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Microgenetic analysis of young children's shifts of attention in arithmetic tasks: underlying dynamics of change in phases of seemingly stable task performance

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Abstract
A key aim of mathematics teaching is for children to develop appropriate and efficient strategies for solving tasks. The analysis presented in this paper moves beyond the exploration of changes in the strategies that children employ to solve tasks and extends to observation and exploration of changes that occur when their overall solving approach remains seemingly stable. We present an analysis of data from two qualitative studies, each of which combined a microgenetic design with task-based interviews, to examine changes that occur in 5–6-year-old children's verbal reports when solving an additive task, and in 9–10-year-old children's verbal reports when solving fraction word problems. Children's verbal reports were analysed through the lens of the theory of shifts of attention. We found that phases of stability are underlain by dynamic changes in how the same strategy is communicated and conceptualised over a number of sessions and these changes appear to be accounted for by changes related to shifts in the object and structure of children's attention, i.e. *what* children attend to and *how*, when reporting on their solving approach. The paper extends the theory by revealing and studying microqualities that underlie different learners' structure of attention during phases of stability in arithmetic tasks. The findings provide new, significant insights for understanding qualitative dynamics of change in learning. Sensitivity to differences and changes in learners' shifts of attention is essential for teachers to make sense of what learners experience and identify opportunities conducive to further learning.

Keywords Shifts of attention · Arithmetic · Primary education · Addition · Fractions · Verbal reports · Microgenetic

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1 Introduction

Researchers have long acknowledged that the process of change has to be examined by employing methods that allow direct observation of learning as this occurs (Chen & Siegler, 2000; Chinn, 2006; Siegler, 2006). Microgenetic methods involve close examination of children's behaviour, repeatedly, over short periods of time when children's knowledge of a particular concept or task is in transition. This allows researchers to address the challenge of accounting for *how* change and stable strategy use occurs and capture the transition phases and mechanisms that may include progression as well as regression and periods of stability as different individuals' knowledge and understanding advances (Granott & Parziale, 2002).

On the premise that the process of change involves progressions, regressions, as well as phases of stability in problem solving performance (Siegler, 2007), the data and analysis presented in this paper focus on phases of stable strategy use; that is, phases during which individual children employ the same problem solving approach continually over a number of sessions. By zooming into such phases of problem solving performance, we aim to examine changes that may occur in how children report on their approach when solving the same type of task over a number of sessions. Phases of stable performance have been considered to be part of the path of change (Siegler, 1995). While the occurrence and frequency of such phases of stability have been noted as steps of a process, qualitative changes in the nature of such phases for different children have not previously been closely examined before.

The majority of microgenetic research is quantitative in nature, with emphasis placed on analysing and interpreting differences and relations between quantified aspects of overt problem solving behaviour, including quantified data of verbal reports and explanations, mainly in the context of solving one-step arithmetic questions. For example, analyses often focus on data associated with the percentage of trials solved by using particular strategies, the frequency with which a type of explanation is used for different types of problems, task solution times and accuracy of performance (e.g., Fazio, DeWolf, & Siegler, 2016; Lemaire & Siegler, 1995). Qualitative differences have been seen as differences in strategy type and have been studied through an analysis of the frequency with which different kinds of strategies (in terms of their sophistication) are used and the particular points of their emergence (e.g., Luwel, Siegler, & Verschaffel, 2008). Qualitative microgenetic studies that analyse qualitative data from children's problem solving are sparse. Examples are: Saada-Robert's (1992) study of young children's microgenetic construction of numerical representations; Schoenfeld, Smith, and Arcavi's (1993) study of a 16-year-old student's developing understanding of graphs and equations of simple algebraic functions; and more recently, Finesilver's (2017) study of low-attaining 11–15-year-old students' multiplicative thinking when working with 3D array tasks. These studies have sought to trace and infer changes occurring in students' knowledge structure and problem solving approaches through analysis of sequences of problem solving behaviours and verbal reports as part of individual task-based interviews. Analysis of data from task-based interviews constitutes one of the fundamental methods used in research that aims at exploring, in depth, the development of individual students' knowledge, reasoning, problem solving and understanding of mathematics concepts through a qualitative lens (Ginsburg, 2009). In contrast to qualitative studies that aim to provide rich descriptions of individual differences or particular aspects and patterns of understanding and reasoning in different mathematics domains through analysis of single sessions with a given task, or data from sessions that take place over an extended period of time (e.g., Cooley, Trigueros, & Baker, 2007; DeBellis & Goldin, 2006; Singh, 2000), qualitative microgenetic studies specifically focus on tracing and documenting points of change in learning through analysis of sequences of problem solving and verbal behaviours that occur close in time. This entails analysis of qualitative data resulting from high density of observations.

In line with Sfard's (2008) communicative position that changes in the way students communicate about mathematics (including verbal and written communication as well as gestures) imply changes in students' thinking, the current paper presents an interpretative analysis of data stemming from two microgenetic qualitative studies that examined changes in children's verbal reports of their approach for solving a multiple-step additive task and fraction word problems respectively. The paper aims to make an empirical contribution by exploring potential "underlying dynamics" (diSessa, 2017, p. 4) of the process of change, which have not been analysed before and in doing so, also highlighting the potential of microgenetic qualitative methods for uncovering hidden growth in learning during phases of stable task performance that can be highly instructive for educators.

We analyse changes in children's verbal reports through the lens of Mason's (1989, 2008, 2010) theory of shifts of attention to explore whether any differences and changes observed in how children communicate their solving approaches can be accounted for by differences and shifts in the object of children's attention (i.e. *what* children attend to when reporting on their solving approach) and the structure of their attention (i.e. *how* children attend to elements of their approach and of the task in hand). Mason and Davis (1988, p. 488) define the notion of a shift of attention as a "shift" "in which one becomes aware that what used to be attended to was only part of a larger whole, which is at once, more complex, and more simple". Mason (2008, p. 43) states that "awareness and attention are closely related. Someone may be attending to something in a particular way but unaware explicitly of the what and the how". Because implicit awareness and attention are not directly accessible, in the context of our analysis, we operationalise the notion of attention as "being or becoming explicitly aware" and we therefore seek to explore shifts of children's explicit (i.e. verbalised) attention to particular elements and to relationships between elements of the task and of their own solution, in the course of a sequence of trials where children's overall approach and task performance remains seemingly stable. Exploring a potential theoretical account for inter- and intra-individual differences and changes in phases of stability in arithmetic tasks is significant for advancing researchers' understanding of the dynamics of change in learning. It is also highly significant for raising educators' sensitivity to individual learners' trajectories of change and to learners' perceived experience so that teachers can offer relevant direction and guidance to a student's attention to task elements and relationships when seeking to support fruitful avenues for reasoning.

The current study addresses the following research questions: Are there qualitative differences and changes observed in the content of verbal reports provided by different children who use the same approach for solving a task over a number of problem solving sessions? If so, what do such differences and changes suggest about *what* children attend to and *how* they attend to elements of the task and their solving approach?

2 Background

2.1 Examining the process of change with the use of microgenetic methods

As Siegler (2006, p. 468) states, "… the only way to find out how children learn is to study them closely while they are learning". Traditional longitudinal and cross-sectional designs may not capture different types of transition and odd phases of transition in behaviour that are often indicative and predictive of change (Flynn, Pine, & Lewis, 2006). In contrast, the use of the microgenetic method is underpinned by the premise that engaging individuals in a particular learning experience (for example, a problem solving situation) in an intensive way over a period of time is likely to accelerate the process of change and is therefore more likely for researchers to capture the change/stability as it happens (Kuhn, 1995). Providing learners with increased-in-density opportunities to discover more advanced strategies and concepts allows for detailed examination of the processes that give rise to quantitative and qualitative aspects of change (Luwel et al., 2008). Highly-concentrated observation of learning behaviour over a specific and, usually, short period of time entails a process of intense data collection that is followed by trial-by-trial analysis of observed behaviours (Siegler, 2006). While microgenetic designs yield very rich data, such approaches entail particularly time- and effort-consuming phases of data collection and analysis. As a result, microgenetic studies generally involve a small number of participants or small number of sessions and, as diSessa (2017, p. 6) notes, entail an analysis that is “a complex, painstaking process with many details”.

In the domain of mathematics, microgenetic methods have been used in studies that have examined children’s learning and development in areas such as children’s strategy discovery and use in single-digit addition and additive problem solving (e.g., Siegler & Crowley, 1991), single-digit multiplication strategies (e.g., van der Ven, Kroesbergen, & Leseman, 2012), college students’ strategies for fraction magnitude comparison (e.g., Fazio et al., 2016), adaptive strategy choices in algebraic problem solving (e.g., Nussbaumer, Schneider, & Stern, 2014) and children’s understanding of mathematical equivalence (e.g., Fyfe, Rittle-Johnson, & DeCaro, 2012). The research by Siegler and his associates represents one of the most prominent series of studies of the path, rate, breadth, source and variability of strategy use and change in children’s mathematics development (for a comprehensive discussion of Siegler and associates’ work, see Lemaire, 2018).

Trial-by-trial analysis in microgenetic research has revealed great inter- and intra-individual variability in strategy use across different trials with the same type of problem presented in sessions that are close in time and also within the same trial (Tunteler, Pronk, & Resing, 2008). On the basis of evidence showing that children’s performance in problem solving does not progress in an orderly manner, Siegler (2007) notes that, in development, periods of high variability alternate with periods of low variability and stability and that for research, the implication is that “the changes with age and expertise that are observed in any particular study depend on the part of the cycle that is observed” (p. 107). Rather than analysing verbal reports as a way of accurately classifying different strategies, the aim of the analysis presented in this paper is to scrutinise differences in the content of verbal reports that children provide and trace related underlying qualitative changes that may occur in phases where children employ the same solving approach to a task.

2.2 The theory of shifts of attention

Mason (2008, p. 34) defines attention as “the medium through which observation takes place” and argues for its significance in human experience. Mason and Davis (1988) argue that shifts of attention may occur suddenly or gradually and that several shifts of attention may often be required in the process of thinking mathematically. Mason (2008) distinguishes two dimensions that need to be considered in the study of attention; the object or focus of attention (i.e.

what people attend to) and the form or structure of attention (i.e. how people attend). The structure of attention encompasses five different ways of attending: holding wholes, discerning details, recognising relationships, perceiving properties and reasoning on the basis of agreed properties.

"Holding wholes" refers to gazing at a mathematical object with or without focusing on particular elements. This may take only few seconds before one starts discerning details. "Discerning details" is a structure of attention whereby one is marking distinctions. "Your attention is caught by some detail in the task or activity in hand, allowing you to distinguish some aspect from some other aspect, and these distinctions participate in and contribute to subsequent attending" (Mason, 2008, p. 37). An individual's capability to discern important details, and realise what is important, is seen as key in the process of learning mathematics. "Recognising relationships" refers to recognising specific relations between elements. These could be, for example, elements of a mathematics example or model, a particular calculation, diagram, equation, etc. This structure of attention refers to "attention on relationships between parts or between part and whole, among aspects, features and attributes discerned" (Mason & Johnston-Wilder, 2004, p. 60). This often occurs automatically from discerning details and the theory postulates that, if details are not discerned, it is difficult for individuals to become aware of relationships between elements. Mason (2008) posits that a feature of relationships within this structure, or form, of attention is that these are recognised in the particular, i.e. the individual is recognising specific relationships between specific elements of a particular instance. Moving one's attention from attending to relationships between specific elements to "perceiving properties" constitutes an important shift. Perceiving properties refers to an individual's awareness of a possible relationship and looking for elements to fit it, thus leading to generalisation (Mason & Johnston-Wilder, 2004). In this case, "particular relations are seen as instances of general properties or abstract concepts" (Mason, 2008, p. 38). Finally, "reasoning on the basis of agreed properties" is a form of attention that refers to formal mathematical reasoning that is based on identified properties that may be already deduced and proved or on axioms. This structure of attention involves attending to already deduced, independent of particular objects, properties, as the only basis for further reasoning (Mason & Johnston-Wilder, 2004; Mason, 2008).

Mason (2010, p. 24) views learning as "the transformation of attention", that is, "a process that necessarily involves shifts in the form as well as the focus of attention". He postulates that engaging with, and experiencing, a learning situation, such as a problem solving activity, is not enough for learning to occur. Rather, learning requires the integration of experience and making sense of it. Making sense of an experience necessitates some transformation of attention, whereby the individual's attention is drawing back from immersion in the task in hand and moves from a goal-directed activity to a broader perspective. Students have to experience shifts in what they attend to and how they attend in order to "internalize and exploit the new concept or approach" (Mason, 2010, p. 42). Within this theorization of learning, Mason (2008, 2010) sees the role of teachers as offering *direction* (our emphasis) to learners' shifts of attention. He is clear in pointing out that "shifts are NOT something you do to someone else" (Mason & Davis, 1988, p. 490; emphasis in the original). Teachers' awareness of shifts in learners' attention supports their decision making for actions that can be taken to promote the necessary shifts and create a productive classroom environment (Mason & Davis, 1988).

Mason's theorisation of the notion of attention has been used to support research on teachers' learning about teaching mathematics, and within this, the need for them to learn how to notice *what* and *how* their students attend to when engaging with mathematics tasks (Mason, 2002). The theory also aims at accounting for students' learning of mathematics, based on the premise that students have to experience shifts of attention for learning to occur (Mason, 2010). Empirical research has employed the idea of teachers' noticing to analyse teachers' shifts when attending to students' mathematical thinking (Jacobs, Lamb, & Philipp, 2010) and the idea of students' noticing and focus of attention on elements of a given task to explore differences in reasoning and interactions that take place in two different classrooms (Lobato, Hohensee, & Rhodehamel, 2013).

In particular, Lobato et al. (2013) adopted the notion of attention, and students' "noticing", as a macro-level lens for the analysis of the impact that observed differences in students' focus of attention during classroom instruction has on their reasoning. In the context of classroom-based research, the researchers inferred students' "centres of focus" on the basis of students' verbal reports, gestures and written work in the classroom. In subsequent research, Hohensee (2016) applied the same framework and methods as those employed by Lobato et al. (2013) to explore pre-/post-instruction changes in students' emerging centres of focus and the influence that instruction about new concepts may have on students' previous reasoning about linear functions. One of the significant differences between that research and the two studies discussed in this paper is that the research by Lobato et al. (2013) and Hohensee (2016) analysed the classroom discourse among the students and their teacher and found it to be the element that gives rise to the emergence of specific foci for students' attention on mathematical features, regularities and objects. As such, the researchers have argued for causality in exploring links between students' attention and classroom interaction in a macro-level view of the classroom. This is beyond the scope and aims of the research that we report here which is not classroom-based and which aims at studying observed shifts and microqualities of students' attention as an explanatory theoretical lens for inter-individual differences and intra-individual changes in phases of stable task performance. In this endeavour, we extend our observations beyond exploring differences and changes in focus/object of attention to exploring how students may differ or change in *how* they attend to particular objects of attention (i.e. the structure of attention). For this reason, Mason's (2010) theorisation of shifts in the object as well as structure of attention provides an appropriate framework for analysis.

One example of previous research on students' shifts in relation to the object as well as the structure of attention that the authors are aware of is Palatnik and Koichu's (2014, 2015) analysis of a 9th-grade student's shifts of attention before an insightful solution to a challenging task occurred. In this paper, rather than employing the construct of students' shifts of attention to analyse the reasoning that students engage with when working towards achieving an initial solution to a task, we utilise the notion of shifts of attention to trace and account for changes that may occur in how students communicate and conceptualise their own solving approach after they have achieved an initial solution to the task and work on the same type of task more than once.

3 Approach to analysis

For this paper, we analysed qualitative data obtained during two studies, each of which combined a microgenetic design (Kuhn, 1995) with video-recorded, individual, task-based interviews (Maher & Sigley, 2014). This section aims to explain the process and theoretical assumptions that underpinned the analysis of data from both studies. Both studies had been conducted with the same primary aim of documenting changes in children's problem solving approaches, albeit in two different areas of mathematics learning. Study 1 explored changes in 5–6-year-old children's problem solving strategies when solving a multiple-step additive task. Study 2 examined 9–10-year-old children's strategies when solving partitive-quotient fraction word problems. A detailed presentation and discussion of the overall findings on children's understanding of the related concepts and changes related to the employed strategies have been provided, separately for each study, elsewhere (see George, 2017; Voutsina, 2012a, b). In this paper, we focus on a new object of study, that of phases of stable solving performance, and we analyse data that relate to such phases.

Consistent with the characteristics of qualitative research, both studies included a small sample (10 children participated in study 1; 9 participated in study 2). Furthermore, selecting to work with a small, rather than extensive, sample is based on the aim to conduct dense observation and intensive analysis of qualitative data and changes in each child's problem solving behaviour across a number of trials (Kuhn, 1995). In line with the recommendations made by Brock and Taber (2017), we adopt a "qualitative conception to validity" with a microgenetic framework whereby analysis is supported by clearly stated theoretical assumptions and a rich, in detail, reporting of the methods and data (Brock & Taber, 2017, pp. 67–68).

We carefully examined transcripts of the task-based interviews to identify particular instances where a child's problem solving approach to the task(s) included a phase during which the same solving approach was applied across more than one session. A trial-by-trial analysis of interview transcripts was conducted using an initially inductive qualitative content analysis approach to identify meaning and emerging patterns in the data through an interpretative lens (Bryman, 2016). The analysis involved contrasting the verbal reports provided by individual children within single trials and sessions, as well as across subsequent trials and problem solving sessions, with the aim of tracing particular points of change in children's word use and reports of their solving approach. The interview transcriptions and analysed data also included descriptions of children's hand movements and pointing actions which were scrutinised to support analytical inferences. A subsequent phase of analysis focused on exploring whether changes in children's reports indicated shifts related to the object and structure of children's attention. For this purpose, we adopted a deductive approach to analysis. Verbal reports and changes occurring in these were associated with inferences related to *what* children were attending to (i.e. the particular elements of the strategy and of the task that children pointed out, verbally as well as physically through pointing or jotting on paper) and shifts related to *how* children were attending to different elements. As it is not possible to capture the point where a child's implicit awareness of certain task elements or conceptual relationships emerges, in this analysis, we report points at which shifts in verbalised attention emerged within a sequence of more than one repeated trial with the same type of task.

Mason (2008, p. 43) notes that there is always a level of "ambiguity inherent in interpreting someone's utterances as statements of generality or of particularity". In recognising and accepting this challenge, we applied to our analysis the adopted theoretical lens systematically, within necessary parameters of process and assumptions. The microgenetic process of analysis focused on multiple episodes that occurred close in time where utterances were captured and analysed alongside aspects of overt behaviour (as explained above). As such, interpretations related to *what* and *how* children were attending were formed and supported by data related to more than one instance in children's engagement with the tasks. Our analysis is based on

Mason's (2008) theoretical position that a structure of attention may be an almost automatic development from another structure (e.g., when "recognising relationships" develops from "discerning details"), and therefore, a clear-cut distinction of moments that correspond to distinct ways of attending is at times difficult or impossible. Our assumption is also that in sessions that involved continuous engagement and reporting on a task, children attended to elements (e.g., numbers involved in the question or in their solutions) by "holding wholes", that is, by gazing, at a certain point, before starting focusing on particular components and providing verbal indication of discerned details. Such moments, which "may last a few microseconds" (Mason, 2008, p. 37), did not constitute part of our analysis as our data focused on children's verbalisations and, as such, did not systematically capture children's gazing before discerning particular elements.

In our analysis, we considered the following indications in children's verbal reports and accompanying overt behaviour as being associated with different structures of attention:

- Children *discerned details* when they were making reference (by noting verbally and/or pointing) to particular elements and components of the task or of their own produced solutions (e.g., noting that in their solution the first step produces always the same number combination—study 1, or noting the number of items and people involved in a sharing situation as part of a fraction word problem—study 2).
- Children *recognised relationships* when they referred to whether, and how, the noted elements were linked or related to each other (e.g., when indicating that there is an emerging numerical pattern in their solutions—study 1, or indicating a relationship between numbers involved in the problem and the resulting fraction—study 2).
- Children *perceived properties* when they referred consistently to previously recognised relationships as part of their reported strategy in subsequent trials, looked in subsequent trials for elements that fitted previously recognised possible relationships and began to extend the reported approach from the particular to the general by expressing relationships as rules. This was considered as an indication of moving from a particular trial that involved specific numbers and the relationship between these specific numbers to a more generalised conception of the properties underpinning the reported approach (e.g., when characterising and naming a previously recognised relationship in an emerging numerical pattern—study 1 or when justifying their solving approach for the task in hand by reference to similar relationships recognised in previous tasks with different numbers and referring to these as a rule—study 2).
- Finally, in the context of the tasks used in these studies, we considered that children *reasoned on the basis of agreed properties* when making reference to abstract, already known or deduced concepts (e.g., the concept of commutativity in addition—study 1 or the concept of equivalent fractions or extrapolated relations between multiples—study 2).

The data provided us with a varied picture in terms of the richness of changes observed in children's verbal reports and inferred shifts of attention during stable task performance. By session-2 of study 1, four out of 10 children from the study had reached a phase of stable solving approach. In study 2, by the third task, the nine children had settled into stable strategy use. We have selected to present the trajectories of four children (two from each study) who used the same approach to their task as pivotal examples of qualitative inter- and intra-individual changes accounted for through the theory of shifts of attention. On the basis that the adopted theoretical framework is not a stage-like developmental model that assumes a particular or typical order of 'stages' or 'levels' of reasoning, the selected examples of individual trajectories of shifts of attention are not typical examples in that a different trajectory of changes and inferred shifts of attention might be observed in other children's engagement with the same tasks. The examples analysed here provide a basis for comparison (because the selected children from each study used the same approach) and theoretical generalisation for how observed differences and changes in verbal reports at periods of stable performance can be accounted for, and explained, theoretically.

4 Study 1

4.1 Design of the study

Ten 5–6-year-old children (five girls and five boys) participated in five individual task-based clinical interviews, each of which lasted 35 min (approximately) and took place over consecutive days. The aim was to explore the types of knowledge and strategies that children combine to solve an additive task that involved more than one step and the ways in which children developed their overall approach as they continued their engagement with the task following initial success. Children who were using addition strategies with confidence, and were willing to share their thinking verbally with others, were selected to participate in the study so that pre- and post-success problem solving and verbal behaviour could be observed within a specific number of sessions, as per the microgenetic design. The children's and their parents'/carers' consent was obtained. The individual interviews were video-recorded and took place in a quiet room that was adjacent to the children's class. The interviews were conducted by the first author of this paper and took place during school hours. The children's teacher was not present at the interviews. Before the start of the data collection, the researcher had spent time in the classroom working with different groups of children and assisting them in their work as part of their daily mathematics lesson. This allowed the children to become familiar with the researcher's presence, and it also allowed the researcher to know the children and build a rapport with them.

In each session, the children engaged with solving an additive task that required them to find all possible, additive, two number combinations that result in a specific number, the 'target number'. The number combinations were referred to as 'number bonds', a term that is most often used in English schools when referring to such additive whole-number combinations. At the time when the study was conducted, children had worked in the classroom on working out and recalling different, separate number bonds to 10 but had never worked on a task that required individual children to think of, and produce, all possible numbers bonds for a number such as 10 (or any other such number). The target numbers used in the study ranged between 6 and 19 and increased gradually over the course of the five sessions. Each child was presented with a pile of cards, such as illustrated in Fig. 1, and tasked to write the first addend (to go in the square) and the second addend (to go in the triangle) to complete the number sentence in all possible ways. The formatting of the cards was familiar to children because they had used them in the classroom for their work on number bonds to 10.

During the task-based interview, the researcher asked questions designed to prompt children to describe how they completed each number bond and the task, for example "How did you work that out?" and "Why did you choose these numbers?".

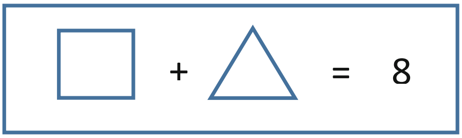

Fig. 1 Example of cards used for the additive task

4.2 Qualitative differences and changes in verbal reports

The trajectories of qualitative changes of two children, Leo and Elsa (pseudonyms), are presented here. Initially, both children approached the production of each number combination as a separate step in the overall solution process. Both combined different strategies and types of knowledge to solve the task such as recall of facts from memory, counting and their knowledge of addition principles (e.g., commutativity). As the children continued working on the task, they started separating their overall approach into two parts. A first set of numbers bonds was produced by following an order (either ascending or descending) to identify the first and second addend of each new number combination and a second set of number bonds was produced by "swapping"; that is, changing around the position of addends in number bonds that had been already produced in the first part of the solution. Both children reached a phase of stability in their overall approach to the task early on (Leo in session-2 and Elsa at the end of session-1) and did not introduce subsequent changes to their overall approach to the task. Tables 1 and 2 present the episodes where points of qualitative changes were observed in the reports that Leo (L) and Elsa (E) provided in response to the researcher's (R) questions over the five sessions. This is followed by an analysis of changes observed in both children's trajectories through the theory of shifts of attention. In Tables 1 and 2, the number bonds that are presented in brackets e.g., [4 + 5]) are the number bonds that the children produced in writing.

In session-1, Leo combined his knowledge of addition facts, counting and addition concepts (in this case knowledge of commutativity in addition when applying swapping) to produce number bonds. In session-2, his verbal reports indicated that he had a "way" for solving the task and that way allowed him to anticipate subsequent solution steps. His verbal reports of his strategy at this point involved the verbal repetition of the written number bonds. Leo applied the same overall approach from session-2 onwards. In one of the trials in session-3, he referred to his way of solving the task by saying "I am going downwards". This was the first time that he specified the order in which he was producing number bonds by describing the order of numbers used as first addends. His utterance with a question "Upwards?" when asked about the order that the second addends followed indicated that he quite possibly only noticed or defined the kind of order that the second addends were following when prompted by the researcher's question. He did not explain why the numbers followed that order and did not provide an explanation of the point of shift to the second part of the solution process. Similar verbal behaviour was repeated in session-4. In session-5, Leo's approach and verbal reports when producing additive combinations for 100 showed that he was primarily focused on the mechanism of putting first and second numbers in an ascending and descending order. The insertion of the add sign at the end seems to indicate a rather automatic approach to producing lines of ascending and descending numbers.

Similarly to Leo, in session-1, Elsa (Table 2) combined her knowledge of addition facts, counting and addition concepts (i.e. commutativity, when applying swapping) to produce number bonds.

When encountering target number 14 in session-1, Elsa applied and verbally reported the approach of producing number bonds "in order" for the first time. She referred to and pointed

Table 1 Study 1—extracts from Leo's verbal reports

Session 1–Target 8	Session 2–Target 9	Session 3–Target 11	Session 4–Target 12	Session 5–Target 14
[4+4] [0+8] [2+6] [1+7] [3+5] **L:** (After the first two) I knew it. **L:** (After 2+6) I was counting in my head. [8+0] [5+3] [6+2] [7+1] **L:** I change them around. (Leo keeps trying to find another number bond. After a while the researcher lets him know that the task has been completed.)	[0+9] [1+8] [2+7] [3+6] [4+5] **L:** I know it (reporting on 0+9). [6+3] [5+4] [7+2] [8+1] [9+0] (Produces next four combinations in two steps: writes down first addend and then counts on to find the second addend. He did not provide an explanation for the specification of first addend. Produces second set rapidly and reports 'swapping'.) **Target 8** [8+0] [1+7] [2+6] **R:** Why did you start with that one (shows 8 + 0)? **L:** Because… I am doing the first one like 10 and 0, or 8 and 0. ..... I… I'm going like… this way. 1 and 7, 2 and 6, 3 and 5 (shows below the 2 as first addend and the 6 as second addend correspondingly.) **L:** (looks at second addends) 7, 6, (continues): 5, 4… 3, 2 and then 1. **R:** And how does this line go? (shows first addends). **L:** 1, 2, 3, 4, 5, 6, 7. **R:** Why do you prefer doing it like that? **L:** It's so quicker... I know.	[0+11] [10+1] [9+2] [8+3] [7+4] [6+5] **L:** (After writing down 9+2) Now I'm going downwards 0 and 11, 10 and 1, 9 and 2. [1+10] [5+6] [4+7] [3+8] [2+9] [11+0] (Keeps looking at the cards. Reports the use of 'swapping' but does not explain why he shifts to swapping at this point.) **R:** Which number bonds did you find by going "downwards"? **L:** This and this and this (shows first addend in first six number bonds). **R:** How do these go? (shows second addends from 1 up to 5). **L:** Upwards? **R:** How come this line goes down and this line goes up? **L:** (Thinks for a while) I don't know.	[12+0] [11+1] [10+2] [9+3] [8+4] [7+5] [6+6] **L:** I am going downwards. **R:** What is going downwards? **L:** 12, 11, 10, 9… **R:** How about the second number in each one, how do you know what the second number should be? **L:** It goes…. upwards? **R:** Why is that? **L:** I don't know. [5+7] [4+8] [3+9] [2+10] [1+11] [0+12]	[14+0] [13+1] [12+2] [11+3] [10+4] [9+5] [8+6] [7+7] **L:** I am doing it like going downwards. I'll change them around now. **R:** Why are you starting changing around now? **L:** Because I can't think of any more different numbers. **Target 100** (The researcher writes the first combination on a blank page.) [0+100] 1 2 3 (Leo writes numbers 1 to 3 underneath 0). (Then he writes three numbers underneath 100.) 1 99 2 98 3 97 **L:** Uhm…I am going downwards. (Writes 96, 95 underneath 97 then 4, 5 underneath 3). 1 + 99 2 + 98 3 + 97 (Leo then inserts the add sign in-between).

separately at each of the columns of first and second addends when saying "going that way and the others going that way". In session-2, Elsa's verbal reports became more explicit when she explained that for the number combination [4 + 7], 4 was chosen because it is "after 3" (showing the first addend of the previous number bond [3 + 8]) and 7 had to be the second added so that she "could go down". The use of the words "after" and "down" indicated the

Table 2 Study 1—extracts from Elsa's verbal reports

Session 1–Target 6	Session 2–Target 11	Session 3–Target 16	Session 4–Target 17	Session 5–Target 100
[3+3] [1+5] [2+4] [0+6] [6+0] [5+1] [4+2] **El:** I know one, 3 and 3. (Counts on with fingers for next two combinations) **El:** I didn't see a zero. **R:** How did you think of 6? Did you count? **El:** No. Because I know I'm thinking of it. 0 and 6 makes 6. I think of it and then I write it down. ……. **El:** I've swapped them around… cause it's… the same numbers (explains last three combinations). **Target 14** [1+13] [2+12] [3+11] **El:** Well, this one is… this one is easy (shows the 3+11) and I put in… all the numbers in order (shows first addends) going that way, and the others going that way (shows second addends top to bottom).	[1+10] [2+9] [3+8] [4+7] [5+6] [10+1] [9+2] [8+3] [7+4] [6+5] [11+0] [0+11] (Explains for 4+7) **El:** Because that's after 3 (shows 3 as first addend). **R:** I saw you looking at the cards …before writing 7 down. **El:** Yes, so that I could… go down (shows second addends top to bottom). **R:** I see. **El:** That one (shows first addends top to bottom) is going in order but that one is just going down (shows second numbers top to bottom). And it's going 10, 9, 8, 7. **R:** How come this happens? **El:** (….)	[0+16] [1+15] [2+14] [3+13] [4+12] [5+11] [6+10] [7+9] [8+8] [16+0] [15+1] [14+2] [13+3] [12+4] [11+5] [10+6] [9+7] **El:** I look at the… card before the other card and I try to find which number is next (shows below the 4 as first addend) and what's going less (shows below the 12 as second addend). **R:** So, for which line of numbers are you trying to find which number is 'next'? **El:** This line (shows first addends). **R:** For which line are you trying to find numbers that are 'less'? **El:** This line (shows column of second addends). It doesn't go next (shows first addends) and next (shows second addends) or less (shows first addends) and less (shows second addends), they are not going the same. **R:** Why don't they go the same? **El:** (…) **R:** Would it be correct if they were both going 'next' or if they were both going 'less'? **El:** (nods 'no'). **R:** Why not? **El:** (looking at the cards) Because when you are writing a sum out, ehm… if you put them in… order, they will go less and more and… if you put them the wrong way around… ehm… they will be…. They won't be the same but they'll be…(does not complete her sentence).	**El:** Uhm… I just do it in order and it goes up and down. [17+0] [16+1] [15+2] [14+3] They are not going the same way (shows with her pencil top to bottom) but they are not going in the same… in the same… direction. It goes up and down (moves pencil vertically first on the left then on the right side of the table), or down and up (moves pencil vertically first on the left then on the right side of the table), and then I do the changing around.	**R:** What if I write the first one, [100+0]. **El:** It's going 1… [… + 1] and… 99 up there? **R:** How did you work it out? **El:** I think the number after 0 and then that's going to go… less… here (shows underneath 100]. **R:** Why it has to be less? **El:** Because…it's going down. Cause this … goes up and this… goes less it's like that. **R:** Why is it like that? **El:** To make 100. The interviewer asks Elsa to find the missing number. [9 + 11 = 20] [7 + … = 20] **El:** (Looks at the numbers and says): 12, 13. It's 13. **R:** How did you find it? **El:** Because they are going in two (shows the 9 and 7 as first addends).

ascending and descending order that she followed for deriving new number bonds with reference to the previous step of the solution process.

While previously-used words such as “next” and “after” denoted the order-based selection of first addends, the change to the phrase “going less” in session-3 is considered as a significant point of change because for the first time, Elsa indicated an arithmetic, quantity-based relationship between two numbers involved in consecutive number combinations and steps of the solution process. Within the same trial in session-3, Elsa explained: “….if you put them in order…. they will go less and more”. The use of the words “less and more” in her attempt to explain the differing order of numbers indicated that Elsa connected the descending and ascending order with a decreasing and increasing quantity-based relationship between numbers.

In session-4, Elsa’s verbal reports included her observation that the first and second addends could go “up and down or down and up”, indicating flexibility in strategy use and verbalised knowledge that a combination of either ascending/descending or descending/ascending number order would work. In session-5, Elsa justified why the numbers go “next” and “less” by explaining that they have “To make 100”, thus referring to the need to conserve the overall number. The most notable point in that session was that Elsa quantified the relationship between addends of successive number bonds in specific terms (“Because they are going in two”). Her explanation indicates explicit, verbalised understanding that the relationship between the first addends of the two number bonds defines the relationship between the second addends of the two number combinations.

4.3 Shifts in the object and structure of attention—study 1

In Table 3, we present an analytical account of the above reported trajectories of change in children’s reports with a focus on shifts related to the object and structure of Leo’s and Elsa’s attention when reporting on their solving approach. The table is structured in two vertical sections (one for each child) that present the focus of each child’s attention for each part of the task solution across the episodes presented earlier and the researchers’ inferences about shifts of attention that occur, with supporting examples of verbal data.

Verbal reports related to the second part of both children’s solution remained unchanged and indicated that children’s focus of attention remained on the reverse positioning of addends. Reports related to the second part of children solution suggested that there were no shifts in the structure of children’s attention either, in that their reports indicated consistently that they reasoned on the basis of a previously known concept (additive commutativity) when reporting the use of swapping for the production of number bonds.

When reporting on the first part of their solution, the ordered sequence of numbers appeared to remain the object of children’s attention throughout the phase of stable strategy use. However, notable differences and shifts were observed in children’s structure of attention, that is, in *how* they were attending to the produced sequence of number bonds. It is noteworthy that, in analysing children’s verbal reports, it was very difficult to recognise the two structures of discerning details and recognising relationships as distinct moments and ways of attending. This is because, in pointing out the details that children were discerning, there was also indication that they recognised a pattern and relationship between successive number bonds, at a perhaps implicit level, initially. For this reason, in Table 3, the structure of discerning details is not presented as distinct from some awareness of the underlying pattern and relationship. Subsequently, both children noted explicitly in their reports the existence of a pattern in the sequence of produced number bonds. The instance where both children indicated that they recognised the underlying

Table 3 Shifts in the object and structure of attention—study 1

Leo	**Focus of attention** (What is attended to?)	**Structure and Micro qualities of attention** (How is one attending?)	Elsa	**Focus of attention** (What is attended to?)	**Structure and Micro qualities of attention** (How is one attending?)
	Session-2 (Emergence and consistent use of approach)			**Session-1** (Emergence and consistent use of approach)	
	Part-1 of solution Ordered pattern of first and second addends.	**Part-1 of solution (Trgt 8)** *Discerning Details and showing awareness of underlying relationship* Notes the first number-bond in solution and emerging pattern **without characterising** it. (*"I am doing the first one like 10 and 0, or 8 and 0", "I'm going like… this way."*) *Recognising Relationships* Between: successive number bonds **by repeating** the observed sequence **without characterising** it. (*"I'm going like this way. 1 and 7, 2 and 6, 3 and 5."*)		**Part-1 of solution** Ordered pattern of first and second addends.	**Part-1 of solution (Trgt 14)** *Discerning Details and showing awareness of underlying relationship* Notes pattern of numbers **without characterising** it. (*"that way"*) *Recognising Relationships* Between: first and second addends **by characterising** the pattern. (*"I put in… all the numbers in order."*)
	Part-2 of solution Reverse position of addends.	**Part-2 of solution** *Reasoning on basis of agreed properties* Reports 'swapping'. Use of commutativity.		**Part-2 of solution** Reverse position of addends.	**Part-2 of solution** *Reasoning on basis of agreed properties* Reports 'swapping'. Use of commutativity.
	Sessions-3 & 4			**Session-2**	
	Part-1 of solution Ordered pattern of first and second addends.	**Part-1 of solution** *Perceiving Properties* Characterises relationship between second addends with **order-based terms** (*"downwards"*). Characterises relationship between first addends with **order-based terms** (*"upwards"*) after prompt. Reports and applies with bigger target numbers.		**Part-1 of solution** Ordered pattern of first and second addends.	**Part-1 of solution** *Perceiving Properties* Characterises relationship between first and second addends with **order-based terms** (*"after" / "before"*). Reports and applies with bigger target numbers.
	Part-2 of solution Reverse position of addends	**Part-2 of solution** *Reasoning on basis of agreed properties* Reports 'swapping'. Use of commutativity.		**Part-2 of solution** Reverse position of addends	**Part-2 of solution** *Reasoning on basis of agreed properties* Reports 'swapping'. Use of commutativity.

Table 3 (continued)

Focus of attention (What is attended to?)	**Structure and Micro qualities of attention** (How is one attending?)	**Focus of attention** (What is attended to?)	**Structure and Micro qualities of attention** (How is one attending?)
	Session-5		**Session-3**
Part-1 of solution Ordered pattern of first and second addends	**Part-1 of solution** *Perceiving properties* Characterises relationship with **order-based terms** (*"downwards"*). Reports his **general rule** with bigger target number, reversing the order.	**Part-1 of solution** Order of first and second addends	**Part-1 of solution** *Perceiving properties* Characterises relationship with **quantity-based terms** (*"less" / "more"*). Refers to **general rule** that relates ordering with observed quantitative-based relationship of less and more. (*"…if you put them in order they will go less and more…"*)
Part-2 of solution. Reverse position of addends.	**Part-2 of solution** *Reasoning on basis of agreed properties* Reports 'swapping'. Use of commutativity.	**Part-2 of solution**. Reverse position of addends	**Part-2 of solution** *Reasoning on basis of agreed properties* Reports 'swapping'. Use of commutativity.
			Sessions-4 & 5
		Part-1 of solution Order of first and second addends Focus on conserving target number	**Part-1 of solution** *Reasoning on basis of agreed properties* **Quantifies relationship** between numbers of successive number bonds in specific terms. (*"going in two"*) Reports and applies general deduced rule with new, bigger target numbers **and steps bigger than one to conserve target.**

relationship between numbers differed in that Leo indicated his awareness of the relationship by repeating the produced number bonds and predicting the next number bond of the sequence, while Elsa characterised the observed pattern with the phrase "all the numbers in order". Children started applying and reporting the use of the ordered pattern in subsequent trials with larger target numbers. This is considered as an indication that, for both children, the "recognised relationship" had become a "perceived property" that was applied as a generalised rule and was referred to consistently when the children reported how they were solving the task.

Nonetheless, children's reports revealed notable qualitative differences within the structure of perceiving properties. Leo characterised the relationship between successive number bonds using solely order-based terms, while Elsa shifted from an order-based characterisation of the pattern to a quantity-based characterisation of the underlying relationship of numbers. She explicitly linked the order of the number bonds with the underlying quantity-based relations and expressed this as a general rule. The above observations are considered as indication that the same structure of attention, perceiving properties, was qualitatively different in the case of the two children. In Elsa's trajectory of changes, her application and report of quantity-based relationships were further extended to steps bigger than 1 between addends, in a situation that slightly differed than the usual task. This is considered as an indication of reasoning on the basis of agreed properties, whereby Elsa only used a previously deduced property for further reasoning.

5 Study 2

5.1 Design of the study

Nine 9–10-year-old children (three girls and six boys) participated in a sequence of eight individual task-based interviews that took place over a 6-week period and involved them tackling eight partitive-quotient word problems. The children, generally, engaged with two tasks per week depending on their availability due to school commitments. The maximum duration of interviews was 30 min. The study aimed at examining the strategies that children, who had only been taught the part-whole fraction sub-construct, used for finding the fraction associated with solving partitive quotient problems, a novel type of problem for them. Therefore, children's previous formal learning of fractions in the classroom (based on information drawn from the curriculum materials used and a discussion with the teacher) constituted a key criterion for selecting the class from which the sample would be drawn. Children who were more likely to verbalise their thinking when working on problems were selected. The children's and their parents'/carers' consent was obtained. The interviews were conducted by the second author of this paper. They took place in the school library and were video-recorded. As in study 1, prior to the data collection, the researcher had spent time in the classroom in order for the children to become familiar with her presence and for the researcher to build a positive relationship with them.

The partitive quotient is exemplified by the solving of problems associated with sharing *a* number of continuous items among *b* people. Tasks were adapted from previous empirical work (Charles & Nason, 2000; Streefland, 1991). Table 4 shows a task exemplar and the number of items and people that each of the eight tasks (T01–T08) involved.

Table 4 Task exemplar from study 2

Share 2 cakes among 3 children so that each child gets the same amount of cake and no cake is left over? (First solution).

(i) How much cake would each child get?

(ii) After the first solution has been given, how else can you share the same 2 cakes among the 3 children? (Subsequent solution(s))

Tasks	T01	T02	T03	T04	T05	T06	T07	T08
Number of Items	2	4	3	4	2	3	2	3
Number of people	3	3	5	6	7	6	5	8

Informed by previous research showing that children are able to work more easily with the bar model represented by a rectangle, as compared to the circular model (e.g., Keijzer & Terwel, 2001), the rectangular region model was selected to represent cakes and pizzas in the tasks. The researcher prompted children to report on their strategies by asking questions such as "How did you figure that out?" and "How do you know?" After providing a solution to the problem, children were prompted to think of other ways for sharing the items. To elicit different solutions, the researcher asked the question: "How else can you share [number of items] among [number of people]?".

5.2 Qualitative differences and changes in verbal reports

The trajectories of qualitative changes of two children, David and Mary (pseudonyms), are presented here. Both children settled in the use of a solving approach that allowed them to work out the fraction that showed the amount of slices that each person would get when all the cakes or pizzas were shared fairly among the people. David introduced and applied the same approach across the eight tasks with no further changes of strategy. Mary initially adopted a mix of approaches for identifying a fraction or number of partitions that might work in task-1 (T01) and task-2 (T02). The approach that she settled on emerged when she tried to offer a second possible solution to T02, as a result of an observation that she made in relation to how her attempted solution-2 related to her solution-1. For this reason, the analysis of Mary's trajectory of qualitative changes and phase of stable strategy use starts with her attempt in working out an alternative solution for T02. Tables 5 and 6 present the episodes where changes in David's and Mary's verbal reports were noted.

Table 5 Study 2—extracts from David's verbal reports

T01: SHARE 2 CAKES AMONG 3 PEOPLE	T02: SHARE 4 CAKES AMONG 3 PEOPLE	T03: SHARE 3 CAKES AMONG 5 PEOPLE	T04: SHARE 4 CAKES AMONG 6 PEOPLE
Solution-1 **D:** (Looks at diagrams) Wait! Actually, since there are three children and two cakes I'm going to separate the cakes into thirds, cause there are chi–, three children. (Partitions each diagram into three).  And so then... make the thirds. So then, each person would get ... so would be two out of six. So, two sixths [2/6] **Solution-2** **D:** Well you can also put it into sixths, into sixths and ninths as well. **R:** Why are you choosing those numbers? **D:** Because there, it's really multiples of three and since there's three children and I got to separate them into three and each kid got three, get, got equal amount of cake, um, it should do the same for multiples, other multiples of three. [4/12]	**Solution-1** **D:** Ok. Like the last session we had, am, like yesterday, uhm, we have three children but this time we have four cakes. So right, like yesterday I would... separate them into thirds? Separate into thirds.  So since, there I separated 4 cakes into 3, 4 times 3 I guess would beee 12 and I'm just gonna check again. (Counts partitions silently). Yep. So it would be four twelfths. **Solution-2** **D:** So I guess then, like I said yesterday we would do it in multiples of 3 so it would work like 3, 6, 9, 12 and so on. **R:** Could you show me then how you would do this? **D:** Well, since I did 3, 6 and 9 yesterday let's see if I can try it with twelves. [16/48]	**Solution-1** **D:** Ok. So this time there are five children. So that means since, I, last time I shared it into threes, into thirds for the three children, now I'm gonna do it into fifths for the five children. [3/15]  **Solution-2** **D:** You can also do it with any other multiple of 5. So example, 10^ths^, 15^ths^, 20^ths^ and so on and so forth. You can even share it into 100^ths^, a hundred cakes, a 100 pieces of cake if you like. **D:** So, if you would be let's say 10 it would be 30 because 10 times three would be 30. And then we count how much each child has gotten. … And I counted and so then for the 10^ths^ now, each of them would get two pieces. Because you cut the cake into a half, am, another, into halves of five. So, like I guess it would be 10^ths^? You cut it into 10^ths^ so each of them should get two pieces of cake. And so, since I- so two times three it would be six. So, they get six pieces of cake each, from each cake. And so, since 10 times three is 30 it would be six 30^ths^.	**Solution-1** **D:** So right now since there are six children this time, then now I'm going to separate it into six so each child gets one piece from one cake. So, I'm just gonna do it in my head. I'm not really going to use the diagram right now. [4/24] **Alternative solutions:** [8/48, 12/72] **T08: SHARE 3 PIZZAS AMONG 8 PEOPLE** **Solution-1** **D:** I'm just going to share it into eights because there are eight children. [3/24] **Solution-2** **D:** Well, I was thinking of doing it into halves, but then I would need 4, 4 pizzas. Sooo, I can't use half really because then I would need another pizza as well. I would also need another pizza. So a solution I guess we'll just use another multiple of 8. So, then I will use 16.

Table 6 Study 2—extracts from Mary's verbal reports

T02: SHARE 4 CAKES AMONG 3 PEOPLE	T03: SHARE 3 CAKES AMONG 5 PEOPLE
M: One and a third might work (partitions the fourth diagram into three and leaves the other three diagrams unpartitioned). One, two, three (counts each of three partitions in fourth diagram) and three (touches first diagram). One (points on first diagram) and one-third (shades first partition in fourth diagram and writes '1' above shaded partition), that's one child. One (points on second diagram) and a third (shades second partition in fourth diagram and writes '2' above shaded partition) that's two children. One (points on third diagram) and a third, (shades third partition in fourth diagram and writes '3' above shaded partition) three children. [1 ⅓] **Solution 2** **M:** Twelve could go... but I wonder... I don't think it would work because I don't, really know how to share four cakes into the number I'm thinking of. Okay. Four cakes. Three children. One, two, three; one, two, three; one, two, three; one, two, three (partitions each diagram into three). But if I did that I would have the same thing as one and a third.	**M**: I think I could try fifteen (referring to the total number of partitions for all three diagrams). Because three fives a fifteen. So each cake would have five. (Writes 3/15 to show each child would get 3 out of 15, in total, slices). **R:** How else can we share the three cakes among the five children? **M:** (Long pause). I think thirty. **R:** Tell me why you chose thirty. **M:** Because since fifteen can work, fifteen times two is thirty. **R:** So if you were to explain to somebody how to share three cakes among five children what would you say to them? **M:** I think I would tell them that they would have to try to find the first number you would, you would get when you multiply three by five because you share, you are sharing three cakes among five children so first you have to find out what number that you can actually share it in with the two numbers you have. 'Cause three fives a fifteen.

T05: SHARE 2 CAKES AMONG 7 PEOPLE	T06: SHARE 3 PIZZAS AMONG 6 PEOPLE	T07: SHARE 2 PIZZAS AMONG 5 PEOPLE
M: … two cakes into fourteen pieces and each child would get two pieces of cake. [2/14] **R:** Why did you chose fourteen? **M:** Because two times seven is fourteen, so I chose it. **R:** How else can you share the two cakes among seven children? **M:** Well I'd just have to add fourteen [Total # of pieces in first solution] or seven ... so I think I would have to share the two pieces of cake into twenty eight and each child would get four pieces of cake.	**M:** [3/18] I got the fraction by multiplying three by six, which is eighteen. **R:** How else can you share the same three pizzas among the family of six? **M:** Each child would get six out of, each person would get six out of thirty six slices of pizza. **R:** …how you got that six out of thirty six? **M:** Well, eighteen plus eighteen is thirty six and still six three's am, six sixes are thirty six so all I had to do was, ... am, … all I had to do was just multiply each number by two so I got six over thirty six.	**M**: Each child would get two out of ten slices [2/10] The reason why I chose ten was because two fives a ten. **R:** How else can we share the two pizzas among the five children? **M:** Well I got my answer. I can share, give each child four slices of, of twenty slices of pizza. [Writes 4/20]. And the reason why I chose twenty was because two tens are twenty and two fours are four. I mean two twos are four. So four over twenty would work good because twenty, you would find it in the five times table and in the ten times table.

From task-1 (T01) onwards, David's approach was to identify the number of pieces that *each* item should be partitioned into, based on the number of people involved in the problem. The fraction that he subsequently provided as solution was the fraction that represented the number of slices that each person would get out of the total number of slices, that is, out of *all the cake* (e.g., 2/6 in T01). The denominator corresponded to the total number of slices that resulted from multiplying the number of objects to be shared by the number of partitions made in each object. David ascertained each person's share from the total number of slices by counting slices, thus identifying the numerator of the fraction. In T01, he justified his solution-2 by stating that an alternative number of slices per cake should be a multiple of 3 because there were three people in the problem. For T02, David's report made reference to specific examples of multiples that could be used. A notable change was observed in his verbal reports for T03 where, compared to T02, he extrapolated subsequent solutions to beyond "near" multiples ("So example 10ths, 15ths, 20ths and so on and so forth. You can even share it into 100ths") to what a 9-year-old child would consider to be *n*. In that task, David did not only explain the number of partitions and resulting denominator of his fraction but also provided an explicit account, in the form of a rule, for how he was working out both the numerator and denominator of the fraction. David explained that, for alternative fractions/solutions where the denominator is a multiple of the denominator of the first solution, one needs to identify the numerator by working out how many slices each person would get out of each cake and then multiplying that number by the number of objects.

Between T04 and T07, David's verbal reports indicated a step away from drawing partitions in the diagrams. A notable change in his reports in T08 involved his attempt to provide an alternative solution by considering the use of a factor rather than a multiple of the number of people in the task as a possible number of partitions for each object. He quickly rejected this thought. This is notable because David moved beyond just explaining why his rule worked to explaining why other numbers would not work for the partitioning. Table 6 presents the episodes where changes in Mary's verbal reports were noted.

In task-1 (T01) and for the first solution in task-2 (T02), Mary adopted a trial-and-error approach for identifying a possible fraction. In T02 (just before stability), Mary responded to the prompt for an alternative solution by shifting her attention from trying to think of a possible fraction first to thinking of the total number of slices that all of the cake could be cut into. In the process of checking whether 12 slices would work, her utterances indicated that she made the following observations. First, that 12 could work because there are "Four cakes, Three children", and second, that 12 slices would lead to the same fraction that had worked for her first solution ("I would have the same thing as one and a third"). It is considered that these two observations and connections that Mary made between the numbers in the task, the number of total slices and the fraction, shaped the approach that she adopted consistently from that point on.

For the first solution to T03, Mary's thinking directly focused on what should be the total number of slices and she identified that number first. She justified her response on the basis that 15 was the result of the multiplication "Number of cakes × Number of slices per cake" ("Because three fives a fifteen. So each cake would have five"). She then identified each person's share as 3/15. Further to this very brief verbalisation ("So each cake would have five"), at the end of the same task, Mary appeared to generalise this for the first solution as follows: "Total number of pieces = Number of cakes × Number of people", when reporting how she would explain to someone else how they could solve the problem. She therefore appeared to move beyond the problem-specific observation that each cake is cut into 5, in expressing a generalised approach ("you have to find out what number that you can actually share it in with the two numbers you have"). It is considered that noticing the relationship between the numbers involved in the task during the previous session underpinned Mary's reports in T03. At the end of that task (T03), Mary offered a justification for why her alternative solution works with reference to her knowledge of multiplication facts in addition to further explicating how she would explain her solution to someone else. For the alternative solution in T05, Mary suggested that the number of people sharing (7) could be added to the first solution of 14 to give an alternative number of total slices that would work. Mary's thinking at that point is not entirely clear. One possible interpretation could be that when referring to seven she was trying to make a connection with the multiplication fact $3 \times 7 = 21$ (i.e. $3 \times$ Number of people = Number of slices that could be shared equally), having previously provided as first solution 14 slices which results from the multiplication: $2 \times$ Number of people.

From T06 onwards, Mary returned to the explanation provided in T03 for solution-2 (i.e. multiply solution-1 by 2) but, for the first time, she explicitly referred to creating another, equivalent fraction as an alternative solution (6/36). Her utterance "six sixes are thirty six" (T06) could be seen as another indication that she was making connections between the numbers involved in the task and numbers emerging in her solutions, based on her knowledge of times tables. For T07, Mary justified her approach explicitly on the basis of her knowledge of the times tables.

5.3 Shifts in the object and structure of attention—study 2

In Table 7, we present an analytical account of the above reported trajectory of changes in children's reports with a focus on shifts related to the object and structure of David's and Mary's attention when reporting on their solving approach. The table captures the phase that started with the emergence of the approach that each child used consistently and refers to the episodes where changes in verbal reports were captured within this phase.

David's initial object of attention for solution-1 in each task was the number of people in the problem which, as he reported, determined the number of slices in *each* object. Determining the total number of slices was underpinned by recognising the relationship between the number of slices per object and the number of objects. Each person's fair share was determined by counting the partitions drawn in corresponding diagrams. On the other hand, Mary's focus of attention was on the number of people as well as the number of objects in the task. From the point when she recognised the relationship between these and the resulting *total* number of slices for her attempted solution-2/T02, her focus of attention centred on these three numbers. For solution-1/T03, Mary's reports indicated a shift of attention to perceiving properties as she moved from recognising relationships between elements in a particular task where the number of objects was more than the number of people to looking for elements in a new task (where the number of people was more than the number of objects) that fitted a previously recognised relationship. For solution-1/T03, she reasoned and reported on the basis of the multiplicative relationship, to determine the total number of slices directly, without reference to the diagrams. For solution-2/T03, Mary attended by perceiving properties when reporting the use of multiples as alternative possible numbers for the total number of slices that *all* the objects could be partitioned into. Similarly, David shifted from recognising relationships to perceiving properties when reporting the use of multiples as alternative possible numbers for the total number of slices that *each* of the objects could be partitioned into (solution-2/T02).

Table 7 Shifts in the object and structure of attention—study 2

David Focus of attention (What is attended to?)	**David** Structure and Micro qualities of attention (How is one attending?)	**Mary** Focus of attention (What is attended to?)	**Mary** Structure and Micro qualities of attention (How is one attending?)
T01		**T02**	
Solution-1 Number of people. Number of slices per object.	**Solution-1** *Discerning Details* Number of people = Number of slices per cake (*"Wait! Actually since there are three children…"*) *Recognising Relationships* Between: number of people - number of slices per cake - fraction showing size of partitions in each cake - fraction showing fair share of slices**.** (*"I'm going to separate the cakes into thirds, cause there are chi–, three children", "would be two out of six."*)	**Attempt for solution 2** Number of objects. Number of people. Total number of slices.	**Attempt for solution-2** *Discerning Details* Total number of slices for all cake = Number of objects × Number of people. (*"Twelve could go", "Four cakes. Three children."*) *Recognising Relationships* Between: number of people - number of objects - total number of slices. The total number of slices results to the same fraction as the fraction provided for solution-1. (*"But if I did that I would have the same thing as one and a third."*)
Solution-2 Number of slices per object.	**Solution-2** *Recognising Relationships* Between: number of people - number of slices Solution-1 and Solution-2. (*"Because there…it's really multiples of three and since there's three children."*)		
T02		**T03**	
Solution-1 Number of people. Number of slices per object.	**Solution-1** *Discerning Details and Recognising Relationships* Between: number of people in T01 and number of people in T02. (*"Like yesterday, uhm… we have three children but this time we have four cakes."*)	**Solution-1** Number of people. Number of objects. Total number of slices.	**Solution-1** *Perceiving Properties* Looks for elements that fit recognised relationship. (*"I think I could try fifteen. Because three fives a fifteen."*)
Solution-2 Number of slices per object.	**Solution-2** *Perceiving properties* Refers to **multiples** for alternative total number of slices. (*"Well, since I did 3, 6 and 9 yesterday let's see if I can try it with... Twelves."*)	**Solution-2** Alternative total number of slices.	**Solution-2** *Perceiving Properties* Refers to a **multiple** for alternative total number of slices. (*"I think thirty", "Because since fifteen can work, fifteen times two is thirty."*)

Table 7 (continued)

Focus of attention (What is attended to?)	**Structure and Micro qualities of attention** (How is one attending?)	**Focus of attention** (What is attended to?)	**Structure and Micro qualities of attention** (How is one attending?)
T03		**T05**	
Solution-1 Number of people. Number of slices per object.	**Solution-1** *Perceiving properties* Looks for elements that fit recognised relationships. (*"…last time I shared it into threes, into thirds for the three children, now I'm gonna do it into fifths for the five children.*")	**Solution-1** Number of objects. Total number of slices.	**Solution-1** *Perceiving Properties* Provides as answer the total number of slices and resulting fraction. **Justification based on multiplicative relation** between numbers. (*"Because two times seven is fourteen, so I chose it."*)
Solution-2 Number of slices per object.	**Solution-2** *Reasoning on Basis of Agreed Properties* Refers to general rule by extending beyond near multiples. (*"You can also do it with any other multiple of 5. So example, 10^{ths}, 15^{ths}, 20^{ths} and so on and so forth. You can even share it into 100^{ths}.*")	**Solution-2** Alternative total number of slices.	**Solution-2** *Perceiving Properties* Reports alternative solution of adding lots of 7 (number of people). Also perceives this as using multiples. Moves away from always doubling the number of slices in solution-1. (*"I'd just have to add fourteen [Total # of pieces in first solution] or seven."*)
T04-08		**T06-07**	
Solution-1 Number of people. Number of slices per object.	**Solution-1/T04** *Perceiving properties* Looks for elements that fit recognised relationships. (*"I think I could try fifteen…Because three fives a fifteen."*)	**Solution-1** Number of people. Number of objects. Resulting total number of slices.	**Solution-1** *Perceiving Properties* Justification based on **multiplicative relation** between numbers. (*"I got the fraction by multiplying three by six, which is eighteen."*)
Solution-2 Number of slices per object.	**Solution-2/T04** *Reasoning on Basis of Agreed Properties* **Generalisation of use of multiples for denominator.** ("*Well, you can use it- in any other multiple of six.*")	**Solution-2** Equivalent fractions.	**Solution-2** *Reasoning on Basis of Agreed Properties* **Justification based on equivalent fractions** based on solution-1. Explicit reference to knowledge of times tables. (*"…all I had to do was just multiply each number by two so I got six over thirty six."*) – T06. (*"…twenty would work good because twenty, you would find it in the five times table and in the ten times table.*") – T07.
Solution 2 Number of partitions	**Solution-2/T08** *Perceiving properties* Recognises elements that fit and also do not fit possible relationships. (*"I can't use half really because then I would need another pizza as well."*)		

A notable shift in David's structure of attention occurred in T03 when, in reporting his approach for solution-2, he extended his response to beyond near multiples to what a 9-year-old child would consider to be *n*. This is considered as a shift to reasoning on the basis of agreed properties whereby David expressed and used a general rule, suggesting reasoning, on the basis of deduced properties. An interesting shift from reasoning on the basis of agreed properties to perceiving properties was observed in solution-2/T08 where David deviated from his general rule and considered the use of a factor of the number of slices per object as part of an alternative solution. In T05, Mary appeared to attend by perceiving properties when she directly determined and justified the denominator of the fraction on the basis of the multiplicative relationship between number of objects and number of people. She moved away from always doubling the denominator for an alternative solution to exploring other possible multiples that would fit the recognised relationships. A notable shift occurred for solution-2/T06 onwards, where Mary's reports suggested reasoning with direct reference to relationships between equivalent fractions when multiplying the numerator and denominator of the first solution by 2 in order to identify an alternative solution.

The following three points are noteworthy. Firstly, both children appeared to shift from recognising relationships to perceiving properties despite the fact that recognised relationships in elements of the task were not fruitful or were not used in a fruitful way to lead to a correct fraction solution to the question of *how much* cake each person would get. Children's prior knowledge and understanding of the concept of fractions may have had a bearing on this. Secondly, both children gradually indicated shifts between perceiving properties to reasoning on the basis of agreed properties when reporting different aspects of their approach to the tasks. Thirdly, while both children extended and extrapolated their thinking about the task, they did so in a different way in terms of the kind of generalised, deduced properties that they used in further reasoning. On the one hand, David's attention stayed fixed on the number of people in the task and corresponding partitions per object. Therefore, he extrapolated his thinking and moved to reasoning on the basis of deduced properties related to the use of multiples, as pertained to one step of his solving approach. On the other hand, Mary's reasoning on the basis of agreed properties is considered to be different in that she reasoned on the basis of known concepts that were more distant from the information provided in the task when she reported for solution-2 reasoning solely on the basis of fractions equivalent to the fraction obtained as part of solution-1.

6 Discussion

Our analysis of the verbal reports of children, who developed the same approach to the task and applied it continuously over a number of sessions, shows that children changed and developed their own verbal reports and conceptualisation of the same solving approach over a number of sessions. On the basis of our analysis, we argue that differences between individuals and changes within individual trajectories can be accounted for by differences and shifts in the object and/or structure of children's attention.

Mason, Stephens and Watson (2009) argue that children's appreciation of relation or structure in mathematics may be underpinned by "different awarenesses" in that "the way they describe what they are doing sometimes suggests not only what they are attending to, but different ways in which they are attending, whether to the particular, or through the particular to the general, or at the particular through the general" (p. 21). Through analysing qualitative data from children of different ages who engaged with very different types of task, the present paper supports the above position by highlighting that differences observed in the verbal reports that individual children provide indicate differences in *what* children attend to and/or *how* they attend, when reporting on their own solutions. While learners' object of attention may be the same, the way in which they are attending (that is, the structure of their attention) may be different, and, vice versa, learners may apply the same structure of attention when their focus and object of attention differs. This paper further extends this point by showing that changes are observed in the way in which the same child attends to the same task and strategy when they are given the opportunity to continue working on the same task after having provided an initial solution.

In line with the point made by Mason (2008) that trying "to distinguish between explicit and implicit awareness is fraught with difficulty" (p. 40) and that "someone may be attending to something in a particular way but unaware explicitly of the what and the how" (p. 42), the analysis here does not aim to provide a clear-cut distinction or capture *all* structures of attention that learners may have applied. Also, we do not propose that children's focus of attention on a particular element or relationship emerged at the point when the verbalisation occurred. Our aim has been to illustrate different objects and structures of explicit attention that interrelate and through which individual children view and gradually change the way they view, communicate and conceptualise their solving approach. The analysis is based on children's verbalised (i.e. explicit) awareness (or awarenesses) that became gradually more explicit during the course of engaging with the tasks. On this basis, we argue that such qualitative changes in shifts of attention in how one views the same strategy and a known task are demonstrations of learning that occurs either through extrapolation of previous thinking or the explicitation of previously implicit conceptualizations that occur during phases where learners' performance remains stable. Therefore, a child's stable performance or consistent delivery of the same type of answer (correct or incorrect) does not necessarily mean that learning is not growing. The qualitative approach adopted here uncovers examples of such hidden growth that can be highly instructive for educators and their assessment of children's learning. Notwithstanding the potential that qualitative data in microgenetic research offer, exploration of differences and changes in children's attention based on analysis of verbal utterances and observation of overt behaviour is accompanied by limitations in that it offers a basis of inferring shifts of attention in an indirect way. Future research could potentially endeavour to capture shifts of attention, as a cognitive mechanism that underlies phases of stable task performance, in a more direct way, perhaps through the use of experimental methods which were beyond the scope of the two studies reported here.

Threlfall (2002) posits that strategies emerge from interaction between what is being noticed about the numbers in a task and the individual's knowledge and previous experience. He therefore maintains that "each solution 'method' is in a sense unique to that case" (p. 42). The qualitative differences and changes identified in this paper highlight the unique individual way in which children not only develop new strategies but also view and communicate already-applied strategies during a period of apparent stable performance. On this basis, we propose that phases of stable strategy use and, by extension, the path of change, can be qualitatively different between individuals, across qualitative dimensions that are beyond strategy choice and use. The implication of this for the theory and research on change is that a model of the path of strategy change needs to expand beyond changes related to strategy type that have emerged from quantitative microgenetic explorations of change. A model of change needs also to account for a nuanced depiction of the path of change when referring to "qualitative distinct understandings" (Siegler, 1995, p. 228), because these may exist and develop within the same strategy use and not only when individuals move between different strategy types. This is significant for capturing and developing an all-encompassing understanding of the notion of change in learning.

In study 1, there were notable qualitative differences in how children applied the same structure of attention because of qualitative differences in the kind of relationships and properties that were recognised and perceived, that is, in the kind of relationships between discerned details that different children might "see" and recognise (order-based relationships vs quantitative-based relationships between numbers). In study 2, children's object of attention differed and from this children discerned slightly different details. Nevertheless, both children shifted (at different points) through the same structures of attention when they recognised relationships and from these perceived properties and, at points, extended their thinking with reasoning on the basis of agreed properties. In study 2 as well, notable differences in how children applied the same structure of attention were observed, in that the "properties" that David deduced, generalised and reasoned upon pertained to one step of his solving approach (abstracted rule of using any multiple of number of people to ascertain alternative number of partitions) while Mary's reports suggested a shift to reasoning solely on the basis of a known concept (equivalent fractions), independent from information provided in the task. Such qualitative differences suggest that different "microqualities" may pertain to the same structure of attention as the relationships or properties the children recognise or generalise and abstract concepts they employ may be of different quality, sophistication or degree of "independence from particular objects" (Mason, 2008) and are influenced by children's prior knowledge.

Mason (2008) uses the notion of "microquality" as a synonym of the notion of "structure" when referring to *how* one attends. On the basis of the observation that learners may attend to relationships or properies in qualitatively different ways, we propose an adaption and the assignment of a distinct meaning to the two aforementioned notions, whereby the term "microqualities of structures of attention" is used to denote the qualitatively different ways in which learners may apply the same structure of attention. The idea that different microqualities may pertain to the same structure of attention across individuals and awareness of underlying microqualities that underlie how different learners perceive a mathematical object or experience is significant for educational practice that aims at honing the students' attention for the advancement of their mathematics learning.

In arguing about the interplay between knowledge and performance, Sophian (1997) maintains that "performance is intrinsically interactive, and therefore dynamic" (p. 292). She proposes that "performance can change children's knowledge as well as that knowledge shapes performance" and describes "a dynamic system in which change is a natural consequence of the children's interaction with the world" (Sophian, 1997, p. 292). While the examination of any causal links between knowledge and performance was beyond the scope of the analysis presented here, viewing our data through the lens of Sophian's aforementioned argument, and Sfard's (2008) position (mentioned earlier in this paper) that changes in how students communicate about mathematics imply changes in their thinking, provides the basis for querying how the process of change in relation to performance is predominantly conceptualised. In light of our observations, we propose that seemingly unchanged performance can also be viewed as a "dynamic" notion that can be underlain by processes of change in how children think about the same task and the same strategy when they are prompted to explain their own solutions in an interactive context. Undoubtedly, further work is needed to evaluate this point and to disentangle what factors might impact on the kind of individual qualitative differences discussed here and how.

Moreover, further work is needed to ascertain the conditions and contexts of interaction that may trigger and support such underlying processes of qualitative change during phases of stable performance. Children participating in the two studies were prompted to report on their own solving approaches, including alternative solutions to the task. As Koichu and Harel (2007, p. 352) note, a clinical task-based interview is "a situation in which an interviewer and a subject interact on a task." Therefore, the inferred shifts of children's attention were changes in students' ways of attending in interaction with the interviewer and in response to the interviewer's prompts. While the sensitivity of the observed behaviours to the particular situational aspects may pose limitations for the present study, we contend that the situated nature of children's shifts of attention provides fruitful avenues for further exploration. Siegler (1995, p. 265) maintained that "people can learn more than usual if they are induced to think about the task more deeply than usual" and, in writing about mathematical abstraction, Mason (1989, p. 7) highlighted the importance that the role of the teacher has for explicitly helping students to "draw back from detail to shift into a more reflective contemplation of what they are doing". Future research could examine whether and how the aforementioned conditions and requirement for verbally reporting and explaining one's own solution and alternative solutions can have measurable effects on learning.

7 Conclusion

In this paper, we contribute to empirical research by providing evidence of between- and within-children variabilities and changes observed in verbal reports within phases of seemingly stable strategy use in arithmetic tasks. We argue that to address the challenge of describing *how* change occurs, and to examine the nature of transition phases and mechanisms that may include progression, regression and stability (Granott & Parziale, 2002), it is necessary that research captures the qualitative variations and advances in thinking that occur, as these render apparent phases of stability in problem solving performance qualitatively different and unique to the individual's path of change. In this endeavour, analysis of microgenetic qualitative data can offer key insights and a nuanced understanding of qualitative dimensions of change that is fundamental for fully comprehending what instructional conditions are most conducive to change for individual learners.

We have found the theory of shifts of attention to be a powerful analytical lens for capturing and explaining nuanced inter- and intra-individual changes that pertain to children's communication and conceptualisation of solving approaches. In this paper, we have explored and illustrated individual differences in the nature of such shifts that can further advance the theorisation of the notion of 'structures' of attention. The implication of this study for mathematics education is that adopting a conceptualisation of learning as a change of learners' attention in interaction with tasks and others in learning situations and as a process that integrates learners' experience with the act of making sense of their experience (Mason, 2010) necessitates that students are offered the opportunity to continue working on a task after having provided an initial solution. This allows them to reflect on their solving approaches and articulate their thinking. Awareness of hidden, often, qualitative aspects of change during phases of stable performance is an inextricable part of understanding individual learners' trajectories of change. Such understanding can support teachers in making informed pedagogical decisions on how to guide individual learners towards meaningful shifts of attention and avenues for reasoning, within productive classroom environments that allow students to step back from the detail, articulate their thinking and thus make sense of their experiences.

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