closely related to the material part of the system, since they are moving particles carrying charge. In this section we discuss only the first term because we would like to focus on the electromagnetic properties of the medium in the case of absence of source terms. We deal with the currents in section 3.4.1.

Similarly to the classical (non-covariant) description of electromagnetism in media [64, 65, 66], we introduce the electromagnetic excitation tensor H^{ab} which is defined through

$$H^{ab} = -2\frac{\partial \mathcal{L}_{\mathrm{EM}}}{\partial F_{ab}}. \tag{3.3.3}$$

This expression is similar to the definition (3.2.10) of momenta of the previous section and provides a covariant constitutive relation between F_{ab} and H^{ab}. The antisymmetry of the Faraday tensor indicates through the definition above that the excitation tensor is antisymmetric as well. The definition (3.3.3) holds for all materials, and is not limited to Lagrangians of the form (3.3.2)[6]. Using the definition (3.3.3) for the Lagrangian of linear media given in equation (3.3.2) we obtain

$$H^{ab} = \frac{1}{2}\chi^{abcd}F_{cd}, \tag{3.3.4}$$

which is the equation where linearity becomes obvious. The excitation tensor is related linearly to the Faraday tensor through a multiplication factor which is the constitutive tensor. Combining the definition (3.3.3) and the equaton above we may write the Lagrangian given in equation (3.3.2) as

$$\mathcal{L}_{\mathrm{EM}}^{\mathrm{lin}} = -\frac{1}{4}H^{ab}F_{ab} + j_{\mathrm{F}}^{a}A_{a}. \tag{3.3.5}$$

As shown in a following section, writing the Lagrangian of a system using the above form is possible only in the linear case.

Following the classical description of electromagnetism we introduce an additional field, the magnetisation-polarisation tensor M^{ab}, that is related to H^{ab} and F^{ab}

[6] For that reason we use $\mathcal{L}_{\mathrm{EM}}$ in the definition rather than $\mathcal{L}_{\mathrm{EM}}^{\mathrm{lin}}$ which implies the linear case.

through

$$H^{ab} = F^{ab} + M^{ab}. \tag{3.3.6}$$

This equation holds for all materials, both linear and non-linear [64, 65, 67, 68] and can be interpreted in the following way. The excitation tensor will always contain the linear vacuum part (described by the first term in equation 3.3.6) and a second part, either linear or non-linear (described by the second term). The magnetisation-polarisation tensor is also antisymmetric, following from the antisymmetry of F^{ab} and H^{ab}. Similarly the electromagnetic field the "free" current j^a_{F} is related to the total j^a and "bound" j^a_{B} currents through

$$j^a = j^a_{\mathrm{F}} + j^a_{\mathrm{B}}. \tag{3.3.7}$$

The term "bound", as in classical electromagnetism, implies that the current is assembled of a large number of microscopic currents. Each of them is confined to move in a microscopic region of spacetime according to the properties of medium. Hence, these microscopic currents do not flow freely in the medium, and from the macroscopic point of view their motion collectively appears as a macroscopic current, which is referred to as the "bound" current. In contrast, the "free" current is free to flow in the medium on a macroscopic scale and for that reason it is characterised as "free".

As mentioned at the beginning of section 3.3.1 the constitutive tensor χ^{abcd} contains all the information about the electric and magnetic properties of the medium under consideration. We may obtain the constitutive tensor of the Lagrangian given in equation (3.3.2) through

$$\chi^{abcd} = 2\frac{\partial H^{ab}}{\partial F_{cd}} = -4\frac{\partial^2 \mathcal{L}_{EM}}{\partial F_{ab}\partial F_{cd}}. \tag{3.3.8}$$

The equations above as discussed in a subsequent section also serve as definitions for the non-linear constitutive tensor in non-linear electromagnetic media. We also have the symmetries

$$\chi^{abcd} = -\chi^{abdc}, \tag{3.3.9}$$

and

$$\chi^{abcd} = -\chi^{bacd}, \tag{3.3.10}$$

which arise by the symmetries of F_{ab} and H^{ab} and equation (3.3.8). These two symmetries reduce the number of independent components of χ^{abcd} to 36. Additionally the commutativity of partial differentiation

$$\frac{\partial^2 \mathcal{L}_{EM}}{\partial F_{ab} \partial F_{cd}} = \frac{\partial^2 \mathcal{L}_{EM}}{\partial F_{cd} \partial F_{ab}}, \tag{3.3.11}$$

in definition (3.3.8) provides the following symmetry

$$\chi^{abcd} = \chi^{cdab}, \tag{3.3.12}$$

which reduces the independent components of the constitutive tensor to 21 [63]. Following equation (3.3.6) in the case of linear media we may write χ^{abcd} as a sum of two tensors

$$\chi^{abcd} = \chi_0^{abcd} + \varsigma^{abcd}, \tag{3.3.13}$$

where χ_0^{abcd} is the constitutive tensor for vacuum given by

$$\chi_0^{abcd} = g^{ac} g^{bd} - g^{ad} g^{bc}, \tag{3.3.14}$$

and ς^{abcd} is the susceptibility tensor of the medium which has the same symmetries as the constitutive tensor. Using equation (3.3.13), equation (3.3.4) becomes

$$H^{ab} = \frac{1}{2}\chi_0^{abcd} F_{cd} + \frac{1}{2}\varsigma^{abcd} F_{cd} = F^{ab} + \frac{1}{2}\varsigma^{abcd} F_{cd}, \tag{3.3.15}$$

where for the second equality we used the vacuum constitutive tensor given in equation (3.3.14). It follows that the magnetisation-polarisation tensor is related to the Faraday tensor through the linear relation

$$M^{ab} = \frac{1}{2}\varsigma^{abcd} F_{cd}. \tag{3.3.16}$$

3.3.2 The decomposition of electromagnetic quantities

Here we show the decomposition of the various electromagnetic quantities that have been introduced are decomposed into temporal and spatial parts. The decomposition of the Faraday tensor with respect to an observer with 4-velocity u_a, into the electric and magnetic parts is given by

$$F_{ab} = 2u_{[a} E_{b]} + \epsilon_{abcd} u^c B^d, \tag{3.3.17}$$

where $E_a = F_{ab}u^b$ is the electric field, $B^a = -\frac{1}{2}\epsilon^{abcd}u_b F_{cd}$ is the magnetic field. In a similar way, the excitation tensor decomposes as

$$H_{ab} = 2u_{[a}D_{b]} + \epsilon_{abcd}u^c H^d, \tag{3.3.18}$$

where $D_a = H_{ab}u^b$ is the electric displacement and $H^a = -\frac{1}{2}\epsilon^{abcd}u_b H_{cd}$ is called, since we have already used the name "magnetic field" for B_a, "magnetic intensity"[7]. In the same manner the magnetisation-polarisation tensor decomposes as

$$M_{ab} = 2u_{[a}P_{b]} - \epsilon_{abcd}u^c M^d, \tag{3.3.19}$$

where $P_a = M_{ab}u^b$ is the polarisation and $M^a = \frac{1}{2}\epsilon^{abcd}u_b M_{cd}$ is the magnetisation. Note here that there is a different sign in the definition of the magnetisation compared with the definitions of the magnetic field and H^a field. This sign difference arises from the fact that equations involving the magnetisation should resemble the equations as treated in textbooks of classical electromagnetism [64, 65]. Additionally, all the decomposed fields in equations (3.3.17), (3.3.18), and (3.3.19) are orthogonal to the observer's 4-velocity. The orthogonality arises from the antisymmetry of the tensors F_{ab}, H_{ab}, and M_{ab} respectively.
The 4-currents given in equation (3.3.7) decompose as

$$j^a_{\mathrm{F}} = \hat{\sigma}_{\mathrm{F}}u^a + J^a_{\mathrm{F}}, \tag{3.3.20a}$$

$$j^a_{\mathrm{B}} = \hat{\sigma}_{\mathrm{B}}u^a + J^a_{\mathrm{B}}, \tag{3.3.20b}$$

$$j^a = \hat{\sigma}u^a + J^a, \tag{3.3.20c}$$

where $\hat{\sigma}_{\mathrm{F}} = -u^a j^a_{\mathrm{F}}$ is the 'free' charge density and $J^a_{\mathrm{F}} = h^a{}_b j^b_{\mathrm{F}}$ is the spatially projected 'free' 3-current. The 'bound' $\hat{\sigma}_{\mathrm{B}}$, and total $\hat{\sigma}$ charge densities and spatially projected 3-currents are defined in a similar way. It follows from equation (3.3.7), that the projected 3-currents are related through

$$J^a = J^a_{\mathrm{B}} + J^a_{\mathrm{F}}, \tag{3.3.21a}$$

and

$$\hat{\sigma} = \hat{\sigma}_{\mathrm{B}} + \hat{\sigma}_{\mathrm{F}}. \tag{3.3.21b}$$

[7] There is an occasional conflict in the names of B^a and H^a due to the fact that the various scientific communities use different names for the fields. For example in astrophysics B^a is usually the magnetic field while in experimental material physics the magnetic field is H^a.

The constitutive tensor is also decomposed into temporal and spatial parts through,

$$\chi^{abcd} = 4\, u^{[a}\mathcal{E}^{b][c}u^{d]} + \epsilon^{ab}{}_{e}\epsilon^{cd}{}_{m}\mathcal{M}^{em} + 2\epsilon^{ab}{}_{e}\mathcal{X}^{e[c}u^{d]} + 2u^{[a}\mathcal{Y}^{b]e}\epsilon_{e}{}^{cd}, \tag{3.3.22}$$

where ϵ^{abc} is defined through equation (2.1.16), and $\mathcal{E}^{ab}$ is the relativistic electric permittivity tensor given by

$$\mathcal{E}^{ac} = -\chi^{abcd}u_b u_d, \tag{3.3.23}$$

$\mathcal{M}^{ac}$ is the relativistic inverse magnetic permeability tensor given by

$$\mathcal{M}^{ac} = \frac{1}{4}\epsilon^{a}{}_{ef}\epsilon^{c}{}_{km}\chi^{efkm}, \tag{3.3.24}$$

and $\mathcal{X}^{ac}$, $\mathcal{Y}^{ac}$ are cross-permittivities given by

$$\mathcal{X}^{ac} = -\frac{1}{2}\epsilon^{a}{}_{ef}\chi^{efcd}u_d, \tag{3.3.25}$$

and

$$\mathcal{Y}^{ac} = \frac{1}{2}\epsilon^{c}{}_{ef}\chi^{abef}u_b. \tag{3.3.26}$$

This decomposition is similar to the decomposition of the Riemann tensor into "gravito-electric", "gravito-magnetic" and mixed "gravito-electromagnetic" parts as discussed in [69]. By contracting one free index of each (3.3.23)-(3.3.26) with the 4-velocity and using properties (3.3.9), (3.3.10), it is obvious to show that these tensors are orthogonal to u^a. Additionally, symmetry (3.3.12) implies that $\mathcal{E}^{ab} = \mathcal{E}^{ba}$, $\mathcal{M}^{ab} = \mathcal{M}^{ba}$ and $\mathcal{Y}^{ab} = -\mathcal{X}^{ba}$. Using equation (3.3.22) with (3.3.17) and (3.3.18) we obtain the decomposition of equation (3.3.4) which provides the relativistic relations between the electric displacement and magnetic intensity with respect to the electric and magnetic fields, in a form similar to that of ordinary electromagnetism,

$$D^a = \mathcal{E}^{ab}E_b + \mathcal{Y}^{ab}B_b, \tag{3.3.27}$$

and

$$H^a = \mathcal{M}^{ab}B_b + \mathcal{X}^{ab}E_b. \tag{3.3.28}$$

As it can be seen from the last two equations, the linear relation (3.3.4) allows for possible coupling between the electric and the magnetic field through the cross-permittivities $\mathcal{Y}^{ab}$ and $\mathcal{X}^{ab}$.

Due to the symmetries given by equations (3.3.9), (3.3.10) and (3.3.12) there are only two non-vanishing and independent traces of the constitutive tensor. These are ${\chi_a}^{bad}$ and ${\chi^{ab}}_{ab}$. Using equation (3.3.22) the two traces in terms of the projected tensors are

$$ {\chi_a}^{bad} = \mathcal{E}^{bd} - {\mathcal{E}^a}_a u^b u^d + {\mathcal{M}^a}_a h^{bd} - \mathcal{M}^{bd} - \epsilon^{aeb}\mathcal{Y}_{ae}u^d - \epsilon^{aed}\mathcal{Y}_{ae}u^b, \tag{3.3.29} $$

where we also used the symmetry of the cross-permittivities $\mathcal{Y}^{ab} = -\mathcal{X}^{ba}$. The other trace is given by

$$ {\chi^{ab}}_{ab} = 2({\mathcal{E}^a}_a + {\mathcal{M}^a}_a). \tag{3.3.30} $$

Note also that there are two scalar invariants of the Faraday tensor, namely I and K given through

$$ I = 2F^{ab}F_{ab} = \chi_0^{abcd}F_{ab}F_{cd} = 4\left(B^2 - E^2\right), \tag{3.3.31} $$

where $E^2 = E_aE^a$ and $B^2 = B_aB^a$, and

$$ K = \epsilon^{abcd}F_{ab}F_{cd} = 8E_aB^a. \tag{3.3.32} $$

3.3.3 Linear media

In this section we demonstrate specific cases of linear media, namely the classical vacuum, the case of isotropic media and the case of media with spontaneous excitation.

3.3.3.1 Classical vacuum

The simplest case of a linear medium is the classical vacuum. The Lagrangian density is given by

$$ \mathcal{L}^0_{\mathrm{EM}} = -\frac{1}{4}F^{ab}F_{ab} = -\frac{1}{8}\chi_0^{abcd}F_{ab}F_{cd}. \tag{3.3.33} $$

where χ_0^{abcd} is given by equation (3.3.14). As already implied in equation (3.3.15) the excitation tensor is related to the Faraday tensor through

$$ H^{ab} = \frac{1}{2}\chi_0^{abcd}F_{cd} = F^{ab}. \tag{3.3.34} $$

Using equations (3.3.23)-(3.3.26) we find that the permittivity and permeability tensors acquire the trivial form

$$\underset{0}{\mathcal{E}}^{ab} = h^{ab}, \tag{3.3.35}$$

$$\underset{0}{\mathcal{M}}^{ab} = h^{ab}. \tag{3.3.36}$$

As was anticipated, since there is no coupling between the electric and magnetic field in vacuum, the cross-permittivities $\underset{0}{\mathcal{X}}^{ac}$, $\underset{0}{\mathcal{Y}}^{ac}$ vanish. Using equations (3.3.35) and (3.3.36) we can show that the relations between the D^a, H^a and E^a, B^a take the familiar form

$$D^a = E^a, \tag{3.3.37}$$

and

$$H^a = B^a. \tag{3.3.38}$$

3.3.3.2 Isotropic media

Isotropic media are characterised by the scalar electrical permitivity ε and the scalar magnetic permeability μ. Using this as starting point along with the form of excitation fields in classical vacuum (which is the simplest isotropic medium) we can work backwards to obtain the constitutive tensor. For an observer at rest with respect to isotropic medium we have [64]

$$D^a = \varepsilon E^a, \tag{3.3.39}$$

and

$$H^a = \mu B^a, \tag{3.3.40}$$

where ε and μ are the scalar permittivity and permeability of the medium respectively. Therefore the permittivity and permeability tensors should read

$$\mathcal{E}^{ab} = \varepsilon h^{ab}, \tag{3.3.41}$$

and

$$\mathcal{M}^{ab} = \mu h^{ab}, \tag{3.3.42}$$

while the cross-permittivities are zero. It follows that $\mathcal{E}^a{}_a = 3\varepsilon$ and $\mathcal{M}^a{}_a = 3\mu$. The constitutive tensor then, with the respect to the observer at rest, obtains the simple form

$$\chi^{abcd} = 4\varepsilon\, u^{[a}h^{b][c}u^{d]} + \mu\left(h^{ac}h^{bd} - h^{ad}h^{bc}\right). \tag{3.3.43}$$

The two traces of the constitutive tensor become

$$\chi_a{}^{bad} = \varepsilon g^{bd} - 2\varepsilon u^b u^d + 2\mu h^{bd}, \tag{3.3.44}$$

and

$$\chi^{ab}{}_{ab} = 6(\varepsilon + \mu). \tag{3.3.45}$$

3.3.3.3 Media with spontaneous excitation field

In the previous sections we discussed linear media which do not possess any spontaneous excitation field. This means, as can be seen from equation (3.3.4), that in order for a medium to have non-zero excitation tensor there has to exist a non-zero electromagnetic field. In this paragraph we consider media with a spontaneous excitation field. This means there is a non-zero excitation tensor even if the Faraday tensor is vanishing. The spontaneous excitation field is formulated through an additional term added to the Lagrangian (that shown in equation (3.3.2)) given by

$$\mathcal{L}_{\mathrm{EM}}^{\mathrm{sp}} = -\frac{1}{2} \overset{\scriptscriptstyle 0}{H}{}^{ab} F_{ab}, \tag{3.3.46}$$

where $\overset{\scriptscriptstyle 0}{H}{}^{ab}$ is the spontaneous excitation field. The total Lagrangian for linear media with spontaneous electromagnetic excitation is then given by $\mathcal{L}_{\mathrm{EM}}^{\mathrm{lin,Sp}} = \mathcal{L}_{\mathrm{EM}}^{\mathrm{sp}} + \mathcal{L}_{\mathrm{EM}}^{\mathrm{lin}}$. Using the definition (3.3.3) we get for the excitation tensor the following expression

$$H^{ab} = \overset{\scriptscriptstyle 0}{H}{}^{ab} + \frac{1}{2}\chi^{abcd} F_{cd}, \tag{3.3.47}$$

and by decomposing this equation we get additional terms for the spontaneous electric displacement $\overset{\scriptscriptstyle 0}{D}{}^a$ and magnetic intensity $\overset{\scriptscriptstyle 0}{H}{}^a$ in equations (3.3.27) and (3.3.28) given in this case by

$$D^a = \overset{\scriptscriptstyle 0}{D}{}^a + \mathcal{E}^{ab} E_b + \mathcal{Y}^{ab} B_b, \tag{3.3.48}$$

and

$$H^a = \overset{\scriptscriptstyle 0}{H}{}^a + \mathcal{M}^{ab} B_b + \mathcal{X}^{ab} E_b. \tag{3.3.49}$$

Pyroelectric media exhibit spontaneous electric displacement [66] and thus are described by equation (3.3.48).

3.3.4 Non-linear Electromagnetic Media

So far we looked into linear media where the theory is well established. In this section we examine the electromagnetic properties of more general types of media. The reason we introduce non-linear media is that in some cases, for example in ferromagnetic materials, equation (3.3.4) does not hold. In these cases the excitation tensor is related to the Faraday tensor through a non-linear relation and as a result the constitutive tensor, still defined by (3.3.8), is not constant but a function of the Faraday tensor. Starting from this point the question to ask is: Does equation (3.3.5) still hold for a Lagrangian density describing a non-linear medium?
To work our way to the answer we first have to understand what equation (3.3.5) implies. The source free part of this equation (i.e the part that does not include the 4-current) using definition (3.3.3) is

$$\mathcal{L}_{\mathrm{EM}}^{\mathrm{lin,f}} = \frac{1}{2}\frac{\partial \mathcal{L}_{\mathrm{EM}}^{\mathrm{lin,f}}}{\partial F_{ab}} F_{ab}. \tag{3.3.50}$$

This is very similar to the second term (the first term of the series would be the Lagrangian evaluated at zero F_{ab}) of a Maclaurin series expansion for $\mathcal{L}_{\mathrm{EM}}^{\mathrm{lin,f}}$. The difference is that the partial derivative $\partial\mathcal{L}_{\mathrm{EM}}^{\mathrm{lin,f}}/\partial F_{ab}$ is not evaluated for any specific value of the Faraday tensor, and there is an extra factor $(\frac{1}{2})$ not matching the Maclaurin series. If this expression was indeed the second term of some approximating series for the Lagrangian we could assume that it holds exactly for linear media (since the constitutive tensor is independent of F_{ab} and the Lagrangian is quadratic in F_{ab} as given by equation (3.3.2)) while it could be a first order approximation for non-linear media. Thus the generalisation to non-linear media would just be the inclusion of higher order terms of the expansion to the desired accuracy. Infinite series are used in non-linear optics to express the magnetic intensity and electric displacement [67, 68] where the various non-linear effects are incorporated in the various terms of the expansion. However, in non-linear optics the Lagrangian formulation is not used and the starting point is the previously mentioned series expansions that serve as constitutive relations for the magnetic intensity and the electric displacement. In this work, since we are interested in results that arise from first principles, it is rather inconvenient to use the phenomenological treatment which is the usual approach in non-linear optics. Moreover, equation (3.3.50) is not trivially generalised to non-linear media (since it is not the largest term of some series expansion). Therefore, providing here the respective expression for the Lagrangian in the case of non-linear media will give us some insight into how to manipulate these media. The MacLaurin expansion for a general Lagrangian which is a function of the Faraday tensor is

$$\begin{aligned}\mathcal{L}_{\mathrm{EM}} = \mathcal{L}_{\mathrm{EM}}(0) + \left.\frac{\partial \mathcal{L}_{\mathrm{EM}}}{\partial F_{ab}}\right|_0 F_{ab} + \frac{1}{2!}\left.\frac{\partial^2 \mathcal{L}_{\mathrm{EM}}}{\partial F_{ab}\partial F_{cd}}\right|_0 F_{ab}F_{cd} \\ + \frac{1}{3!}\left.\frac{\partial^3 \mathcal{L}_{\mathrm{EM}}}{\partial F_{ab}\partial F_{cd}\partial F_{ef}}\right|_0 F_{ab}F_{cd}F_{ef} - \dots,\end{aligned} \tag{3.3.51}$$

Using equation (A.2.40) in Appendix A.2 it follows that any Lagrangian can be written as a function of the Faraday tensor and the derivatives with respect to the Faraday tensor as

$$\begin{aligned}\mathcal{L}_{\mathrm{EM}} = \mathcal{L}(0) + \frac{\partial \mathcal{L}_{\mathrm{EM}}}{\partial F_{ab}} F_{ab} - \frac{1}{2!}\frac{\partial^2 \mathcal{L}_{\mathrm{EM}}}{\partial F_{ab}\partial F_{cd}} F_{ab}F_{cd} \\ + \frac{1}{3!}\frac{\partial^3 \mathcal{L}_{\mathrm{EM}}}{\partial F_{ab}\partial F_{cd}\partial F_{ef}} F_{ab}F_{cd}F_{ef} - \dots,\end{aligned} \tag{3.3.52}$$

or in compact form as

$$\mathcal{L}_{\mathrm{EM}} = \mathcal{L}_{\mathrm{EM}}(0) + \sum_{N=1}^{+\infty}\left[\frac{(-1)^{N+1}}{N!}\,\underbrace{\frac{\partial^N \mathcal{L}_{\mathrm{EM}}}{\partial F_{ab}\partial F_{cd}\dots\partial F_{ef}}}_{N\ \text{terms}}\,\underbrace{F_{ab}F_{cd}\dots F_{ef}}_{N\ \text{terms}}\right]. \tag{3.3.53}$$

In Appendix A.2 we show how the previous formula is derived for a single variable function. Here we have considered the straightforward generalization of equation (A.2.40) for tensor fields.[8] Equation (3.3.53) is the generalisation of equation (3.3.50) for non-linear media. We would also like to mention here that equation (3.3.53) can be used when we prefer to avoid the calculation of the derivatives for specified values of the variables, in contrast with the Maclaurin series that restricts the derivative at a specific point. It does require, though, the knowledge of the value of the function at a specific point. Since we mainly calculate variations of the Lagrangian densities the term $\mathcal{L}_{\mathrm{EM}}(0)$ is not important and we will assume that it vanishes.
In the linear case the Lagrangian consists of, as stated previously, one term quadratic in F_{ab}. In this special case, all terms except the first and second order derivatives in

[8]The idea behind this generalization, which might seem to contain a logical leap, is that the Faraday tensor F_{ab}, as any other tensor, using the abstract index notation behaves as a single variable in the sense that the components do not appear explicitly as independent variables. We only have the field "F", and that is similar to having one variable. To give another example where tensors are treated as a single variable function, we mention the formula for the Taylor expansion of scalar functions of tensors, which is the same as the formula for single variable scalar functions (see [70]).

equation (3.3.51) vanish. As a result, the Lagrangian can be written by just using the second term multiplied with a constant as shown in equation (3.3.5). Conversely, if the Lagrangian density is not a quadratic monomial of the Faraday tensor [9] then there can not be a linear relation between the excitation and Faraday tensor as in equation (3.3.4) and eventually the Lagrangian cannot be written as in the form of equation (3.3.5).

In the analysis above we assumed a Lagrangian which is in general a function of F_{ab} and expanded it in series with respect to that tensor. We could have assumed that it is a function of the 4-vector potential and performed a similar expansion for it as well. The reason we did not consider this option is that the 4-vector potential is assumed in all cases to appear in the Lagrangian only as a source term, in the form of the second term in the right hand side of equation (3.3.2) . Hence, non-linearity as treated here is characterized entirely by the functional dependence of the Lagrangian to the Faraday tensor. Finally, equations (3.3.3) and (3.3.6) imply that in all cases the Lagrangian of electromagentism is of the form

$$\mathcal{L}_{\mathrm{EM}} = \mathcal{L}^{0}_{\mathrm{EM}} + \mathcal{L}_{\mathrm{Med}}, \tag{3.3.54}$$

and as a consequence the magnetisation-polarisation tensor is defined through

$$M^{ab} = -2\frac{\partial \mathcal{L}_{\mathrm{Med}}}{\partial F_{ab}}. \tag{3.3.55}$$

In the equations above, the $\mathcal{L}_{\mathrm{Med}}$ part of the Lagrangian contains any terms that are related to coupling between matter and the electromagnetic field including both the linear and the non-linear cases.

3.3.4.1 The Quantum Electrodynamics Lagrangian

A simple example model of a non-linear Lagrangian is that of Quantum Electrodynamics (QED). In the case of extremely intense electromagnetic fields, QED theory provides corrections [71, 72] to the electromagnetic Lagrangian and as a result it deviates from that given in equation (3.3.33). In that sense electromagnetism in vacuum with the QED corrections can be perceived as a non-linear medium. Such a Lagrangian will in general consist of $\mathcal{L}^{\mathrm{EM}}_{0}$ plus a non-linear correction which we

[9] In general any Lagrangian that is a monomial power product of the Faraday tensor can be treated in a way similar to that of the linear case. All these cases fall into the non-linear case though as we consider that the linear case is strictly that of section 3.3.1.

denote as $\mathcal{L}_1^{\mathrm{EM}}$. That is

$$\mathcal{L}_{\mathrm{QED}}^{\mathrm{EM}} = \mathcal{L}_0^{\mathrm{EM}} + \mathcal{L}_1^{\mathrm{EM}}. \tag{3.3.56}$$

Additionally, since the Lagrangian has to be Lorentz invariant [71] it will be a function of the invariants of the Faraday tensor[10] given by equations (3.3.31) and (3.3.32). We have used the decomposition for the Faraday tensor, given by equation (3.3.17), to derive the last equalities in terms of the electric and the magnetic field. Using the definition of the excitation tensor (3.3.3) and equation (3.3.33) we get

$$H^{ab} = \frac{1}{2}\chi_0^{abcd}F_{cd} - 2\left(\frac{\partial \mathcal{L}_1^{\mathrm{EM}}}{\partial I}\frac{\partial I}{\partial F_{ab}} + \frac{\partial \mathcal{L}_1^{\mathrm{EM}}}{\partial K}\frac{\partial K}{\partial F_{ab}}\right). \tag{3.3.57}$$

The derivatives of the Faraday invariants with respect to the Faraday tensor are given by

$$\frac{\partial I}{\partial F_{ab}} = 2\chi_0^{abcd}F_{cd}, \tag{3.3.58}$$

and

$$\frac{\partial K}{\partial F_{ab}} = 2\epsilon^{abcd}F_{cd}, \tag{3.3.59}$$

and by substituting equations (3.3.58) and (3.3.59) in equation (3.3.57) we get

$$H^{ab} = \frac{1}{2}\chi_0^{abcd}F_{cd} - 4\left(\frac{\partial \mathcal{L}_1^{\mathrm{EM}}}{\partial I}\chi_0^{abcd} + \frac{\partial \mathcal{L}_1^{\mathrm{EM}}}{\partial K}\epsilon^{abcd}\right)F_{cd}. \tag{3.3.60}$$

The respective non-linear constitutive tensor, which is given by equation (3.3.8), is

$$\begin{aligned}\chi_{\mathrm{QED}}^{abcd} = \chi_0^{abcd} - 8 \Bigg[& 2\frac{\partial^2 \mathcal{L}_1^{\mathrm{EM}}}{\partial I^2}\chi_0^{absp}\chi_0^{cdnt}F_{sp}F_{nt} + 2\frac{\partial^2 \mathcal{L}_1^{\mathrm{EM}}}{\partial K^2}\ \epsilon^{absp}\epsilon^{cdnt}F_{sp}F_{nt} \\ & +2\,\frac{\partial^2 \mathcal{L}_1^{\mathrm{EM}}}{\partial K \partial I}\left(\chi_0^{absp}\epsilon^{cdnt} + \epsilon^{absp}\chi_0^{cdnt}\right)F_{sp}F_{nt} \\ & +\frac{\partial \mathcal{L}_1^{\mathrm{EM}}}{\partial I}\chi_0^{abcd} + \frac{\partial \mathcal{L}_1^{\mathrm{EM}}}{\partial K}\ \epsilon^{abcd}\Bigg].\end{aligned} \tag{3.3.61}$$

We proceed now with the calculation of the non-linear permittivity, permeability and cross-permittivities. Using equations (3.3.23)-(3.3.26) we get

$$\begin{aligned}\mathcal{E}_{\mathrm{QED}}^{ab} = h^{ab} - 8\,\frac{\partial \mathcal{L}_1^{\mathrm{EM}}}{\partial I}h^{ab} - 64 \Bigg[& \frac{\partial^2 \mathcal{L}_1^{\mathrm{EM}}}{\partial I^2}E^aE^b + \frac{\partial^2 \mathcal{L}_1^{\mathrm{EM}}}{\partial K^2}B^aB^b \\ & -\frac{\partial^2 \mathcal{L}_1^{\mathrm{EM}}}{\partial I \partial K}\left(E^aB^b + B^aE^b\right)\Bigg],\end{aligned} \tag{3.3.62}$$

[10]Note that the classical vacuum part of the Lagrangian $\mathcal{L}_{\mathrm{EM}}^0$ is a linear function of the I invariant. $\mathcal{L}_{\mathrm{EM}}^0 = -\frac{1}{8}I$

and

$$\begin{aligned}\mathcal{M}^{ab}_{\text{QED}} = h^{ab} - 8\,\frac{\partial \mathcal{L}_1^{\text{EM}}}{\partial I} h^{ab} - 64\left[\frac{\partial^2 \mathcal{L}_1^{\text{EM}}}{\partial I^2} B^a B^b + \frac{\partial^2 \mathcal{L}_1^{\text{EM}}}{\partial K^2} E^a E^b\right.\\ \left. + \frac{\partial^2 \mathcal{L}_1^{\text{EM}}}{\partial I \partial K}\left(E^a B^b + B^a E^b\right)\right],\end{aligned} \tag{3.3.63}$$

for the permittivity and permeability. For the cross-permittivities we obtain

$$\begin{aligned}\mathcal{X}^{ab}_{\text{QED}} = -8\,\frac{\partial \mathcal{L}_1^{\text{EM}}}{\partial K} h^{ab} + 64\left[\frac{\partial^2 \mathcal{L}_1^{\text{EM}}}{\partial I^2} B^a E^b - \frac{\partial^2 \mathcal{L}_1^{\text{EM}}}{\partial K^2} E^a B^b\right.\\ \left. + \frac{\partial^2 \mathcal{L}_1^{\text{EM}}}{\partial I \partial K}\left(E^a E^b - B^a B^b\right)\right],\end{aligned} \tag{3.3.64}$$

and

$$\begin{aligned}\mathcal{Y}^{ab}_{\text{QED}} = 8\,\frac{\partial \mathcal{L}_1^{\text{EM}}}{\partial K} h^{ab} - 64\left[\frac{\partial^2 \mathcal{L}_1^{\text{EM}}}{\partial I^2} E^a B^b - \frac{\partial^2 \mathcal{L}_1^{\text{EM}}}{\partial K^2} B^a E^b\right.\\ \left. + \frac{\partial^2 \mathcal{L}_1^{\text{EM}}}{\partial I \partial K}\left(E^a E^b - B^a B^b\right)\right].\end{aligned} \tag{3.3.65}$$

As anticipated, the previous tensors have the symmetries described in section 3.3.2. In a following section we will use a similar Lagrangian accounting for the material part as well, to model an isotropic medium with non-linear electromagnetic properties.

3.3.5 Probing non-linear electromagnetic relations using the material derivative

The present part is based on the analogy between Carter's relativistic elasticity and electromagnetism in matter [23]. Since both phenomena arise in the presence of a material it makes sense to generalize the linear constitutive relations of electromagnetism in a way similar to the generalization of general relativistic elasticity. In all media, the excitation tensor is given, as stated previously, by equation (3.3.3). Since in the non-linear case there is not a linear relation between the H_{ab} and F_{ab} we have to generalise equation (3.3.4) so it holds for all cases. In order to obtain a relation between those quantities we start by assuming that the excitation tensor is a function of the Faraday tensor. Thus, we have the following relation between the material derivatives of these tensors

$$\not{\mathcal{L}}_u H_{ab} = \frac{\partial H_{ab}}{\partial F_{cd}} \not{\mathcal{L}}_u F_{cd} = \frac{1}{2} \chi_{ab}{}^{cd} \not{\mathcal{L}}_u F_{cd}, \tag{3.3.66}$$

where we used equation (3.3.8) to substitute the non-linear constitutive tensor. Of course this relation should reduce to equation (3.3.4) in the case of the classical vacuum. Additionally, calculating the derivatives of both sides of equation (3.3.6) yields

$$\begin{aligned} \not{\mathcal{L}}_u H_{ab} &= \not{\mathcal{L}}_u F_{ab} + \not{\mathcal{L}}_u M_{ab} \\ \frac{1}{2} \chi_{ab}{}^{cd} \not{\mathcal{L}}_u F_{cd} &= \frac{1}{2} \overset{0}{\chi}_{ab}{}^{cd} \not{\mathcal{L}}_u F_{cd} + \frac{\partial M_{ab}}{\partial F_{cd}} \not{\mathcal{L}}_u F_{cd} \\ \frac{1}{2} \chi_{ab}{}^{cd} \not{\mathcal{L}}_u F_{cd} &= \frac{1}{2} \overset{0}{\chi}_{ab}{}^{cd} \not{\mathcal{L}}_u F_{cd} + \frac{1}{2} \varsigma_{ab}{}^{cd} \not{\mathcal{L}}_u F_{cd}, \end{aligned} \tag{3.3.67}$$

where we have substituted, in analogy to equation (3.3.8), the definition of the susceptibility tensor

$$\varsigma_{ab}{}^{cd} = 2 \frac{\partial M_{ab}}{\partial F_{cd}}. \tag{3.3.68}$$

Equation (3.3.67) implies that for a non-linear medium the constitutive quantities are still related through

$$\chi_{ab}{}^{cd} = \overset{0}{\chi}_{ab}{}^{cd} + \varsigma_{ab}{}^{cd}, \tag{3.3.69}$$

which is the same as equation (3.3.13). Using equations (3.3.17), (3.3.18) and (3.3.22) we decompose equation (3.3.66) it into temporal and spatial parts. This provides us with the following non-linear constitutive relations between the fields D_a, H^a and the electric and magnetic fields

$$\not{\mathcal{L}}_u D_a = \mathcal{E}_a{}^b \not{\mathcal{L}}_u E_b + \mathcal{Y}_{ab} \not{\mathcal{L}}_u B^b + \Theta \mathcal{Y}_{ab} B^b, \tag{3.3.70}$$

and

$$\not{\mathcal{L}}_u H^a = \mathcal{M}^a{}_b \not{\mathcal{L}}_u B^b + \mathcal{X}^{ab} \not{\mathcal{L}}_u E_b + \Theta \mathcal{M}^a{}_b B^b - \Theta H^a, \tag{3.3.71}$$

where $\Theta = \nabla_a u^a$ is the volume expansion expansion scalar which measures the average separation between close distance observers [73]. It is straightforward to show that, in absence of coupling between the electric displacement D_a and the magnetic field, as well as between the H^a field and the electric field, these equations reduce to

$$\not{\mathcal{L}}_u D_a = \mathcal{E}_a{}^b \not{\mathcal{L}}_u E_b, \tag{3.3.72}$$

and

$$\not{\mathcal{L}}_u H^a = \mathcal{M}^a{}_b \not{\mathcal{L}}_u B^b + \Theta \left(\mathcal{M}^a{}_b B^b - H^a \right). \tag{3.3.73}$$

It is interesting that equation (3.3.72) simplifies more than equation (3.3.73), as the latter contains terms related to Θ. These peculiar terms exist due to the fact that the 3D volume form ϵ_{abc} is not materially constant along the worldlines of the 4-velocity. The non-linear relations (3.3.70) and (3.3.71) should of course satisfy equations (3.3.27) and (3.3.28) in the case of a linear material. Substituting the right-hand-side of the latter in the left-hand-side of the former we get

$$\begin{aligned} &\left(\not{\mathcal{L}}_u \mathcal{E}_a{}^b\right) E_b + \mathcal{E}_a{}^b \not{\mathcal{L}}_u E_b + \left(\not{\mathcal{L}}_u \mathcal{Y}_{ab}\right) B^b + \mathcal{Y}_{ab} \not{\mathcal{L}}_u B^b \\ =&\mathcal{E}_a{}^b \not{\mathcal{L}}_u E_b + \mathcal{Y}_{ab} \not{\mathcal{L}}_u B^b + \Theta \mathcal{Y}_{ab} B^b, \end{aligned} \tag{3.3.74}$$

and

$$\begin{aligned} &\left(\not{\mathcal{L}}_u \mathcal{M}^a{}_b\right) B^b + \mathcal{M}^a{}_b \not{\mathcal{L}}_u B^b + \left(\not{\mathcal{L}}_u \mathcal{X}^{ab}\right) E_b + \mathcal{X}^{ab} \not{\mathcal{L}}_u E_b = \\ &\mathcal{M}^a{}_b \not{\mathcal{L}}_u B^b + \mathcal{X}^{ab} \not{\mathcal{L}}_u E_b + \Theta \mathcal{M}^a{}_b B^b - \Theta H^a, \end{aligned} \tag{3.3.75}$$

which reduce to

$$\left(\not{\mathcal{L}}_u \mathcal{E}_a{}^b\right) E_b + \left(\not{\mathcal{L}}_u \mathcal{Y}_{ab} - \Theta \mathcal{Y}_{ab}\right) B^b = 0, \tag{3.3.76}$$

for the electric properties of the medium and

$$\left(\not{\mathcal{L}}_u \mathcal{M}^a{}_b\right) B^b + \left(\not{\mathcal{L}}_u \mathcal{X}^{ab}\right) E_b - \Theta \left(\mathcal{M}^a{}_b B^b - H^a\right) = 0, \tag{3.3.77}$$

for the magnetic properties of the medium. These two equations describe the evolution of the decomposed constitutive tensors in terms of the material derivative for the case of a linear medium. In case the coupling tensors $\mathcal{Y}_{ab}$ and $\mathcal{X}^{ab}$ are zero the previous equations simplify to

$$\left(\not{\mathcal{L}}_u \mathcal{E}_a{}^b\right) E_b = 0, \tag{3.3.78}$$

for the electric properties of the medium and

$$\left(\not{\mathcal{L}}_u \mathcal{M}^a{}_b\right) B^b = 0, \tag{3.3.79}$$

for its magnetic properties. If we assume that the above equations hold for all electric and magnetic fields we get

$$\not{\mathcal{L}}_u \mathcal{E}_a{}^b = 0, \tag{3.3.80}$$

and

$$\not{\mathcal{L}}_u \mathcal{M}^a{}_b = 0, \tag{3.3.81}$$

for the electric and magnetic properties, respectively. It is straightforward to show that for the case of classical vacuum equations (3.3.35) and (3.3.36) satisfy the previously mentioned equations since the projection tensor ${h_a}^b$ is materially constant.

We should mention here the following. In this section we used the material derivative to produce a relation between the excitation tensor and the Faraday tensor in analogy with the linear case. By choosing the material derivative for the calculation, we have ensured that equations (3.3.80) and (3.3.81) are satisfied for the case of vacuum which provides some additional consistency to the argument. However, we could have used any other derivative for which the chain rule of differentiation holds and additionally the differentiated quantity is still a tensor. In these cases though, we possibly have to make additional assumptions, such that the equations inferred by that other derivative regarding the linear media –which will be equivalent to equations (3.3.80) and (3.3.81)– hold.

3.3.6 The Maxwell equations

In this section we will derive the two Maxwell equations in covariant form. The first Maxwell equation arises from the antisymmetry of the Faraday tensor (see Appendix A.1.4) and is given by

$$\nabla_{[a}F_{bc]} = 0. \tag{3.3.82}$$

Since we assumed a symmetric Christoffel symbol this equation can be written with a partial instead of a covariant derivative. The second Maxwell equation, is slightly more complicated to derive, and arises by varying the Lagrangian with respect to the vector potential A_a. Following the variational derivation demonstrated in equation (3.0.7) we assume a Lagrangian[11] of the form $\mathcal{L}_{\mathrm{EM}}(F_{ab}, A_a)$. The variation of this Lagrangian yields

$$\begin{aligned}\delta\mathcal{L}_{\mathrm{EM}} &= \frac{\partial\mathcal{L}_{\mathrm{EM}}}{\partial F_{ab}}\delta F_{ab} + \frac{\partial\mathcal{L}_{\mathrm{EM}}}{\partial A_b}\delta A_b \\ &= 2\frac{\partial\mathcal{L}_{\mathrm{EM}}}{\partial F_{ab}}\delta\left(\nabla_a A_b\right) + \frac{\partial\mathcal{L}_{\mathrm{EM}}}{\partial A_b}\delta A_b.\end{aligned} \tag{3.3.83}$$

[11] From equation (3.3.1) for the Faraday tensor it follows that the variation with respect to F_{ab} as an independent field is equivalent to the variation with respect to the gradient of the 4-vector potential. See [63] for an elaborate discussion on this matter.

Now, the variation commutes with the covariant derivative (since it is a variation of the electromagnetic field alone) and so we have

$$\begin{aligned}\delta\mathcal{L}_{\mathrm{EM}} &= 2\frac{\partial\mathcal{L}_{\mathrm{EM}}}{\partial F_{ab}}\nabla_a\delta A_b + \frac{\partial\mathcal{L}_{\mathrm{EM}}}{\partial A_b}\delta A_b \\ &= \left[\nabla_a\left(-2\frac{\partial\mathcal{L}_{\mathrm{EM}}}{\partial F_{ab}}\right) + \frac{\partial\mathcal{L}_{\mathrm{EM}}}{\partial A_b}\right]\delta A_b + \nabla_a\left(2\frac{\partial\mathcal{L}_{\mathrm{EM}}}{\partial F_{ab}}\delta A_b\right),\end{aligned} \tag{3.3.84}$$

where the last term is a divergence of a term that contains the variation. This term vanishes after transformed to a boundary term (see equation (3.0.6)) as was demonstrated in the beginning of this chapter. Additionally, the variation of the action (see equation 3.0.2) is independent of the variation of A_a and so the first term vanishes as well, which means that

$$\nabla_a\left(-2\frac{\partial\mathcal{L}_{\mathrm{EM}}}{\partial F_{ba}}\right) = \frac{\partial\mathcal{L}_{\mathrm{EM}}}{\partial A_b}, \tag{3.3.85}$$

which is the second Maxwell equation. As we have stated previously, we will consider only cases where the dependence to the vector potential is described by a term of the form $j^a_{\mathrm{F}}A_a$. Substituting this function for the source term, and the definition (3.3.3) of the excitation tensor, we get

$$\nabla_b H^{ab} = j^a_{\mathrm{F}}, \tag{3.3.86}$$

We find that the "free" current is covariantly conserved

$$\nabla_a j^a_{\mathrm{F}} = 0, \tag{3.3.87}$$

which follows by application of the covariant derivative with the free index contracted on equation (3.3.86). Projecting the Maxwell equations (3.3.82) and (3.3.86) along and orthogonally to the observer's 4-velocity we get the general relativistic counterparts of the classical Maxwell equations. Equation (3.3.82) decomposes into the relativistic Faraday equation

$$h_{ab}\dot{B}^b + \epsilon_{abc}\bar{\nabla}^b E^c = -\epsilon_{acb}\dot{u}^c E^b + B^c\bar{\nabla}_c u_a - B_a\bar{\nabla}_c u^c, \tag{3.3.88}$$

and relativistic Gauss law for the magnetic field

$$\bar{\nabla}^a B_a = -\epsilon^{abc}E_a\bar{\nabla}_b u_c, \tag{3.3.89}$$

while the second equation (3.3.86) decomposes into the relativistic Ampére law

$$h_a{}^b \dot{D}_b - \epsilon_{abc}\bar{\nabla}^b H^c + J^a_{\mathrm{F}} = D^b \bar{\nabla}_b u_a - D_a \bar{\nabla}^b u_b + \epsilon_{abc}\dot{u}^b H^c, \tag{3.3.90}$$

and the relativistic Gauss law for the electric displacement

$$\bar{\nabla}^a D_a - \hat{\sigma}_{\mathrm{F}} = \epsilon^{abc} H_c \bar{\nabla}_a u_b. \tag{3.3.91}$$

In the equations above we have moved to the left-hand side the relativistic counterparts of the terms existing in classical equations while the right-hand side contains terms that do not have classical equivalents. These extra terms contain the covariant derivative of the 4-velocity and are of geometric origin.

3.4 Electromagnetism in multifluid media

Up to this point we have treated fluid dynamics and electromagnetism separately in order to provide a clear description for both. In this section we combine electromagnetism in media with the multifluid description. The idea behind this synthesis is that the medium, as we discussed in section 3.3, is related to the material part of the system under consideration. Hence, our task is to present a model that accounts for both electromagnetic and hydrodynamic phenomena of multifluids.

3.4.1 The Lagrangian source term $j^a_{\mathrm{F}} A_a$

We will look into the coupling between electromagnetism and matter that arises from the 4-current. Electromagentic currents consist of moving charged particles and thus posses both fluid and electromagnetic properties.

We assume that each charged fluid component carries a single, either positive or negative, unit of charge (i.e. e or $-e$). Neutral fluid components such as neutrons carry zero charge. For each component the unit of charge is shown as q_{x}. For example, if the system consists of protons and electrons the two components will be $q_{\mathrm{p}} = e$ and $q_{\mathrm{e}} = -e$ respectively (if the system consists of neutrons we also have $q_{\mathrm{n}} = 0$). The free current of each species $j^a_{\mathrm{F,x}}$ is given by

$$j^a_{\mathrm{F,x}} = q_{\mathrm{x}} n^a_{\mathrm{x}}, \tag{3.4.1}$$

while the total free 4-current is the sum of the above-stated currents

$$j_{\mathrm{F}}^{a}=\sum_{\mathrm{x}} j_{\mathrm{F,x}}^{a}. \tag{3.4.2}$$

It is apparent through the previous two equations that for a two-component medium with opposite charges, if $n_{\mathrm{x}}^{a}=n_{\mathrm{y}}^{a}$ (with $\mathrm{x} \neq \mathrm{y}$) then the total free 4-current vanishes. Additionally, the number density current conservation given by equation (3.2.4) ensures that j_{F}^{a} is divergent-free

$$\nabla_{a} j_{\mathrm{F}}^{a}=0. \tag{3.4.3}$$

We proceed with calculating the variation of the source term given by

$$\mathcal{L}_{\mathrm{EM}}^{\mathrm{src}}=j_{\mathrm{F}}^{a} A_{a}, \tag{3.4.4}$$

with respect to n_{x}^{a} and the metric tensor. The variation with respect to the 4-potential has already been calculated in section 3.3.6 and provides the right-hand side of the equation (3.3.86). We have

$$\delta\left(\mathcal{L}_{\mathrm{EM}}^{\mathrm{src}} \sqrt{-g}\right)=\sum_{\mathrm{x}}\left[q_{\mathrm{x}}\left(\delta n_{\mathrm{x}}^{a} A_{a} \sqrt{-g}+n_{\mathrm{x}}^{a} A_{a} \delta \sqrt{-g}\right)\right], \tag{3.4.5}$$

and using equations (3.2.5) and (3.1.8) we obtain the following expression

$$\delta\left(\mathcal{L}_{\mathrm{EM}}^{\mathrm{src}} \sqrt{-g}\right)=\sqrt{-g} \sum_{\mathrm{x}}\left(q_{\mathrm{x}} A_{a} n_{\mathrm{x}}^{b} \nabla_{b} \xi_{\mathrm{x}}^{a}-q_{\mathrm{x}} A_{a} \xi_{\mathrm{x}}^{b} \nabla_{b} n_{\mathrm{x}}^{a}-q_{\mathrm{x}} A_{a} n_{\mathrm{x}}^{a} \nabla_{b} \xi_{\mathrm{x}}^{b}\right), \tag{3.4.6}$$

where the last term of equation (3.2.5) cancels with the term which emerges from the variation of the metric. We have already encountered an expression similar to that of equation (3.4.6) in section 3.2. Using the same process we finally get for the variation of the current term

$$\begin{aligned}\delta\left(\mathcal{L}_{\mathrm{EM}}^{\mathrm{src}} \sqrt{-g}\right)= & \sqrt{-g} \sum_{\mathrm{x}}\left(2 \xi_{\mathrm{x}}^{a} q_{\mathrm{x}} n_{\mathrm{x}}^{b} \nabla_{[a} A_{b]}\right) \\ & +\sqrt{-g} \sum_{\mathrm{x}} \nabla_{b}\left(\frac{1}{2} \xi_{\mathrm{x}}^{a} n_{e f a}^{\mathrm{x}} \epsilon^{b e f c} A_{c}\right).\end{aligned} \tag{3.4.7}$$

The last term is a vanishing boundary term while the first term contains the parts of relativistic Lorentz force $f_{a}^{\mathrm{L,x}}$ for each charged component

$$f_{a}^{\mathrm{L,x}}=q_{\mathrm{x}} n_{\mathrm{x}}^{b} F_{a b}, \tag{3.4.8}$$

which provides additional terms to the equations of motion (3.2.19) of the medium. Collectively all these terms constitute the relativistic Lorentz force given by

$$f_a^{\mathrm{L}} = \sum_{\mathrm{x}} q_{\mathrm{x}} n_{\mathrm{x}}^b F_{ab}. \tag{3.4.9}$$

3.4.2 An isotropic non-linear electromagnetic fluid medium

In this section we consider an isotropic, non-linear[12] electromagnetic fluid medium. An example of such a medium can be a part of the neutron star core where matter can be treated as a multifluid while it may exhibit ferromagnetic behaviour at densities a few times the nuclear saturation density [31, 30, 32, 33, 74]. Ferromagnetism is usually treated microscopically since most of the properties related to it require a quantum description. However, since our treatment is mesoscopic (by this we mean that the scales are small enough that we can adequately refer to infinitesimal regions, but not so small that quantum manipulation would be necessary) we will focus on the non-linear electromagnetic behaviour of ferromagnetic media which appears at larger scales. In order to do this we assume that the part of the Lagrangian describing the electromagnetic properties has a similar role to the one describing the fluid in section 3.2. So, in some sense it is an "equation of state" for the electromagnetic properties of the medium just as $\mathcal{L}_{\mathrm{F}}$ is perceived as the equation of state for the fluid.

We assume that the system is described by a Lagrangian density $\mathcal{L}_{\mathrm{fer}}$ which is a function of the number density currents n_{x}^a, the Faraday tensor, the 4-vector potential and the metric tensor. Additionally, as in sections 3.2 and 3.3.4.1 we consider the case that the Lagrangian does not explicitly depend on these fields but to the related scalar invariants in order the Lagrangian to be Lorentz invariant. Thus, we have

$$\mathcal{L}_{\mathrm{fer}} = \mathcal{L}_{\mathrm{EM}}^{0} + \mathcal{L}_{\mathrm{EM}}^{\mathrm{src}} + \mathcal{L}_{\mathrm{F}}\left(n_{\mathrm{x}}^2, n_{\mathrm{xy}}^2\right) + \mathcal{L}_{\mathrm{NL}}\left(I, K, n_{\mathrm{x}}^2, n_{\mathrm{xy}}^2, F_{\mathrm{xy}}\right), \tag{3.4.10}$$

where the last term is a function of the invariants $I, K, n_{\mathrm{x}}^2, n_{\mathrm{xy}}^2, F_{\mathrm{xy}}$, coupling the fluid and electromagnetic properties of the medium. The quantity F_{xy} is an invariant given by

$$F_{\mathrm{xy}} = F_{ab} n_{\mathrm{x}}^a n_{\mathrm{y}}^b, \tag{3.4.11}$$

[12]The analysis here applies to linear electromagnetic media, as well.

which by using equation (3.2.3) is decomposed in the frame of u^a as

$$F_{\mathrm{xy}} = \gamma_{\mathrm{x}}\gamma_{\mathrm{y}} \left[E_a \left(v^a_{\mathrm{x}} - v^a_{\mathrm{y}}\right) + \epsilon_{abc} v^a_{\mathrm{x}} v^b_{\mathrm{y}} B^c\right]. \tag{3.4.12}$$

This quantity obeys the symmetry $F_{\mathrm{xy}} = -F_{\mathrm{yx}}$ which implies that if $\mathrm{x} = \mathrm{y}$ then it vanishes. If the components x, y are comoving then it vanishes as well. Note the components x, y related to this scalar should carry some charge for the expression to be meaningful. Neutral components should not be able to interact directly with the electromagnetic field.
In order to provide the complete set of equations for the system we first have to manipulate the last in equation (3.4.10). From the definition (3.3.3) we get

$$\begin{aligned} H^{ab} =& \frac{1}{2}\chi_0^{abcd} F_{cd} \\ &- 2\left(\frac{\partial \mathcal{L}_{\mathrm{NL}}}{\partial I}\frac{\partial I}{\partial F_{ab}} + \frac{\partial \mathcal{L}_{\mathrm{NL}}}{\partial K}\frac{\partial K}{\partial F_{ab}} + \frac{1}{2}\sum_{\mathrm{x}}\sum_{\mathrm{x}\neq\mathrm{y}} \frac{\partial \mathcal{L}_{\mathrm{NL}}}{\partial F_{\mathrm{xy}}}\frac{\partial F_{\mathrm{xy}}}{\partial F_{ab}}\right), \end{aligned} \tag{3.4.13}$$

which by using equations (3.3.58) and (3.3.59) and the results of Appendix A.1.5 yield

$$H^{ab} = \frac{1}{2}\chi_0^{abcd} F_{cd} + \mathcal{I}\,\chi_0^{abcd} F_{cd} + \mathcal{K}\,\epsilon^{abcd} F_{cd} + \frac{1}{2}\sum_{\mathrm{x}}\sum_{\mathrm{x}\neq\mathrm{y}} \mathcal{F}_{\mathrm{xy}}\, n^{[a}_{\mathrm{x}} n^{b]}_{\mathrm{x}}, \tag{3.4.14}$$

where in the same fashion as in section 3.2 we defined the quantities

$$\mathcal{I} = -4\frac{\partial \mathcal{L}_{\mathrm{NL}}}{\partial I}, \quad \mathcal{K} = -4\frac{\partial \mathcal{L}_{\mathrm{NL}}}{\partial K}, \quad \text{and} \quad \mathcal{F}_{\mathrm{xy}} = -2\frac{\partial \mathcal{L}_{\mathrm{NL}}}{\partial F_{\mathrm{xy}}}, \tag{3.4.15}$$

that are similar to the quantities defined in equation (3.2.8). To obtain the hydrodynamical 4-momenta of the system we calculate the partial derivative of the Lagrangian with respect to the number density current through

$$\begin{aligned} \mu^a_{\mathrm{x}} =& \frac{\partial \mathcal{L}^{\mathrm{src}}_{\mathrm{EM}}}{\partial n^a_{\mathrm{x}}} + \frac{\partial(\mathcal{L}_{\mathrm{F}} + \mathcal{L}_{\mathrm{NL}})}{\partial n^2_{\mathrm{x}}}\frac{\partial n^2_{\mathrm{x}}}{\partial n^a_{\mathrm{x}}} \\ &+ \sum_{\mathrm{x}\neq\mathrm{y}}\left(\frac{\partial(\mathcal{L}_{\mathrm{F}} + \mathcal{L}_{\mathrm{NL}})}{\partial n^2_{\mathrm{xy}}}\frac{\partial n^2_{\mathrm{xy}}}{\partial n^a_{\mathrm{x}}} + \frac{\partial \mathcal{L}_{\mathrm{NL}}}{\partial F_{\mathrm{xy}}}\frac{\partial F_{\mathrm{xy}}}{\partial n^a_{\mathrm{x}}}\right). \end{aligned} \tag{3.4.16}$$

where we considered only the terms of equation (3.4.10) that contain non vanishing quantities. So the previous equation becomes

$$\mu_a^{\mathrm{x}} = \left(\mathcal{B}^{\mathrm{x}} + \mathcal{D}^{x}\right) n_a^{\mathrm{x}} + \sum_{\mathrm{x}\neq\mathrm{y}} \left[\left(\mathcal{A}^{\mathrm{xy}} + \mathcal{C}^{\mathrm{xy}}\right) n_a^{\mathrm{y}} + \mathcal{F}_{\mathrm{xy}} F_{ab} n_{\mathrm{y}}^{b}\right] + q_{\mathrm{x}} A_a, \tag{3.4.17}$$

The additional term involving $\mathcal{F}_{\mathrm{xy}}$ in the previous equation arises from the contribution of F_{xy} to the momentum. In analogy to section 3.2 $\mathcal{D}^{x}$ and C^{xy} are given by

$$\mathcal{D}^{\mathrm{x}} = -2\frac{\partial \mathcal{L}_{\mathrm{NL}}}{\partial n_{\mathrm{x}}^2} \quad \text{and} \quad \mathcal{C}^{\mathrm{xy}} = -\frac{\partial \mathcal{L}_{\mathrm{NL}}}{\partial n_{\mathrm{xy}}^2}. \tag{3.4.18}$$

Instead of considering two separate terms that contain the hydrodynamical information in the Lagrangian we could have assumed only one. In that case the quantities $\mathcal{B}^{\mathrm{x}}$ and $\mathcal{D}^{x}$ as well as $\mathcal{A}^{\mathrm{xy}}$ and $\mathcal{C}^{\mathrm{xy}}$ would have been merged. Following the calculation of section 3.2 regarding the variation with respect to the number density currents, the equations of motion for component x are given by equation (3.2.19). The vorticity ω_{ab}^{x} is calculated by equation (3.2.18) with respect to the momentum given in equation (3.4.17). That is

$$\begin{aligned} &2q_{\mathrm{x}} n_{\mathrm{x}}^{b} \nabla_{[b} A_{a]} + 2n_{\mathrm{x}}^{b} \nabla_{[b} \left[n_{a]}^{\mathrm{x}} \left(\mathcal{B}^{\mathrm{x}} + \mathcal{D}^{x}\right)\right] + \sum_{\mathrm{x}\neq\mathrm{y}} 2\nabla_{[b} \left[n_{a]}^{\mathrm{y}} \left(\mathcal{A}^{\mathrm{xy}} + \mathcal{C}^{\mathrm{xy}}\right)\right] \\ &+ \sum_{\mathrm{x}\neq\mathrm{y}} 2\nabla_{[b} \left(F_{a]c} \mathcal{F}_{\mathrm{xy}} n_{\mathrm{y}}^{c}\right) = 0, \end{aligned} \tag{3.4.19}$$

and by using equations (3.3.1) and (3.3.82) we get

$$\begin{aligned} &j_{\mathrm{F,x}}^{b} F_{ba} + 2n_{\mathrm{x}}^{b} \nabla_{[b} \left[n_{a]}^{\mathrm{x}} \left(\mathcal{B}^{\mathrm{x}} + \mathcal{D}^{x}\right)\right] + \sum_{\mathrm{x}\neq\mathrm{y}} 2\nabla_{[b} \left[n_{a]}^{\mathrm{y}} \left(\mathcal{A}^{\mathrm{xy}} + \mathcal{C}^{\mathrm{xy}}\right)\right] \\ &+ \sum_{\mathrm{x}\neq\mathrm{y}} 2F_{[ac} \nabla_{b]} \left(\mathcal{F}_{\mathrm{xy}} n_{\mathrm{y}}^{c}\right) = 0. \end{aligned} \tag{3.4.20}$$

Finally, we will consider the variations with respect to the metric tensor in order to obtain the stress-energy-momentum tensor of the medium, which is given by equation (3.2.20). Starting by calculating the derivative of $\mathcal{L}_{\mathrm{fer}}$ with respect to the metric we obtain

$$\begin{aligned} \frac{\partial \mathcal{L}_{\mathrm{fer}}}{\partial g_{ab}} =& \frac{\partial \mathcal{L}_{\mathrm{fer}}}{\partial I} \frac{\partial I}{\partial g_{ab}} + \frac{\partial \mathcal{L}_{\mathrm{fer}}}{\partial K} \frac{\partial K}{\partial g_{ab}} \\ &+ \sum_{\mathrm{x}} \left(\frac{\partial \mathcal{L}_{\mathrm{fer}}}{\partial n_{\mathrm{x}}^2} \frac{\partial n_{\mathrm{x}}^2}{\partial g_{ab}} + \frac{1}{2} \sum_{x\neq y} \frac{\partial \mathcal{L}_{\mathrm{fer}}}{\partial n_{\mathrm{xy}}^2} \frac{\partial n_{\mathrm{xy}}^2}{\partial g_{ab}} + \frac{\partial \mathcal{L}_{\mathrm{fer}}}{\partial F_{\mathrm{xy}}} \frac{\partial F_{\mathrm{xy}}}{\partial g_{ab}} \right), \end{aligned} \tag{3.4.21}$$

which by using the results of appendix A.1.5 we have

$$\begin{aligned}\frac{\partial \mathcal{L}_{\mathrm{fer}}}{\partial g_{ab}} =& \frac{1}{2}F^{ac}F^{b}{}_{c} + \frac{1}{2}\sum_{\mathrm{x}}\left[\left(\mathcal{B}^{\mathrm{x}}+\mathcal{D}^{\mathrm{x}}\right)n_{\mathrm{x}}^{a}n_{\mathrm{x}}^{b} + \sum_{\mathrm{x}\neq\mathrm{y}}\left(\mathcal{A}^{\mathrm{xy}}+\mathcal{C}^{\mathrm{xy}}\right)n_{\mathrm{x}}^{a}n_{\mathrm{y}}^{b}\right] \\ &+ \mathcal{I}F^{ac}F^{b}{}_{c} + \frac{1}{8}\mathcal{K}Kg^{ab}.\end{aligned} \tag{3.4.22}$$

Note that the $\mathcal{F}_{\mathrm{xy}}$ term does not appear as the related derivative vanishes. We point out that a part of the variation with respect to the metric is given through the number density current variation in equation (3.2.5). This part corresponds to the subtracted quantity in the first term in the following equation. The energy momentum tensor is therefore given by

$$\begin{aligned}T^{ab}_{\mathrm{fer}} =& \left(\mathcal{L}_{\mathrm{fer}} - \mathcal{L}_{\mathrm{src}} - \sum_{\mathrm{x}}\mu^{\mathrm{x}}_{c}n^{c}_{\mathrm{x}} + \frac{1}{4}\mathcal{K}K\right)g^{ab} + F^{ac}F^{b}{}_{c} + 2\mathcal{I}F^{ac}F^{b}{}_{c} \\ &+ \sum_{\mathrm{x}}\left[\left(\mathcal{B}^{\mathrm{x}}+\mathcal{D}^{\mathrm{x}}\right)n_{\mathrm{x}}^{a}n_{\mathrm{x}}^{b} + \sum_{\mathrm{x}\neq\mathrm{y}}\left(\mathcal{A}^{\mathrm{xy}}+\mathcal{C}^{\mathrm{xy}}\right)n_{\mathrm{x}}^{a}n_{\mathrm{y}}^{b}\right].\end{aligned} \tag{3.4.23}$$

The first term as in section 3.2, corresponds to a generalized pressure of the system through

$$\Psi_{\mathrm{fer}} = \mathcal{L}_{\mathrm{fer}} - \mathcal{L}_{\mathrm{src}} - \sum_{\mathrm{x}}\mu^{\mathrm{x}}_{a}n^{a}_{\mathrm{x}} + \frac{1}{4}\mathcal{K}K. \tag{3.4.24}$$

We also observe that the coupling of the electric and magnetic field adds a term to the pressure Ψ_{fer}. Note that in both equations (3.4.23) and (3.4.24) we have subtracted the $\mathcal{L}_{\mathrm{src}}$ term since, as it was found in section 3.4.1, it does not contribute to the energy momentum tensor.

3.4.3 The ideal magnetohydrodynamic approximation

In this section we will discuss the ideal magnetohydrodynamic (MHD) approach. According to the classical (Newtonian) theory the ideal MHD is a simplifying approximation for the description of plasmas [75, 76] which, among other cases, is relevant in astrophysical configurations.

Assuming a two fluid plasma consisting of positive ions and negative electrons the ideal MHD approximation follows, by imposing a set of simplifying assumptions. The system is described by a single fluid and is perfectly conducting, which means that the electric field with respect to the observer moving with the fluid vanishes. In the following sections, we start from the multifluid perspective and by introducing a set of approximations we obtain the equations of single fluid ideal MHD, following [77, 78].

3.4.3.1 Relativistic ideal MHD from the variational principle

To derive the single fluid ideal MHD we start by assuming that the system is described by the Lagrangian

$$\mathcal{L}_{\mathrm{MHD}} = \mathcal{L}^{0}_{\mathrm{EM}} + \mathcal{L}^{\mathrm{src}}_{\mathrm{EM}} + \mathcal{L}_{\mathrm{F}}, \tag{3.4.25}$$

where the first term is given by equation (3.3.33), while the source term for the electromagnetic field i.e. $\mathcal{L}^{\mathrm{src}}_{\mathrm{EM}}$, introduces the 4-current. The fluid term $\mathcal{L}_{\mathrm{F}}\left(n^2_{\mathrm{x}}\right)$ is discussed in section 3.2 with $\mathrm{x} = \{\mathrm{i}, \mathrm{e}, \mathrm{s}\}$, i.e. positive ions, electrons and entropy. We assume that ions and entropy form a single two-constituent fluid in the sense of section 3.2.1, while the electrons are considered as a separate fluid. We have not assumed any n^2_{xy} terms and therefore the fluids are not coupled through entrainment. The number density currents for the ions and entropy are given by equations (3.2.22) and (3.2.23) respectively, while for the electrons is given by equation (3.2.2). Furthermore, this system is a special case of the one described in section 3.4.2 as we see by comparing the Lagrangians (3.4.10) and (3.4.25).

As discussed previously the system is descirbed by the following system of equations. We have the conservation laws for n^a_{i}, n^a_{s}, n^a_{e} given by equation (3.2.4). Following the definition (3.2.10) the momenta of the ions and entropy arise from the $\mathcal{L}^{\mathrm{src}}_{\mathrm{EM}}, \mathcal{L}_{\mathrm{F}}$ and $\mathcal{L}_{\mathrm{F}}$ parts of the Lagrangian respectively, and are given by

$$\mu^{\mathrm{i}}_a = \mathcal{B}^{\mathrm{i}} n_{\mathrm{i}} u^{\mathrm{i}}_a + q_{\mathrm{i}} A_a, \tag{3.4.26a}$$

$$\mu^{\mathrm{s}}_a = \mathcal{B}^{\mathrm{s}} n_{\mathrm{s}} u^{\mathrm{i}}_a. \tag{3.4.26b}$$

The momentum of the electrons is given by

$$\mu^{\mathrm{e}}_a = \mathcal{B}^{\mathrm{e}} n_{\mathrm{e}} u^{\mathrm{e}}_a + q_{\mathrm{e}} A_a, \tag{3.4.27}$$

and is calculated by the $\mathcal{L}^{\mathrm{src}}_{\mathrm{EM}}, \mathcal{L}_{\mathrm{F}}$ terms of the Lagrangian. The equations of motion for the fluid are then given by

$$q_{\mathrm{i}} n^b_{\mathrm{i}} \nabla_{[b} A_{a]} + n^b_{\mathrm{i}} \nabla_{[b} \left(n^{\mathrm{i}}_{a]}\, \mathcal{B}^{\mathrm{i}}\right) + n^b_{\mathrm{s}} \nabla_{[b} \left(n^{\mathrm{s}}_{a]}\, \mathcal{B}^{\mathrm{s}}\right) = 0, \tag{3.4.28}$$

for the ion-entropy fluid. Note that the equation above is essentially equation (3.2.27) with the additional (first) term that accounts for the interaction between the matter and the electromagnetic field. Similarly the equation of motion for electrons is

$$q_{\mathrm{e}} n_{\mathrm{e}}^{b} \nabla_{[b} A_{a]} + n_{\mathrm{e}}^{b} \nabla_{[b} \left(n_{a]}^{\mathrm{e}} \mathcal{B}^{\mathrm{e}} \right) = 0. \tag{3.4.29}$$

Adding the two equation provides a single equation of motion for the system

$$f_{a}^{\mathrm{L}} + n_{\mathrm{i}}^{b} \nabla_{[b} \left(n_{a]}^{\mathrm{i}} \mathcal{B}^{\mathrm{i}} \right) + n_{\mathrm{s}}^{b} \nabla_{[b} \left(n_{a]}^{\mathrm{s}} \mathcal{B}^{\mathrm{s}} \right) + n_{\mathrm{e}}^{b} \nabla_{[b} \left(n_{a]}^{\mathrm{e}} \mathcal{B}^{\mathrm{e}} \right) = 0, \tag{3.4.30}$$

where f_{a}^{L} is the Lorentz force, given by equation (3.4.9). We also have the two Maxwell equations (3.3.82) and (3.3.86). Additionally, the energy momentum tensor of the system is given by

$$T_{\mathrm{MHD}}^{ab} = \left(\mathcal{L}_{\mathrm{EM}}^{0} + \mathcal{L}_{\mathrm{F}} - \sum_{\mathrm{x}} \mu_{c}^{\mathrm{x}} n_{\mathrm{x}}^{c} \right) g^{ab} + F^{ac} F^{b}{}_{c} + \sum_{\mathrm{x}} \left(\mathcal{B}^{\mathrm{x}} n_{\mathrm{x}}^{a} n_{\mathrm{x}}^{b} \right). \tag{3.4.31}$$

As done in the previous sections, we can assign the generalised pressure of the system as

$$\Psi_{\mathrm{MHD}} = \left(\mathcal{L}_{\mathrm{EM}}^{0} + \mathcal{L}_{\mathrm{F}} - \sum_{\mathrm{x}} \mu_{c}^{\mathrm{x}} n_{\mathrm{x}}^{c} \right). \tag{3.4.32}$$

It is useful here to separate the purely electromagnetic part from the fluid part of the energy-momentum tensor. Note that this separation is not always possible[13] as for example in the energy-momentum tensor of section 3.4.2. The fluid part is

$$T_{\mathrm{MHD,\,F}}^{ab} = \left(\mathcal{L}_{\mathrm{F}} - \sum_{\mathrm{x}} \mu_{c}^{\mathrm{x}} n_{\mathrm{x}}^{c} \right) g^{ab} + \sum_{\mathrm{x}} \left(\mathcal{B}^{\mathrm{x}} n_{\mathrm{x}}^{a} n_{\mathrm{x}}^{b} \right), \tag{3.4.33}$$

while the electromagnetic part is

$$T_{\mathrm{MHD,\,EM}}^{ab} = \mathcal{L}_{\mathrm{EM}}^{0} g^{ab} + F^{ac} F^{b}{}_{c}, \tag{3.4.34}$$

which is the standard energy-momentum tensor for the vacuum case of the electromagnetic field [43]. Obviously, the fluid pressure is

$$\Psi_{\mathrm{MHD,\,F}} = \left(\mathcal{L}_{\mathrm{F}} - \sum_{\mathrm{x}} \mu_{c}^{\mathrm{x}} n_{\mathrm{x}}^{c} \right). \tag{3.4.35}$$

[13] Actually, this is the subject of the long standing Abraham-Minkowski controversy. For a brief review see [79] and references therein.

3.4.3.2 The single fluid approximation of electrons and ions

Up to this point we have not made any assumption for the system other than that it consists of two fluids and has three components. Our purpose is to describe the system in a way such that it is characterised by a single fluid with one 4-velocity. For this reason we consider the following simplifications. We assume an observer with 4-velocity u^a such that the drift velocities (i.e. the velocities of the fluid components with respect to u^a) given in equation (3.2.3) are small compared to the speed of light. This assumption, though, does not constrain the u^a itself, which may still be relativistic. This 'low relative-velocity' assumption implies that the Lorentz factors are small i.e. that $\gamma_{\rm x} \simeq 1$ and that $v_{\rm x}v_{\rm y} \simeq 0$, where x, y are any of $\{{\rm i, e, s}\}$ and $v_{\rm x}$ is the magnitude of the drift-velocity vector $v^a_{\rm x}$.
Under these assumptions the fluid part of energy momentum tensor is given by

$$\begin{aligned} T^{ab}_{\rm MHD,\,F} = {} & \Psi_{\rm MHD,\,F} g^{ab} + (n_{\rm i}\mu_{\rm i} + n_{\rm s}\mu_{\rm s} + n_{\rm e}\mu_{\rm e})\, u^a u^b \\ & + \left[(n_{\rm i}\mu_{\rm i} + n_{\rm s}\mu_{\rm s})\, v^a_{\rm i} + n_{\rm e}\mu_{\rm e} v^a_{\rm e}\right] u^b + \left[(n_{\rm i}\mu_{\rm i} + n_{\rm s}\mu_{\rm s})\, v^b_{\rm i} + n_{\rm e}\mu_{\rm e} v^b_{\rm e}\right] u^a. \end{aligned} \tag{3.4.36}$$

Contracting both indices with u^a and using equation (3.4.35) we get the fluid energy density $\rho_{\rm MHD,\,F}$ given by

$$\rho_{\rm MHD,\,F} = T^{ab}_{\rm MHD,\,F} u_a u_b = -\mathcal{L}_{\rm F}. \tag{3.4.37}$$

The fluid energy-momentum flux $q^a_{\rm MHD,\,F}$ in the frame of u^a is given by [38]

$$q^a_{\rm MHD,\,F} = -h^a_c T^{cb}_{\rm MHD,\,F} u_b = (n_{\rm i}\mu_{\rm i} + n_{\rm s}\mu_{\rm s})\, v^a_{\rm i} + n_{\rm e}\mu_{\rm e} v^a_{\rm e}. \tag{3.4.38}$$

Since in ideal MHD the fluid under consideration is perfect, the fluid energy-momentum tensor in the frame of u^a should only contain the density and the isotropic pressure terms. To implement this we assume that u^a is comoving with the Landau-Lifshitz frame [80], that is a frame where the above mentioned energy-momentum flux vanishes. Therefore, for this observer we have

$$(n_{\rm i}\mu_{\rm i} + n_{\rm s}\mu_{\rm s})\, v^a_{\rm i} + n_{\rm e}\mu_{\rm e} v^a_{\rm e} = 0, \tag{3.4.39}$$

and thus the fluid energy momentum tensor takes the form of a perfect fluid

$$T^{ab}_{\rm MHD,\,F} = \Psi_{\rm MHD,\,F} g^{ab} + \rho_{\rm MHD,\,F} u^a u^b. \tag{3.4.40}$$

with energy density given by equation (3.4.37) and pressure given by (3.4.35). In ideal MHD the plasma is usually assumed to be locally neutral [76, 75] and therefore we have $n_{\mathrm{e}} = n_{\mathrm{i}}$. Furthermore, we have $\mu_{\mathrm{i}} \gg \mu_{\mathrm{e}}$ which means that the chemical potential of ions is much larger than that of electrons. This is a reasonable assumption, when the electrons are not relativistic, as in most astrophysical systems. By non-relativistic here, we mean that the electrons are not characterised by velocities at the microscopic scale (i.e. at a scale that is much smaller than the scale of the fluid element considered in the analysis above) that are comparable to the speed of light. Additionally, this assumption is in agreement with the Newtonian theory where the equivalent of the chemical potential is the mass.
Using these assumptions, in equation (3.4.39) we obtain

$$v_{\mathrm{i}}^{a} = -\frac{n_{\mathrm{e}}\mu_{\mathrm{e}}}{(n_{\mathrm{i}}\mu_{\mathrm{i}} + n_{\mathrm{s}}\mu_{\mathrm{s}})} v_{\mathrm{e}}^{a} \simeq 0, \tag{3.4.41}$$

where we have used that $n_{\mathrm{s}}\mu_{\mathrm{s}} > 0$. This inequality is justified since the entropy density n_{s} is positive and the related chemical potential is the absolute temperature, which is also always positive. The equation above means that the drift velocity of ions is approximately zero and therefore $u_{\mathrm{i}}^{a} \simeq u^{a}$. The 4-current given by equation (3.4.2) becomes

$$j^{a} = e\, n_{\mathrm{i}} \left(v_{\mathrm{i}}^{a} - v_{\mathrm{e}}^{a}\right) \simeq -e\, n_{\mathrm{i}} v_{\mathrm{e}}^{a}. \tag{3.4.42}$$

Furthermore, for the conservation laws of the number density currents of ions and electrons given by equations (3.2.4) we have

$$\begin{aligned} \nabla_{a}\left(n_{\mathrm{i}}^{a} + n_{\mathrm{e}}^{a}\right) &\simeq 2\nabla_{a}\left(n_{\mathrm{i}} u^{a}\right) - \frac{1}{e}\nabla_{a} j^{a} \\ \nabla_{a}\left(n_{\mathrm{i}}^{a} + n_{\mathrm{e}}^{a}\right) &\simeq \nabla_{a}\left(n_{\mathrm{i}} u^{a}\right) = 0, \end{aligned} \tag{3.4.43}$$

where we have used the previously mentioned low drift velocity approximation, equation (3.4.41), we substituted v_{e}^{a} from equation (3.4.42) and also used the 4-current conservation law given by equation (3.3.87). Furthermore, equation (3.4.42) means that the 4-current is orthogonal to the 4-velocity. This result states that within the range of our approximations the two conservation laws for the number density currents of ions and electrons can be approximately substituted by a one. Generally in multifluid systems, the existence of a frame such that both the energy momentum tensor and the conservation of number density current are those of a perfect single fluid, is not guaranteed [77].

3.4.3.3 The Ohm's law for perfect conductors

The Ohm's law for ideal MHD is also referred to as the "perfect conductivity" law for the following reason. Ohm's law for an observer moving with an electric conductor is

$$J^{\star}_a = \varsigma E^{\star}_a, \tag{3.4.44}$$

where $J^{\star}_a$ and $E^{\star}_a$ are the projected spatial current and electric field with respect to observer co-moving with the electric conductor and ς is the isotropic scalar electrical conductivity[14]. Assuming that $\varsigma \to +\infty$, then in order to have a finite current, the electric field necessarily is zero. Therefore, a medium with "perfect" (i.e. tending to infinity) conductivity experiences zero electric field.

It is possible to begin with the derivation of the generalised Ohm's law in relativistic context and then derive the classical Ohm's and perfect conductivity laws as simplifying approach [81, 78, 77]. Although, as discussed in [78], this derivation of perfect conductivity is less straightforward to obtain in relativity than in the Newtonian context.

In this work, we regard the perfect conductivity law as an assumption of our system given by

$$F_{ab} u^b = E_a = 0. \tag{3.4.45}$$

where u^a is the 4-velocity of the Landau-Lifshitz frame introduced in section 3.4.3.2 and E_a is the electric field with respect to that frame. The condition above has the following implications on the system. Using equations (3.3.33), (3.3.31), (3.4.34) along with the decomposition of the Faraday tensor (3.3.17) we find that the electromagnetic part of the energy momentum tensor is given by

$$T^{ab}_{\text{MHD, EM}} = \frac{1}{2} B^2 g^{ab} + B^2 u^a u^b - B_a B_b. \tag{3.4.46}$$

The Faraday and Gauss law for the magnetic field given in equations (3.3.88) and (3.3.89) become

$$h_{ab} \dot{B}^b - B^c \bar{\nabla}_c u_a + B_a \bar{\nabla}_c u^c = 0, \tag{3.4.47}$$

and

$$\bar{\nabla}^a B_a = 0. \tag{3.4.48}$$

[14] In general conductivity is anisotropic described by a second rank tensor, $\varsigma_a{}^b$. Here we consider the case that $\varsigma_a{}^b = \varsigma \delta_a{}^b$

The Ampére law given by equation (3.3.90) takes the form

$$J_a = \epsilon_{abc}\dot{u}^b B^c + \epsilon_{abc}\bar{\nabla}^b B^c, \tag{3.4.49}$$

while the relativistic Gauss law for the electric field given by (3.3.91) is

$$\hat{\sigma} = -\epsilon^{abc} B_c \bar{\nabla}_a u_b. \tag{3.4.50}$$

3.4.3.4 The system of equations for single fluid relativistic ideal MHD

In the previous two sections we worked towards the single fluid approximation of a multifluid and introduced the perfect conductivity law. Here we will complete the description of the relativistic ideal MHD with the system of equations. The collective energy-momentum tensor putting together the terms of equations (3.4.40) and (3.4.46) is given by

$$T^{ab}_{\text{MHD}} = \left(\Psi_{\text{MHD, F}} + \frac{1}{2}B^2\right) g^{ab} + \left(\rho_{\text{MHD, F}} + B^2\right) u^a u^b - B_a B_b. \tag{3.4.51}$$

Below we derive the conservation equation for ρ_{MHD} and the Euler equation of the fluid. To do so we project equation (3.1.23) along and orthogonally to u^a respectively. Additionally, in the case the electromagnetic part of the energy-momentum tensor is given by equation (3.4.34) we have (see Appendix section A.1.6 for the derivation)

$$\nabla_a T^{ab}_{\text{MHD,EM}} = -F_{ab} j^b, \tag{3.4.52}$$

where, as mentioned before, the right-hand-side is the relativistic Lorentz force. We substitute the equation above in the conservation equation (3.1.23) in order to simplify the latter. It follows that the energy conservation is given by

$$\dot{\rho}_{\text{MHD,F}} + \nabla_a u^a \left(\rho_{\text{MHD,F}} + \Psi_{\text{MHD,F}}\right) = 0. \tag{3.4.53}$$

This is the equivalent of the Newtonian continuity equation. The Euler equation is given by

$$\left(\rho_{\text{MHD,F}} + \Psi_{\text{MHD,F}}\right) u^b \nabla_b u^a + h^{ab} \nabla_b \Psi_{\text{MHD,F}} - \epsilon^{abc} J^b B^c = 0, \tag{3.4.54}$$

and is analogous to the Newtonian Euler equation. Substituting the spatial current J^a from Maxwell equation (3.4.49) and after some manipulation the Euler equation takes the form

$$\begin{aligned}&(\rho_{\mathrm{MHD,F}}+\Psi_{\mathrm{MHD,F}})\,u^b\nabla_b u^a+h^{ab}\nabla_b\Psi_{\mathrm{MHD,F}}+B_cB^cu^b\nabla_b u^a+h^{ab}B_c\nabla_b B^c\\&-B^aB_cu^b\nabla_b u^c-h^a_{\ b}B^c\nabla_c B^b=0.\end{aligned}\tag{3.4.55}$$

The evolution of the magnetic field is given by equation (3.4.47), which serves as the relativistic counterpart of the induction equation. The system of equations is completed with adiabatic condition (3.2.30). The specific entropy Σ serves as an equation of state and is assumed to be a function of the fluid energy density $\rho_{\mathrm{MHD,F}}$ and the pressure $\Psi_{\mathrm{MHD,F}}$. The fact that the specific entropy is a function of both the pressure and energy density is in agreement with the consideration of section 3.2.1 where the Lagrangian (which serves as an equation of state, as well) is a function of two independent variables, namely the number density and entropy density. In that sense we have chosen a different equation of state (namely Σ) but still with two independent variables (the pressure and the energy density).
The system of equation is now complete. Euler equation and induction equation have three independent components because they are orthogonal to the 4-velocity. Therefore we have four equations with eight independent components in total for eight unknowns, namely three components of the 4-velocity (note that the fourth component is not independent of the rest due to the $u^a u_a = -1$ normalisation), three components of the magnetic field, the fluid energy density and the fluid pressure. In the following chapter we will use this set of equations in order to look into some phenomena of ideal MHD, and we will eventually describe the the magnetorotational instability [36].

Before moving on, it is worth noting that the Gauss law for the magnetic field given by equation (3.4.48) should be satisfied identically and while the respective law for the electric field (equation (3.4.50)) indicates a non-zero free-electric charge of kinematic origin when the right-hand side of the respective equation is not vanishing. Also, it is evident we have not used the momentum conservation equation (3.4.30) but instead we used the relativistic Euler equation (3.4.55) which is equivalent. Additionally, instead of using the conservation law for the approximated single fluid and the entropy density current given by equations (3.4.43) and (3.2.4) we use the adiabatic condition (3.2.30). This condition is justified since entropy and ions are co-moving, as is shown in section 3.2.1.

CHAPTER 4

Perturbations

In this part we perform a first order perturbation analysis using the geometric optics approach. This method is a special case of the "WKB" or "WKBJ" approximation, after the physicists G. Wentzel, H. Kramers, L. Brillouin and mathematician H. Jeffreys. It is also sometimes called the "two-timing" method [82, 83, 84]. Although the name "geometric optics" seems to refer to optics, it is used in the context of wave propagation in general, as well. In this chapter we look into various systems which are not limited to electromagnetism so we use the method in a broader sense. Assuming an initial background solution to the system under consideration this analysis provides some insight into the stability of the system when perturbed.

4.1 The geometric optics approximation

In this section we introduce geometric optics in order to look into linear perturbations. We introduce the notion of "fast" and "slow" quantities and discuss stability criteria, as well.

4.1.1 Zero and first order terms

To calculate first order perturbations for a system of equations, we assume for any physical quantity, say V^a, a solution of the form

$$V^a = V_{0}^{a} + \delta V^a, \tag{4.1.1}$$

where V_{0}^{a} is the background term. We assume that the first order perturbation, δV^a, is given by the formal asymptotic expansion [82]

$$\delta V^a = \bar{\delta}\left(\sum_{q=0}^{+\infty} \bar{\varepsilon}^q \bar{V}_{\mathrm{q}}^a\right) e^{i\frac{S}{\bar{\varepsilon}}}, \tag{4.1.2}$$

where $\bar{\delta}, \bar{\varepsilon}$ are small dimensionless book-keeping parameters such that

$$0 < \bar{\varepsilon} < \bar{\delta} \ll 1, \tag{4.1.3}$$

and $\bar{V}_{\mathrm{q}}^a$ are vector coefficients of the expansion[1]. The expression above may be written as

$$\delta V^a = \bar{\delta}\bar{V}^a e^{i\frac{S}{\bar{\varepsilon}}}, \tag{4.1.4}$$

describing a locally plane wave with amplitude $\bar{V}^a$ (which is equal to the term in the parenthesis in equation (4.1.2)) and phase S. The quantity $\bar{\delta}$ provides a relative measure between the scale (i.e. magnitude) of the background terms and the scale of the perturbed terms through

$$\frac{|\delta V^a|}{|V_{0}^{a}|} \sim \bar{\delta}, \tag{4.1.5}$$

where $|V_{0}^{a}|$ and $|\delta V^a|$ are the norms of the respective quantities, while $\bar{\varepsilon}$ measures how "fast" or "slow" the various quantities of the system are (as we will discuss in the following section).

Substituting the solution (4.1.1) into the system of equations we get various orders in terms of powers of $\bar{\delta}$ and $\bar{\varepsilon}$. It follows from inequality (4.1.3) that in descending magnitude (i.e. from largest to smallest) these are

$$\begin{aligned} &\bar{\delta}^0\bar{\varepsilon}^0 > \bar{\delta}^1\bar{\varepsilon}^0 > \bar{\delta}^0\bar{\varepsilon}^1 > \bar{\delta}^1\bar{\varepsilon}^1 > \dots \\ &> \bar{\delta}^m\bar{\varepsilon}^m > \bar{\delta}^{m+1}\bar{\varepsilon}^m > \bar{\delta}^m\bar{\varepsilon}^{m+1} > \bar{\delta}^{m+1}\bar{\varepsilon}^{m+1} > \dots\ . \end{aligned} \tag{4.1.6}$$

[1] Note that there is no Einstein summation for q.

Since we are employing the geometric optics approximation, we will consider only two orders. The "zero order" (i.e. the $\bar{\delta}^0\varepsilon^0$) also referred to as the "background", which is assumed to satisfy the system of equations alone, and "first-order" (i.e. the $\delta^1\varepsilon^0$ terms) also referred to as "perturbed order" or "linearised order", which again is assumed to satisfy the system of equations. Higher order terms in $\bar{\varepsilon}$ consist "post-geometric optics" approximations, while higher orders in $\bar{\delta}$ imply non-linear perturbations [41, 82, 83].

In the following sections we will focus on the case where the metric is fixed and so the metric perturbations are vanishing i.e. $\bar{g}_{ab} = 0$. This is usually referred to as the Cowling approximation. This assumption is valid when the background spacetime is curved but the origin of curvature is not due to the configuration we examine. Therefore, perturbing the matter and electromagnetic field under consideration does not introduce a perturbation in the metric. An example of a physical situation where this assumption applies is the perturbation of a configuration (possibly a gas-dust disc) in the proximity of a massive object (a neutron star, for example). We assume that the curvature and any perturbation of spacetime at a point in the disk induced by the disk itself is negligible compared to that induced by the presence of the neutron star. In that sense

$$\frac{|\delta g_{ab}|}{|g^0_{ab}|} \sim \bar{\delta}^2. \tag{4.1.7}$$

4.1.2 "Fast" and "slow" quantities

In order to be able to describe more complicated configurations, we introduce the "fast" and "slow" characterisation of the various quantities [82]. Within this formalism all the quantities that appear in our equations are assumed to be either "fast" or "slow" varying. We assume the existence of coordinates[2] x^μ such that a "slow" component (in this coordinate basis) of some quantity is of the form[3]

$$V^{\mathrm{s}}_\mu = V^{\mathrm{s}}_\mu\left(\bar{\varepsilon}x^\nu\right). \tag{4.1.8}$$

[2]Note that Latin indices are abstract while Greek indices are concrete taking values $0\ldots3$.

[3]Here we mean that a "slow" quantity is of the funtional form (4.1.8). This statement expressed with mathematical rigor would read $V^{\mathrm{s}}_\mu = \tilde{\tilde{V}}_\mu\left(\bar{\varepsilon}x^\nu\right)$, i.e. that components V^{s}_μ in this specific coordinate basis have the functional dependence of "slow" functions, namely $\tilde{\tilde{V}}_\mu\left(\bar{\varepsilon}x^\nu\right)$. Nevertheless in physics literature the same statement is usually expressed as given in form (4.1.8) i.e. by using the same symbol in both sides. In order to avoid introducing new symbols in equations (4.1.8), (4.1.9) and (4.1.10) we use the latter.

For the components of the "fast" varying quantities (in the same coordinate basis) we have

$$V^{\rm f}_{\mu} = V^{\rm f}_{\mu}\left(x^{\nu}\right). \tag{4.1.9}$$

Although we have used co-vectors as an example, the above formulas are directly generalised for higher rank tensors. "Slow" quantities have components which are functions of $\bar{\varepsilon}x^{\mu}$ while "fast" components do not obey that assumption. By introducing a coordinate system which satisfies the assumptions above, we are diverting from the covariant approach which we have followed up to this point. To clarify things regarding this diversion we discuss the various cases of "slow" quantities and how they affect the covariance[4] of the various quantities.

The "most" covariant (or more precisely, closest to covariance) case is when all components (with respect to a coordinate system) of some tensorial quantity is given by either of the above forms and additionally this happens for a number of coordinate systems. The reason behind the requirement "for a number of coordinate systems" (rather than just one coordinate system) where the assumption holds is that it ensures (in some sense) the independence of the frame. In the extreme case where the above forms hold for any coordinate system we have covariance.

In most cases some of the components of the tensorial quantities of the system are "slow" with respect to some of the coordinates (in a given coordinate system) and "fast" with respect to the rest coordinates (of the same coordinate system). In such case a mixed "fast" and "slow" component has the functional form

$$V_{\mu} = V_{\mu}\left(\bar{\varepsilon}x^{\nu}, x^{\rho}\right), \quad \nu \neq \rho, \tag{4.1.10}$$

while other components of the same tensorial quantity may have different "slow" and "fast" dependence with respect to the coordinates. In the following sections we work in a specific coordinate frame and we make assumptions about the "fast"-ness and "slow"-ness of the various components of the quantities with respect to each of the coordinates of the frame. The motivation behind this strategy, as will become apparent subsequently, is that we try to make contact with the respective Newtonian calculations which are carried out in specific frames. Additionally, choosing which components are "fast" and "slow" is basically choosing in a qualitative manner the background configuration. The only constraint on these choices is that they have to make sense physically.

Below we discuss the physical intuition behind the notion of "fast" and "slow" de-

[4] By "covariance" here we refer to the property of tensors being independent of frames.

pendence as well as some details of the formulation.
If we consider a small region of spacetime (sufficiently small for our linear perturbation analysis to hold but large enough so that the hydrodynamic description is still valid) then to the orders we are considering here i.e. $\bar{\delta}^0\varepsilon^0$ and $\bar{\delta}^1\varepsilon^0$ the "slow" components (given by equation (4.1.8)) are assumed to be approximately constant, while the "fast" components (given by equation (4.1.9)) vary within this region. In that sense, $\bar{\varepsilon}$ provides a relative measure between the gradients of "slow" and "fast" quantities. In analogy with equation (4.1.5) we would write

$$\frac{|\partial_a(\text{``slow''})|}{|\partial_b(\text{``fast''})|} \sim \bar{\varepsilon}. \tag{4.1.11}$$

Looking at the same idea from a different perspective, the "slow" components introduce a lengthscale (and a timescale[5]) L such that they do not vary significantly within a region characterised by this scale. Then $\bar{\varepsilon}$ is related to this scale through

$$\frac{1}{L} \sim \bar{\varepsilon}. \tag{4.1.12}$$

The "fast" components, on the other hand, will vary within that region. In the intuitive argument regarding the relation between the scales of the system and $\bar{\varepsilon}$ we have assumed, for simplicity, that the "slow" components are purely "slow" i.e. have functional form given by equation (4.1.8). If we instead consider components given by equation (4.1.10) the same argument regarding the lengthscale holds but only for the coordinates there is an $\bar{\varepsilon}x^b$ functional dependence.
In general, background quantities (i.e. the quantities which are of zero order in $\bar{\delta}$), may have either "fast" or "slow" (or mixed given by equation (4.1.10)) components depending on the specific system under consideration. Meanwhile, the barred quantities (i.e. the first order in $\bar{\delta}$) and the phase S always have only purely "slow" components (given by equation (4.1.8)).

4.1.2.1 Metric with "slow" components

The components of the metric are one of the three types mentioned above, since the metric is a background quantity. Nevertheless, there are some implications on the Christoffel symbols and subsequently on the covariant derivative that are worth mentioning. The assumption that the background metric components are "slow",

[5] Since we use geometrised units the respective timescale, which would be cL, is again L.

means where the spacetime appears to be almost flat within the region that the approximation (is valid in the coordinate system we have chosen).
A metric with purely "slow" components with functional dependence given in equation (4.1.8) implies for the Christoffel symbols, using equation (2.1.18),

$$\begin{aligned}\Gamma^{\mu}{}_{\nu\rho}\left(\bar{\varepsilon}x^{\kappa}\right) =&\frac{1}{2}g^{\mu\sigma}\left(\bar{\varepsilon}x^{\kappa}\right)\left[\partial_{\rho}g_{\sigma\nu}\left(\bar{\varepsilon}x^{\kappa}\right)+\partial_{\nu}g_{\sigma\rho}\left(\bar{\varepsilon}x^{\kappa}\right)-\partial_{\sigma}g_{\nu\rho}\left(\bar{\varepsilon}x^{\kappa}\right)\right]\\ =&\frac{1}{2}g^{\mu\sigma}\left(X^{\kappa}\right)\left[\frac{\partial X^{\lambda}}{\partial x^{\rho}}\frac{\partial g_{\sigma\nu}\left(X^{\kappa}\right)}{\partial X^{\lambda}}+\frac{\partial X^{\lambda}}{\partial x^{\nu}}\frac{\partial g_{\sigma\rho}\left(X^{\kappa}\right)}{\partial X^{\lambda}}\right.\\ &\left.-\frac{\partial X^{\lambda}}{\partial x^{\sigma}}\frac{\partial g_{\nu\rho}\left(X^{\kappa}\right)}{\partial X^{\lambda}}\right]\\ =&\bar{\varepsilon}\,\frac{1}{2}g^{\mu\sigma}\left(X^{\kappa}\right)\left[\frac{\partial g_{\sigma\nu}\left(X^{\kappa}\right)}{\partial X^{\rho}}+\frac{\partial g_{\sigma\rho}\left(X^{\kappa}\right)}{\partial X^{\nu}}-\frac{\partial g_{\nu\rho}\left(X^{\kappa}\right)}{\partial X^{\sigma}}\right]\\ =&\bar{\varepsilon}\,\hat{\Gamma}^{\mu}{}_{\nu\rho}\left(X^{\kappa}\right),\end{aligned} \tag{4.1.13}$$

where we have used the coordinate transformation $X^a = \varepsilon x^a$. Note that, in order to avoid confusion, '∂_μ' denotes partial differentiation with respect to coordinates x^a while partial differentiation with respect to X^a is denoted with '$\frac{\partial}{\partial X^a}$'. The $\hat{\Gamma}^{\mu}{}_{\nu\rho}$ components are of order unity since partial derivatives of any "slow" component (given by equation (4.1.8)) with respect to X^a are of order unity. Thus, equation (4.1.13) yields that the Christoffel symbols with respect to the x^a coordinates are of order $\bar{\varepsilon}$. Equation (3.1.14) then implies that the components of the Riemann tensor (in x^a coordinates) are of the order $\bar{\varepsilon}^2$ and that means that the spacetime is approximately flat.
The components of the covariant derivative (see definition (2.1.17)) of a co-vector is given by

$$\nabla_\mu V^{\mathrm{s}}_\nu = \bar{\varepsilon}\left(\frac{\partial V^{\mathrm{s}}_\nu}{\partial X^\mu} - \hat{\Gamma}^{\sigma}_{\mu\nu}V^{\mathrm{s}}_\sigma\right), \tag{4.1.14}$$

where the component V^{s}_ν is "slow" (i.e. having the functional form 4.1.8). For first order perturbation terms, given in equation (4.1.4), we have

$$\nabla_\mu \delta V_\nu = \bar{\varepsilon}\left(\frac{\partial \bar{V}_\nu}{\partial X^\mu} - \hat{\Gamma}^{\sigma}_{\mu\nu}\bar{V}_\sigma\right)e^{i\frac{S}{\bar{\varepsilon}}} + ik_\mu \delta V_\nu, \tag{4.1.15}$$

where k_μ is the 4-wavevector defined through

$$k_\mu = \partial_\mu\left(\frac{S}{\bar{\varepsilon}}\right) = \frac{\partial S}{\partial X^\mu}, \tag{4.1.16}$$

which is normal to the constant S surfaces. The 4-wavevector is the 4-dimensional generalisation of the 3-dimensional wavevector and denotes the direction of the plane

wave in spacetime. Since the wavevector is a gradient of a scalar, the definition above for the 4-wavevector is independent of the "slow"-ness or "fast"-ness of the components of the metric. Additionally, the components of the 4-wavevector are of order unity since they are partial derivatives with respect to X^a. This implies, through the approximation (4.1.12), that the wavelength of the plane wave is smaller than the characteristic lengthscale L. Alternatively stated, there is enough space for the plane wave to oscillate within L.
Finally, for the "fast" components of a co-vector we have

$$\nabla_\mu V^{\mathrm{f}}_\nu = \frac{\partial V^{\mathrm{f}}_\nu}{\partial x^\mu} - \bar{\varepsilon}\hat{\Gamma}^\sigma_{\mu\nu} V^{\mathrm{f}}_\sigma. \tag{4.1.17}$$

From equations (4.1.14) and (4.1.17) we observe that the components of the covariant derivative of a co-vector, to order $\bar{\varepsilon}^0$, is either zero or equal to the partial differentiation with the same indices. Additionally, equation (4.1.15) implies that for a first order perturbation co-vector it is equivalent (again to order $\bar{\varepsilon}^0$) to substitute the operator ∇_μ with ik_μ. Although we have used co-vectors for the analysis above, the generalisation to any kind of tensor is straightforward.

4.1.2.2 Metric with mixed "slow" and "fast" components

In the previous section we discussed the case where the metric components have the functional form (4.1.8). Here we discuss the implications of a metric that may have some components "slow" with respect to some coordinate(s) of the coordinate system x^a (i.e. the components are of the functional form 4.1.10). In this case only some of the Christoffel symbols will be of order $\bar{\varepsilon}$, given in equation (4.1.13). The remaining Christoffel symbols will be of order unity. This consideration, that the metric has mixed "fast" and "slow" dependence means that we are restricted to a specifically chosen spacetime. Although some Christoffel symbols are of order unity, flat spacetime is still a possible choice. This is the case because flat spacetime may be expressed in coordinate systems, such as the cylindrical polar coordinates, where some of the Christoffel symbols are not zero.
Therefore, for the Christoffel symbols that are of order unity the components of the covariant derivative of a co-vector are

$$\nabla_\mu V^{\mathrm{s}}_\nu = \bar{\varepsilon}\frac{\partial V^{\mathrm{s}}_\nu}{\partial X^\mu} - \Gamma^\sigma_{\mu\nu} V^{\mathrm{s}}_\sigma, \tag{4.1.18}$$

where the co-vector component $V^{\rm s}_\nu$ has the functional form (4.1.8). The respective equation when the co-vector component has the functional form (4.1.9) is

$$\nabla_\mu V^{\rm f}_\nu = \frac{\partial V^{\rm f}_\nu}{\partial x^\mu} - \Gamma^\sigma_{\mu\nu} V^{\rm f}_\sigma. \tag{4.1.19}$$

Finally a first order perturbation co-vector we have

$$\nabla_\mu \delta V_\nu = \bar{\varepsilon} e^{i\frac{S}{\bar{\varepsilon}}} \frac{\partial \bar{V}_\nu}{\partial X^\mu} - \Gamma^\sigma_{\mu\nu}\delta\bar{V}_\sigma + ik_\mu \delta V_\nu, \tag{4.1.20}$$

where the 4-wavevector is given by the definition (4.1.16). The components of the covariant derivative of a co-vector with components of mixed "slow" and "fast" dependence along the coordinates, i.e. of the form (4.1.10), are given by equations (4.1.18) or (4.1.19), depending on the "fast" or "slow"-ness of each co-vector component along each of the coordinates.

In contrast to the previous section, we find that the components of the covariant derivative may not be negligible since they contain Christoffel symbols of order unity. Allowing the metric to have "fast" components has also the following implication. A tensor quantity, say a vector with the index upstairs, that has all components and with respect to all coordinates of the coordinate system "slow" (i.e. having the functional form 4.1.8) may have "fast" components with respect to some coordinate(s) when the index is lowered. This means that we should take into account the type (i.e. which indices are upstairs and downstairs) of the various tensor quantities when making the assumptions regarding the "fast" and "slow"-ness of the components.

4.1.3 The causality and stability criteria

In this section we discuss the causality and stability criteria arising by using the geometric optics method. In the analysis below we consider that the fundamental observer (in the sense of Chapter 3) is a background quantity and therefore denoted as u^a_0. The 4-wavevector given by definition (4.1.16) decomposes with respect to this observer as

$$k_a = k\left(v_{\rm ph} u^0_a + \hat{k}_a\right), \tag{4.1.21}$$

where $v_{\rm ph}$ is the phase velocity, $\hat{k}_a$ is the spatial (i.e. $\hat{k}_a u^a_0 = 0$) unit 3-wavevector (i.e $\hat{k}^a\hat{k}_a = 1$), and k is the spatial wavenumber. The vector $k^\perp_a = h^0{}_a{}^b k_b = k\hat{k}_a$ is the spatial part of the wavevector that is usually mentioned in textbooks discussing wave propagation in optics and acoustics. [85]. The angular frequency of the plane wave is given by

$$\omega = v_{\mathrm{ph}}\, k. \tag{4.1.22}$$

The 4-wavevector is a spacelike or null vector [41] since it is normal to the S isosurfaces which are timelike or null. An S isosurface is timelike or null because, by definition, it has everywhere the same value and for that reason there should be a timelike separation between two distinct points on the surface. Conversely, two points on a phase isosurface have to be causally connected (this is also called the local causality condition [86]) in order for the surface to have the same value everywhere. Using equations (4.1.21), (4.1.22), and that k_a is spacelike or null i.e. $k_a k^a \geq 0$ we get the causality criterion

$$\begin{aligned} k^2 - \omega^2 &\geq 0 \\ v_{\mathrm{ph}}^2 &\leq 1. \end{aligned} \tag{4.1.23}$$

When the 4-wavevector is null the wave is propagating with the speed of light. We see that the phase velocity is less than or equal to the speed of light. This is in contrast to Newtonian theory where the group velocity is constrained by the speed of light. The derivation of a relativistic constraint involving the group velocity in analogy with the Newtonian analysis discussed in [82], requires different consideration of wave propagation and is beyond the scope of this thesis.

In order to provide the stability criterion we first have to explain the next steps in the perturbation analysis. After considering the assumptions of the previous sections, we obtain an algebraic system, which is a homogeneous system of linear equations with perturbation amplitudes serving as variables. The coefficient matrix of this system contains only background quantities and components of the 4-wavevector. Since the solution of this system should not depend on specific values of the perturbation amplitudes we demand that the determinant of the coefficient matrix should be zero. By doing this we obtain the characteristic equation (this is also referred to as the dispersion relation [87]) ,

$$P(\omega, k, \text{background terms}) = 0, \tag{4.1.24}$$

which is a polynomial that relates the angular frequency, the 3-wavevector and the rest of the background quantities.

Following the Newtonian theory [87] we consider that the independent variable of the characteristic polynomial is the angular frequency ω and that the spatial wavenumber k is real. This latter assumption, as will become obvious later, implies that any instability we find is an absolute instability. A root of the characteristic polynomial ω_{r} can be purely real, purely imaginary or complex. In order to get some intuition of the relation between the type of the root and the linear stability of the system we work as follows. Assuming that $S(\bar{\varepsilon}x^a)$ has derivatives in x^a of all orders and using equation (A.2.40) with respect to the $X^a = \bar{\varepsilon}x^a$ coordinates we get

$$\begin{aligned} S(X^a) =& S(\vec{0}) + \frac{\partial S}{\partial X^b}X^b - \frac{1}{2!}\frac{\partial S}{\partial X^b X^c}X^b X^c + \frac{1}{3!}\frac{\partial S}{\partial X^b X^c X^d}X^b X^c X^d - \ldots \\ =& S(\vec{0}) + \bar{\varepsilon}\frac{\partial S}{\partial X^b}x^b - \frac{\bar{\varepsilon}^2}{2!}\frac{\partial S}{\partial X^b X^c}x^b x^c + \frac{\bar{\varepsilon}^3}{3!}\frac{\partial S}{\partial X^b X^c X^d}x^b x^c x^d - \ldots, \end{aligned} \tag{4.1.25}$$

where $\vec{0}$ is the origin of the X^a coordinates. Dividing the equation above by $\bar{\varepsilon}$ and using the definition (4.1.16) we obtain

$$\frac{S\left(\bar{\varepsilon}x^a\right)}{\bar{\varepsilon}} = \frac{S(\vec{0})}{\bar{\varepsilon}} + k_b x^b - \frac{\bar{\varepsilon}}{2!}\frac{\partial k_b}{\partial X^c}x^b x^c + \frac{\bar{\varepsilon}^2}{3!}\frac{\partial k_b}{\partial X^c X^d}x^b x^c x^d - \ldots. \tag{4.1.26}$$

It follows, by keeping terms of order unity and larger, that the exponential part of the perturbation terms given in equation (4.1.4) can by approximated through

$$e^{i\frac{S}{\bar{\varepsilon}}} \approx c_1 e^{ik_a x^a}, \tag{4.1.27}$$

where c_1 is a constant which can be neglected[6]. Although the coordinates x^a do not have the properties of vectors (since they do not transform as vectors), we see that up to order of unity the exponential part of the perturbation is approximated by the relativistic analogue of a Newtonian plane wave. Note that, since k_a is not calculated at a specific point, the approximation above holds for any point.

We assume a complex solution of the form $\omega_{\mathrm{r}} = \omega_{\mathrm{r}}^{\mathrm{Re}} + i\omega_{\mathrm{r}}^{\mathrm{Im}}$ (where $\omega_{\mathrm{r}}^{\mathrm{Re}}$ and $\omega_{\mathrm{r}}^{\mathrm{Im}}$ are real) for the characteristic polynomial given by equation (4.1.24) and substitute the decomposition given in equation (4.1.21) in the right-hand-side of the equation above

$$\exp\left[i\left(\omega_{\mathrm{r}}^{\mathrm{Re}}u_a x^a + k_a^{\perp}x^a\right)\right]\exp\left[-\omega_{\mathrm{r}}^{\mathrm{Im}}\left(u_a x^a\right)\right], \tag{4.1.28}$$

where we also used equation (4.1.22) and the assumption that k is real. Since projection along the 4-velocity provides temporal components of the projected quantities the right exponential is in some sense the equivalent of the Newtonian $e^{-\omega_{\mathrm{r}}^{\mathrm{Im}}t_{\mathrm{Newt}}}$ (where t_{Newt} is the Newtonian time). Therefore, the term $\exp\left[-\omega_{\mathrm{r}}^{\mathrm{Im}}\left(u_a x^a\right)\right]$ implies

[6] The c_1 constant appears only in equations of order $\bar{\delta}^1\bar{\varepsilon}^0$ and is multiplied by all terms. Therefore it can be factored out.

an exponential growth of the perturbation along the observer's temporal part of the coordinates if the exponent is positive and a decay if the exponent is negative. In most cases the polynomial (4.1.24) has only real coefficients and therefore for every complex root the respective complex conjugate of the root (i.e. $\omega_{\mathrm{r}}^{\dagger} = \omega_{\mathrm{r}}^{\mathrm{Re}} - i\omega_{\mathrm{r}}^{\mathrm{Im}}$) will also be a root [88]. It follows that in order to have a stable solution, that is to avoid terms of exponential growth, the imaginary part of the angular frequency $\omega_{\mathrm{r}}^{\mathrm{Im}}$ has to vanish. This means that for stability the angular frequency should be

$$\begin{aligned} \omega^2 &\geq 0 \\ v_{\mathrm{ph}}^2 &\geq 0, \end{aligned} \tag{4.1.29}$$

which in turn means that the system is stable when the polynomial given in equation (4.1.24) has only real roots. The criterion above along with that given by inequality (4.1.23) constrain the phase velocity through

$$0 \leq v_{\mathrm{ph}}^2 \leq 1. \tag{4.1.30}$$

In the analysis we did not consider complex spatial wavenumbers. Considering such would introduce terms of exponential growth or decay along the observer's spatially projected part of the coordinates. These kind of instabilities or decays would be the equivalent of the Newtonian amplifying and evanescent waves [87] and are beyond the scope of this thesis.

Finally, the instabilities we calculate with this method are local instabilities. This means that the analysis and the results refer to a small region of spacetime where the background solution is valid. Nevertheless, this can be any small spacetime region around any point (of those that satisfy the background solution) and therefore the results are in some sense holistic.

4.1.4 An intuitive argument regarding the choice of observer

Before we move on there is one more thing to discuss regarding geometric optics and stability. The introduction of "fast" and "slow" components in specific coordinate systems is frame dependent as discussed in section 4.1.2. A component of a projected tensor with respect to some observer, say the energy density given in equation (3.4.37), that is "slow" along some coordinate(s) may not be "slow" if it is calculated with respect to another observer along the same coordinate(s) (in the same coordinate system). Thus, the choice of observer matters in the characterisation of stability for the system.

In order to avoid unphysical choices of observers we choose the observer co-moving with the fluid, i.e. the fundamental observer u_0^a (as in section 3.4.3.2). The intuitive argument behind this consideration is the following. As we discussed in section 2.3 an observer co-moving with the fluid has "attached" a material element along their worldline which consists of a specific number of particles. Therefore, an instability with respect to this observer is transferred to the matter space and to the particles consisting the material element. If instead we choose a different observer (i.e. one that is not co-moving with the fluid) then the particles along their worldline do not correspond to conserved material elements in matter space, and thus instabilities might be related to the specific choice of this (not co-moving) observer.

4.2 A Newtonian interlude

Before proceeding to describe the application of the geometric optics method in a relativistic context we will derive some well-known results in the Newtonian framework. The reasons for this Newtonian interlude in an otherwise relativistic thesis are two. Firstly, obtaining already known results provides more insight and understanding of the method. This will help in providing a more accurate interpretation of the results that we obtain in the relativistic context. Secondly, some of the known results discussed in the following section are usually derived using different approaches. In that sense it is interesting to show how to obtain these results using the geometric optics method.

The covariant expression for the Euler equation of motion for a fluid in the Newtonian case contains the covariant derivative of the fluid velocity [40]. In this section it is possible to assume that the Christoffel symbols will be of order of $\bar{\varepsilon}$ but we need to use coordinates where this assumption holds. As discussed previously this assumption will not hold for any coordinate system (e.g. in cylindrical polar coordinates). The following examples are either in Cartesian coordinates where the Christoffel symbols vanish completely or in cylindrical coordinates where we have retained the terms of at least the order of unity.

4.2.1 The Newtonian framework

The results we derive here are either in the context of hydrodynamics or magnetohydrodynamics [75]. The equations presented here are the Newtonian versions of the equation discussed in section 3.4.3.4. The description of a single fluid in the Newtonian framework employs the continuity equation given by

$$\frac{\partial \rho}{\partial t} + \rho \nabla \cdot \boldsymbol{v} + (\boldsymbol{v} \cdot \nabla)\rho = 0, \tag{4.2.1}$$

where $\boldsymbol{v}$ is the fluid velocity and ρ is the density. We also have the Euler (momentum conservation) equation

$$\frac{\partial \boldsymbol{v}}{\partial t} + (\boldsymbol{v} \cdot \nabla)\boldsymbol{v} + \frac{1}{\rho}\nabla P + \nabla \Phi + \underbrace{\frac{1}{4\pi\rho}\boldsymbol{B} \times (\nabla \times \boldsymbol{B})}_{\text{MHD Lorentz force}} = 0, \tag{4.2.2}$$

where Φ is the gravitational potential and $\boldsymbol{B}$ is the magnetic field and P is the pressure. For a purely hydrodynamical system the MHD Lorentz force (i.e. the under-braced term) vanishes. In order to describe MHD systems we have to introduce one more equation, the magnetic field induction equation

$$\frac{\partial \boldsymbol{B}}{\partial t} - \nabla \times (\boldsymbol{v} \times \boldsymbol{B}) = 0. \tag{4.2.3}$$

Additionally, the Gauss law for the magnetic field $\nabla \cdot \boldsymbol{B} = 0$ should be identically satisfied (so that it does not overdetermine the system). Obviously for pure hydrodynamic systems this equation is not needed. We also have the adiabatic condition

$$\frac{d\Sigma}{dt} = \frac{\partial \Sigma}{\partial t} + (\boldsymbol{v} \cdot \nabla)\Sigma = 0, \tag{4.2.4}$$

where Σ is the specific entropy. The operator $\frac{d}{dt}$ is the Newtonian convective derivative. This equation means that the entropy is conserved along the flow lines which happens in the case of adiabatic flows. We assume that the entropy is a function of the pressure and the density

$$\Sigma = \Sigma\left(P, \rho\right), \tag{4.2.5}$$

that serves as an equation of state for the system. Applying the chain rule of differentiation to the functional form of the specific entropy (4.2.5) and using the adiabatic condition we get

$$\begin{aligned}\frac{d\Sigma}{dt} &= 0 \\ \left.\frac{\partial\Sigma}{\partial P}\right|_{\rho}\frac{dP}{dt} + \left.\frac{\partial\Sigma}{\partial\rho}\right|_{P}\frac{d\rho}{dt} &= 0 \\ \frac{dP/dt}{d\rho/dt} &= -\frac{\partial\Sigma/\partial\rho|_P}{\partial\Sigma/\partial P|_\rho} \\ \left.\frac{\partial P}{\partial\rho}\right|_{\Sigma} &= -\frac{\partial\Sigma/\partial\rho|_P}{\partial\Sigma/\partial P|_\rho}.\end{aligned} \tag{4.2.6}$$

The speed of sound is defined through

$$c_{\mathrm{s}}^2 = \left.\frac{\partial P}{\partial\rho}\right|_{\Sigma}, \tag{4.2.7}$$

and describes the speed of propagation for acoustic perturbations [62]. Using this definition for speed of sound, equation (4.2.4) in terms of P and ρ becomes

$$\frac{\partial P}{\partial t} + (\boldsymbol{v}\cdot\nabla)P - c_{\mathrm{s}}^2\left[\frac{\partial\rho}{\partial t} + (\boldsymbol{v}\cdot\nabla)\rho\right] = 0. \tag{4.2.8}$$

4.2.2 Linear perturbations (Newtonian framework)

The linear perturbations in the Newtonian frame are formulated similarly to the relativistic case, discussed in section 4.1. We substitute all quantities of the system using equation (4.1.1). As previously mentioned, we assume that components of all first order perturbation amplitudes are "slow". Additionally, we assume that the perturbation of the gravitational potential is zero ($\delta\Phi = 0$). The components of the background quantities are either "slow" or "fast" according to the system under consideration. The phase of the plane wave is given by

$$S = \varepsilon(\boldsymbol{k}\cdot\boldsymbol{r} - \omega t), \tag{4.2.9}$$

where $\boldsymbol{k}$ is the wavevector, $\boldsymbol{r}$ is the position vector, ω the frequency and t the time. Following equation (4.1.4) and since the perturbations amplitudes of the various quantities are "slow" (as discussed in section 4.1.2), the perturbation of the fluid velocity is given by

$$\delta\boldsymbol{v} = \bar{\delta}\bar{\boldsymbol{v}}\,(\bar{\varepsilon}t, \bar{\varepsilon}\boldsymbol{r})\,e^{i\frac{S}{\bar{\varepsilon}}}. \tag{4.2.10}$$

The perturbations of the pressure and density and entropy are given by equivalent expressions through

$$\delta\rho = \bar{\delta}\bar{\rho}\left(\bar{\varepsilon}t, \bar{\varepsilon}\boldsymbol{r}\right) e^{i\frac{S}{\bar{\varepsilon}}}, \tag{4.2.11}$$

and

$$\delta P = \bar{\delta}\bar{P}\left(\bar{\varepsilon}t, \bar{\varepsilon}\boldsymbol{r}\right) e^{i\frac{S}{\bar{\varepsilon}}}. \tag{4.2.12}$$

Finally, the perturbation of the background magnetic field is given by

$$\delta\boldsymbol{B} = \bar{\delta}\bar{\boldsymbol{B}}\left(\bar{\varepsilon}t, \bar{\varepsilon}\boldsymbol{r}\right) e^{i\frac{S}{\bar{\varepsilon}}}. \tag{4.2.13}$$

The background terms (i.e. those of order $\bar{\delta}^0\bar{\varepsilon}^0$) of the system are given by equations (4.2.1)-(4.2.5) with the various quantities considered only in the $\bar{\delta}^0$ order[7]. Keeping terms of order $\bar{\delta}^1\bar{\varepsilon}^0$ in the general case where all background quantities are "slow", the continuity equation (4.2.1) yields

$$-i\omega\bar{\rho} + \bar{\rho}\left(\nabla\cdot\boldsymbol{v}_0\right) + i\rho_0\left(\boldsymbol{k}\cdot\bar{\boldsymbol{v}}\right) + i\left(\boldsymbol{v}_0\cdot\boldsymbol{k}\right)\bar{\rho} + \bar{\boldsymbol{v}}\cdot\nabla\rho_0 = 0. \tag{4.2.14}$$

Similarly, the Euler equation (4.2.2) obtains the following form

$$\begin{aligned} &- i\omega\bar{\boldsymbol{v}} + \left(\bar{\boldsymbol{v}}\cdot\nabla\right)\boldsymbol{v}_0 + i\left(\boldsymbol{v}_0\cdot\boldsymbol{k}\right)\bar{\boldsymbol{v}} + \left(\boldsymbol{v}_0\cdot\nabla\right)\bar{\boldsymbol{v}} - \frac{\bar{\rho}}{\rho_0^2}\nabla P_0 + i\frac{\boldsymbol{k}}{\rho_0}\bar{P} \\ &\underbrace{-\frac{\bar{\rho}}{4\pi\rho_0^2}\boldsymbol{B}_0\times\left(\nabla\times\boldsymbol{B}_0\right) + \frac{1}{4\pi\rho_0}\bar{\boldsymbol{B}}\times\left(\nabla\times\boldsymbol{B}_0\right) + \frac{1}{4\pi\rho_0}\boldsymbol{B}_0\times\left(\nabla\times\bar{\boldsymbol{B}}\right)}_{\text{Perturbed MHD Lorentz force terms}} = 0. \end{aligned} \tag{4.2.15}$$

The perturbed induction equation is

$$-i\omega\bar{\boldsymbol{B}} + \bar{\boldsymbol{B}}\left(\nabla\cdot\boldsymbol{v}_0\right) + i\boldsymbol{B}_0\left(\boldsymbol{k}\cdot\bar{\boldsymbol{v}}\right) - \left(\bar{\boldsymbol{B}}\cdot\nabla\right)\boldsymbol{v}_0 - \left(\boldsymbol{B}_0\cdot\nabla\right)\bar{\boldsymbol{v}} + \left(\boldsymbol{v}_0\cdot\nabla\right)\bar{\boldsymbol{B}} = 0. \tag{4.2.16}$$

Using the chain rule of differentiation the perturbation of equation (4.2.5) provides a relation between the perturbation of the entropy and the perturbations of pressure and density

$$\begin{aligned} \delta\Sigma &= \left.\frac{\partial\Sigma_0}{\partial P}\right|_{\rho_0}\delta P + \left.\frac{\partial\Sigma_0}{\partial\rho}\right|_{P_0}\delta\rho \\ \bar{\Sigma} &= \left.\frac{\partial\Sigma_0}{\partial P}\right|_{\rho_0}\bar{P} + \left.\frac{\partial\Sigma_0}{\partial\rho}\right|_{P_0}\bar{\rho}. \end{aligned} \tag{4.2.17}$$

[7] The background is basically equations (4.2.1)-(4.2.5) with all quantities subscripted zero

In the equation above the partial derivatives are calculated at the values of the background. This is justified because the specific entropy, a dependent variable of the system, may have different perturbation values for different background values of pressure and density. The perturbation of entropy conservation is given by

$$
\begin{aligned}
& -i\omega\bar{\Sigma} + \bar{\boldsymbol{v}}\cdot\nabla\Sigma_0 + i\boldsymbol{v}_0\cdot\boldsymbol{k}\bar{\Sigma} + \boldsymbol{v}_0\cdot\nabla\bar{\Sigma} = 0 \\
& i\left(\boldsymbol{v}_0\cdot\boldsymbol{k}-\omega\right)\left(\left.\frac{\partial\Sigma_0}{\partial P}\right|_{\rho_0}\bar{P} + \left.\frac{\partial\Sigma_0}{\partial\rho}\right|_{P_0}\bar{\rho}\right) + \left(\bar{\boldsymbol{v}}\cdot\nabla P_0\right)\left.\frac{\partial\Sigma_0}{\partial P}\right|_{\rho_0} \\
& +\left(\bar{\boldsymbol{v}}\cdot\nabla\rho_0\right)\left.\frac{\partial\Sigma_0}{\partial\rho}\right|_{P_0} + \boldsymbol{v}_0\cdot\nabla\bar{\Sigma} = 0 \qquad (4.2.18) \\
& i\left(\boldsymbol{v}_0\cdot\boldsymbol{k}-\omega\right)\left(\bar{P}-c_{\mathrm{s}}^2\bar{\rho}\right) + \bar{\boldsymbol{v}}\cdot\nabla P_0 - c_{\mathrm{s}}^2\,\bar{\boldsymbol{v}}\cdot\nabla\rho_0 \\
& +\boldsymbol{v}_0\cdot\nabla\left(\bar{P}-c_{\mathrm{s}}^2\bar{\rho}\right) + \frac{\bar{P}-c_{\mathrm{s}}^2\bar{\rho}}{\partial\Sigma_0/\partial P|_{\rho_0}}\boldsymbol{v}_0\cdot\nabla\left(\left.\frac{\partial\Sigma_0}{\partial P}\right|_{\rho_0}\right) = 0,
\end{aligned}
$$

hence

$$
\begin{aligned}
& i\left(\boldsymbol{v}_0\cdot\boldsymbol{k}-\omega\right)\left(\bar{P}-c_{\mathrm{s}}^2\bar{\rho}\right) + \bar{\boldsymbol{v}}\cdot\nabla P_0 - c_{\mathrm{s}}^2\,\bar{\boldsymbol{v}}\cdot\nabla\rho_0 - \bar{\rho}\boldsymbol{v}_0\cdot\nabla c_{\mathrm{s}}^2 \\
& + \frac{\bar{P}-c_{\mathrm{s}}^2\bar{\rho}}{\partial\Sigma_0/\partial P|_{\rho_0}}\boldsymbol{v}_0\cdot\nabla\left(\left.\frac{\partial\Sigma_0}{\partial P}\right|_{\rho_0}\right) = 0,
\end{aligned} \qquad (4.2.19)
$$

where we divided the second line by $\partial\Sigma_0/\partial P|_{\rho_0}$, used the definition of sound waves (4.2.7) and substituted equation (4.2.17). We also removed the gradients and partial time derivatives of the perturbations of pressure and density. Since $\bar{\rho}$ and $\bar{P}$ are scalars (no Christoffel symbols involved in the gradients) and are always "slow" (see equations 4.2.11 and 4.2.12) the partial derivatives will always be of order $\bar{\varepsilon}^1$ and therefore not considered. Furthermore, the speed of sound is a function of P, ρ and the term $\partial\Sigma_0/\partial P|_{\rho_0}$ is a function of P (since it is calculated at the fixed value ρ_0). Using the chain rule, equation (4.2.19) takes the form

$$
\begin{aligned}
& i\left(\boldsymbol{v}_0\cdot\boldsymbol{k}-\omega\right)\left(\bar{P}-c_{\mathrm{s}}^2\bar{\rho}\right) + \bar{\boldsymbol{v}}\cdot\nabla P_0 - c_{\mathrm{s}}^2\,\bar{\boldsymbol{v}}\cdot\nabla\rho_0 \\
& -\bar{\rho}\left(\left.\frac{\partial c_{\mathrm{s}}^2}{\partial P}\right|_{\rho_0}\boldsymbol{v}_0\cdot\nabla P_0 + \left.\frac{\partial c_{\mathrm{s}}^2}{\partial\rho}\right|_{P_0}\boldsymbol{v}_0\cdot\nabla\rho_0\right) \\
& + \frac{\bar{P}-c_{\mathrm{s}}^2\bar{\rho}}{\partial\Sigma_0/\partial P|_{\rho_0}}\left(\left.\frac{\partial^2\Sigma_0}{\partial P^2}\right|_{\rho_0}\right)\boldsymbol{v}_0\cdot\nabla P_0 = 0.
\end{aligned} \qquad (4.2.20)
$$

In this section we assumed that all background quantities have "fast" components since this consideration provides the most general form of the perturbed (i.e. of order $\bar{\delta}^1\bar{\varepsilon}^0$) equations. If some of the components are "slow" in the system under consideration, the respective terms are of order $\bar{\varepsilon}^1$ and therefore vanish.

4.2.3 The elimination of sound waves

In the following sections we find that characteristic polynomials of the different configurations include, expect for the terms specific to each configuration, additionally terms related to sound waves. In order to investigate stability in many of these cases we separate the solutions specific to the system from the sound waves. To do this we bring the polynomial given in equation (4.1.24) to the form

$$P_1\left(\omega, k, \text{background terms}\right) + c_\mathrm{s}^2 P_2\left(\omega, k, \text{background terms}\right) = 0, \tag{4.2.21}$$

where P_1 and P_2 are polynomials in ω. We divide the equation above by c_s^2 (assuming that the speed of sound is not zero) and multiply by v_ph^2. The quantity defined through $\mathcal{M} = v_\mathrm{ph}/c_\mathrm{s}$ is the "Mach number" [54, 82] and provides a comparison between the speed of propagation of the plane wave and the speed of sound. Low Mach numbers imply that the phase velocity is small compared to the speed of sound, while large values imply the opposite. The characteristic polynomial now has the form

$$\mathcal{M}^2\, P_1\left(\omega, k, \text{background terms}\right) + v_\mathrm{ph}^2\, P_2\left(\omega, k, \text{background terms}\right) = 0, \tag{4.2.22}$$

and by assuming $\mathcal{M}^2 \sim \bar{\varepsilon}^\alpha$ with $\alpha \geq 1$, we obtain after eliminating the first term

$$v_\mathrm{ph}^2\, P_2\left(\omega, k, \text{background terms}\right) = 0. \tag{4.2.23}$$

This equation is the characteristic polynomial[8] with sound waves removed. This is a reasonable approximation since most of the times sound waves are much faster than the propagation speed of the rest modes and thus the exclusion of sound waves does not alter the qualitative behaviour of the system regarding the stability.

The method we used here for eliminating sound waves has the advantage that it can be directly used in relativistic context. In the Newtonian framework where the various speeds of the system are not constrained, we would obtain equation (4.2.23) by merely diving equation (4.2.21) by c_s^2 and then considering that $c_\mathrm{s}^2 \to +\infty$, instead of introducing the Mach number. In contrast, in relativity this consideration

[8] The factor v_ph^2 in equation (4.2.23) introduces two additional trivial $\omega = 0$ roots to the set of solutions.

is invalid because the speed of sound is constrained by the speed of light, $c_s^2 < 1$. Since it is not possible to assign an arbitrarily large value to the speed of sound, we assume that c_s is large in comparison to the speed of wave propagation v_{ph} (which is constrained too, as shown in 4.1.30).

4.2.4 Newtonian applications of geometric optics (Cartesian coordinates)

Using the perturbed equations from section 4.2.2 we derive some results known in the literature by choosing appropriately the scale of variation for the background quantities. We discuss sound waves, the Taylor-Rayleigh instability [89, 90], the Kelvin-Helmholtz instability [89] and the Alfvén waves [75]. In the first three examples the magnetic field is zero since these are purely hydrodynamic phenomena. However, the magnetic field enters in the derivation of the Alfvén waves. Working in Cartesian coordinates in an orthonormal frame[9] the Cartesian coordinates are (x, y, z). The ∇ operator in the Cartesian orthonormal frame is given by

$$\nabla = \hat{\boldsymbol{x}}\frac{\partial}{\partial x} + \hat{\boldsymbol{y}}\frac{\partial}{\partial y} + \hat{\boldsymbol{z}}\frac{\partial}{\partial z}, \tag{4.2.24}$$

where $(\hat{\boldsymbol{x}}, \hat{\boldsymbol{y}}, \hat{\boldsymbol{z}})$ are the orthonormal basis vectors.

4.2.4.1 The sound waves

The simplest result we can produce using the geometric optics approximation is to derive the sound waves for a single fluid at rest ($\boldsymbol{v}_0 = \boldsymbol{0}$), where the background pressure, density and gravitational potential are "slow" along all directions. Under these assumptions equation (4.2.14) becomes

$$ik_x\rho_0\bar{v}_x + ik_y\rho_0\bar{v}_y + ik_z\rho_0\bar{v}_z - i\omega\bar{\rho} = 0, \tag{4.2.25}$$

the components of the linearised Euler equation are

$$-i\omega\bar{v}_x + i\frac{k_x}{\rho_0}\bar{P} = 0, \tag{4.2.26}$$

[9] An orthonormal frame is a frame where all basis vectors, say $\boldsymbol{e}_i$ with $i = 1, 2, 3$, are of unit length ($\boldsymbol{e}_i \cdot \boldsymbol{e}_i = 1$) and orthogonal to each other ($\boldsymbol{e}_i \cdot \boldsymbol{e}_j = \delta_{ij}$).

$$-i\omega\bar{v}_y + i\frac{k_y}{\rho_0}\bar{P} = 0, \tag{4.2.27}$$

$$-i\omega\bar{v}_z + i\frac{k_z}{\rho_0}\bar{P} = 0. \tag{4.2.28}$$

for the x, y and z component respectively. The perturbed entropy conservation (given in equation (4.2.20)) is

$$\omega\left(\bar{P} - c_{\mathrm{s}}^2\bar{\rho}\right) = 0. \tag{4.2.29}$$

The characteristic equation of the system above is

$$\omega^3\left(c_{\mathrm{s}}^2 k^2 - \omega^2\right) = 0, \tag{4.2.30}$$

which has a triple root $\omega = 0$ and additionally the solution

$$\omega^2 = c_{\mathrm{s}}^2 k^2. \tag{4.2.31}$$

Using equation (4.1.22) this implies that the phase velocity of the perturbations travel with the speed of sound $v_{\mathrm{ph}}^2 = c_{\mathrm{s}}^2$.

4.2.4.2 The Taylor-Rayleigh instability

The Taylor-Rayleigh instability [90] arises when a fluid of some density is superimposed over a less dense fluid in the presence of a gravitational field that varies (or when the fluids are accelerated) along the direction normal to the interface of the fluids. Here we derive a single-fluid version of this instability where instead of an interface with a discontinuity we have a density gradient. Following the original derivation where the fluids are assumed to be at rest, the background velocity in the present consideration is vanishing, i.e. $\boldsymbol{v}_0 = \boldsymbol{0}$. The pressure and density and gravitational potential are assumed "slow" in x, y directions and "fast" in the z direction, i.e. $P_0 = P_0\left(\bar{\varepsilon}t, \bar{\varepsilon}x, \bar{\varepsilon}y, z\right)$, $\rho_0 = \rho_0\left(\bar{\varepsilon}t, \bar{\varepsilon}x, \bar{\varepsilon}y, z\right)$ and $\Phi_0 = \Phi_0\left(\bar{\varepsilon}t, \bar{\varepsilon}x, \bar{\varepsilon}y, z\right)$. Assuming that the gravitational potential has a uniform gradient we have

$$\nabla\Phi_0 = -\boldsymbol{g}_0, \tag{4.2.32}$$

where $\boldsymbol{g} = -g_0\hat{\boldsymbol{z}}$, with $g_0 > 0$ the gravitational acceleration. The Euler equation (4.2.2) for the background (we remind the reader that this is of order $\bar{\delta}^0\bar{\varepsilon}^0$) yields the hydrostatic equilibrium equation

$$\frac{1}{\rho_0}\frac{\partial P_0}{\partial z} = -g_0. \tag{4.2.33}$$

As in the original work we also assume that the wavevector is orthogonal to the z-axis and given by $\boldsymbol{k} = k_x\hat{\boldsymbol{x}} + k_y\hat{\boldsymbol{y}}$. Under these assumptions the linearised continuity equation (4.2.14) becomes

$$(ik_x\rho_0)\,\bar{v}_x + (ik_y\rho_0)\,\bar{v}_y + \frac{\partial\rho_0}{\partial z}\bar{v}_z - i\omega\bar{\rho} = 0 \tag{4.2.34}$$

while the linearised Euler equation (4.2.15) becomes

$$-\,i\omega\bar{v}_x + i\frac{k_x}{\rho_0}\bar{P} = 0 \tag{4.2.35}$$

for the x component,

$$-\,i\omega\bar{v}_y + i\frac{k_y}{\rho_0}\bar{P} = 0 \tag{4.2.36}$$

for the y component, and

$$-\,i\omega\bar{v}_z - \frac{1}{\rho_0}\frac{\partial P_0}{\partial z}\bar{\rho} = 0 \tag{4.2.37}$$

for the z component. Similarly, entropy conservation equation (4.2.20) becomes

$$\left(\frac{\partial P_0}{\partial z} - c_s^2\frac{\partial\rho_0}{\partial z}\right)\bar{v}_z - i\omega\bar{P} + i\omega c_s^2\bar{\rho} = 0. \tag{4.2.38}$$

Setting the determinant of the coefficients of the system (4.2.34-4.2.38) to zero we obtain the following characteristic equation

$$\rho_0^2\omega^4 - \frac{\partial P_0}{\partial z}\frac{\partial\rho_0}{\partial z}\omega^2 - k^2\left(\frac{\partial P_0}{\partial z}\right)^2 + k^2c_s^2\left(\frac{\partial P_0}{\partial z}\frac{\partial\rho_0}{\partial z} - \rho_0^2\omega^2\right) = 0. \tag{4.2.39}$$

along with the $\omega = 0$ solution. We are interested in the case where sound waves are eliminated. Following the low Mach number argument of section 4.2.3 we obtain the reduced characteristic equation

$$\left(\omega^2\rho_0^2 - \frac{\partial P_0}{\partial z}\frac{\partial\rho_0}{\partial z}\right) = 0, \tag{4.2.40}$$

having the pair of solutions

$$\omega^2 = \frac{1}{\rho_0^2}\frac{\partial P_0}{\partial z}\frac{\partial\rho_0}{\partial z}. \tag{4.2.41}$$

Using the background equation (4.2.33) the previous result becomes

$$\omega^2 = -\frac{g_0}{\rho_0}\frac{\partial \rho_0}{\partial z}, \qquad (4.2.42)$$

and therefore, since $\rho_0 > 0$ always, the background equilibrium is unstable if $\frac{\partial \rho_0}{\partial z} > 0$, which means that the instability occurs when the density is increasing opposite to the direction of the Newtonian gravitational force (here gravity points towards the negative z-direction), as in the original derivation by Rayleigh. The result we have provided is qualitatively the same but not identical to that in [90]. The reason behind this is that our analysis, as mentioned before, is local in the sense that it describes a small region of space using continuous quantities while the original calculation uses a discontinuous density field to describe two separate fluids. Apart from that technical difference in the formulation we see that this kind of instability can be formulated in Newtonian context with a single fluid using the geometric optics approximation along with the "fast" and "slow" consideration of background quantities.

4.2.4.3 The Kelvin-Helmholtz instability

The Kelvin-Helmholtz instability occurs when two adjacent fluids with distinctive interface, flow parallel to each other, uniformly, but in opposite directions. This configuration has a velocity discontinuity across the interface and additionally is unstable. In this section we formulate the continuous version of the Kelvin-Helmholtz instability using the geometric optics approximation. We assume that the background velocity is related to a scalar f_0 through $\boldsymbol{v}_0 = \nabla f_0$. This scalar is of the form $f_0 = f_0(\bar{\varepsilon}t, x, y, \bar{\varepsilon}z)$ and therefore the velocity has a "slow" z component and "fast" x and y components given through

$$v_{0,x} = \partial f_0 / \partial x, \qquad (4.2.43)$$

and

$$v_{0,y} = \partial f_0 / \partial y, \qquad (4.2.44)$$

while $v_{0,z}$ is of the order $\bar{\varepsilon}$. It follows by the commutativity of partial derivatives that

$$\frac{\partial v_{0,x}}{\partial y} = \frac{\partial v_{0,y}}{\partial x} = \frac{\partial^2 f_0}{\partial y \partial x}. \qquad (4.2.45)$$

We also assume that $\partial v_{0,x}/\partial x = 0 = \partial v_{0,y}/\partial y$ so that the fluid is incompressible (i.e. $\nabla \cdot \boldsymbol{v}_0 = 0$). Additionally, we consider that all other background quantities are "slow" along all directions, except for the gravitational potential which is $\Phi_0 = \Phi_0(\bar{\varepsilon}t, x, y, \bar{\varepsilon}z)$. The reason that the gravitational potential has this form is that it serves as balance term in the background Euler equation (4.2.2). The wavevector has only a z component, $\boldsymbol{k} = k\hat{\boldsymbol{z}}$. Under these assumptions, the linearised continuity equation (4.2.14) becomes

$$ik_z\rho_0\bar{v}_z - i\omega\bar{\rho} = 0, \tag{4.2.46}$$

the x component of the linearised Euler equation (4.2.15) is

$$-i\omega\bar{v}_x + \frac{\partial v_{0,x}}{\partial y}\bar{v}_y = 0, \tag{4.2.47}$$

the y component is

$$-i\omega\bar{v}_y + \frac{\partial v_{0,y}}{\partial x}\bar{v}_x = 0, \tag{4.2.48}$$

and the z component is

$$-i\omega\bar{v}_z + i\frac{k}{\rho_0}\bar{P} = 0. \tag{4.2.49}$$

Finally, the perturbed entropy conservation equation is the same as in equation (4.2.29). The characteristic equation (obtained by setting the determinant of the coefficient matrix of the system above equal to zero) is given by

$$\left(\omega^2 - c_{\mathrm{s}}^2k^2\right)\left(\omega^2 + \frac{\partial v_{0,x}}{\partial y}\frac{\partial v_{0,y}}{\partial x}\right) = 0. \tag{4.2.50}$$

Eliminating the sound waves, the reduced characteristic polynomial (along with the double $\omega = 0$ root) is

$$\omega^2 = -\left(\frac{\partial^2 f_0}{\partial y \partial x}\right)^2, \tag{4.2.51}$$

where we used also equation (4.2.45). The solution for ω implies that the configuration is always unstable, since ω^2 is always negative, provided there is shear in the fluid velocity.

This result is the Kelvin-Helmholtz instability in a single fluid with shear. It is similar to the Rayleigh shearing instability that we discuss in a following section.

4.2.4.4 The Alfvén waves

Here we introduce the magnetic field and derive the characteristic equation for the two types of Alfvén waves [75]. The one type of waves, the compressional mode or the magnetosonic mode, consists of compression and de-compression of magnetic field lines. These waves are similar to the sound waves in the sense that they are disturbances of the magnetic pressure as the sound waves are disturbances of the (thermodynamic) pressure P. The other type of waves is the shear mode or Alfvén mode [91], which involves more complicated disturbances (e.g. shearing motion) of the magnetic field lines[10]. The system of equations are those of sections 4.2.1 and 4.2.2 including the terms related to the magnetic field.
We assume that the background fluid velocity vanishes and there is a background "slow" magnetic field along the z-axis $\boldsymbol{B}_0 = B_0\left(\bar{\varepsilon}x, \bar{\varepsilon}y, \bar{\varepsilon}z\right)\hat{\boldsymbol{z}}$. The gravitational potential, the pressure and the density are "slow" in the background along all directions (i.e. $P_0 = P_0\left(\bar{\varepsilon}t, \bar{\varepsilon}x, \bar{\varepsilon}y, \bar{\varepsilon}z\right)$, $\rho_0 = \rho_0\left(\bar{\varepsilon}t, \bar{\varepsilon}x, \bar{\varepsilon}y, \bar{\varepsilon}z\right)$ and $\Phi_0 = \Phi_0\left(\bar{\varepsilon}t, \bar{\varepsilon}x, \bar{\varepsilon}y, \bar{\varepsilon}z\right)$). To the order $\bar{\delta}^1\bar{\varepsilon}^0$ the continuity equation is the same as in the sound waves section given by equation (4.2.25). The components of Euler equation (4.2.15) are

$$-i\omega\bar{v}_x + i\frac{k_x}{\rho_0}\bar{P} - i\frac{B_0 k_z}{4\pi\rho_0}\bar{B}_x + i\frac{B_0 k_x}{4\pi\rho_0}\bar{B}_z = 0, \tag{4.2.52}$$

$$-i\omega\bar{v}_y + i\frac{k_y}{\rho_0}\bar{P} - i\frac{B_0 k_z}{4\pi\rho_0}\bar{B}_y + i\frac{B_0 k_y}{4\pi\rho_0}\bar{B}_z = 0, \tag{4.2.53}$$

and

$$-i\omega\bar{v}_z + i\frac{k_z}{\rho_0}\bar{P} = 0, \tag{4.2.54}$$

for the x, y and z components respectively. The components of the perturbed induction equation (4.2.16) are given by

$$-iB_0 k_z\bar{v}_x - i\omega\bar{B}_x = 0, \tag{4.2.55}$$

for the x component,

$$-iB_0 k_z\bar{v}_y - i\omega\bar{B}_y = 0, \tag{4.2.56}$$

for the y component, and

$$iB_0 k_x\bar{v}_x + iB_0 k_y\bar{v}_y - i\omega\bar{B}_z = 0, \tag{4.2.57}$$

[10] The two modes are also called fast and slow respectively. We have not used this terminology in order to avoid confusion with the notions of "fast" and "slow" quantities as they are introduced in this chapter.

for the z component. Finally, the perturbed entropy equation is the same as in the derivation of sound waves, given in equation (4.2.29). Combining the equations above, the characteristic equation of the system (along with a double $\omega = 0$ root) is

$$\left[\omega^4 - k^2\left(c_{\rm s}^2 + v_{\rm A}^2\right)\omega^2 + c_{\rm s}^2 k^2 k_z^2 v_{\rm A}^2\right]\left(\omega^2 - k_z^2 v_{\rm A}^2\right) = 0, \tag{4.2.58}$$

where $v_{\rm A}^2 = B_0^2/4\pi\rho_0$ is the Alfvén velocity. The two modes, the compressional and the shear, appear coupled because we have allowed non-zero pressure perturbations. If the pressure perturbations vanish then the two modes are derived separately using different assumption for each one. For the compressional mode we have the solution

$$\omega_\pm^2 = \frac{1}{2}\left[k^2\left(v_{\rm A}^2 + c_{\rm s}^2\right) \pm \sqrt{k^4\left(v_{\rm A}^2 + c_{\rm s}^2\right)^2 - 4k^2 k_z^2 v_{\rm A}^2 c_{\rm s}^2}\right], \tag{4.2.59}$$

while for the shear mode we have

$$\omega^2 = v_{\rm A}^2 k_z^2. \tag{4.2.60}$$

If the wave is propagating only along the z direction i.e. $\boldsymbol{k} = k_z \hat{\boldsymbol{z}}$, then equation (4.2.59) reduces to either equation (4.2.60) or to the purely acoustic mode along z direction

$$\omega^2 = c_{\rm s}^2 k_z^2. \tag{4.2.61}$$

The two solutions for ω^2 given in equation (4.2.59) are both positive (the first term is always larger or equal than the radical and additionally the radicand is always positive) and therefore the system is always stable.

4.2.5 Newtonian applications of geometric optics (Cylindrical polar coordinates)

In this section we discuss, in a cylindrical polar frame, the sound waves and the inertial modes in a rotating fluid, the Rayleigh shearing instability [89, 90] and the Magnetorotational instability [36]. We use an orthonormal frame $\hat{\boldsymbol{R}}, \hat{\boldsymbol{z}}, \hat{\boldsymbol{\phi}}$ and the respective coordinates (R, z, ϕ). Below in all cases we assume that all quantities are axisymmetric i.e. they do not depend on the ϕ coordinate (though, we can still have vector components along the $\hat{\boldsymbol{\phi}}$).

4.2.5.1 The sound waves (again) and the inertial modes of a rotating fluid

In this section we derive the characteristic equation of the sound waves along with the inertial modes for a fluid flowing around the z-axis. We assume that the fluid velocity is $\boldsymbol{v}_0 = \Omega(\bar{\varepsilon}R)R\hat{\boldsymbol{\phi}}$, where Ω is the ("slow") angular velocity of the fluid[11]. We assume that the background density and pressure are "slow" along all directions while the gravitational potential is of the form $\Phi_0 = \Phi_0(\bar{\varepsilon}t, R, \bar{\varepsilon}z)$. The "fast" dependence of the R coordinate in the gravitational potential provides a balance force term in the R component of the background Euler equation (4.2.2)

$$\Omega^2 R = \frac{\partial \Phi_0}{\partial R}. \tag{4.2.62}$$

The equation above means that the centrifugal force is balanced by the gravity and therefore it implies that the fluid is a Keplerian flow. It is the only equation of the order $\bar{\delta}^0\bar{\varepsilon}^0$. The wavevector is $\boldsymbol{k} = k_R\hat{\boldsymbol{R}} + k_z\hat{\boldsymbol{z}}$. Using the vector calculus formulas of section A.1.7 for axisymmetric configurations, the linearised continuity equation (4.2.14) becomes

$$\rho_0\left(\frac{1}{R_0} + ik_R\right)\bar{v}_R + i\rho_0 k_z\bar{v}_z - i\omega\bar{\rho} = 0. \tag{4.2.63}$$

The three components of the Euler equation (4.2.15) are

$$-i\omega\bar{v}_R - 2\Omega\bar{v}_\phi + i\frac{k_R}{\rho_0}\bar{P} = 0, \tag{4.2.64}$$

$$-i\omega\bar{v}_z + i\frac{k_z}{\rho_0}\bar{P} = 0, \tag{4.2.65}$$

and

$$2\Omega\bar{v}_R - i\omega\bar{v}_\phi = 0, \tag{4.2.66}$$

for the R, z and ϕ components respectively. The perturbation of the entropy conservation equation is the same as in the Cartesian sound wave derivation, given by equation (4.2.29). The system of equations contains the term $\frac{1}{R_0}$ which appears because the cylindrical coordinates are curvilinear and thus there are nonvanishing Christoffel symbols. The characteristic equation of the above system is given by

$$-\frac{1}{R}c_s^2\omega^3 k_R + i\,\omega\left[\omega^4 - \omega^2\left(4\Omega^2 + k^2c_s^2\right) + 4\Omega^2c_s^2k_z^2\right] = 0. \tag{4.2.67}$$

[11] Since in our analysis we do not consider perturbations on the anglular velocity Ω or the R coordinate we do not present this quantities with subscript zero or barred.

Keeping the $1/R$ terms, the resulting characteristic polynomial is complex[12]. In order to avoid complex coefficients, we assume that R is large enough so that $1/R$ is effectively zero. A suitable choice for this assumption is $1/R \sim \bar{\varepsilon}^{\beta}$, with $\beta > 1$. We have chosen $\beta > 1$ rather than specifying β precisely because we want some freedom regarding the order of $1/R$. We have already made a similar choice for the Mach number in section 4.2.3 and we intend to avoid the possibility that terms of the form $\mathcal{M}^n\, R^m \sim \bar{\varepsilon}^{n\alpha - m\beta}$ (with n, m positive integers to be of order unity). Under this simplification we get the $\omega = 0$ solution along with the characteristic equation

$$\omega^4 - \left(4\Omega^2 + c_{\mathrm{s}}^2 k^2\right)\omega^2 + 4c_{\mathrm{s}}^2\Omega^2 k_z^2 = 0, \tag{4.2.68}$$

The two positive solutions for ω^2 are

$$\omega_{\pm}^2 = \frac{1}{2}\left(4\Omega^2 + k^2 c_{\mathrm{s}}^2 \pm \sqrt{\left(4\Omega^2 + c_{\mathrm{s}}^2 k^2\right)^2 - 16\Omega^2 c_{\mathrm{s}}^2 k_z^2}\right). \tag{4.2.69}$$

We observe that ω is a combination of sound waves and the inertial modes of the fluid. If we set $\Omega = 0$ then we only get the sound wave solution $\omega^2 = k^2 c_{\mathrm{s}}^2$. In order to obtain the inertial modes alone we eliminate the sound waves in equation (4.2.68) following the method of section 4.2.3. The characteristic polynomial then becomes

$$k^2\omega^2 - 4\Omega^2 k_z^2 = 0. \tag{4.2.70}$$

Therefore the inertial modes are given by

$$\omega^2 = 4\Omega^2 \frac{k_z^2}{k^2}, \tag{4.2.71}$$

and are always stable.

4.2.5.2 The Rayleigh shearing instability

The next is to examine the case where the background angular velocity is "fast". We assume that the fluid velocity is $\boldsymbol{v}_0 = \Omega(R)R\hat{\boldsymbol{\phi}}$, which is an differentially rotating, axisymmetric flow. The density is "slow" with respect to all coordinates,the pressure and gravitational potential are of the form $P_0 = P_0\left(\bar{\varepsilon}t, R, z\right)$ and $\Phi_0 = \Phi\left(\bar{\varepsilon}t, R, z\right)$.

[12]In general polynomials with complex coefficients may have real solutions which would imply stability. Nevertheless our analysis, as discussed in section 4.1.3, is restricted to polynomials with real coefficients.

Using these assumptions the background equations to the order $\bar{\delta}^0\bar{\varepsilon}^0$ are the R and z components of the Euler equation (4.2.2)

$$\Omega^2 R = \frac{\partial \Phi_0}{\partial R} + \frac{1}{\rho_0}\frac{\partial P_0}{\partial R}, \tag{4.2.72}$$

and

$$\frac{\partial \Phi_0}{\partial z} + \frac{1}{\rho_0}\frac{\partial P_0}{\partial z} = 0. \tag{4.2.73}$$

The rest of the background equations of section 4.2.1 are either of higher order in $\bar{\varepsilon}$ or vanishing. The reason we have considered a "fast" dependence with respect to R and z for the pressure and gravitational potential is the following. We want to be able to consider general flows for the background and therefore we need more freedom in the force balance equation. This extra freedom is introduced by the gradient of the pressure. In contrast, assuming that the pressure is "slow" along the R coordinate (as we have done in section 4.2.5.1) implies a Keplerian flow and therefore a specific functional form for $\Omega(R)$.

We assume that the wavevector is the same as in the previous case. The linearised continuity equation is given by equation (4.2.63). For the linearised Euler equations we have

$$-i\omega\bar{v}_R - 2\Omega\bar{v}_\phi - \frac{1}{\rho_0^2}\frac{\partial P_0}{\partial R}\bar{\rho} + i\frac{k_R}{\rho_0}\bar{P} = 0, \tag{4.2.74}$$

$$-i\omega\bar{v}_z - \frac{1}{\rho_0^2}\frac{\partial P_0}{\partial z}\bar{\rho} + i\frac{k_z}{\rho_0}\bar{P} = 0, \tag{4.2.75}$$

and

$$\left(2\Omega + R\frac{d\Omega}{dR}\right)\bar{v}_R - i\omega\bar{v}_\phi = 0, \tag{4.2.76}$$

for the R, z and ϕ components respectively. The entropy conservation equation (4.2.20) takes the form

$$\frac{\partial P_0}{\partial R}\bar{v}_R + \frac{\partial P_0}{\partial z}\bar{v}_z - i\omega\bar{P} + i\omega c_s^2\bar{\rho} = 0. \tag{4.2.77}$$

Note that the directional derivatives $\boldsymbol{v}_0 \cdot \nabla$ of scalars, as in equation (4.2.20), are vanishing. This happens because the fluid velocity has a single component along $\hat{\boldsymbol{\phi}}$ and axisymmetric quantities do not have a ϕ dependence. The full characteristic equation (along with the root $\omega = 0$) is given by

$$\left(\omega^2 k^2 - k_z^2 \kappa^2\right) c_s^2 + \left[-\omega^4 + \omega^2 \kappa^2 + \frac{1}{\rho_0^2} \left(\frac{\partial P_0}{\partial z} k_R - \frac{\partial P_0}{\partial R} k_z \right)^2 \right] \\ + \frac{1}{R} \left[\omega^2 \left(\frac{1}{\rho_0} \frac{\partial P_0}{\partial R} - i c_s^2 k_R \right) + \frac{i}{\rho_0^2} \frac{\partial P_0}{\partial z} \left(\frac{\partial P_0}{\partial R} k_z - \frac{\partial P_0}{\partial z} k_R \right) \right] = 0, \tag{4.2.78}$$

where κ is the epicyclic frequency given by

$$\kappa^2 = 4\Omega^2 + 2R\Omega \frac{d\Omega}{dR}. \tag{4.2.79}$$

After removing the sound waves and assuming that terms multiplied by $1/R_0$ are of the same order as in the previous section i.e. $1/R_0 \sim \bar{\varepsilon}^\beta$ with $\beta > 1$, the reduced characteristic equation is given by the first term of equation (4.2.78). The roots in ω are

$$\omega^2 = \kappa^2 \frac{k_z^2}{k^2}, \tag{4.2.80}$$

which are stable provided that $\kappa^2 > 0$. Another form of the Rayleigh stability criterion for circular shearing flow is the following. If $\Omega > 0$, then multiplying the definition of the epicyclic frequency (4.2.79) by $1/\left(2\Omega^2 R\right)$ we have

$$\begin{aligned} &\frac{1}{\Omega} \frac{d\Omega}{dR} + \frac{2}{R} > 0 \\ &\frac{d}{dR} \left(\log \Omega\right) + \frac{d}{dR} \left(\log R^2\right) > 0 \\ &\frac{d}{dR} \log \left(\Omega R^2\right) > 0 \\ &\frac{d}{dR} \left(R^2 \Omega\right) > 0. \end{aligned} \tag{4.2.81}$$

4.2.5.3 The Magnetorotational Instability

In this section we introduce the magnetic field in the configuration of the previous section and examine the stability of the system. It appears that the magnetic field gives rise to a different kind of instability, the magnetorotational instability (MRI) [36]. As in the Rayleigh shearing instability we assume that the background density is "slow" along all directions while pressure and the gravitational potential are $P_0 = P_0\left(\bar{\varepsilon}t, R, z\right)$ and $\Phi_0 = \Phi\left(\bar{\varepsilon}t, R, z\right)$. The fluid velocity is circular $\boldsymbol{v}_0 = \Omega(R) R \hat{\boldsymbol{\phi}}$. The background magnetic field is along the z and ϕ directions and has the form $\boldsymbol{B}_0 = B_{0,z}\left(\bar{\varepsilon}t, \bar{\varepsilon}R, \bar{\varepsilon}z\right) \hat{\boldsymbol{z}} + B_{0,\phi}\left(\bar{\varepsilon}t, \bar{\varepsilon}R, \bar{\varepsilon}z\right) \hat{\boldsymbol{\phi}}$. Also, the wavevector is axisymmetric,

$\boldsymbol{k} = k_R\hat{\boldsymbol{R}} + k_z\hat{\boldsymbol{z}}$. The main difference between this analysis and Balbus' original paper [36] is that we use the full continuity equation instead of $\nabla \cdot \boldsymbol{v} = 0$. By using the full continuity equation 4.2.1 we do not implicitly impose additional conditions on the background and perturbed density. Also we avoid the assumption that isobaric and isochoric surfaces coincide which may be somewhat restrictive.

The non-vanishing background equations are, as in the Rayleigh shearing instability, the R and z components of the Euler equation (4.2.2). The z component is given by equation (4.2.73) while for the R component we have

$$\Omega^2 R = \frac{\partial \Phi_0}{\partial R} + \frac{1}{\rho_0}\frac{\partial P_0}{\partial R} + \frac{B_{0,\phi}^2}{R}, \tag{4.2.82}$$

where the last term appears due to the MHD Lorentz force. The linearised continuity equation and entropy conservation are same as in the Rayleigh shearing instability, given by equation (4.2.63) and (4.2.77) respectively. The R component of the linearised Euler equation (4.2.15) is

$$\begin{aligned} &- i\omega\bar{v}_R - 2\Omega\bar{v}_\phi - \left(\frac{1}{\rho_0^2}\frac{\partial P_0}{\partial R} + \frac{1}{R_0}\frac{B_\phi^2 k_z}{4\pi\rho_0^2}\right)\bar{\rho} + i\frac{k_R}{\rho_0}\bar{P} - i\frac{B_z k_z}{4\pi\rho_0}\bar{B}_R \\ &+ i\frac{B_z k_R}{4\pi\rho_0}\bar{B}_z + \frac{B_\phi}{4\pi\rho_0}\left(\frac{2}{R_0} + ik_R\right)\bar{B}_\phi = 0, \end{aligned} \tag{4.2.83}$$

the z component is

$$- i\omega\bar{v}_z + i\frac{k_z}{\rho_0}\bar{P} - \frac{1}{\rho_0^2}\frac{\partial P_0}{\partial z}\bar{\rho} + i\frac{B_\phi k_z}{4\pi\rho_0}\bar{B}_\phi = 0, \tag{4.2.84}$$

and the ϕ component is

$$\left(2\Omega + R_0\frac{d\Omega}{dR}\right)\bar{v}_R - i\omega\bar{v}_\phi - \frac{1}{R_0}\frac{B_\phi}{4\pi\rho_0}\bar{B}_R - i\frac{B_z k_z}{4\pi\rho_0}\bar{B}_\phi = 0. \tag{4.2.85}$$

The components of the linearised induction equation (4.2.16) are

$$- iB_z k_z\bar{v}_R - i\omega\bar{B}_R = 0, \tag{4.2.86}$$

$$B_z\left(\frac{1}{R_0} + ik_R\right)\bar{v}_R - i\omega\bar{B}_z = 0. \tag{4.2.87}$$

and

$$iB_\phi k_R\bar{v}_R + iB_\phi k_z\bar{v}_z - iB_z k_z\bar{v}_\phi - R_0\frac{d\Omega}{dR}\bar{B}_R - i\omega\bar{B}_\phi = 0, \tag{4.2.88}$$

for the R,z and ϕ components respectively. The full characteristic polynomial (along with a double $\omega = 0$ root) is given by

$$
\begin{aligned}
&\left[\omega^4 k^2 - \omega^2 k_z^2 \left(\kappa^2 + 2k^2 v_{\mathrm{Az}}^2\right) + k_z^4 v_{\mathrm{Az}}^2 \left(\kappa^2 - 4\Omega^2 + k^2 v_{\mathrm{Az}}^2\right)\right] c_{\mathrm{s}}^2 \\
&- \left\{\omega^6 - \omega^4 \left[\kappa^2 + k_R^2 \left(v_{\mathrm{Az}}^2 + v_{\mathrm{A}\phi}^2\right) - k_z \left(2v_{\mathrm{Az}}^2 + v_{\mathrm{A}\phi}^2\right]\right)\right. \\
&+ \frac{\omega^2}{\rho_0^2}\left[2\frac{\partial P_0}{\partial R}\frac{\partial P_0}{\partial z}k_R k_z + k_R^2\left(\rho_0^2 k_z^2 v_{\mathrm{Az}}^2\left(v_{\mathrm{Az}}^2 + v_{\mathrm{A}\phi}^2\right) - \left(\frac{\partial P_0}{\partial z}\right)^2\right)\right. \\
&\left.+k_z^2\left(\rho_0^2\left(v_{\mathrm{Az}}^2\left(v_{\mathrm{A}\phi}^2 k_z^2 + \kappa^2 - 4\Omega^2\right) + v_{\mathrm{Az}}^4 k_z^2 + v_{\mathrm{A}\phi}^2\kappa^2\right) - \left(\frac{\partial P_0}{\partial R}\right)^2\right)\right] \\
&\left.+\frac{\omega}{\rho_0}\left[4\Omega k_z^2 v_{\mathrm{Az}} v_{\mathrm{A}\phi}\left(k_z\frac{\partial P_0}{\partial R} - k_R\frac{\partial P_0}{\partial z}\right)\right] + \frac{1}{\rho_0}\left(k_z\frac{\partial P_0}{\partial R} - k_R\frac{\partial P_0}{\partial z}\right)^2 k_z^2 v_{\mathrm{Az}}^2\right\} \\
&- \frac{1}{R\rho_0^2}\left\{i\left(\omega^2 - k_z^2 v_{\mathrm{Az}}^2\right)\left[k_R\left(\rho_0^2\left(c_{\mathrm{s}}^2\left(\omega^2 - v_{\mathrm{Az}}^2 k_z^2\right) + \omega^2 v_{\mathrm{Az}}^2\right) + \left(\frac{\partial P_0}{\partial z}\right)^2\right)\right.\right. \\
&\left.-\frac{\partial P_0}{\partial R}\frac{\partial P_0}{\partial z}k_z + i\frac{\partial P_0}{\partial R}\rho_0\omega^2\right] + 2\rho_0\omega\Omega v_{\mathrm{Az}} v_{\mathrm{A}\phi} k_z\left(4\rho_0 c_{\mathrm{s}}^2 k_z^2 + i\frac{\partial P_0}{\partial z}k_z - 3\rho_0\omega^2\right) \\
&+ \rho_0 v_{\mathrm{A}\phi}^2\left[4\rho_0^2\omega\Omega v_{\mathrm{Az}} v_{\mathrm{A}\phi}^3 k_z^3 + 2\frac{\partial P_0}{\partial R}k_z^2\left(v_{\mathrm{Az}}^2 k_z^2 + \omega^2\right)\right. \\
&\left.\left.+ik_R\left(v_{\mathrm{Az}}^2 k_z^2\left(-\rho_0\omega^2 + 2i\frac{\partial P_0}{\partial z}k_z\right) + 2i\frac{\partial P_0}{\partial z}\omega^2 k_z + \rho_0\omega^4\right)\right]\right\} \\
&+ \frac{v_{\mathrm{A}\phi}^2}{R^2\rho_0^2}\left\{\left(v_{\mathrm{Az}}^2 k_z^2 + \omega^2\right)\left[\rho_0\omega^2 - k_z\left(\rho_0 k_z\left(v_{\mathrm{A}\phi}^2 + 2c_{\mathrm{s}}^2\right) + i\frac{\partial P_0}{\partial z}\right)\right]\right\} = 0,
\end{aligned}
\tag{4.2.89}
$$

where $v_{\mathrm{Az}}^2 = \frac{B_{0,z}^2}{4\pi\rho_0}$ and $v_{\mathrm{A}\phi}^2 = \frac{B_{0,\phi}^2}{4\pi\rho_0}$. This is obviously a very complicated characteristic equation. We may simplify the expression by introducing the assumptions of the previous sections. Eliminating the terms of order $1/R$ and $1/R^2$ the last five lines of equation (4.2.89) vanish. Additionally, if we eliminate the sound waves all lines of the same equation vanish apart from the first. The simplified characteristic polynomial now reads

$$
\frac{k^2}{k_z^2}\omega^4 - \left(\kappa^2 + 2k^2 v_{\mathrm{Az}}^2\right)\omega^2 + k_z^2 v_{\mathrm{Az}}^2\left(\kappa^2 - 4\Omega^2 + k^2 v_{\mathrm{Az}}^2\right) = 0, \tag{4.2.90}
$$

This characteristic equation is identical to the one derived in [92] (if we consider a wavevector with z component only). It is a convex quadratic polynomial (since the coefficient of ω^4 is positive) in ω^2. The discriminant is $k_z^4\left(\kappa^4 + 16k^2\Omega^2 v_{\mathrm{Az}}^2\right)$ which is always positive and therefore the two roots of the polynomial are real. Additionally the two ω^2 roots are positive and thus the system is stable if the coefficient of ω^2

is negative and the constant term is positive. The first condition implies that the minimum of the polynomial occurs at positive ω^2 and the second condition implies that the polynomial intersects the $\omega^2 = 0$ axis at a positive value. These two conditions read

$$\kappa^2 + 2k^2 v_{\mathrm{Az}}^2 \geq 0, \tag{4.2.91}$$

and

$$\kappa^2 - 4\Omega^2 + k^2 v_{\mathrm{Az}}^2 \geq 0. \tag{4.2.92}$$

Of these two inequalities (provided that $k^2 v_{\mathrm{Az}}^2 > 0$) we only need the second one because if it is satisfied, the first is satisfied as well. Using the definition of the epicyclic frequency from equation (4.2.79) the stability condition obtains the following form

$$\frac{d\Omega^2}{d\ln R} + k^2 v_{\mathrm{Az}}^2 \geq 0, \tag{4.2.93}$$

which is the one derived in [92]. Assuming then that $k^2 v_{\mathrm{Az}}^2$ goes to zero (since we can either have a very small magnetic field or very small wavenumbers) the stability criterion reads

$$\frac{d\Omega^2}{d\ln R} \geq 0. \tag{4.2.94}$$

This inequality implies that a disk is stable if the magnitude of Ω is radially increasing outwards. However, for most physical configurations Ω decreases in magnitude with respect to the radius and so the majority of realistic models should be unstable. A peculiar and interesting aspect of this result is that for a vanishing magnetic field equation (4.2.93) does not coincide with the Rayleigh shearing instability criterion of the previous section as we would anticipate. Physically this means that an arbitrarily small magnetic field would produce an instability in a configuration which would be stable if the magnetic field had not be introduced at all, i.e because we may have $\kappa^2 > 0$ but $\kappa^2 - 4\Omega^2 < 0$. In the following section, we will discuss this issue.

4.2.5.4 The Rayleigh shearing instability limit of the MRI

The condition (4.2.94) arises by taking the limit $k^2 v_{\mathrm{Az}}^2 \to 0$. Instead of taking this limit after having calculated the condition we could assume that $k^2 v_{\mathrm{Az}}^2$ is proportional to some small bookkeeping number $0 < \bar{\zeta} \ll 1$ such that $k^2 v_{\mathrm{Az}}^2 \to \bar{\zeta} k^2 v_{\mathrm{Az}}^2$ [13]

[13] To be more precise we assume that $k^2 v_{\mathrm{Az}}^2 \to \bar{\zeta} \tilde{k}^2 \tilde{v}_{\mathrm{Az}}^2$, assuming the tilded quantities are of order unity and then we rename again $\tilde{k} \to k$ and $\tilde{v}_{\mathrm{Az}} \to v_{\mathrm{Az}}$.

where k^2 and v_{Az}^2 on the right hand side of the arrow are assumed to be of order unity. The characteristic equation (4.2.90) then reads

$$\frac{1}{\cos^2 q}\omega^4 - \left(\kappa^2 + 2\bar{\zeta}k^2 v_{\mathrm{Az}}^2\right)\omega^2 + \bar{\zeta}(\cos^2 q)k^2 v_{\mathrm{Az}}^2\left(\kappa^2 - 4\Omega^2 + \bar{\zeta}k^2 v_{\mathrm{Az}}^2\right) = 0, \quad (4.2.95)$$

where $k_z = k\cos q$ which is the direction cosine for k_z. The equation above rearranged in powers of $\bar{\zeta}$ becomes

$$\begin{aligned}&\left(k^4 v_{\mathrm{Az}}^4 \cos^2 q\right)\bar{\zeta}^2 + k^2 v_{\mathrm{Az}}^2\left(\kappa^2\cos^2 q - 4\Omega^2\cos^2 q - 2\omega^2\right)\bar{\zeta}\\ &+\omega^2\left(\frac{\omega^2}{\cos^2 q} - \kappa^2\right) = 0.\end{aligned} \quad (4.2.96)$$

The above expansion in $\bar{\zeta}$ indicates that there are three cases to be considered. The first case is when $\bar{\zeta}$ and $\bar{\zeta}^2$ terms are not neglected so we have to keep all the terms and the stability condition should be that given by inequality (4.2.93). The $k^2 v_{\mathrm{Az}}^2$ is now included in the final criterion because we have made the assumption that this term is of the same magnitude as the other terms.
The second case happens when the product of the Alfvén speed and the wavenumber is such that the $\bar{\zeta}^2$ terms are sufficiently small to be omitted, but $\bar{\zeta}$ terms are not. In this case equation (4.2.96) reduces to

$$\frac{1}{\cos^2 q}\omega^4 - \left(\kappa^2 + 2k^2 v_{\mathrm{Az}}^2\right)\omega^2 + (\cos^2 q)k^2 v_{\mathrm{Az}}^2\left(\kappa^2 - 4\Omega^2\right) = 0. \quad (4.2.97)$$

The stability criterion for this characteristic equation is given by inequality (4.2.94) i.e. that ultimately obtained in [92].
The third case happens when the product of the Alfvén speed and the wavenumber is such that both the $\bar{\zeta}^2$ and the $\bar{\zeta}$ terms are negligible. In this instance the characteristic equation yields

$$\omega^2\left(\omega^2\frac{1}{\cos^2 q} - \kappa^2\right) = 0, \quad (4.2.98)$$

which is the characteristic equation of the Rayleigh shearing instability given in section 4.2.5.2 with the related stability criterion

$$\kappa^2 > 0. \quad (4.2.99)$$

In the following discussion we consider that the magnitude of the Alfvén speed is controlled only by the magnitude of the magnetic field (i.e. the density is of order unity). Therefore, it appears that by considering the magnitude of the term $k^2 v_{\mathrm{Az}}^2$ in the characteristic equation (4.2.90) of the MRI it is possible to derive the Balbus criterion if the magnetic field (or the wavenumber) is small (keeping $\bar{\zeta}^1$ terms but not $\bar{\zeta}^2$). A very small (tending to zero) magnetic field (or wavenumber) though, reduces the criterion to the Rayleigh shearing instability criterion.

Another way of looking at this result is the following. Suppose there is a number $\bar{\zeta}_\star$ which is the largest possible value of $\bar{\zeta}$ such that the terms proportional to $\bar{\zeta}$ vanish. For all values of $\bar{\zeta} < \bar{\zeta}_\star$ the characteristic equation reduces to the Rayleigh shearing equation (since the $\bar{\zeta}^2$ vanish as well). The value $\bar{\zeta}_\star$ in other words is the largest value for which $\bar{\zeta}$ is effectively zero.
For values $\bar{\zeta}_\star < \bar{\zeta} < \sqrt{\bar{\zeta}_\star}$ (the right bound is the value such that $\bar{\zeta}^2 = \bar{\zeta}_\star$) the linear terms in $\bar{\zeta}$ do not vanish but the $\bar{\zeta}^2$ terms vanish. For this interval, namely $\left(\bar{\zeta}_\star, \sqrt{\bar{\zeta}_\star}\right)$ (note here that the square root is larger than the number itself since $\bar{\zeta} < 1$) the stability condition reduces to that derived by Balbus [36].
Further increase in $\bar{\zeta}$, i.e. $\bar{\zeta} > \sqrt{\bar{\zeta}_\star}$, implies that neither the $\bar{\zeta}$ nor the $\bar{\zeta}^2$ terms are sufficiently small to be considered effectively zero. In this case the stability condition is that given by inequality (4.2.93).

The analysis above has an interesting consequence. Although we have found three qualitatively different intervals for $\bar{\zeta}$ it is possible to always find, for a given Alfvén speed, a wavenumber such that $k^2 v_{\mathrm{Az}}^2 \sim \bar{\zeta}'$ where $\bar{\zeta}_\star < \bar{\zeta}' < \sqrt{\bar{\zeta}_\star}$. It follows that an Alfvén speed of the order, say, $v_{\mathrm{Az}}^2 \sim \bar{\zeta}'^{\beta}$ requires a wavenumber of the order $k^2 \sim \bar{\zeta}'^{1-\beta}$ with $\beta \in \mathbb{R}$. A weak magnetic field (i.e. $\beta > 0$) requires a wavenumber of at least the order unity to trigger the MRI instability (using the criterion 4.2.94). In the opposite limit, a strong magnetic field ($\beta < 0$) gives rises to the MRI (according to criterion 4.2.94) provided that the wavenumber is sufficiently small.
Therefore, using criterion (4.2.93) for the case of an infinitesimal magnetic field, requires large wavenumbers such that they could be unphysical. This is justified in the same manner that wavenumbers (and frequencies) have some upper bound defined by the microscopic properties of media in the case of sound waves. Roughly, the wavelength (i.e. the inverse of wavenumber) cannot be less than the mean free path of the particles consisting the medium. Assuming an upper finite limit for the wavenumber k_{max} the stability of a configuration with a sufficiently small magnetic field (such that $k_{\mathrm{max}}^2 v_{\mathrm{Az}}^2 < \bar{\zeta}_\star$) is characterised by the Rayleigh shearing criterion (4.2.99) rather than criterion (4.2.94).

4.3 The relativistic framework

In this section we discuss using the geometric optics method to explore the relativistic analogues of the sound waves and inertial waves, the Rayleigh shearing instability and the MRI. We first present the system of equations and then we introduce the frame and the metric. We derive the characteristic polynomials for each case and discuss various limits.

4.3.1 The system of equations

The systems we are examining are single fluids in the context of relativistic hydrodynamics or magnetohydrodynamics. In order to be able to compare to the Newtonian cases we use the equations derived in section 3.4.3.4 and rename the symbols used for the physical quantities. The energy conservation equation (3.4.53) is given by

$$u^a \nabla_a \rho + (\rho + P) \nabla_a u^a = 0, \tag{4.3.1}$$

where u^a is 4-velocity of the observer co-moving with the fluid (normalised through $u^a u_a = -1$), ρ is the relativistic energy density and P is the relativistic pressure with respect to this observer[14]. The relativistic Euler equation (3.4.55) is given by

$$\begin{aligned} &(\rho + P) u^b \nabla_b u^a + h^{ab} \nabla_b P \\ &\underbrace{+ B_c B^c u^b \nabla_b u^a + h^{ab} B_c \nabla_b B^c - B^a B_c u^b \nabla_b u^c - h^{ab} B^c \nabla_c B_b}_{\text{relativisic MHD Lorentz force}} = 0, \end{aligned} \tag{4.3.2}$$

where B^a is the magnetic field with respect to the observer. The relativistic induction equation (3.4.47) is

$$h^{ab} u^c \nabla_c B_b - B^b \nabla_b u^a + B^a \nabla_b u^b = 0. \tag{4.3.3}$$

Note that the Euler equation and induction equation are orthogonal to u^a and therefore these equations have three independent components each. The adiabatic condition (3.2.30) becomes

$$u^a \nabla_a \Sigma = 0. \tag{4.3.4}$$

[14]Note that in section 3.4.3.4 energy was denoted by $\rho_{\text{MHD,F}}$ and pressure by $\Psi_{\text{MHD,F}}$

As in the Newtonian case we assume that the specific entropy Σ is an equation of state for the system of the form

$$\Sigma = \Sigma\left(P, \rho\right). \tag{4.3.5}$$

Writing the condition (4.3.4) in the form

$$\frac{d\Sigma}{d\tau} = 0, \tag{4.3.6}$$

where τ is the proper time along the observer's worldline, it follows, similarly to the Newtonian derivation given in equation (4.2.6), that the relativistic speed of sound is given by the definition (4.2.7). Using this definition along with equation (4.3.5) equation (4.3.4) becomes

$$u^a \nabla_a P - c_{\mathrm{s}}^2\, u^a \nabla_a \rho = 0. \tag{4.3.7}$$

4.3.2 Linear perturbations (relativistic framework)

Here we present the linear perturbations of the system of equations. Using equation (4.1.1) we obtain the background and perturbed terms of the system of equations (i.e. of the order $\bar{\delta}^0 \bar{\varepsilon}^0$ and $\bar{\delta}^1 \bar{\varepsilon}^0$ respectively). The background equations are obtained by substituting all quantities with the respective subscripted zero quantities. As discussed in section 4.1.2 scalar perturbations (such as $\bar{P}$ and $\bar{\rho}$) are "slow" i.e $\bar{P} = \bar{P}\left(\bar{\varepsilon} x^\nu\right)$, $\bar{\rho} = \bar{\rho}\left(\bar{\varepsilon} x^\nu\right)$ and therefore the respective gradients are of order $\bar{\varepsilon}^1$. Assuming that the perturbation of the metric is vanishing the perturbed energy conservation equation (4.3.1) is given by

$$\bar{u}^a \nabla_a \rho_0 + i u_0^a k_a + \left(\bar{\rho} + \bar{P}\right) \nabla_a u_0^a + i k_a \bar{u}^a \left(\rho_0 + P_0\right) + \left(\rho_0 + P_0\right) \nabla_a \bar{u}^a = 0, \tag{4.3.8}$$

where the 4-wavevector is defined through equation (4.1.16). The Euler equation (4.3.2) becomes

$$\begin{aligned}
&\left(\bar{\rho}+\bar{P}\right) u_{0}^{b}\nabla_{b}u_{0}^{a} + \left(\rho_{0}+P_{0}\right)\bar{u}^{b}\nabla_{b}u_{0}^{a} + \left(\rho_{0}+P_{0}\right)u_{0}^{b}\nabla_{b}\bar{u}^{a}\\
&+ i\left(\rho_{0}+P_{0}\right)u_{0}^{b}k_{b}\bar{u}^{a} + \bar{u}^{a}u_{0}^{b}\nabla_{b}P_{0} + u_{0}^{a}\bar{u}^{b}\nabla_{b}P_{0} + ih_{0}^{ab}k_{b}\bar{P}\\
&+ 2\bar{B}^{c}B_{c}^{0}u_{0}^{b}\nabla_{b}u_{0}^{a} + B_{c}^{0}B_{0}^{c}\bar{u}^{b}\nabla_{b}u_{0}^{a} + B_{c}^{0}B_{0}^{c}u^{b}\nabla_{b}\bar{u}^{a} + iB_{c}^{0}B_{0}^{c}u_{0}^{b}k_{b}\bar{u}^{a}\\
&+ \bar{u}^{a}u_{0}^{b}B_{c}^{0}\nabla_{b}B_{0}^{c} + u_{0}^{a}\bar{u}^{b}B_{c}^{0}\nabla_{b}B_{0}^{c} + h_{0}^{ab}\bar{B}_{c}\nabla_{b}B_{0}^{c} + h_{0}^{ab}B_{c}^{0}\nabla_{b}\bar{B}^{c}\\
&+ ih_{0}^{ab}B_{c}^{0}k_{b}\bar{B}^{c} - \bar{B}^{a}B_{c}^{0}u_{0}^{b}\nabla_{b}u_{0}^{c} - B_{0}^{a}B_{c}^{0}\bar{u}^{b}\nabla_{b}u_{0}^{c} - B_{0}^{a}u_{0}^{b}B_{c}^{0}\nabla_{b}\bar{u}^{c}\\
&- iB_{0}^{a}u_{0}^{b}k_{b}B_{c}^{0}\bar{u}^{c} - B_{0}^{a}\bar{B}_{c}u_{0}^{b}\nabla_{b}u_{0}^{c} - \bar{u}^{a}u_{b}^{0}B_{0}^{c}\nabla_{c}B_{0}^{b} - u_{0}^{a}\bar{u}_{b}B_{0}^{c}\nabla_{c}B_{0}^{b}\\
&- h_{0}^{ab}\bar{B}^{c}\nabla_{c}B_{b}^{0} - h_{0}^{ab}B_{0}^{c}\nabla_{c}\bar{B}_{b} - ih_{0}^{ab}B_{0}^{c}k_{c}\bar{B}_{b} = 0,
\end{aligned} \tag{4.3.9}$$

where we have substituted the perturbation of the projection tensor through $\bar{h}_{ab} = \bar{u}_{a}u_{b} + u_{a}\bar{u}_{b}$. The terms including the magnetic field and perturbations of the magnetic field are related to the relativistic MHD Lorentz force. The perturbation of the induction equation (4.3.3) is given by

$$\begin{aligned}
&\bar{u}^{a}u_{b}^{0}u_{0}^{c}\nabla_{c}B_{0}^{b} + u_{0}^{a}\bar{u}_{b}u_{0}^{c}\nabla_{c}B_{0}^{b} + h_{0}^{ab}\bar{u}^{c}\nabla_{c}B_{b}^{0} + h_{0}^{ab}u_{0}^{c}\nabla_{c}\bar{B}_{b} + ih_{0}^{ab}u_{0}^{c}k_{c}\bar{B}_{b}\\
&- \bar{B}^{b}\nabla_{b}u_{0}^{a} - B_{0}^{b}\nabla_{b}\bar{u}^{a} - iB_{0}^{b}k_{b}\bar{u}^{a} + \bar{B}^{a}\nabla_{c}u_{0}^{c} + iB_{0}^{a}k_{c}\bar{u}^{c} + B_{0}^{a}\nabla_{c}\bar{u}^{c} = 0.
\end{aligned} \tag{4.3.10}$$

In analogy with the Newtonian case, the perturbation of the specific entropy is related to those of density and pressure through equation (4.2.17). Using this along with the definition of the speed of sound (4.2.7), the perturbation of the adiabatic conservation equation (4.3.4) becomes

$$\begin{aligned}
&\bar{u}^{a}\nabla_{a}\Sigma_{0} + ik_{a}u_{0}^{a}\bar{\Sigma} + u_{0}^{a}\nabla_{a}\bar{\Sigma} = 0\\
&\bar{u}^{a}\left(\nabla_{a}P_{0} - c_{\mathrm{s}}^{2}\nabla_{a}\rho_{0}\right) + ik_{a}u_{0}^{a}\left(\bar{P} - c_{\mathrm{s}}^{2}\bar{\rho}\right) + u_{0}^{a}\nabla_{a}\left(\frac{\bar{\Sigma}}{\partial\Sigma_{0}/\partial P|_{\rho_{0}}}\right)\\
&- \bar{\Sigma}u_{0}^{a}\nabla_{a}\left(\frac{1}{\partial\Sigma_{0}/\partial P|_{\rho_{0}}}\right) = 0\\
&\bar{u}^{a}\left(\nabla_{a}P_{0} - c_{\mathrm{s}}^{2}\nabla_{a}\rho_{0}\right) + ik_{a}u_{0}^{a}\left(\bar{P} - c_{\mathrm{s}}^{2}\bar{\rho}\right)\\
&+ u_{0}^{a}\nabla_{a}\bar{P} - c_{\mathrm{s}}^{2}u_{0}^{a}\nabla_{a}\bar{\rho} - \bar{\rho}u_{0}^{a}\nabla_{a}c_{\mathrm{s}}^{2} + \frac{\bar{P} - c_{\mathrm{s}}^{2}\bar{\rho}}{\partial\Sigma_{0}/\partial P|_{\rho_{0}}}u_{0}^{a}\nabla_{a}\left(\left.\frac{\partial\Sigma_{0}}{\partial P}\right|_{\rho_{0}}\right) = 0,
\end{aligned} \tag{4.3.11}$$

hence

$$\begin{aligned}&\bar{u}^a\left(\nabla_a P_0 - c_s^2\nabla_a\rho_0\right) + ik_a u_0^a\left(\bar{P} - c_s^2\bar{\rho}\right)\\&-\bar{\rho}u_0^a\left[\left(\left.\frac{\partial c_s^2}{\partial P}\right|_{\rho_0}\right)\nabla_a P_0 + \left(\left.\frac{\partial c_s^2}{\partial\rho}\right|_{P_0}\right)\nabla_a\rho_0\right]\\&+\frac{\bar{P} - c_s^2\bar{\rho}}{\partial\Sigma_0/\partial P|_{\rho_0}}\left(\left.\frac{\partial^2\Sigma_0}{\partial P^2}\right|_{\rho_0}\right)u_0^a\nabla_a P_0 = 0.\end{aligned} \tag{4.3.12}$$

We also have the perturbation of the 4-velocity normalisation

$$\bar{u}^a u_a^0 = 0, \tag{4.3.13}$$

which means that $\bar{u}^a$ has three independent components. The orthogonality of the magnetic field and the 4-velocity yield

$$u_0^a B_a^0 = 0, \tag{4.3.14}$$

for the background, and for the first order

$$\bar{u}^a B_a^0 + u_0^a \bar{B}_a = 0, \tag{4.3.15}$$

imply that B_0^a and $\bar{B}^a$ have three independent components each. Therefore, as in the Newtonian case, along with $\bar{P}$ and $\bar{\rho}$, there are eight unknowns (i.e. barred quantities).

Furthermore, the orthogonality of the non-linear Euler and induction equations to the observer 4-velocity indicates that equations (4.3.9) and (4.3.10) have three independent components each. It is straightforward to show this result if we write the orthogonality condition as $u_a f^a = 0$, where f^a is the left-hand-side of either equation (4.3.2) or (4.3.3). The first order perturbation of this equation is $u_a^0 \bar{f}^a + \bar{u}_a f_0^a = 0$ where f_0^a denotes the background left-hand-side terms of the equations and $\bar{f}^a$ the left-hand-side terms of either equation (4.3.9) or (4.3.10). Thus, in total we have, along with equations (4.3.8) and (4.3.12) eight independent components in the system of linearised equations.

In this section we have shown the equations in abstract index notation since they hold in all coordinate frames. Nevertheless, in the following sections we find it convenient introduce a coordinate frame and a specific metric.

4.3.3 Relativistic applications of geometric optics

In this section, working in a coordinate basis with cylindrical polar coordinates $x^\nu = (t, R, z, \phi)$ we derive the relativistic equivalents of section 4.2.5 (i.e. sound waves along with the inertial modes, the relativistic Rayleigh shearing instability and the MRI). Additionally, we obtain limits of these results that approximate the respective Newtonian ones. In analogy to the Newtonian consideration, we assume that all quantities are axisymmetric, that is they do not have ϕ dependence. However, it is possible to have vector components along the ϕ direction. The calculations and the characteristic polynomials are calculated with Mathematica [93] using the RGTC package [94].

4.3.3.1 The choice of metric, the plane wave, and the background observer

Since we are working in axisymmetry, the background metric should satisfy this assumption. In order to allow for some freedom for the background spacetime we choose the following functional form of the metric

$$g_{\mu\nu} = \mathrm{diag}\left[-g_{tt}\left(R, \bar{\varepsilon} z\right), g_{II}\left(R, \bar{\varepsilon} z\right), g_{II}\left(R, \bar{\varepsilon} z\right), g_{\phi\phi}\left(R, \bar{\varepsilon} z\right)\right], \tag{4.3.16}$$

which is a diagonal matrix with $g_{RR} = g_{zz} = g_{II} > 0$, $g_{tt} > 0$ and $g_{\phi\phi} > 0$. The components of the metric with indices upstairs (i.e. $g^{\mu\nu}$), since the above matrix is diagonal, are given by a diagonal matrix with elements the inverses of $g_{\mu\nu}$ components. The Christoffel symbols to the order $\bar{\varepsilon}^0$ of this metric are[15]

$$\begin{aligned}
&\Gamma^t{}_{tR} = \Gamma^t{}_{Rt} = \frac{1}{2g_{tt}}\frac{\partial g_{tt}}{\partial R}, \quad \Gamma^R_{tt} = \frac{1}{2g_{II}}\frac{\partial g_{tt}}{\partial R}, \\
&\Gamma^R_{RR} = \Gamma^z{}_{Rz} = \Gamma^z{}_{zR} = -\Gamma^R_{zz} = \frac{1}{2g_{II}}\frac{\partial g_{II}}{\partial R}, \\
&\Gamma^R_{\phi\phi} = -\frac{1}{2g_{II}}\frac{\partial g_{\phi\phi}}{\partial R}, \quad \Gamma^\phi{}_{R\phi} = \Gamma^\phi{}_{\phi R} = \frac{1}{2g_{\phi\phi}}\frac{\partial g_{\phi\phi}}{\partial R}.
\end{aligned} \tag{4.3.17}$$

[15]For brevity we suppressed the arguments of the metric components in the equation (4.3.17) and we do the same in the following equations. In any case, if the "fast" and "slow" dependence are to be considered different from those already assumed in equation (4.3.16) we will explicitly state this.

This metric describes a static, axially symmetric spacetime. It is in some sense the Weyl metric [95, 96], if we choose the metric components appropriately. We have chosen "fast" dependence of the components along the R direction and "slow" along the z direction and this means that we focus on the radial direction of the various configurations.
From a physical point of view, where the source of gravity is a large spherical mass at the origin of the coordinate system, this consideration has the following implication. We are looking into a spacetime region around the equatorial plane, where the dependence on z direction can be omitted. This kind of coordinate dependence introduces a simplification in comparison to the Newtonian case (where we considered fast dependence along both R and z coordinates). The logic behind this assumption is that astrophysical disks, where the kind of instabilities we are examining are most likely to occur, are usually thin (i.e. the R dimension is much larger than the z dimension of the disk [97]) and mostly close to a region around the equatorial plane.

We also assume that the phase is of the form $S = S(\bar{\varepsilon}t, \bar{\varepsilon}z)$, which means that the 4-wavevector has only t and z components

$$k_\mu = (k_t, 0, k_z, 0)\,. \tag{4.3.18}$$

Following the Newtonian analysis we introduce a background observer with a circular velocity field

$$u_{\scriptscriptstyle 0}^\mu = (\gamma, 0, 0, \gamma\Omega)\,, \tag{4.3.19}$$

where the scalar γ is calculated using the normalisation equation $u_{\scriptscriptstyle 0}^a u_a^{\scriptscriptstyle 0} = -1$ and the metric defined in equation (4.3.16) through

$$\gamma = \frac{1}{\sqrt{g_{tt} - \Omega^2 g_{\phi\phi}}}, \tag{4.3.20}$$

with Ω the angular velocity[16] of the fluid (with respect to an observer at rest at infinity). Constant angular velocity implies uniform rotation. On the other hand, "fast" dependence of Ω along the R coordinate suggests that the fluid is differentially rotating (while "slow" dependence along R means that the differential rotation is of order $\bar{\varepsilon}^1$).
The 4-wavevector is given through equation (4.3.18) with respect to the cylindrical

[16] Although Ω is a quantity of the background, similarly to the Newtonian case, we do not show this quantity with a zero subscript.

coordinate frame. Nevertheless, the characteristic polynomial described in equation (4.1.24) requires the angular frequency and wavenumber with respect to the background observer. Using the decomposition of k_a given in equation (4.1.21) and equations (4.3.18), (4.3.19) we find, by contraction with the 4-velocity, that k_t is related to the angular frequency through

$$k_t = -\frac{\omega}{\gamma} = -\omega\sqrt{g_{tt} - \Omega^2 g_{\phi\phi}}. \tag{4.3.21}$$

To derive the respective relation for k_z we use the definition of the spatial wavenumber $k^2 = h_0^{ab} k_a k_b$ (see section 4.1.3). The projection tensor, using the definition (2.1.6) with indices upstairs, related to the background metric and observer is given by

$$h_0^{\mu\nu} = \begin{pmatrix} \frac{1}{g_{tt}-g_{\phi\phi}\Omega^2} - \frac{1}{g_{tt}} & 0 & 0 & \frac{\Omega}{g_{tt}-g_{\phi\phi}\Omega^2} \\ 0 & \frac{1}{g_{II}} & 0 & 0 \\ 0 & 0 & \frac{1}{g_{II}} & 0 \\ \frac{\Omega}{g_{tt}-g_{\phi\phi}\Omega^2} & 0 & 0 & \frac{g_{tt}}{g_{\phi\phi}\left(g_{tt}-g_{\phi\phi}\Omega^2\right)} \end{pmatrix}. \tag{4.3.22}$$

The definition of the spatial wavevector then yields

$$k_z^2 = g_{II}\left(k^2 - \omega^2 \frac{g_{\phi\phi}\Omega}{g_{tt}}\right). \tag{4.3.23}$$

The components of the perturbation amplitude for the 4-velocity is given by

$$\bar{u}^\mu = \left(\frac{g_{\phi\phi}\Omega}{g_{tt}}\bar{u}^\phi\left(\bar{\varepsilon}x^\nu\right), \bar{u}^R\left(\bar{\varepsilon}x^\nu\right), \bar{u}^z\left(\bar{\varepsilon}x^\nu\right), \bar{u}^\phi\left(\bar{\varepsilon}x^\nu\right)\right). \tag{4.3.24}$$

Also, we used equation (4.3.13) to eliminate the $\bar{u}^t$ component of the perturbation amplitude. For the magnetic field we have

$$\bar{B}^\mu = \left(\bar{B}^t, \bar{B}^R\left(\bar{\varepsilon}x^\nu\right), \bar{B}^z\left(\bar{\varepsilon}x^\nu\right), \bar{B}^\phi\left(\bar{\varepsilon}x^\nu\right)\right), \tag{4.3.25}$$

where the $\bar{B}^t$ component may be eliminated using equation (4.3.15). Since this component depends on background quantities it may be “fast” along the R direction[17].

[17] The fact that the components $\bar{u}^t$ and $\bar{B}^t$ may be fast does not have any implication since these components are not independent of the rest.

4.3.3.2 Low velocity, flat metric, and the elimination of sound waves

As in the Newtonian case we may introduce simplifications to the characteristic polynomial (4.1.24). In particular, it is possible to eliminate the sound waves in the way discussed in section 4.2.3.
We further simplify the characteristic polynomial by assuming that the metric is flat. We substitute the components of the background metric (4.3.16) and the respective derivatives with those of the Minkowski metric $\eta_{\mu\nu}$ which in cylindrical coordinates is

$$\eta_{\mu\nu} = \mathrm{diag}\left[-1, 1, 1, R^2\right]. \tag{4.3.26}$$

This consideration removes the implications of the curvature of spacetime and provides a reduced characteristic polynomial that is subject only to the laws of special relativity.
Additionally, we may also consider a "low speed" approximation. Additionally, we consider that the various velocities involved (in the most general case considered here, these are the phase velocity v_{ph}, the angular velocity Ω and the Alfvén velocity v_{Az}) are proportional to a small bookkeeping parameter $\bar{\zeta}$. As in section 4.2.5.4 we substitute the velocities through $v_{\mathrm{ph}} \to \bar{\zeta} v_{\mathrm{ph}}$, $\Omega \to \bar{\zeta}\Omega$ and $v_{\mathrm{Az}} \to \bar{\zeta} v_{\mathrm{Az}}$, where the tilde quantities are of order unity[18]. By keeping various powers of $\bar{\zeta}$ we obtain different "low speed" limits of the characteristic equation. The rest of the background quantities appearing in the characteristic polynomial are assumed to be of the order unity. As we show in the following sections using these simplifying assumptions it is possible to obtain the Newtonian results of section 4.2.5.

4.3.3.3 The relativistic sound waves and the inertial modes

In order to derive the sound waves and the inertial modes we assume that all the background quantities are "slow" along all coordinates (i.e. $\Omega = \Omega(\bar{\varepsilon}R)$, $P_0 = P_0(\bar{\varepsilon}R, \bar{\varepsilon}z)$, $\rho_0 = \rho_0(\bar{\varepsilon}R, \bar{\varepsilon}z)$). Additionally this is a purely hydrodynamical configuration and thus the magnetic field is zero. The background equations for this system vanish identically except for the R component of the Euler equation (4.3.2)

$$g'_{tt} = \Omega^2 g'_{\phi\phi}, \tag{4.3.27}$$

[18] As in section 4.2.5.4 to avoid crumbled expressions after introducing $\bar{\zeta}$ we have changed to intermediate tilded quantities and then again to the original through $\tilde{v}_{\mathrm{ph}} \to v_{\mathrm{ph}}$, $\tilde{\Omega} \to \Omega$, $\tilde{v}_{\mathrm{Az}} \to v_{\mathrm{Az}}$.

where the prime means differentiation with respect to the R coordinate. The equation above is in some sense the relativistic analogue of the Keplerian condition if we assume that $g'_{tt} \simeq 2\Phi'_{\rm Newt}$, with $\Phi_{\rm Newt}$ the Newtonian graviational potential [41]. The perturbed continuity equation (4.3.8) becomes

$$\begin{aligned}&\frac{(P_0+\rho_0)\left[g_{II}g_{\phi\phi}g'_{tt}+g_{tt}\left(2g_{\phi\phi}g'_{II}+g_{II}g'_{\phi\phi}\right)\right]}{2g_{II}g_{tt}g_{\phi\phi}}\bar{u}^R+ik_z\left(P_0+\rho_0\right)\bar{u}^z\\&+ik_tg_{\phi\phi}\Omega\frac{P_0+\rho_0}{g_{tt}}\bar{u}^\phi+i\gamma k_t\bar{\rho}=0,\end{aligned}\tag{4.3.28}$$

The components of the perturbed Euler equation (4.3.9) are

$$ik_t\left(P_0+\rho_0\right)\bar{u}^R+\Omega\left(P_0+\rho_0\right)\frac{g_{\phi\phi}g'_{tt}-g_{tt}g'_{\phi\phi}}{g_{II}g_{tt}}\bar{u}^\phi=0,\tag{4.3.29}$$

$$i\gamma k_t\left(P_0+\rho_0\right)\bar{u}^z+i\frac{k_z}{g_{II}}\bar{P}=0,\tag{4.3.30}$$

and

$$\frac{\left(P_0+\rho_0\right)\gamma g'_{tt}}{g_{tt}}\bar{u}^R+i\Omega\gamma k_t\frac{g_{\phi\phi}}{g_{tt}}\left(P_0+\rho_0\right)\bar{u}^\phi+ik_t\left(\gamma^2-\frac{1}{g_{tt}}\right)\bar{P}=0,\tag{4.3.31}$$

for the R, z and ϕ components respectively. Note that the background equation (4.3.27) implies that γ' is vanishing

$$\gamma'=-\frac{1}{2}\gamma^3\left(g'_{tt}-\Omega^2g'_{\phi\phi}\right)=0.\tag{4.3.32}$$

The perturbation of the adiabatic condition (4.3.12) is

$$-c_{\rm s}^2\bar{\rho}+\bar{P}=0.\tag{4.3.33}$$

Note that the gradients of scalars contracted with the background 4-velocity in equation (4.3.12) vanish due to axisymmetry and stationarity.
The full characteristic polynomial using equations (4.3.20), (4.3.21), (4.3.23) and (4.3.27) is given by

$$\begin{aligned}&\Omega^2g_{II}\left(g'_{\phi\phi}\right)^2\left\{g_{tt}\left[c_{\rm s}^2\left(2k^2+\omega^2\right)-2\omega^2\right]-\omega^2\Omega^2c_{\rm s}^2g_{\phi\phi}\right\}\\&+2\omega^2g_{II}^2g_{tt}^2g_{\phi\phi}\left(\omega^2-k^2c_{\rm s}^2\right)+2\omega^2\,\Omega^2\,c_s^2\,g_{tt}\,g_{\phi\phi}\,g'_{II}\,g'_{\phi\phi}=0.\end{aligned}\tag{4.3.34}$$

In the absence of rotation (i.e. $\Omega = 0$) we get the sound waves alone, $\omega^2 = c_s^2 k^2$. On the other hand the inertial waves are given by the polynomial

$$\omega^2 \left[-2k^2 g_{II}^2 g_{tt}^2 g_{\phi\phi} + \Omega^2 g_{II} g_{tt} \left(g'_{\phi\phi} \right)^2 + 2\Omega^2 g_{tt} g_{\phi\phi} g'_{II} g'_{\phi\phi} \right.$$
$$\left. -\Omega^4 g_{II} g_{\phi\phi} \left(g'_{\phi\phi} \right)^2 \right] + 2k^2 \Omega^2 g_{II} g_{tt} \left(g'_{\phi\phi} \right)^2 = 0. \tag{4.3.35}$$

Assuming that the metric components are given by the flat metric (4.3.26) and taking the low velocity approximation of section 4.3.3.2 we obtain the following polynomial in $\bar{\zeta}$

$$2\bar{\zeta}^6 \omega^2 \Omega^4 R^2 - 2\bar{\zeta}^4 \omega^2 \Omega^2 + \bar{\zeta}^2 k^2 \left(\omega^2 - 4\Omega^2 \right) = 0. \tag{4.3.36}$$

The smallest degree polynomial in $\bar{\zeta}$ of the equation above that also includes ω (that is the $\bar{\zeta}^2$ term in this case) reproduces exactly the Newtonian result of equation (4.2.71) provided that the Newtonian wavevector has only a z component.

4.3.3.4 The relativistic Rayleigh shearing instability

To derive the relativistic Rayleigh shearing instability we assume that the angular velocity of the fluid and the background pressure are "fast" along the R coordinate (i.e. $\Omega = \Omega(R)$, $P_0 = P_0(R, \bar{\varepsilon} z)$). The background density is assumed to be "slow" along all directions (i.e. $\rho_0 = \rho_0(\bar{\varepsilon} R, \bar{\varepsilon} z)$). As in the previous section the only non vanishing background equation is the R component of equation (4.3.2)

$$\frac{1}{2\left(g_{tt} - \Omega^2 g_{\phi\phi}\right)} \Omega^2 g'_{\phi\phi} = \frac{1}{2\left(g_{tt} - \Omega^2 g_{\phi\phi}\right)} g'_{tt} + \frac{1}{P_0 + \rho_0} P'_0. \tag{4.3.37}$$

which is the relativistic counterpart of equation (4.2.72). As in the Newtonian case the pressure gradient allows for some freedom in the choice of the profile of the angular velocity. The perturbed continuity equation is same as in the previous section and given by equation (4.3.28). The R and z components of the perturbed Euler are also same, and therefore are given by equations (4.3.29) and (4.3.30), respectively. The ϕ component is given by

$$\left[\frac{\left(P_0 + \rho_0\right)\left(g_{tt}\gamma\right)'}{g_{tt}} + \gamma P'_0 \right] \bar{u}^R + i\Omega\gamma k_t \frac{g_{\phi\phi}}{g_{tt}} \left(P_0 + \rho_0\right) \bar{u}^\phi$$
$$+ ik_t \left(\gamma^2 - \frac{1}{g_{tt}} \right) \bar{P} = 0, \tag{4.3.38}$$

where the partial derivative of γ with respect to R coordinate is

$$\gamma' = -\frac{1}{2}\gamma^3 \left(g'_{tt} - \Omega^2 g'_{\phi\phi} - 2\Omega\,\Omega' g_{\phi\phi}\right). \tag{4.3.39}$$

The perturbed adiabatic condition takes the form

$$P'_0 \bar{u}^R - ik_t c_s^2 \gamma \bar{\rho} + ik_t \gamma \bar{P} = 0. \tag{4.3.40}$$

The characteristic polynomial is, after we remove the sound waves

$$\begin{aligned} &\omega^2 \Big\{-4k^2 g_{II}^2 g_{tt}^2 g_{\phi\phi} \left(g_{tt} - \Omega^2 g_{\phi\phi}\right)^2 + 2 g_{tt} g_{\phi\phi} g'_{II} \left(g_{tt} - \Omega^2 g_{\phi\phi}\right) \left[\Omega^2 g_{tt} g'_{\phi\phi}\right. \\ &\left.+ g'_{tt}\left(g_{tt} - 2\Omega^2 g_{\phi\phi}\right)\right] + g_{II}\left[\Omega^2 g_{tt} g'_{\phi\phi} + g'_{tt}\left(g_{tt} - 2\Omega^2 g_{\phi\phi}\right)\right]\left[g_{tt} g'_{\phi\phi}\left(g_{tt}\right.\right. \\ &\left.\left. -3\Omega^2 g_{\phi\phi}\right) + g_{\phi\phi}\left(g'_{tt}\left(g_{tt} + \Omega^2 g_{\phi\phi}\right) - 2\Omega g_{tt} g_{\phi\phi}\Omega'\right)\right]\Big\} \\ &+ 4k^2 \Omega g_{II} g_{tt}\left(g_{tt} g'_{\phi\phi} - g_{\phi\phi} g'_{tt}\right)\left[g_{tt}\left(g_{\phi\phi}\Omega' + \Omega g'_{\phi\phi}\right) - \Omega g_{\phi\phi} g'_{tt}\right] = 0, \end{aligned} \tag{4.3.41}$$

where we have used equations (4.3.20),(4.3.21),(4.3.23) and (4.3.37). The last of these equations is used to eliminate the P'_0 from the final expression. The full polynomial can be found in appendix A.1.8. The flat metric and low velocity approximation discussed in section 4.3.3.2 for the polynomial above yield

$$\begin{aligned} &\bar{\zeta}^6 \left(k^2\omega^2\Omega^4 R^4 + 3\omega^2\Omega^4 R^2\right) + \bar{\zeta}^5 \left(\omega^2\Omega^3 R^3 \Omega'\right) + \bar{\zeta}^4 \left(-2k^2\omega^2\Omega^2 R^2 - \omega^2\Omega^2\right) \\ &+ \bar{\zeta}^2 \left(k^2\omega^2 - 4k^2\Omega^2\right) - \bar{\zeta}\left(2k^2\Omega R \Omega'\right) = 0. \end{aligned} \tag{4.3.42}$$

The largest term in $\bar{\zeta}$ (i.e. $\bar{\zeta}^1$ term) alone does not provide a characteristic equation[19] in ω. Therefore by additionally considering the $\bar{\zeta}^2$ term and eliminating the book-keeping parameter (i.e. $\bar{\zeta} = 1$) we obtain the result of equation (4.2.80) provided that the Newtonian wavevector has only a z component.

4.3.3.5 The relativistic MRI

To obtain the relativistic magnetorotational instability we assume that the background magnetic field only has a z component which is also "slow" along the R and z directions (i.e. $B_0^\mu = (0, 0, B_0^z(\bar{\varepsilon}R, \bar{\varepsilon}z), 0)$). We do not consider a ϕ component for the magnetic field as we did in the Newtonian case because we want to describe the simplest possible configuration that reproduces the MRI. From an astrophysical

[19]Here we mean that ω does not explicitly appear in the $\bar{\zeta}^1$ term. Therefore it is not possible to obtain a characteristic equation in ω by considering this term alone.

point of view (where the magnetic field is assumed to be a dipole originating from the neutron star) this implies that we are examining a region at distance from the star such that the z component of the magnetic field is much larger compared to the other components. The fluid angular velocity Ω, and the background pressure P_0, are assumed to be "fast" along the R direction and "slow" along the z direction. The background density ρ_0 is "slow" along both R and z. As in both previous sections the only non vanishing background equation is the R component of equation (4.3.2)

$$\begin{aligned}\frac{\Omega^2 g'_{\phi\phi}}{2\left(g_{tt}-\Omega^2 g_{\phi\phi}\right)} &= \frac{g'_{tt}}{2\left(g_{tt}-\Omega^2 g_{\phi\phi}\right)}+\frac{P'_0}{P_0+\rho_0} \\ &+\frac{B^z_0}{P_0+\rho_0}\frac{2g_{tt}g'_{II}-2\Omega^2 g_{\phi\phi}g'_{II}+g_{II}\left(g'_{tt}-\Omega^2 g'_{\phi\phi}\right)}{2\left(g_{tt}-\Omega^2 g_{\phi\phi}\right)},\end{aligned} \tag{4.3.43}$$

which is the relativistic analogue of equation (4.2.82). The perturbed continuity equation for this system is given by equation (4.3.28). The R, z and ϕ components of the linearised Euler equation (4.3.9) are

$$\begin{aligned}& i\gamma k_t\left[P_0+\rho_0+\left(B^z_0\right)^2 g_{II}\right]\bar{u}^R+\gamma\Omega\left[P_0+\rho_0+\left(B^z_0\right)^2 g_{II}\right]\frac{g_{\phi\phi}g'_{tt}-g_{tt}g'_{\phi\phi}}{g_{II}g_{tt}}\bar{u}^\phi \\ &+\gamma^2\frac{g'_{tt}-\Omega^2 g'_{\phi\phi}}{2g_{II}}\left(\bar{\rho}+\bar{P}\right)-iB^z_0 k_z\bar{B}^R \\ &+\frac{B^z_0}{g_{II}}\left[2g'_{II}+\gamma^2 g_{II}\left(g'_{tt}-\Omega^2 g'_{\phi\phi}\right)\right]\bar{B}^z=0,\end{aligned} \tag{4.3.44}$$

$$i\gamma k_t\left(P_0+\rho_0\right)\bar{u}^z+i\frac{k_z}{g_{II}}\bar{P}-\frac{B^z_0}{2g_{II}}\left[2g'_{II}+\gamma^2 g_{II}\left(g'_{tt}-\Omega^2 g'_{\phi\phi}\right)\right]\bar{B}^R=0, \tag{4.3.45}$$

and

$$\begin{aligned}&\left[\frac{\left(P_0+\rho_0\right)\left(g_{tt}\gamma\right)'+\left(B^z_0\right)^2\left(\gamma g_{tt}g_{II}\right)'}{g_{tt}}+\gamma P'_0\right]\bar{u}^R \\ &+i\left(B^z_0\right)^2\frac{g_{II}k_z}{\gamma g_{tt}}\left(\gamma^2 g_{tt}-1\right)\bar{u}^z \\ &+i\Omega\gamma k_t\frac{g_{\phi\phi}}{g_{tt}}\left[P_0+\rho_0+\left(B^z_0\right)^2 g_{II}\right]\bar{u}^\phi+ik_t\left(\gamma^2-\frac{1}{g_{tt}}\right)\bar{P} \\ &+iB^z_0\frac{g_{II}k_t}{g_{tt}}\left(\gamma^2 g_{tt}-1\right)\bar{B}^z-i\Omega B^z_0 k_z\frac{g_{\phi\phi}}{g_{tt}}\bar{B}^\phi=0,\end{aligned} \tag{4.3.46}$$

for the R, z and ϕ components, respectively, and γ' is given by equation (4.3.39). Similarly, the linearised induction equation (4.3.10) yields

$$-iB^z_0 k_z\bar{u}^R+i\gamma k_t\bar{B}^R=0, \tag{4.3.47}$$

for the R component,

$$\frac{1}{2}B_0^z\left(2\frac{g_{II}'}{g_{II}}+\frac{g_{tt}'}{g_{tt}}+\frac{g_{\phi\phi}'}{g_{\phi\phi}}\right)\bar{u}^R+i\Omega B_0^z k_t\frac{g_{\phi\phi}}{g_{tt}}\bar{u}^\phi+i\gamma k_t\bar{B}^z=0, \tag{4.3.48}$$

for the z component, and

$$\begin{aligned}&-iB_0^z k_t\frac{g_{II}}{g_{tt}}\left(\gamma^2 g_{tt}-1\right)\bar{u}^z-i\Omega B_0^z k_z\frac{g_{\phi\phi}}{g_{tt}}\bar{u}^\phi\\&+\frac{1}{2}\left[-2\gamma'+\gamma^3\left(\Omega^2 g_{\phi\phi}'-g_{tt}'\right)\right]\bar{B}^R+i\gamma k_t\Omega\frac{g_{\phi\phi}}{g_{tt}}\bar{B}^\phi=0,\end{aligned} \tag{4.3.49}$$

for the ϕ component. The perturbed adiabatic conservation is given by equation (4.3.40).
Using equation (4.3.43) to eliminate the P_0' term, along with equations (4.3.20), (4.3.21), (4.3.23) and (4.3.43) the characteristic equation, having removed the sound waves, is given by

$$\begin{aligned}&4\left(k^2-\omega^2\right)g_{II}^4 g_{tt}g_{\phi\phi}\left(g_{tt}-\Omega^2 g_{\phi\phi}\right)^2\bigl[\omega^4 g_{\phi\phi}^2\Omega^4+\omega^2\left(\omega^2-2k^2\right)g_{tt}g_{\phi\phi}\Omega^2\\&+\left(k^4-k^2\omega^2\right)g_{tt}^2\bigr]v_{\mathrm{Az}}^4+g_{II}^3\bigl\{-2\omega^4 g_{\phi\phi}^4 v_{\mathrm{Az}}^2 g_{tt}'g_{\phi\phi}'\Omega^8+2\omega^2 g_{tt}g_{\phi\phi}^3 v_{\mathrm{Az}}^2\bigl[-2\omega^2 g_{\phi\phi}\Omega' g_{\phi\phi}'\Omega^3\\&+2\left(k^2+\omega^2\right)g_{tt}'g_{\phi\phi}'\Omega^2+\left(\omega^2-k^2\right)\left(g_{tt}'\right)^2\bigr]\Omega^4-g_{tt}^2 g_{\phi\phi}^2\bigl[-4\omega^4\left(\omega^2-2k^2\right)g_{\phi\phi}^2\bigl(v_{\mathrm{Az}}^2\\&-1\bigr)\Omega^4-8k^2\omega^2 g_{\phi\phi}v_{\mathrm{Az}}^2\Omega' g_{\phi\phi}'\Omega^3+v_{\mathrm{Az}}^2\Bigl(\omega^2\left(3\omega^2-k^2\right)\left(g_{\phi\phi}'\right)^2\Omega^4+2\bigl(k^4+3\omega^2k^2\\&+\omega^4\bigr)g_{tt}'g_{\phi\phi}'\Omega^2+\left(-2k^4+\omega^2k^2+\omega^4\right)\left(g_{tt}'\right)^2\Bigr)\Bigr]\Omega^2+8k^2\omega^2\left(k^2-\omega^2\right)g_{tt}^5 g_{\phi\phi}\left(v_{\mathrm{Az}}^2-1\right)\\&-2g_{tt}^4\bigl[-2\omega^2\left(-4k^4+2\omega^2k^2+\omega^4\right)\Omega^2\left(v_{\mathrm{Az}}^2-1\right)g_{\phi\phi}^2+2k^2\left(\omega^2-k^2\right)\Omega v_{\mathrm{Az}}^2\Omega' g_{\phi\phi}'g_{\phi\phi}\\&+k^2 v_{\mathrm{Az}}^2 g_{\phi\phi}'\bigl(\left(k^2-\omega^2\right)g_{tt}'-\left(k^2-2\omega^2\right)\Omega^2 g_{\phi\phi}'\bigr)\bigr)+g_{tt}^3 g_{\phi\phi}\bigl(8\omega^2\bigl(k^4+\omega^2k^2\end{aligned}$$

$$
\begin{aligned}
&-\omega^4) g_{\phi\phi}^2 (v_{\mathrm{Az}}^2 - 1) \Omega^4 - 4 (k^4 + \omega^2 k^2 - \omega^4) g_{\phi\phi} v_{\mathrm{Az}}^2 \Omega' g'_{\phi\phi} \Omega^3 + v_{\mathrm{Az}}^2 ((\omega^2 k^2 + 4\omega^4 \\
&-k^4) (g'_{\phi\phi})^2 \Omega^4 + 2 (k^4 + 2\omega^2 k^2 - \omega^4) g'_{tt} g'_{\phi\phi} \Omega^2 + (k^2\omega^2 - k^4) (g'_{tt})^2)] \} v_{\mathrm{Az}}^2 \\
&- 2\omega^2 g_{tt}^2 g_{\phi\phi} (g_{tt} - \Omega^2 g_{\phi\phi}) (v_{\mathrm{Az}}^2 - 1) g'_{II} \{2\omega^2 g_{\phi\phi} [2\Omega^2 g_{\phi\phi} v_{\mathrm{Az}}^2 g'_{II} - (v_{\mathrm{Az}}^2 - 1) g'_{tt}] \Omega^2 \\
&+4k^2 g_{tt}^2 v_{\mathrm{Az}}^2 g'_{II} - g_{tt} [4 (k^2 + \omega^2) \Omega^2 g_{\phi\phi} v_{\mathrm{Az}}^2 g'_{II} - \omega^2 (v_{\mathrm{Az}}^2 - 1) (g'_{\phi\phi} \Omega^2 + g'_{tt})] \} \\
&- g_{II} g_{tt} \{2\omega^4 g_{\phi\phi}^3 (v_{\mathrm{Az}}^2 - 1) g'_{tt} [2\Omega^2 g_{\phi\phi} v_{\mathrm{Az}}^2 g'_{II} - (v_{\mathrm{Az}}^2 - 1) g'_{tt}] \Omega^4 \\
&+\omega^2 g_{tt} g_{\phi\phi}^2 [8 g_{\phi\phi}^2 v_{\mathrm{Az}}^2 g'_{II} ((\omega^2 - k^2) \Omega v_{\mathrm{Az}}^2 g'_{II} - \omega^2 (v_{\mathrm{Az}}^2 - 1) \Omega') \Omega^3 \\
&-4 g_{\phi\phi} (v_{\mathrm{Az}}^2 - 1) (\Omega v_{\mathrm{Az}}^2 g'_{II} (g'_{tt} k^2 + 4\omega^2 \Omega^2 g'_{\phi\phi}) - \omega^2 (v_{\mathrm{Az}}^2 - 1) \Omega' g'_{tt}) \Omega \\
&+ (v_{\mathrm{Az}}^2 - 1)^2 g'_{tt} (7\omega^2 g'_{\phi\phi} \Omega^2 + (4k^2 - \omega^2) g'_{tt})] \Omega^2 + 4k^2 g_{tt}^4 v_{\mathrm{Az}}^2 g'_{II} [(v_{\mathrm{Az}}^2 - 1) g'_{\phi\phi} \omega^2 \\
&+2 (k^2 - \omega^2) g_{\phi\phi} v_{\mathrm{Az}}^2 g'_{II}] + g_{tt}^2 g_{\phi\phi} [8 g_{\phi\phi}^2 v_{\mathrm{Az}}^2 g'_{II} ((k^2 + \omega^2) (v_{\mathrm{Az}}^2 - 1) \Omega' \omega^2 + (k^4 \\
&+\omega^2 k^2 - 2\omega^4) \Omega v_{\mathrm{Az}}^2 g'_{II}) \Omega^3 + 2\omega^2 g_{\phi\phi} (v_{\mathrm{Az}}^2 - 1) (\Omega v_{\mathrm{Az}}^2 g'_{II} ((8k^2 + 11\omega^2) \Omega^2 g'_{\phi\phi} \\
&- (2k^2 + \omega^2) g'_{tt}) - (v_{\mathrm{Az}}^2 - 1) \Omega' (\omega^2 g'_{\phi\phi} \Omega^2 + (2k^2 + \omega^2) g'_{tt})) \Omega + \omega^2 (v_{\mathrm{Az}}^2 \\
&-1)^2 (-3\omega^2 (g'_{\phi\phi})^2 \Omega^4 - 4 (2k^2 + \omega^2) g'_{tt} g'_{\phi\phi} \Omega^2 + \omega^2 (g'_{tt})^2)] \\
&+g_{tt}^3 [(v_{\mathrm{Az}}^2 - 1)^2 g'_{\phi\phi} (g'_{tt} \omega^2 + (4k^2 + \omega^2) \Omega^2 g'_{\phi\phi}) \omega^2 - 2 g_{\phi\phi} (v_{\mathrm{Az}}^2 \\
&-1) (2\Omega\Omega' g'_{\phi\phi} k^2 + v_{\mathrm{Az}}^2 (g'_{II} ((10k^2 + 3\omega^2) g'_{\phi\phi} \Omega^2 + (\omega^2 - 4k^2) g'_{tt}) \\
&-2k^2 \Omega\Omega' g'_{\phi\phi})) \omega^2 + 8\Omega g_{\phi\phi}^2 v_{\mathrm{Az}}^2 g'_{II} ((-2k^4 + \omega^2 k^2 + \omega^4) \Omega v_{\mathrm{Az}}^2 g'_{II} \\
&-k^2 \omega^2 (v_{\mathrm{Az}}^2 - 1) \Omega')] \} + g_{II}^2 \{-2\omega^4 g_{\phi\phi}^4 v_{\mathrm{Az}}^2 g'_{tt} [2\Omega^2 g_{\phi\phi} v_{\mathrm{Az}}^2 g'_{II} - (v_{\mathrm{Az}}^2 - 1) g'_{tt}] \Omega^6 \\
&+\omega^2 g_{tt} g_{\phi\phi}^3 v_{\mathrm{Az}}^2 [-8\omega^2 g_{\phi\phi}^2 v_{\mathrm{Az}}^2 \Omega' g'_{II} \Omega^3 + 4 g_{\phi\phi} ((v_{\mathrm{Az}}^2 - 1) \Omega' \omega^2 \\
&+2 (k^2 + \omega^2) \Omega v_{\mathrm{Az}}^2 g'_{II}) g'_{tt} \Omega - (v_{\mathrm{Az}}^2 - 1) g'_{tt} (3\omega^2 g'_{\phi\phi} \Omega^2 + (2k^2 + 5\omega^2) g'_{tt})] \Omega^4 \\
&+g_{tt}^2 g_{\phi\phi}^2 v_{\mathrm{Az}}^2 [16 k^2 \omega^2 g_{\phi\phi}^2 v_{\mathrm{Az}}^2 \Omega' g'_{II} \Omega^3 + 2 g_{\phi\phi} ((v_{\mathrm{Az}}^2 - 1) \Omega' (\omega^2 g'_{\phi\phi} \Omega^2 \\
&+ (\omega^2 - 4k^2) g'_{tt}) \omega^2 + \Omega v_{\mathrm{Az}}^2 g'_{II} (\omega^2 (3k^2 - 5\omega^2) g'_{\phi\phi} \Omega^2 + (-2k^4 \\
&-9\omega^2 k^2 + \omega^4) g'_{tt})) \Omega + \omega^2 (v_{\mathrm{Az}}^2 - 1) (6\omega^2 (g'_{\phi\phi})^2 \Omega^4 + (3k^2 + 5\omega^2) g'_{tt} g'_{\phi\phi} \Omega^2 \\
&+ (7k^2 + \omega^2) (g'_{tt})^2)] \Omega^2 + 4k^2 g_{tt}^5 (g_{\phi\phi} [v_{\mathrm{Az}}^2 - 1)^2 \omega^4 + (\omega^2 - k^2) v_{\mathrm{Az}}^4 g'_{II} g'_{\phi\phi}] \\
&+g_{tt}^3 g_{\phi\phi} [4 g_{\phi\phi}^2 (-2k^2 \Omega v_{\mathrm{Az}}^2 \omega^4 + k^2 \Omega \omega^4 + v_{\mathrm{Az}}^4 (k^2 \omega^4 \Omega - 2 (k^4 + \omega^2 k^2 \\
&-\omega^4) \Omega' g'_{II})) \Omega^3 - 2 g_{\phi\phi} v_{\mathrm{Az}}^2 (\Omega v_{\mathrm{Az}}^2 g'_{II} ((3k^4 - 7\omega^4) g'_{\phi\phi} \Omega^2 \\
&+ (-5k^4 - 2\omega^2 k^2 + 3\omega^4) g'_{tt}) - (v_{\mathrm{Az}}^2 - 1) \Omega' (\omega^2 (k^2 - 3\omega^2) g'_{\phi\phi} \Omega^2 \\
&+ (2k^4 - \omega^2 k^2 - \omega^4) g'_{tt})) \Omega - \omega^2 v_{\mathrm{Az}}^2 (v_{\mathrm{Az}}^2 - 1) (2 (3k^2 + 4\omega^2) (g'_{\phi\phi})^2 \Omega^4 \\
&+ (6k^2 + \omega^2) g'_{tt} g'_{\phi\phi} \Omega^2 + (2k^2 - \omega^2) (g'_{tt})^2)] + g_{tt}^4 [8k^2 \Omega (2\Omega v_{\mathrm{Az}}^2 \omega^4 \\
&-\omega^4 \Omega - v_{\mathrm{Az}}^4 (\Omega \omega^4 + (\omega^2 - k^2) \Omega' g'_{II})) g_{\phi\phi}^2 - 2 v_{\mathrm{Az}}^2 (v_{\mathrm{Az}}^2 (2 (k^2 - 2\omega^2) \Omega\Omega' g'_{\phi\phi} k^2 \\
&+g'_{II} ((-5k^4 + 5\omega^2 k^2 + 2\omega^4) g'_{\phi\phi} \Omega^2 + 3 (k^4 - k^2 \omega^2) g'_{tt})) \\
&-2k^2 (k^2 - 2\omega^2) \Omega\Omega' g'_{\phi\phi}) g_{\phi\phi} + \omega^2 v_{\mathrm{Az}}^2 (v_{\mathrm{Az}}^2 - 1) g'_{\phi\phi} ((9k^2 + \omega^2) g'_{\phi\phi} \Omega^2 \\
&+ (\omega^2 - 3k^2) g'_{tt})] \} = 0,
\end{aligned}
\tag{4.3.50}
$$

where v_{Az} is the relativistic Alfvén velocity [98], defined through

$$v_{\mathrm{Az}}^2 = \frac{(B_0^z)^2}{(B_0^z)^2 + P_0 + \rho_0}. \tag{4.3.51}$$

The full polynomial can be found in appendix A.1.8. The flat metric and low speed limit yields the following polynomial in $\bar{\zeta}$

$$\begin{aligned}
&- R^8\omega^6\Omega^8 v_{\mathrm{Az}}^4\bar{\zeta}^{18} + \left(k^2R^8\omega^4v_{\mathrm{Az}}^4\Omega^8 + 2R^6\omega^6v_{\mathrm{Az}}^4\Omega^6\right)\bar{\zeta}^{16} - 2R^7\omega^4\Omega^7v_{\mathrm{Az}}^4\Omega'\bar{\zeta}^{15}\\
&+ \left[-k^2R^6\omega^4v_{\mathrm{Az}}^4\Omega^6 - R^6\omega^6v_{\mathrm{Az}}^2\Omega^6 - R^4\omega^4\left(\omega^2 - 3\Omega^2\right)v_{\mathrm{Az}}^4\Omega^4\right]\bar{\zeta}^{14}\\
&+ R^5\omega^4\Omega^5v_{\mathrm{Az}}^4\Omega'\bar{\zeta}^{13} + \left[2R^4\Omega^4v_{\mathrm{Az}}^2\omega^6 - 2k^2R^4\Omega^4v_{\mathrm{Az}}^4\omega^4 - R^2\Omega^4v_{\mathrm{Az}}^4\omega^4\right.\\
&+2k^2R^6\Omega^6v_{\mathrm{Az}}^2\omega^4 - 6R^4\Omega^6v_{\mathrm{Az}}^2\omega^4 - 2k^4R^6\Omega^6v_{\mathrm{Az}}^4\omega^2\\
&\left.+k^2R^4\Omega^4\left(\omega^2 + \Omega^2\right)v_{\mathrm{Az}}^4\omega^2\right]\bar{\zeta}^{12} + \left(4k^2R^5\omega^2\Omega^5v_{\mathrm{Az}}^4\Omega' - R^5\omega^4\Omega^5v_{\mathrm{Az}}^2\Omega'\right)\bar{\zeta}^{11}\\
&+ \left[-R^2\Omega^2v_{\mathrm{Az}}^2\omega^6 - 4k^2R^4\Omega^4v_{\mathrm{Az}}^2\omega^4 + 2R^2\Omega^4v_{\mathrm{Az}}^2\omega^4 + 4k^4R^4\Omega^4v_{\mathrm{Az}}^4\omega^2\right.\\
&\left.-2k^2R^2\Omega^4v_{\mathrm{Az}}^4\omega^2 + k^2R^2\Omega^2\left(\omega^2 - 3\Omega^2\right)v_{\mathrm{Az}}^4\omega^2\right]\bar{\zeta}^{10}\\
&+ \left(R^3\omega^4\Omega^3v_{\mathrm{Az}}^2\Omega' - k^2R^3\omega^2\Omega^3v_{\mathrm{Az}}^4\Omega'\right)\bar{\zeta}^9 + \left[R^4\Omega^4v_{\mathrm{Az}}^4k^6 - R^2\omega^2\Omega^2v_{\mathrm{Az}}^4k^4\right.\\
&-R^2\Omega^2\left(\omega^2 + \Omega^2\right)v_{\mathrm{Az}}^4k^4 - 2R^4\omega^2\Omega^4v_{\mathrm{Az}}^2k^4 + R^4\omega^4\Omega^4k^2 + \omega^2\Omega^2v_{\mathrm{Az}}^4k^2\\
&\left.+6R^2\omega^2\Omega^4v_{\mathrm{Az}}^2k^2 + 2R^2\omega^4\Omega^2v_{\mathrm{Az}}^2k^2 + 3R^2\omega^4\Omega^4 + \omega^4\Omega^2v_{\mathrm{Az}}^2\right]\bar{\zeta}^8 + \left(R^3\Omega^3\Omega'\omega^4\right.\\
&\left.-k^2R^3\Omega^3v_{\mathrm{Az}}^2\Omega'\omega^2 - 2k^4R^3\Omega^3v_{\mathrm{Az}}^4\Omega'\right)\bar{\zeta}^7 + \left(-2R^2\Omega^2v_{\mathrm{Az}}^4k^6 + 2\Omega^2v_{\mathrm{Az}}^4k^4\right.\\
&\left.+4R^2\omega^2\Omega^2v_{\mathrm{Az}}^2k^4 - 2R^2\omega^4\Omega^2k^2 - \omega^2\Omega^2v_{\mathrm{Az}}^2k^2 - \omega^4\Omega^2\right)\bar{\zeta}^6\\
&+ \left(v_{\mathrm{Az}}^4k^6 - 2\omega^2v_{\mathrm{Az}}^2k^4 + \omega^4k^2 - 4\omega^2\Omega^2k^2\right)\bar{\zeta}^4\\
&+ \left(2k^4R\Omega v_{\mathrm{Az}}^2\Omega' - 2k^2R\omega^2\Omega\Omega'\right)\bar{\zeta}^3 = 0.
\end{aligned} \tag{4.3.52}$$

Considering only the smallest power of $\bar{\zeta}$ (i.e. $\bar{\zeta}^3$) we obtain

$$\omega^2 - v_{\mathrm{Az}}^2k^2 = 0, \tag{4.3.53}$$

which is the characteristic polynomial for the Alfvén waves. Retaining the $\bar{\zeta}^4$ term as well and assuming $\bar{\zeta} = 1$ we obtain the Newtonian MRI characteristic equation (4.2.90) assuming for the Newtonian wavevector $k_z = k$.

4.3.3.6 A remark on the characteristic polynomials and a discussion on more realistic models

In the previous sections we obtained the characteristic equations that provide information on the stability of each system. More specifically, the full (general relativistic) characteristic equations (i.e. including the sound waves) are of degree four in ω for the inertial waves (section 4.3.3.3) and the Rayleigh shearing instability (section 4.3.3.4). After we remove the sound waves, the polynomials are of degree two in ω for both cases. Regarding the MRI, the full characteristic equation is of degree six in ω while the reduced equation is of degree four.

The coefficients of the characteristic equations are too involved to be reduced analytically to general, physically intuitive conditions, similar the respective Newtonian conditions obtained in section 4.2.5. However, the characteristic equations acquire practical use, if instead of deriving general conditions, we consider specific forms for the various background quantities involved. Using explicit functions or tabulated results of numerical simulations we may investigate the stability of models in full general relativity as long as these background solutions satisfy the "fast" and "slow" assumptions we have imposed. Since our assumptions about the background are general, there is substantial freedom regarding the background solutions that may be probed as possible stable solutions of the system. Such solutions for astrophysical disks are, for example, discussed in [97].

Apart from the investigation of stability in the context of general relativity it is possible to obtain results in special relativity, as well. The fully special relativistic polynomials are obtained by setting $\bar{\zeta} = 1$ in equations (4.3.36), (4.3.42), and (4.3.52). Furthermore, we may obtain post-Newtonian corrections by considering the terms providing the Newtonian results, plus extra terms (i.e. higher powers of $\bar{\zeta}$) of the previously mentioned equations.

Moreover, in the applications of the general relativistic case in section 4.3.3 we considered a static, axially symmetric spacetime of the form (4.3.16) as discussed in [95, 96]. However, more realistic models of rotating configurations in general relativity would require the consideration of spacetime framedragging [48]. To implement such physical process we need to consider a more general metric than that provided in equation (4.3.16), which will contain off diagonal components. More specifically, framedragging in our analysis introduces the off diagonal $g_{t\phi}$ background metric component, which may be chosen either "fast" or "slow" as discussed in section 4.1.2.2.

CHAPTER 5

Conclusions

In this thesis we have examined the dynamical behaviour of electromagnetic fluid media in the context of general relativity. In the first part, using the variational approach, we derived the Einstein equations, the Euler-Lagrange equations for a multicomponent fluid and the equations for the electromagnetic field. Starting with the covariant description of electromagnetism in linear media we took steps towards the non-linear case. In analogy with the expression for the Lagrangian used in linear media we provided the respective formula for non-linear media in terms of an infinite series. The linear case is given by the first two terms of the series. We also provided a set of propagation equations in terms of the material derivative for the permittivity and permeability tensors. We continued with the description of a model for a general fluid consisting of multiple components with possible non-linear electromagnetic properties. Such a medium may be found in the crust or in the core of a neutron star. This description is quite general and it allows the derivation of the the single fluid ideal magnetohydrodynamic limit of the general medium. The derivation parallels the Newtonian derivation of the same limit.

In the second part we examined the dynamical behaviour of systems relevant to astrophysical environments. We performed a first order perturbation analysis using the geometric optics method assuming that the perturbations have harmonic dependence, in the form of a plane wave. Additionally, in order to be able to model various physical configurations we introduced the notion of “fast” and “slow” quantities and we discussed the stability and causality criteria that constrain the phase velocity of the plane waves. The method is general in the sense that we can model various configurations and also is not limited to ideal MHD. It can also be applied in Newtonian context and thus, in order to gain insight, we calculated the modes and instabilities of specific Newtonian systems. More precisely, working in a Cartesian framework we derived in pure hydrodynamical systems the sound waves, the continuous versions of the Taylor-Rayleigh and Kelvin-Helmholtz instabilities, and the Alfvén waves in MHD. Taking a step towards more realistic astrophysical configurations we worked in cylindrical polar coordinates and obtained the inertial waves, the Rayleigh shearing instability and the magnetorotational instability. We also demonstrated the vanishing magnetic field limit of the latter and argued the circumstances under which it reduces to the Rayleigh shearing instability. Although this limit has been discussed in literature [99, 100, 97], most of the times the arguments favouring the difference between the vanishing magnetic field MRI and the Rayleigh shearing instability are vague. In this work we provided a quantitative analysis of this limit, using expansions of the MRI characteristic polynomial and discussed the various possible cases. In that sense, we provided a clearer picture regarding the stability condition in the vanishing magnetic field limit of the MRI. Subsequently, we considered an axisymmetric spacetime in cylindrical coordinates and worked out the characteristic equations of the inertial waves, and the Rayleigh shearing and magnetorotational instabilities. We also demonstrated that these equations reduce to the respective Newtonian results provided that the background metric is flat and the various velocities of the system are small. This Newtonian-like limit of the characteristic equations provide insight into the Newtonian behaviour of relativistic systems.

The thesis would be incomplete if we did not mention the various paths that may be followed starting with this work. Regarding the first part, it is possible to extend the propagation equations of the permittivity and permeability tensors in order to account for dependence on number density currents in addition to the electromagnetic field. Also, following the long Abraham-Minkowski controversy on the essence of the spatial electromagnetic flux vector it would be interesting to calculate the two forms for the general medium presented here.

Regarding the second part, it would be interesting to look into the possibility of regarding the notion of “fast” and “slow” quantities in such a way that the calculations remain covariant, in the sense of a 1+3 decomposition. Additionally, we may extend the existing relativistic approach to account for “fast” quantities along z axis. This consideration will result in a characteristic polynomial that will be valid in regions far from the equatorial plane. Furthermore, using a different coordinate system we may obtain solutions in three spatial dimensions. Finally, since the geometric optics method may be used in many physical configurations, it is possible to perform a stability analysis of systems involving electromagnetic media and multifluids as discussed in the first part of the thesis.

APPENDIX A

Mathematical formulas

A.1 Definitions and additional calculations

A.1.1 Formulas for the Levi-Civita tensor

From equation (2.1.14), contracting a pair of indices yields

$$\epsilon_{abcd}\epsilon^{afpq} = -3!\delta_{[b}^{\ f}\delta_{c}^{\ p}\delta_{d]}^{\ q}, \tag{A.1.1}$$

while contracting two pairs leads to

$$\epsilon_{abcd}\epsilon^{abpq} = -2!\delta_{[c}^{\ p}\delta_{d]}^{\ q}. \tag{A.1.2}$$

For three pair contraction we have

$$\epsilon_{abcd}\epsilon^{abcq} = -6\delta_{d}^{\ q}. \tag{A.1.3}$$

For the spatial Levi-Civita tensor we have

$$\epsilon_{abc}\epsilon^{def} = 3!h_{[a}{}^{d}h_{b}{}^{e}h_{c]}{}^{f}, \tag{A.1.4}$$

while contracting one pair of indices implies

$$\epsilon_{abc}\epsilon^{aef} = 2!h_{[b}{}^{e}h_{c]}{}^{f}, \tag{A.1.5}$$

while two pair contracting yields

$$\epsilon_{abc}\epsilon^{abf} = 2h_{c}{}^{f}, \tag{A.1.6}$$

and finally contracting all indices provides

$$\epsilon_{abc}\epsilon^{abc} = 3!. \tag{A.1.7}$$

A.1.1.1 Lagrangian variation

The Lagrangian variation of the metric tensor is calculated starting with equation (2.2.5)

$$\begin{aligned}\Delta g_{ab} &= \delta g_{ab} + \pounds_{\xi} g_{ab} \\ &= \delta g_{ab} + \xi^c \nabla_c g_{ab} + g_{cb}\nabla_a \xi^c + g_{ac}\nabla_b \xi^c \\ &= \delta g_{ab} + \nabla_a \xi_b + \nabla_b \xi_a \\ &= \delta g_{ab} + 2\nabla_{(a}\xi_{b)}.\end{aligned} \tag{A.1.8}$$

If ξ^a is a Killing vector then the last term vanishes and the Lagrangian and Eulerian variations for the metric are the same. Since the Lagrangian variation of the Kronecker delta is zero the variations of the metric with indices upstairs and downstairs are related through

$$\Delta g^{ab} = -g^{ac}\, g^{bd}\, \Delta g_{cd}. \tag{A.1.9}$$

A.1.2 The variation of the Levi-Civita tensor

To calculate the Eulerian variation of the Levi-Civita tensor we start with equations (2.1.14) and (2.1.12)

$$\begin{aligned}\delta\left(\epsilon_{abcd}\epsilon^{efgh}\right) =&0\\ \epsilon^{efgh}\delta\epsilon_{abcd} =&-\epsilon_{abcd}\left(\delta\epsilon_{pqrs}\right)g^{pe}g^{qf}g^{rg}g^{sh}-\epsilon_{abcd}\epsilon_{pqrs}\delta\left(g^{pe}g^{qf}g^{rg}g^{sh}\right)\\ \delta\epsilon_{abcd} =&\frac{1}{4!}\epsilon_{efgh}\epsilon_{abcd}\left(\delta\epsilon_{pqrs}\right)g^{pe}g^{qf}g^{rg}g^{sh}\\ &+\frac{1}{4!}\epsilon_{efgh}\epsilon_{abcd}\epsilon_{pqrs}\delta\left(g^{pe}g^{qf}g^{rg}g^{sh}\right),\end{aligned} \tag{A.1.10}$$

we now multiply this expression with ϵ^{abcd} and we contract the respective indices and so we get

$$\begin{aligned}\epsilon^{abcd}\delta\epsilon_{abcd} &= -\epsilon_{efgh}\left(\delta\epsilon_{pqrs}\right)g^{pe}g^{qf}g^{rg}g^{sh}-\epsilon_{efgh}\epsilon_{pqrs}\delta\left(g^{pe}g^{qf}g^{rg}g^{sh}\right)\\ &=-\frac{1}{2}\epsilon_{efgh}\epsilon_{pqrs}\delta\left(g^{pe}g^{qf}g^{rg}g^{sh}\right).\end{aligned} \tag{A.1.11}$$

We now substitute the first term of equation (A.1.10) with the result of the previous equation and we have

$$\begin{aligned}\delta\epsilon_{abcd} &= \frac{1}{2\cdot 4!}\epsilon_{abcd}\epsilon_{efgh}\epsilon_{pqrs}\delta\left(g^{pe}g^{qf}g^{rg}g^{sh}\right)\\ &=-\frac{2}{4!}\epsilon_{abcd}\epsilon_{efgh}\epsilon_{p}{}^{fgh}g^{pm}g^{en}\delta g_{mn}\\ &=-\frac{2}{4!}\epsilon_{abcd}\epsilon_{efgh}\epsilon^{mfgh}g^{en}\delta g_{mn}\\ &=\frac{1}{2}\epsilon_{abcd}g^{mn}\delta g_{mn}.\end{aligned} \tag{A.1.12}$$

This formula is the same for the Lagrangian variation of the Levi-Civita tensor that is

$$\Delta\epsilon_{abcd} = \frac{1}{2}\epsilon_{abcd}g^{mn}\Delta g_{mn}. \tag{A.1.13}$$

A.1.3 The variation of multi-fluid Lagrangian

We will show the manipulation of the terms related to the infinitesimal spacetime displacement in the variation of the Lagrangian for the multi-fluid. Starting with equation (3.2.14) and omitting the summation symbol for clarity the terms in the brackets can be written as

$$\begin{aligned}&\mu^{\rm x}_a n^b_{\rm x}\nabla_b\xi^a_{\rm x} - \mu^{\rm x}_a\xi^b_{\rm x}\nabla_b n^a_{\rm x} - \mu^{\rm x}_a n^a_{\rm x}\nabla_b\xi^b_{\rm x} = \\ &\nabla_b\left(\mu^{\rm x}_a\xi^a_{\rm x}n^b_{\rm x}\right) - \underbrace{\mu^{\rm x}_a\xi^a_{\rm x}\nabla_b n^b_{\rm x}}_{=0} - n^b_{\rm x}\xi^a_{\rm x}\nabla_b\mu^{\rm x}_a - \nabla_b\left(\mu^{\rm x}_a n^a_{\rm x}\xi^b_{\rm x}\right) + n^a_{\rm x}\xi^b_{\rm x}\nabla_b\mu^{\rm x}_a.\end{aligned} \tag{A.1.14}$$

The second term in the right-hand side vanishes due to the conservation of each fluid number density current while the third and fifth in the right-hand side, by renaming indices, are

$$-n^b_{\rm x}\xi^a_{\rm x}\nabla_b\mu^{\rm x}_a + n^a_{\rm x}\xi^b_{\rm x}\nabla_b\mu^{\rm x}_a = -2\xi^b_{\rm x}n^a_{\rm x}\,\nabla_{[a}\mu^{\rm x}_{b]}. \tag{A.1.15}$$

Finally, the first and fourth terms in the right-hand side of equation (A.1.15) using equations (2.1.14), (2.3.4) and (3.2.12) is

$$\nabla_b\left(\mu^{\rm x}_a\xi^a_{\rm x}n^b_{\rm x}\right) - \nabla_b\left(\mu^{\rm x}_a n^a_{\rm x}\xi^b_{\rm x}\right) = \nabla_b\left(\frac{1}{2}\mu^{bef}_{\rm x}n^{\rm x}_{efa}\xi^b_{\rm x}\right). \tag{A.1.16}$$

A.1.4 The first Maxwell equation

In order to derive the first Maxwell's equation (3.3.82) we will use the definition of the covariant derivative for a 2nd rank covariant tensor S_{ab}. That is

$$\nabla_c S_{ab} = \partial_c S_{ab} - \Gamma^d{}_{ac}S_{db} - \Gamma^d{}_{bc}S_{ad}. \tag{A.1.17}$$

The left-hand part of equation (3.3.82) expands as

$$\nabla_{[a}F_{bc]} = \frac{1}{6}\left(\nabla_a F_{bc} - \nabla_a F_{cb} - \nabla_b F_{ac} + \nabla_b F_{ca} - \nabla_c F_{ba} + \nabla_c F_{ab}\right). \tag{A.1.18}$$

Using equation (A.1.17) and the antisymmetry of the Faraday tensor, this result is equivalent to

$$\begin{aligned}\nabla_{[a}F_{bc]} = \frac{1}{6}\Big[&2\partial_a F_{bc} + 2\partial_b F_{ca} + 2\partial_c F_{ab} + 2\left(\Gamma^d{}_{ac} - \Gamma^d{}_{ca}\right)F_{bd} \\ &+2\left(\Gamma^d{}_{bc} - \Gamma^d{}_{cb}\right)F_{da} + 2\left(\Gamma^d{}_{ba} - \Gamma^d{}_{ab}\right)F_{cd}\Big],\end{aligned} \tag{A.1.19}$$

and since the connection is symmetric the related terms cancel and we obtain

$$\nabla_{[a}F_{bc]} = \frac{1}{3}\left(\partial_a F_{bc} + \partial_b F_{ca} + \partial_c F_{ab}\right). \tag{A.1.20}$$

The Faraday tensor given in equation (3.3.1) can be written as

$$F_{ab} = \partial_a A_b - \Gamma^c{}_{ba} A_c - \partial_b A_a + \Gamma^c_{ab} A_c = \partial_a A_b - \partial_b A_a. \tag{A.1.21}$$

Substituting the previous equation in equation (A.1.20) we get

$$\nabla_{[a} F_{bc]} = \frac{1}{3} \left[\partial_a \partial_b A_c - \partial_a \partial_c A_b + \partial_b \partial_c A_a - \partial_b \partial_a A_c + \partial_c \partial_a A_b - \partial_c \partial_b A_a \right], \tag{A.1.22}$$

and finally, the commutativity of the partial differentiation yields

$$\nabla_{[a} F_{bc]} = 0. \tag{A.1.23}$$

A.1.5 Calculation of derivatives of invariants

Here we will provide the derivatives of the invariants with respect to the fundamental fields that were used in section 3.4.2. The second derivatives of the the electromagnetic invariants I and K are (since the first are already given in equations 3.3.58) and 3.3.59)

$$\frac{\partial^2 I}{\partial F_{ab} \partial F_{cd}} = 2\chi_0^{abcd}, \tag{A.1.24}$$

and

$$\frac{\partial^2 K}{\partial F_{ab} \partial F_{cd}} = 2\epsilon^{abcd}, \tag{A.1.25}$$

For F_{xy} we have

$$\frac{\partial F_{\mathrm{xy}}}{\partial F_{ab}} = \delta_{[c}{}^a \delta_{d]}{}^b n_{\mathrm{x}}^c n_{\mathrm{y}}^d = n_{\mathrm{x}}^{[a} n_{\mathrm{y}}^{b]}, \tag{A.1.26}$$

while the second derivative with respect to the Faraday tensor vanishes. The derivatives of the electromagnetic invariants I, K with respect to the number density current are zero since these quantities depend only on the electromagnetic field. For F_{xy} we have

$$\frac{\partial F_{\mathrm{xy}}}{\partial n_{\mathrm{x}}^a} = F_{cd} n_{\mathrm{y}}^d \delta_a{}^c = F_{ad} n_{\mathrm{y}}^d, \tag{A.1.27}$$

and for the second derivative we have

$$\frac{\partial^2 F_{\mathrm{xy}}}{\partial n_{\mathrm{x}}^a \partial n_{\mathrm{y}}^b} = F_{ad} \delta_b{}^d = F_{ab}, \tag{A.1.28}$$

while the second derivative with respect to number density current and the Faraday tensor is

$$\frac{\partial^2 F_{\mathrm{xy}}}{\partial n_{\mathrm{x}}^a \partial F_{bc}} = \delta_e{}^{[b}\delta_d{}^{c]} n_{\mathrm{y}}^d \delta_a{}^e = \delta_a{}^{[c} n_{\mathrm{y}}^{d]}. \tag{A.1.29}$$

The derivatives of I, K with respect to the the metric are

$$\frac{\partial I}{\partial g_{ab}} = F_{sp}F_{cd}\frac{\partial}{\partial g_{ab}}\left(g^{sc}g^{pd} - g^{sd}g^{pc}\right) = -4F^a{}_c F^{bc}, \tag{A.1.30}$$

where we have used the vacuum constitutive tensor given in equation (3.3.14) and used equation (3.1.10). The derivative of the Levi-Civita tensor with respect to the metric is obtained by a process similar to that of appendix A.1.2 given by

$$\frac{\partial K}{\partial g_{ab}} = -\frac{1}{2}Kg^{ab}. \tag{A.1.31}$$

Finally, the derivative of F_{xy} with respect to the metric is vanishing since number density currents are defined with indices upstairs while the Faraday tensor is defined with indices downstars.

A.1.6 Relation between the Lorenz force and the electromagnetic energy-momentum tensor in vacuum

Here we derive the $\nabla^a T_{ab}^{\mathrm{EM}} = -F_{ab}j^b$ relation, where T_{ab}^{EM} is given by equation (3.4.34). We have

$$\begin{aligned}
\nabla^a T_{ab}^{\mathrm{EM}} &= \nabla^b\left(-F_{ac}F_b^c - \frac{1}{4}F_{cd}F^{cd}g_{ab}\right) \\
&= -F_{ac}\left(\nabla^b F_b^c\right) - \nabla^b F_{ac}F_b^c - \frac{1}{4}g_{ab}\nabla^b\left(F_{cd}F^{cd}\right) \\
&= -F_{ac}j^c - F^{cb}\nabla_b F_{ac} - \frac{1}{2}F^{cd}\nabla_a F_{cd}
\end{aligned}$$

where we substituted j^a by the Maxwell equation (3.3.86)

$$\begin{aligned}
&= -F_{ac}j^c - F^{cd}\left(\nabla_c F_{da} + \frac{1}{2}\nabla_a F_{cd}\right) \\
&= -F_{ac}j^c - \frac{1}{2}F^{cd}\left(\nabla_c F_{da} - \nabla_c F_{ad} + \nabla_a F_{cd}\right) \\
&= -F_{ac}j^c - \frac{1}{2}F^{cd}\left(\nabla_c F_{da} + \nabla_d F_{ac} + \nabla_a F_{cd}\right) \\
&= -F_{ac}j^c - \frac{3}{2}F^{cd}\nabla_{[a}F_{cd]} \\
&= -F_{ac}j^c.
\end{aligned} \tag{A.1.32}$$

The second term in the second to last line vanished because of the Maxwell equation (3.3.82).

A.1.7 The axisymmetric Newtonian ∇ operator in a cylindrical polar orthonormal frame

We consider the orthonormal basis vectors $\hat{\boldsymbol{R}}, \hat{\boldsymbol{z}}, \hat{\boldsymbol{\phi}}$ and the respective coordinates (R, z, ϕ). For the axissymetric quantities $f(R,z)$, $\boldsymbol{A} = A_R\,(R,z)\,\hat{\boldsymbol{R}} + A_z\,(R,z)\,\hat{\boldsymbol{z}} + A_\phi\,(R,z)\,\hat{\boldsymbol{\phi}}$ and $\boldsymbol{B} = B_R\,(R,z)\,\hat{\boldsymbol{R}} + B_z\,(R,z)\,\hat{\boldsymbol{z}} + B_\phi\,(R,z)\,\hat{\boldsymbol{\phi}}$ we have

$$\nabla f = \frac{\partial f}{\partial R}\hat{\boldsymbol{R}} + \frac{\partial f}{\partial z}\hat{\boldsymbol{z}}, \tag{A.1.33}$$

$$\nabla \cdot \boldsymbol{A} = \frac{1}{R}A_R + \frac{\partial A_R}{\partial R} + \frac{\partial A_z}{\partial z}, \tag{A.1.34}$$

$$\nabla \times \boldsymbol{A} = -\frac{\partial A_\phi}{\partial z}\hat{\boldsymbol{R}} + \left(\frac{\partial A_R}{\partial z} - \frac{\partial A_z}{\partial R}\right)\hat{\boldsymbol{\phi}} + \left(\frac{\partial A_\phi}{\partial R} + \frac{A_\phi}{R}\right)\hat{\boldsymbol{z}}, \tag{A.1.35}$$

and

$$\begin{aligned}
(\boldsymbol{A}\cdot\nabla)\,\boldsymbol{B} = &\left(A_R\frac{\partial B_R}{\partial R} + A_z\frac{\partial B_R}{\partial z} - \frac{A_\phi B_\phi}{R}\right)\hat{\boldsymbol{R}} \\
&+ \left(A_R\frac{\partial B_\phi}{\partial R} + A_z\frac{\partial B_\phi}{\partial z} + \frac{A_\phi B_R}{R}\right)\hat{\boldsymbol{\phi}} \\
&+ \left(A_R\frac{\partial B_z}{\partial R} + A_z\frac{\partial B_z}{\partial z}\right)\hat{\boldsymbol{z}}.
\end{aligned} \tag{A.1.36}$$

A.1.8 The full relativistic characteristic polyomials

The full characteristic equation (i.e. including the speed of sound) for the relativistic Rayleigh shearing instability (of section 4.3.3.4) is given by

$$
\begin{aligned}
&4\omega^2 g_{II}^2 g_{tt}^2 g_{\phi\phi} \left(\omega^2 - k^2 c_{\mathrm{s}}^2\right) \left(g_{tt} - \Omega^2 g_{\phi\phi}\right)^2 + g_{II} \Big\{ \Omega^2 g_{tt}^2 \left(g'_{\phi\phi}\right)^2 \big[g_{tt} \left(c_{\mathrm{s}}^2 \left(4k^2 + \omega^2\right) - 5\omega^2\right) \\
&\quad -\Omega^2 g_{\phi\phi} \left(3\omega^2 c_{\mathrm{s}}^2 + k^2 - 4\omega^2\right)\big] + g_{tt} g'_{\phi\phi} \big[g'_{tt} \big(2\Omega^2 g_{tt} g_{\phi\phi} \left(-2c_{\mathrm{s}}^2 \left(2k^2 + \omega^2\right) + k^2 + 2\omega^2\right) \\
&\quad +\omega^2 \left(c_{\mathrm{s}}^2 + 1\right) g_{tt}^2 + \omega^2 \Omega^4 \left(7c_{\mathrm{s}}^2 - 3\right) g_{\phi\phi}^2\big) + 2\Omega g_{tt} g_{\phi\phi} \Omega' \big(-2 g_{tt} \left(\omega^2 - k^2 c_{\mathrm{s}}^2\right) \\
&\quad -\omega^2 \Omega^2 \left(c_{\mathrm{s}}^2 - 1\right) g_{\phi\phi}\big)\big] + g_{\phi\phi} g'_{tt} \big[2\Omega g_{tt} g_{\phi\phi} \Omega' \left(g_{tt} \left(\omega^2 - c_{\mathrm{s}}^2 \left(2k^2 + \omega^2\right)\right) + 2\omega^2 \Omega^2 c_{\mathrm{s}}^2 g_{\phi\phi}\right) \\
&\quad -g'_{tt} \left(g_{tt}^2 \left(k^2 - \omega^2 c_{\mathrm{s}}^2\right) + \Omega^2 g_{tt} g_{\phi\phi} \left(c_{\mathrm{s}}^2 \left(\omega^2 - 4k^2\right) + \omega^2\right) + 2\omega^2 \Omega^4 c_{\mathrm{s}}^2 g_{\phi\phi}^2\right)\big]\Big\} \\
&\quad + 2\omega^2 g_{tt} g_{\phi\phi} g'_{II} \left(g_{tt} - \Omega^2 g_{\phi\phi}\right) \left\{\Omega^2 \left(c_{\mathrm{s}}^2 - 1\right) g_{tt} g'_{\phi\phi} + g'_{tt} \left[\left(c_{\mathrm{s}}^2 + 1\right) g_{tt} - 2\Omega^2 c_{\mathrm{s}}^2 g_{\phi\phi}\right]\right\} = 0.
\end{aligned}
\tag{A.1.37}
$$

Similarly the full characteristic polynomial for the relativistic MRI (of section 4.3.3.5) is given by

$$
\begin{aligned}
&-\frac{v_{\mathrm{Az}}^2}{1 - v_{\mathrm{Az}}^2} \Big\{4\omega^2 g_{tt}^2 \big[\omega^2 \left(\omega^2 + \left(\omega^2 - 2k^2\right) c_{\mathrm{s}}^2\right) g_{\phi\phi} \Omega^2 + 2\left(k^2 - \omega^2\right) \big(k^2 c_{\mathrm{s}}^2 \\
&-\omega^2\big) g_{tt}\big] g_{\phi\phi} \left(g_{tt} - \Omega^2 g_{\phi\phi}\right)^2 g_{II}^3 + \Big[2\omega^4 c_{\mathrm{s}}^2 g_{\phi\phi}^4 \left(g'_{tt}\right)^2 \Omega^6 + \omega^2 g_{tt} g_{\phi\phi}^3 g'_{tt} \big(\omega^2 (1 \\
&-3c_{\mathrm{s}}^2\big) g'_{\phi\phi} \Omega^2 + 4\omega^2 c_{\mathrm{s}}^2 g_{\phi\phi} \Omega' \Omega + \left(\omega^2 - \left(2k^2 + 5\omega^2\right) c_{\mathrm{s}}^2\right) g'_{tt}\big) \Omega^4 + \omega^2 g_{tt}^2 g_{\phi\phi}^2 \big(- \big(k^2 + \omega^2 \\
&-6\omega^2 c_{\mathrm{s}}^2\big) \left(g'_{\phi\phi}\right)^2 \Omega^4 + \left(k^2 - 5\omega^2 + \left(3k^2 + 5\omega^2\right) c_{\mathrm{s}}^2\right) g'_{tt} g'_{\phi\phi} \Omega^2 + 2 g_{\phi\phi} \Omega' \big(\big(c_{\mathrm{s}}^2 \\
&+1\big) \omega^2 g'_{\phi\phi} \Omega^2 + \left(\omega^2 + \left(\omega^2 - 4k^2\right) c_{\mathrm{s}}^2\right) g'_{tt}\big) \Omega + \left(\left(7k^2 + \omega^2\right) c_{\mathrm{s}}^2 - 4\omega^2\right) \left(g'_{tt}\right)^2\big) \Omega^2 \\
&+g_{tt}^3 g_{\phi\phi} \Big(\big(k^4 - \omega^2 k^2 + 8\omega^4 - 2\left(4\omega^4 + 3k^2\omega^2\right) c_{\mathrm{s}}^2\big) \left(g'_{\phi\phi}\right)^2 \Omega^4 - \big(2k^4 - 2\omega^2 k^2 - 5\omega^4 \\
&+ \left(\omega^4 + 6k^2\omega^2\right) c_{\mathrm{s}}^2\big) g'_{tt} g'_{\phi\phi} \Omega^2 + 2 g_{\phi\phi} \Omega' \big(\omega^2 \left(-k^2 + \omega^2 + \left(k^2 - 3\omega^2\right) c_{\mathrm{s}}^2\right) g'_{\phi\phi} \Omega^2 \\
&+ \left(k^2 - \omega^2\right) \left(\left(2k^2 + \omega^2\right) c_{\mathrm{s}}^2 - \omega^2\right) g'_{tt}\big) \Omega
\end{aligned}
$$

$$
\begin{aligned}
&+\left(k^4-\omega^2k^2+\omega^4+\left(\omega^4-2k^2\omega^2\right)c_{\mathrm{s}}^2\right)\left(g_{tt}'\right)^2\Big)\\
&+g_{tt}^4\left(\omega^2\left(\left(k^2-9\omega^2+\left(9k^2+\omega^2\right)c_{\mathrm{s}}^2\right)g_{\phi\phi}'\Omega^2+\left(-k^2+3\omega^2+\left(\omega^2-3k^2\right)c_{\mathrm{s}}^2\right)g_{tt}'\right)\right.\\
&\left.-4\left(k^2-2\omega^2\right)\Omega\left(k^2c_{\mathrm{s}}^2-\omega^2\right)g_{\phi\phi}\Omega'\right)g_{\phi\phi}'\big]\,g_{II}^2\\
&+2\omega^2g_{tt}\left(g_{tt}-\Omega^2g_{\phi\phi}\right)g_{II}'\left[2\omega^2c_{\mathrm{s}}^2g_{\phi\phi}^3g_{tt}'\Omega^4-2g_{tt}g_{\phi\phi}^2\left(-\omega^2g_{\phi\phi}'\Omega^2+2\omega^2c_{\mathrm{s}}^2g_{\phi\phi}\Omega'\Omega\right.\right.\\
&+\left.\ c_{\mathrm{s}}^2\left(4\omega^2g_{\phi\phi}'\Omega^2+\left(k^2-\omega^2\right)g_{tt}'\right)\right)\Omega^2+g_{tt}^2g_{\phi\phi}\left(\left(k^2-8\omega^2\right.\right.\\
&+\left.\left(8k^2+3\omega^2\right)c_{\mathrm{s}}^2\right)g_{\phi\phi}'\Omega^2-4\left(\omega^2-k^2c_{\mathrm{s}}^2\right)g_{\phi\phi}\Omega'\Omega+\left(-k^2+4\omega^2\right.\\
&+\left.\left.\left(\omega^2-4k^2\right)c_{\mathrm{s}}^2\right)g_{tt}'\right)+2\left(\omega^2-k^2c_{\mathrm{s}}^2\right)g_{tt}^3g_{\phi\phi}'\big]\,g_{II}+8\omega^2g_{tt}^2\left[\omega^2\Omega^2g_{\phi\phi}c_{\mathrm{s}}^2\right.\\
&+\left.\left(\omega^2-k^2c_{\mathrm{s}}^2\right)g_{tt}\right]g_{\phi\phi}\left(g_{tt}-\Omega^2g_{\phi\phi}\right)^2\left(g_{II}'\right)^2\Big\}-\frac{v_{\mathrm{Az}}^4}{\left(v_{\mathrm{Az}}^2-1\right)^2}g_{II}\left[\omega^2\Omega^2g_{\phi\phi}c_{\mathrm{s}}^2\right.\\
&+\left.\left(\omega^2-k^2c_{\mathrm{s}}^2\right)g_{tt}\right]\Big\{4\left(k^2-\omega^2\right)g_{tt}g_{\phi\phi}\left(g_{tt}-\Omega^2g_{\phi\phi}\right)^2\left[\left(k^2-\omega^2\right)g_{tt}-\omega^2\Omega^2g_{\phi\phi}\right]g_{II}^3\\
&+\Big[2\omega^2g_{\phi\phi}^3g_{tt}'g_{\phi\phi}'\Omega^6-2g_{tt}g_{\phi\phi}^2\Big(-2\omega^2g_{\phi\phi}\Omega'g_{\phi\phi}'\Omega^3+\left(k^2+2\omega^2\right)g_{tt}'g_{\phi\phi}'\Omega^2\\
&+\left(\omega^2-k^2\right)\left(g_{tt}'\right)^2\Big)\Omega^2+g_{tt}^2g_{\phi\phi}\Big(-g_{\phi\phi}'\left(4g_{\phi\phi}\Omega'k^2+\left(k^2-3\omega^2\right)\Omega g_{\phi\phi}'\right)\Omega^3\\
&+2\left(k^2+\omega^2\right)g_{tt}'g_{\phi\phi}'\Omega^2+\left(\omega^2-k^2\right)\left(g_{tt}'\right)^2\Big)+2g_{tt}^3\left(\left(k^2-2\omega^2\right)g_{\phi\phi}'\Omega^2\right.\\
&+\left.2\left(k^2-\omega^2\right)g_{\phi\phi}\Omega'\Omega+\left(\omega^2-k^2\right)g_{tt}'\right)g_{\phi\phi}'\Big]\,g_{II}^2-2\left(g_{tt}-\Omega^2g_{\phi\phi}\right)g_{II}'\left[2\omega^2g_{\phi\phi}^3g_{tt}'\Omega^4\right.\\
&+2g_{tt}g_{\phi\phi}^2\left(2\omega^2\Omega g_{\phi\phi}\Omega'-\left(k^2+\omega^2\right)g_{tt}'\right)\Omega^2+g_{tt}^2g_{\phi\phi}\left(\left(5\omega^2-3k^2\right)g_{\phi\phi}'\Omega^2\right.\\
&+\left.\left.4\left(\omega^2-k^2\right)g_{\phi\phi}\Omega'\Omega+3\left(k^2-\omega^2\right)g_{tt}'\right)+2\left(k^2-\omega^2\right)g_{tt}^3g_{\phi\phi}'\right]g_{II}\\
&-8\left(k^2-\omega^2\right)g_{tt}^2g_{\phi\phi}\left(g_{tt}-\Omega^2g_{\phi\phi}\right)^2\left(g_{II}'\right)^2\Big\}-\omega^2g_{tt}\Big\{4\left(\omega^2-k^2c_{\mathrm{s}}^2\right)g_{II}^2g_{tt}^2g_{\phi\phi}\big(g_{tt}\\
&-\Omega^2g_{\phi\phi}\big)^2\omega^2+2g_{tt}g_{\phi\phi}\left(g_{tt}-\Omega^2g_{\phi\phi}\right)g_{II}'\left[g_{tt}\left(\left(c_{\mathrm{s}}^2-1\right)g_{\phi\phi}'\Omega^2+\left(c_{\mathrm{s}}^2+1\right)g_{tt}'\right)\right.\\
&-\left.2\Omega^2c_{\mathrm{s}}^2g_{\phi\phi}g_{tt}'\right]\omega^2+g_{II}\Big[-2\omega^2c_{\mathrm{s}}^2g_{\phi\phi}^3\left(g_{tt}'\right)^2\Omega^4+g_{tt}g_{\phi\phi}^2g_{tt}'\left(\left(7c_{\mathrm{s}}^2-3\right)\omega^2g_{\phi\phi}'\Omega^2\right.\\
&+\left.4\omega^2c_{\mathrm{s}}^2g_{\phi\phi}\Omega'\Omega-\left(\omega^2+\left(\omega^2-4k^2\right)c_{\mathrm{s}}^2\right)g_{tt}'\right)\Omega^2\\
&-g_{tt}^2g_{\phi\phi}\Big(\left(k^2-4\omega^2+3\omega^2c_{\mathrm{s}}^2\right)\left(g_{\phi\phi}'\right)^2\Omega^4\\
&+2\left(-k^2-2\omega^2+2\left(2k^2+\omega^2\right)c_{\mathrm{s}}^2\right)g_{tt}'g_{\phi\phi}'\Omega^2+2g_{\phi\phi}\Omega'\left(\left(c_{\mathrm{s}}^2-1\right)\omega^2g_{\phi\phi}'\Omega^2\right.\\
&+\left.\left(\left(2k^2+\omega^2\right)c_{\mathrm{s}}^2-\omega^2\right)g_{tt}'\right)\Omega+\left(k^2-\omega^2c_{\mathrm{s}}^2\right)\left(g_{tt}'\right)^2\Big)+g_{tt}^3g_{\phi\phi}'\left(\left(c_{\mathrm{s}}^2+1\right)g_{tt}'\omega^2\right.\\
&-\left.4\Omega\left(\omega^2-k^2c_{\mathrm{s}}^2\right)g_{\phi\phi}\Omega'+\left(\left(4k^2+\omega^2\right)c_{\mathrm{s}}^2-5\omega^2\right)\Omega^2g_{\phi\phi}'\right)\Big]\Big\}=0.
\end{aligned}
\tag{A.1.38}
$$

A.2 An alternative form of Maclaurin series

In this section we present a Taylor-like series for a real single variable function. Let $f : R \to R$ be a real analytic function. The MacLaurin series (Taylor series around $x_0 = 0$) are [1]

$$f(x) = \sum_{n=0}^{+\infty} \left(\frac{f_0^{(n)}}{n!} x^n \right). \tag{A.2.1}$$

The MacLaurin series for the k-th derivative of f are given by

$$f^{(k)}(x) = \sum_{n=0}^{+\infty} \left(\frac{f_0^{(n+k)}}{n!} x^n \right). \tag{A.2.2}$$

We will now calculate the following infinite series

$$\begin{aligned} A &= f_0 + x f^{(1)} - \frac{1}{2!} x^2 f^{(2)} + \ldots + \frac{(-1)^{k+1}}{k!} x^k f^{(k)} \ldots \\ &= f_0 + \sum_{k=1}^{+\infty} \left(\frac{(-1)^{k+1}}{k!} f^{(k)} x^k \right). \end{aligned} \tag{A.2.3}$$

The manipulation of the previous expression is easier if we substitute equation (A.2.2). Then we obtain

$$A = f_0 + \sum_{k=1}^{+\infty} \left(\frac{(-1)^{k+1}}{k!} x^k \sum_{n=0}^{+\infty} \left(\frac{f_0^{(n+k)}}{n!} x^n \right) \right), \tag{A.2.4}$$

[1] Here the subscripts denote the point where the function is evaluated i.e. $f_0 = f(0)$ and the superscripts the order of the derivative i.e. $f^{(k)} = f^{(k)}(x) = \frac{d^k f}{dx^k}$. Note that $f_0 = f_0^{(0)} = f(0)$. Since we manipulate single variable functions we do not show the variable in some occasions.

and we split the sum into three individual sums

$$
\begin{aligned}
A = f_0 + & \underbrace{\sum_{k=1}^{N} \left(\frac{(-1)^{k+1}}{k!} x^k \sum_{n=0}^{N-k} \left(\frac{f_0^{(n+k)}}{n!} x^n \right) \right)}_{A_0} \\
+ & \underbrace{\sum_{k=1}^{N} \left(\frac{(-1)^{k+1}}{k!} x^k \sum_{n=N-k+1}^{+\infty} \left(\frac{f_0^{(n+k)}}{n!} x^n \right) \right)}_{A_2} \\
+ & \underbrace{\sum_{k=N+1}^{+\infty} \left(\frac{(-1)^{k+1}}{k!} x^k \sum_{n=0}^{+\infty} \left(\frac{f_0^{(n+k)}}{n!} x^n \right) \right)}_{A_1} .
\end{aligned}
\tag{A.2.5}
$$

Starting with the second term A_0 we have

$$
\begin{aligned}
A_0 &= \sum_{k=1}^{N} \left(\frac{(-1)^{k+1}}{k!} x^k \sum_{n=0}^{N-k} \left(\frac{f_0^{(n+k)}}{n!} x^n \right) \right) \\
&\overset{m=n+1}{=} \sum_{k=1}^{N} \sum_{m=1}^{N-k+1} \left(\frac{(-1)^{k+1}}{k!(m-1)!} f_0^{(m-1+k)} x^{m-1+k} \right) \\
&\overset{s=m-1+k}{=} \sum_{k=1}^{N} \sum_{s=k}^{N} \left(\frac{(-1)^{k+1}}{k!(s-k)!} f_0^{(s)} x^s \right) .
\end{aligned}
\tag{A.2.6}
$$

Using now the property for double sums [101]

$$
\sum_{k=1}^{N} \sum_{n=k}^{N} c(k,n) = \sum_{n=1}^{N} \sum_{k=1}^{n} c(k,n),
\tag{A.2.7}
$$

equation (A.2.4) takes the following form

$$
\begin{aligned}
A_0 &= \sum_{k=1}^{N} \sum_{s=k}^{N} \left(\frac{(-1)^{k+1}}{k!(s-k)!} f_0^{(s)} x^s \right) \\
&= \sum_{s=1}^{N} \sum_{k=1}^{s} \left(\frac{(-1)^{k+1}}{k!(s-k)!} f_0^{(s)} x^s \right) \\
&= \sum_{s=1}^{N} \left(\frac{f_0^{(s)}}{s!} x^s \right) .
\end{aligned}
\tag{A.2.8}
$$

So we have shown that

$$A_0 = \sum_{k=1}^{N} \left(\frac{(-1)^{k+1}}{k!} x^k \sum_{n=0}^{N-k} \left(\frac{f_0^{(n+k)}}{n!} x^n \right) \right) = \sum_{s=1}^{N} \left(\frac{f_0^{(s)}}{s!} x^s \right). \tag{A.2.9}$$

Calculating the limit of the previous equation for $N \to +\infty$ yields

$$\lim_{N\to+\infty} A_0 = \lim_{N\to+\infty} \sum_{s=1}^{N} \left(\frac{f_0^{(s)}}{s!} x^s \right) = \sum_{s=1}^{+\infty} \left(\frac{f_0^{(s)}}{s!} x^s \right), \tag{A.2.10}$$

which by using equation (A.2.1) is found to take the value

$$\lim_{N\to+\infty} A_0 = f(x) - f_0. \tag{A.2.11}$$

We will now calculate the last term of equation (A.2.3). We assume that $\left| f_0^{(n+k)} \right| \leq \varepsilon$ for all n, k where ε is assumed to be finite. We know that a series $\sum_n a(n)$ converges if $\sum_n |a(n)|$ converges. So, we will calculate the related absolute series (namely A_1' and A_2') for terms A_1 and A_2. Starting with A_1 we have

$$\begin{aligned} A_1' = \sum_{k=N+1}^{+\infty} \left(\left| \frac{(-1)^{k+1}}{k!} x^k \sum_{n=0}^{+\infty} \left(\frac{f_0^{(n+k)}}{n!} x^n \right) \right| \right) \leq \\ \sum_{k=N+1}^{+\infty} \left(\frac{\left|(-1)^{k+1}\right|}{k!} \left|x^k\right| \sum_{n=0}^{+\infty} \left(\frac{\left|f_0^{(n+k)}\right|}{n!} \left|x^n\right| \right) \right). \end{aligned} \tag{A.2.12}$$

The absolute value of x^k for $k \in \mathbb{Z}$ is given through

$$\left|x^k\right| = \begin{cases} x^k & , x \geq 0 \\ (-1)^k x^k & , x < 0 \end{cases} \tag{A.2.13}$$

Since the expressions for positive and negative x are different we will consider them separately. Starting with $x \geq 0$ equation (A.2.12) becomes

$$\begin{aligned} \sum_{k=N+1}^{+\infty} \left(\frac{\left|(-1)^{k+1}\right|}{k!} \left|x^k\right| \sum_{n=0}^{+\infty} \left(\frac{\left|f_0^{(n+k)}\right|}{n!} \left|x^n\right| \right) \right) \leq \varepsilon \sum_{k=N+1}^{+\infty} \left(\frac{x^k}{k!} \sum_{n=0}^{+\infty} \frac{x^n}{n!} \right) = \\ \varepsilon \sum_{k=N+1}^{+\infty} \left(\frac{x^k}{k!} \right) = \varepsilon e^{2x} \left(1 - \frac{\Gamma(N+1, x)}{\Gamma(N+1)} \right), \end{aligned} \tag{A.2.14}$$

where $\Gamma(x)$ is the Gamma function and $\Gamma(n,x)$ is the upper incomplete Gamma function. Using the series definition of the later [102]

$$\Gamma(n,x) = (n-1)!e^{-x}\sum_{s=0}^{n-1}\frac{x^s}{s!}, \tag{A.2.15}$$

equation (A.2.14) obtains the following form

$$\sum_{k=N+1}^{+\infty}\left(\frac{\left|(-1)^{k+1}\right|}{k!}\left|x^k\right|\sum_{n=0}^{+\infty}\left(\frac{\left|f_0^{(n+k)}\right|}{n!}\left|x^n\right|\right)\right) \leq \varepsilon e^{2x}\left(1-e^{-x}\sum_{s=0}^{N}\frac{x^s}{s!}\right). \tag{A.2.16}$$

Similarly, equation (A.2.12) for $x<0$ becomes

$$\begin{aligned}
&\sum_{k=N+1}^{+\infty}\left(\frac{\left|(-1)^{k+1}\right|}{k!}\left|x^k\right|\sum_{n=0}^{+\infty}\left(\frac{\left|f_0^{(n+k)}\right|}{n!}\left|x^n\right|\right)\right) \leq \\
&\varepsilon\sum_{k=N+1}^{+\infty}\left(\frac{(-1)^k x^k}{k!}\sum_{n=0}^{+\infty}\frac{(-1)^n x^n}{n!}\right) = \\
&\varepsilon e^{-x}\sum_{k=N+1}^{+\infty}\left(\frac{(-1)^k x^k}{k!}\right) = \varepsilon e^{-2x}(-1)^N(-x)^{-N}x^N\left(1-\frac{\Gamma(N+1,-x)}{\Gamma(N+1)}\right),
\end{aligned} \tag{A.2.17}$$

and by using equation (A.2.15) we get

$$\begin{aligned}
&\sum_{k=N+1}^{+\infty}\left(\frac{\left|(-1)^{k+1}\right|}{k!}\left|x^k\right|\sum_{n=0}^{+\infty}\left(\frac{\left|f_0^{(n+k)}\right|}{n!}\left|x^n\right|\right)\right) \\
&\leq \varepsilon e^{-2x}\left(1-e^{x}\sum_{s=0}^{N}\frac{(-x)^s}{s!}\right).
\end{aligned} \tag{A.2.18}$$

Putting together the result of equations (A.2.16), (A.2.18) and (A.2.12) we get

$$\begin{aligned}
A_1' = &\sum_{k=N+1}^{+\infty}\left(\left|\frac{(-1)^{k+1}}{k!}x^k\sum_{n=0}^{+\infty}\left(\frac{f_0^{(n+k)}}{n!}x^n\right)\right|\right) \\
&\leq \begin{cases} \varepsilon e^{2x}\left(1-e^{-x}\sum_{s=0}^{N}\frac{x^s}{s!}\right) & ,x\geq 0 \\ \varepsilon e^{-2x}\left(1-e^{x}\sum_{s=0}^{N}\frac{(-x)^s}{s!}\right) & ,x<0 \end{cases}.
\end{aligned} \tag{A.2.19}$$

Taking now the limits of both sides of the previous piecewise equation as $N \to +\infty$ we get

$$\begin{gathered}\lim_{N\to+\infty} \sum_{k=N+1}^{+\infty} \left(\left| \frac{(-1)^{k+1}}{k!} x^k \sum_{n=0}^{+\infty} \left(\frac{f_0^{(n+k)}}{n!} x^n \right) \right| \right) \leq \\ \begin{cases} \lim_{N\to+\infty} \left(\varepsilon e^{2x} \left(1 - e^{-x} \sum_{s=0}^{N} \frac{x^s}{s!} \right) \right) & , x \geq 0 \\ \lim_{N\to+\infty} \left(\varepsilon e^{-2x} \left(1 - e^{x} \sum_{s=0}^{N} \frac{(-x)^s}{s!} \right) \right) & , x < 0 \end{cases}\end{gathered} \tag{A.2.20}$$

Both limits in the right-hand-side of equation (A.2.20) are zero and so we have

$$\lim_{N\to+\infty} \sum_{k=N+1}^{+\infty} \left(\left| \frac{(-1)^{k+1}}{k!} x^k \sum_{n=0}^{+\infty} \left(\frac{f_0^{(n+k)}}{n!} x^n \right) \right| \right) = 0. \tag{A.2.21}$$

Note that since all terms in the left-hand-side of equation (A.2.21) are positive the "<" symbol is dropped. We will now calculate the A_2' term

$$\begin{gathered}A_2' = \sum_{k=1}^{N} \left(\left| \frac{(-1)^{k+1}}{k!} x^k \sum_{n=N-k+1}^{+\infty} \left(\frac{f_0^{(n+k)}}{n!} x^n \right) \right| \right) \leq \\ \sum_{k=1}^{N} \left(\frac{\left|(-1)^{k+1}\right|}{k!} \left|x^k\right| \sum_{n=N-k+1}^{+\infty} \left(\frac{\left|f_0^{(n+k)}\right|}{n!} \left|x^n\right| \right) \right).\end{gathered} \tag{A.2.22}$$

As before we consider first the case $x \geq 0$

$$\begin{gathered}\sum_{k=1}^{N} \left(\frac{\left|(-1)^{k+1}\right|}{k!} \left|x^k\right| \sum_{n=N-k+1}^{+\infty} \left(\frac{\left|f_0^{(n+k)}\right|}{n!} \left|x^n\right| \right) \right) \leq \\ \varepsilon \sum_{k=1}^{N} \left(\frac{1}{k!} x^k \sum_{n=N-k+1}^{+\infty} \left(\frac{1}{n!} x^n \right) \right) = \varepsilon e^x \sum_{k=1}^{N} \left[\frac{x^k}{k!} \left(1 - \frac{\Gamma(N+1-k, x)}{\Gamma(N+1-k)} \right) \right],\end{gathered} \tag{A.2.23}$$

and by using equation (A.2.15) we get

$$\begin{aligned}&\sum_{k=1}^{N}\left(\frac{\left|(-1)^{k+1}\right|}{k!}\left|x^{k}\right|\sum_{n=N-k+1}^{+\infty}\left(\frac{\left|f_0^{(n+k)}\right|}{n!}\left|x^{n}\right|\right)\right)\leq\\&\varepsilon e^{x}\sum_{k=1}^{N}\left(\frac{x^{k}}{k!}\right)-\varepsilon\sum_{k=1}^{N}\left(\frac{x^{k}}{k!}\sum_{s=0}^{N-k}\frac{x^{s}}{s!}\right)=\\&\overset{m=k+s}{=}\varepsilon e^{x}\sum_{k=1}^{N}\left(\frac{x^{k}}{k!}\right)-\varepsilon\sum_{k=1}^{N}\left(\sum_{m=k}^{N}\left(\frac{x^{m}}{k!(m-k)!}\right)\right)\\&=\varepsilon e^{x}\sum_{k=1}^{N}\left(\frac{x^{k}}{k!}\right)-\varepsilon\sum_{m=1}^{N}\left(\sum_{k=1}^{m}\left(\frac{x^{m}}{k!(m-k)!}\right)\right).\end{aligned}\tag{A.2.24}$$

So after some manipulation we have

$$\begin{aligned}&\sum_{k=1}^{N}\left(\frac{\left|(-1)^{k+1}\right|}{k!}\left|x^{k}\right|\sum_{n=N-k+1}^{+\infty}\left(\frac{\left|f_0^{(n+k)}\right|}{n!}\left|x^{n}\right|\right)\right)\\&\leq\varepsilon e^{x}\sum_{k=1}^{N}\left(\frac{x^{k}}{k!}\right)-\varepsilon\sum_{m=1}^{N}\left(\frac{2^{m}-1}{m!}x^{m}\right).\end{aligned}\tag{A.2.25}$$

We calculate equation (A.2.22) for $x<0$

$$\begin{aligned}&\sum_{k=1}^{N}\left(\frac{\left|(-1)^{k+1}\right|}{k!}\left|x^{k}\right|\sum_{n=N-k+1}^{+\infty}\left(\frac{\left|f_0^{(n+k)}\right|}{n!}\left|x^{n}\right|\right)\right)\leq\\&\varepsilon\sum_{k=1}^{N}\left(\frac{(-1)^{k}}{k!}x^{k}\sum_{n=N-k+1}^{+\infty}\left(\frac{(-1)^{n}}{n!}x^{n}\right)\right)=\\&\varepsilon\sum_{k=1}^{N}\left(\frac{(-1)^{k}}{k!}x^{k}(-1)^{N+2-k}e^{-x}(-x)^{k-N}x^{N-k}\left(1-\frac{\Gamma(N+1-k,-x)}{\Gamma(N+1-k)}\right)\right)=\\&\varepsilon e^{-x}\sum_{k=1}^{N}\left(\frac{(-1)^{k}}{k!}x^{k}\right)-\varepsilon e^{-x}\sum_{k=1}^{N}\left(\frac{(-1)^{k}}{k!}x^{k}\frac{\Gamma(N+1-k,-x)}{\Gamma(N+1-k)}\right),\end{aligned}\tag{A.2.26}$$

and by using the definition of the incomplete Gamma function (A.2.15) we get

$$\begin{aligned} &\sum_{k=1}^{N}\left(\frac{\left|(-1)^{k+1}\right|}{k!}\left|x^k\right|\sum_{n=N-k+1}^{+\infty}\left(\frac{\left|f_0^{(n+k)}\right|}{n!}\left|x^n\right|\right)\right) \leq \\ &\leq \varepsilon e^{-x}\sum_{k=1}^{N}\left(\frac{(-1)^k}{k!}x^k\right) - \varepsilon\sum_{k=1}^{N}\left(\sum_{s=0}^{N-k}\frac{(-1)^{k+s}}{k!s!}x^{k+s}\right) = \\ &\overset{m=s+k}{=} \varepsilon e^{-x}\sum_{k=1}^{N}\left(\frac{(-1)^k}{k!}x^k\right) - \varepsilon\sum_{k=1}^{N}\left(\sum_{m=k}^{N}\frac{(-1)^m}{k!(m-k)!}x^m\right) = \\ &= \varepsilon e^{-x}\sum_{k=1}^{N}\left(\frac{(-1)^k}{k!}x^k\right) - \varepsilon\sum_{m=1}^{N}\left(\frac{(-1)^m(2^m-1)}{m!}x^m\right). \end{aligned} \tag{A.2.27}$$

The piecewise inequality for A_2' using equations (A.2.25), (A.2.26) and (A.2.22) is

$$\begin{aligned} A_2' = &\sum_{k=1}^{N}\left(\left|\frac{(-1)^{k+1}}{k!}x^k\sum_{n=N-k+1}^{+\infty}\left(\frac{f_0^{(n+k)}}{n!}x^n\right)\right|\right) \leq \\ &\begin{cases} \varepsilon e^{x}\sum_{k=1}^{N}\left(\frac{x^k}{k!}\right) - \varepsilon\sum_{m=1}^{N}\left(\frac{2^m-1}{m!}x^m\right), x \geq 0 \\ \varepsilon e^{-x}\sum_{k=1}^{N}\left(\frac{(-1)^k}{k!}x^k\right) - \\ \varepsilon\sum_{m=1}^{N}\left(\frac{(-1)^m(2^m-1)}{m!}x^m\right), x < 0 \end{cases}. \end{aligned} \tag{A.2.28}$$

Calculating the limit as $N \to +\infty$ on both sides of equation (A.2.29) we find that the right-hand-side limits are zero and thus

$$\lim_{N\to+\infty}\sum_{k=1}^{N}\left(\left|\frac{(-1)^{k+1}}{k!}x^k\sum_{n=N-k+1}^{+\infty}\left(\frac{f_0^{(n+k)}}{n!}x^n\right)\right|\right) = 0. \tag{A.2.29}$$

We will now calculate the values of A_1 and A_2. For A_1 we have

$$|A_1| \leq A_1' \tag{A.2.30}$$

and thus, we obtain

$$\begin{aligned} &\left|\sum_{k=N+1}^{+\infty}\left(\frac{(-1)^{k+1}}{k!}x^k\sum_{n=0}^{+\infty}\left(\frac{f_0^{(n+k)}}{n!}x^n\right)\right)\right| \\ &\leq \sum_{k=N+1}^{+\infty}\left(\left|\frac{(-1)^{k+1}}{k!}x^k\sum_{n=0}^{+\infty}\left(\frac{f_0^{(n+k)}}{n!}x^n\right)\right|\right). \end{aligned} \tag{A.2.31}$$

Taking the limit as $N \to +\infty$ on both sides of the above equation and using equation (A.2.21) we get

$$\lim_{N\to+\infty} |A_1| = \lim_{N\to+\infty} \left| \sum_{k=N+1}^{+\infty} \left(\frac{(-1)^{k+1}}{k!} x^k \sum_{n=0}^{+\infty} \left(\frac{f_0^{(n+k)}}{n!} x^n \right) \right) \right| = 0. \tag{A.2.32}$$

For A_1 we have

$$-|A_1| \leq A_1 \leq |A_1| \tag{A.2.33}$$

and therefore,

$$\begin{aligned} -\left| \sum_{k=N+1}^{+\infty} \left(\frac{(-1)^{k+1}}{k!} x^k \sum_{n=0}^{+\infty} \left(\frac{f_0^{(n+k)}}{n!} x^n \right) \right) \right| \leq A_1 \leq \\ \left| \sum_{k=N+1}^{+\infty} \left(\frac{(-1)^{k+1}}{k!} x^k \sum_{n=0}^{+\infty} \left(\frac{f_0^{(n+k)}}{n!} x^n \right) \right) \right|, \end{aligned} \tag{A.2.34}$$

which is a condition that is in general true for any real number. Calculating the limit as $N \to +\infty$ of the previous inequality and using the result of equation (A.2.32) we get

$$0 \leq \lim_{N\to+\infty} A_1 \leq 0, \tag{A.2.35}$$

and therefore

$$\lim_{N\to+\infty} A_1 = 0. \tag{A.2.36}$$

Similarly using the same arguments the value of A_2 as $N \to +\infty$ is

$$\lim_{N\to+\infty} A_2 = 0. \tag{A.2.37}$$

Finally, we calculate the limit of equation (A.2.5) as $N \to +\infty$ using equations (A.2.11), (A.2.36) and (A.2.37)

$$\lim_{N\to+\infty} A = f_0 + \lim_{N\to+\infty} A_0 + \lim_{N\to+\infty} A_1 + \lim_{N\to+\infty} A_2 \tag{A.2.38}$$

which reduces to

$$\lim_{N\to+\infty} A = f(x), \tag{A.2.39}$$

and by substituting A from the definition (A.2.3) we find

$$f(x) = f_0 + \sum_{k=1}^{+\infty} \left(\frac{(-1)^{k+1}}{k!} f^{(k)} x^k \right). \tag{A.2.40}$$

A.2.1 Mathematical results of the modified MacLaurin series

Calculating the series given by equation (A.2.40) for $\arctan(x)$ and then calculating for $x = 1$ we obtain in an analytic way the following BBP type formula [103] which was originally obtained numerically in [104]

$$\pi = \sum_{N=0}^{+\infty} \left[\frac{(-1)^N}{2^{2N-1}} \cdot \frac{20N^2 + 21N + 5}{32N^3 + 48N^2 + 22N + 3} \right]. \tag{A.2.41}$$

Additionally the error function $erf(x)$ which is defined through

$$\mathrm{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} \mathrm{d}t \tag{A.2.42}$$

by using equation (A.2.40) obtains the following form

$$\mathrm{erf}(x) = \frac{2}{\sqrt{\pi}} e^{-x^2} \sum_{N=1}^{+\infty} \left[\frac{x^N}{N!} H_{N-1}(x) \right], \tag{A.2.43}$$

where $H_{\mathrm{N}}(x)$ are the Hermitte polynomials [102]. Using the Rodrigues' formula for the Hermitte polynomials given by

$$H_N(x) = (-1)^N e^{x^2} \frac{d^N}{dx^N} \left(e^{-x^2} \right), \tag{A.2.44}$$

along with equation (A.2.40) we get the following identity

$$\sum_{N=1}^{+\infty} \left[\frac{x^N}{N!} H_N(x) \right] = e^{x^2}, \tag{A.2.45}$$

which is a special case of an exponential generating function [102].

Bibliography

[1] A. Hewish, S. J. Bell, J. D. H. Pilkington, P. F. Scott and R. A. Collins, *Observation of a Rapidly Pulsating Radio Source*, *Nature* **217** (Feb., 1968) 709–713.

[2] F. Özel and P. Freire, *Masses, Radii, and the Equation of State of Neutron Stars*, *Annual Review of Astronomy and Astrophysics* **54** (Sept., 2016) 401–440.

[3] The LIGO Scientific Collaboration and the Virgo Collaboration, *GW170817: Measurements of neutron star radii and equation of state*, *ArXiv e-prints* (May, 2018) arXiv:1805.11581, [1805.11581].

[4] N. Chamel, P. Haensel, J. L. Zdunik and A. F. Fantina, *On the Maximum Mass of Neutron Stars*, *International Journal of Modern Physics E* **22** (July, 2013) 1330018, [1307.3995].

[5] N. K. Glendenning, ed., *Compact stars : nuclear physics, particle physics, and general relativity.* 2000.

[6] S. L. Shapiro and S. A. Teukolsky, *Black holes, white dwarfs, and neutron stars: The physics of compact objects.* 1983.

[7] M. A. Alpar and Cheng Ho, *Expectancy of large pulsar glitches.*, MNRAS **204** (Aug., 1983) 655–667.

[8] D. Pines and M. A. Alpar, *Superfluidity in neutron stars*, Nature **316** (July, 1985) 27–32.

[9] N. Andersson, K. Glampedakis, W. C. G. Ho and C. M. Espinoza, *Pulsar Glitches: The Crust is not Enough*, Phys. Rev. Lett. **109** (Dec., 2012) 241103, [1207.0633].

[10] C. Thompson, R. C. Duncan, P. M. Woods, C. Kouveliotou, M. H. Finger and J. van Paradijs, *Physical Mechanisms for the Variable Spin-down and Light Curve of SGR 1900+14*, ApJ **543** (Nov., 2000) 340–350, [astro-ph/9908086].

[11] S. Mereghetti, *The strongest cosmic magnets: soft gamma-ray repeaters and anomalous X-ray pulsars*, *Astronomy and Astrophysics Review* **15** (July, 2008) 225–287, [0804.0250].

[12] G. Baym, C. Pethick and D. Pines, *Superfluidity in Neutron Stars*, Nature **224** (Nov., 1969) 673–674.

[13] G. Baym and C. Pethick, *Neutron stars.*, *Annual Review of Nuclear and Particle Science* **25** (Jan., 1975) 27–77.

[14] J. A. Sauls, *Superfluidity in the Interiors of Neutron Stars*. Springer Netherlands, Dordrecht, 1989.

[15] P. Mészáros, *High-Energy Radiation from Magnetized Neutron Stars*. Theoretical Astrophysics. University of Chicago Press, 1992.

[16] F. Özel, *Surface Emission Properties of Strongly Magnetic Neutron Stars*, ApJ **563** (Dec., 2001) 276–288, [astro-ph/0103227].

[17] D. Lai and W. C. G. Ho, *Transfer of Polarized Radiation in Strongly Magnetized Plasmas and Thermal Emission from Magnetars: Effect of Vacuum Polarization*, ApJ **588** (May, 2003) 962–974, [astro-ph/0211315].

[18] W. Heisenberg and H. Euler, *Folgerungen aus der Diracschen Theorie des Positrons*, *Zeitschrift fur Physik* **98** (Nov., 1936) 714–732.

[19] J. Schwinger, *On Gauge Invariance and Vacuum Polarization*, *Physical Review* **82** (June, 1951) 664–679.

[20] S. L. Adler, *Photon splitting and photon dispersion in a strong magnetic field.*, *Annals of Physics* **67** (Jan., 1971) 599–647.

[21] P. Haensel, A. Y. Potekhin and D. G. Yakovlev, eds., *Neutron Stars 1 : Equation of State and Structure*, vol. 326 of *Astrophysics and Space Science Library*, 2007.

[22] N. Chamel and P. Haensel, *Physics of Neutron Star Crusts*, *Living Reviews in Relativity* **11** (Dec., 2008) , [0812.3955].

[23] B. Carter and H. Quintana, *Foundations of General Relativistic High-Pressure Elasticity Theory*, *Royal Society of London Proceedings Series A* **331** (Nov., 1972) 57–83.

[24] M. Karlovini and L. Samuelsson, *Elastic stars in general relativity: I. Foundations and equilibrium models*, *Classical and Quantum Gravity* **20** (Aug., 2003) 3613–3648.

[25] R. Beig and B. G. Schmidt, *Relativistic elasticity*, *Classical and Quantum Gravity* **20** (Mar., 2003) 889–904, [gr-qc/0211054].

[26] C. Gundlach, I. Hawke and S. J. Erickson, *A conservation law formulation of nonlinear elasticity in general relativity*, *Classical and Quantum Gravity* **29** (Jan., 2012) 015005, [1107.2551].

[27] B. Carter, *Rheometric Structure Theory, Convective Differentiation and Continuum Electrodynamics*, *Proceedings of the Royal Society A: Mathematical, Physical and Engineering Sciences* **372** (1980) 169–200.

[28] C. J. Pethick and A. Y. Potekhin, *Liquid crystals in the mantles of neutron stars*, *Physics Letters B* **427** (May, 1998) 7–12, [astro-ph/9803154].

[29] P. G. de Gennes and J. Prost, *International Series of Monographs on Physics, Volume 83, The Physics of Liquid Crystals, Second Edition*, vol. 72. Feb., 1995, 10.1021/ed072pA42.7.

[30] S. D. Silverstein, *Criteria for Ferromagnetism in Dense Neutron Fermi Liquids-Neutron Stars*, *Phys. Rev. Lett.* **23** (Jul, 1969) 139–141.

[31] D. H. Brownell and J. Callaway, *Ferromagnetic transition in superdense matter and neutron stars*, *Il Nuovo Cimento B Series 10* **60** (1969) 169–188.

[32] P. Bernardos, S. Marcos, R. Niembro and M. L. Quelle, *Magnetic properties of strongly asymmetric nuclear matter in a Dirac-Hartree-Fock approach*, *Physics Letters B* **356** (Feb., 1995) 175–180.

[33] A. A. Isayev, *Competition between Ferromagnetic and Antiferromagnetic Spin Ordering in Nuclear Matter*, *Soviet Journal of Experimental and Theoretical Physics Letters* **77** (Mar., 2003) 251–255, [nucl-th/0301101].

[34] J. E. Pringle and M. J. Rees, *Accretion Disc Models for Compact X-Ray Sources*, A&A **21** (Oct., 1972) 1.

[35] P. Ghosh and F. K. Lamb, *Disk accretion by magnetic neutron stars.*, *ApJ* **223** (July, 1978) L83–L87.

[36] S. A. Balbus and J. F. Hawley, *A powerful local shear instability in weakly magnetized disks. I - Linear analysis.*, *ApJ* **376** (July, 1991) 214–233.

[37] J. Marsden and A. Tromba, *Vector Calculus.* W. H. Freeman, 2003.

[38] G. Ellis, R. Maartens and M. MacCallum, *Relativistic Cosmology.* Cambridge University Press, 2012.

[39] C. G. Tsagas, A. Challinor and R. Maartens, *Relativistic cosmology and large-scale structure*, Phys. Rep. **465** (Aug., 2008) 61–147, [0705.4397].

[40] R. Abraham, J. Marsden and T. Ratiu, *Manifolds, Tensor Analysis, and Applications.* Applied Mathematical Sciences. Springer New York, 1993.

[41] C. W. Misner, K. S. Thorne and J. A. Wheeler, *Gravitation.* 1973.

[42] B. F. Schutz, *Geometrical Methods of Mathematical Physics.* Cambridge University Press, 1980.

[43] R. Wald, *General Relativity.* University of Chicago Press, 2010.

[44] A. Papapetrou, *Lectures on General Relativity.* Springer Netherlands, 2012.

[45] S. W. Hawking and G. F. R. Ellis, *The large-scale structure of space-time.* 1973.

[46] R. K. Sachs and H. H. Wu, *General Relativity for Mathematicians.* 1977.

[47] J. L. Friedman and B. F. Schutz, *Lagrangian perturbation theory of nonrelativistic fluids*, ApJ **221** (May, 1978) 937–957.

[48] J. L. Friedman and N. Stergioulas, *Rotating Relativistic Stars.* 2013.

[49] J. E. Marsden, T. J. R. Hughes and D. E. Carlson, *Mathematical Foundations of Elasticity*, vol. 51. 1984, 10.1115/1.3167757.

[50] N. Andersson and G. L. Comer, *A covariant action principle for dissipative fluid dynamics: from formalism to fundamental physics*, *Classical and Quantum Gravity* **32** (Apr., 2015) 075008, [1306.3345].

[51] N. Andersson, G. L. Comer and I. Hawke, *A variational approach to resistive relativistic plasmas*, *Classical and Quantum Gravity* **34** (June, 2017) 125001.

[52] N. Andersson and G. L. Comer, *Relativistic Fluid Dynamics: Physics for Many Different Scales*, *Living Reviews in Relativity* **10** (Jan., 2007) 1, [gr-qc/0605010].

[53] S. M. Carroll, *Spacetime and geometry. An introduction to general relativity.* 2004.

[54] M. Alcubierre, *Introduction to 3+1 Numerical Relativity.* 2008.

[55] W. Greiner and J. Reinhardt, *Field Quantization:.* Springer, 1996.

[56] J. Synge and A. Schild, *Tensor calculus.* Mathematical expositions.

University of Toronto Press, 1949.

[57] B. Carter, *(Lecture Notes in Mathematics vol 1385) ed A Anile and M Choquet-Bruhat*, .

[58] L. Tisza, *Transport Phenomena in Helium II*, Nature **141** (May, 1938) 913.

[59] L. D. Landau, *Theory of superfluidity of Helium-II*, *Zhurnal Eksperimentalnoi i Teoreticheskoi Fiziki* **11** (Dec., 1941) 592.

[60] R. Prix, *Variational description of multifluid hydrodynamics: Uncharged fluids*, Phys. Rev. D **69** (Feb., 2004) 043001, [`physics/0209024`].

[61] R. Prix, G. L. Comer and N. Andersson, *Slowly rotating superfluid Newtonian neutron star model with entrainment*, A&A **381** (Jan., 2002) 178–196, [`astro-ph/0107176`].

[62] L. Rezzolla and O. Zanotti, *Relativistic Hydrodynamics.* Oxford University Press, 2018.

[63] E. J. Post, *Formal Structure of Electromagnetics: General Covariance and Electromagnetics.* 1962.

[64] J. D. Jackson, *Classical electrodynamics.* Wiley, New York, NY, 3rd ed. ed., 1999.

[65] D. Griffiths, *Introduction to Electrodynamics.* Prentice Hall, 1999.

[66] L. Landau, E. Lifshitz, E. Lifshitz and L. Pitaevskii, *Electrodynamics of continuous media:.* Pergamon international library of science, technology, engineering, and social studies. Pergamon, 1984.

[67] R. Boyd, *Nonlinear Optics.* Electronics & Electrical. Academic Press, 2003.

[68] Y. Shen, *The Principles of Nonlinear Optics.* Pure & Applied Optics Series: 1-349. Wiley, 1984.

[69] M. A. G. Bonilla and J. M. M. Senovilla, *Some Properties of the Bel and Bel-Robinson Tensors*, *General Relativity and Gravitation* **29** (Jan., 1997)

91–116.

[70] R. F. Tinder, *Tensor Properties of Solids.* Morgan & Claypool, 2007.

[71] V. B. Berestetskii, E. M. Lifshitz and V. B. Pitaevskii, *Relativistic quantum theory. Pt.2.* Oxford: Pergamon Press, 1974.

[72] J. S. Heyl and L. Hernquist, *The Birefringence and Dichroism of the QED Vacuum*, *Journ. Phys. A* **30** (1997) 6485–6492, [9705367v1].

[73] E. Poisson, *A Relativist's Toolkit: The Mathematics of Black-hole Mechanics.* Cambridge University Press, 2014.

[74] J. P. W. Diener, *Ferromagnetic Phase Transitions in Neutron Stars.* PhD thesis, 2013.

[75] P. Bellan, *Fundamentals of Plasma Physics.* Cambridge University Press, 2014.

[76] M. Goossens, *An Introduction to Plasma Astrophysics and Magnetohydrodynamics.* Astrophysics and Space Science Library. Springer Netherlands, 2003.

[77] N. Andersson, K. Dionysopoulou, I. Hawke and G. L. Comer, *Beyond ideal magnetohydrodynamics: resistive, reactive and relativistic plasmas*, *Classical and Quantum Gravity* **34** (2017) 125002.

[78] N. Andersson, *Resistive relativistic magnetohydrodynamics from a charged multifluids perspective*, Phys. Rev. D **86** (Aug., 2012) 043002, [1204.2695].

[79] K. McDonalds, *Bibliography on the Abraham-Minkowski Debate*, .

[80] L. D. Landau and E. M. Lifshitz, *The classical theory of fields.* 1975.

[81] A. Kandus and C. G. Tsagas, *Generalized Ohm's law for relativistic plasmas*, *Monthly Notices of the Royal Astronomical Society* **385** (2008) 883–892.

[82] A. M. Anile, *Relativistic Fluids and Magneto-fluids: With Applications in Astrophysics and Plasma Physics.* Cambridge Monographs on Mathematical

Physics. Cambridge University Press, 1990, 10.1017/CBO9780511564130.

[83] G. Whitham, *Linear and Nonlinear Waves.* Pure and Applied Mathematics: A Wiley Series of Texts, Monographs and Tracts. Wiley, 2011.

[84] V. Perlick, *Ray Optics, Fermat's Principle, and Applications to General Relativity.* Lecture Notes in Physics Monographs. Springer Berlin Heidelberg, 2003.

[85] M. Born and E. Wolf, *Principles of Optics: Electromagnetic Theory of Propagation, Interference and Diffraction of Light.* Cambridge University Press, 1999.

[86] B. Carter, *Speed of Sound in a High-Pressure General-Relativistic Solid*, *Physical Review D* **7** (Mar., 1973) 1590–1593.

[87] D. Swanson, *Plasma Waves, 2nd Edition.* Series in Plasma Physics. Taylor & Francis, 2003.

[88] W. Ledermann, *Complex Numbers.* Library of Mathematics. Springer US, 2012.

[89] S. Chandrasekhar, *Hydrodynamic and Hydromagnetic Stability.* Dover Books on Physics. Dover Publications, 2013.

[90] Rayleigh, *Investigation of the Character of the Equilibrium of an Incompressible Heavy Fluid of Variable Density*, *Proceedings of the London Mathematical Society* **s1-14** (1882) 170–177.

[91] H. Alfvén, *Existence of Electromagnetic-Hydrodynamic Waves*, *Nature* **150** (Oct., 1942) 405–406.

[92] S. A. Balbus, *Magnetohydrodynamics of Protostellar Disks*, *ArXiv e-prints* (June, 2009) arXiv:0906.0854, [0906.0854].

[93] W. R. Inc., "Mathematica, Version 11.3."

[94] S. Bonanos, *Capabilities of the Mathematica Package 'riemannian Geometry and Tensor Calculus'*, in *Recent Developments in Gravity* (K. D. Kokkotas

and N. Stergioulas, eds.), pp. 174–182, 2003. DOI.

[95] J. Griffiths and J. Podolskỳ, *Exact Space-Times in Einstein's General Relativity*. Cambridge Monographs on Mathematical Physics. Cambridge University Press, 2012.

[96] D. Kramer, H. Stephani, M. MacCullum and E. Herlt, *Exact Solutions of Einstein's Equations*. Cambridge Monographs on Mathematical Physics. Cambridge University Press, 1981.

[97] P. Armitage, *Astrophysics of Planet Formation*. Cambridge University Press, 2010.

[98] M. Gedalin, *Linear waves in relativistic anisotropic magnetohydrodynamics*, Phys. Rev. E **47** (June, 1993) 4354–4357.

[99] J. Pringle and A. King, *Astrophysical Flows*. Cambridge University Press, 2007.

[100] S. A. Balbus and W. J. Potter, *Surprises in astrophysical gasdynamics*, *Reports on Progress in Physics* **79** (2016) 066901.

[101] K. Rosen, *Handbook of Discrete and Combinatorial Mathematics, Second Edition*. Discrete Mathematics and Its Applications. Taylor & Francis, 2016.

[102] G. Arfken, H. Weber and F. Harris, *Mathematical Methods for Physicists: A Comprehensive Guide*. Elsevier, 2012.

[103] P. B. David Bailey and S. Plouffe, *On the rapid computation of various polylogarithmic constants*, *Mathematics of Computation* **66** (1997) 903–913.

[104] V. Adamchik and S. Wagon, *A Simple Formula for* π, *The American Mathematical Monthly* **104** (1997) 852–855.