UNIVERSITY OF SOUTHAMPTON

FACULTY OF PHYSICAL SCIENCES AND ENGINEERING

Physics and Astronomy

Amplitudes in $\mathcal{N} = 4$ super Yang-Mills

An exploration of kinematical limits

by

Stefan-Gheorghe Druc

Thesis for the degree of Doctor of Philosophy

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ABSTRACT

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In this thesis we explore aspects of scattering amplitudes in planar $\mathcal{N} = 4$ super Yang-Mills. In particular we shall focus on studying the mathematical structure of scattering amplitudes in different kinematical limits. First we use linear combinations of differential operators and the properties of multiple polylogarithms to solve for a differential equation obeyed by a 2-loop, 5-point dual conformal scalar integral in a coplanar kinematical limit. Next we dedicate the bulk of this thesis to planar amplitudes in multi-Regge kinematics (MRK) and we exploit the simplifications due to this limit to completely classify their mathematical structure.

We show that in MRK, the singularity structure of the amplitude corresponds to finite cluster algebras and thus may be described entirely by single-valued multiple polylogarithms. We then present a factorised form for the amplitude expressed as a Fourier-Mellin dispersion integral and proceed to derive novel results at leading logarithmic accuracy (LLA) for both MHV and non-MHV configurations. Specifically we show that amplitudes at L loops are determined by amplitudes with $L + 4$ legs and classify their leading singularities in MRK. Next we go beyond LLA by using 2-loop, 7-point data to extract corrections to the BFKL central emission vertex which is the only quantity in the dispersion integral not known to all orders. Finally we utilise the corrections to the central emission vertex to conjecture a finite coupling expression and thus extend the dispersion integral for amplitudes in MRK to all orders as well as all multiplicities and helicity configurations.

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Declaration of Authorship

I, Stefan-Gheorghe Druc, declare that the thesis entitled *Amplitudes in $\mathcal{N} = 4$ super Yang-Mills* and the work presented in the thesis are both my own, and have been generated by me as the result of my own original research. I confirm that:

- this work was done wholly or mainly while in candidature for a research degree at this University;
- where any part of this thesis has previously been submitted for a degree or any other qualification at this University or any other institution, this has been clearly stated;
- where I have consulted the published work of others, this is always clearly attributed;
- where I have quoted form the work of others, the source is always given. With the exception of such quotations, this thesis is entirely my own work;
- I have acknowledged all main sources of help;
- where the thesis is based on work done by myself jointly with others, I have made clear exactly what was done by others and what I have contributed myself;
- parts of this work have been published as:
  - V. Del Duca, S. Druc, J. Drummond, C. Duhr, F. Dulat, R. Marzucca, G. Papathanasiou, and B. Verbeek, *Multi-Regge kinematics and the moduli space of Riemann spheres with marked points*, JHEP 08 (2016) 152, [arXiv:1606.0880].
  - V. Del Duca, S. Druc, J. Drummond, C. Duhr, F. Dulat, R. Marzucca, G. Papathanasiou, and B. Verbeek, *The seven-gluon amplitude in multi-Regge kinematics beyond leading logarithmic accuracy*, JHEP 06 (2018) 116, [arXiv:1801.1060].

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CHAPTER 1

Introduction

Our most successful theories regarding the constituents of matter and the way they interact with each other are quantum field theories (QFT). By marrying quantum mechanics with the symmetries of space-time, QFTs have been able to both guide and explain experiments probing matter at ever smaller scales. From the highly accurate predictions of QED to the complexity of the standard model we have made many discoveries regarding the nature of fundamental particles, with the Higgs boson being a most notable recent addition. However for experiments to be able to detect new physical phenomena, it is essential to have an understanding of how likely they are compared to the background processes. This likelihood may be calculated theoretically and is encapsulated in scattering amplitudes, thus making them principal objects of study in any QFT, linking theory with prediction. Indeed, by calculating them we may form scattering cross sections which determine the probabilities of processes at high energy colliders.

In the standard perturbative approach to quantum field theories established in the 1970s, processes are broken up into all possible interactions, and their combinations, allowed by the theory under consideration. Each individual scenario, drawn pictorially as a Feynman diagram, is then treated separately. The procedure is to use the Feynman rules derived from the Lagrangian of the theory and associate to each graph an integral, either over position or momenta. While conceptually straightforward the integrals quickly become notoriously difficult to evaluate and they are often divergent in either the low or high momentum limits known as the IR and UV respectively. To make sense of these infinities we require regularisation and renormalisation which have led to new insights

about the nature of the standard model. Indeed confinement or asymptotic freedom, a major features of QCD, have their origins in the need to treat the divergent nature of scattering amplitudes. Thus it is fair to say that calculation of scattering amplitudes has been a major driving force in the development of our understanding of QFTs.

Although it has taken us far, the Lagrangian approach to scattering amplitudes may not be the best or most efficient way of doing calculations. The first hints of this fact were found in the work of Parke and Taylor [1], where, by using a particular parametrisation, the sum of hundreds of Feynman diagrams was seen to collapse to a single line. The reasons for this remarkable simplification are two fold. Due to locality, our most realistic theories possess non-abelian gauge symmetries, however they are not maintained by the expansion in Feynman diagrams. Secondly the Lagrangian formulation of QFTs forces us to deal with off-shell quantities that simplify significantly when we restrict ourselves to the physical space. Scattering amplitudes are non-local, gauge invariant quantities that are manifestly on shell. Therefore the Feynman approach to scattering amplitudes, while systematic, introduces a lot of complications in the intermediate steps. Another surprising feature that contributes to their simplicity is the fact that amplitudes often possess symmetries that are not evident at the level of the Lagrangian. With these considerations there has been a lot of impetus to make on-shell amplitudes the starting point rather than the final step in a calculation. These on-shell approaches focus on finding the most efficient ways of parametrising the amplitude such that they incorporate its analytic and symmetry properties. Consequently these parametrisations shall be a recurring theme throughout this thesis.

There has been remarkable progress in recent years in both the understanding and calculation of gauge theory scattering amplitudes. There have been results at high multiplicity and new mathematical structures discovered. The acquired mathematical sophistication is largely due to the remarkable properties of $\mathcal{N} = 4$ Super-Yang-Mills (SYM). It is a maximally supersymmetric theory in four dimensions that can be cast as a reduction of a ten dimensional $\mathcal{N} = 1$ super-Yang-Mills. Indeed $\mathcal{N} = 4$ SYM is not only supersymmetric but conformal, with a vanishing Beta function, and thus dependent only on the single renormalisation free t'Hooft coupling λ. All of these features are of great theoretical interest, which is further justified by comprising the CFT in the AdS/CFT correspondence. Unfortunately, for the most part $\mathcal{N} = 4$ SYM is not a realistic theory of the world we see around us, however this need not be an obstacle to progress. It has long been the strategy of the physicist when confronted with a complex problem, such as QCD, to find a similar yet simpler problem and adapt to the former the lessons learned from the latter. In the context of scattering amplitudes the role of the simpler theory is beautifully fulfilled by $\mathcal{N} = 4$ SYM and it rightly deserves the focus we shall accord it in this thesis. Of course simple here should be interpreted more as evidence of the mathematical richness of the theory rather than a lack of features.

In the context of colour ordered planar amplitudes, that shall be of primary interest, $\mathcal{N} = 4$ possesses a dual conformal symmetry [2–4] which closes with the usual superconformal algebra to form an infinite dimensional Yangian algebra [5]. The Yangian provides an infinite set of charges in involution and thus is a strong indication of integrability, and has prompted much work in investigating this property (see [6] for a review). In fact the scaling dimensions of local operators in $\mathcal{N} = 4$ stem from the thermodynamic Bethe ansatz and have been solved exactly. The origin of this unexpected integrability in a four dimensional theory lies in its duality with a type IIB string theory living on $AdS_5 \times S_5$. The latter is prone to integrability by virtue of being a non linear sigma model on a symmetric coset space, then the CFT inherits this property via the AdS/CFT correspondence. At the QFT level the dual conformal invariance is broken by IR divergences, however this happens in a controlled manner and we may restrict our attention to a natural IR finite dual conformal invariant part. Thus the analytic structure of scattering amplitudes is constrained and the four point and five point amplitudes are completely fixed, which agrees with an earlier ansatz due to Bern, Dixon and Smirnov (BDS) [7]. This ansatz packaged all information up to five points into a single exponential form containing the one loop amplitudes and the cusp anomalous dimension (known exactly from integrability). The BDS ansatz also contains the divergent part of all amplitudes and is usually factored out to leave finite functions of the cross ratios known as remainder functions.

Although linked to integrability, a separate avenue of progress has stemmed from the interplay between kinematics and geometry. Notably the kinematics of the amplitude may be efficiently encoded in terms of momentum twistors, which are points in $\mathbb{CP}^3$. They simultaneously resolve the momentum conservation and null momenta conditions as well as linearise the action of the dual conformal symmetry generators. Furthermore the kinematics may be organised into cluster algebras [8, 9] which describe the space of totally positive matrices whose minors are given by the Plücker coordinates made from the momentum twistors [10]. Evidence suggests that given an initial cluster we may use a process called mutation to generate the singularity structure of amplitudes up to seven points in general kinematics. Once the singularity structure is known the entire functional space of the amplitude may be constructed and then bootstrapped using physical considerations such as soft and collinear limits [11–13]. At eight points and beyond however this approach breaks down, non-cluster letters start to appear and eventually the singularity structure may not be enough to fix the amplitudes. However as we shall see, in certain kinematic regimes it is possible to use finite cluster algebras to determine the analytic structure of amplitudes at all points.

This is of course not the only way to proceed and there has been great progress in a variety of techniques involving twistors. In [14, 15] it was shown that by considering the factorisation of scattering amplitudes in certain limits it was possible to construct all tree

level amplitudes. Crucially this is a completely on-shell method that relies on a deformation of the amplitude and the residue theorem to recursively build amplitudes from products of amplitudes with fewer legs. At loop level this is complicated by the branch but it is still possible to build the integrand by using unitarity methods. While this does not provide one with the amplitude directly it is nevertheless highly desirable to be able to decompose the amplitude in terms of a general basis of integrals that may already be known. In the so called generalised unitarity method [16–18] we may be able to determine the coefficients in this decomposition by cutting successive propagators. Furthermore, by analysing generalised unitarity in momentum twistor space it was possible to express integrands of amplitudes as the volume of a remarkable generalised polytope, over a Grassmannian, known as the amplituhedron [19–21]. This interesting geometrisation of scattering amplitudes does not obfuscate the Yangian symmetry, because locality and unitarity are no longer external criteria but emergent phenomena.

Through the Yangian, dual conformal symmetry has played a central role in the progress made thus far, remarkably this prominence may be understood geometrically via the AdS/CFT correspondence. By considering the boundary conditions of the saddle point solution to the string scattering equations it is possible to establish a duality between scattering amplitudes in $\mathcal{N} = 4$ SYM and minimal surfaces bound by null polygonal Wilson loops [22–25]. In this context the dual conformal symmetry is just the canonical conformal symmetry of the Wilson loop and is related by T-duality to that of the amplitude. The Wilson loop/amplitude duality has led to the development of some very powerful techniques. Most notably the Wilson loop OPE [26–31] where a collinear limit is taken to decompose the loop in terms of polygons with fewer edges. In this way it is possible to tessellate the polygon with a series of squares or overlapping pentagons. For planar $\mathcal{N} = 4$ SYM it we may interpret the two segments of each square belonging to the original polygon as sourcing a colour-electric flux tube. Integrability manifests itself here as well, as the excitations of this flux tube correspond to an integrable spin chain [28–31]. The OPE is then an expansion in the states propagating across these squares with special transition function taking the state from one square to the other. Although for most of this work we do not make explicit use of the Wilson loop OPE it shall however be a valuable point of comparison.

Instead we focus on the the functional space of the simplest scattering amplitudes, which is expected to be that of iterated integrals over some one forms. These one forms are determined by the configuration space of points in complex projective space $\mathrm{Conf}(\mathbb{CP}^3)$. We shall be working with iterated integrals over rational one forms, otherwise know as multiple polylogarithms (MPLs), which are believed to describe all maximally helicity violating (MHV) and next-to-MHV (NMHV) [21]. MPLs and iterated integrals in general are of deep mathematical interest. They have connections with algebraic geometry and number theory, areas of mathematics which have a remarkably large overlap with the

study of scattering amplitudes. MPLs have a plethora of properties, stemming from their Hopf algebra structure, and obey many functional relations among themselves. Furthermore they admit an algebraic representation called the symbol [32–34], which is in some sense simpler and more unique. The symbol has been used with great success to simplify results that would have taken 17 pages to write [35, 36] and has proven to be a very efficient tool with which to build scattering amplitudes. In particular we know the symbols of all two-loop MHV amplitudes [37] and that of the three-loop seven point amplitude [11]. It has been difficult to go to higher multiplicity however, with reasons including the infinity of the cluster algebra as well as the expected appearance of new classes of functions not expressible as MPLs.

This thesis is organised as follows. Section 2 is not original and is intended as a brief overview of some of the more relevant topics in scattering amplitudes as well as an introduction to some subjects that would be of use later on. In particular we spend some time going through the principles of iterated integrals and their Hopf algebra. In Section 3 we explore what can be learned from applying differential operators to a scalar conformal 2-loop, 5-point integral. We find that in a coplanar limit of the kinematics we may write a differential equation that has a solution in terms of single valued MPLs. This section is based on unpublished work done in collaboration with my supervisor James Drummond. In Section 4 we investigate scattering amplitudes in multi-Regge kinematics (MRK) at leading log approximation (LLA). We show that in MRK the cluster algebra of the kinematics decomposes into two copies of finite type, which can be represented by the Dynkin diagrams for A_n. Thus we are able to describe the amplitude in terms of iterated integrals on the moduli space of Riemann spheres with marked points $\mathfrak{M}_{0,n}$. After proposing a Fourier-Mellin dispersion integral for amplitudes at all loops and points we proceed to investigate how the amplitude factorises. More precisely MHV amplitudes at L loops are completely determined by MHV amplitudes with up to (L+4) external legs. Furthermore by considering convolutions with a helicity flip kernel the results are extended to NMHV. In Section 5 we develop the results of Section 4 beyond LLA. To do so we shall promote the known symbol of the two loop heptagon to a function and fix divergent terms in the integral by considering soft limits. After obtaining the function we use its holomorphic part to completely fix the NLO corrections to the central emission block via a map from the Taylor expansion of MPLs to the space of single valued Fourier-Mellin integrals. Once the corrections are obtained we proceed to generate predictions through a mixture of nested sum algorithms and convolution methods. Finally we obtain further corrections to the central emission vertex and promote them to a Wilson loop OPE inspired all order conjecture that is consistent with symmetries and seven points data. Thus we complete our proposal for a finite coupling dispersion integral valid for all multiplicities and helicity configurations. The work of the preceding two sections was done in collaboration with Vittorio Del Duca, James Drummond, Claude Duhr, Falko Dulat, Robin Marzucca, Georgios Papathanasiou and Bram Verbeek [38, 39].

CHAPTER 2

Overview

2.1 Colour and Spin

Given that our most realistic QFTs are non-abelian gauge theories with gauge group $SU(N)$ it is natural to expect additional structure in the amplitude, besides the kinematical dependence. However, as we shall see, in the planar limit of large rank $(N-1)$ the gauge group dependence or colour structure will become mostly decorative. That is to say given that our fields are Lie algebra valued, we demonstrate that the amplitude can be expressed as a series of single traces of the generators. Furthermore, even if the theories we consider are not entirely made of massless particles it is nevertheless consistent with collider processes to assume a hyper-relativistic limit in which the particles have zero effective mass. Thus the full spin structure can be reduced to unitary representations of the little group that are parametrised by the helicity of the particle. Following [40, 41] we proceed to describe the colour decomposition of amplitudes and the spinor helicity formalism.

2.1.1 Colour decomposition

Colour decomposition techniques for scattering amplitudes were introduced in [42] for open string theory computations. Although the colour gauge group of QCD is $SU(3)$ the colour structure can be readily generalised to the group $SU(N_c)$. We define the Lie

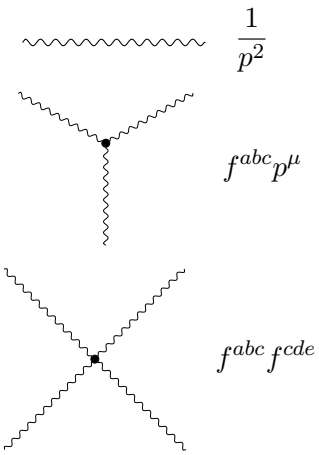


Figure 2.1.1: Yang-Mills Feynman rules

algebra $\mathfrak{su}(N_c)$ of the gauge group as an algebra with the following bilinear operation

$$[T^a, T^b] = i\sqrt{2}f_{abc}T^c \qquad a, b, c = 1 \dots N_c^2 - 1 \tag{2.1.1}$$

where the T^a are called the generators of the algebra and, for $\mathfrak{su}(N_c)$, are represented by traceless hermitian $N_c \times N_c$ matrices. In a non abelian gauge theory and specifically in QCD, the Feynman rules for the pure gauge vertices are expressed in terms of the structure constants f_{abc} (fig. 2.1.1), while the quark-gauge-anti-quark interactions are in terms of the generators. This is due to the fact that the generators have three separate indices $(T^a)^{\bar{i}}_{j}$ for the group, the fundamental and the anti-fundamental. Our aim is to homogenise the Feynman rules by transitioning from the structure constant basis, to the smaller generator basis. To do this we normalise the generators $\mathrm{Tr}\left(T^aT^b\right) = \delta^{ab}$, and combine this with (2.1.1)

$$i\sqrt{2}f^{abc} = \mathrm{Tr}\left(T^aT^bT^c\right) - \mathrm{Tr}\left(T^aT^cT^b\right) \tag{2.1.2}$$

As can be seen from (fig. 2.1.2) this has the effect of turning every pure colour vertex into an internal fermionic loop. Thus for multi gluonic amplitudes the colour factors are products of traces. If there are external fermions present then the amplitude will contain strings of generators with fundamental indices $(T^{a_1} \dots T^{a_m})^{\bar{j}}_{i}$. For diagrams with internal gluon lines there will be terms of the form $\mathrm{Tr}\left(\dots T^a \dots\right) \mathrm{Tr}\left(\dots T^a \dots\right)$. We may treat these terms by using the Fierz identity

$$(T^a)^{\bar{j}_1}_{i_1} (T^a)^{\bar{j}_2}_{i_2} = \delta^{\bar{j}_2}_{i_1}\delta^{\bar{j}_1}_{i_2} - \frac{1}{N_c}\delta^{\bar{j}_1}_{i_1}\delta^{\bar{j}_2}_{i_2} \tag{2.1.3}$$

In terms of the diagram this splits the colour factor of two fermionic lines that are joined by an internal gluon. To prove the above identity we augment the $SU(N_c)$ with a $U(1)$ generator corresponding to a photon. Note that the new generator is proportional to

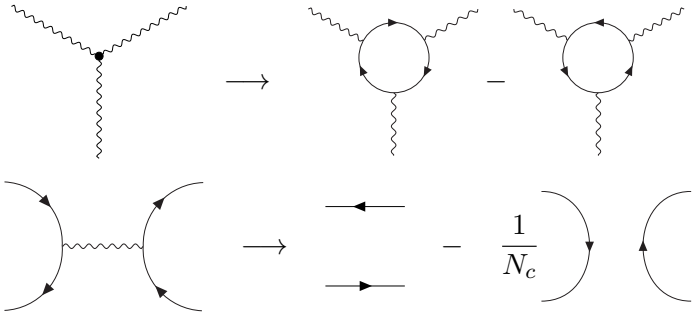


Figure 2.1.2: Colour decomposition

the identity $(T^{a_{U(1)}})_i^{\bar{j}} = \frac{1}{\sqrt{N}}\delta_i^{\bar{j}}$ and thus does not couple to the gluons or alternatively $f^{abc_{U(1)}} = 0$. Taking these generators together they form a basis for all hermitian $N_c \times N_c$ matrices. A canonical basis is given by [43]

$$\left(e^{ij}\right)_{kl} = \delta_k^i \delta_l^j \qquad (i,j,k,l = 1 \dots N_c) \tag{2.1.4}$$

For which we can write a positive definite inner product that implies the completeness relation

$$\mathrm{Tr}[e^{ij}(e^{kl})^\dagger] = \delta^{ik}\delta^{jl} \quad \rightarrow \quad \left(T^A\right)_{i_1}^{\bar{j}_1}\left(T^A\right)_{i_2}^{\bar{j}_2} = \delta_{i_1}^{\bar{j}_2}\delta_{i_2}^{\bar{j}_1} \tag{2.1.5}$$

Where $A = 1 \dots N_c^2$. Moving the $U(1)$ generators to the right hand side of (2.1.5) we recover (2.1.3). After applying all of the above identities to the colour factors of the n-gluon tree amplitude we find that the $1/N_c$ terms drop out and we can write its colour decomposition

$$\mathcal{A}_n^{tree}(\{k_i, \lambda_i, a_i\}) = g^{n-2} \sum_{\sigma \in S_n/Z_n} \mathrm{Tr}\left(T^{a_{\sigma(1)}} \dots T^{a_{\sigma(n)}}\right) A_n^{tree}\left(\sigma(1^{\lambda_1}), \dots, \sigma(n^{\lambda_n})\right) \tag{2.1.6}$$

Here the k_i denote the gluon momenta and λ_i their helicities. Due to the properties of the trace, σ must belong to the coset of all cyclically independent permutations S_n/Z_n. This is in line with the intuition that rotating the amplitude should not affect its properties. The quantities A_n are known as partial amplitudes and given that they have a fixed ordering among the momenta, are simpler than the full amplitude. This restricts the possible singularity structure of the amplitude and it can only have poles in channels of adjacent momenta

$$s_{ij} \equiv (p_i + p_{i+1} + \dots + p_{j-1} + p_j)^2 \tag{2.1.7}$$

A similar procedure is applied to loop amplitudes and up to first order we have single

and double trace terms

$$\mathcal{A}_n^{1-loop}(\{k_i,\lambda_i,a_i\}) = g^n \sum_{\sigma\in S_n/Z_n} \Big\{ N_c \,\mathrm{Tr}\left(T^{a_{\sigma(1)}}\dots T^{a_{\sigma(n)}}\right) A_{n;1}\left(\sigma(1^{\lambda_1}),\dots,\sigma(n^{\lambda_n})\right) + \sum_{i=2}^{\lfloor\frac{n}{2}\rfloor+1} \mathrm{Tr}\left(T^{a_{\sigma(1)}}\dots T^{a_{\sigma(i-1)}}\right)\mathrm{Tr}\left(T^{a_{\sigma(i)}}\dots T^{a_{\sigma(n)}}\right) A_{n;i}\left(\sigma(1^{\lambda_1}),\dots,\sigma(n^{\lambda_n})\right)\Big\} \tag{2.1.8}$$

Where the brackets $\lfloor\cdot\rfloor$ in the limit denote the floor of the enclosed quantity and the $A_{n;i}$ can be constructed from permutations of $A_{n;1}$. In calculating the cross section we need to square the amplitude and average over all the colours, which produces factors of N_c. From (2.1.8) we can see that the dominant powers of N_c will come from gluing amplitudes with the same ordering of the colour indices. Thus in the large N_c limit only planar graphs will contribute to the cross section.

2.1.2 Spinor Helicity Formalism

As mentioned previously when treating spin we can take helicity as a convenient basis for particles in the ultra-relativistic limit. Fermion lines conserve helicity whereas gauge bosons do not. The complexity of the amplitudes can be linked to the degree of helicity violation (fig. 2.1.3), with the simplest amplitudes having the maximal degree (MHV) and the next simplest being next-to-MHV (NMHV) etc. Fortunately we can make use of the parity symmetry (2.1.23) to relate the amplitudes such that we only have $\lfloor\frac{n}{2}\rfloor$ independent helicity configurations to consider. Similarly to the colour ordering case we must re-express our amplitude in terms of quantities with the correct indices. To do so we will have to consider complexified four dimensional Minkowski space $\mathbb{M}_{\mathbb{C}}$.

$$ds^2 = \eta_{ab}dx^a dx^b \qquad x^a \in \mathbb{C}^4 \tag{2.1.9}$$

Note that signature has no meaning in $\mathbb{M}_{\mathbb{C}}$ as that will change depending on what subset of $\mathbb{C}^4$ we restrict ourselves to. Thus we can view complexified Minkowski space as an analytic continuation of all flat spacetimes. Although we are ultimately interested in real spacetimes it can be extremely useful to express scattering amplitudes in terms of complex spinors and twistors.

A momentum vector on $\mathbb{M}_{\mathbb{C}}$ transforms in the $(\frac{1}{2},\frac{1}{2})$ representation of $SL(2,\mathbb{C})\times SL(2,\mathbb{C})$. This is because the Lie algebra of the spin group $\mathfrak{so}(4,\mathbb{C})$ of complexified spacetime is isomorphic to $\mathfrak{sl}(2,\mathbb{C})\times\mathfrak{sl}(2,\mathbb{C})$. Thus we can decompose a spacetime index into two

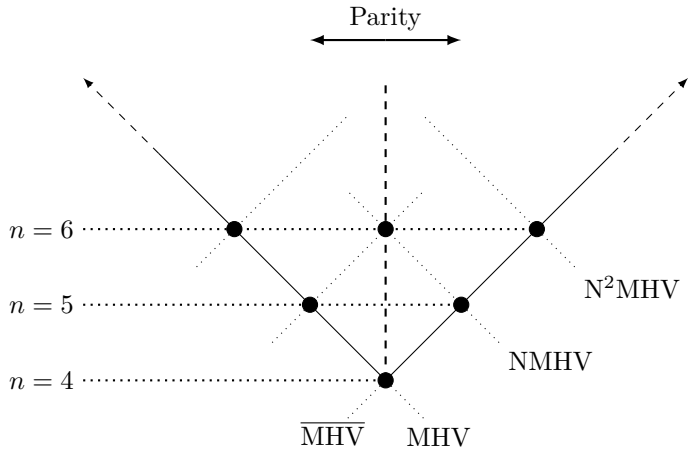


Figure 2.1.3: Helicity classification of amplitudes based on multiplicity. Amplitudes with massless particles and with less than two helicities that are distinct from the rest will vanish

conjugate spinor indices of opposite chirality by using the Pauli matrices $\sigma_\mu^{\alpha\dot{\alpha}} = (\mathbb{I}, \vec{\sigma})$,

$$\sigma_1^{\alpha\dot{\alpha}} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \sigma_2^{\alpha\dot{\alpha}} = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \sigma_3^{\alpha\dot{\alpha}} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{2.1.10}$$

The map to the spinor representation and its inverse is given by

$$p^{\alpha\dot{\alpha}} \equiv p^\mu \sigma_\mu^{\alpha\dot{\alpha}} = \begin{pmatrix} p^0 + p^3 & p^1 - ip^2 \\ p^1 + ip^2 & p^0 - p^3 \end{pmatrix} \qquad p^\mu = \frac{1}{2}\,\mathrm{Tr}(\bar{\sigma}^\mu p) \tag{2.1.11}$$

where $\bar{\sigma}_\mu^{\alpha\dot{\alpha}} = (\mathbb{I}, -\vec{\sigma})$. Note that this map may be easily generalised to any number of indices

$$T^{abc\ldots} \to T^{\alpha\dot{\alpha}\beta\dot{\beta}\gamma\dot{\gamma}\ldots} \tag{2.1.12}$$

From (2.1.11) we can see that the norm of the momentum vector is given by the determinant of the corresponding spinor representation

$$\eta_{\mu\nu} p^\mu p^\nu = \frac{1}{4}\eta_{\mu\nu}\bar{\sigma}^\mu_{\alpha\dot{\alpha}}\bar{\sigma}^\nu_{\beta\dot{\beta}} p^{\alpha\dot{\alpha}} p^{\beta\dot{\beta}} = \frac{1}{2}\varepsilon_{\alpha\beta}\varepsilon_{\dot{\alpha}\dot{\beta}} p^{\alpha\dot{\alpha}} p^{\beta\dot{\beta}} = \det(p^{\alpha\dot{\alpha}}) \tag{2.1.13}$$

Where $\varepsilon_{\alpha\beta} = -\varepsilon^{\alpha\beta}$ is the Levi-Civita tensor and in the second equality we used another formulation of the completeness relation (2.1.5). For massless particles the right hand side of (2.1.13) vanishes and the corresponding matrix is singular. Consequently the null on-shell momenta can be expressed in terms of two spinors of opposite chirality.

$$p^{\alpha\dot{\alpha}} = \lambda^\alpha \tilde{\lambda}^{\dot{\alpha}} \equiv |p\rangle^\alpha [p|^{\dot{\alpha}} \tag{2.1.14}$$

However unlike the usual fermionic spinors which anti-commute these spinors are not Grassmann valued. We use the Levi-Civita tensor to raise and lower indices as well as

contract spinors.

$$|p\rangle_\alpha = \varepsilon_{\alpha\beta}\langle p|^\beta \quad [p|_{\dot\alpha} = \varepsilon_{\dot\alpha\dot\beta}|p]^{\dot\beta}$$

$$\varepsilon_{\alpha\beta}\lambda_i^\alpha\lambda_j^\beta = \langle ij\rangle \quad \varepsilon_{\dot\alpha\dot\beta}\tilde\lambda_i^{\dot\alpha}\tilde\lambda_j^{\dot\beta} = [ij] \tag{2.1.15}$$

We can easily see that the helicity spinors satisfy the massless Weyl equation

$$p^{\alpha\dot\alpha}|p\rangle_\alpha = 0 \quad [p|_{\dot\alpha}p^{\alpha\dot\alpha} = 0 \tag{2.1.16}$$

Where we have used the anti-symmetry of the spinor contraction $\langle ij\rangle = -\langle ji\rangle$. However the helicity spinors for a given momentum are not unique and (2.1.14) will be invariant under the following transformations

$$\lambda^\alpha \to t\lambda^\alpha \quad \tilde\lambda^{\dot\alpha} \to t^{-1}\tilde\lambda^{\dot\alpha} \quad t \in \mathbb{C}^* \tag{2.1.17}$$

Thus we can see that the complex null momenta can be described by a cone over $\mathbb{CP}^1 \times \mathbb{CP}^1$. For real momenta the two spinors are related by complex conjugation and are thus forced to transform under the little group $SO(2) \sim U(1)$ restricting t to a phase

$$e^{i\phi}\tilde\lambda^{\dot\alpha} = \left(e^{-i\phi}\lambda^\alpha\right)^* \tag{2.1.18}$$

And in this case we can give an explicit realisation

$$\lambda^\alpha = \frac{1}{\sqrt{p^0+p^3}}\begin{pmatrix} p^0+p^3 \\ p^1+ip^2\end{pmatrix} \quad \tilde\lambda^{\dot\alpha} = \frac{1}{\sqrt{p^0+p^3}}\begin{pmatrix} p^0+p^3 \\ p^1-ip^2\end{pmatrix} \tag{2.1.19}$$

Next we note two important identities. The first is the Schouten identity which comes from the fact that only two spinors are required to form a basis for $\mathbb{C}^2$

$$\langle 12\rangle\langle 3a\rangle + \langle 23\rangle\langle 1a\rangle + \langle 31\rangle\langle 2a\rangle = 0 \tag{2.1.20}$$

Where λ_a is arbitrary and a similar identity holds for the $\tilde\lambda$. The second identity is a re-expression of the momentum conservation condition for amplitudes

$$\sum_{i=1}^n p_i^{\alpha\dot\alpha} = 0 \quad \Longleftrightarrow \quad \sum_{i=1}^n \langle ai\rangle[ib] = 0 \tag{2.1.21}$$

It would be convenient if the amplitudes we consider did not care about whether we swapped an incoming particle for an outgoing one (crossing -symmetry). Thus we use the convention of taking all particles as outgoing and take the parity conjugate quantum numbers of the incoming particles when writing down the amplitude.

The spinor helicity formalism will prove to be a very natural language in which the amplitude can take a surprisingly simplified form. Intuitively this comes from the converse of (2.1.14), i.e. by choosing the spinors λ and $\tilde{\lambda}$ we automatically construct the null subspace of the kinematics. Thus by using these variables we avoid having to artificially impose the massless on-shell conditions upon our amplitudes. In fact determining the most efficient variables in which to express our on-shell kinematics has been a critical source of progress in investigating the structure of scattering amplitudes.

Here we give some general properties of the colour ordered partial amplitudes that impose powerful constraints and provide simplifications.

Cyclicity:

$$A(1,2,\ldots,n) = A(n,1,\ldots,n-1) \tag{2.1.22}$$

This is a simple consequence of the cyclicity of the trace, and as already stated leads to a simpler analytical structure.

Parity:

$$A(1,2,\ldots,n) = A(\bar{1},\bar{2},\ldots,\bar{n}) \tag{2.1.23}$$

Where we have inverted the helicity of all the legs to find the real amplitude invariant. This demonstrates that the positive and negative helicities are conjugate labels as evidenced by the spinor-helicity formalism.

Charge conjugation:

$$A(1_q,2_{\bar{q}},3\ldots,n) = -A(1_{\bar{q}},2_q,3\ldots,n) \tag{2.1.24}$$

Where we have exchanged the helicity of a quark line. It can be seen from the Feynamn rules that the amplitudes are equivalent up to a minus sign.

Reflection:

$$A^{tree}(1,2,\ldots,n) = (-1)^n A^{tree}(n,n-1,\ldots,1) \tag{2.1.25}$$

This is due to n-point amplitudes having $n+2L-2$ three-point vertices, which are antisymmetric under reflection. For quarks this holds only at tree level.

Photon Decoupling:

$$\sum_{\sigma\in Z_{n-1}} A_n^{tree}(1,\sigma(2,3,\ldots,n)) = 0 \tag{2.1.26}$$

This can seen by putting a $U(1)$ generator into (2.1.6) and collecting terms with equal

colour factors. Since the photon doesn't couple to the gluon the amplitude must vanish and we recover (2.1.26). At loop level this relates planar and non-planar partial amplitudes [44]

Collinear limit:

$$A_n^{(L)}(1,\dots,(n-1)^{h_{n-1}},n^{h_n}) \to \sum_{l=0}^{L}\sum_{h} A_{n-1}^{(L-l)}(1,\dots,k^h)\text{Split}_{-h}^{l}((n-1)^{h_{n-1}},n^{h_n}) \quad (2.1.27)$$

Where L is the number of loops and we take the adjacent momenta of the n and $(n-1)$ particle to be collinear. The l-loop splitting amplitudes are universal functions of the helicities of the collinear particles and the momentum k of the intermediate parton.

Soft limit:

$$A^{tree}(1^+,2,\dots,n) \to \frac{\langle n2\rangle}{\langle n1\rangle\langle 12\rangle}A^{tree}(2,\dots,n) \quad (2.1.28)$$

Where we take the first momentum as soft. Finally, not only are partial amplitudes simpler analytically but they are individually gauge invariant, which means that we may study them as seperate quantities.

2.1.3 A three-point example

As an illustration of the techniques discussed so far we will use the little group scaling to determine a three gluon amplitude. First we note that the amplitude transforms under the scaling (2.1.17)

$$\mathcal{A}_n(\dots,\{t_i\lambda_i,t_i^{-1}\tilde{\lambda}_i,h_i\},\dots) = t_i^{-2h_i}\mathcal{A}_n(\dots,\{\lambda_i,\tilde{\lambda}_i,h_i\},\dots) \quad (2.1.29)$$

A priori the three point amplitude $\mathcal{A}_3(1^{h_1},2^{h_2},3^{h_3})$ could depend on spinors of both chiralities. However by choosing both $a,b=3$ in the conservation of momentum identity (2.1.21), we can infer that one of the following conditions must be true

$$\lambda_3 \propto \lambda_2 \propto \lambda_1 \quad \text{or} \quad \tilde{\lambda}_3 \propto \tilde{\lambda}_2 \propto \tilde{\lambda}_1 \quad (2.1.30)$$

Thus for three-point kinematics either all square or all angle bracket contractions must vanish, and we can express the amplitude solely in terms of one or the other. Note that this is only possible if we take the momenta as complex. We choose to work with angle brackets and proceed by making an ansatz

$$\mathcal{A}_3(1^{h_1},2^{h_2},3^{h_3}) = c\langle 12\rangle^{r_{12}}\langle 13\rangle^{r_{13}}\langle 23\rangle^{r_{23}} \quad (2.1.31)$$

By considering the transformation of the spinors under scalings we obtain a linear system of equations $-2h_1 = r_{12} + r_{13}$ etc. which can be easily solved to find

$$\mathcal{A}_3(1^{h_1}, 2^{h_2}, 3^{h_3}) = c\langle 12\rangle^{h_3-h_2-h_1}\langle 13\rangle^{h_2-h_1-h_3}\langle 23\rangle^{h_1-h_3-h_2} \tag{2.1.32}$$

Similar to the scaling dimension of a CFT, scaling under the little group has been used to completely fix the three particle amplitude. We could have proceded similarly with the square brackets to find two possible solutions for the MHV 3-gluon tree amplitude

$$\mathcal{A}_3(g_1^-, g_2^-, g_3^+) = \begin{cases} g\frac{\langle 12\rangle^3}{\langle 13\rangle\langle 23\rangle} \\ \\ g'\frac{[13][23]}{[12]^3} \end{cases} \tag{2.1.33}$$

However by dimensional analysis the square bracket amplitude would have had to come from a non-local Lagrangian (assuming a Yang-Mills like dynamic term), and we can disregard it. An alternate way of fixing the 3-particle amplitude is by using the helicity operator

$$\hat{h}_i = -\frac{1}{2}\lambda_i^\alpha\frac{\partial}{\partial\lambda_i^\alpha} + \frac{1}{2}\tilde{\lambda}_i^{\dot{\alpha}}\frac{\partial}{\partial\tilde{\lambda}_i^{\dot{\alpha}}} \tag{2.1.34}$$

For which the amplitude is an eigenfunction $\hat{h}_i\mathcal{A} = h_i\mathcal{A}$ and the spinors have helicity $\pm\frac{1}{2}$. Despite their simplicity the three point amplitude are quite important, they are in a sense a fundamental building block. As was shown by BCFW [14, 15], all tree-level amplitudes at arbitrary helicity can be constructed by recursively gluing the three-point amplitudes together. We shall review this process for the supersymmetric case in Section (2.2.2). One may reasonably think that the simple form of (2.1.33) is mainly due to the small number of particles under consideration. It was found early on however [1] and later proved in [45] as well as BCFW that the MHV tree-level amplitudes have a remarkably simple all-point structure.

$$\mathcal{A}_n(\ldots, i^-, \ldots, j^-, \ldots) = \frac{\langle ij\rangle^4}{\langle 12\rangle\ldots\langle n1\rangle} \tag{2.1.35}$$

The degree of simplicity becomes even more striking if we consider the number of Feynman diagrams (for a theory with tri-valent and quad-valent vertices) that contribute to this final form [46]

n	2	3	4	5	6	7
no. diagrams	1	3	10	38	154	654

The reason why we can express the sum of so many diagrams in a single line is easy to guess. When using the path integral formulation we take the momenta to be off-shell and we spoil the gauge invariance by expanding the generating function in terms of the interaction vertices. The simple form of the Parke-Taylor amplitude (2.1.35) engendered

the hope that perhaps these difficult calculations could be avoided.Thus it served as the initial inspiration for the search of efficient on-shell and gauge independent methods that has led to the field of modern scattering amplitudes.

2.2 Amplitudes in $\mathcal{N} = 4$ Super Yang-Mills

So far we have been discussing amplitudes in the context of Yang-Mills or QCD in the hyper-relativistic limit. These theories have proven themselves to be realistic and thus form an integral part of the standard model. The question is then why should we wish to study a theory that is manifestly unrealistic. Despite the simplifications encountered so far at tree level, calculations in QCD remain notoriously complex. In situations like these we turn to toy models for insight. These should be simple enough to solve, but complex enough to learn something about the original problem and ideally should be realistic in some approximation. The theory that surpasses all other in these respects is $\mathcal{N} = 4$ SYM. It is a maximally supersymmetric conformal theory in 4 dimensions that possesses an infinite Yangian symmetry in the planar limit and is believed to be integrable. Physically the theory can be thought of as a supersymmetric analogue of QCD and for purely gluonic amplitudes at tree level the matter content drops out and both theories produce exactly the same result. Furthermore it is thought that planar $\mathcal{N} = 4$ SYM gives the maximally transcendental part of QCD. It is closely linked to the AdS/CFT correspondence and string theory allowing for results at strong coupling and then extended to finite coupling via integrability. Lastly due to its rich mathematical structure it is an effective testing ground for novel techniques and has significant overlap with purely mathematical topics from number theory to algebraic topology. In the following we shall rely on the references [41, 47, 48].

2.2.1 General properties

The 4 dimensional $\mathcal{N} = 4$ SYM is in fact a reduction of $\mathcal{N} = 1$ SYM in 10 dimensions, which is in turn a low energy limit of type I superstring theory. Its action is given in terms of a real vector and a Majorana-Weyl spinor in the adjoint representation.

$$S_{\mathcal{N}=1} = \int d^{10}x \,\mathrm{Tr}\left\{ -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + i\bar{\Psi}\Gamma^{\mu}D_{\mu}\Psi \right\} \tag{2.2.1}$$

Where Γ^{μ} are part of the 10d Clifford algebra, $D_{\mu} = \partial_{\mu} - igA^{\mathrm{ad}}_{\mu}$ is the usual covariant derivative and $F^{\mu\nu}$ is the associated curvature. The reduction of the action to 4 dimensions is performed by compactifying the $x^4, x^5, \ldots, x^9$ directions and discarding massive modes, The vector then is divided into one 4 dimensional gauge potential plus 6 scalars. The spinor is also divided into four Weyl spinors thus extending the supersymmetry to

$\mathcal{N}=4$ and ultimately giving the action

$$S_{\mathcal{N}=4} = \frac{1}{g_{YM}^2}\int d^4x \,\mathrm{Tr}\Big\{ -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - (D_\mu\phi_{AB})(D^\mu\phi^{AB}) - \frac{1}{2}[\phi_{AB},\phi_{CD}][\phi^{AB},\phi^{CD}] \\ + i\bar{\psi}^A\sigma_\mu D^\mu\psi_A - \frac{i}{2}\psi_A[\phi^{AB},\psi_B] - \frac{i}{2}\bar{\psi}_A[\phi_{AB},\bar{\psi}^B]\Big\} \tag{2.2.2}$$

All the fields in (2.2.2) remain in the adjoint since the reduction from (2.2.1) does not affect the gauge indices. The scalar fields ϕ^{AB} are part of the totally antisymmetric representation of the R-Symmetry group $SU(4)$, which is isomorphic to the fundamental of $SO(6)$ (a subgroup of the full $SO(1,9)$ of the 10d action). An immediate feature of (2.2.2) is that there are only two free parameters, namely the coupling and the gauge group which we shall take to be $SU(N)$ in the planar limit. Less evident features are that it has a super conformal symmetry $PSU(2,2|4)$ which does not develop anomalies upon quantisation, and the coupling does not run. Consequently scattering amplitudes are UV finite, however they still suffer from IR divergences. If we introduce the Grassmann variables η^A for $A=1,\ldots,4$ and define the partial derivatives

$$\partial_\alpha \equiv \frac{\partial}{\partial\lambda^\alpha} \qquad \tilde{\partial}_{\dot\alpha} \equiv \frac{\partial}{\partial\tilde{\lambda}^{\dot\alpha}} \qquad \partial_A \equiv \frac{\partial}{\partial\eta^A} \tag{2.2.3}$$

Then the generators of $\mathfrak{psu}(2,2|4)$ are given by, beginning with the standard generators of the super Poincaré sub-algebra

$$\begin{aligned} &p^{\alpha\dot\alpha} = \lambda^\alpha\tilde{\lambda}^{\dot\alpha} \qquad q^{\alpha A} = \lambda^\alpha\eta^A \qquad \tilde{q}^{\dot\alpha}_A = \tilde{\lambda}^{\dot\alpha}\partial_A \\ &m_{\alpha\beta} = \lambda_{(\alpha}\partial_{\beta)} \qquad \bar{m}_{\dot\alpha\dot\beta} = \tilde{\lambda}_{(\dot\alpha}\tilde{\partial}_{\dot\beta)} \qquad r^A_B = \eta^A\partial_B - \frac{1}{4}\delta^A_B\eta^C\partial_C \end{aligned} \tag{2.2.4}$$

The special conformal generators are

$$s_{\alpha A} = \partial_\alpha\partial_A \qquad \bar{s}^A_{\dot\alpha} = \eta^A\tilde{\partial}_{\dot\alpha} \qquad k_{\alpha\dot\alpha} = \partial_\alpha\tilde{\partial}_{\dot\alpha} \tag{2.2.5}$$

Finally the dilatation and central charge generators

$$d = \frac{1}{2}\left[\lambda^\alpha\partial_\alpha + \tilde{\lambda}^{\dot\alpha}\tilde{\partial}_{\dot\alpha} + 1\right] \qquad c = 1 + \frac{1}{2}\left[\lambda^\alpha\partial_\alpha - \tilde{\lambda}^{\dot\alpha}\tilde{\partial}_{\dot\alpha} - \eta^A\partial_A\right] \tag{2.2.6}$$

There are 8 bosonic and 8 fermionic on-shell degrees of freedom that are part of a single PCT self-conjugate multiplet, which can be organised into a single on-shell superfield

$$\Phi(\eta) = G_+ + \eta^A\lambda_A + \frac{1}{2!}\eta^A\eta^B S_{AB} + \frac{1}{3!}\eta^A\eta^B\eta^C\varepsilon_{ABCD}\bar{\lambda}^D + \frac{1}{4!}(\eta)^4 G_- \tag{2.2.7}$$

Where $G_\pm$ are the gluon states of opposite helicity, S_{AB} are the six scalars and $\lambda_A, \bar{\lambda}^D$ are the gluinos. We assign a helicity of $+\frac{1}{2}$ to η and write the supersymmetric analogue

of the helicity generator (2.1.34)

$$\hat{h} = \frac{1}{2}\left[-\lambda^{\alpha}\partial_{\alpha} + \tilde{\lambda}^{\dot{\alpha}}\tilde{\partial}_{\dot{\alpha}} + \eta^{A}\partial_{A}\right] = 1 - c \tag{2.2.8}$$

thus we see that the superfield has a helicity of one ($\hat{h}\Phi = \Phi$). Gien that $\mathcal{N} = 4$ SYM is PCT self-conjugate we can combine all n-particle amplitudes into one super-amplitude

$$\mathcal{A}(\Phi_1, \ldots, \Phi_n) = \sum_{k=0}^{n-4} \mathcal{A}^{N^k MHV}(\{\lambda_i, \tilde{\lambda}_i, h_i\}) \tag{2.2.9}$$

Where we have expanded with respect to the degree of helicity violation. Because the superfield (2.2.7) is just a polynomial expansion in the Grassmann variables with the states as coefficients, we can select a particular amplitude by differentiation (or integration) with respect to η, e.g:

$$A_n\left(S_{12}, S_{34}, 3^{-}, \ldots, n^{+}\right) = \left(\frac{\partial}{\partial\eta_1^1}\frac{\partial}{\partial\eta_1^2}\right)\left(\frac{\partial}{\partial\eta_2^3}\frac{\partial}{\partial\eta_2^4}\right)\left(\prod_{A=1}^{4}\frac{\partial}{\partial\eta_3^A}\right)\mathcal{A}(\Phi_1, \ldots, \Phi_n)\Big|_{\eta_i=0} \tag{2.2.10}$$

Note that the top limit is not the maximal n, which is a consequence of the fact that all the amplitudes with n or $(n-1)$ positive helicities vanish. To see this we note that the amplitude is annihilated by the sum of the super-symmetry generators over all particles (2.2.4)-(2.2.6) i.e.

$$\begin{aligned} Q^{\alpha A} &= \sum_{i=1}^{n} \lambda_i^{\alpha}\eta_i^{A} && \rightarrow && Q^{\alpha A}\mathcal{A}_n = 0 \\ \tilde{Q}^{\dot{\alpha}A} &= \sum_{i=1}^{n} \tilde{\lambda}_i^{\dot{\alpha}}\partial_{iA} && \rightarrow && \tilde{Q}^{\dot{\alpha}A}\mathcal{A}_n = 0 \\ &\vdots && \rightarrow && \vdots \end{aligned} \tag{2.2.11}$$

Where the ellipses denote sums over particles of the remaining super-symmetry generators. These conditions are enough to constrain the general form of the amplitude

$$\mathcal{A}_n(\{\lambda_i, \tilde{\lambda}_i, h_i\}) = \frac{\delta^{(4)}(p)\delta^{(8)}(Q)}{\langle 12\rangle\langle 23\rangle \ldots \langle n1\rangle} P_n(\{\lambda_i, \tilde{\lambda}_i, h_i\}) \tag{2.2.12}$$

Where we have adopted the convention of extracting an overall factor of the MHV amplitude, which carries all the helicity information. thus leaving P_n with vanishing helicity for every particle. The first delta function imposes the usual conservation of momentum and the second is the analogous conservation of supermomentum. For Grassmann variables the delta function is simply

$$\delta(\eta) = \eta \tag{2.2.13}$$

and we can write $\delta^{(8)}(Q)$ as a polynomial in the η_i

$$\delta^{(8)}\left(Q\right)=\frac{1}{2^4}\prod_{A=1}^{4}Q^{\alpha A}Q_{\alpha}^{A}=\frac{1}{2^4}\prod_{A=1}^{4}\sum_{i,j=1}^{n}\langle ij\rangle\eta_i^A\eta_j^A \tag{2.2.14}$$

Because the first term of the function P_n is 1 by construction we see how the conservation of supermomentum requires that the amplitude has a minimum Grassmann degree of eight. Consequently, it must contain at least two pairs of legs along which helicity is conserved. The absence of a $\delta(\tilde{Q})$ in (2.2.12) is due to our choice of representation for the supersymmetry generators $\tilde{q}_A^{\dot{\alpha}}$ as a differential operator, which breaks the symmetry between Q and $\tilde{Q}$. However $\delta(\tilde{Q})$ is not necessary to ensure the supersymmetry constraints (2.2.11) are obeyed. This is due to the fact that if we have overall momentum conservation then

$$\tilde{Q}^{\dot{\alpha}A}\delta^{(8)}(Q)=0 \tag{2.2.15}$$

Of course if we were to consider the parity conjugated amplitude then $\delta(\tilde{Q})$ would be needed to impose supermomentum conservation. Although QCD does not posses the supersymetry described in this section the pure gluon part of $\mathcal{N}=4$ SYM is identical to that of QCD. Because the MHV condition still holds if we restrict ourselves to amplitudes containing only gluons, we can expect the same vanishing amplitudes in QCD. This is indeed the case and is indicative of a hidden symmetry at tree level.

2.2.2 BCFW and tree amplitudes

As mentioned in the section on the three-point amplitude, the entire tree-level structure can be determined by the use of the BCFW recursion relations. Although the general framework can be applied to higher dimensions and massive theories without supersymmetry we shall focus on the setting relevant to $\mathcal{N}=4$ SYM. The essential concept of BCFW is taking complex momenta and studying the effect of deforming them for two of the particles.

$$p_1\rightarrow\hat{p}_1=(\lambda_1-z\lambda_n)\tilde{\lambda}_1\qquad p_n\rightarrow\hat{p}_n=\lambda_n(\tilde{\lambda}_n+z\tilde{\lambda}_1) \tag{2.2.16}$$

Note that this does not spoil the null momentum condition $\hat{p}_1^2=\hat{p}_2^2=0$ and momentum conservation $\hat{p}_1+\hat{p}_n=p_1+p_2$. However if we wish to maintain supermomentum conservation then we must deform the Grassmann variables as well

$$\eta_1\rightarrow\hat{\eta_1}=\eta_1\qquad\eta_n\rightarrow\hat{\eta_n}=\eta_n+z\eta_1 \tag{2.2.17}$$

Thus we establish a one parameter deformation of the amplitude $\mathcal{A}_n(z)$. The tree amplitudes are rational functions of the spinors and we can only have simple poles in the

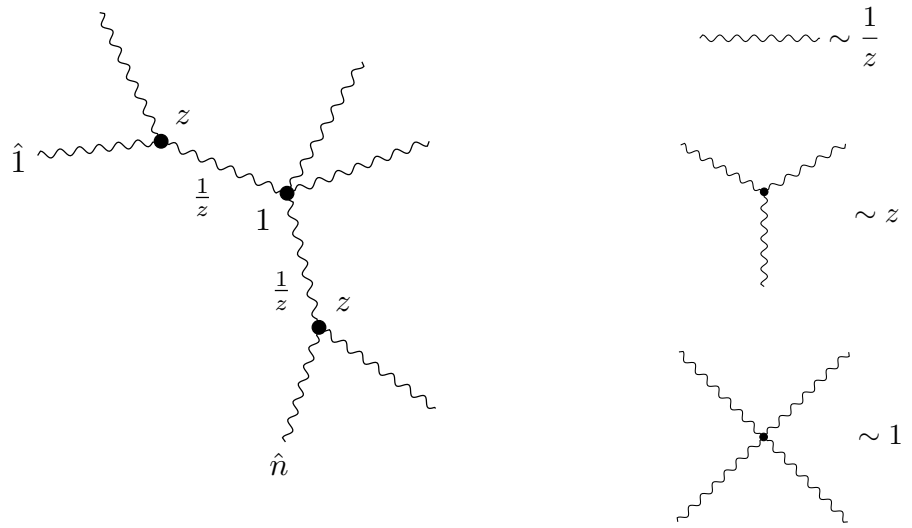


Figure 2.2.1: Maximal z scaling of tree level scattering amplitude with deformed $\hat{1}$ and $\hat{n}$ momenta

parameter z of the form

$$\frac{1}{\hat{P}_i^2} = \frac{1}{(\hat{p}_1 + p_2 + \ldots + p_{i-1})^2} = \frac{1}{P_i^2 - z\langle n|P_i|1]} \tag{2.2.18}$$

If we introduce a pole at $z = 0$ then the non-shifted amplitude will be contained in the following contour integral

$$\oint \frac{\mathcal{A}_n(z)}{z} = \mathcal{A}_n + \sum_i \text{Res}\left(\frac{\mathcal{A}_n(z)}{z}, z_{P_i}\right) = \text{Res}\left(\frac{\mathcal{A}_n(z)}{z}, \infty\right) \tag{2.2.19}$$

Where z_{P_i} is the pole at which $\hat{P}_i^2$ goes on shell.

$$z_{P_i} = \frac{P_i^2}{\langle n|P_i|1]} \tag{2.2.20}$$

At these poles the amplitude factorises into two lower point pieces.

$$\lim_{z \to z_{P_i}} \mathcal{A}_n(z) \sim \frac{1}{z - z_{P_i}} \left(\frac{-1}{\langle n|P_i|1]}\right) \sum_s \mathcal{A}_L^s \left(\hat{1}(z_{P_i}), \ldots, i-1, -\hat{P}(z_{P_i})\right) \times \mathcal{A}_R^{\bar{s}} \left(\hat{P}(z_{P_i}), i, \ldots, i-1, \hat{n}(z_{P_i})\right) \tag{2.2.21}$$

Where we sum over the helicity s of all possible intermediate states. All that is left to consider is the behaviour of the amplitude at infinity, fortunately in $\mathcal{N} = 4$ SYM the residue at infinity vanishes [49]. Without going into too much detail we can motivate this by noticing from the Feynman rules that each three-point vertex is $\mathcal{O}(z)$, each four-point vertex is $\mathcal{O}(z^0)$ and, as we have seen, each internal propagator goes like $\mathcal{O}(z^{-1})$

as $z \to \infty$ (figure 2.2.1). Given that there is always one more vertex than propagators we conclude that the dominant amplitudes are $\mathcal{O}(z)$. Next we include the effects of the momentum dependent polarization vectors

$$\begin{aligned} \epsilon_{1,+} &= \frac{\tilde{\lambda}_1 \mu}{\langle \hat{1} \mu \rangle} \sim \frac{1}{z} & \qquad \epsilon_{n,+} &= \frac{\hat{\tilde{\lambda}}_n \mu}{\langle n \mu \rangle} \sim z \\ \epsilon_{1,-} &= \frac{\hat{\lambda}_1 \tilde{\mu}}{[1 \tilde{\mu}]} \sim z & \qquad \epsilon_{n,-} &= \frac{\lambda_n \tilde{\mu}}{[\hat{n} \tilde{\mu}]} \sim \frac{1}{z} \end{aligned} \tag{2.2.22}$$

where $\mu, \tilde{\mu}$ are arbitrary spinors afforded by gauge symmetry. Ostensibly the results are not too encouraging and depending on the helicities of the shifted momenta the dominant behaviour as $z \to \infty$ ranges from $\mathcal{O}(z^{-1})$ to $\mathcal{O}(z^3)$. However in [50] the background field method was used to show that there are enhanced spin symmetries that further suppress the behaviour of the $(++)$ and $(--)$ cases to $\mathcal{O}(z^{-1})$ rather than $\mathcal{O}(z)$.

$$\mathcal{A}(+-) \sim \frac{1}{z} \quad \mathcal{A}(++) \sim \frac{1}{z} \quad \mathcal{A}(--) \sim \frac{1}{z} \quad \mathcal{A}(-+) \sim z^3 \tag{2.2.23}$$

Finally we can take advantage of supersymmetry to set two of the Grassmann variables to zero and relate the z- dependence of the superamplitude to the $(--)$ component. Thus for $\mathcal{N}=4$ there is no residue at infinity and we can freely relate scattering amplitudes to products of amplitudes with fewer particles. Consequently it is possible to solve the recursion relations and write an expression for the entire tree-level S-matrix [51]

$$\begin{aligned} \mathcal{A}_n^{N^p MHV} = & \int \frac{d^4 \eta_{\hat{P}}}{P^2} \mathcal{A}_3^{\overline{MHV}}(z_P) \mathcal{A}_{n-1}^{N^p MHV}(z_P) \\ & + \sum_{m=0}^{p-1} \sum_i \int \frac{d^4 \eta_{\hat{P}_i}}{P_i^2} \mathcal{A}_i^{N^m MHV}(z_{P_i}) \mathcal{A}_{n-i+2}^{N^{p-m-1} MHV}(z_{P_i}) \end{aligned} \tag{2.2.24}$$

Note that due to there being a single superfield the sum of the states has been replaced by an integral over the Grassmann variables. Moreover we can write down closed form expressions for the amplitudes. For example we give the P_n^{NMHV} term in (2.2.12)

$$P_n^{NMHV} = \sum_{2 \le s < t \le n-1} R_{n:st} \tag{2.2.25}$$

Where the $R_{n:st}$ are descriptively called R-invariants and first appeared in [52]. They may be expressed in the following way

$$R_{n:st} = \frac{\langle s\, s-1 \rangle \langle t\, t-1 \rangle \delta^{(4)}(\langle n | x_{ns} x_{st} | \theta_{tn}^A \rangle + \langle n | x_{nt} x_{ts} | \theta_{sn}^A \rangle)}{x_{st}^2 \langle n | x_{ns} x_{st} | t \rangle \langle n | x_{ns} x_{st} | t-1 \rangle \langle n | x_{nt} x_{ts} | s \rangle \langle n | x_{nt} x_{ts} | s-1 \rangle} \tag{2.2.26}$$

Where we define the change of variables

$$x_{ij}^{\alpha\dot{\alpha}} = \sum_{k=i}^{j-1} p_k^{\alpha\dot{\alpha}}$$
$$\theta_{ij}^{\alpha A} = \sum_{k=i}^{j-1} q_k^{\alpha A} \tag{2.2.27}$$

The higher N^{k}MHV amplitudes can be similarly expressed in terms of nested sums of R-invariants generalised to many pairs of labels $\{s_i, t_i\}$. Note that in expressing the NMHV amplitude we have introduced spurious non-local poles of the type.

$$\frac{1}{\langle n|x_{ns}x_{st}|t\rangle} \tag{2.2.28}$$

Which continue to persist in other helicity configurations. Of course these cancel in the full amplitude as the theory only has local poles, however their appearance is quite natural due to the extra symmetries of the amplitudes described in the next section.

2.2.3 Dual coordinates and symmetries

We begin this section by establishing the dual conformal symmetry, which forms a unique and central feature of scattering amplitudes in $\mathcal{N} = 4$ SYM. Note that we can invert the change of variables (2.2.27) and define the dual coordinates

$$p_i^{\alpha\dot{\alpha}} = x_{i\,i+1}^{\alpha\dot{\alpha}} = x_i^{\alpha\dot{\alpha}} - x_{i+1}^{\alpha\dot{\alpha}} \qquad q_i^{\alpha A} = \theta_{i\,i+1}^{\alpha A} = \theta_i^{\alpha A} - \theta_{i+1}^{\alpha A} \tag{2.2.29}$$

The null-momentum conditions require that adjacent $\{x, \theta\}$ are null separated. The dual coordinates are essentially telescopic solutions to the total (super)momentum conditions. Thus the delta functions in (2.2.12) become

$$\delta^{(4)}(p)\delta^{(8)}(Q) = \delta^{(4)}(x_{n+1} - x_1)\delta^{(8)}(\theta_{n+1} - \theta_1) \tag{2.2.30}$$

Therefore if we make the identification $x_1 \sim x_{n+1}$ ($\theta_1 \sim \theta_{n+1}$) then we can trivialise the (super)momentum conditions. The identification effectively arranges the dual coordinates into the vertices of a polygon with light-like edges (figure 2.2.2). However this is not the entire extent of their usefulness. It is easy to see that Poincaré transformations of the dual coordinates leave the amplitudes invariant and we can treat them as honest coordinates for a dual copy of spacetime. They also furnish their own copy of superconformal symmetry independent from that of the Lagrangian. For example under inversion I:

$$I : x_i^{\alpha\dot{\alpha}} \to (x_i^{\alpha\dot{\alpha}})^{-1} = \frac{x_i^{\alpha\dot{\alpha}}}{x_i^2}$$
$$I : \theta_i^{\alpha A} \to (\theta_i^{\alpha A})^{-1} = (x_i^{-1}\theta_i)^{\dot{\alpha}A} \tag{2.2.31}$$

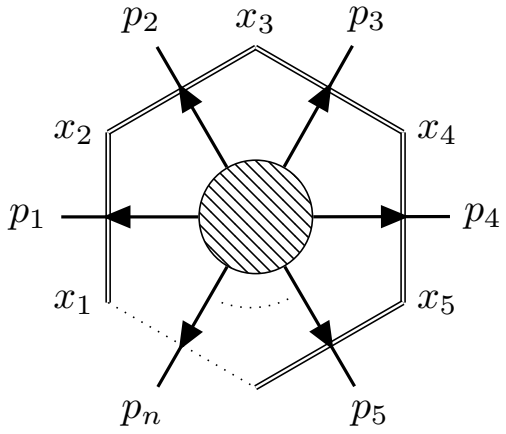


Figure 2.2.2: Dual Coordinates

The dual coordinates are linearly related to the spinor-helicity variables via (2.2.29) and thus they must also transform under the inversion

$$x_{ij}^{\alpha\dot{\alpha}} \to \frac{(x_i(x_i - x_j)x_j)^{\alpha\dot{\alpha}}}{x_i^2 x_j^2} = (x_i^{-1} x_{ij} x_j^{-1}) \tag{2.2.32}$$

Note that the null separation is not spoiled by inversion and defines an automorphism in the space of polygons with light-like edges. By setting $j = i + 1$ we can infer

$$\lambda^\alpha \to (x_i^{-1}\lambda_i)^{\dot{\alpha}} \qquad \tilde{\lambda}_i^{\dot{\alpha}} \to (x_{i+1}^{-1}\tilde{\lambda}_i)^\alpha \tag{2.2.33}$$

However this is not unique and from (2.2.29) we have

$$(x_i - x_{i+1})^{\alpha\dot{\alpha}}\lambda_\alpha = 0 \qquad \to \qquad x_i^{\alpha\dot{\alpha}}\lambda_\alpha = x_{i+1}^{\alpha\dot{\alpha}}\lambda_\alpha \tag{2.2.34}$$

From this we can determine that spinor contractions behave like

$$\langle i\, i+1\rangle \to \langle i|x_i^{-1}x_{i+1}^{-1}|i+1\rangle = \frac{\langle i\, i+1\rangle}{x_i^2} \tag{2.2.35}$$

Thus we if we take the inversion weight of the delta function into account can see that the MHV tree amplitude is covariant under the dual conformal symmetry

$$\mathcal{A}_n^{MHV} \to (x_1^2 \dots x_n^2)\mathcal{A}_n^{MHV} \tag{2.2.36}$$

For NMHV we note that the delta function is also covariant

$$\delta^{(4)}(\langle n|x_{ns}x_{st}|\theta_{tn}^A\rangle + \langle n|x_{nt}x_{ts}|\theta_{sn}^A\rangle) = \delta^{(4)}(\langle n|x_{ns}x_{st}|\theta_t^A\rangle + \langle n|x_{nt}x_{ts}|\theta_s^A\rangle + x_{st}^2\langle n\theta_n^A\rangle) \tag{2.2.37}$$

Together with the previous transformations it is easy to determine that the R-invariants

have zero weight under inversion and thus are dual-conformal invariant as well. The extension to dual superconformal symmetry is not as straightforward, since it leaves $R_{n:st}$ invariant but not its generalisations that appear beyond NMHV. These however manifest themselves only in nested sums which are in combination dual super-conformally invariant [53] as one would expect from the BCFW expansion (2.2.24). Thus we see that the dual symmetry extends to all tree level amplitudes.

The dual conformal generator is given by

$$K_{\alpha\dot{\alpha}} = \sum_i [x_{i\alpha}^{\dot{\beta}} x_{i\dot{\alpha}}^{\beta} \partial_{i\beta\dot{\beta}} + x_{i\dot{\alpha}}^{\beta} \theta_{i\alpha}^{A} \partial_{i\beta A} + x_{i\dot{\alpha}}^{\beta} \lambda_{i\alpha} \partial_{i\beta} + x_{i+1\,\alpha}^{\dot{\beta}} \tilde{\lambda}_{i\dot{\alpha}} \partial_{i\dot{\beta}} + \tilde{\lambda}_{i\dot{\alpha}} \theta_{i+1\alpha}^{A} \partial_{iA}] \quad (2.2.38)$$

this differs from the expected form of the generator due to the restriction that it must commute with the conditions (2.2.29). The rest of the generators for $\mathfrak{psu}(2,2|4)$ can be obtained in a similar fashion [52]. Comparing with (2.2.5) we see that the symmetries are distinctly generated, however the two dilatation operators coincide up to a minus sign due to the linear relationship between the dual coordinates and the momenta. Furthermore if we re-express all the generators from the original variables $\{\lambda, \tilde{\lambda}, \eta\}$ by using (2.2.27) then we find that P and Q become trivial, due to delta functions, while $\bar{S}$, $\bar{Q}$ coincide with $\bar{q}$,$\bar{s}$ respectively.

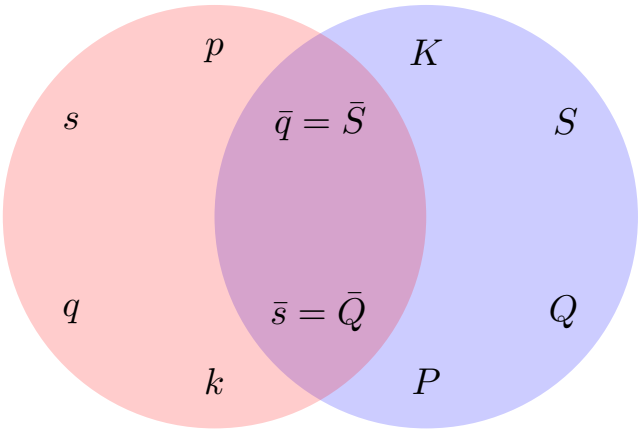


Figure 2.2.3: Overlapping of the two superconformal symmetries

Summarily the dual superconformal symmetry generators have the following action on the superamplitude

$$D\mathcal{A}_n = n\mathcal{A}_n \quad C\mathcal{A}_n = n\mathcal{A}_n \quad K^{\alpha\dot{\alpha}}\mathcal{A}_n = -\sum_i x_i^{\alpha\dot{\alpha}}\mathcal{A}_n \quad S^{\alpha A}\mathcal{A}_n = -\sum_i \theta_i^{\alpha A}\mathcal{A}_n \quad (2.2.39)$$

While all the other generators annihilate the amplitude. Note that we can simply redefine the generators in (2.2.39) so that they also act trivially on the amplitude. Remarkably both symmetries can be elegantly combined into a single Yangian symmetry $Y(\mathfrak{psu}(2,2|4))$. The Yangian $Y(\mathfrak{g})$ of a simple Lie algebra was introduced by Drin-

feld [54, 55] and is the Hopf algebra of the graded deformation of the loop-algebra. Its generators are $J_a^{(n)}$ where $n \in \mathbb{N}$ denotes the level. However it is not necessary to consider all generators as $Y(\mathfrak{g})$ is spanned by $J_a^{(0)}$ and $J_a^{(1)}$ which obey the commutation relations

$$[J_a^{(0)}, J_b^{(0)}] = f_{ab}^c J_c^{(0)} \qquad [J_a^{(0)}, J_b^{(1)}] = f_{ab}^c J_c^{(1)} \tag{2.2.40}$$

Given that our generators have a Grassmann degree we must use the graded general commutator

$$[O_1, O_2] = O_1 O_2 - (-1)^{|O_1||O_2|} O_2 O_1 \tag{2.2.41}$$

The higher order generators can be constructed from commutators of $J_a^{(1)}$ which are in turn constrained by the Serre relations

$$[J_a^{(1)}, [J_b^{(1)}, J_c^{(0)}]] - [J_a^{(0)}, [J_b^{(1)}, J_c^{(1)}]] = g_{abc}{}^{def} \{J_d^{(0)}, J_e^{(0)}, J_f^{(0)}\} \tag{2.2.42}$$

Where $g_{abc}{}^{def} = \frac{1}{24} f_{ai}^d f_{bj}^e f_{ck}^f f^{ijk}$ and $\{J_d^{(0)}, J_e^{(0)}, J_f^{(0)}\}$ is the totally symmetric sum over products of the three terms. For completeness we also quote the action of the coproduct on the generators which is part of the Hopf algebra structure.

$$\Delta(J_a^{(0)}) = J_a^{(0)} \otimes 1 + 1 \otimes J_a^{(0)} \qquad \Delta(J_a^{(1)}) = J_a^{(1)} \otimes 1 + 1 \otimes J_a^{(1)} + \frac{1}{2} f_a^{bc} J_b^{(0)} \otimes J_c^{(0)} \tag{2.2.43}$$

This non-trivial coaction is indicative that the symmetries act non-locally on multi-particle states. We may relate the $J_a^{(0)}$ generators to the standard conformal algebra of $\mathfrak{psu}(2,2|4)$

$$J_a^{(0)} = \sum_{k=1}^{n} j_{ka}^{(0)} \tag{2.2.44}$$

where the $j_{ka}^{(0)}$ are single particle generators. It turns out that the $J^{(1)}$ generators can be explicitly given by

$$J_a^{(1)} = f_a{}^{cb} \sum_{1 \le k < k' \le n} j_{kb}^{(0)} j_{k'c}^{(0)} \tag{2.2.45}$$

Note that the level one generators are manifestly non local. For (2.2.45) to hold in the gluon supermultiplet representation of the Yangian that is of interest for $\mathcal{N}=4$ SYM, it is sufficient to show that dual superconformal symmetry implies that the level one generators annihilate the amplitude. This was demonstrated in [5] by identifying the level one Yangian generator $q^{(1)}{}_\alpha^A$ constructed via (2.2.45) and the dual superconformal symmetry generator S_α^A. We may obtain all other level one generators from $q^{(1)}{}_\alpha^A$ via the commutation relations (2.2.40), thus establishing a new infinite dimensional algebra that annihilates the tree amplitude

$$y\mathcal{A}_n^{tree} = 0 \qquad y \in Y(\mathfrak{psu}(2,2|4)) \tag{2.2.46}$$

Given that the original context of the Yangian are integrable 2-dimensional theories or

discrete spin chain models, its appearance in $\mathcal{N} = 4$ SYM is quite notable. Indeed in the planar limit the single trace operators that appear in the colour decomposed amplitude can be mapped to an integrable spin chain, with the dilatation operator acting as the Hamiltonian. Integrability is also expected at strong coupling as SYM is related to the AdS_5 sigma model which is classically integrable [56]. Beyond tree level the Yangian symmetry described here is broken even for infrared finite quantities. However it is possible to restore exactness by incorporating the anomalies into an overall symmetry [57]. Ultimately the Yangian is a powerful non-local symmetry that can be used to completely determine the S-matrix. Finally we note that its generators are most elegantly expressed in terms of yet another set of variables called momentum twistors $\mathcal{Z}_i^{\mathcal{A}} = (\lambda_i^{\alpha}, \mu_i^{\dot{\alpha}}, \chi_i^{A})$ [58]

$$J^{(0)\mathcal{A}}{}_{\mathcal{B}} = \sum_i \mathcal{Z}_i^{\mathcal{A}} \frac{\partial}{\partial \mathcal{Z}_i^{\mathcal{B}}} \tag{2.2.47}$$

$$J^{(1)\mathcal{A}}{}_{\mathcal{B}} = \sum_{i<j} (-1)^{\mathcal{C}} \left[\mathcal{Z}_i^{\mathcal{A}} \frac{\partial}{\partial \mathcal{Z}_i^{\mathcal{C}}} \mathcal{Z}_j^{\mathcal{C}} \frac{\partial}{\partial \mathcal{Z}_i^{\mathcal{B}}} - (i,j) \right] \tag{2.2.48}$$

Where both expressions are understood to have their supertrace removed. In contrast to (2.2.44) and (2.2.45) the roles of the generators are reversed and the dual-conformal symmetry generators are identified with the level-zero generators of the Yangian. Furthermore we note that these Yangian generators annihilate only the MHV normalised amplitudes

$$JP_n = J^{(1)}P_n = 0 \tag{2.2.49}$$

For completeness we add that there are ordinary twistor variables for which the Yangian generators are not reversed, take a similar form and annihilate the full amplitude. However in the next section we shall focus on momentum twistor variables as not only do they simplify the Yangian but the amplitude in general.

2.2.4 Momentum Twistors

Following [59] we take twistors space $\mathbb{PT}$ as an open subset of the complex projective space $\mathbb{CP}^3$. Exactly which subset depends on which slice of complexified Minkowski space $\mathbb{M}_{\mathbb{C}}$ we want. We may describe twistors via the homogenoeous coordinates

$$Z^A = (Z^1, Z^2, Z^3, Z^4) \neq (0,0,0,0) \quad Z^A \sim rZ^A \quad r \in \mathbb{C}^* \tag{2.2.50}$$

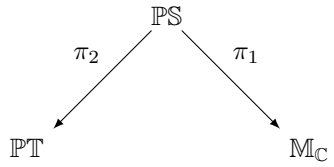
They transform in the fundamental of the complexified conformal group $PGL(4,\mathbb{C}) = SL(4,\mathbb{C})/\mathbb{Z}_4$ and can be divided into two Weyl spinors of opposite chirality.

$$Z^A = (\lambda_\alpha, \mu^{\dot{\alpha}}) \tag{2.2.51}$$

By doing so we can establish a connection between twistor space and space-time via the incidence relation

$$\mu^{\dot{\alpha}} = x^{\alpha\dot{\alpha}}\lambda_\alpha \tag{2.2.52}$$

Which can be represented as a double fibration of the projective spinor bundle $\mathbb{PS}$



Where $\mathbb{PS} \cong \mathbb{M}_{\mathbb{C}} \times \mathbb{CP}^1$ and has coordinates $(x^{\alpha\dot{\alpha}}, \lambda_\beta)$. The projection π_1 just isolates the spacetime point $\pi_1 : (x^{\alpha\dot{\alpha}}, \lambda_\beta) \to x^{\alpha\dot{\alpha}}$ and the second projection implements the incidence relation $\pi_2 : (x^{\alpha\dot{\alpha}}, \lambda_\beta) \to (x^{\alpha\dot{\alpha}}\lambda_\alpha, \lambda_\beta)$. These relations are quite central in the formulation of gauge theories in twistor space. Their usefulness stems from the fact that they relate local points in space-time to a non-local line $\mathbb{CP}^1 \subset \mathbb{CP}^3$. The line is isomorphic to the Riemann sphere and any linear holomorphic embedding of $\mathbb{CP}^1$ in twistor space can be shown to be of the form (2.2.52). Let $\sigma_a = (\sigma_0, \sigma_1)$ denote the homogeneous coordinates of the Riemann sphere. Then we define the map $(r^{\dot{\alpha}a}, s^a_\alpha)$

$$\mu^{\dot{\alpha}} = r^{\dot{\alpha}a}\sigma_a \quad \lambda_\alpha = s^a_\alpha\sigma_a \tag{2.2.53}$$

Naively the map has 8 components however we make use of $SL(2, \mathbb{C})$ and projective scaling to trivialise s^a_α, thus obtaining

$$\mu^{\dot{\alpha}} = r^{\dot{\alpha}a}\sigma_a \quad \lambda_\alpha = \delta^a_\alpha\sigma_a \tag{2.2.54}$$

Furthermore by setting $r^{\dot{\alpha}a} = x^{\alpha\dot{\alpha}}$ we recover the incidence relation. Formulating gauge theories in twistor space allows for solutions to zero rest-mass fields equations via the Penrose transform. For our purposes however, they are a very convenient way to link the dual coordinates and the helicity spinors such that both the null-momentum condition and conservation of momentum are trivialised. To see this we consider the point in twistor space Z as the intersection of two embeddings of the Riemann sphere which we denote by X and X'.

$$X \cap X' = Z \in \mathbb{PT} \quad \Longrightarrow \quad \mu^{\dot{\alpha}} = x^{\alpha\dot{\alpha}}\lambda_\alpha \quad \mu^{\dot{\alpha}} = x'^{\alpha\dot{\alpha}}\lambda_\alpha \quad x, x' \in \mathbb{M}_{\mathbb{C}} \tag{2.2.55}$$

If we simply subtract the two relations from each other we obtain

$$(x - x')^{\alpha\dot{\alpha}}\lambda_\alpha = 0 \tag{2.2.56}$$

It is easy to see that this equation is solved exactly by the definition of the dual coordi-

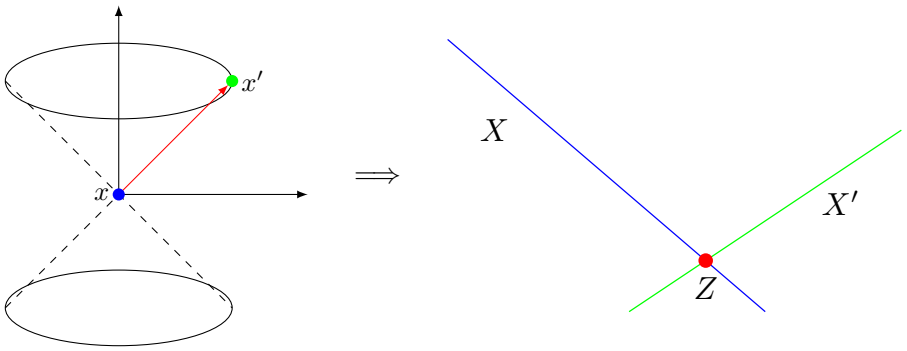


Figure 2.2.4: Null separated points in space-time correspond to intersecting lines in twistor space

nates

$$(x - x')^{\alpha\dot{\alpha}} = \lambda^{\alpha}\tilde{\lambda}^{\dot{\alpha}} \tag{2.2.57}$$

Thus if two points in $\mathbb{M}_{\mathbb{C}}$ are null separated then their corresponding lines in twistor space intersect. If we do not restrict $\tilde{\lambda}^{\dot{\alpha}}$ then (2.2.57) defines a two-plane in $\mathbb{M}_{\mathbb{C}}$ and we see that non-locality extends to both sides of the correspondence. That is for every point in space-time we have a corresponding line in twistor space and for every point in twistor space we have a plane in space-time. Therefore if we wish to trivialise the conditions on our kinematics then we necessarily have to consider non-local quantities. Having established the correspondence then all that is required to completely determine our kinematics is to choose an arbitrary set of $\{Z_i\}$ of rank n for which the embedding maps are given by

$$x_i^{\alpha\dot{\alpha}} = \frac{\mu_{i-1}^{\dot{\alpha}}\lambda_i^{\alpha} - \lambda_{i-1}^{\alpha}\mu_i^{\dot{\alpha}}}{\langle i\, i-1\rangle} \tag{2.2.58}$$

We can use the twistors to neatly re-express Lorentz invariants

$$x_{ij}^2 = \frac{\langle i-1\, i\, j-1\, j\rangle}{\langle I\, i-1\, i\rangle\langle I\, j-1\, j\rangle} \tag{2.2.59}$$

where we define the $SL(4,\mathbb{C})$ invariant Plücker coordinates

$$\langle i\, j\, k\, l\rangle = \varepsilon_{ABCD}\, Z_i^A Z_j^B Z_k^C Z_l^D \tag{2.2.60}$$

and the infinity twistor denoted by I is used to project out the helicity spinors

$$I^{AB} = \begin{pmatrix} \varepsilon^{\alpha\beta} & 0 \\ 0 & 0 \end{pmatrix} \quad \Rightarrow \quad \langle I\, i\, j\rangle = \langle i\, j\rangle \tag{2.2.61}$$

It is easy to recover the momenta from the twistor representation

$$\tilde{\lambda}_i^{\dot{\alpha}} = \frac{\langle i+1\, i\rangle\mu_{i-1}^{\dot{\alpha}} + \langle i-1\, i+1\rangle\mu_i^{\dot{\alpha}} + \langle i\, i-1\rangle\mu_{i+1}^{\dot{\alpha}}}{\langle i-1\, i\rangle\langle i\, i+1\rangle} \tag{2.2.62}$$

It is worth mentioning that twistors, besides solving for the kinematical restrictions, are also natural variables to describe conformal structures. If we define six homogeneous coordinates on $\mathbb{CP}^5$ organised in the anti-symmetric tensor representation X^{AB}, then a conformal structure on compactified complex space-time is given by the condition

$$\varepsilon_{ABCD}X^{AB}X^{CD}=0 \tag{2.2.63}$$

This condition can be solved by a skew tensor of rank two

$$X=Z_1\wedge Z_2 \qquad Z_1,Z_2\in\mathbb{PT} \tag{2.2.64}$$

And the incidence relation is the trivial

$$X_{AB}Z^B_{1,2}=0 \tag{2.2.65}$$

We can see that the bitwistor X^{AB} encodes a point in space-time $\mathbb{M}_{\mathbb{C}}$ and reach our previous definition of the incidence relation by writing

$$X^{AB}=\begin{pmatrix}\frac{1}{2}\varepsilon^{\dot\alpha\dot\beta}x^2 & x^{\dot\alpha}_{\beta}\\ -x^{\dot\beta}_{\alpha} & \varepsilon_{\alpha\beta}\end{pmatrix} \tag{2.2.66}$$

Note that twistor space is morally analogous to the spinor helicity formalism for null momenta. Extending the twistor formalism to the supersymmetric case is straightforward and we define the Grassmann incidence relation

$$\chi^A=\theta^{\alpha A}\lambda_\alpha \tag{2.2.67}$$

Where χ^A transform in the fundamental of the $SU(4)$ R-symmetry. We simply append these variables to our previous construction and define super-momentum twistors

$$\mathcal{Z}^{\mathcal{A}}=(\lambda_\alpha,\mu^{\dot\alpha},\chi^A) \tag{2.2.68}$$

Similar to (2.2.62) we can recover the original Grassmann variables from the ones defined by the incidence relation

$$\eta_i^A=-\frac{\chi^A_{i-1}\langle i\,i+1\rangle+\chi^A_i\langle i+1\,i-1\rangle+\chi^A_{i+1}\langle i-1\,i\rangle}{\langle i-1\,i\rangle\langle i\,i+1\rangle} \tag{2.2.69}$$

We have already seen how twistors simplify the form of the Yangian generators and we can give a flavour of their effect on the amplitude by using them to write the R-invariants

$$[i,j,k,l,m]\equiv\frac{\delta^{(4)}\left(\chi^A_i\langle jklm\rangle+\text{cyclic}\right)}{\langle ijkl\rangle\langle jklm\rangle\langle klmi\rangle\langle lmij\rangle\langle mijk\rangle} \tag{2.2.70}$$

Where $R_{n:st}=[n,s-1,s,t-1,t]$ are relevant for the NMHV amplitude. Given that (2.2.70)

is constructed purely from $SL(4,\mathbb{C})$ invariants, dual-conformal invariance is explicitly manifest. Remarkably it is possible to use momentum twistors to construct R-invariants by defining a Grassmannian contour integral [60, 61]

$$R_{k,n} \equiv \frac{1}{(2\pi i)^{k(n-k)}} \oint_{\Gamma \subset G(k,n)} d\mu \prod_{r=1}^{k} \delta^{(4|4)} \left(\sum_{i=1}^{n} T_i^r \mathcal{Z}_i \right) \tag{2.2.71}$$

Where $R_{k,n}$ is a $\mathrm{N^kMHV}$ n-particle invariant. The T_i^r are $(k \times n)$ complex homogeneous coordinates of the Grassmannian $\mathrm{Gr}(k,n)$ and the $\mathcal{Z}_i$ are the momentum twistors parametrising the kinematics. The measure is a $k(n-k)$ top form

$$d\mu \equiv \frac{D^{k(n-k)}T}{(12\ldots k)(23\ldots k+1)\ldots(n1\ldots k-1)} \tag{2.2.72}$$

The brackets in the denominator represent the $k \times k$ minors made from the columns of T_i^r. Besides Yangian invariants the contour integral (2.2.71) also produces identities among them, at 6-points for example

$$[12345] - [23456] + [34561] - [45612] + [56123] - [61234] = 0 \tag{2.2.73}$$

Which ensures that the 6-point NMHV amplitude is cyclically symmetric. Finally we note that these Grassmannian contour integrals are the starting point of the amplituhedron program [19–21].

2.2.5 Wilson Loops and Scattering Amplitudes

In the following we shall be making frequent reference to the review [62]. One could ask what is the physical origin of the dual symmetries discussed so far. The answer is that there is a surprising duality between scattering amplitudes in $\mathcal{N} = 4$ SYM and a polygonal Wilson loop with light like edges whose vertices are naturally parametrised by the dual coordinates. This remarkable relationship was first noticed at strong coupling [22, 23] where the AdS/CFT correspondence can be used to calculate scattering amplitudes of open strings by considering a minimal surface ending on a null polygon at the boundary. The two cases are linked by a T-duality where the dual superconformal symmetry of the scattering amplitude is the T-dual of the standard superconformal symmetry of the Wilson loop and vice-versa. While the original statement was formulated for strong interactions it turns out that the equivalence is independent of the coupling and we can conjecture a perturbative relation [24, 25]

$$\log\left(1 + \sum_l a^l P_{l;n}^{MHV}(\{p_i\})\right) = \log\left(1 + \sum_l a^l W_{l;n}(\{x_i\})\right) + \mathcal{O}(\epsilon) \qquad a = \frac{g^2 N}{8\pi^2} \tag{2.2.74}$$

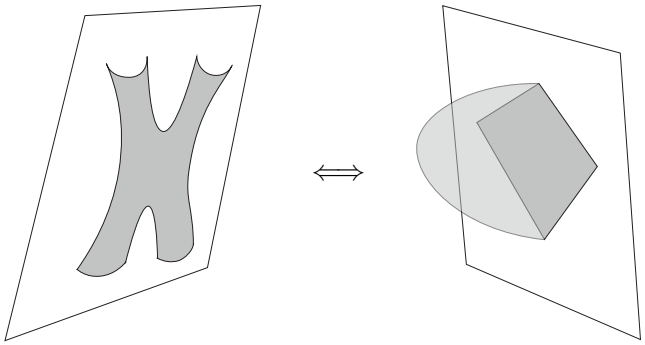


Figure 2.2.5: Open strings scattering are T-dual to a minimal area in AdS

Where the right hand side is a sum over loop corrections to the Wilson loop expectation

$$W_n = \langle \mathcal{P} \exp \oint_{C_n} d\tau \, \dot{x}^\mu A_\mu \rangle \tag{2.2.75}$$

The action used to evaluate the expectation is for $\mathcal{N} = 4$ SYM and the contour C_n is the union of the contours C_i given in terms of the dual coordinates x_i^μ and the affine parameters $\tau_i \in [0, 1]$

$$C_n = \bigcup_{i=1}^{n} C_i \quad C_i = \{x^\mu(\tau_i) = x_i^\mu + \tau_i(x_{i+1}^\mu - x_i^\mu) = x_i^\mu - \tau_i p_i^\mu\} \tag{2.2.76}$$

At lowest order in perturbation theory the Wilson loop expectation is given by

$$\langle W_n \rangle = 1 + \frac{1}{2}(ig)^2 C_F \int_{C_n} d\tau \int_{C_n} d\tilde{\tau} \, \dot{x}^\mu(\tau)\dot{x}^\mu(\tau) G_{\mu\nu}(x(\tau_i) - x(\tau_2)) + \mathcal{O}(g^4) \tag{2.2.77}$$

Where $C_F = (N^2 - 1)/(2N)$ is the fundamental quadratic Casimir of $SU(N)$ and $G_{\mu\nu}$ is the gluon propagator. It is straightforward to evaluate the one loop expectation by breaking it down in terms of a gluon propagator linking various edges. Thus the basic integral to consider is

$$I_{ij} = -\int_0^1 d\tau_i \int_0^1 d\tau_j \frac{(p_i \cdot p_j)\Gamma(1-\epsilon)(\pi\mu^2)^\epsilon}{[(-(x_i - x_j - \tau_i p_i + \tau_j p_j)^2 + i\epsilon]^{1-\epsilon}} \tag{2.2.78}$$

The case where the gluon is emitted and absorbed on the same edge vanishes as $I_{ii} \propto p_i^2 = 0$. If we consider adjacent edges however, we have a divergent contribution from the discontinuities at the cusp of the polygon. Thus they are called ultraviolet divergences

$$I_{i\,i+1} = -\int_0^1 d\tau_i \int_0^1 d\tau_{i+1} \frac{(p_i \cdot p_{i+1})\Gamma(1-\epsilon)(\pi\mu^2)^\epsilon}{[-2(p_i \cdot p_{i+1})(1-\tau_i)\tau_{i+1}]^{1-\epsilon}} = (-x_{i\,i+2}^2 \pi\mu^2)^\epsilon \frac{\Gamma(1-\epsilon)}{2\epsilon^2} \tag{2.2.79}$$

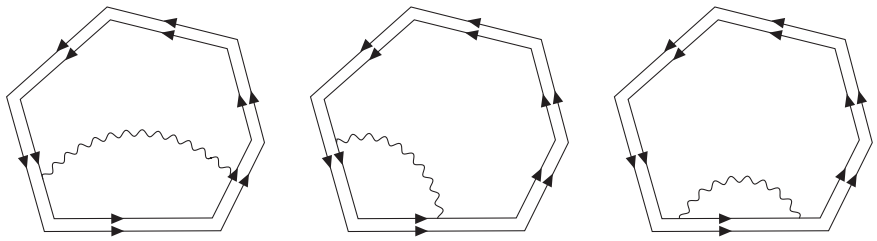


Figure 2.2.6: The three types of one-loop contribution to the Wilson loop expectation

Finally for all other edges we have a finite integral which, remarkably, can be evaluated to give [25]

$$I_{ij} = \frac{1}{8\pi^2}\left[\mathrm{Li}_2(1-qs_{j+1\,i-1}) + \mathrm{Li}_2(1-qs_{i+1\,j-1}) - \mathrm{Li}_2(1-qs_{j\,i-1}) - \mathrm{Li}_2(1-qs_{i\,j-1})\right] \tag{2.2.80}$$

where we have introduced the generalised Mandelstam invariant $s_{ij} \equiv (p_i + \ldots + p_j)^2$ and

$$q = \frac{s_{j+1\,i-1} + s_{i+1\,j-1} - s_{j\,i-1} - s_{i\,j-1}}{s_{j+1\,i-1}s_{i+1\,j-1} - -s_{j\,i-1}s_{i\,j-1}} \tag{2.2.81}$$

The function Li_2 is the second order polylogarithm defined by the integral

$$\mathrm{Li}_2(z) = -\int_0^z dt\,\frac{\log(1-t)}{t} \tag{2.2.82}$$

For the simplest Wilson loop W_4 however, (2.2.80) simplifies to just a product of logarithms

$$I_{13}^{W_4} = -\frac{1}{4}\left[\log^2\left(\frac{s}{t}\right) + \pi^2\right] \tag{2.2.83}$$

This is achieved by means of the identity

$$\mathrm{Li}_2(z) + \mathrm{Li}_2\left(\frac{1}{z}\right) + \frac{1}{2}\log^2(-z) + \frac{\pi^2}{6} = 0 \tag{2.2.84}$$

As we shall see such identities are ubiquitous in scattering amplitudes and will develop methods for generating them. Now that we have evaluated all the integrals that can appear, it is straightforward to express the one-loop expectation value by simply summing over all the edges linked by the gluon propagator

$$\langle W_n\rangle = 1 - \frac{g^2 C_F}{4\pi^2}\sum_{1\le i\le j\le n} I_{ij} + \mathcal{O}(g^4) \tag{2.2.85}$$

As was mentioned earlier the left hand side of the correspondence (2.2.74) is IR divergent while the right hand side has ultraviolet divergences arising from the discontinuities at the cusps of the polygon. In order to make the duality precise we must first remove these divergences by defining suitable finite quantities. We can easily see that at one-loop the

finite and divergent parts are separable and, remarkably, this continues to all orders

$$\log\langle W_n\rangle = -\sum_{i=1}^{n}\sum_{L}\frac{1}{4}a^L(-x^2_{i\,i+2}\mu^2)^{L\epsilon}\left[\frac{\Gamma^{(L)}_{\text{cusp}}}{(L\epsilon)^2}+\frac{\Gamma^{(L)}_{\text{col}}}{L\epsilon}\right]+F_n(\{x_i\}) \tag{2.2.86}$$

Where F_n are finite functions independent of μ and the quantities $\Gamma^{(L)}_{\text{cusp}}$ and $\Gamma^{(L)}_{\text{col}}$ are the cusp and collinear anomalous dimensions respectively. We encountered the one loop value of the former in (2.2.79) however due to integrability its value is known up to all orders for planar SYM [63]. The latter also arises from integrals of the type $I_{i\,i+1}$ in the limit where the gluon propagates parallel to one of the polygonal edges. We can extract the desired finite part F_n by dividing out the divergent part, which universally factorises into pieces dependent on only one Mandelstam invariant.

$$H_n^{WL} = \log\left(\frac{W_n(\{x^2_{ij}\})}{\left[\prod_{i=1}^{n}W_4(\{x^2_{i\,i+2},x^2_{i\,i+2}\})\right]^{\frac{1}{4}}}\right) \tag{2.2.87}$$

Having removed the divergences for the Wilson loop we can follow a similar procedure for the amplitude and safely identify the two finite quantities

$$H_n^{WL} = H_n^{\mathcal{A}} \tag{2.2.88}$$

The weak coupling correspondence was first observed for four points at lowest order in perturbation theory [24] and then generalised to n points [25]. Two loop calculations [64–68] further confirmed the validity of the duality, thus suggesting that this is a coupling independent phenomenon. On the amplitude side explicit calculation of four point amplitudes up to three loops led Bern, Dixon and Smirnov [7] to postulate a general solution in terms of the one loop result

$$F_n^{\text{BDS}}(p_1,\ldots,p_n;a) = \Gamma_{\text{cusp}}(a)\mathcal{F}_n^{1-\text{loop}}(p_1,\ldots,p_n;a) \tag{2.2.89}$$

Explicitly at four points

$$F_4^{\text{BDS}} = \frac{1}{4}\Gamma_{\text{cusp}}(a)\log^2\left(\frac{x^2_{13}}{x^2_{24}}\right)+\text{const.} \tag{2.2.90}$$

Which can be easily seen to match the Wilson loop result (2.2.83). The BDS anzatz was also shown to hold at five points.

$$F_5^{\text{BDS}} = -\frac{1}{8}\Gamma_{\text{cusp}}(a)\sum_{i=1}^{5}\log\left(\frac{x^2_{i\,i+2}}{x^2_{i\,i+3}}\right)\log\left(\frac{x^2_{i+1\,i+3}}{x^2_{i+2\,i+4}}\right)+\text{const.} \tag{2.2.91}$$

Note that the entire coupling dependence is encapsulated in the cusp anomalous dimension. This is consistent with dual conformal symmetry, which acts at the level of

the Lagrangian for the Wilson loop. We may write a Ward identity for the conformal generator K^μ [65]

$$K^\mu F_n = \frac{1}{2}\Gamma_{\text{cusp}}(a)\sum_{i=1}^{n} x^\mu_{i\,i+1}\log\left(\frac{x^2_{i\,i+2}}{x^2_{i-1\,i+1}}\right) \tag{2.2.92}$$

Thus the BDS ansatz is the solution to the conformal Ward identity at four and five points. The natural question to ask is whether the BDS ansatz holds for arbitrary number of gluons. Without doing the explicit calculations we could consider a Wilson loop corresponding to an amplitude made of many gluons that alternate from positive to negative energies.

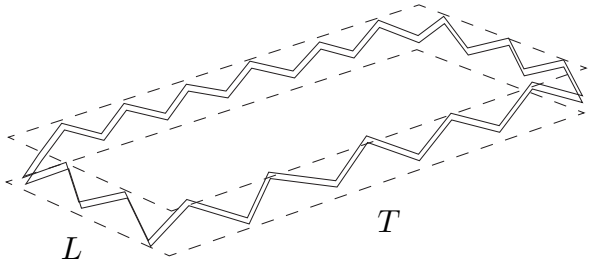


Figure 2.2.7: Zigzag Wilson loop approaching the four sided polygon

As the number of gluons tends to infinity the expectation value approaches that of a rectangular Wilson loop which can be calculated at strong coupling [23]

$$\log\langle W_{\text{rect}}\rangle = \frac{\sqrt{\lambda}}{4}\frac{16\pi^2}{\Gamma\left(\frac{1}{4}\right)^4}\frac{T}{L} \tag{2.2.93}$$

Where $\lambda = g^2N$ is the t'Hooft coupling and T/L denote the euclidean time and length in one of the spatial directions of the Wilson loop, which are taken to be large and $T \gg L$. We compare this result to the strong coupling, infinite gluon limit of the finite part BDS ansatz (2.2.89)

$$\lim_{n\to\infty} F_n^{\text{BDS}} = \frac{\sqrt{\lambda}}{4}\frac{T}{L} \tag{2.2.94}$$

The discrepancy arises because there is an ambiguity in the solution to the conformal Ward identity. Beyond five points (2.2.92) is only defined up to a function of conformal cross-ratios

$$u_{ijkl} = \frac{x^2_{ij}x^2_{kl}}{x^2_{ik}x^2_{jl}} \tag{2.2.95}$$

Thus we must correct the BDS solution by adding a conformally invariant remainder function R_n

$$F_n = F_n^{\text{BDS}} + R_n(\{u_i\}) \tag{2.2.96}$$

In order to calculate the number of independent cross ratios we note that in D dimen-

sions the conformal group reduces the number of independent variables to

$$Dn - \frac{(D+2)(D+1)}{2} \tag{2.2.97}$$

Which in four dimensions makes $(4n-15)$ independent cross ratios, however given that our points are arranged along a null polygon we must subtract the number of edges to obtain $(3n-15)$. Thus we see that the hexagon is the first quantity with non trivial cross-ratio dependence and the BDS ansatz is no longer sufficient. The discussion so far has only been concerned with the duality between Wilson loops and MHV amplitudes. At strong coupling the minimal surface in AdS provides the leading part of all amplitudes and the helicity structure is secondary. At weak coupling however we require an appropriate supersymmetrisation of the Wilson loop [69] in order to describe the much richer non-MHV structure of the amplitudes.

2.2.6 Wilson loop OPE

Perturbative corrections to the remainder function are the principal objects of study throughout much of this thesis. However at times we shall draw inspiration from a non-perturbative Wilson loop approach that is heuristically linked to the OPE of conformal field theory. The analogue of the sum over fields will be given by an expansion in terms of the excitations of a flux tube that ends on two light-like lines. These states have appeared as excitations of the infinite spin limit of the GKP string [70] which can be computed exactly using integrabilty. The first step is to isolate a reference square by placing a couple of imaginary cuts across the Wilson loop linking non adjacent edges. In order to perform the OPE we wish to expand the Wilson loop in terms of the states propagating from the bottom edge of the square to the top. This turns out to be equivalent to taking a collinear limit by using the symmetries associated to each individual square. That is we may parametrise the states that propagate across the square by the quantum numbers associated to a rank three subset of the $SO(2,4)$ symmetry, which leaves the square invariant.

In order to expose this subset let us choose a particular square and place it in the (x^0, x^1) plane with one vertex at the origin, two at past and future null infinity and the final one at spacelike infinity ($x^1 \to \infty$ with x^0 fixed). Then in this configuration the symmetries of the square are dilatations D, boosts in the (x^0, x^1) plane M_{01} and rotations in the transverse (x^2, x^3) plane M_{23}. We can give an explicit representation of these symmetries beginning with $e^{\lambda D} = e^{\lambda I}$

$$e^{-\xi M_{01}} = \begin{pmatrix} \cosh\xi & -\sinh\xi & 0 & 0 \\ -\sinh\xi & \cosh\xi & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad e^{-i\phi M_{23}} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \cos\phi & -\sin\phi \\ 0 & 0 & -\sin\phi & \cos\phi \end{pmatrix} \tag{2.2.98}$$

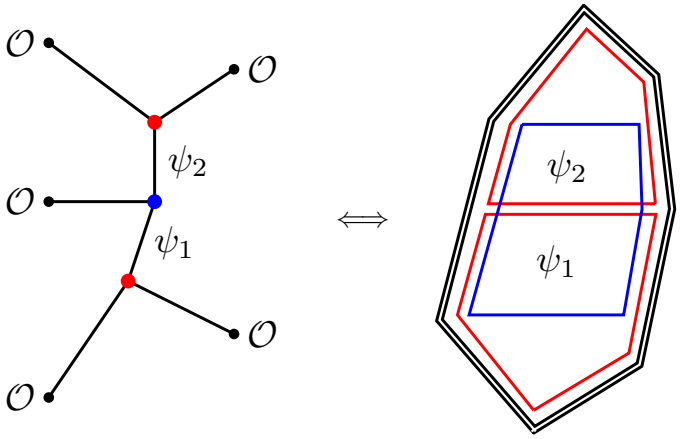


Figure 2.2.8: Comparison between the conformal theory OPE and Wilson loop OPE

which can be seen to leave the square invariant. By considering the other generators of $SO(2,4)$ we can see that there are no other symmetries that leave the square invariant. The special conformal generator, for example, would move the point at spatial infinity. We can take particular combinations of the above generators that have the effect of translating along the two directions of the square. For example the combination $(D - M_{01})$ can be thought of as a twist or "Hamiltonian" operator that leaves points along x^+ invariant but scales the transverse directions $x^- \to \lambda^2 x^-$ and $x^{2,3} \to \lambda x^{2,3}$. We note that because the lines of the square joining the sides of the Wilson loop are space-like separated this is a Euclidean "Hamiltonian". Similarly the combination $(D + M_{01})$ acts as a "momentum" operator providing translations along x^-. Having established the symmetry group we can consider its action on the cusps bellow the square. If we take $\tau(D - M_{01})$ in the limit $\tau \to \infty$

$$\left(\lim_{\tau\to\infty} e^{-\tau(D-M_{01})}\right) \cdot x^\mu = \frac{1}{2}\begin{pmatrix} x^+ \\ x^+ \\ 0 \\ 0 \end{pmatrix} \tag{2.2.99}$$

We can bring a cusp to lie on the edge linking the origin to future null-infinity. By acting in this manner on all the cusps bellow the square we flatten out the bottom of the Wilson loop and reach its collinear limit. We can think of the two sides of the square that belong to the Wilson loop as two high energy quarks that source a flux tube. Therefore we can decompose the Wilson loop in terms of the excitations of the flux tube. The great advantage of studying the symmetries of the square is that the excitations are eigenstates of the generators and we can write schematically,

$$\langle W \rangle = \int dn e^{-\tau E_n + i p_n \sigma + i m_n \phi} C_n^{\text{top}} C_n^{\text{bottom}} \tag{2.2.100}$$

Where C_n^{top} and C_n^{bottom} denote the overlap between top and bottom parts of the polygon and the intermediate excitations. This decomposition around the collinear limit is the Wilson loop equivalent of the expansion over local operators of the usual OPE. Note that instead of a sum over states we have an integral in (2.2.100), this is due to the momentum operator being non-compact and thus having a continuous spectrum. At this point we would like to observe that the derivation of the Wilson loop OPE was only dependent on the symmetries of the square and thus it exists for any conformal field theory where the flux is conserved. For non-planar $\mathcal{N}=4$ SYM, the approach is still valid if we consider lines in the fundamental because the flux vacuum and its excitations are well defined. For the adjoint case, however, the flux can be screened and the OPE will require modifications. Fortunately for the planar case not only is the flux conserved but the excitations are in direct correspondence with the states of an integrable spin chain. We can express these states as single trace operators with the vacuum given by

$$\text{Tr}\left(ZD_+^S Z\right), \qquad D_+ = D_0 + D_1 \tag{2.2.101}$$

Where Z is a complex combination of two scalars of the theory and $D_{0,1}$ denote their respective covariant derivatives. We excite the vacuum by inserting fundamental fields Φ

$$\text{Tr}\left(ZD_+^{S_1}\Phi D_+^{S_2} Z\right) \tag{2.2.102}$$

The corresponding energy of the excitation can be found by considering

$$E(p) = (D - M_{01})_1 - (D - M_{01})_{\text{vac}} = 1 + \sum_l^\infty \lambda^l E^{(l)}(p) \tag{2.2.103}$$

Which is known to all loops through integrability [27]. As we saw in the previous section UV divergence can spoil the conformal symmetry that has been instrumental in deriving the Wilson loop OPE. In order to restore the symmetry we instead define the finite ratio

$$r = \log\left[\frac{\langle W\rangle\langle W_{\text{square}}\rangle}{\langle W_{\text{top}}\rangle\langle W_{\text{bottom}}\rangle}\right] \tag{2.2.104}$$

Where W_{top} (W_{bottom}) corresponds to the polygon obtained by flattening all the cusps bellow (above) the square. To this ratio we can safely attribute the OPE decomposition (2.2.100). As stated previously, our object of interest is the remainder function (2.2.96), which can be broken down into several contributions obtained from the Wilson loop

$$R = R_{\text{top}} + R_{\text{bottom}} + r - r_{U(1)} \tag{2.2.105}$$

Where R_{top} and R_{bottom} are the remainder functions of W_{top} and W_{bottom} respectively and $r_{U(1)}$ is defined exactly as in (2.2.104) except for an underlying $U(1)$ theory. In fact

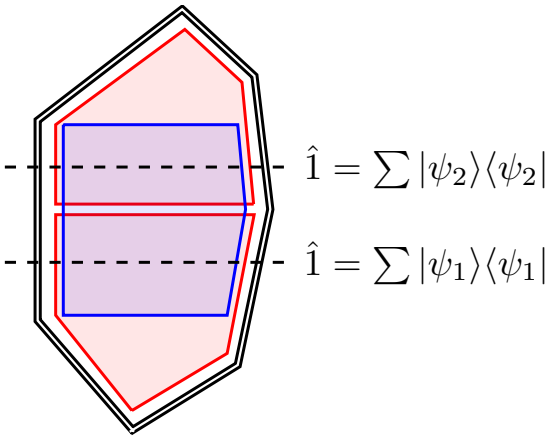


Figure 2.2.9: Expansions in terms of states propagating through the cut

we can make a connection with the BDS ansatz by noting

$$r^{\text{BDS}} = \frac{1}{4}\Gamma_{\text{cusp}} r_{U(1)} \tag{2.2.106}$$

From (2.2.105) we can see that the OPE provides a way to express the remainder function in terms of the remainders for lower order polygons. It is therefore natural to further break up the Wilson loop into squares so that we factorise the remainder function completely. The procedure begins by placing more cuts across the polygon such that we decompose the polygon into $(n-3)$ squares, with adjacent squares forming a pentagon. If we associate the squares at the extremities with the GKP vacuum then we are left with $(n-5)$ middle squares and $(n-4)$ pentagons. Generalising our discussion for the single square we may use the individual symmetries to parametrise our problem $\{\tau_i, \sigma_i, \phi_i\}$. Given that there are $3n-15$ of these parameters it is possible to map them to the 4d conformal cross ratios that appear in the remainder function. The three cross ratios that appear for the hexagon, for example, are given by

$$\begin{aligned}
\frac{1}{u_2} &= 1 + e^{2\tau} \\
\frac{1}{u_3} &= 1 + (e^{-\tau} + e^{\sigma+i\phi})(e^{-\tau} + e^{\sigma-i\phi}) \\
u_1 &= e^{2\sigma+2\tau} u_2 u_3
\end{aligned} \tag{2.2.107}$$

For a general polygon it is possible to use momentum twistors to define the generic map [29] for higher points. In order to remove the UV divergences we generalise the ratio (2.2.104) by multiplying the expectation of the full Wilson loop by that of the middle squares and dividing by the expectation of the pentagons

$$\mathcal{W}_n \equiv \langle W \rangle \frac{\prod_{i=1}^{n-5} \langle W_i^{\square} \rangle}{\prod_{i=1}^{n-4} \langle W_i^{\pentagon} \rangle} \tag{2.2.108}$$

Having established a well defined object we proceed to investigate the dynamics. Beginning with the GKP vacuum at the bottom we evolve it all the way to the top where it is reabsorbed into the vacuum. Along the way the flux tube is decomposed in terms of GKP eigenstates ψ_i which propagate trivially in their respective squares for a time τ_i. Before it can propagate across to the next square the state undergoes a pentagon transition $\mathcal{P}$ after which the decomposition is performed again. Remarkably this entire process can be captured in a single factorised form

$$\mathcal{W}_n = \sum_{\psi_i} e^{\sum_j (-E_j \tau_j + i p_j \sigma_j + i m_j \phi_j)} \mathcal{P}(0|\psi_1)\mathcal{P}(\psi_1|\psi_2)\dots\mathcal{P}(\psi_{n-5}|0) \tag{2.2.109}$$

The GKP excitations may be fermions, gluons, scalars or bound-states of fundamental fields. If we denote the type of excitations for the n particles by the set $\mathbf{a} = \{a_1, \dots, a_n\}$ and their Bethe rapidities $\mathbf{u} = \{u_1, \dots u_n\}$ then we can re-write the sum (2.2.109) in a similar manner to (2.2.100). For example the hexagon and heptagon are given by

$$\begin{aligned}
\mathcal{W}_6 &= \sum_{\mathbf{a}} \int d\mathbf{u}\, P_{\mathbf{a}}(0|\mathbf{u}) P_{\mathbf{a}}(\bar{\mathbf{u}}|0) e^{-E(\mathbf{u})\tau + i p(\mathbf{u})\sigma + i m \phi} \\
\mathcal{W}_7 &= \sum_{\mathbf{a},\mathbf{b}} \int d\mathbf{u}\, d\mathbf{v}\, P_{\mathbf{a}}(0|\mathbf{u}) P_{\mathbf{ab}}(\bar{\mathbf{u}}|\mathbf{v}) P_{\mathbf{b}}(\bar{\mathbf{v}}|0) e^{-E(\mathbf{u})\tau_1 + i p(\mathbf{u})\sigma_1 + i m_1 \phi_1} e^{-E(\mathbf{v})\tau_2 + i p(\mathbf{v})\sigma_2 + i m_2 \phi_2}
\end{aligned} \tag{2.2.110}$$

Where $\bar{\mathbf{u}} = \{-u_n, \dots, -u_1\}$ and the measure is given by

$$d\mathbf{u} = N_{\mathbf{a}} \prod_{j=1}^{n} \mu_{a_j}(u_j) \frac{du_j}{2\pi} \qquad \operatorname*{Res}_{v=u} P_{aa}(u|v) = \frac{i}{\mu_a(u)} \tag{2.2.111}$$

Where $N_{\mathbf{a}}$ is a symmetry factor for the particle configurations. The equality between the residue of the pentagon transition and the measure relates the heptagon and hexagon expansions. It comes from the fact that the residue is conformally equivalent to flattening the pentagon to a square. The pentagon transitions are analogues to the structure constants of local operators independent of the global geometry. Fora single particle we may define three main axioms beginning with reflection

$$P(-u|-v) = P(v|u) \tag{2.2.112}$$

Which comes from the fact that flipping both momenta results in a reflection of the pentagon. The second is called the fundamental relation

$$P(u|v) = S(u,v) P(v|u) \tag{2.2.113}$$

Where $S(u,v)$ is the GKP S-matrix and can be computed exactly using integrability. The final axiom describes how to move the excitations between different edges of the pen-

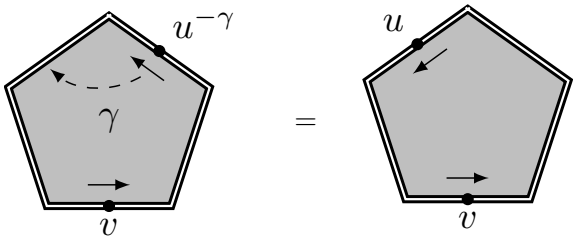


Figure 2.2.10: Mirror symmetry of the pentagon

tagon

$$P(u^{-\gamma}|v) = P(v|u) \tag{2.2.114}$$

Where $u^{-\gamma}$ denotes the mirror transformations that exchanges the GKP space and time direction with the effect

$$E(u^{-\gamma}) = -ip(u) \qquad p(u^{-\gamma}) = -iE(u) \tag{2.2.115}$$

Although the effect is general, the precise nature of the transformation is dependent on the particular excitation being considered. Furthermore we note that the axioms (2.2.112)-(2.2.114) were written for scalar excitations, however they still hold for all fundamental fields with minor modifications. Of course we must also consider multi-particle transitions. These obey the same axioms as the single particle cases and in [28] they were given a conjectural form

$$P(\mathbf{u}|\mathbf{v}) = \frac{\prod_{i,j} P(u_i|v_j)}{\prod_{i>j} P(u_i|u_j) \prod_{i<j} P(v_i|v_j)} \tag{2.2.116}$$

However it turns out that the most relevant pentagon transitions are those for single-particle states. They are the lightest states and thus dominate the collinear limit at generic coupling and form a basis for building up heavier more intricate states. In other words they decay the slowest in the multi-collinear limit and their contributions may be simply separated from the rest of the excitations and are thus able to give a good approximation for the Wilson loop in general kinematics. In contrast to the perturbative approach the OPE is organised in terms of the states propagating across the cut. The comparison of the sum over states versus the sum over orders is a rich source of constraints and checks between the OPE and direct approaches to scattering amplitudes. For the hexagon the leading contribution is given by

$$r = 2\cos\phi e^{-\tau} \int_{-\infty}^{\infty} \frac{du}{2\pi} \mu_1(u) e^{-\gamma_1(u)\tau + ip_1(u)\sigma} + \mathcal{O}(e^{-2\tau}) \tag{2.2.117}$$

The remainder function associated to the above ratio was calculated in [71]

$$R = 2\cos\phi e^{-\tau}\sum_{l=1}^{\infty} g^{2l}\left[\sum_{n=0}^{l-1}\tau^n h_n^{(l)}(\sigma) + \frac{\Gamma_{\text{cusp}}^l}{4}\left[e^{-\sigma}\log(1+e^{2\sigma}) + e^{\sigma}\log(1+e^{-2\sigma})\right]\right] + \mathcal{O}(e^{-2\tau}) \tag{2.2.118}$$

where the functions

$$h_n^{(l)}(\sigma) = \sum_{s,r,m_i} c^{\pm}_{s,m_1,\ldots,m_r} e^{\pm\sigma}\sigma^s H_{m_1,\ldots,m_r}\left(-e^{-2\sigma}\right) \qquad m_i \geq 1 \tag{2.2.119}$$

are given as an expansion in terms of a special group of functions called the Harmonic polylogarithms $H_{m_1,\ldots,m_r}(z)$. These functions are a subset of a more general class of transcendental functions called multiple polylogarithms, which we proceed to describe in the next section.

2.3 Multiple Polylogarithms

In our discussion on scattering amplitudes in $\mathcal{N}=4$ SYM all the explicit results encountered so far have featured either logarithms or classical polylogarithms.

$$\log z = \int_1^z \frac{dt}{t} \qquad \text{Li}_n(z) = \int_0^z \frac{dt}{t}\,\text{Li}_{n-1}(t) \tag{2.3.1}$$

Their appearance is not restricted only to the supersymmetric case however, but a large class of Feynman integrals may also be expressed in terms of these functions. Their generalisations, called multiple polylogarithms, will be of central importance to this work and we dedicate the next section to their description. We begin by noting that the functions in (2.3.1) are defined as iterated integrals. Let M be a smooth manifold and γ a smooth map $\gamma : [0,1] \to M$. Then by defining a set of $\omega_1, \ldots \omega_n$ smooth differential one-forms on M we can write their iterated integral along γ as

$$\int_\gamma \omega_n \ldots \omega_1 = \int_{0\leq t_1 \leq \ldots \leq t_n \leq 1} f_n(t_n)dt_n \ldots f_1(t_1)dt_1 \tag{2.3.2}$$

Where we have taken the pullback of the one-forms $\gamma^*(\omega_i) = f_i(t)dt$ to the interval $[0,1]$. For example if we draw our forms from the set $\{\omega_0, \omega_1\}$ and take γ as a smooth path in $\mathbb{C}\backslash\{0,1\}$ with end point $\gamma(1) = z$ then logarithms and classical polylogarithms may be expressed as

$$\frac{1}{n!}\log^n z = \int_\gamma \omega_0^n \qquad \text{Li}_n(z) = -\int_\gamma \omega_0^{n-1}\omega_1 \tag{2.3.3}$$

In order to agree with (2.3.1) the pullbacks must be of the form $\omega_0 = \frac{dt}{t}$ and $\omega_1 = \frac{dt}{t-1}$. We can extend the set of one-forms to include multiple variables and thus construct the

multiple polylogarithms

$$\mathrm{Li}_{m_1,\ldots,m_k}(x_1,\ldots,x_k) = (-1)^k \int_0^1 \omega_0^{m_k-1}\omega_k \ldots \omega_0^{m_1-1}\omega_1 \tag{2.3.4}$$

and the pullbacks of the ω_i are related to the x_i via

$$\omega_0 = \frac{dt}{t} \qquad \omega_{i\geq 1} = \frac{dt}{t - \frac{1}{x_1 x_2 \ldots x_i}} \tag{2.3.5}$$

Note that apart from the end points all dependence on the path must drop out of the integral if we are to have a well defined multiple polylogarithm. This means that if we take another path γ' which is homotopic to γ then the iterated integral must be invariant. Paths are homotopic if the end points coincide $\gamma'(1) = \gamma(1)$, $\gamma'(0) = \gamma(0)$ and one path may be continuously deformed into the other. For a one fold integral, homotopy invariance requires that

$$\int_{\gamma'} \omega = \int_{\gamma} \omega \tag{2.3.6}$$

Which is true if and only if ω is closed. For iterated integrals however we must impose more complex constraints on the combination of one-forms. In Chen's seminal [72] work the homotopy invariant integrals were constructed by considering linear combinations of one-forms. The initial step is to observe that for closed one-forms ω_1 and ω_2 we then have for some one-from ω_{12}

$$\omega_1 \wedge \omega_2 + d\omega_{12} = 0 \quad \Longrightarrow \quad \left(\int_{\gamma'} - \int_{\gamma}\right) \omega_1\omega_2 + \omega_{12} = 0 \tag{2.3.7}$$

We can extend this result by considering a map D from the tensor products of smooth one-forms (not closed) $T(\Omega^1(M))$ to the space of tensor products of all forms $T(\Omega(M))$ given by

$$\begin{aligned} D(\omega_1 \otimes \ldots \otimes \omega_n) = &\sum_{i=1}^{n} \omega_1 \otimes \ldots \otimes d\omega_j \otimes \ldots \otimes \omega_n \\ &+ \sum_{i=1}^{n-1} \omega_1 \otimes \ldots \otimes \omega_j \wedge \omega_{j+1} \otimes \ldots \otimes \omega_n \end{aligned} \tag{2.3.8}$$

Further defining the kernel of D through linear combinations of tensor products up to rank m

$$B_m(\Omega) = \left\{ \xi = \sum_{l=0}^{m} \sum_{i_1,\ldots,i_l} c_{i_1,\ldots,i_l}\omega_1 \otimes \ldots \otimes \omega_l \,\middle|\, D\xi = 0 \right\} \tag{2.3.9}$$

Where $c_{i_1,\ldots,i_l}$ are just constant coefficients. This is called the space of integrable words and the equation $D\xi = 0$ is known as the integrability condition. Chen's theorem asserts that mapping $B_m(\Omega)$ to an iterated integral by simply integrating the words defines an

isomorphism to the space of homotopy invariant iterated integrals of up to length m.

$$\xi \quad \rightarrow \quad \sum_{l=0}^{m} \sum_{i_1,\ldots,i_l} c_{i_1,\ldots,i_l} \int_\gamma \omega_1 \ldots \omega_l \tag{2.3.10}$$

The reverse is also true; that is a homotopically invariant integral corresponds to an integrable word. We shall revisit the concept of integrable words in our discussion of the symbol of a multiple polylogarithm. For now we proceed to describe iterated integrals and their properties.

2.3.1 General properties of iterated integrals

Although we could continue in the established formalism for iterated path integrals, in the physics literature however, there are two main types of iterated integrals. The most prominent type are due to Goncharov [73] and are defined recursively

$$G(a_1, \ldots, a_n; z) = \int_0^z \frac{dt}{t - a_1} G(a_2, \ldots, a_n; z) \tag{2.3.11}$$

Where the parameters $\{a_i\}$ can be taken as constants or variables and $G(z) \equiv 1$. It is easy to see from the definition of the multiple polylogarithm that we can express them in the Goncharov notation as

$$\mathrm{Li}_{m_1,\ldots,m_k}(x_1, \ldots, x_k) = (-1)^k G\left(0, \ldots, 0, \frac{1}{x_k}, \ldots, 0, \ldots, 0, \frac{1}{x_1 \ldots x_k}; 1\right) \tag{2.3.12}$$

Given how similar their definitions are we shall use the term multiple polylogarithm (MPL) to interchangeably mean both (2.3.4) and (2.3.11). A second more general notation is

$$I(a_0; a_1, \ldots, a_n; a_{n+1}) = \int_{a_0}^{a_{n+1}} \frac{dt}{t - a_n} I(a_0; a_1, \ldots, a_{n-1}; t) \tag{2.3.13}$$

This is a similar definition to the previous version except that it allows for a general basepoint. For both cases the number of parameters defines the weight, that is (2.3.13) is an iterated integral of weight n. It is easy to define a map going from the Goncharov notation to the general one

$$G(a_1, \ldots, a_n; z) = I(0; a_n, \ldots, a_1; z) \tag{2.3.14}$$

The inverse relation is slightly more complicated however, due to the necessity of changing every basepoint in the iterated integral. We can nevertheless deduce it recursively by applying the identity

$$I(a_0; a_1; a_2) = \int_{a_0}^{a_2} \frac{dt}{t - a_1} = \int_0^{a_2} \frac{dt}{t - a_1} - \int_0^{a_0} \frac{dt}{t - a_1} = G(a_1; a_2) - G(a_1; a_0) \tag{2.3.15}$$

In fact we can use this property of the integral to deduce much a more general result. Let u and v be two paths from the unit interval to M with end points $u(0) = a_0$, $u(1) = v(0) = x$ and $v(1) = a_{n+1}$. Then the iterated integral along the composed path from a_0 to a_{n+1} is given by

$$I(a_0; a_1, \ldots, a_n; a_{n+1}) = \sum_{k=0}^{n} I(a_0; a_1, \ldots, a_k; x) I(x; a_{k+1}, \ldots, a_n; a_{n+1}) \tag{2.3.16}$$

This is the path composition formula. Alternatively it may be viewed as the decomposition of a path γ into two constituent paths u and v. We combine (2.3.16) with the following inversion formula

$$I(a_{n+1}; a_n, \ldots, a_1; a_0) = (-1)^n I(a_0; a_1, \ldots, a_n; a_{n+1}) \tag{2.3.17}$$

To obtain a map from the general iterated integral to the Goncharov MPLs

$$I(a_0; a_1, \ldots, a_n; a_{n+1}) = \sum_{k=0}^{n} (-1)^k G(a_1, \ldots, a_k, a_0) G(a_n, \ldots, a_{k+1}; a_{n+1}) \tag{2.3.18}$$

We could further use the path decomposition property to determine the monodromy of an iterated integral. From the iterated integral definition it is easy to see that MPLs have singularities whenever a path passes through one of the $\{a_i\}$. The monodromy at a point a_i is obtained by considering integrals over paths that encircle one singular point. Let γ be a path from 0 to 1 and γ' be its deformation that encircles the singular point only once, then [74]

$$I_{\gamma'}(0; , a_1, \ldots, a_n; 1) - I_{\gamma}(0; , a_1, \ldots, a_m; 1) = 2\pi i I(0; a_1, \ldots, a_{i-1}; a_i) I(a_i; a_{i+1}, \ldots, a_n; 1) \tag{2.3.19}$$

This can be seen from the fact that for a set of rational functions $f_1 \ldots f_n$, the difference between the above paths is given by

$$\frac{1}{2\pi i} \left(\int_{\gamma'} - \int_{\gamma} \right) (d \log f_1 \ldots d \log f_n) =$$
$$\sum_{k=1}^{n} v_\delta(f_k) \left(\int_{\alpha_1} d \log f_1 \ldots d \log f_{k-1} \right) \left(\int_{\alpha_2} d \log f_{k+1} \ldots d \log f_n \right) \tag{2.3.20}$$

Where the composed path $\alpha_2 \alpha_1$ is given by $0 \to a_i \to 1$. Furthermore $v_\delta(f_k) \in \mathbb{Z}$ is an index defined by an integral around the path δ which encircles the singular point

$$v_\delta(f) = \frac{1}{2\pi i} \int_\delta d \log f \tag{2.3.21}$$

Given that no other rational function vanishes at the singular point then only $v_\delta(f_i) \neq 0$ and we recover (2.3.19). A further central property of the iterated integrals is that they

form a shuffle algebra. If we take two MPLs of weight p and q with the same basepoint and endpoint then their product is given by

$$I(a_0;a_1,\ldots,a_p;z)I(a_0;a_{p+1},\ldots,a_{q+p};z)=\sum_{\sigma\in\Sigma(p,q)}I(a_0;a_{\sigma(1)},\ldots,a_{\sigma(p+q)};z) \tag{2.3.22}$$

Where $\Sigma(p,q)$ is a subset of the symmetric group S_n and denotes the set of all shuffles of $(p+q)$ elements

$$\Sigma(p,q)=\{\sigma\in S_{p+q}|\sigma^{-1}(1)<\ldots<\sigma^{-1}(p)\quad\text{and}\quad\sigma^{-1}(p+1)<\ldots<\sigma^{-1}(p+q)\} \tag{2.3.23}$$

For example take two Goncharov MPLs

$$\begin{aligned}G(a_1,a_2;z)G(b_1,b_2;z)=&G(a_1,a_2,b_1,b_2;z)+G(a_1,b_1,a_2,b_2;z)+G(b_1,a_1,a_2,b_2;z)\\&+G(b_1,a_1,b_2,a_2;z)+G(a_1,b_1,b_2,a_2;z)+G(b_1,b_2,a_1,a_2;z)\end{aligned} \tag{2.3.24}$$

Similar to the path decomposition case the proof of the shuffle relations relies on the recursive application of the following identity

$$\int_{a_0}^{z}\frac{dt_1}{t_1-a_1}\int_{a_0}^{z}\frac{dt_1}{t_2-a_2}=\int_{a_0}^{z}\frac{dt_1}{t_1-a_1}\int_{a_0}^{t_1}\frac{dt_2}{t_2-a_2}+\int_{a_0}^{z}\frac{dt_2}{t_2-a_2}\int_{a_0}^{t_2}\frac{dt_1}{t_1-a_1} \tag{2.3.25}$$

Which is the decomposition of an integral over a square into the two constituent triangles.

Let $a_n\neq 0$ then MPLs are invariant under rescaling by $k\in\mathbb{C}^*$

$$G(ka_1,\ldots,ka_n;kz)=G(a_1,\ldots,a_n;z) \tag{2.3.26}$$

To see why it is necessary to have the condition $a_n\neq 0$ we note that by the shuffle relations

$$G(a_1,\ldots,a_{n-1},0;z)=G(0;z)G(a_1,\ldots,a_{n-1};z)+\ldots \tag{2.3.27}$$

Which does not obey the scale invariance property because

$$G(0;kz)=\log kz-\log k0\neq G(0;z) \tag{2.3.28}$$

For the same reason $G(a_1,\ldots,a_n;z)$ is only analytic at $z=0$ if $a_n\neq 0$. Now assuming that all the $\{a_i\}$ are non-zero then the total derivative of an iterated integral is given by

$$dI(a_0;a_1,\ldots,a_n;a_{n+1})=\sum_{i=1}^{m}I(a_0;a_1,\ldots,\hat{a}_i,\ldots,a_n;a_{n+1})d\log\frac{(a_{i+1}-a_i)}{(a_i-a_{i-1})} \tag{2.3.29}$$

Where the hat indicates omission of that parameter from the integral. To see how (2.3.29)

comes about we restrict ourselves to the case where all the parameters are different and differentiate with respect to a_i

$$\int_{a_0}^{a_{n+1}} \frac{dt_n}{t_n - a_n} \dots \int_{a_0}^{t_{i+1}} \frac{dt_i}{(t_i - a_i)^2} I(a_0; a_1, \dots, a_{i-1}; t_i) \tag{2.3.30}$$

By rewriting the differential with respect to a_i as a differential with respect to t_i inside the integral and integrating by parts we obtain

$$\begin{aligned} &\int_{a_0}^{a_{n+1}} \frac{dt_n}{t_n - a_n} \dots \int_{a_0}^{t_{i+1}} \frac{dt_i}{(t_i - a_i)(t_i - a_{i-1})} I(a_0; a_1, \dots, a_{i-1}; t_i) \\ &\quad - \int_{a_0}^{a_{n+1}} \frac{dt_n}{t_n - a_n} \dots \int_{a_0}^{t_{i+2}} \frac{dt_{i+1}}{(t_{i+1} - a_{i+1})(t_{i+1} - a_i)} I(a_0; a_1, \dots, a_{i-1}; t_{i+1}) \end{aligned} \tag{2.3.31}$$

We use partial fraction decomposition in order to separate the denominators

$$\frac{1}{(t_i - a_i)(t_i - a_{i-1})} = \frac{1}{(a_i - a_{i-1})} \left(\frac{1}{(t_i - a_i)} - \frac{1}{(t_i - a_{i-1})} \right) \tag{2.3.32}$$

$$\frac{1}{(t_{i+1} - a_{i+1})(t_{i+1} - a_i)} = \frac{1}{(a_{i+1} - a_i)} \left(\frac{1}{(t_{i+1} - a_{i+1})} - \frac{1}{(t_{i+1} - a_i)} \right) \tag{2.3.33}$$

Performing the remaining integrations leads to

$$\begin{aligned} &\frac{1}{(a_i - a_{i-1})} \big(I(a_0; \dots, \hat{a}_{i-1}, \dots; a_{n+1}) - I(a_0; \dots, \hat{a}_i, \dots; a_{n+1}) \big) \\ &\quad - \frac{1}{(a_{i+1} - a_i)} \big(I(a_0; \dots, \hat{a}_i, \dots; a_{n+1}) - I(a_0; \dots, \hat{a}_{i+1}, \dots; a_{n+1}) \big) \end{aligned} \tag{2.3.34}$$

Which when summed over the da_i give (2.3.19). The more general case where some of the $\{a_i\}$ are allowed to be zero can also be incorporated. The procedure is simply to treat all the zero entries as standard parameters when performing the total derivative. Once this is done we restore the original values of the parameters (i.e. allow them to be zero again). The only subtlety is that we may have $a_i = a_{i+1} = 0$, in which case we take the 1-form $d\log(a_i - a_{i+1})$ to be zero. This property is known as canonical regularisation and we may use it to treat divergent integrals of the form

$$I(0, 0^p, a_1, \dots, a_m, 1^q; 1) \to I(\varepsilon, 0^p, a_1, \dots, a_m, 1^q; 1 - \varepsilon) = \sum_{k=0} \log^k \varepsilon \, f_k(\varepsilon) \tag{2.3.35}$$

Where 0^p and 1^q denote sequences of length p and q respectively. Thus we see that the divergent part of the integral is a polynomial in $\log \varepsilon$ with regular coefficients $f_k(0) \neq 0$. Canonical regularisation then, is just keeping the finite part f_0, which can be zero. Note that with this regularisation scheme in mind we may identify

$$G(0^n; z) = \frac{\log^n z}{n!} \tag{2.3.36}$$

The total derivative of a multiple polylogarithm leads to a interesting and useful quantity known as the symbol, which we proceed to study in the next section.

2.3.2 Symbols of multiple polylogarithms

One remarkable property of multiple polylogarithms, is that there are many non-trivial functional identities among them and we have already seen an example in (2.2.84). There are also various ways we could produce these identities, for example some may be derived from the shuffle algebra like (2.3.24) others via the Holder convolution for all $p \in \mathbb{C}^*$ and $a_1 \neq 1$, $a_n \neq 0$.

$$G(a_1, \ldots, a_n; 1) = \sum_{k=0}^{n} (-1)^k G\left(1 - a_k, \ldots, 1 - a_1; 1 - \frac{1}{p}\right) G\left(a_{k+1}, \ldots, a_n; \frac{1}{p}\right) \quad (2.3.37)$$

And other functional relations are more complex still. It is therefore entirely possible that a long and complicated linear combination of MPLs obtained from the computation of a scattering amplitude may reduce via these identities to something much simpler. Indeed this is the context in which symbols first appeared in physics [32] where they were used to dramatically simplify the six-point remainder function [35, 36] given in terms of MPLs. The central concept is to map the space of polylogarithms to a tensor algebra over the group of rational functions. Beginning with the total differential of a transcendental function $F_n(x_1, \ldots, x_k)$

$$dF_n = \sum_i F_{i,n-1} d\log R_i \quad (2.3.38)$$

Where $F_{i,n-1}$ are transcendental functions of weight $(n-1)$ and R_i are rational functions in the x_i. Thus we define the symbol recursively in the weight

$$\mathcal{S}(F_n) = \sum_i \mathcal{S}(F_{i,n-1}) \otimes R_i \quad (2.3.39)$$

Note that for MPLs the rational functions are given in (2.3.29). One of the simpler class of symbols belongs to the classical polylogarithms

$$\mathcal{S}(\mathrm{Li}_n(z)) = -1 - z \otimes \overbrace{z \otimes \ldots \otimes z}^{n-1} \quad (2.3.40)$$

We can see that the symbol must inherit a few properties from the logarithm, namely it is additive with respect to products of the entries

$$\ldots \otimes ab \otimes \ldots = \ldots \otimes a \otimes \ldots + \ldots \otimes b \otimes \ldots \quad (2.3.41)$$

Furthermore we have

$$\ldots \otimes c \otimes \ldots = 0 \quad (2.3.42)$$

Where c is any numerical constant, and particularly when we have a 0 entry, by canonical regularisation. However we must take care in the case of vanishing entries, if for instance one of the factors in (2.3.41) tends to zero then we see that the two sides may have different limits. To recover the finite part of the limit we must first expand the factors

$$\lim_{a\to 0}(\ldots \otimes ab \otimes \ldots) = \ldots \otimes b \otimes \ldots \tag{2.3.43}$$

Combining these properties we may determine the symbol of an MPL by identifying all the rational functions and then expanding the factors. Often in practical scenarios there is a finite set of R_i which is called the alphabet and the individual R_i are called the letters. Because of the multiplicative property an alphabet is not unique and two alphabets $\{R'_i\}$ and $\{R_i\}$ are equivalent if there exists an invertible matrix M_{ij} over the rationals such that

$$\log R_i = \sum_j M_{ij} \log R'_j \tag{2.3.44}$$

The symbol is a linear map and takes the product of transcendental functions to the shuffle product of their symbols.

$$\mathcal{S}(F_p G_q) = \mathcal{S}(F_p) \sqcup\!\sqcup \mathcal{S}(G_q) \tag{2.3.45}$$

Where F_p and G_q are two transcendental functions of weight p and q respectively and $\sqcup\!\sqcup$ denotes the shuffle product, defined similarly to the iterated integral case

$$R_1 \otimes \ldots \otimes R_p \sqcup\!\sqcup R_{p+1} \otimes \ldots \otimes R_{p+q} = \sum_{\sigma\in\Sigma(p,q)} R_{\sigma(1)} \otimes \ldots \otimes R_{\sigma(p)} \otimes R_{\sigma(p+1)} \otimes \ldots \otimes R_{\sigma(p+q)} \tag{2.3.46}$$

Unlike the iterated integral, however, there are no basepoint conditions and we may take the shuffle product of symbols freely, for example

$$\mathcal{S}(\log z \log y) = z \sqcup\!\sqcup y = z \otimes y + y \otimes z \tag{2.3.47}$$

An unfortunate consequence of the property (2.3.42) is that any function that is proportional to $i\pi$ will vanish, and thus the symbol map is not injective. Furthermore because the symbol only admits weight one entries then any function containing the transcendental constants ζ_n will vanish as well. Naturally this complicates the problem of finding the inverse map from symbols to polylogarithms. In addition a general symbol does not necessarily correspond to a function, and we must impose an integrability condition for this to be the case. By continuing the recursion formula (2.3.39) a general symbol may be written as

$$\mathcal{S}(F_n) = \sum_{i_1,\ldots,i_n} F_{0,i_1,\ldots i_n} R_{i_1} \otimes \ldots \otimes R_{i_n} \tag{2.3.48}$$

And the integrability condition is defined as

$$\sum_{j=1}^{n-1}\sum_{i_1,\ldots,i_n} F_{0,i_1,\ldots i_n} R_{i_1}\otimes\ldots\otimes d\log R_{i_j}\wedge d\log R_{i_j+1}\otimes\ldots\otimes R_{i_n}=0 \tag{2.3.49}$$

Which is the same as the previous integrability condition (2.3.8) for closed forms ($d\omega_i = 0$). As we have seen (2.3.47) is a necessary condition for homotopy invariance, however it also implies the natural property of functions that partial derivatives commute at all weights. To see this we note that the partial derivative of a symbol is determined by its last entry

$$\mathcal{S}\left(\frac{\partial}{\partial x_i}F_n\right)=\sum_{i_1,\ldots,i_n} F_{0,i_1,\ldots i_n} R_{i_1}\otimes\ldots\otimes R_{i_{n-1}}\frac{\partial}{\partial x_i}\log R_{i_n} \tag{2.3.50}$$

By applying another partial derivative we may produce a symbol of weight $(n-2)$ which must be the same regardless of the order of differentiation. Thus by taking successive partial derivatives and requiring that they commute we arrive at (2.3.49). Similarly the first entry determines the monodromy of the symbol

$$\mathcal{S}(\mathcal{M}_{x_i=a}F_n)=\sum_{i_1,\ldots,i_n} F_{0,i_1,\ldots i_n}(\mathcal{M}_{x_i=a}\log R_{i_1})R_{i_2}\otimes\ldots\otimes R_{i_n} \tag{2.3.51}$$

Where

$$\mathcal{M}_{x_i=a}\log R=\begin{cases} 2\pi i & R\big|_{x_i=a}=0 \\ 0 & \text{otherwise}\end{cases} \tag{2.3.52}$$

It is relatively straightforward to compute the symbol of a function however it is often the case that we wish to find the function corresponding to a symbol. The first step is to make sure the symbol does indeed belong to a function, that is it must satisfy the integrability condition. Next we choose a particular path along which to integrate the symbol and obtain a MPL representation [72]. Thus given a set of k variables $\{x_i\}$ on which the letters of our symbol depend $R(x_1,\ldots,x_k)$ we shall choose an integration path γ which will be a composition of paths γ_i that allow only one variable to vary at a time. In other words the map is an integral over the edges of a k-dimensional hypercube where γ_i is the path along the edge that goes from the basepoint a_0 to x_k. Of course there are multiple paths that we may take determined by the order of integration

$$\gamma(\sigma)=\bigcup_{i=1}^{k}\gamma_{\sigma(i)} \tag{2.3.53}$$

Where σ denotes a permutation of the γ_i. Note that as long as two paths integrate up to the same point (i.e. they both end with the same γ_i) then, subject to some normal crossing conditions, the corresponding MPL representations are homotopy equivalent to each other. However we are not constrained to always integrate up to the same variable and we may obtain a representation of the same symbol in terms of MPLs that have

different top argument. Given that all the functions thus obtained are related through the symbol map then the choice of permutation is similar to choosing a gauge. Having selected the composition of paths we would wish to integrate our symbol along we may construct the corresponding MPL explicitly via the path decomposition formula

$$\begin{aligned}\tilde{\mathcal{S}}_{\sigma}^{-1}(R_1 \otimes \ldots \otimes R_n) = \sum_{0\leq i_1 \leq i_2 \ldots i_k \leq n} \int_{\gamma_{\sigma(1)}} d\log R_1 \ldots d\log R_{i_1} \\ \times \int_{\gamma_{\sigma(2)}} d\log R_{i_1+1} \ldots d\log R_{i_2} \ldots \int_{\gamma_{\sigma(k)}} d\log R_{i_k+1} \ldots d\log R_n\end{aligned} \tag{2.3.54}$$

Where $\int_{\gamma_{\sigma(j)}} d\log R_{i_j+1} d\log R_{i_j} = 1$ and thus drops out of the integral. Note that for a particular γ_i all variables except x_i are treated as constants and take values

$$x_i\big|_{\gamma_j} = \begin{cases} x_i & i < j \\ 0 & i > j \end{cases} \tag{2.3.55}$$

Where yet again we use canonical regularisation for any vanishing $R_i|_{\gamma j}$. Although the map (2.3.50) has been defined on a single sequence of tensor products we stress that it is only valid for an integrable symbol of the form (2.3.48). As already mentioned the solution to the inverse map is not unique. Apart from the obvious multitude of paths, the MPL representation is defined up to any functions that disappear under the symbol map, such as functions of non-maximal weight or those multiplied by transcendental constants.

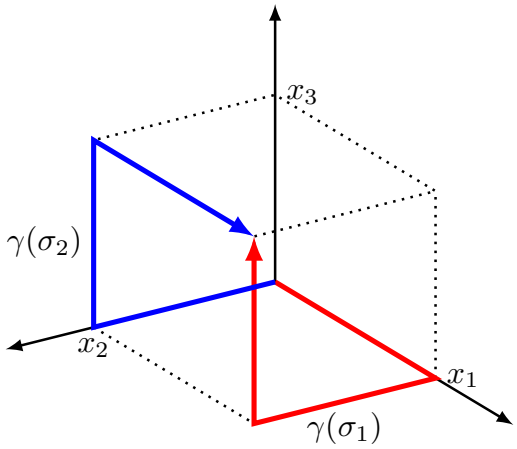


Figure 2.3.1: Two different paths we may choose to integrate our symbol along

As an example take the symbol of $G(x_1, x_2; x_3)$

$$\begin{aligned}\mathcal{S}(G(x_1,x_2;x_3)) =& (x_1-x_3)\otimes\frac{(x_1-x_2)}{x_2} + (x_2-x_3)\otimes\frac{(x_1-x_3)}{(x_1-x_2)} \\ &+x_1\otimes\frac{x_2}{(x_1-x_2)} + x_2\otimes\frac{(x_1-x_2)}{(x_1-x_3)}\end{aligned} \tag{2.3.56}$$

Note that the symbol is automatically integrable because it is derived from a function. If we choose the path starting at the origin $\gamma(\sigma_1)$ in figure 2.3.1 then we recover $G(x_1, x_2; x_3)$

$$\tilde{\mathcal{S}}^{-1}_{\sigma_1}(\mathcal{S}(G(x_1, x_2; x_3))) = G(x_1, x_2; x_3) \tag{2.3.57}$$

Similarly we would obtain the same result if we would permute x_1 with x_2 in the path since the top argument would still be x_3. Alternatively if we choose the path $\gamma(\sigma_2)$ we obtain

$$\begin{aligned}\tilde{\mathcal{S}}^{-1}_{\sigma_2}(\mathcal{S}(G(x_1, x_2; x_3))) = G(x_2, x_3; x_1) - G(x_2, 0; x_1) + G(0, x_2; x_3) + G(x_2; x_3)G(x_3; x_1) \\ - G(x_2; x_1)G(x_2; x_3) + G(0; x_3)G(x_2; x_1)\end{aligned} \tag{2.3.58}$$

By reviewing all the properties above it is clear that the strength of the symbol lies in its ability to elucidate functional relationships between MPLs. However there is a substantial caveat in that it determines functional equivalence only up to terms proportional to multiple zeta values, which carry transcendental weight but vanish under the symbol map. In order to address this issue one could use an alternative definition of the symbol, which is obtained from summing over dissections of a rooted and decorated polygon [34]. This method has the advantage that its combinatoric nature does not distinguish between constants and variables. Another way would to use the fact that symbols and MPLs form a Hopf algebra [75] and use the coproduct which allows for the incorporation of zeta values.

2.3.3 Hopf algebras

Although we have already described an extensive list of properties for MPLs our manipulation of symbols and iterated integrals shall rely on their more general algebraic structure. The structure in question is that of Hopf algebra and its applications range from integrability to the study of sub-divergences of a Feynman graph. However before establishing the relationship with MPLs we begin by reviewing the notions of algebras and co-algebras. An algebra over a field k (k-algebra) is a k-vector space A together with two linear maps

$$m : A \otimes_k A \to A \quad \text{and} \quad u : k \to A \tag{2.3.59}$$

Where $\otimes_k$ denotes the tensor product over the field k which we shall leave implicit in the following. The first operation is called multiplication and is associative. If we take two elements $a, b \in A$ and define $m(a \otimes b) \equiv a \cdot b$ then

$$m(\mathrm{id} \otimes m)(a \otimes b \otimes c) = a \cdot (b \cdot c) = m(m \otimes \mathrm{id})(a \otimes b \otimes c) = (a \cdot b) \cdot c \tag{2.3.60}$$

where we define for two functions f and g

$$(f \otimes g)(a \otimes b) = f(a) \otimes g(b) \tag{2.3.61}$$

the second is called the unit element and embeds the field into the vector space

$$m(\mathrm{id} \otimes u)(a \otimes k) = ka = m(u \otimes \mathrm{id})(k \otimes a) \tag{2.3.62}$$

Thus this implies the existence of the unit element for the k-algebra

$$u(1_k)A = 1_A A = A \tag{2.3.63}$$

We note that this is not the usual definition of an algebra which is defined on $A \times A$ rather than on the tensor product. However the two definitions are completely compatible. At this point we can see that the previous description of MPLs may be recast in terms of an algebra over the rationals where multiplication is given by the shuffle product. The weight gives a natural grading that is preserved by the shuffle product and we can write the entire algebra of multiple polylogarithms as

$$A_{MPL} = \bigoplus_{n=0}^{\infty} A_n \qquad m(A_{n_1} \otimes A_{n_2}) \subset A_{n_1+n_2} \tag{2.3.64}$$

Where A_0 corresponds to the field of rational functions over which we have defined our algebra. Next we proceed to define the coalgebra by reversing the definitions above. A k-coalgebra is a k-vector space C with two linear maps

$$\Delta : C \to C \otimes C \qquad \text{and} \qquad \varepsilon : C \to k \tag{2.3.65}$$

The map Δ is called the co-product and obeys the co-associativity condition

$$(\Delta \otimes \mathrm{id})\Delta = (\mathrm{id} \otimes \Delta)\Delta \tag{2.3.66}$$

and the co-unit satisfies

$$(\mathrm{id} \otimes \varepsilon)\Delta C = C \otimes 1 \tag{2.3.67}$$

Conversely to the multiplication of elements the co-multiplication axiom amounts to a type of decomposition and the coassociativity condition states that the order of this decomposition is irrelevant. To expand more on this we introduce the Sweedler notation for coproducts

$$\Delta(a) = \sum_i a_i^{(1)} \otimes a_i^{(2)} \tag{2.3.68}$$

Where $a, a_i^{(j)} \in C$ are some elements of the co algebra. We can apply the coproduct again and obtain the two sides of (2.3.67)

$$(\Delta \otimes \mathrm{id})\Delta(a) = \sum_{i,j} a_{i,j}^{(1,1)} \otimes a_{i,j}^{(1,2)} \otimes a_i^{(2)} \tag{2.3.69}$$

$$(\mathrm{id} \otimes \Delta)\Delta(a) = \sum_{i,j} a_i^{(1)} \otimes a_{i,j}^{(2,1)} \otimes a_{i,j}^{(2,2)} \tag{2.3.70}$$

The striking feature of coassociativity is that while the above equations are in general different, for a co-algebra they will be equal. Thus as a consequence there is essentially only one way to iterate the coproduct. We define an algebra homomorphism as a linear map $\phi : A \to B$ between two k-algebras if it satisfies

$$\phi(m(A \otimes A)) = m(\phi(A) \otimes \phi(A)) \quad \phi(u_A(k)) = u_B(k) \tag{2.3.71}$$

Similarly a coalgebra homomorphism $\phi : C \to D$ is one which commutes with the coproduct

$$\Delta(\phi(C)) = (\phi \otimes \phi)\Delta(C) \quad \varepsilon_D(\phi(C)) = \varepsilon_C(C) \tag{2.3.72}$$

We define a bialgebra A as a k-vector space that is simultaneously an algebra and a co-algebra . That is a bialgebra is the tuple $A = (A, m, u, \Delta, \varepsilon)$ where (A, m, u) is an algebra and (A, Δ, ε) is a coalgebra such that the linear maps on one are homomorphism on the other.

$$\Delta(m_A(a \otimes b)) = m_{A\otimes A}(\Delta(a) \otimes \Delta(b)) \tag{2.3.73}$$

In general if we take two k-bialgebras A and B then their tensor product is also one. We can see this by noting

$$A \otimes B \otimes A \otimes B \xrightarrow{\mathrm{id}\otimes\tau\otimes\mathrm{id}} A \otimes A \otimes B \otimes B \xrightarrow{m_A \otimes m_B} A \otimes B \tag{2.3.74}$$

where τ is the flip operator $\tau(a \otimes b) = b \otimes a$. And similarly for the coproduct

$$A \otimes B \xrightarrow{\Delta_A \otimes \Delta_B} A \otimes A \otimes B \otimes B \xrightarrow{\mathrm{id}\otimes\tau\otimes\mathrm{id}} A \otimes B \otimes A \otimes B \tag{2.3.75}$$

Note that the presence of the inversion step implies that

$$m_{A\otimes B} = (m_A \otimes m_B)(\mathrm{id} \otimes \tau \otimes \mathrm{id}) \iff (a_1 \otimes b_1)(a_2 \otimes b_2) = (a_1 a_2 \otimes b_1 b_2) \tag{2.3.76}$$

For iterated integrals the coproduct is non trivially given by

$$\Delta(I(a_0;a_1,\ldots,a_n;a_{n+1})) = \sum_{0=i_1<i_2\ldots<i_k<i_{k+1}=n} I(a_0;a_{i_1},\ldots,a_{i_k};a_{n+1}) \otimes \left[\prod_{p=0}^{k} I(a_{i_p};a_{i_p+1},\ldots,a_{i_{p+1}-1};a_{i_{p+1}})\right] \tag{2.3.77}$$

In the case that the entries are not generic we replace all quantities on the right hand side by their canonically regularised forms. As already mentioned, the coproduct may be used to simplify expressions in a similar way to the symbol. For example let us take the coproduct for the logarithms and classical polylogarithms

$$\Delta(\log^n z) = \sum_{k=0}^{n} \binom{n}{k} \log^k z \otimes \log^{n-k} z \tag{2.3.78}$$

$$\Delta(\mathrm{Li}_n(z)) = 1 \otimes \mathrm{Li}_n(z) + \mathrm{Li}_n(z) \otimes 1 + \sum_{k=1}^{n-1} \mathrm{Li}_{n-k}(z) \otimes \frac{\log^k z}{k!} \tag{2.3.79}$$

We can see that the iterative application of the coproduct together with the coassociativity property leads to an object similar to the symbol. The marked difference however is that the coproduct has a non trivial action on the transcendental constants. Specifically the coproduct of the odd zeta values is given by

$$\Delta(\zeta_{2n+1}) = \Delta(\mathrm{Li}_{2n+1}(1)) = 1 \otimes \zeta_{2n+1} + \zeta_{2n+1} \otimes 1 \tag{2.3.80}$$

Even though $\mathrm{Li}_{2n}(1) = \zeta_{2n}$, the even zeta values are not independent and if we consider the homomorphism property of the coproduct we can see that the above equation is not consistent any more. Instead we have to modify the coproduct [76]

$$\Delta(\zeta_{2n}) = \zeta_{2n} \otimes 1 \quad \Longleftrightarrow \quad \Delta(\pi) = \pi \otimes 1 \tag{2.3.81}$$

That is, the coproduct for MZVs becomes a coaction by taking the second entry in the tensor product modulo ζ_2, thus defining the μ algebra

$$\Delta : A \to A \otimes \mu \qquad \mu \equiv A/\zeta_2 \tag{2.3.82}$$

Note that the second relation in (2.3.81) comes from the fact that $\zeta_2 \propto \pi^2$. Of course these are the simplest zeta values and analogously to polylogarithms the multiple zeta values also form a bialgebra. To finish our definition of a Hopf algebra we require one more ingredient: a linear endomorphism S called the antipode such that for $a \in A$

$$\varepsilon(a) = m(\mathrm{id} \otimes S)\Delta(a) = m(S \otimes \mathrm{id})\Delta(a) \tag{2.3.83}$$

The antipode may be determined recursively for MPLs by noting that the counit is given by $\varepsilon(1) = 1$ and is 0 for all other iterated integrals. Thus

$$\mu(S \otimes \mathrm{id})\Delta(G(\vec{a}; z))) = \mu(\mathrm{id} \otimes S)\Delta(G(\vec{a}; z))) = 0\,, \text{if } |\vec{a}| \geq 1\,, \tag{2.3.84}$$

Taking the classical polylogarithms as an example we have the relations

$$\begin{aligned} 1 =& S(1) \\ 0 =& S(1)\,\mathrm{Li}_1(z) + S(\mathrm{Li}_1(z)) \\ 0 =& S(1)\,\mathrm{Li}_2(z) + S(\mathrm{Li}_2(z)) + S(\mathrm{Li}_1(z))\log(z) \\ &\vdots \end{aligned} \tag{2.3.85}$$

Given that the antipode is an endomorphism it obeys

$$S(ab) = S(a)S(b) \quad \varepsilon(S(a)) = \varepsilon(a) \quad \Delta(S(a)) = (S \otimes S)\tau\Delta(a) \tag{2.3.86}$$

As we shall see later the antipode will play a central role in our study of single valued MPLs, i.e. iterated integrals with trivial monodromies. Given that we are always interested in the monodromies and differentials of MPLs we end this section by showing how these operations commute with the coproduct. Thus for an MPL $F_n(x_1, \ldots, x_k)$

$$\Delta\left(\frac{\partial}{\partial x_i} F_n\right) = \left(\mathrm{id} \otimes \frac{\partial}{\partial x_i}\right) \Delta(F_n) \tag{2.3.87}$$

$$\Delta\left(\mathcal{M}_{x_i=a} F_n\right) = \left(\mathcal{M}_{x_i=a} \otimes \mathrm{id}\right) \Delta(F_n) \tag{2.3.88}$$

These relations were conjectured in [75] and proved for generic arguments. Note that these operators act on the coproduct in the same way as they do on the symbol. As has become evident there are numerous similarities between symbol and coproduct. Indeed it is possible to make precise the relationship between them

$$\mathcal{S} \equiv \Delta_{1,\ldots,1} \bmod \pi \tag{2.3.89}$$

Where $\Delta_{1,\ldots,1}$ is defined as the maximum iteration of the coproduct.

$$\Delta_{1,\ldots,1}(F_n) = (\Delta \otimes \overbrace{\mathrm{id} \otimes \ldots \otimes \mathrm{id}}^{n-1}) \ldots (\Delta \otimes \mathrm{id})\Delta(F_n) \tag{2.3.90}$$

Iterated integrals possess a rich mathematical structure, only a subset of which has been described above. It shall be the main focus of this thesis to further elucidate their algebraic properties and their application to scattering amplitudes. Thus in the next chapter we proceed to study their role in solving a two loop five point conformal integral.

CHAPTER 3

Differential equations for scalar integrals

Although the multiple polylogarithms are a mathematically interesting and versatile class of functions they only become of relevance if we can use them to describe our amplitudes. It is often quite challenging to recast a given solution to a Feynman integral in terms of MPLs so that we may make use of their properties. For example when studying interesting regimes such as Regge limits or for numerical evaluation. Furthermore it is clearly more desirable to avoid any intermediary steps and try to solve the integral directly in terms of MPLs or even their symbols. Having outlined the iterated integral structure in the previous section a natural approach to consider would be to study what are the particular differential equations that our integral obeys.

Indeed, the use of differential equations for scattering amplitudes is well established. The usual approach is to differentiate the integral with respect to masses and kinematical invariants to obtain a combination of already evaluated Feynman integrals and then solve the differential equation with appropriate boundary conditions. The method was developed in [77] as an extension of the integration by parts identities which uses differentiation with respect to the loop momenta under the integral to derive algebraic relations between integrals. Thus they are used to express and evaluate an amplitude as a linear combination of a preferred set of "master integrals", which are an entire field of study in themselves.

Although the above methods are completely general our focus shall be on the differential equations obeyed by scalar integrals with conformal symmetry. By combining differential operators acting on various legs we intend to exhibit the use of MPLs as a

general technology for scattering amplitudes. This shall be done by changing variables and exploiting the symbols in such a way that we may easily integrate the differential equation obeyed by the integral. Finally in order to resolve the ambiguities in the integration we shall introduce single valued MPLs and impose that our solution be single valued as well. This is a general physical constraint and single valuedness shall play an important role in subsequent chapters where we treat amplitudes in the multi Regge limit of the kinematics.

3.1 The integral

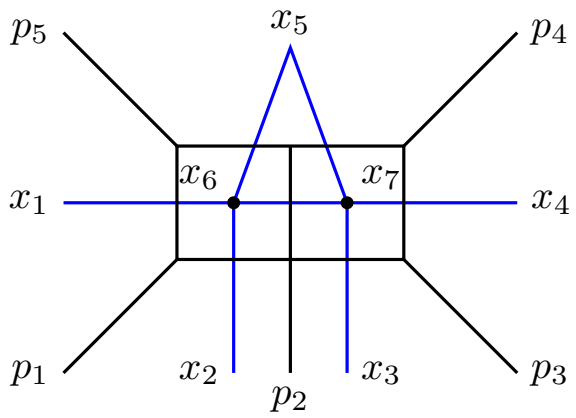


Figure 3.1.1: 2-loop , 5-point massive scalar conformal integral

In [78–80] it was shown that it is possible to obtain two and three-point graphs as the finite limit of four point graphs that obey certain differential equations. In this section we wish to generalise these ideas to higher points, by analysing the behaviour of a 5-point, massive 2-loop integral under various differential operators. Let us begin by writing the integral depicted in 3.1.1 in terms of the dual coordinates

$$I_{12;34;5} = \frac{1}{\pi^4} \int \frac{d^4x_6 d^4x_7}{x_{16}^2 x_{26}^2 x_{37}^2 x_{47}^2 x_{67}^2 x_{56}^2 x_{57}^2} \tag{3.1.1}$$

By using the inversion of the dual coordinates

$$I : x_{ij}^2 \to \frac{x_{ij}^2}{x_i^2 x_j^2} \tag{3.1.2}$$

we see that the integrand transforms in a covariant manner

$$\frac{d^4x_6 d^4x_7}{x_{16}^2 x_{26}^2 x_{37}^2 x_{47}^2 x_{67}^2 x_{56}^2 x_{57}^2} \to (x_5^4 x_1^2 x_2^2 x_3^2 x_4^2) \frac{d^4x_6 d^4x_7}{x_{16}^2 x_{26}^2 x_{37}^2 x_{47}^2 x_{67}^2 x_{56}^2 x_{57}^2} \tag{3.1.3}$$

Note that the factors coming from the inversion of the loop dual coordinates x_6 and x_7 are precisely canceled by the Jacobian of the corresponding change of variables. We can

exploit this conformal symmetry and write the integral as

$$I_{12;34;5} = \frac{f(s_1, \ldots, s_5)}{x_{15}^2 x_{25}^2 x_{34}^2} \tag{3.1.4}$$

Where f is a function of the five multiplicatively independent conformal cross-ratios that one can construct from the five external points,

$$s_1 = \frac{x_{12}^2 x_{35}^2 x_{45}^2}{x_{34}^2 x_{15}^2 x_{25}^2}\,, \quad s_2 = \frac{x_{13}^2 x_{45}^2}{x_{34}^2 x_{15}^2}\,, \quad s_3 = \frac{x_{14}^2 x_{35}^2}{x_{34}^2 x_{15}^2}\,, \quad s_4 = \frac{x_{23}^2 x_{45}^2}{x_{34}^2 x_{25}^2}\,, \quad s_5 = \frac{x_{24}^2 x_{35}^2}{x_{34}^2 x_{25}^2} \tag{3.1.5}$$

Next we would like to separate the function f in terms of a purely transcendental and rational piece. To do so we need to calculate the leading singularity of the integral which gives the rational prefactor. The leading singularity is obtained by taking the residues of the integral on each of its global poles evaluated on a contour describing the maximal torus surrounding that pole. In order to demonstrate the technique it is useful to consider the example [81]

$$g_{1234} = \int \frac{d^4 x_5}{x_{15}^2 x_{25}^2 x_{35}^2 x_{45}^2} \tag{3.1.6}$$

We then make a change of coordinates $p_i = x_{i5}^2$ and compute the Jacobian

$$J = \det\left(\frac{\partial p_i}{\partial x_5^\mu}\right) \tag{3.1.7}$$

This can be rendered more transparent by considering the square of the Jacobian

$$J^2 = \det(4 x_{i5} \cdot x_{j5}) = 16 \det(x_{ij}^2 - x_{i5}^2 - x_{j5}^2) \tag{3.1.8}$$

Thus the integral (3.1.7) now becomes

$$g_{1234} = \int \frac{d^4 p_i}{p_1 p_2 p_3 p_4 J} \tag{3.1.9}$$

Which has the following residue on the global pole $p_1 = p_2 = p_3 = p_4 = 0$

$$g_{1234} \to \frac{1}{4\sqrt{\det(x_{ij}^2)}} = \frac{1}{4\lambda_{1234}} \tag{3.1.10}$$

Where by convention we have ignored any contour that may encircle the pole produced by the Jacobian. This may be justified by the fact that the integrals posses no pole at infinity and any residue produced from the Jacobian factor is equivalent to the one already considered. Going back to our five point integral we can see that it may be separated into two parts

$$I_{12;34;5} = \frac{1}{\pi^4} \int \frac{d^4 x_6}{x_{16}^2 x_{26}^2 x_{56}^2} \int \frac{d^4 x_7}{x_{37}^2 x_{47}^2 x_{57}^2 x_{67}^2} \tag{3.1.11}$$

Using the previous example it is easy to see that upon taking residues we may reduce

this expression to

$$\frac{1}{\pi^4}\int \frac{d^4x_6}{x_{16}^2 x_{26}^2 x_{56}^2, \lambda_{3456}} \tag{3.1.12}$$

Now we must calculate the Jacobian with respect to the new denominator

$$J = \det\left(\frac{\partial(x_{16}^2, x_{26}^2, x_{56}^2, \lambda_{3456})}{\partial x_6^\mu}\right) \tag{3.1.13}$$

The procedure then is essentially the same as before, and the leading singularity is obtained by considering the square of the Jacobian under the global residue $x_{16}^2 = x_{26}^2 = x_{56}^2 = 0$ which also influences the remaining pole $\lambda_{3456} = x_{36}^2 x_{45}^2 - x_{35}^2 x_{46}^2 = 0$, from which we infer that $x_{36}^2 = x_{46}^2 = 0$ and we obtain

$$J^2 = 16\det\begin{pmatrix} x_{ij}^2 & x_{i3}^2 x_{45}^2 - x_{i4}^2 x_{35}^2 \\ x_{j3}^2 x_{45}^2 - x_{j4}^2 x_{35}^2 & -2x_{34}^2 x_{35}^2 x_{45}^2 \end{pmatrix} \tag{3.1.14}$$

Finally having obtained the leading singularity we may use our cross ratios to separate the function $f(s_1, \ldots, s_5)$ into a rational and purely transcendental function $\hat{f}$ of degree four.

$$f(s_1, s_2, s_3, s_4, s_5) = \frac{1}{\sqrt{(s_2 - s_3 - s_4 + s_5)^2 - 4s_1}}\hat{f}(s_1, s_2, s_3, s_4, s_5) \tag{3.1.15}$$

In order to simplify the problem we would like to make use of the conformal symmetry in order to remove one of the points by taking it to infinity. There are two ways of obtaining a topologically distinct four-point integral by taking a point to infinity. First we may take x_5 to infinity

$$\lim_{x_5\to\infty} x_5^4 I_{12;34;5} = H_{12;34} = \frac{1}{\pi^4}\int \frac{d^4x_6 d^4x_7}{x_{16}^2 x_{26}^2 x_{37}^2 x_{47}^2 x_{67}^2} = \frac{f(t_1, \ldots, t_5)}{x_{34}^2} \tag{3.1.16}$$

Here we have defined t_i to be the limit of the s_i cross-ratios

$$t_1 = \frac{x_{12}^2}{x_{34}^2}, \qquad t_2 = \frac{x_{13}^2}{x_{34}^2}, \qquad t_3 = \frac{x_{14}^2}{x_{34}^2}, \qquad t_4 = \frac{x_{23}^2}{x_{34}^2}, \qquad t_5 = \frac{x_{24}^2}{x_{34}^2} \tag{3.1.17}$$

Alternatively we may take one of the other points, x_1 say, to infinity

$$\lim_{x_1\to\infty} x_1^2 I_{12;34;5} = I_{2;34;5} = \frac{1}{\pi^4}\int \frac{d^4x_6 d^4x_7}{x_{16}^2 x_{26}^2 x_{37}^2 x_{47}^2 x_{67}^2} \tag{3.1.18}$$

We will focus on the former limit for the moment, as the integral has more obvious symmetries. Indeed we have the relations

$$H_{12;34} = H_{21;34} = H_{34;12} \tag{3.1.19}$$

Which imply the following relations for the f function

$$f(t_1,t_2,t_3,t_4,t_5) = f(t_1,t_4,t_5,t_2,t_3) = \frac{1}{t_1} f\left(\frac{1}{t_1},\frac{t_2}{t_1},\frac{t_4}{t_1},\frac{t_3}{t_1},\frac{t_5}{t_1}\right) \tag{3.1.20}$$

Next we proceed to derive a solvable differential equation for the purely transcendental part of f in terms of MPLs.

3.2 Differential equations

We can derive differential equations for the integral by acting with the Laplace operator on external points with a only a single propagator attached. The Laplace operator produces a delta function which, upon integration, reduces the loop order by one [2,78,80].

$$\Box_i \frac{1}{x_{ij}^2} = -4\pi^2 \delta^4(x_i - x_j) \tag{3.2.1}$$

The integral H therefore obeys the differential equations

$$\begin{aligned} \Box_1 H_{12;34} &= -\frac{4}{x_{12}^2} I_{134} & \Box_2 H_{12;34} &= -\frac{4}{x_{12}^2} I_{234} \\ \Box_3 H_{12;34} &= -\frac{4}{x_{34}^2} I_{123} & \Box_4 H_{12;34} &= -\frac{4}{x_{34}^2} I_{124} \end{aligned} \tag{3.2.2}$$

Similar equations arise after using integration by parts to produce a Laplace operator acting on the intermediate propagator

$$(\partial_1+\partial_2)^2 H_{12;34} = (\partial_3+\partial_4)^2 H_{12;34} = -4 I_{1234} \tag{3.2.3}$$

Where I_{134} and I_{1234} are the well-known one-loop graphs

$$I_{134} = \frac{1}{\pi^2}\int \frac{d^4x_7}{x_{17}^2 x_{37}^2 x_{47}^2} = \frac{1}{x_{34}^2}\Phi^{(1)}(t_2,t_3) \tag{3.2.4}$$

$$I_{1234} = \frac{1}{\pi^2}\int \frac{d^4x_7}{x_{17}^2 x_{27}^2 x_{37}^2 x_{47}^2} = \frac{1}{x_{12}^2 x_{34}^2}\Phi^{(1)}\left(\frac{t_2 t_5}{t_1},\frac{t_3 t_4}{t_1}\right) \tag{3.2.5}$$

The function on the right hand side $\Phi^{(1)}$ is known as the one-loop ladder function

$$\Phi^{(1)}\big((1-x)(1-\bar{x}),x\bar{x}\big) = -\frac{1}{x-\bar{x}}\phi^{(1)}\left(\frac{x}{x-1},\frac{\bar{x}}{\bar{x}-1}\right) = \frac{1}{x-\bar{x}}\phi^{(1)}(x,\bar{x})\,, \tag{3.2.6}$$

And $\phi^{(1)}$ is the the Bloch-Wigner dilogarithm

$$\phi^{(1)}(x,\bar{x}) = -\log(x\bar{x})\big(\mathrm{Li}_1(x)-\mathrm{Li}_1(\bar{x})\big) + 2\big(\mathrm{Li}_2(x)-\mathrm{Li}_2(\bar{x})\big) \tag{3.2.7}$$

The variables $x, \bar{x}$ that appear above, are indeed complex conjugates of each other, thus making the $\phi^{(1)}$ single-valued as one would expect from the four point graphs. The single valued property can be seen from the symbol

$$\mathcal{S}(\phi^{(1)}(x,\bar{x})) = (1-x)(1-\bar{x}) \otimes x/\bar{x} - x\bar{x} \otimes (1-\bar{x})/(1-x) \tag{3.2.8}$$

Recalling that the monodromy operator acts on the first entry of the symbol we can easily check that the symbol is single valued. We begin our analysis by focusing on the relations generated by $\Box_1$, $\Box_2$ and $(\partial_1+\partial_2)^2$. From the first two differential equations in (3.2.2) we obtain the following equations for f

$$\begin{aligned}\mathcal{O}_1 f(t_1,t_2,t_3,t_4,t_5) &= -\frac{1}{t_1}\Phi^{(1)}(t_2,t_3)\\ \mathcal{O}_2 f(t_1,t_2,t_3,t_4,t_5) &= -\frac{1}{t_1}\Phi^{(1)}(t_4,t_5)\end{aligned} \tag{3.2.9}$$

Where $\mathcal{O}_1$ and $\mathcal{O}_2$ are the operators $\Box_1$, $\Box_2$ expressed in terms of $\partial_i = \frac{\partial}{\partial t_i}$

$$\begin{aligned}\mathcal{O}_1 =& 2(\partial_1+\partial_2+\partial_3) + t_1\partial_1^2 + t_2\partial_2^2 + t_3\partial_3^2\\ &+ (t_1+t_2-t_4)\partial_1\partial_2 + (t_1+t_3-t_5)\partial_1\partial_3 + (t_2+t_3-1)\partial_2\partial_3 \qquad (3.2.10)\\ \mathcal{O}_2 =& 2(\partial_1+\partial_4+\partial_5) + t_1\partial_1^2 + t_4\partial_4^2 + t_5\partial_5^2\\ &+ (t_1+t_4-t_2)\partial_1\partial_4 + (t_1+t_5-t_3)\partial_1\partial_5 + (t_4+t_5-1)\partial_4\partial_5 \qquad (3.2.11)\end{aligned}$$

Equation (3.2.3) becomes

$$\mathcal{O}_{12} f(t_1,t_2,t_3,t_4,t_5) = -\frac{1}{t_1}\Phi^{(1)}\left(\frac{t_2t_5}{t_1}, \frac{t_3t_4}{t_1}\right) \tag{3.2.12}$$

However it is more convenient to work with the operator $\tilde{\mathcal{O}}_{12} = \mathcal{O}_{12} - \mathcal{O}_1 - \mathcal{O}_2$ which represents the action of $2\partial_1 \cdot \partial_2$ in the x variables. The operator $\tilde{\mathcal{O}}_{12}$ is then given by

$$\begin{aligned}\tilde{\mathcal{O}}_{12} =& -4\partial_1 - 2t_1\partial_1^2 + (t_4-t_1-t_2)\partial_1\partial_2 + (t_5-t_1-t_3)\partial_1\partial_3\\ &+ (t_2-t_4-t_1)\partial_1\partial_4 + (t_3-t_1-t_5)\partial_1\partial_5 + (t_2+t_4-t_1)\partial_2\partial_4\\ &+ (t_3+t_4-t_1-1)\partial_2\partial_5 + (t_2+t_5-t_1-1)\partial_3\partial_4 + (t_3+t_5-t_1)\partial_3\partial_5\end{aligned} \tag{3.2.13}$$

Solving the above equations directly in terms of polylogarithmic functions is not straightforward. However we may do so if we restrict ourselves to a special case of two dimensional kinematics, where we choose to make the points coplanar. Therefore we intro-

duce a new set of coordinates

$$\begin{aligned} t_1 &= (a_1 - a_2)(\bar{a}_1 - \bar{a}_2) + y \\ t_2 &= a_1\bar{a}_1 \\ t_3 &= (1 - a_1)(1 - \bar{a}_1) \\ t_4 &= a_2\bar{a}_2 \\ t_5 &= (1 - a_2)(1 - \bar{a}_2) \end{aligned} \tag{3.2.14}$$

Where the coordinate y is treated as a deformation away from coplanarity. By setting $y = 0$ then we may identify the plane with the complex numbers and make use of translational and rotational symmetry to set $x_3 = 0$ and $x_4 = 1$ with $x_1 = a_1$ and $x_2 = a_2$. In terms of the variables $\{a_1, \bar{a}_1, a_2, \bar{a}_2, y\}$ the operators become,

$$\begin{aligned} \mathcal{O}_1 =& \frac{1}{a_1 - \bar{a}_1}\partial_{a_1}\partial_{\bar{a}_1}(a_1 - \bar{a}_1) - \frac{y}{a_1 - \bar{a}_1}\partial_y(\partial_{a_1} - \partial_{\bar{a}_1}) + \frac{a_2 - \bar{a}_2}{a_1 - \bar{a}_1}\partial_y y \partial_y \\ \mathcal{O}_2 =& \frac{1}{a_2 - \bar{a}_2}\partial_{a_2}\partial_{\bar{a}_2}(a_2 - \bar{a}_2) - \frac{y}{a_2 - \bar{a}_2}\partial_y(\partial_{a_2} - \partial_{\bar{a}_2}) + \frac{a_1 - \bar{a}_1}{a_2 - \bar{a}_2}\partial_y y \partial_y \\ \tilde{O}_{12} =& \left(-2\partial_y + \frac{1}{a_2 - \bar{a}_2}(\partial_{a_1} - \partial_{\bar{a}_1}) + \frac{1}{a_1 - \bar{a}_1}(\partial_{a_2} - \partial_{\bar{a}_2})\right) y\partial_y \\ & - \frac{y}{(a_1 - \bar{a}_1)(a_2 - \bar{a}_2)}(\partial_{a_1} - \partial_{\bar{a}_1})(\partial_{a_2} - \partial_{\bar{a}_2}) + \partial_{a_1}\partial_{\bar{a}_2} + \partial_{\bar{a}_1}\partial_{a_2} \end{aligned} \tag{3.2.15}$$

However we may not restrict them to the planar case individually due to the $\partial_y y$ terms which do not commute with the coplanar condition and may introduce inconsistencies. In order to remedy this we may consider a linear combination $\mathcal{O} = c_1\mathcal{O}_1 + c_2\mathcal{O}_2 + c_{12}\tilde{\mathcal{O}}_{12}$ of all three operators. Thus we find that the resultant operator contains solely products of $y\partial_y$ terms if the following condition holds

$$c_{12} = \frac{1}{2}\left(c_1\frac{a_2 - \bar{a}_2}{a_1 - \bar{a}_1} + c_2\frac{a_1 - \bar{a}_1}{a_2 - \bar{a}_2}\right) \tag{3.2.16}$$

Imposing the above condition then allows us to restrict to the coplanar case $y = 0$. Furthermore if we choose $c_1 = (a_1 - \bar{a}_1)^2$ and $c_2 = (a_2 - \bar{a}_2)^2$ we find the operator $\mathcal{O}$ takes a particularly convenient factorised form

$$\begin{aligned} \mathcal{O} =& \big[(a_1 - \bar{a}_1)\partial_{a_1} + (a_2 - \bar{a}_2)\partial_{a_2}\big]\big[(a_1 - \bar{a}_1)\partial_{\bar{a}_1} + (a_2 - \bar{a}_2)\partial_{\bar{a}_2} - 1\big] \\ =& \big[(a_1 - \bar{a}_1)\partial_{a_1} + (a_2 - \bar{a}_2)\partial_{a_2}\big]\frac{1}{a_1 - \bar{a}_1}\big[(a_1 - \bar{a}_1)\partial_{\bar{a}_1} + (a_2 - \bar{a}_2)\partial_{\bar{a}_2}\big](a_1 - \bar{a}_1) \end{aligned} \tag{3.2.17}$$

We recall that the operator $\mathcal{O}$ acts on the function f. If we take into account the form of the leading singularity we find (making a choice for the branch of the square root)

$$\frac{1}{\sqrt{(t_2 - t_3 - t_4 + t_5)^2 - 4t_1}} = \frac{1}{a_1 - \bar{a}_1 - a_2 + \bar{a}_2} \tag{3.2.18}$$

Hence by commuting the operator past the rational term we obtain

$$\mathcal{O}f(a_1,\bar{a}_1,a_2,\bar{a}_2)=\frac{1}{a_1-\bar{a}_1-a_2+\bar{a}_2}\hat{\mathcal{O}}\hat{f}(a_1,\bar{a}_1,a_2,\bar{a}_2) \tag{3.2.19}$$

with

$$\hat{\mathcal{O}}=(a_1-\bar{a}_1)^2\partial_{a_1}\partial_{\bar{a}_1}+(a_1-\bar{a}_1)(a_2-\bar{a}_2)(\partial_1\partial_{\bar{a}_2}+\partial_{a_2}\partial_{\bar{a}_1})+(a_2-\bar{a}_2)^2\partial_{a_2}\partial_{\bar{a}_2} \tag{3.2.20}$$

We can again rewrite this operator in a factorised form, $\hat{\mathcal{O}}=\hat{\mathcal{O}}_1\hat{\mathcal{O}}_2$, where

$$\hat{\mathcal{O}}_1=\left[(a_1-\bar{a}_1)\partial_{a_1}+(a_2-\bar{a}_2)\partial_{a_2}-1\right] \tag{3.2.21}$$

$$\hat{\mathcal{O}}_2=\left[(a_1-\bar{a}_1)\partial_{\bar{a}_1}+(a_2-\bar{a}_2)\partial_{\bar{a}_2}\right] \tag{3.2.22}$$

Thus the purely transcendental part of the integral must satisfy the following equation

$$\begin{aligned}\hat{\mathcal{O}}\hat{f}(a_1,\bar{a}_1,a_2,\bar{a}_2)=-\frac{a_1-\bar{a}_1-a_2+\bar{a}_2}{(a_1-a_2)(\bar{a}_1-\bar{a}_2)}\Bigg[&(c_1-c_{12})\Phi^{(1)}(t_2,t_3)\\&+c_{12}\Phi^{(1)}\Big(\frac{t_2t_5}{t_1},\frac{t_3t_4}{t_1}\Big)\\&+(c_2-c_{12})\Phi^{(1)}(t_4,t_5)\Bigg]\end{aligned} \tag{3.2.23}$$

Putting in the choices we made above for c_1 and c_2 and recalling the constraint (3.2.16) we find that the equation we have to solve is given by

$$\begin{aligned}\hat{\mathcal{O}}\hat{f}(a_1,\bar{a}_1,a_2,\bar{a}_2)=&-\frac{(a_1-\bar{a}_1-a_2+\bar{a}_2)^2(a_1-\bar{a}_1)}{(a_1-a_2)(\bar{a}_1-\bar{a}_2)}\Phi^{(1)}(t_2,t_3)\\&-\frac{(a_1-\bar{a}_1-a_2+\bar{a}_2)(a_1-\bar{a}_1)(a_2-\bar{a}_2)}{(a_1-a_2)(\bar{a}_1-\bar{a}_2)}\Phi^{(1)}\Big(\frac{t_2t_5}{t_1},\frac{t_3t_4}{t_1}\Big)\\&+\frac{(a_1-\bar{a}_1-a_2+\bar{a}_2)^2(a_2-\bar{a}_2)}{(a_1-a_2)(\bar{a}_1-\bar{a}_2)}\Phi^{(1)}(t_4,t_5)\end{aligned} \tag{3.2.24}$$

Using the definition of the one loop ladder function we can make explicit the rational and pure parts of the equation

$$\begin{aligned}\hat{\mathcal{O}}\hat{f}(a_1,\bar{a}_1,a_2,\bar{a}_2)=&-\frac{(a_1-\bar{a}_1-a_2+\bar{a}_2)^2}{(a_1-a_2)(\bar{a}_1-\bar{a}_2)}[\phi^{(1)}(a_1,\bar{a}_1)-\phi^{(1)}(a_2,\bar{a}_2)]\\&-\frac{(a_1-\bar{a}_1-a_2+\bar{a}_2)(a_1-\bar{a}_1)(a_2-\bar{a}_2)}{a_1(1-a_2)(\bar{a}_1-\bar{a}_2)-\bar{a}_1(1-\bar{a}_2)(a_1-a_2)}\phi^{(1)}\left(\frac{a_1(1-a_2)}{a_1-a_2},\frac{\bar{a}_1(1-\bar{a}_2)}{\bar{a}_1-\bar{a}_2}\right)\end{aligned} \tag{3.2.25}$$

3.3 Solving the Differential Equation

In the previous section we have obtained an equation for the pure part of the integral in the planar limit $\hat{\mathcal{O}}\hat{f}(a_1,\bar{a}_1,a_2,\bar{a}_2)=X$ where X is given by the RHS of (3.2.25). Since

the operator $\hat{\mathcal{O}}$ factorises $\hat{\mathcal{O}} = \hat{\mathcal{O}}_1\hat{\mathcal{O}}_2$, in the fist instance we need to solve the equation $\hat{\mathcal{O}}_1 g = X$ where we expect g to be a degree three combination of MPLs. As has been the case previously the key to tackling the problem is choosing appropriate variables. First we choose variables suitable for rewriting $\hat{\mathcal{O}}_1$. Let us define s and t via

$$a_1 - \bar{a}_1 = st\,, \qquad a_2 - \bar{a}_2 = \frac{s}{t}\,. \tag{3.3.1}$$

In terms of the variables $s, t, \bar{a}_1, \bar{a}_2$ the operator $\hat{\mathcal{O}}_1$ takes the simple form

$$\hat{\mathcal{O}}_1 = s^2\partial_s\frac{1}{s}\,. \tag{3.3.2}$$

Now let us consider reproducing the first line of the RHS of (3.2.25). We look for a solution of

$$\begin{aligned} s^2\partial_s\frac{1}{s}g_1(s,t,\bar{a}_1,\bar{a}_2) &= -\frac{(a_1-\bar{a}_1-a_2+\bar{a}_2)^2}{(a_1-a_2)(\bar{a}_1-\bar{a}_2)}[\phi^{(1)}(a_1,\bar{a}_1)-\phi^{(1)}(a_2,\bar{a}_2)] \\ &= -\frac{s^2(t-1/t)^2}{(st+\bar{a}_1-s/t-\bar{a}_2)(\bar{a}_1-\bar{a}_2)}[\phi^{(1)}(a_1,\bar{a}_1)-\phi^{(1)}(a_2,\bar{a}_2)] \end{aligned} \tag{3.3.3}$$

To obtain a solution in terms of G-functions all we need to do is to rewrite the pure functions appearing in the square brackets as iterated integrals with s appearing in the outermost upper integration limit. This is simple to achieve as the letters $\{a_1, 1-a_1, a_2, 1-a_2\}$ are all linearly related to s and we may take the symbol of a function and choose a contour γ ending at s. We note that starting from the origin $\{0,0,0,0\}$ in the variables $\{a_1,a_2,\bar{a}_1,\bar{a}_2\}$ corresponds to starting from $\{0,1,0,0\}$ in the variables $\{s,t,\bar{a}_1,\bar{a}_2\}$. For example,

$$\begin{aligned} \mathrm{Li}_2(a_1) &= -G(0,1;a_1) \\ &= \int_0^{a_1}\frac{dv_1}{v_1}\int_0^{v_1}\frac{dv_2}{1-v_2} \\ &= -\int_\gamma [1-a_1, a_1] \\ &= -\int_\gamma [1-st-\bar{a}_1, st+\bar{a}_1] \\ &= -\int_\gamma \big([s-(1-\bar{a}_1)/t, s+\bar{a}_1/t] + [t, s+\bar{a}_1/t] + [s-(1-\bar{a}_1)/t, t] + [t,t]\big) \end{aligned} \tag{3.3.4}$$

Where in the third line we have gone from the one-form representation to the symbol and have expressed the tensor product as an array. We now use the freedom to choose the contour along the axes in the coordinates $\{s,t,\bar{a}_1,\bar{a}_2\}$, first along the $\bar{a}_2$ direction (where, in this example, nothing happens since nothing depends on this variable), then along the $\bar{a}_1$ direction (with $s=0$ and $t=1$ still), then along the t direction (with $s=0$ still) and then finally along the s direction. Having defined our contour we may then

apply the map (2.3.54) taking us from the symbol to the function

$$\mathrm{Li}_2(a_1) = -G\left(-\frac{\bar{a}_1}{t}, \frac{1-\bar{a}_1}{t}; s\right) - G(1;\bar{a}_1)G\left(-\frac{\bar{a}_1}{t}; s\right) - G(0,1,\bar{a}_1) \tag{3.3.5}$$

In this manner integrating with respect to s becomes simply a matter of adjoining the appropriate letter to the beginning of the weight vector. Going through the above procedure for the full inhomogeneous term we find

$$\begin{aligned}\phi^{(1)}(s,t,\bar{a}_1) - \phi^{(1)}(s,t,\bar{a}_2) = {} & 2G(0;\bar{a}_1)\,G\left(\frac{1-\bar{a}_1}{t}; s\right) - 2G(1;\bar{a}_1)\,G\left(-\frac{\bar{a}_1}{t}; s\right) \\ & + 2G(1;\bar{a}_2)\,G(-\bar{a}_2 t; s) - 2G(0;\bar{a}_2)\,G(t(1-\bar{a}_2); s) \\ & + G\left(\frac{1-\bar{a}_1}{t}, -\frac{\bar{a}_1}{t}; s\right) - G\left(-\frac{\bar{a}_1}{t}, \frac{1-\bar{a}_1}{t}; s\right) \\ & + G(-\bar{a}_2 t, t(1-\bar{a}_2); s) - G(t(1-\bar{a}_2), -\bar{a}_2 t; s)\end{aligned} \tag{3.3.6}$$

re-expressing the rational prefactor in terms of s and t

$$-\frac{(a_1-\bar{a}_1-a_2+\bar{a}_2)^2}{(a_1-a_2)(\bar{a}_1-\bar{a}_2)} = -\frac{s^2(t-1/t)^2}{(s(t-1/t)+\bar{a}_1-\bar{a}_2)(\bar{a}_1-\bar{a}_2)} \tag{3.3.7}$$

we find that we can integrate and obtain a solution for g_1

$$\partial_s\Big[\frac{1}{s}g_1(s,t,\bar{a}_1,\bar{a}_2)\Big] = -\frac{(t-1/t)}{\bar{a}_1-\bar{a}_2}\frac{1}{s+\frac{\bar{a}_1-\bar{a}_2}{t-1/t}}[\phi^{(1)}(s,t,\bar{a}_1) - \phi^{(1)}(s,t,\bar{a}_2)] \tag{3.3.8}$$

for convenience let us introduce the variable

$$q = -\frac{\bar{a}_1-\bar{a}_2}{t-1/t} = -\frac{s(\bar{a}_1-\bar{a}_2)}{a_1-\bar{a}_1-a_2+\bar{a}_2} \tag{3.3.9}$$

as outlined previously we may integrate straightforwardly to find a solution to (3.3.3)

$$\begin{aligned}g_1(s,t,\bar{a}_1,\bar{a}_2) = -\frac{s(t-1/t)}{\bar{a}_1-\bar{a}_2}\Big[& 2G(0;\bar{a}_1)\,G\left(q, \frac{1-\bar{a}_1}{t}; s\right) - 2G(1;\bar{a}_1)\,G\left(q, -\frac{\bar{a}_1}{t}; s\right) \\ & + 2G(1;\bar{a}_2)\,G(q, -\bar{a}_2 t; s) - 2G(0;\bar{a}_2)\,G(q, t(1-\bar{a}_2); s) \\ & + G\left(q, \frac{1-\bar{a}_1}{t}, -\frac{\bar{a}_1}{t}; s\right) - G\left(q, -\frac{\bar{a}_1}{t}, \frac{1-\bar{a}_1}{t}; s\right) \\ & + G(q, -\bar{a}_2 t, t(1-\bar{a}_2); s) - G(q, t(1-\bar{a}_2), -\bar{a}_2 t; s)\Big]\end{aligned} \tag{3.3.10}$$

Fortunately we may proceed in a similar way for the more complicated terms

$$s^2\partial_s\frac{1}{s}g_2(s,t,\bar{a}_1,\bar{a}_2) = -\frac{(a_1-\bar{a}_1-a_2+\bar{a}_2)(a_1-\bar{a}_1)(a_2-\bar{a}_2)}{a_1(1-a_2)(\bar{a}_1-\bar{a}_2)-\bar{a}_1(1-\bar{a}_2)(a_1-a_2)}\phi^{(1)}\left(\frac{a_1(1-a_2)}{a_1-a_2}, \frac{\bar{a}_1(1-\bar{a}_2)}{\bar{a}_1-\bar{a}_2}\right). \tag{3.3.11}$$

From the form of the Block-Wigner dilog we can see that the inhomogeneous term contains letters of the form

$$1-\frac{a_1(1-a_2)}{a_1-a_2}=\frac{a_2(a_1-1)}{a_1-a_2}. \tag{3.3.12}$$

These still factorise in terms of factors linear in s and we treat the rational prefactor

$$\begin{aligned}-\frac{(a_1-\bar{a}_1-a_2+\bar{a}_2)(a_1-\bar{a}_1)(a_2-\bar{a}_2)}{a_1(1-a_2)(\bar{a}_1-\bar{a}_2)-\bar{a}_1(1-\bar{a}_2)(a_1-a_2)}&=\frac{s^2(1-t)(1+t)}{(\bar{a}_1-\bar{a}_2)ts-(1-\bar{a}_1)\bar{a}_1+(1-\bar{a}_2)\bar{a}_2t^2}\\&=-\frac{s^2(t-1/t)}{\bar{a}_1-\bar{a}_2}\frac{1}{s-r}\end{aligned} \tag{3.3.13}$$

where

$$r=\frac{\bar{a}_1(1-\bar{a}_1)/t-\bar{a}_2(1-\bar{a}_2)t}{\bar{a}_1-\bar{a}_2}=\frac{\bar{a}_1(1-\bar{a}_1)(a_2-\bar{a}_2)-\bar{a}_2(1-\bar{a}_2)(a_1-\bar{a}_1)}{s(\bar{a}_1-\bar{a}_2)} \tag{3.3.14}$$

Again we simply adjoin the letter r to the beginning of each weight vector in the s-dependent version of (3.3.10)

$$\begin{aligned}
g_2\left(s,t,\bar{a}_1,\bar{a}_2\right)=&-\frac{s\left(t-1/t\right)}{\bar{a}_1-\bar{a}_2}\left[2G\left(r,\frac{1-\bar{a}_1}{t};s\right)G\left(0;\bar{a}_1\right)-2G\left(0;\bar{a}_2\right)G\left(r,\frac{1-\bar{a}_1}{t};s\right)\right.\\
&+2G\left(1;\bar{a}_2\right)G\left(r,\frac{1-\bar{a}_1}{t};s\right)-2G\left(\bar{a}_2;\bar{a}_1\right)G\left(r,\frac{1-\bar{a}_1}{t};s\right)-2G\left(1;\bar{a}_1\right)G\left(r,-\frac{\bar{a}_1}{t};s\right)\\
&+2G\left(\bar{a}_2;\bar{a}_1\right)G\left(r,-\frac{\bar{a}_1}{t};s\right)+2G\left(0;\bar{a}_1\right)G\left(r,-\bar{a}_2t;s\right)-2G\left(0;\bar{a}_2\right)G\left(r,-\bar{a}_2t;s\right)\\
&+2G\left(1;\bar{a}_2\right)G\left(r,-\bar{a}_2t;s\right)-2G\left(\bar{a}_2;\bar{a}_1\right)G\left(r,-\bar{a}_2t;s\right)-2G\left(1;\bar{a}_1\right)G\left(r,t-\bar{a}_2t;s\right)\\
&-2G\left(0;\bar{a}_1\right)G\left(r,-\frac{\left(\bar{a}_1-\bar{a}_2\right)t}{t^2-1};s\right)+2G\left(0;\bar{a}_2\right)G\left(r,-\frac{\left(\bar{a}_1-\bar{a}_2\right)t}{t^2-1};s\right)\\
&+2G\left(1;\bar{a}_1\right)G\left(r,-\frac{\left(\bar{a}_1-\bar{a}_2\right)t}{t^2-1};s\right)-2G\left(1;\bar{a}_2\right)G\left(r,-\frac{\left(\bar{a}_1-\bar{a}_2\right)t}{t^2-1};s\right)\\
&+G\left(r,\frac{1-\bar{a}_1}{t},-\frac{\bar{a}_1}{t};s\right)+G\left(r,\frac{1-\bar{a}_1}{t},t-\bar{a}_2t;s\right)\\
&-G\left(r,-\frac{\bar{a}_1}{t},\frac{1-\bar{a}_1}{t};s\right)-G\left(r,-\frac{\bar{a}_1}{t},-\bar{a}_2t;s\right)+G\left(r,-\frac{\bar{a}_1}{t},-\frac{\left(\bar{a}_1-\bar{a}_2\right)t}{t^2-1};s\right)\\
&+G\left(r,-\bar{a}_2t,-\frac{\bar{a}_1}{t};s\right)+G\left(r,-\bar{a}_2t,t-\bar{a}_2t;s\right)-G\left(r,-\bar{a}_2t,-\frac{\left(\bar{a}_1-\bar{a}_2\right)t}{t^2-1};s\right)\\
&-G\left(r,t-\bar{a}_2t,\frac{1-\bar{a}_1}{t};s\right)-G\left(r,t-\bar{a}_2t,-\bar{a}_2t;s\right)+G\left(r,t-\bar{a}_2t,-\frac{\left(\bar{a}_1-\bar{a}_2\right)t}{t^2-1};s\right)\\
&+G\left(r,-\frac{\left(\bar{a}_1-\bar{a}_2\right)t}{t^2-1},\frac{1-\bar{a}_1}{t};s\right)-G\left(r,-\frac{\left(\bar{a}_1-\bar{a}_2\right)t}{t^2-1},-\frac{\bar{a}_1}{t};s\right)\\
&+G\left(r,-\frac{\left(\bar{a}_1-\bar{a}_2\right)t}{t^2-1},-\bar{a}_2t;s\right)-G\left(r,-\frac{\left(\bar{a}_1-\bar{a}_2\right)t}{t^2-1},t-\bar{a}_2t;s\right)\\
&\left.+2G\left(\bar{a}_2;\bar{a}_1\right)G\left(r,t-\bar{a}_2t;s\right)-G\left(r,\frac{1-\bar{a}_1}{t},-\frac{\left(\bar{a}_1-\bar{a}_2\right)t}{t^2-1};s\right)\right]
\end{aligned} \tag{3.3.15}$$

Remarkably the prefactors of the weight 3 pure functions in g_1 and g_2 are the same so that the two pieces add up simply and we have

$$g = g_1 + g_2 = -\frac{s(t-1/t)}{\bar{a}_1 - \bar{a}_2}\hat{g}, \tag{3.3.16}$$

where $\hat{g}$ is a pure function of degree 3. Now that we have integrated with respect to $\mathcal{O}_1$ we may proceed to the next operator $\mathcal{O}_2$, however this time there is only one function to consider.

$$\hat{\mathcal{O}}_2\hat{f}(a_1, a_2, \bar{a}_1, \bar{a}_2) = g(a_1, a_2, \bar{a}_1, \bar{a}_2) \tag{3.3.17}$$

We use the same variables s and t as before

$$a_1 - \bar{a}_1 = st\,, \qquad a_2 - \bar{a}_2 = \frac{s}{t}\,, \tag{3.3.18}$$

except that this time we consider the coordinates $\{a_1, a_2, s, t\}$ (i.e. we eliminate the barred variables). In these coordinates the operator $\hat{\mathcal{O}}_2$ takes the form

$$\hat{\mathcal{O}}_2 = -s\partial_s\,. \tag{3.3.19}$$

Hence we need to solve

$$\partial_s\hat{f} = \frac{(t-1/t)}{\bar{a}_1 - \bar{a}_2}\hat{g}\,. \tag{3.3.20}$$

Expressing the denominator in the coordinates $\{a_1, a_2, s, t\}$ we have

$$\partial_s\hat{f} = \frac{(t-1/t)}{a_1 - a_2 - st + s/t}\hat{g} = \frac{1}{-s + \frac{a_1-a_2}{t-1/t}}\hat{g} = -\frac{1}{s-p}\hat{g} \tag{3.3.21}$$

where

$$p = \frac{a_1 - a_2}{t - 1/t}\,. \tag{3.3.22}$$

To integrate again we must first express the weight 3 function $g = g_1 + g_2$ found above in terms of the variables $\{a_1, a_2, s, t\}$. Remarkably we find that all letters are again at most linear in s so the procedure is analogous to the one already employed. Then we can express the function g as an iterated integral successively along the a_2, a_1, t and finally s directions. Thus integrating to obtain a particular solution for $\hat{f}$ may be easily achieved by adjoining a p to the beginning of the weight vector in the s-dependent MPLs.

By analysing the symbol of the particular solution we are able to make a list of the letters appearing there. Finally we obtain an iterated integral over a weight 4 integrable word

in the following letters:

$$\begin{aligned}&\{a_1, \bar{a}_1, a_2, \bar{a}_2, 1-a_1, 1-\bar{a}_1, 1-a_2, 1-\bar{a}_1, a_1-a_2, \bar{a}_1-\bar{a}_2, a_1-\bar{a}_1, a_2-\bar{a}_2,\\ &a_1-\bar{a}_2-a_2+\bar{a}_2, a_1\bar{a}_2-a_2\bar{a}_1, a_1-\bar{a}_1-a_2+\bar{a}_1a_2+\bar{a}_2-a_1\bar{a}_2,\\ &\bar{a}_1a_2-a_1\bar{a}_1a_2-a_1\bar{a}_2+a_1\bar{a}_1\bar{a}_2+a_1a_2\bar{a}_2-\bar{a}_1a_2\bar{a}_2\}\end{aligned} \tag{3.3.23}$$

Note that the solution obtained so far is not the most general one as we still have some ambiguity due to the boundary conditions of the differential equation. Because the discontinuities of physical amplitudes must coincide with the unitary and local nature of the underlying theory, we shall have to impose that our solution be single-valued.

3.4 Single-valued hyperlogarithms from Picard-Fuchs equations

In this section we discuss the construction of single-valued hyperlogarithms, following ref. [82, 83]. We review the construction in detail, because the techniques introduced in the hyperlogarithm case can be extended to the KZ equation on the moduli space of Riemann spheres with marked points $\mathfrak{M}_{0,n}$ that shall be central in the next chapter.

Consider a set of complex constants, $\Sigma = \{\sigma_1, \ldots, \sigma_n\}$. We denote the shuffle algebra of all hyperlogarithms with singularities in Σ by L_Σ, see eq. (2.3.11). In the following it will be useful to take a more abstract viewpoint. Let $X = \{x_1, \ldots, x_n\}$, and $\mathbb{C}\langle X\rangle$ is the complex vector space generated by all words with letters from X, and the multiplication is the shuffle product. We start by defining the *universal algebra of hyperlogarithms* $\mathcal{HL}_\Sigma$ as the algebra $\mathbb{C}\langle X\rangle$, but with rational functions (with poles at most at $z \in \Sigma$) as coefficients, and a derivation ∂ which acts on rational functions as $\partial/\partial z$ and on words as

$$\partial(x_i w) = \frac{1}{z-\sigma_i}\, w \tag{3.4.1}$$

$\mathcal{HL}_\Sigma$ is an abstract algebra (with a derivation) which has exactly the same properties as the algebra L_Σ (shuffle and differentiation). A *realisation of* $\mathcal{HL}_\Sigma$ is then an algebra morphism $\rho : \mathcal{HL}_\Sigma \to A$ that preserves the derivative. In particular, the hyperlogarithms L_Σ are a realisation of $\mathcal{HL}_\Sigma$. We will in the following refer to this realisation as the *standard realisation*,

$$\rho_G : \mathcal{HL}_\Sigma \to L_\Sigma,\ \ w \mapsto G(w; z) \tag{3.4.2}$$

where we made a slight abuse of notation: if the word is $w = x_{i_1} \ldots x_{i_{|w|}}$, with $|w|$ the length of the word w, then we define $G(w; z) \equiv G(\sigma_{i_1}, \ldots, \sigma_{i_{|w|}}; z)$. In the following also the dual of $\mathcal{HL}_\Sigma$ will be important. The dual of $\mathbb{C}\langle X\rangle$ is the space $\mathbb{C}\langle\langle X\rangle\rangle$ of formal power series in words.

Next, consider a realisation ρ of $\mathcal{HL}_\Sigma$, and consider the generating series

$$F_\rho = \sum_{w\in X^*} \rho(w)\, w \tag{3.4.3}$$

F_ρ satisfies a Picard-Fuchs-type equation

$$\frac{\partial}{\partial z} F_\rho = \sum_{i=1}^{n} \frac{x_i}{z-\sigma_i} F_\rho \tag{3.4.4}$$

Conversely, every solution to eq. (3.4.4) gives rise to a realisation of $\mathcal{HL}_\Sigma$. Moreover, one can check that if F' is any other solution to eq. (3.4.4), then there is a constant series $T \in \mathbb{C}\langle\langle X\rangle\rangle$ such that $F' = F_\rho T$. Finally, it is easy to see that we can find n solutions L^{σ_i} such that close to the singularity $z \sim \sigma_i$ we have $L^{\sigma_i} \sim (z-\sigma_i)^{x_i}$. Hence, we conclude that there are constant series $Z_{ij} \in \mathbb{C}\langle\langle X\rangle\rangle$ such that

$$L^{\sigma_j} = L^{\sigma_i} Z_{ij} \tag{3.4.5}$$

We refer to the Z_{ij} as *associators* and we note that $Z_{ij}Z_{jk} = Z_{ik}$, and their inverses are $Z_{ij}^{-1} = Z_{ji}$. The associators can be obtained as the shuffle-regularised values of F_ρ at the singular points [83]. In particular, if $\Sigma = \{0,1\}$, we have $Z_{01} = \Phi(x_0, x_1)$, where Φ is the Drinfeld associator

From now on we will always identify one of the singular points with 0, say $\sigma_0 = 0$ (this is always possible using $SL(2,\mathbb{C})$ transformations). We define

$$L(z) \equiv L^{\sigma_0}(z) = \sum_{w\in X^*} \rho_G(w)\, w = \sum_{w\in X^*} G(w;z)\, w \tag{3.4.6}$$

and we write the associators as $Z^{\sigma_i} \equiv Z_{0i}$, so that $L^{\sigma_i}(z) = L(z)\, Z^{\sigma_i}$.

Due to the presence of the singularities in eq. (3.4.4), the solutions to eq. (3.4.4) will in general have discontinuities with branch points at $z = \sigma_i$. We denote by $\mathcal{M}_{\sigma_i} F_\rho$ the monodromy obtained by analytically continuing F_ρ along a small loop encircling $z = \sigma_i$. It is easy to check that $\mathcal{M}_{\sigma_i} F_\rho$ is still a solution to eq. (3.4.4), and so there must be a constant series M_{σ_i} such that $\mathcal{M}_{\sigma_i} F_\rho = F_\rho M_{\sigma_i}$. We obtain

$$\mathcal{M}_{\sigma_j} L^{\sigma_j} = L^{\sigma_i}\, e^{2\pi i x_j} \;\text{ and }\; \mathcal{M}_{\sigma_j} L = L\,(Z^{\sigma_j})^{-1}\, e^{2\pi i x_j}\, Z^{\sigma_j} \tag{3.4.7}$$

One of the main results of ref. [83] is that there is always a solution to eq. (3.4.4) with a prescribed monodromy. More precisely, if we are given n (grouplike) elements $A_1, \ldots, A_n \in \mathbb{C}\langle\langle X\rangle\rangle$, then there is always a realisation $\rho : \mathcal{HL}_\Sigma \to L_\Sigma \overline{L}_\Sigma$, with $\overline{L}_\Sigma$ the complex conjugate of L_Σ, such that $\mathcal{M}_{\sigma_i} F_\rho = F_\rho A_i$. There are two particular cases of this:

1. If we choose $A_k = (Z^{\sigma_k})^{-1}\, e^{2\pi i x_k}\, Z^{\sigma_k}$, $\forall 1 \leq k \leq n$, we obtain the standard realisation ρ_G.

2. We may also choose $A_k = 1$, $\forall 1 \leq k \leq n$ and we see that there is a realisation of $\mathcal{HL}_\Sigma$ that is single-valued.

It is possible to write down a generating function for the single-valued realisation, similar to the generating series $L(z)$ for the standard realisation. Consider two alphabets $X = \{x_1, \ldots, x_n\}$ and $Y = \{y_1, \ldots, y_n\}$ and two generating functions

$$L_X(z) = \sum_{w \in X^*} G(w; z)\, w \text{ and } \widetilde{\overline{L}}_Y(\bar{z}) = \sum_{w \in Y^*} G(\bar{w}; \bar{z})\, \tilde{w} \tag{3.4.8}$$

where $\tilde{w}$ is the word w with all its letters in reversed order. We define

$$\mathcal{L}_X(z) \equiv L_X(z) \widetilde{\overline{L}}_Y(\bar{z}) \tag{3.4.9}$$

$\mathcal{L}_X(z)$ is a solution of eq. (3.4.4), because

$$\frac{\partial}{\partial z}\mathcal{L}_X(z) = \frac{\partial}{\partial z} L_X(z) \widetilde{\overline{L}}_Y(\bar{z}) = \sum_{i=1}^{n} \frac{x_i}{z - \sigma_i}\, L_X(z) \widetilde{\overline{L}}_Y(\bar{z}) = \sum_{i=1}^{n} \frac{x_i}{z - \sigma_i}\, \mathcal{L}(z) \tag{3.4.10}$$

The monodromies of $\mathcal{L}_X(z)$ are

$$\mathcal{M}_{\sigma_k} \mathcal{L}_X(z) = L_X(z)\, M_{\sigma_k} \widetilde{\overline{L}}_Y(\bar{z}) \tag{3.4.11}$$

with

$$M_{\sigma_k} = Z^{\sigma_k}(X)^{-1}\, e^{2\pi i x_k}\, Z^{\sigma_k}(X)\, \widetilde{\overline{Z}}^{\sigma_k}(Y)\, e^{-2\pi i y_k}\, \widetilde{\overline{Z}}^{\sigma_k}(Y)^{-1} \tag{3.4.12}$$

Obviously $\mathcal{L}_X(z)$ is single-valued if $M_{\sigma_k} = 1$, $\forall 1 \leq k \leq n$, which implies that the letters in X and Y are not independent. Infinitesimally, this condition becomes

$$\widetilde{\overline{Z}}^{\sigma_k}(Y)\, y_k\, \widetilde{\overline{Z}}^{\sigma_k}(Y)^{-1} = Z^{\sigma_k}(X)^{-1}\, x_k\, Z^{\sigma_k}(X) \tag{3.4.13}$$

This equation can be solved perturbatively in the length of the words. While solving the constraints (3.4.13) is conceptually very algorithmic, explicitly constructing the solutions order-by-order in the length of the words quickly becomes very tedious. Below we construct an explicit solution to the constraints (3.4.13). Before doing so, however, it will be useful to introduce some more notation.

Let us for now assume that we have obtained the solution to eq. (3.4.13) to any desired order. If we substitute this solution into the definition of $\mathcal{L}_X$, we obtain in this way the

single-valued realisation ρ_{SV} of $\mathcal{HL}_\Sigma$,

$$\mathcal{L}_X(z) \equiv \sum_{w\in X^*} \rho_{SV}(w)\, w \tag{3.4.14}$$

Some comments are in order: First, in the case where $\Sigma = \{0,1\}$, the single-valued realisation corresponds to the single-valued harmonic polylogarithms of ref. [82]. Second, the solution for Y in terms of X is unique order-by-order in the length of the words, and so the single-valued realisation is unique. Finally, ρ_{SV} and ρ_G are really just two different realisations of the same abstract algebra $\mathcal{HL}_\Sigma$ (just like an abstract group may have different representations). In other words, the standard and single-valued realisations have *exactly* the same properties. In particular, they form a shuffle algebra and have the same behaviour under holomorphic differentiation. We stress, however, that the behaviour under anti-holomorphic differentiation is less obvious. We will address this issue in Section 4.2.4.2.

In the following we write $\mathcal{G}(w;z) \equiv \mathcal{G}_w(z) \equiv \rho_{SV}(w)$. Let us denote the algebra generated by the functions $\mathcal{G}(w;z)$ by L_Σ^{SV}. We can define a linear map

$$\mathbf{s}_\Sigma : L_\Sigma \to L_\Sigma^{SV} \quad G(w;z) \mapsto \mathcal{G}(w;z) \tag{3.4.15}$$

As L_Σ and L_Σ^{SV} are just different realisations of $\mathcal{HL}_\Sigma$, $\mathbf{s}_\Sigma$ preserves the multiplication,

$$\mathbf{s}_\Sigma(a\cdot b) = \mathbf{s}_\Sigma(a)\cdot\mathbf{s}_\Sigma(b) \tag{3.4.16}$$

In the following we denote by $\mathcal{Z}$ the algebra of multiple zeta values, and by $\mathcal{Z}^{SV}$ the algebra of their single-valued analogues. It is possible to construct explicitly a homomorphism $\mathbf{s}_\zeta : \mathcal{Z} \to \mathcal{Z}^{SV}$ [84]. One can check that if $G(w,z) \in L_\Sigma$, then its regularised version at some singularity reduces to a linear combination of hyperlogarithms with one singularity less and with MZVs as coefficients. In other words, we have

$$\text{Reg}_{z=\sigma_k} G(w;z) \in \mathcal{Z}\otimes L_{\Sigma_k} \tag{3.4.17}$$

with $\Sigma_k = \Sigma/\{\sigma_k\}$ and where we see elements of L_{Σ_k} as functions of σ_k. We denote by $\hat{\mathbf{s}}_\Sigma$ the natural map

$$\hat{\mathbf{s}}_\Sigma \equiv \mathbf{s}_\zeta \otimes \mathbf{s}_\Sigma : \mathcal{Z}\otimes L_\Sigma \to \mathcal{Z}^{SV}\otimes L_\Sigma^{SV} \tag{3.4.18}$$

The single-valued maps preserve the multiplication, and so they commute with shuffle-regularisation,

$$\hat{\mathbf{s}}_{\Sigma_k}\left[\text{Reg}_{z=\sigma_k} G(w;z)\right] = \text{Reg}_{z=\sigma_k}\left[\mathbf{s}_\Sigma(G(w;z))\right] = \text{Reg}_{z=\sigma_k}\mathcal{G}(w;z) \tag{3.4.19}$$

Using these definitions, we can explicitly solve the constraints (3.4.13). We claim that the

solution for y_k to eq. (3.4.13) is obtained by conjugating x_k by the single-valued analogue of the associator $Z^{\sigma_k}(X)$,

$$y_k = \hat{\mathbf{s}}_{\Sigma_k}\left(Z^{\sigma_k}(X)\right)^{-1} x_k \, \hat{\mathbf{s}}_{\Sigma_k}\left(Z^{\sigma_k}(X)\right) \tag{3.4.20}$$

Equation (3.4.20) states that the single-valued analogues of the hyperlogarithms, and thus the map $\mathbf{s}_\Sigma$, can be constructed recursively in the number of singularities σ_k. The recursion starts with the single-valued harmonic polylogarithms, in which case the associator involves only MZVs, and so the map $\hat{\mathbf{s}}_{\Sigma_k}$ reduces to $\mathbf{s}_\zeta$.

In order to see why eq. (3.4.20) holds, let us cast the constraints (3.4.13) in the form

$$\begin{aligned} y_k &= \widetilde{\overline{Z}}^{\sigma_k}(Y_X)^{-1} Z^{\sigma_k}(X)^{-1} x_k Z^{\sigma_k}(X) \widetilde{\overline{Z}}^{\sigma_k}(Y_X) \\ &= \left(Z^{\sigma_k}(X) \widetilde{\overline{Z}}^{\sigma_k}(Y_X)\right)^{-1} x_k \left(Z^{\sigma_k}(X) \widetilde{\overline{Z}}^{\sigma_k}(Y_X)\right) \end{aligned} \tag{3.4.21}$$

where we write Y_X instead of Y in order to indicate that this identity holds on the solution to the constraints (3.4.13), i.e., we have inserted the solution to eq. (3.4.13) into the right-hand side of eq. (3.4.21). The right-hand-side then only depends on the letters x_i, and so eq. (3.4.21) is a formal solution to the constraints. Comparing eq. (3.4.21) and eq. (3.4.20), we need to show that

$$\hat{\mathbf{s}}_{\Sigma_k}\left(Z^{\sigma_k}(X)\right) = Z^{\sigma_k}(X) \widetilde{\overline{Z}}^{\sigma_k}(Y_X) \tag{3.4.22}$$

This relation is in fact a generalisation of the relation between Deligne's and Drinfeld's associators in the case where $\Sigma = \{0, 1\}$ [84]. We start from the fact that the associator can be written as the shuffle regularised version of $L_X(z)$ at the point $z = \sigma_k$,

$$Z^{\sigma_k}(X) = \mathrm{Reg}_{z=\sigma_k} L_X(z) \text{ and } \overline{Z}^{\sigma_k}(Y) = \mathrm{Reg}_{\bar{z}=\bar{\sigma}_k} \overline{L}_Y(\bar{z}) \tag{3.4.23}$$

We assume that we have constructed all single-valued hyperlogarithms with a certain number of singularities, and we want to add one more singularity, i.e., we assume that we know how to construct all the $\mathbf{s}_{\Sigma_k}$, and we want to construct $\mathbf{s}_\Sigma$. We have

$$\begin{aligned} \hat{\mathbf{s}}_{\Sigma_k}(Z^{\sigma_k}(X)) &= \hat{\mathbf{s}}_{\Sigma_k}\left[\mathrm{Reg}_{z=\sigma_k} L_X(z)\right] \\ &= \mathrm{Reg}_{z=\sigma_k}\left[\mathbf{s}_\Sigma(L_X(z))\right] \\ &= \mathrm{Reg}_{z=\sigma_k}\left[L_X(z)\widetilde{\overline{L}}_{Y_X}(\bar{z})\right] \\ &= \left[\mathrm{Reg}_{z=\sigma_k} L_X(z)\right]\left[\mathrm{Reg}_{\bar{z}=\bar{\sigma}_k} \widetilde{\overline{L}}_{Y_X}(\bar{z})\right] \end{aligned} \tag{3.4.24}$$

The first factor immediately gives an associator, $\mathrm{Reg}_{z=\sigma_k} L_X(z) = Z^{\sigma_k}(X)$. The second factor also gives an associator. Indeed, the solution Y_X is independent of z, and so the shuffle regularisation does not act on the letters y_i and it commutes with the reversal

of words. Hence, $\mathrm{Reg}_{\bar{z}=\bar{\sigma}_k}\widetilde{\overline{L}}_{Y_X}(\bar{z}) = \widetilde{\overline{Z}}^{\sigma_k}(Y_X)$, which finishes the proof. Note that at the same time we have proved the identity

$$\hat{\mathbf{s}}_{\Sigma_k}(Z^{\sigma_k}(X)) = \mathrm{Reg}_{z=\sigma_k}\mathcal{L}_X(z) \tag{3.4.25}$$

In practice, it is often easier to use this last relation to construct the single-valued associators than constructing the standard associators and then acting with the single-valued map.

3.4.1 Single-valued check of the integral

Before proceeding to study the monodromies of our integral we note that it belongs to a more general class than the one described in the previous section. The alphabet of symbol entries mixes holomorphic and antiholomorphic variables and thus may couple the a_i to the $\bar{a}_i$. However using the above we may still compute the monodromies of our integral around the singular points of (3.2.14) and ensure that the solution is single valued in those cases.

That is we must ensure that the integral is physical and does not have any discontinuities when approaching kinematical limits from different directions. As shown previously, in order to calculate the monodromy around a point σ_i we must first write the solution L^{σ_i} to the Picard-Fuchs equation at that point. We may do so by using associators to translate $L(z)$, however let us instead decompose L^{σ_i} into

$$Z^{\sigma_i} = f_i(z)\exp(x_k\log(z-\sigma_i)) \tag{3.4.26}$$

The function $f_i(z)$ has a well defined limit $f_i(\sigma_i) = 1$ and it is holomorphic on a subset of the complex plane $\mathbb{C}\backslash\cup_{k\neq i}\ell(\sigma_i)$. Where $\ell(\sigma_i)$ signifies the closed half line beginning at σ_i and does not cross any other such half line for the other singular points. Of course $f_i(z)$ is also a generating series and we may write an explicit formula for its coefficients. Thus for every word not ending in x_i and of the form $w = x_i^{n_r}x_{k_r}x_i^{n_{r-1}}x_{k_{r-1}}\dots x_i^{n_1}x_{k_1}$ we may write

$$\begin{aligned} L_w^{(\sigma_i)}(z) = \sum_{1\leq<m_1<\ldots<m_r} \frac{(-1)^r}{m_1^{n_1+1}\dots m_r^{n_r+1}}&\left(\frac{z-\sigma_i}{\sigma_{k_1}-\sigma_i}\right)^{m_1} \\ &\times\left(\frac{z-\sigma_i}{\sigma_{k_2}-\sigma_i}\right)^{m_2-m_1}\dots\left(\frac{z-\sigma_i}{\sigma_{k_r}-\sigma_i}\right)^{m_r-m_{r-1}} \end{aligned} \tag{3.4.27}$$

Functions of this form are easily seen to be regular in the limit $z\to\sigma_i$ and satisfy 3.4.4.

This sum representation may be analytically continued in a straightforward manner to an iterated integral whose entries are described by the word w and its basepoint is taken to be σ_i. In fact the form 3.4.27 may be considered a Taylor expansion of the integral and as such may be defined for points not contained in Σ and for any word.

Thus in order to find the monodromies of our solution to the differential equations and check that it is single valued we must translate it to different singular points. This may be achieved via the composition of paths property of iterated integrals to re-express an iterated integral at one basepoint in terms of another. Taking an example from the set $\Sigma = \{0, 1\}$

$$G(0,1,1,0;z) = G(0,1,1,0;1) + G(1,1,0;1)I(1;0;z) + G(1,0,1)I(1;1,0,z) \\ +G(0;1)I(1;1,1,0;z) + I(1;0,1,1,0;z) \quad (3.4.28)$$

Although this follows from a straighforward decomposition of paths we may see the terms coming from the Drinfeld associator if we shuffle out the trailing zeros in the G-functions

$$G(0,1,1,0;z) = I(1;0,1,1,0;z) - \zeta_2 I(1;1,0;z) + \zeta_3 I(1;0;z) - \frac{5}{4}\zeta_4 \quad (3.4.29)$$

Where we have made the identification $G(0^{n_1-1},1,0^{n_2-1},1,\ldots,0^{n_r-1},1;1) = \zeta_{n_1,\ldots,n_r}$ and have regularised the G-functions such that $G(0,0) = G(0,1) = \zeta_1 = 0$. Note however that not all iterated integrals in the above formula are of the form (3.4.27). Specifically, because $I(1;1,0;z)$ begins with a 1 (note the reversal of words in comparison with $L_w^{(\sigma_i)}(z)$) it should have an explicit logarithm around 1

$$I(1;1,0;z) = I(1;1;z)I(1;0;z) - I(1;0,1;z) \quad (3.4.30)$$

Thus we see that the monodromy of $G(0,1,1,0,z)$ around 1 is given by

$$\mathcal{M}_1 G(0,1,1,0,z) = -2\pi i\, \zeta_2 I(1;0;z) \quad (3.4.31)$$

Of course we may extend this analysis to any Σ and in the particular context of our integral we checked the monodromies of a_1, a_2 around $0, 1$ as well as each and found them to be trivial. Thus we may conclude that our solution to the differential equation (3.2.25) is single valued in these cases and we present the result in Appendix A.

CHAPTER 4

Scattering amplitudes in multi Regge kinematics

It has been known since the early days of QCD that in the Regge limit $s \gg |t|$ scattering amplitudes exhibit a rich analytic structure. The paradigm example is the BFKL equation in QCD, which resums the radiative corrections in $\log(s/|t|)$ to parton-parton scattering at leading logarithmic accuracy (LLA) [85–87] and next-to-LLA (NLLA) [88–90]. The building blocks of the BFKL resummation at LLA are the multi-gluon amplitudes, which are evaluated in multi-Regge kinematics (MRK), i.e., in the approximation of a strong rapidity ordering of the outgoing gluons. The multi-Regge limit is thus the kinematic cornerstone of the BFKL resummation at LLA. In establishing the BFKL equation, the gluon rapidities are then integrated out, and the BFKL equation is reduced to a two-dimensional problem in terms of purely transverse degrees of freedom: i.e., the evolution of a t-channel gluon ladder in transverse momentum space and Mellin moment.

The aim of this section is to study the multi-Regge limit of scattering amplitudes in $\mathcal{N} = 4$ SYM. We can completely describe the geometry in the transverse space as a configuration of points in the complex plane, and hence we can completely classify all the iterated integrals that appear in the final result. In other words, MRK is described by the geometry of the moduli space $\mathfrak{M}_{0,n}$ of Riemann spheres with n marked points. The geometry of $\mathfrak{M}_{0,n}$ is well understood. In particular, the cluster algebra associated to $\mathfrak{M}_{0,n}$ is always of finite type and corresponds to the Dynkin diagram A_{n-3}. The algebra of iterated integrals on this space can also be completely described: they are iterated integrals of $d\log$-forms with singularities when some of the marked points coincide. We study scattering amplitudes in planar $\mathcal{N} = 4$ SYM in MRK for any number N of external legs and

arbitrary helicity configurations. In this chapter we we restrict ourselves to LLA and work with the leading order central emission block describing the emission of a gluon along the t-channel ladder. Beyond LLA corrections shall be explored in subsequent chapters.

4.1 Multi-Regge kinematics

The focus of this chapter are planar colour-ordered scattering amplitudes in $\mathcal{N} = 4$ SYM in a special kinematic limit of 2-to-$(N-2)$-gluon scattering, the so-called multi-Regge kinematics (MRK) [91]. In order to define this limit, it is convenient to work in conventions where all momenta are taken as outgoing. We define lightcone and (complex) transverse momenta

$$p^{\pm} \equiv p^0 \pm p^z \quad \mathbf{p}_k \equiv p_{k\perp} = p_k^x + i p_k^y \tag{4.1.1}$$

Using this decomposition, the scalar product between two four vectors p and q is given by

$$2p \cdot q = p^+ q^- + p^- q^+ - \mathbf{p}\bar{\mathbf{q}} - \bar{\mathbf{p}}\mathbf{q} \tag{4.1.2}$$

Without loss of generality we may choose a reference frame such that the momenta of the initial state gluons p_1, p_2 lie on the z-axis with $p_2^z = p_2^0$, which implies $p_1^+ = p_2^- = \mathbf{p}_1 = \mathbf{p}_2 = 0$. Then the multi-Regge limit is defined as the limit where the outgoing gluons with momenta p_i, $i \geq 3$ are strongly ordered in the lightcone coordinates while having comparable transverse momenta

$$p_3^+ \gg p_4^+ \gg \dots p_{N-1}^+ \gg p_N^+ \qquad |\mathbf{p}_3| \simeq \dots \simeq |\mathbf{p}_N| \tag{4.1.3}$$

The mass-shell condition $p_i^2 = p_i^+ p_i^- - |\mathbf{p}_i|^2 = 0$ implies that

$$p_N^- \gg p_{N-1}^- \gg \dots p_4^- \gg p_3^- \tag{4.1.4}$$

The ordering between the lightcone coordinates in (4.1.3) implies the following hierarchy between the Lorentz invariants,

$$\begin{aligned} s_{12} \gg s_{3\dots N-1}, s_{4\dots N} \gg s_{3\dots N-2}, s_{4\dots N-1}, s_{5\dots N} \gg \cdots \\ \dots \gg s_{34}, \dots, s_{N-1N} \gg -t_1, \cdots, -t_{N-3} \end{aligned} \tag{4.1.5}$$

with t_i held fixed, where

$$s_{i(i+1)\dots j} \equiv (p_i + p_{i+1} + \dots + p_j)^2 = x_{(i-1)j}^2 \tag{4.1.6}$$

$$t_{i+1} \equiv q_i^2 \quad q_i \equiv -p_2 - \dots - p_{i+3} = x_{(i+3)1} \tag{4.1.7}$$

We briefly sketch how the hierarchy in (4.1.5) follows from the strong ordering in light-cone coordinates, (4.1.3). In MRK momentum conservation can be written in the form

$$p_1^- = -\sum_{i=3}^{N} p_i^- \simeq -p_N^- \quad p_2^+ = -\sum_{i=3}^{N} p_i^+ \simeq -p_3^+ \quad 0 = \sum_{i=3}^{N} \mathbf{p}_i \tag{4.1.8}$$

and the two-particle invariants in MRK become modulo (4.1.8)

$$s_{ij} = 2p_i \cdot p_j \simeq p_i^+ p_j^- \quad i < j \leq N \tag{4.1.9}$$

From this, it is evident that all Mandelstam invariants made of k consecutive final state momenta $s_{ii+1\ldots i+k} \simeq s_{ii+k}$ will be comparable in size, and much larger than invariants made of $k-1$ consecutive momenta. For the scale separation between s-like and t-like variables, we use (4.1.3)-(4.1.7) and (4.1.8) to infer that $q_i^+ \simeq p_{i+4}^+$, $q_i^- \simeq -p_{i+3}^-$ and therefore $-q_i^+ q_i^- \ll p_{i+3}^+ p_{i+3}^- \simeq |\mathbf{q}_i|^2$. In other words, the q_i are dominated by their transverse components, $q_i^2 \simeq -|\mathbf{q}_i|^2$.

The analysis of MRK thus far only relied on Lorentz symmetry. We could also make use of the dual conformal invariance and write the kinematical dependence of the amplitude in terms of conformal cross ratios.

$$U_{ij} \equiv \frac{x_{i+1j}^2 x_{ij+1}^2}{x_{ij}^2 x_{i+1j+1}^2} \quad p_i = x_i - x_{i-1} \tag{4.1.10}$$

with indices cyclically identified, $i + N \simeq i$. As mentioned before there are $3N - 15$ of them in four dimensions and following ref. [92, 93], from the set of all the U_{ijkl} we can pick a particular algebraically independent set.

$$u_{1i} = \frac{x_{i+1,i+5}^2 x_{i+2,i+4}^2}{x_{i+1,i+4}^2 x_{i+2,i+5}^2} \qquad u_{2i} = \frac{x_{N,i+3}^2 x_{1,i+2}^2}{x_{N,i+2}^2 x_{1,i+3}^2} \qquad u_{3i} = \frac{x_{1\,i+4}^2 x_{2,i+3}^2}{x_{1,i+3}^2 x_{2,i+4}^2} \tag{4.1.11}$$

The three conformally invariant cross ratios (u_{1i}, u_{2i}, u_{3i}) of (4.1.11) can be associated to the t-channel invariants (4.1.7), which have transverse momentum $|\mathbf{q}_i|^2$ [92, 93]. In MRK these cross ratios take the form

$$\begin{aligned} u_{1i} &= 1 - \delta_i \frac{|\mathbf{k}_i + \mathbf{k}_{i+1}|^2}{|\mathbf{k}_{i+1}|^2} + \mathcal{O}(\delta_i^2) \\ u_{2i} &= \delta_i \frac{|\mathbf{q}_{i-1}|^2}{|\mathbf{q}_i|^2} + \mathcal{O}(\delta_i^2) \\ u_{3i} &= \delta_i \frac{|\mathbf{q}_{i+1}|^2 |\mathbf{k}_i|^2}{|\mathbf{q}_i|^2 |\mathbf{k}_{i+1}|^2} + \mathcal{O}(\delta_i^2) \end{aligned} \tag{4.1.12}$$

where $k_i \equiv p_{i+3}$, $1 \leq i \leq N-4$, denote the momenta of the gluons emitted along the t-channel ladder, and we define the ratio $\delta_i \equiv k_{i+1}^+ / k_i^+$. From (4.1.3) it is evident that in MRK we have $\delta_i \to 0$, and so we see that all the u_{1i} tend to 1 at the same speed as the u_{2i}

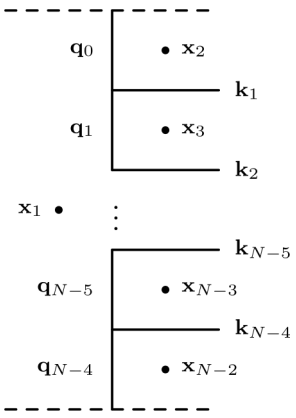


Figure 4.1.1: The dual coordinates in the transverse space. Dashed lines indicate the forward momenta with zero transverse momentum, which are strictly speaking absent in the transverse momentum space because they are orthogonal to it.

and u_{3i} vanish. It is convenient to define the reduced cross ratios [92, 93]

$$\begin{aligned}\tilde{u}_{2i} &= \frac{u_{2i}}{1-u_{1i}} = \frac{|\mathbf{q}_{i-1}|^2\,|\mathbf{k}_{i+1}|^2}{|\mathbf{q}_i|^2\,|\mathbf{k}_i+\mathbf{k}_{i+1}|^2} + \mathcal{O}(\delta_i)\\ \tilde{u}_{3i} &= \frac{u_{3i}}{1-u_{1i}} = \frac{|\mathbf{q}_{i+1}|^2\,|\mathbf{k}_i|^2}{|\mathbf{q}_i|^2\,|\mathbf{k}_i+\mathbf{k}_{i+1}|^2} + \mathcal{O}(\delta_i)\end{aligned} \tag{4.1.13}$$

We now introduce dual coordinates in the transverse space $\mathbb{CP}^1$ by (see Fig. 4.1.1)

$$\mathbf{q}_i = \mathbf{x}_{i+2} - \mathbf{x}_1 \;\text{ and }\; \mathbf{k}_i = \mathbf{x}_{i+2} - \mathbf{x}_{i+1} \tag{4.1.14}$$

The reduced cross ratios $\tilde{u}_{2i}$ and $\tilde{u}_{3i}$ can then be written as (squares of) cross ratios in $\mathbb{CP}^1$,

$$\tilde{u}_{2i} \simeq |\xi_{2i}|^2 \;\text{ and }\; \tilde{u}_{3i} \simeq |\xi_{3i}|^2 \tag{4.1.15}$$

with

$$\xi_{2i} = \frac{(\mathbf{x}_1-\mathbf{x}_{i+1})\,(\mathbf{x}_{i+3}-\mathbf{x}_{i+2})}{(\mathbf{x}_1-\mathbf{x}_{i+2})\,(\mathbf{x}_{i+3}-\mathbf{x}_{i+1})} \;\text{ and }\; \xi_{3i} = \frac{(\mathbf{x}_1-\mathbf{x}_{i+3})\,(\mathbf{x}_{i+2}-\mathbf{x}_{i+1})}{(\mathbf{x}_1-\mathbf{x}_{i+2})\,(\mathbf{x}_{i+3}-\mathbf{x}_{i+1})} \tag{4.1.16}$$

It is easy to check that

$$\xi_i \equiv \xi_{2i} = 1 - \xi_{3i} \tag{4.1.17}$$

We also introduce the transverse cross ratios

$$z_i \equiv 1 - \frac{1}{\xi_i} = \frac{(\mathbf{x}_1-\mathbf{x}_{i+3})\,(\mathbf{x}_{i+2}-\mathbf{x}_{i+1})}{(\mathbf{x}_1-\mathbf{x}_{i+1})\,(\mathbf{x}_{i+2}-\mathbf{x}_{i+3})} = -\frac{\mathbf{q}_{i+1}\,\mathbf{k}_i}{\mathbf{q}_{i-1}\,\mathbf{k}_{i+1}} \tag{4.1.18}$$

In the literature it is customary to use the variables $w_i \equiv -z_i$.

It is easy to see from Fig. 4.1.1 that the MRK setup has a natural $\mathbb{Z}_2$ symmetry, called *target-projectile symmetry* [92, 93], which acts by reflecting all the points along the horizontal

symmetry axis. This symmetry acts on the points $\mathbf{x}_i$ via

$$\mathbf{x}_i \mapsto \begin{cases} \mathbf{x}_1 & \text{if } i = 1 \\ \mathbf{x}_{N-i} & \text{if } 2 \leq i \leq N-2 \end{cases} \tag{4.1.19}$$

On the cross ratios z_i target-projectile symmetry acts by

$$z_i \mapsto 1/z_{N-4-i} \tag{4.1.20}$$

4.1.1 Scattering amplitudes and cluster algebras

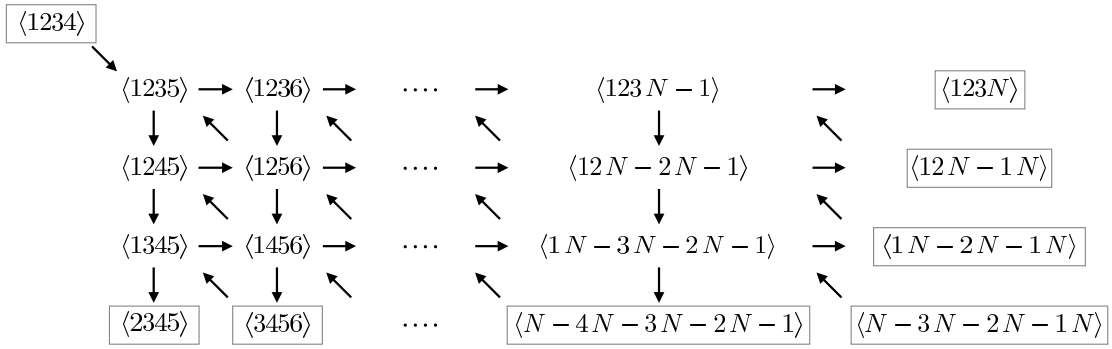


Figure 4.1.2: The $\mathcal{A}$-coordinates for the initial quiver for $\mathrm{Gr}(4, N)$ with frozen nodes in boxes.

As we have seen in previous sections, the kinematics of scattering amplitudes in planar $\mathcal{N} = 4$ SYM are naturally encoded through a configuration of N momentum twistors in three-dimensional projective space $\mathbb{CP}^3$. As momentum twistors are free variables, we can describe the kinematics of colour-ordered partial amplitudes by a configuration of N points in $\mathbb{CP}^3$ [10]. We denote the set of all such configurations by

$$\mathrm{Conf}_N(\mathbb{CP}^3) \simeq \mathrm{Gr}(4, N)/(\mathbb{C}^*)^{N-1} \tag{4.1.21}$$

Naturally associated to the spaces $\mathrm{Conf}_N(\mathbb{CP}^3)$ are cluster algebra structures [8,9,94–96], which play a role in describing the singularity structure of scattering amplitudes or light-like Wilson loops in planar $\mathcal{N} = 4$ SYM theory [10]. The $\mathcal{A}$-coordinates of the cluster algebras are homogeneous polynomials in the Plücker coordinates $\langle ijkl \rangle$. For the cluster algebras associated to $\mathrm{Gr}(4, N)$ one defines an initial cluster given by the quiver diagram in Fig. 4.1.2. Other clusters are obtained by a repeated process called mutation. The $\mathcal{A}$-coordinates in the initial cluster are given by certain Plücker coordinates. The nodes in boxes are called *frozen nodes* and the others are called *unfrozen*. For each unfrozen node one can form $\mathcal{X}$-coordinates by taking the product of all $\mathcal{A}$-coordinates connected by incoming arrows and dividing by the product of all $\mathcal{A}$-coordinates connected by outgoing ones. We label the $\mathcal{X}$-coordinates as $\mathcal{X}_{ij}$ for $i = 1, 2, 3$ and $j = 1, \ldots, N-5$ follow-

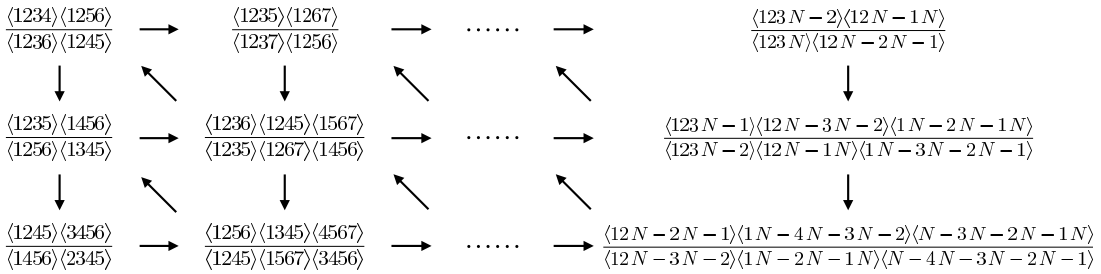


Figure 4.1.3: The $\mathcal{X}$-coordinates for the initial quiver for $\mathrm{Gr}(4,N)$.

ing the obvious structure of Fig. 4.1.3. Explicitly, they are given by

$$\begin{aligned}
\mathcal{X}_{1j} &= \frac{\langle 1\,2\,3\,j+3\rangle\langle 1\,2\,j+4\,j+5\rangle}{\langle 1\,2\,3\,j+5\rangle\langle 1\,2\,j+3\,j+4\rangle}\\
\mathcal{X}_{2j} &= \frac{\langle 1\,2\,3\,j+4\rangle\langle 1\,2\,j+2\,j+3\rangle\langle 1\,j+3\,j+4\,j+5\rangle}{\langle 1\,2\,3\,j+3\rangle\langle 1\,2\,j+4\,j+5\rangle\langle 1\,j+2\,j+3\,j+4\rangle}\\
\mathcal{X}_{3j} &= \frac{\langle 1\,2\,j+3\,j+4\rangle\langle 1\,j+1\,j+2\,j+3\rangle\langle j+2\,j+3\,j+4\,j+5\rangle}{\langle 1\,2\,j+2\,j+3\rangle\langle 1\,j+3\,j+4\,j+5\rangle\langle j+1\,j+2\,j+3\,j+4\rangle}
\end{aligned} \tag{4.1.22}$$

The $\mathcal{X}$-coordinates of any given cluster, in particular the initial one outlined above, form a complete set of coordinates for the kinematical dependence of the scattering amplitude or Wilson loop.

In the remainder of this section we show that there is a very natural geometrical interpretation of MRK in terms of momentum twistors. More precisely, we will show that the dual conformal invariance of planar $\mathcal{N}=4$ SYM implies that the multi-Regge limit defined in (4.1.3) is conformally equivalent to the strongly-ordered multi-soft limit where the momenta p_i, $3\leq i\leq N-3$, are soft, with p_i softer than p_{i+1}.

Before proving the connection between the multi-Regge and soft limits, let us discuss in more detail how to take a single soft limit in momentum twistor space. In terms of dual coordinates, the momentum p_{i+1} is soft if the points x_i and x_{i+1} coincide. As the points x_i correspond to lines in momentum twistor space, the soft limit corresponds to taking the twistor Z_i to lie on the line between Z_{i-1} and Z_{i+1}. This limit leaves two degrees of freedom from the three associated to Z_i. The remaining degrees of freedom can be thought of (using real twistor space as an analogy) as the distance along the line between Z_{i-1} and Z_{i+1} and the angle of approach to the line in which the limit is taken. More generally, let us consider a limit where the twistor Z_i approaches the line between Z_j and Z_k. We parametrise this situation as follows

$$Z_i \to \hat{Z}_i = Z_j + \alpha_i \tfrac{\langle j\,j-1\,k+1\,k+2\rangle}{\langle k\,j-1\,k+1\,k+2\rangle} Z_k + \epsilon_i \tfrac{\langle j\,k\,k+1\,k+2\rangle}{\langle j-1\,k\,k+1\,k+2\rangle} Z_{j-1} - \epsilon_i\beta_i \tfrac{\langle j\,j-1\,k\,k+2\rangle}{\langle k+1\,j-1\,k\,k+2\rangle} Z_{k+1} \tag{4.1.23}$$

and the limit where Z_i, Z_j and Z_k are aligned corresponds to the limit $\epsilon_i\to 0$. The existence of the last two terms in (4.1.23) ensures that $x^2_{i-1i+1}, x^2_{ii+2}\sim\epsilon_i$ as we approach

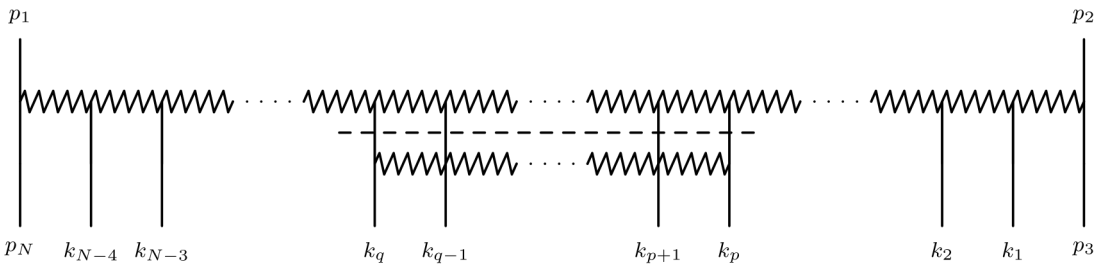


Figure 4.1.4: Diagrammatic representation of the Mandelstam region $[p, q]$. The discontinuity in the $(k_p + \ldots + k_q)^2$ channel is indicated by the dashed line.

the limit, as can be shown from (2.2.59) with the canonical choice of infinity twistor.

The multi-soft limit we wish to consider is one where we sequentially take the momenta p_i, $3 \leq i \leq N-3$, to be soft. This corresponds to taking twistor Z_2 to the line $(Z_1 Z_3)$, then Z_3 to the line $(Z_1 Z_4)$ and so on. This limit reproduces the behaviour of the cross ratios (4.1.11) described in the previous section

$$u_{1i} \to 1 \qquad u_{2i} \to 0 \qquad u_{3i} \to 0 \tag{4.1.24}$$

i.e., the cross ratios behave in the same way as in MRK, cf. (4.1.3). This is, however, still insufficient to conclude that this multi-soft limit is equivalent to MRK, and we still need to show that the cross ratios approach their limiting values at the same speed. Equivalently, we need to show that the reduced cross ratios (4.1.13) are finite in the limit. This is indeed the case, and we find

$$\begin{aligned} \tilde{u}_{2i} &= \frac{u_{2i}}{1-u_{1i}} \to \frac{\alpha_{i+1}\beta_{i+1}}{(1+\alpha_{i+1})(1+\beta_{i+1})} \\ \tilde{u}_{3i} &= \frac{u_{3i}}{1-u_{1i}} \to \frac{1}{(1+\alpha_{i+1})(1+\beta_{i+1})} \end{aligned} \tag{4.1.25}$$

Hence, we conclude that this particular multi-soft limit is conformally equivalent to the multi-Regge limit. Comparing (4.1.25) to (4.1.15) and (4.1.18), we see that we can identify the parameters α_{i+1} and β_{i+1} that describe the reduced cross ratios in the multi-soft limit with the $\mathbb{CP}^1$ cross ratio that appear in MRK

$$\alpha_{i+1} = -1/z_i \text{ and } \beta_{i+1} = -1/\bar{z}_i \tag{4.1.26}$$

4.1.2 Planar SYM amplitudes in multi-Regge kinematics

So far all the considerations were purely kinematical. In this section we present the (conjectural) representation of an amplitude in MRK to leading logarithmic accuracy (LLA). Helicity must be conserved by the gluons going very forward, so that the different helicity configurations are distinguished only by the helicities of the gluons emitted along

the ladder. Denoting these helicities by $h_1, \ldots, h_{N-4}$, we define the ratio

$$e^{i\Phi_{h_1,\ldots,h_{N-4}}}\,\mathcal{R}_{h_1,\ldots,h_{N-4}} \equiv \left[\frac{A_N(-,+,h_1,\ldots,h_{N-4},+,-)}{A_N^{\text{BDS}}(-,+,\ldots,+,-)}\right]_{|\text{MRK, LLA}} \tag{4.1.27}$$

where $A_N(-,+,h_1,\ldots,h_{N-4},+,-)$ is the (colour-ordered) amplitude for the production of $N-4$ gluons emitted along the ladder, and $A_N^{\text{BDS}}(-,+,\ldots,+,-)$ is the corresponding BDS amplitude. The function $\mathcal{R}_{h_1,\ldots,h_{N-4}}$ is finite, and thus dual conformally invariant. It can easily be related to the well-known remainder and ratio functions. Since Regge factorisation holds in the Euclidean region, the ratio in the left-hand side of (4.1.27) tends to a phase in this region. The exact form of this phase is immaterial in the following, because it is simply obtained as the ratio of the corresponding tree amplitudes [97]. We normalise the left-hand side of (4.1.27) such that $\mathcal{R}_{h_1,\ldots,h_{N-4}} = 1$ in the Euclidean region.

If we take a discontinuity corresponding to a consecutive subset of final-state momenta k_l, $l \in [p,q] \subseteq \{1,\ldots,N-4\}$, i.e., a discontinuity with respect to the invariant $(k_p + \ldots + k_q)^2$, then $\mathcal{R}_{h_1,\ldots,h_{N-4}}$ is no longer trivial due to the presence of a Regge cut (see Fig. 4.1.4) [93, 98–105]. In terms of the dual conformal cross ratios taking this discontinuity corresponds to analytically continuing U_{pq+2} around the origin while all other cross ratios U_{ij} are held fixed. In the following we denote the value of the ratio $\mathcal{R}_{h_1,\ldots,h_{N-4}}$ in this so-called *Mandelstam region* $[p,q]$ by $\mathcal{R}^{[p,q]}_{h_1,\ldots,h_{N-4}}$. We conjecture that $\mathcal{R}^{[p,q]}_{h_1,\ldots,h_{N-4}}$ in MRK to LLA can be cast in the form of a multiple Fourier-Mellin integral

$$\begin{aligned}\mathcal{R}^{[p,q]}_{h_1\ldots h_{N-4}}(\{\tau_k, z_k\}_{p\le k\le q-1}) =& 1 + a\, i\pi\, r^{[p,q],(1)} \\ &+ a\, i\pi\, (-1)^{q-p}\mathcal{F}^{[p,q]}\left[\chi_p\left(\prod_{k=p}^{q-2} C_k\right)\chi_{\bar{q}}\,\tau^{[p,q]}\right]\end{aligned} \tag{4.1.28}$$

Where the multiple (inverse) Fourier-Mellin transform $\mathcal{F}^{[p,q]}$ of a function $F(\{\nu_k, n_k\})$ is defined as

$$f(\{z_k\}) = \mathcal{F}^{[p,q]}[F(\{\nu_k,n_k\})] = \prod_{k=p}^{q-1}\sum_{n_k=-\infty}^{+\infty}\left(\frac{z_k}{\bar{z}_k}\right)^{n_k/2}\int_{-\infty}^{+\infty}\frac{d\nu_k}{2\pi}|z_k|^{2i\nu_k}F(\{\nu_k,n_k\}) \tag{4.1.29}$$

The remaining quantities in (4.1.28) are as follows

$$\tau^{[p,q]} = -1 + \prod_{k=p}^{q-1}\tau_k^{aE_{\nu_k n_k}} \tag{4.1.30}$$

We define $\tau_k \equiv \sqrt{u_{2k}u_{3k}}$, and a is the 't Hooft coupling. To LLA, the value of τ_k is independent of k, but we prefer to keep the τ_k different for reasons that will become clear in subsequent sections. The one-loop coefficients $r^{[p,q],(1)}$ receive contributions from both

the Regge pole and cut. They are sums of logarithms whose functional form is irrelevant for the remainder of this chapter. $E_{\nu n}$ is the leading-order (LO) BFKL eigenvalue,

$$E_{\nu n} = -\frac{1}{2}\frac{|n|}{\nu^2+\frac{n^2}{2}} + \psi\left(1+i\nu+\frac{|n|}{2}\right) + \psi\left(1-i\nu+\frac{|n|}{2}\right) - 2\psi(1) \tag{4.1.31}$$

and $\chi^h(\nu,n)$ is the LO impact factor [98, 99]

$$\begin{aligned}\chi_p &= \chi^{h_p}(\nu_p, n_p)\\ \chi_{\bar{q}} &= \chi^{-h_q}(\nu_{q-1}, n_{q-1})\\ \chi^{\pm}(\nu,n) &= \frac{1}{i\nu\pm\frac{n}{2}} = \left[\chi^{\mp}(-\nu,-n)\right]^*\end{aligned} \tag{4.1.32}$$

The central emission block $C_k = C^+(\nu_k, n_k, \nu_{k+1}, n_{k+1})$ for the emission of a positive-helicity gluon is [102]

$$C^+(\nu_k,n_k,\nu_{k+1},n_{k+1}) = \frac{\Gamma(1-i\nu_k-\frac{n_k}{2})\,\Gamma(i\nu_{k+1}+\frac{n_{k+1}}{2})\,\Gamma(i(\nu_k-\nu_{k+1})+\frac{n_{k+1}-n_k}{2})}{\Gamma(1+i\nu_k-\frac{n_k}{2})\,\Gamma(-i\nu_{k+1}+\frac{n_{k+1}}{2})\,\Gamma(1-i(\nu_k-\nu_{k+1})+\frac{n_{k+1}-n_k}{2})} \tag{4.1.33}$$

Conversely for the emission of a negative-helicity gluon it takes the form

$$C^-(\nu,n,\mu,m) = [C^+(-\nu,-n,-\mu,-m)]^* = C^+(\nu,-n,\mu,-m) \tag{4.1.34}$$

The (inverse) Fourier-Mellin integral transform (4.1.28) is invertible and, focusing on the single variable case, its inverse is given by

$$\mathcal{F}^{-1}[f(z)] = \int \frac{d^2z}{\pi}\, z^{-1-i\nu-n/2}\,\bar{z}^{-1-i\nu+n/2}\, f(z) \tag{4.1.35}$$

with the usual metric on the complex plane

$$d^2z = -\frac{dz\wedge d\bar{z}}{2i} = dx\wedge dy = r\,dr\wedge d\varphi \quad \text{for } z = x+iy = re^{i\varphi} \tag{4.1.36}$$

The Fourier-Mellin transform has the property that it maps ordinary products into convolutions. More precisely, if $\mathcal{F}[F] = f$ and $\mathcal{F}[G] = g$, then

$$\mathcal{F}[F\cdot G] = \mathcal{F}[F] * \mathcal{F}[G] = f * g \tag{4.1.37}$$

where the convolution is defined by

$$(f*g)(z) = \frac{1}{\pi}\int \frac{d^2w}{|w|^2}\, f(w)\; g\left(\frac{z}{w}\right) \tag{4.1.38}$$

A proof of the convolution theorem for the Fourier-Mellin transform is given in Ap-

pendix A of [38]. It is easy to see that the convolution product is associative and commutative, and the distribution $\pi\,\delta^{(2)}(1-z)$ is a neutral element.

In order to fully define the expression for $\mathcal{R}^{[p,q]}_{h_1,\ldots,h_{N-4}}$ in (4.1.28) we must specify the contours of integration. The integrals over the ν_k are taken along the real ν_k-axes, however the quantities χ and C have poles on the real axes for certain values of the n_k. Our contour prescription for avoiding these poles is as follows. For $n_p = 0$ we replace

$$\chi^{h_p}(\nu_p, 0) \to \frac{1}{i\nu_p - \epsilon} \tag{4.1.39}$$

For $n_{k-1} = n_k$ we find that $C(\nu_{k-1}, n_{k-1}, \nu_k, n_k)$ exhibits a pole at $\nu_{k-1} = \nu_k$, as can be seen by inspecting the third factor in the numerator of (4.1.33). We avoid this pole by replacing it as follows

$$\frac{1}{i(\nu_{k-1} - \nu_k)} \to \frac{1}{i(\nu_{k-1} - \nu_k) + \epsilon} \tag{4.1.40}$$

For $n_{q-1} = 0$ we replace

$$\chi^{-h_{q-1}}(\nu_{q-1}, 0) \to \frac{1}{i\nu_{q-1} + \epsilon} \tag{4.1.41}$$

In all cases we take ϵ to be an infinitesimal positive number.

The effect of the replacement (4.1.39) is to shift the pole from $\chi^{h_p}(\nu_p, 0)$ slightly into the lower half ν_p plane. The shift (4.1.40) means the pole is slightly shifted into the upper half ν_{k-1} plane (or the lower half ν_k plane). Finally the shift (4.1.41) takes the pole slightly into the upper half ν_{q-1} plane.

We conclude this section by quoting some properties of the Fourier-Mellin space functions that enter (4.1.28). For $n_k = 0$, the BFKL eigenvalue and the central emission block have the following properties [85, 87, 91, 106–109]

$$\lim_{\nu\to 0} E_{\nu 0} = 0 \tag{4.1.42}$$

$$\lim_{\nu\to 0} C^{\pm}(\nu, 0, \mu, m) = \chi^{\pm}(\mu, m) \tag{4.1.43}$$

$$\lim_{\mu\to 0} C^{\pm}(\nu, n, \mu, 0) = -\chi^{\mp}(\nu, n) \tag{4.1.44}$$

$$\mathrm{Res}_{\nu=\mu} C^{\pm}(\nu, n, \mu, n) = (-1)^n\, i \tag{4.1.45}$$

Note that $E_{\nu 0}$ vanishes quadratically as $\nu \to 0$ due to its symmetry under $\nu \leftrightarrow -\nu$. As we will see shortly, the above relations guarantee that (4.1.28) has the correct soft behaviour. In order to prove the last relation (4.1.45), we need the following identity

$$\frac{\sin\pi(\frac{n}{2} + i\nu)}{\sin\pi(\frac{n}{2} - i\nu)} = (-1)^{n+1} \quad n \in \mathbb{Z} \tag{4.1.46}$$

In order to show this identity, let us define

$$S_n = \frac{\sin\pi(\frac{n}{2}+i\nu)}{\sin\pi(\frac{n}{2}-i\nu)} \tag{4.1.47}$$

Obviously, $S_0 = -1$ and $S_1 = 1$. Moreover, S_n satisfies a recursion of order two

$$S_{n+2} = \frac{\sin\left[\pi+\pi(\frac{n}{2}+i\nu)\right]}{\sin\left[\pi+\pi(\frac{n}{2}-i\nu)\right]} = \frac{\sin\pi(\frac{n}{2}+i\nu)}{\sin\pi(\frac{n}{2}-i\nu)} = S_n \tag{4.1.48}$$

Hence, $S_n = (-1)^{n+1}$. Finally, we note the following relation between the central emission block and the impact factor

$$\frac{C^-(\nu,n,\mu,m)}{C^+(\nu,n,\mu,m)} = \frac{\chi^+(\nu,n)\,\chi^-(\mu,m)}{\chi^-(\nu,n)\,\chi^+(\mu,m)} \tag{4.1.49}$$

Equation (4.1.28) reproduces the known Fourier-Mellin representation of the six-point MHV and NMHV amplitudes in MRK to LLA [98,99,102], and also of the seven-point MHV amplitude to LLA [102]. In [38] further support to the conjecture was given by showing that it is consistent with target-projectile symmetry and with the factorisation of the amplitude in infrared limits. Due to the strong ordering in the rapidities (or equivalently, in the (+-) lightcone coordinates), there are no collinear singularities. All the singularities of an amplitude in MRK can therefore be associated to some final-state partons being soft. Indeed an amplitude in MRK has soft singularities only in the limits where one of the momenta k_i, $1 \le i \le N-4$, vanishes. More concretely, in terms of the transverse cross ratios in MRK, the limits and their effect upon $\mathcal{R}$ are described by

$$\begin{aligned}
&z_1 \to 0: && \mathcal{R}_{h_1\ldots}(\tau_1,z_1,\ldots,\tau_{N-5},z_{N-5}) \to \mathcal{R}_{h_2\ldots}(\tau_2,z_2,\ldots,\tau_{N-5},z_{N-5})\\
&z_i \to 0\,,\ z_{i-1}z_i \text{ fixed}: && \mathcal{R}_{\ldots h_i\ldots}(\ldots,\tau_{i-1},z_{i-1},\tau_i,z_i,\ldots) \to \mathcal{R}_{\ldots\hat{h}_i\ldots}(\ldots,\tau_{i-1}\tau_i,-z_{i-1}z_i,\ldots)\\
&z_{N-5} \to \infty: && \mathcal{R}_{\ldots h_{N-4}}(\tau_1,z_1,\ldots,\tau_{N-5},z_{N-5}) \to \mathcal{R}_{\ldots h_{N-5}}(\tau_1,z_1,\ldots,\tau_{N-6},z_{N-6})
\end{aligned} \tag{4.1.50}$$

Furthermore these limits have their analogues in the Fourier-Mellin space and it turns out that the prescribed contours are precisely the ones required to ensure that the conjectured form reproduces the soft limit behaviour. Similarly by considering how the variables z_i and τ_i behave under target-projectile symmetry it is possible to show that the conjecture is consistent with the behaviour of $\mathcal{R}_{h_1,\ldots,h_{N-4}}$.

Finally the function $\mathcal{R}^{[p,q]}_{h_1,\ldots,h_{N-4}}$ is identical to the ratio where all the gluons not present in the discontinuity $[p,q]$ have been removed. In other words, if we know the results for the Mandelstam regions $[1,N-4]$, then we can reconstruct all other cases. Hence, in the following we only discuss this particular case, and we simply write $\mathcal{R}_{h_1,\ldots,h_{N-4}}$ for $\mathcal{R}^{[1,N-4]}_{h_1,\ldots,h_{N-4}}$.

4.1.3 Perturbative expansion of the ratio $\mathcal{R}_{h_1,\ldots,h_{N-4}}$

So far all the considerations were made before the perturbative expansion of the function $\mathcal{R}_{h_1,\ldots,h_{N-4}}$. If we expand the integrand in (4.1.28) perturbatively, then at each order we obtain logarithms of τ_k. The coefficients of these logarithms are the main objects of interest in the rest of this chapter. We write the perturbative expansion of the function $\mathcal{R}_{h_1,\ldots,h_{N-4}}$ as

$$\begin{aligned}\mathcal{R}_{h_1,\ldots,h_{N-4}}(\tau_1,z_1,\ldots,\tau_{N-5},z_{N-5}) &= 1 + a\, i\pi\, r^{(1)}_{h_1,\ldots,h_{N-4}} \\ &+ 2\pi i \sum_{i=2}^{\infty} \sum_{i_1+\ldots+i_{N-5}=i-1} a^i \left(\prod_{k=1}^{N-5} \frac{1}{i_k!}\log^{i_k}\tau_k\right) g^{(i_1,\ldots,i_{N-5})}_{h_1,\ldots,h_{N-4}}(z_1,\ldots,z_{N-5})\end{aligned} \tag{4.1.51}$$

The perturbative coefficients are completely known for $N = 6$ for both MHV and NMHV [100, 101, 103, 110–112], and for all MHV amplitudes at two loops [102, 113, 114]. Comparing the perturbative expansion to (4.1.28), we see that the perturbative coefficients admit a Fourier-Mellin transform representation

$$g^{(i_1,\ldots,i_{N-5})}_{h_1,\ldots,h_{N-4}}(z_1,\ldots,z_{N-5}) = \frac{(-1)^{N+1}}{2}\,\mathcal{F}^{[1,N-4]}\left[\chi_1\left(\prod_{k=1}^{N-6} C_k\right)\chi_{\overline{N-5}}\prod_{k=1}^{N-5} E^{i_k}_{\nu_k n_k}\right] \tag{4.1.52}$$

The poles on the real axis are dealt with by the prescription already outlined in (4.1.39) - (4.1.41).

The symmetries of the ratio $\mathcal{R}_{h_1,\ldots,h_{N-4}}$ induce similar symmetries on the perturbative coefficients,

$$\begin{aligned} g^{(i_1,\ldots,i_{N-5})}_{h_1,\ldots,h_{N-4}}(z_1,\ldots,z_{N-5}) &= g^{(i_1,\ldots,i_{N-5})}_{-h_1,\ldots,-h_{N-4}}(\bar{z}_1,\ldots,\bar{z}_{N-5}) \\ &= g^{(i_{N-5},\ldots,i_1)}_{-h_{N-4},\ldots,-h_1}\left(\frac{1}{z_{N-5}},\ldots,\frac{1}{z_1}\right)\end{aligned} \tag{4.1.53}$$

In the soft limits, the perturbative coefficients must reduce to lower-point functions. The limits where either $\mathbf{k}_1$ or $\mathbf{k}_{N-4}$ vanish are easy to describe: the perturbative coefficients reduce to the corresponding coefficients with the soft momentum removed, except if the corresponding large logarithm is present, in which case the perturbative coefficient vanishes in the limit. More precisely

$$\begin{aligned}\lim_{z_1\to 0} g^{(i_1,\ldots,i_{N-5})}_{h_1,\ldots,h_{N-4}}(z_1,\ldots,z_{N-5}) &= \delta_{i_1 0}\, g^{(i_2,\ldots,i_{N-5})}_{h_2,\ldots,h_{N-4}}(z_2,\ldots,z_{N-5}) \\ \lim_{z_{N-5}\to\infty} g^{(i_1,\ldots,i_{N-5})}_{h_1,\ldots,h_{N-4}}(z_1,\ldots,z_{N-5}) &= \delta_{i_{N-5} 0}\, g^{(i_1,\ldots,i_{N-6})}_{h_1,\ldots,h_{N-5}}(z_1,\ldots,z_{N-6})\end{aligned} \tag{4.1.54}$$

If $\mathbf{k}_j$, with $j \notin \{1, N-4\}$ is soft, then the perturbative coefficients behave like

$$\begin{aligned}\lim_{\substack{(z_{j-1},z_j)\to(\infty,0)\\ z_{j-1}z_j \text{ fixed}}} & g^{(i_1,\ldots,i_{N-5})}_{h_1,\ldots,h_{N-4}}(z_1,\ldots,z_{N-5}) \\ &= g^{(i_1,\ldots,i_{j-1}+i_j,\ldots,i_{N-5})}_{h_1,\ldots,\hat{h}_j,\ldots,h_{N-4}}(z_1,\ldots,-z_{j-1}z_j,\ldots,z_{N-5})\end{aligned} \tag{4.1.55}$$

Indeed, we have

$$\lim_{\mathbf{k}_j\to 0} \mathcal{R}_{h_1,\ldots,h_{N-4}}(\tau_1,z_1,\ldots,\tau_{N-5},z_{N-5}) \tag{4.1.56}$$

$$\begin{aligned}&= 2\pi i \sum_{i=2}^{\infty} \sum_{i_1+\ldots+i_{N-5}=i-1} a^i \left(\prod_{k=1}^{N-5} \frac{1}{i_k!}\log^{i_k}\tau_k\right) g^{(i_1,\ldots,i_{j-1}+i_j,\ldots,i_{N-5})}_{h_1,\ldots,\hat{h}_j,\ldots,h_{N-4}} \\ &= 2\pi i \sum_{i=2}^{\infty} \sum_{i_1+\ldots+i'+\ldots+i_{N-5}=i-1} \sum_{i_{j-1}+i_j=i'} a^i \left(\prod_{k=1}^{N-5} \frac{1}{i_k!}\log^{i_k}\tau_k\right) g^{(i_1,\ldots,i',\ldots,i_{N-5})}_{h_1,\ldots,\hat{h}_j,\ldots,h_{N-4}} \\ &= 2\pi i \sum_{i=2}^{\infty} \sum_{i_1+\ldots+i'+\ldots+i_{N-5}=i-1} a^i \frac{1}{i'!}\log^{i'}(\tau_{j-1}\tau_j) \left(\prod_{\substack{k=1\\ k\notin\{j-1,j\}}}^{N-5} \frac{1}{i_k!}\log^{i_k}\tau_k\right) g^{(i_1,\ldots,i',\ldots,i_{N-5})}_{h_1,\ldots,\hat{h}_j,\ldots,h_{N-4}}\end{aligned} \tag{4.1.57}$$

where the last step follows from the binomial theorem

$$\sum_{i_{j-1}+i_j=i'} \frac{1}{i_{j-1}!i_j!}\log^{i_{j-1}}\tau_{j-1}\log^{i_j}\tau_j = \frac{1}{i'!}\log^{i'}(\tau_{j-1}\tau_j) \tag{4.1.58}$$

See [38] for a more in-depth discussion of soft limits.

4.2 MRK and the moduli space of genus zero curves with marked points

In this section we argue that it is possible to describe the space of functions of scattering amplitudes in planar $\mathcal{N}=4$ SYM in MRK. We start by noting that in MRK the only non-trivial functional dependence is through the transverse momenta. In the previous section we have seen that the kinematics in the transverse space are described by $n \equiv N-2$ dual coordinates $\mathbf{x}_i$. Hence, in the multi-Regge limit the kinematics are described by a configuration of $(N-2)$ points in $\mathbb{CP}^1$. The space of such configurations is equivalent to the moduli space of genus zero curves with $(N-2)$ marked points

$$\text{Conf}_{N-2}(\mathbb{CP}^1) \simeq \mathfrak{M}_{0,N-2} \tag{4.2.1}$$

In Section 4.1.1 we have seen that the cluster algebra attached to the configuration space describing the kinematics of an amplitude is related to the singularities of the amplitude.

From the previous discussion it is thus natural to expect that amplitudes in planar $\mathcal{N}=4$ SYM in MRK can be expressed in terms of iterated integrals on $\mathfrak{M}_{0,N-2}$. We now show that this is indeed the case. More precisely, we show that the cluster algebra associated to $\text{Conf}_N(\mathbb{CP}^3)$ in full kinematics reduces to the cluster algebra of $\mathfrak{M}_{0,N-2}$.

4.2.1 MRK and the moduli space $\mathfrak{M}_{0,N-2}$

We start from the duality between MRK and multi-soft limits discussed in Section 4.1.1. We insert the parametrisation of eq. (4.1.23) into the cluster $\mathcal{X}$-coordinates of eq. (4.1.22) and we take the limit $\epsilon_i \to 0$. We see that all $\mathcal{X}$-coordinates of the form $\mathcal{X}_{2j}$ vanish in the limit, while the others reduce to either holomorphic or anti-holomorphic cross ratios in $\mathbb{CP}^1$

$$\mathcal{X}_{1j} = \frac{(\overline{\mathbf{x}}_2 - \overline{\mathbf{x}}_{j+2})(\overline{\mathbf{x}}_{j+3} - \overline{\mathbf{x}}_{j+4})}{(\overline{\mathbf{x}}_2 - \overline{\mathbf{x}}_{j+4})(\overline{\mathbf{x}}_{j+2} - \overline{\mathbf{x}}_{j+3})} \quad \mathcal{X}_{2j} = 0 \quad \mathcal{X}_{3j} = \frac{(\mathbf{x}_1 - \mathbf{x}_{j+1})(\mathbf{x}_{j+2} - \mathbf{x}_{j+3})}{(\mathbf{x}_1 - \mathbf{x}_{j+3})(\mathbf{x}_{j+1} - \mathbf{x}_{j+2})} \tag{4.2.2}$$

We see that the $\mathcal{X}$-coordinates are singular when two points $\mathbf{x}_i$ coincide, which is precisely the singularity structure of the moduli space $\mathfrak{M}_{0,N-2}$. However, we have obtained two copies of points, a holomorphic and an anti-holomorphic one. This can be understood from the cluster algebra in Fig. 4.1.3. Indeed, in the multi-Regge limit the middle line in the quiver vanishes, and so the cluster algebra splits into two disconnected parts, one which only depends on holomorphic variables and the other one only on anti-holomorphic variables. Each of these two parts is isomorphic to the cluster algebra A_{N-5}, which is the cluster algebra that describes the singularity structure of $\text{Conf}_{N-2}(\mathbb{CP}^1) \simeq \mathfrak{M}_{0,N-2}$. Hence, we conclude that in MRK the cluster algebra of $\text{Conf}_N(\mathbb{CP}^3)$ reduces to the cluster algebra $A_{N-5} \times A_{N-5}$, and the two copies of A_{N-5} are complex conjugate to each other in the case of real 2-to-$(N-2)$ scattering. As a consequence, we expect that planar scattering amplitudes in $\mathcal{N}=4$ SYM in MRK can be expressed through iterated integrals with singularities precisely when the $\mathcal{X}$-coordinates in eq. (4.2.2) are singular, i.e., iterated integrals over integrable words made out of the one-forms $d\log(\mathbf{x}_i - \mathbf{x}_j)$ (and their complex conjugates). Note that scattering amplitudes in MRK are singular whenever one of the final-state gluons is soft, $\mathbf{k}_i \to 0$, which happens precisely when $\mathbf{x}_i = \mathbf{x}_{i+1}$, $2 \leq i \leq N-4$. It is remarkable that the cluster algebra in MRK is of finite type, independently of the number N of external particles. Indeed, it is known that a cluster algebra is of finite type precisely if one of the quivers that represent its seeds is a Dynkin diagram [9]. The cluster algebras associated to the six and seven-point amplitudes are of finite type (the corresponding Dynkin diagrams are A_3 and E_6), but starting from $N=8$ the cluster algebra is infinite [10,94]. Remarkably, the cluster algebra in general kinematics always reduces to a cluster algebra of finite type in MRK.

Scattering amplitudes, however, cannot be arbitrary combinations of iterated integrals

built on $A_{N-5} \times A_{N-5}$, but the branch cuts of the amplitudes are constrained by physical considerations. In particular, massless scattering amplitudes can have branch points at most when a Mandelstam invariant vanishes or becomes infinite, which puts strong constraints on the first letter in the word defining the iterated integral[1] [115]. Dual conformal invariance implies that the first letter of the word must be a cross ratio $d\log U_{ijkl}$. In the Mandelstam region $[p, q]$, however, integrability combined with the first entry condition implies that on this Riemann sheet the branch points are determined by products of cross ratios that become equal to 0, 1 or ∞. In other words, in a Mandelstam region the first letter is either a cross ratio $d\log U_{ijkl}$ or $d\log(1 - \prod_{ijkl} U_{ijkl}^{n_{ijkl}})$. In the following we show that this implies that in MRK the first entries are necessarily absolute values squared of cross ratios in $\mathbb{CP}^1$.

To start, we note that there are $N(N-5)/2$ multiplicatively independent cross ratios, which we may choose as

$$\begin{aligned} u_{1i},\, u_{2i},\, u_{3i} \quad & 1 \leq i \leq N-5 \\ U_{ij} \quad & 2 \leq i \leq j-4 \leq N-5 \end{aligned} \tag{4.2.3}$$

where these cross ratios have been defined in eq. (4.1.11). The multi-Regge limit of (u_{1i}, u_{2i}, u_{3i}) was already analysed in Section 4.1. Using the duality between MRK and the multi-soft limit, it is easy to show that all the U_{ij} tend to 1 in MRK. We introduce new reduced cross ratios which have a finite multi-Regge limit

$$\tilde{U}_{ij} \equiv \frac{1-U_{ij}}{\prod_{k=i-1}^{j-4}(1-u_{1k})} \to \left| \frac{\mathbf{x}_i - \mathbf{x}_{j-1}}{\mathbf{x}_i - \mathbf{x}_{i+2}} \prod_{k=i+1}^{j-3} \frac{\mathbf{x}_k - \mathbf{x}_{k+1}}{\mathbf{x}_k - \mathbf{x}_{k+2}} \right|^2 \tag{4.2.4}$$

From eq. (4.2.4) we see that all the U_{ij} approach 1 at different speeds in the multi-soft limit. Indeed, the multi-soft limit is approached sequentially according to $\epsilon_2 \ll \epsilon_3 \ll \ldots \ll \epsilon_{N-4}$, where ϵ_i are the small parameters introduced in eq. (4.1.23). Since $u_{1i} = 1 + \mathcal{O}(\epsilon_{i+1})$, we see that $U_{ij} = 1 + \mathcal{O}(\epsilon_i \ldots \epsilon_{j-4})$, and so all the U_{ij} approach 1 at a different speed.

We now show that the first entries of the perturbative coefficients reduce to absolute values squared of cross ratios in $\mathbb{CP}^1$ (up to logarithmically divergent terms that are absorbed into the definition of the τ_k). Let us first look at the case where the first letter is $d\log U_{ijkl}$. It is sufficient to analyse the multiplicatively independent cross ratios in eq. (4.2.3). They all tend to 1, except for u_{2i} and u_{3i}, which we may exchange for the corresponding reduced cross ratios $\tilde{u}_{2i}$ and $\tilde{u}_{3i}$. The latter reduce to absolute values squared of cross ratios in $\mathbb{CP}^1$, see eq. (4.1.15).

[1] We note that this condition is independent of whether the iterated integral can be evaluated in terms of multiple polylogarithms.

Next, let us analyse the case of a letter of the type $d\log(1-\prod_{ijkl} U_{ijkl}^{n_{ijkl}})$. It is sufficient to assume that the factors in the product are taken from eq. (4.2.3). If one of the factors goes to zero in MRK, then the claim is true, because we have for example

$$d\log(1-u_{2i}^{n}U) \to \begin{cases} n\,d\log u_{2i} + d\log U & \text{if } n<0 \\ 0 & \text{if } n>0 \end{cases} \tag{4.2.5}$$

where U is any product of cross ratios that tend to 1 in MRK. If all the factors in the product $\prod_{ijkl} U_{ijkl}^{n_{ijkl}}$ tend to 1, then we know that one of the factors tends to one much slower than the others. Hence, up to terms that are power-suppressed in MRK, we only need to keep this factor. The claim then follows from eq. (4.2.4).

The previous discussion implies that the coefficients appearing in the perturbative expansion of scattering amplitudes in planar $\mathcal{N}=4$ SYM are iterated integrals with singularities described by the cluster algebra $A_{N-5}\times A_{N-5}$ and whose first letters are absolute values squared of cross ratios. As the first entries describe the branch points of the function, we conclude that the perturbative coefficients have no branch cuts when seen as functions of the complex points $\mathbf{x}_i$. In other words, these iterated integrals must define single-valued functions on the moduli space of Riemann spheres with $N-2$ marked points. In the remainder of this section we review the theory of single-valued iterated integrals on $\mathfrak{M}_{0,N-2}$. We first discuss ordinary, not necessarily single-valued, iterated integrals on $\mathfrak{M}_{0,N-2}$, and we turn to the construction of their single-valued analogues at the end of the section.

4.2.2 Coordinate systems on $\mathfrak{M}_{0,n}$

In this section we review various coordinate systems on $\mathfrak{M}_{0,n}$ which are useful to study iterated integrals and the multi-Regge limit. As a geometric space, we can describe $\mathfrak{M}_{0,n}$ by configurations of n distinct points on the Riemann sphere. We identify configurations that are related by conformal transformations. As $SL(2,\mathbb{C})$ has complex dimension 3, we immediately see that

$$\dim_{\mathbb{C}} \mathfrak{M}_{0,n} = n-3 \tag{4.2.6}$$

Roughly speaking, since $\mathfrak{M}_{0,n}$ is $SL(2,\mathbb{C})$-invariant, a system of coordinates is given by a set of cross ratios formed out of the points $\mathbf{x}_i$. There is no global coordinate system on $\mathfrak{M}_{0,n}$. One such set of cross ratios is given by the cross ratios z_i defined in eq. (4.1.18). We will refer to these coordinates as *Fourier-Mellin coordinates*. These coordinates are well suited to write down the Fourier-Mellin transforms that describe amplitudes in MRK. These coordinates, however, are not ideal to describe the iterated integrals on $\mathfrak{M}_{0,n}$.

In ref. [116] various local systems of coordinates are discussed that are well suited to study iterated integrals on $\mathfrak{M}_{0,n}$. A particularly simple set of local coordinates are the

simplicial coordinates, obtained by using the $SL(2,\mathbb{C})$ invariance to fix three of the n points to 0, 1 and ∞, e.g.

$$(\mathbf{x}_1,\ldots,\mathbf{x}_n)\to(0,1,\infty,t_1,\ldots,t_{n-3}) \text{ with } t_{i-3}=\frac{(\mathbf{x}_i-\mathbf{x}_1)(\mathbf{x}_2-\mathbf{x}_3)}{(\mathbf{x}_i-\mathbf{x}_3)(\mathbf{x}_2-\mathbf{x}_1)} \quad 4\leq i\leq n \tag{4.2.7}$$

Note that there are $6\binom{n}{3}=n(n-1)(n-2)$ different choices for simplicial coordinates, depending on which three points we fix to $(0,1,\infty)$. Using simplicial coordinates we can describe $\mathfrak{M}_{0,n}$ (roughly speaking) as the space

$$\{(t_1,\ldots,t_{n-3})\in\mathbb{C}^{n-3}|t_i\neq 0,1 \text{ and } t_i\neq t_j\} \tag{4.2.8}$$

While there is in principle no reason to prefer one particular choice of simplicial coordinates over the other, some choices are more suited to MRK than others. In particular, it is useful to choose the coordinates so that they transform nicely under the symmetries of the problem. In our case, we prefer to choose simplicial coordinates on which target-projectile symmetry acts in a simple way. It is easy to check that the simplicial coordinate systems with this property are defined by fixing the points $(\mathbf{x}_1,\mathbf{x}_k,\mathbf{x}_{N-k})$, $2\leq k\leq\lceil\frac{N-1}{2}\rceil$. In addition, for N even the set of simplicial coordinates defined by fixing $(\mathbf{x}_{N/2},\mathbf{x}_k,\mathbf{x}_{N-k})$ also has this property.

There is one particularly nice choice of simplicial coordinates for which the two-loop MHV amplitudes factorise into sums of six-point amplitudes [102, 113, 114]. They are defined by

$$(\mathbf{x}_1,\ldots,\mathbf{x}_{N-2})\to(1,0,\rho_1,\ldots,\rho_{N-5},\infty) \tag{4.2.9}$$

We refer to these coordinates as *simplicial MRK coordinates*. From the previous discussion it follows that simplicial MRK coordinates transform nicely under target projectile symmetry

$$(\rho_1,\ldots,\rho_{N-5})\mapsto(1/\rho_{N-5},\ldots,1/\rho_1) \tag{4.2.10}$$

Simplicial MRK coordinates are related to the Fourier-Mellin coordinates by

$$z_i=\frac{(\rho_i-\rho_{i-1})(\rho_{i+1}-1)}{(\rho_i-\rho_{i+1})(\rho_{i-1}-1)} \tag{4.2.11}$$

with $\rho_0=0$ and $\rho_{N-4}=\infty$. In these coordinates the two-loop MHV amplitude takes a particularly simple form [102, 113]

$$g_{+\ldots+}^{(0,\ldots,0,1,0,\ldots,0)}(\rho_1,\ldots,\rho_{N-5})=\frac{1}{4}\,\log|1-\rho_k|^2\,\log\left|1-\frac{1}{\rho_k}\right|^2 \tag{4.2.12}$$

where k denotes the position of the 1 in $(0,\ldots,0,1,0,\ldots,0)$. Finally, we point out that soft limits are very easy to describe in simplicial MRK coordinates. In the limit where $\mathbf{k}_i$ is soft we have $\rho_{i-1}=\rho_i$ (with $\rho_0=0$ and $\rho_{N-4}=\infty$).

There is another class of simplicial coordinates which will be important in the remainder of this paper. Let us start from the Fourier-Mellin coordinates, and let us single out one of them, say z_i. Then there is always a (non unique) set of simplicial coordinates $(t_1^{(i)}, \ldots, t_{N-5}^{(i)})$ such that $t_i^{(i)} = z_i$. Indeed, from eq. (4.1.18) we see that we can define these coordinates by

$$(\mathbf{x}_1, \ldots, \mathbf{x}_{N-2}) \to (\infty, t_1^{(i)}, \ldots, t_i^{(i)}, 0, 1, \ldots, t_{N-5}^{(i)}) \tag{4.2.13}$$

We will refer to these simplicial coordinates as *simplicial coordinates based at* z_i. They do in general not possess any simple transformation properties under target-projectile symmetry, but they will be essential in order to carry out all the Fourier-Mellin integrations, because they 'interpolate' between the Fourier-Mellin and simplicial MRK coordinates.

Sometimes it is helpful to describe the moduli space $\mathfrak{M}_{0,n}$ in projective terms. To do so we can introduce n elements $r_i \in \mathbb{CP}^1$, that is n two-component complex vectors modulo non-zero complex scalings. We may return to the $\mathbf{x}_i$ coordinates by making use of the scalings so that $r_i = (1, \mathbf{x}_i)$. In the projective language $SL(2,\mathbb{C})$ invariance means that all quantities should be expressed in terms of the $SL(2,\mathbb{C})$ invariant two-brackets

$$(ij) = \epsilon_{ab} r_i^a r_j^b \tag{4.2.14}$$

where ϵ_{ab} is the two-index antisymmetric tensor with $\epsilon_{12} = 1$. Moreover, since we must maintain the projective nature of the r_i we must form only quantities which are homogeneous of degree zero. Such quantities are given by cross-ratios.

If we choose an ordering of our points (corresponding to the one induced by the colour ordering of the scattering amplitude) we may introduce a particular set of cross-ratios, the *dihedral coordinates*

$$v_{ij} = \frac{(i\,j+1)(i+1\,j)}{(ij)(i+1\,j+1)} = \frac{(\mathbf{x}_i - \mathbf{x}_{j+1})(\mathbf{x}_{i+1} - \mathbf{x}_j)}{(\mathbf{x}_i - \mathbf{x}_j)(\mathbf{x}_{i+1} - \mathbf{x}_{j+1})} \tag{4.2.15}$$

where indices are treated modulo n and we have given both projective and coordinate-fixed forms. Note that only $(n-3)$ of the v_{ij} are algebraically independent, since this is the dimension of the moduli space $\mathfrak{M}_{0,n}$. To continue, we pick a dihedral structure η on $\mathfrak{M}_{0,n}$, i.e. a cyclic ordering of the n points r_i modulo reflections . In our case the points $\mathbf{x}_i$, and hence also the r_i, come with a natural dihedral structure induced by the colour ordering and target projectile symmetry. We therefore assume from now on that $\mathfrak{M}_{0,n}$ is equipped with this particular dihedral structure, and we will often omit the dependence on the choice of η explicitly. Dihedral coordinates will play an important role in the next section when defining iterated integrals on $\mathfrak{M}_{0,n}$. Moreover, they allow one to give a nice geometric interpretation of real moduli space $\mathfrak{M}_{0,n}(\mathbb{R})$, which we describe in the

remainder of this section.

In the real moduli space, the region of $\mathfrak{M}_{0,n}$ defined by $0 < v_{ij} < 1$ describes the interior of a Stasheff polytope or associahedron. The full real moduli space is tiled by $n!/(2n)$ such regions, each one corresponding to a different choice of dihedral structure η. The codimension one faces of the polytope are each obtained by taking one of the v_{ij} to zero (while maintaining $0 < v_{ij} < 1$ for the others). One can then continue to codimension two boundaries of the boundary face etc. This process can be continued all the way until one reaches the codimension $(n-3)$ (i.e. dimension zero) vertices.

The combinatorics describing the various boundaries are such that each vertex V of the Stasheff polytope is labelled by a triangulation T_V of an n-sided polygon (which in our case corresponds to the polygon formed by the dual coordinates $\mathbf{x}_i$ in the natural order induced by the color ordering, see Fig. 4.1.1), with the chords $\{i,j\} \in T_V$ defining the triangulation given by the set of v_{ij} that are zero at the vertex V. The other v_{ij} are equal to one at this vertex. This structure is described in detail in ref. [116] and we refer the reader there for more details. Let us note however that two vertices V and V' which are separated by a single edge correspond to two triangulations which differ by a single chord. In other words, to obtain $T_{V'}$ from T_V, one removes some chord $\{i,j\}$ from T_V and replaces it with a crossing chord $\{k,l\}$ such that the result is still a triangulation. The projective and dihedral coordinates will be useful in the discussion of the Knizhnik-Zamolodchikov equation on $\mathfrak{M}_{0,n}$ which follows.

4.2.3 Iterated integrals on $\mathfrak{M}_{0,n}$

In this section we summarise the theory of iterated integrals on $\mathfrak{M}_{0,n}$, before describing their single-valued analogues in the next section. A very helpful way to think about iterated integrals on $\mathfrak{M}_{0,n}$ is to think of them as being described in terms of generating functions which obey the Knizhnik-Zamolodchikov (KZ) equation [116]. The KZ equation on $\mathfrak{M}_{0,n}$ can be written in terms of the projective variables r_i introduced above eq. (4.2.14) as follows,

$$dL = \Omega L \qquad \Omega = \sum_{i<j} \Omega_{ij} \qquad \Omega_{ij} = X_{ij} d\log(ij) \tag{4.2.16}$$

Here the X_{ij} are a collection of formal generators obeying

$$X_{ij} = X_{ji} \qquad X_{ii} = 0 \qquad \sum_i X_{ij} = 0 \qquad [X_{ij}, X_{kl}] = 0 \quad \{i,j,k,l\} \text{ distinct.} \tag{4.2.17}$$

The first two relations in eq. (4.2.17) are conventional, ensuring that there are as many generators as there are one-forms $d\log(ij)$. The third relation ensures that the connection Ω is homogeneous under rescalings of the r_i, so that it is indeed a connection on the

moduli space of points in $\mathbb{CP}^1$. The final relation in eq. (4.2.17) completes a centre-free presentation of the infinitesimal pure braid relations on the X_{ij} and it ensures that the connection Ω obeys

$$\Omega \wedge \Omega = 0 \tag{4.2.18}$$

Since Ω also trivially obeys $d\Omega = 0$, the condition (4.2.18) implies that the connection is flat. We can consider solutions of eq. (4.2.16) which take the form,

$$L = 1 + \text{higher-order terms in the } X_{ij} \tag{4.2.19}$$

Such solutions are formal series in the generators X_{ij}, i.e., they are a sum over all words in the X_{ij} of any length, modulo the relations (4.2.17). The coefficients of the independent words are given by iterated integrals on $\mathfrak{M}_{0,n}$, and hence the solutions L can be viewed as generating functions of the class of A_{n-3} cluster polylogarithms. Iterated integrals form a shuffle algebra, and in the following we denote by B_n the shuffle algebra over $\mathbb{Q}$ of all iterated integrals on $\mathfrak{M}_{0,n}$. As a vector space, B_n is generated by the coefficients of the independent words in L.

The description of the KZ equation given in eq. (4.2.16) and (4.2.17) is manifestly invariant under all permutations of the r_i. In other words it did not depend on our initial choice of ordering $r_1, \ldots, r_n$. It will be useful however to present another description, presented in detail in ref. [116], which manifests only a dihedral symmetry. The construction depends on the choice of dihedral structure, and as before we choose the one induced by the colour ordering. In terms of the dihedral coordinates v_{ij} the KZ equation takes the form

$$dL = \Omega L \qquad \Omega = \sum_{\{i,j\}} \delta_{ij}\, d\log v_{ij} \tag{4.2.20}$$

The sum is over all pairs $\{i, j\}$ where the indices i and j are separated by at least two, with all indices treated modulo n. We can identify a pair $\{i, j\}$ with the corresponding chord of the polygon built on the points r_i, or equivalently $\mathbf{x}_i$ (see Section 4.2.2). The generators δ_{ij} are related to the X_{ij} via

$$X_{ij} = \delta_{i\,j+1} + \delta_{i+1\,j} - \delta_{ij} - \delta_{i+1\,j+1} \tag{4.2.21}$$

and consequently obey

$$[\delta_{i\,j+1} + \delta_{i+1\,j} - \delta_{ij} - \delta_{i+1\,j+1}, \delta_{k\,l+1} + \delta_{k+1\,l} - \delta_{kl} - \delta_{k+1\,l+1}] = 0 \quad \{i, j, k, l\} \text{ distinct.} \tag{4.2.22}$$

We also define $\delta_{ii} = \delta_{i\,i+1} = 0$. Note that the above relations imply that two generators δ_{ij} and δ_{kl} commute if the chords $\{i, j\}$ and $\{k, l\}$ of the polygon do not intersect. This implies in particular that all the δ_{ij} associated to a triangulation, and hence to a vertex V

of the Stasheff polytope, commute

$$[\delta_{ij}, \delta_{kl}] = 0 \qquad \{i,j\},\{k,l\} \in T_V \tag{4.2.23}$$

We may now define canonically normalised solutions L_V to the KZ equation (4.2.20) associated to each vertex V on the boundary of the polytope defining the positive region such that L_V is real-valued in the interior of the Stasheff polytope, i.e., where all v_{ij} obey $0 < v_{ij} < 1$. The solution L_V that we want is chosen to have the following behaviour in a neighbourhood of V

$$L_V = L_{V,\mathrm{an}} \left(\prod_{\{i,j\} \in T_V} v_{ij}^{\delta_{ij}} \right) \tag{4.2.24}$$

where $L_{V,\mathrm{an}}$ is analytic in a neighbourhood of V. To linear order we have

$$L_V = 1 + \sum_{\{i,j\}} \delta_{ij} \log v_{ij} + \ldots \tag{4.2.25}$$

The behaviour (4.2.25) is in fact independent of the choice of V, with the dependence on V arising at quadratic and higher order. We may regard L_V as a shuffle regularised path-ordered exponential in the connection Ω. The coefficients of the independent words in L_V are again iterated integrals on $\mathfrak{M}_{0,n}$. In fact, these coefficients simply provide an alternative set of generators for the shuffle algebra B_n. Note that, although the set of generators depends on the choice of the vertex V used to define the generating function L_V, the shuffle algebra B_n is independent of the vertex V.

Let us discuss how the generators obtained from different choices of V are related. In analogy with the hyperlogarithm case, different solutions of the KZ equation associated to different vertices V and V', are related by a parallel transport by a constant series $\Phi_{V,V'}$,

$$L_{V'} = L_V \, \Phi_{V,V'} \tag{4.2.26}$$

Continuing the analogy, just like Z_{ij} of (3.4.5) the $\Phi_{V,V'}$ are associators. By considering the case where two vertices are connected by a single edge on the boundary of the polytope $\mathfrak{M}_{0,n}(\mathbb{R})$, we find that the constant series is given by the canonical Drinfeld associator, given by a sum over shuffle regularised multiple-zeta values,

$$\Phi(e_0, e_1) = \sum_w w (-1)^{d(w)} \zeta_{\text{Ш}}(w) \tag{4.2.27}$$

where the sum is over all words w in two non-commuting generators e_0 and e_1 and $\zeta_{\text{Ш}}(w)$ is the shuffle regularised multiple zeta value labelled by the word w. The quantity $d(w)$ is the number of e_1 generators in the word w and is present in order to be coherent with the usual definition of multiple zeta values. To complete the relation between

L_V and $L_{V'}$ we still need to determine the values of e_0 and e_1 that enter eq. (4.2.27). In Section 4.2.2 we have seen that to every vertex V of the Stasheff polytope we can associate a triangulation T_V of the polygon formed by the points $\mathbf{x}_i$, and the triangulation associated to two vertices connected by a single edge differ by exactly one chord. Since to every chord $\{i,j\}$ we can associate a letter δ_{ij}, we can determine the e_0 and e_1 from the two chords by which the triangulations differ. More precisely, to move between two adjacent vertices of the polytope we apply the associator $\Phi(\delta,\delta')$ where the arguments δ corresponds to the generators δ_{ij} associated to the codimension one face being left behind and δ' corresponds δ_{kl} associated to the one being moved to. Note that since these two faces do not intersect on the boundary of the Stasheff polytope, the two generators δ and δ' will never commute. This corresponds precisely with the fact that one obtains the triangulation $T_{V'}$ from T_V by removing the chord $\{i,j\}$ and replacing it with a crossing chord $\{k,l\}$.

Iterated integrals are in general not single-valued. The monodromies of L_V around the singularities defined by $v_{ij}=0$ for $\{i,j\}\in T_V$ immediately follow from the asymptotic behaviour of eq. (4.2.24). If we denote the monodromy operator associated with the singularity $v_{ij}=0$ by $\mathcal{M}_{ij}$, we have

$$\mathcal{M}_{ij}L_V = L_V\, e^{2\pi i\delta_{ij}} \qquad \{i,j\}\in T_V \tag{4.2.28}$$

To compute the monodromies around another singularity, one first applies a parallel transport from the vertex V to the vertex V' which sits on that singularity via eq. (4.2.26), then performs the monodromy canonically according to the prescription (4.2.28), and then parallel transports back again

$$\mathcal{M}_{ij}L_V = L_V\,\Phi_{V,V'}\,e^{2\pi i\delta_{ij}}\,\Phi_{V',V} \tag{4.2.29}$$

This formula can be taken for all $\{i,j\}$. It reduces to (4.2.28) in the case where the vertex already sits on the singularity labelled by $\{i,j\}$ since in that case δ_{ij} commutes with $\Phi_{V,V'}$ and

$$\Phi_{V,V'}\,\Phi_{V',V}=1 \tag{4.2.30}$$

In practice it is often useful to work with an explicit basis for the iterated integrals generated by solutions of the KZ equation. The basis we will use is given in terms of hyperlogarithms. We can simply relate this to the previous description of the KZ equation and its solutions as follows. We work in simplicial coordinates of the form

$$\{\mathbf{x}_1,\mathbf{x}_2,\ldots,\mathbf{x}_n\}=\{\infty,0,1,t_1,\ldots,t_{n-3}\} \tag{4.2.31}$$

The KZ connection on $\mathfrak{M}_{0,n}$ becomes

$$\Omega^{(n)} = \sum_{4\leq i\leq n} [X_{2i}\, d\log t_{i-3} + X_{3i}\, d\log(1-t_{i-3})] + \sum_{4\leq i<j\leq n} X_{ij}\, d\log(t_{i-3}-t_{j-3}) \quad (4.2.32)$$

where we have indicated the number n of marked points. We iteratively factorise solutions of KZ in the form

$$L_n = F_n L_{n-1}\,, \quad (4.2.33)$$

where L_{n-1} is a solution of KZ on $\mathfrak{M}_{0,n-1}$

$$dL_{n-1} = \Omega^{(n-1)} L_{n-1} \quad (4.2.34)$$

and $L_3 \equiv 1$. In other words we have a solution of the form

$$L_n = F_n F_{n-1} \dots F_4 \quad (4.2.35)$$

Since $F_n = L_n (L_{n-1})^{-1}$ we find that

$$\begin{aligned} dF_n &= dL_n (L_{n-1})^{-1} + L_n d(L_{n-1})^{-1} \\ &= \Omega^{(n)} F_n - L_n (L_{n-1})^{-1} \Omega^{(n-1)} \end{aligned} \quad (4.2.36)$$

From this it follows that F_n obeys a Picard-Fuchs type equation

$$\frac{dF_n}{dt_{n-3}} = \left(\frac{X_{2n}}{t_{n-3}} + \frac{X_{3n}}{t_{n-3}-1} + \sum_{i=4}^{n-1} \frac{X_{in}}{t_{n-3}-t_{i-3}} \right) F_n \quad (4.2.37)$$

We are interested in the solution of the above equation given by

$$F_n = \sum_w w\, G(\sigma_1, \dots, \sigma_{|w|}; t_{n-3}) \quad (4.2.38)$$

Here the sum is over all words $w \in \langle\langle X_{2n}, \dots, X_{n-1n} \rangle\rangle$ and we denote the 'weight' or the length of the word w by $|w|$. The variables $\sigma_1, \dots, \sigma_w$ are obtained from the word w by the translation of generators X_{in} into letters defined by

$$X_{2n} \mapsto 0 \quad X_{3n} \mapsto 1 \quad X_{in} \mapsto t_{i-3} \text{ for } i \geq 4 \quad (4.2.39)$$

From the above discussion it is clear that the shuffle algebra B_n has a recursive structure. In particular, if we work in simplicial coordinates, this recursive structure reads

$$B_n \simeq B_{n-1} \otimes_{\mathbb{Q}} L_{\{0,1,t_1,\dots,t_{n-4}\}} \quad (4.2.40)$$

where L_Σ denotes the shuffle algebra of hyperlogarithms with singularities at $\sigma_i \in \Sigma$ (the σ_i are complex constants).

The recursion starts with $B_3 \equiv \mathbb{Q}$ (because we cannot form a cross ratio with three points), and B_4 is the algebra of harmonic polylogarithms with singularities at most at 0 and 1. In other words, if we fix an order on the simplicial coordinates t_i, we can describe B_n explicitly as linear combinations of hyperlogarithms with singularities at most at $t_{n-3} \in \{0, 1, t_1, \ldots, t_{n-4}\}$, and the coefficients in the linear combination are iterated integrals on the moduli space $\mathfrak{M}_{0,n-1}$. A vector-space basis for L_Σ is simply given by all hyperlogarithms, and so we can easily obtain a basis for B_n.

We end this discussion by noting that there is an alternative way to construct a basis for B_n. Since $\mathfrak{M}_{0,n} \simeq Gr(2,n)$, we can equally well describe B_n as the algebra of all A_{n-3} cluster polylogarithms [117], and a basis for all A_{n-3} cluster polylogarithms was given in ref. [118].

4.2.4 Single-valued iterated integrals on $\mathfrak{M}_{0,n}$

We have seen that scattering amplitudes in MRK can be expressed through single-valued iterated integrals on $\mathfrak{M}_{0,n}$. In this section we present different ways to construct these functions. The strategy to construct single-valued iterated integrals on $\mathfrak{M}_{0,n}$ is to generalise the results of ref. [82,83] from the Picard-Fuchs equation in the hyperlogarithm case reviewed in the previous chapter to the KZ equation (4.2.16) on $\mathfrak{M}_{0,n}$. In both cases the construction of single-valued functions preserves the algebra structure. Hence, since iterated integrals on $\mathfrak{M}_{0,n}$ can always be written in terms of hyperlogarithms, as a byproduct we find that both constructions give consistent results, and every single-valued iterated integral on $\mathfrak{M}_{0,n}$ can be written in terms of single-valued hyperlogarithms. Finally, inspired by ref. [84, 119], we present a purely algebraic way to define single-valued analogues of hyperlogarithms.

4.2.4.1 Single-valued iterated integrals from a differential equation on $\mathfrak{M}_{0,n}$

In this section we extend the construction of Section 3.4 to iterated integrals on $\mathfrak{M}_{0,n}$. Our goal will be to find single-valued solutions to the KZ equation (4.2.16) on $\mathfrak{M}_{0,n}$ To construct a generating series of single-valued polylogarithms on $\mathfrak{M}_{0,n}$ we first take two copies of the infinitesimal pure braid generators, δ_{ij} and δ'_{ij}, obeying the same relations (4.2.22). We then have two copies of the KZ equation, one based on the δ_{ij} with dihedral coordinates v_{ij} and one based on the δ'_{ij} with coordinates $\bar{v}_{ij}$ respectively. We then choose a vertex V and pick a solution L_V, a formal series in the δ_{ij}, and the corresponding $\bar{L}'_V$, a series in the δ'_{ij}.

Now we consider

$$\mathcal{L}_V = L_V \tilde{\bar{L}}'_V \tag{4.2.41}$$

where the tilde operation means reversing all words in the δ'_{ij} generators. Now if we

impose that the $\bar{v}_{ij}$ coordinates are the complex conjugates of the v_{ij} then we obtain the following results for the general monodromy of $\mathcal{L}_V$

$$\mathcal{M}_{ij}\mathcal{L}_V = L_V\, \Phi_{V,V'}\, e^{2\pi i \delta_{ij}}\, \Phi_{V',V}\, \tilde{\Phi}'_{V',V}\, e^{-2\pi i \delta'_{ij}}\, \tilde{\Phi}'_{V,V'}\, \tilde{\bar{L}}'_V \tag{4.2.42}$$

where V' is again some vertex which sits on the singularity denoted by the pair $\{i,j\}$.

Single valuedness means imposing that there is no such monodromy and hence we have

$$\Phi_{V,V'} e^{2\pi i \delta_{ij}} \Phi_{V',V} \tilde{\Phi}'_{V',V} e^{-2\pi i \delta'_{ij}} \tilde{\Phi}'_{V,V'} = 1. \tag{4.2.43}$$

for all $\{i,j\}$. This provides exactly the right number of conditions to eliminate the δ'_{ij} in terms of the δ_{ij}. For the $\{i,j\}$ in the triangulation T_V the relation (4.2.43) reduces simply to

$$\delta'_{ij} = \delta_{ij} \qquad \{i,j\} \in T_V \tag{4.2.44}$$

for the other $\{i,j\}$ it becomes

$$\delta'_{ij} = \delta_{ij} + \text{ higher order terms involving MZVs,} \qquad \{i,j\} \notin T_V \tag{4.2.45}$$

The series $\mathcal{L}_V$ then becomes a generating series for all single-valued multiple polylogarithms on $\mathfrak{M}_{0,n}$. Since it is real-valued inside the polytope $\mathfrak{M}_{0,n}(\mathbb{R})$ and it has no monodromy, it is real valued everywhere in $\mathfrak{M}_{0,n}$. Expanding $\mathcal{L}_V$ over all words in the δ_{ij} modulo the pure braid relations (4.2.22) gives all the single-valued multiple polylogarithms as coefficients

$$\mathcal{L}_V = \sum_w w \mathcal{L}_{V,w} \tag{4.2.46}$$

The advantage of this construction is that it shows that the construction of single-valued polylogarithms does not rely directly on the decomposition into hyperlogarithms. Since both the generating series of single-valued hyperlogarithms and of single-valued iterated integrals on $\mathfrak{M}_{0,n}$ satisfy the same holomorphic differential equation as their non-single-valued analogues, we can repeat the very same argument given at the end of Section 4.2.3 to conclude that the algebra B_n^{SV} of single-valued iterated integrals on $\mathfrak{M}_{0,n}$ has a recursive structure similar to the recursive structure of B_n (see eq. (4.2.40)). In particular, working with a specific choice of simplicial coordinates, we have

$$B_n^{SV} \simeq B_{n-1}^{SV} \otimes_{\mathbb{Q}} L^{SV}_{\{0,1,t_1,\ldots,t_{n-4}\}} \tag{4.2.47}$$

i.e., for a given choice of simplicial coordinates, every single-valued iterated integral on $\mathfrak{M}_{0,n}$ can be written as a linear combination of products of single-valued hyperlogarithms.

4.2.4.2 A purely algebraic approach to single-valued hyperlogarithms

So far we have seen that it is possible to define single-valued multiple polylogarithms, and thus single-valued iterated integrals on $\mathfrak{M}_{0,n}$, as solutions to a certain Picard-Fuchs equation with trivial monodromy. While the construction of these solutions is algorithmic, it can be desirable to have a purely combinatorial definition of single-valued multiple polylogarithms that does not require any reference to any differential equation. Inspired by ref. [84,119] we present in this section such a purely combinatorial definition. We introduce a map $\mathbf{s}$ that only relies on the Hopf algebra structure of multiple polylogarithms, and we show that the resulting functions satisfy the Picard-Fuchs equation of Section 3.4 and are single-valued. Hence, they must be identical to the single-valued functions of Section 3.4.

Let us now show how we can use the coproduct and the antipode to define single-valued hyperlogarithms. We use the notation of Section 3.4 and we write L_Σ for the shuffle algebra of all hyperlogarithms with singularities in Σ, $\overline{L}_\Sigma$ is its complex conjugate and $L_\Sigma \overline{L}_\Sigma \simeq L_\Sigma \otimes \overline{L}_\Sigma$. Note that each of these algebras is actually a Hopf algebra for the coproduct of MPLs Let us define a map

$$\tilde{S} : L_\Sigma \to \overline{L}_\Sigma\,; \quad G(\vec{a};z) \mapsto (-1)^{|\vec{a}|}\,\overline{S}(G(\vec{a};z)) \tag{4.2.48}$$

where $\overline{S}$ denotes the complex conjugate of the antipode. It is easy to check that $\tilde{S}$ inherits many properties from S. In particular, it is an involution and it satisfies

$$\tilde{S}(a\cdot b) = \tilde{S}(b)\cdot\tilde{S}(a) \;\text{ and }\; \Delta\tilde{S} = (\tilde{S}\otimes\tilde{S})\tau\Delta \tag{4.2.49}$$

Unlike the antipode, $\tilde{S}$ does not satisfy eq. (2.3.84). Rather, the equivalent equation for $\tilde{S}$ defines the single-valued map (see also ref. [84]),

$$\mathbf{s} = \mu(\tilde{S}\otimes \mathrm{id})\Delta \tag{4.2.50}$$

i.e., we claim that $\mathcal{G}(\vec{a};z) = \mathbf{s}(G(\vec{a};z))$ is the single-valued analogue of $G(\vec{a};z)$. Before proving single-valuedness, let us discuss some of the properties of the single-valued map $\mathbf{s}$. Unlike the definition of the map $\mathbf{s}_\Sigma$ of Section 3.4, the definition (4.2.50) is purely combinatorial and does not depend on the set of singularities. It is easy to see that $\mathbf{s}$ is $\mathbb{Q}$-linear and that it preserves the multiplication

$$\mathbf{s}(a\cdot b) = \mathbf{s}(a)\cdot\mathbf{s}(b) \tag{4.2.51}$$

We stress at this point that $\mathbf{s}$ is only linear with respect to rational numbers. In particular, this means that $\mathbf{s}$ may act non-trivially on non-algebraic periods. Indeed, we have [84]

$$\mathbf{s}(i\pi) = 0 \;\text{ and }\; \mathbf{s}(\zeta_n) = 2\zeta_n \;\text{ for } n \text{ odd} \tag{4.2.52}$$

Let us denote by $L_\Sigma^{SV} \subset L_\Sigma\overline{L}_\Sigma$ the image of L_Σ under the map $\mathbf{s}$. We use suggestively the same notation as for the shuffle algebra of single-valued hyperlogarithms from Section 3.4. While L_Σ and $L_\Sigma\overline{L}_\Sigma$ are Hopf algebras, the algebra L_Σ^{SV} is not a sub-Hopf algebra of $L_\Sigma\overline{L}_\Sigma$, but the Hopf algebra structure on $L_\Sigma\overline{L}_\Sigma$ turns L_Σ^{SV} into a graded $L_\Sigma\overline{L}_\Sigma$-comodule, whose coaction agrees with the coproduct on $L_\Sigma\overline{L}_\Sigma$,

$$\Delta : L_\Sigma^{SV} \to L_\Sigma^{SV} \otimes L_\Sigma\overline{L}_\Sigma \tag{4.2.53}$$

Let us now show that $\mathcal{G}(\vec{a};z) = \mathbf{s}(G(\vec{a};z))$ is single-valued. Following Section 3.4 we denote by $\mathcal{M}_\sigma\mathcal{G}(\vec{a};z)$ the result of analytically continuing $\mathcal{G}(\vec{a};z)$ along a small loop (oriented counterclockwise) encircling the singularity $\sigma \in \Sigma$ (and no other singularity). In order to show that $\mathcal{G}(\vec{a};z)$ is single-valued, we need show that

$$\mathcal{M}_\sigma\mathcal{G}(\vec{a};z) = \mathcal{G}(\vec{a};z) \quad \forall\sigma \in \Sigma \tag{4.2.54}$$

or equivalently

$$\mathrm{Disc}_\sigma\mathcal{G}(\vec{a};z) = 0 \quad \forall\sigma \in \Sigma \tag{4.2.55}$$

where the discontinuity operator is $\mathrm{Disc}_\sigma = \mathcal{M}_\sigma - \mathrm{id}$. The proof that $\mathcal{G}(\vec{a};z)$ is single-valued proceeds by induction in the weight. If $|\vec{a}| = 1$, we have

$$\mathcal{G}(a;z) = G(a;z) + \tilde{S}(G(a;z)) = \log\left|1 - \frac{z}{a}\right|^2 \tag{4.2.56}$$

and this function is manifestly single-valued. Let us now assume that all functions $\mathcal{G}$ are single-valued up to a certain weight n, and let us show that a function $\mathcal{G}(\vec{a};z)$ of weight $n+1$ is still single-valued. Since the discontinuity operator only acts in the first factor of the coproduct, $\Delta\mathrm{Disc}_\sigma = (\mathrm{Disc}_\sigma \otimes \mathrm{id})\Delta$, the graded comodule structure of L_Σ^{SV} implies that

$$\Delta\mathrm{Disc}_\sigma(\mathcal{G}(\vec{a};z)) = (\mathrm{Disc}_\sigma \otimes \mathrm{id})\Delta(\mathcal{G}(a;z)) = \mathrm{Disc}_\sigma\mathcal{G}(\vec{a};z) \otimes 1 \tag{4.2.57}$$

From eq. (2.3.84) we obtain

$$0 = \mu(\mathrm{id} \otimes S)\Delta\mathrm{Disc}_\sigma(\mathcal{G}(\vec{a};z)) = \mathrm{Disc}_\sigma(\mathcal{G}(\vec{a};z)) \cdot S(1) = \mathrm{Disc}_\sigma(\mathcal{G}(\vec{a};z)) \tag{4.2.58}$$

and so $\mathcal{G}(\vec{a};z)$ is single-valued.

So far we have shown that $\mathbf{s}$ respects multiplication and that the resulting functions are single-valued. We now show that the functions $\mathcal{G}(\vec{a};z)$ agree with the single-valued realisation ρ_{SV} of $\mathcal{HL}_\Sigma$, see Section 3.4. In order to see this we need to prove that the single-valued map commutes with holomorphic differentiation

$$\partial_z\,\mathbf{s} = \mathbf{s}\,\partial_z \tag{4.2.59}$$

This follows immediately from the fact that derivatives only act in the second factor of the coproduct, $\Delta\partial_z = (\mathrm{id}\otimes\partial_z)\Delta$. We obtain

$$\mathbf{s}\,\partial_z = \mu(\tilde{S}\otimes\mathrm{id})\Delta\partial_z = \mu(\tilde{S}\otimes\partial_z)\Delta = \partial_z\mathbf{s} - \mu(\partial_z\tilde{S}\otimes\mathrm{id})\Delta \tag{4.2.60}$$

where the last step follows from the Leibniz rule, $\partial_z\mu = \mu(\partial_z\otimes\mathrm{id}+\mathrm{id}\otimes\partial_z)$. The claim then follows upon noting that $\tilde{S}(G(\vec{a};z))$ is always anti-holomorphic, and so $\partial_z\tilde{S}=0$. Hence, we have shown that $\mathcal{G}(a,\vec{b};z)$ and $G(a,\vec{b};z)$ behave in the same way under holomorphic differentiation

$$\partial_z\,\mathcal{G}(a,\vec{b};z) = \frac{1}{z-a}\,\mathcal{G}(\vec{b};z) \tag{4.2.61}$$

Moreover, it is easy to check that $\mathcal{G}(a,\vec{b};z)$ vanishes as $z\to 0$, and so the functions $\mathcal{G}(\vec{a};z)$ coincide with the single-valued realisation of $\mathcal{HL}_\Sigma$ defined in Section 3.4. Note, however, that the single-valued map does not commute with anti-holomorphic derivatives, $\bar{\partial}_z\mathbf{s}\neq(\mathbf{s}\partial_z)^*$.

Single-valued hyperlogarithms naturally have both anti-holomorphic and holomorphic parts. Hence, they carry a natural action of complex conjugation. We can again decompose a complex conjugated single-valued hyperlogarithm into standard single-valued hyper-logarithms

$$\mathcal{G}(\vec{a};\bar{z}) = \sum_{\vec{b}} c_{\vec{a},\vec{b}}\,\mathcal{G}(\vec{b};z) \tag{4.2.62}$$

Note that the fact that complex conjugation acts non-trivially on single-valued hyperlogarithms (in the sense that the complex conjugate of a single-valued hyperlogarithm is a linear combination of single-valued hyperlogarithms) is at the origin of why $\mathbf{s}$ does not commute with anti-holomorphic derivatives. The action of complex conjugation on single-valued hyperlogarithms is encoded in the map $\tilde{S}$. If $\bar{\mathbf{s}}$ denotes the complex conjugate of $\mathbf{s}$, we find

$$\bar{\mathbf{s}} = \mathbf{s}\,\tilde{S} \tag{4.2.63}$$

As an example, we have

$$\mathcal{G}(\bar{a},\bar{b};\bar{z}) = \bar{\mathbf{s}}(G(a,b;z)) = \mathcal{G}(b,a;z) + \mathcal{G}(b;a)\,\mathcal{G}(a;z) - \mathcal{G}(a;b)\,\mathcal{G}(b;z) \tag{4.2.64}$$

In the same way, we can also easily compute anti-holomorphic derivatives, because we can reduce the anti-holomorphic derivative to a holomorphic one via the map $\tilde{S}$. For example, we find

$$\bar{\partial}_z\mathcal{G}(a,b;z) = \frac{1}{\bar{z}-\bar{a}}\,\mathcal{G}(b;a) + \frac{1}{\bar{z}-\bar{b}}(\mathcal{G}(a;z)-\mathcal{G}(a;b)) \tag{4.2.65}$$

We conclude this section by commenting on functional equations for single-valued hyperlogarithms. We can of course obtain functional equations by expressing single-

valued hyperlogarithms in terms of ordinary hyperlogarithms, and then applying functional equations to the latter. There is, however, a simpler way to obtain functional equations for single-valued hyperlogarithms: assume we are given a relation between ordinary hyperlogarithms. We can then act with $\mathbf{s}$ on it, and we obtain a relation among single-valued hyperlogarithms. Since the action of $\mathbf{s}$ is, essentially, to replace G by $\mathcal{G}$, we conclude that single-valued hyperlogarithms satisfy the same identities as ordinary hyperlogarithms. Note that eq. (4.2.52) is crucial for this to work. Let us consider an example to see how this works: we start from the following relation among ordinary hyperlogarithms of weight three (valid on some branch for the logarithm)

$$\begin{aligned} G\left(0,1,1;\frac{1}{z}\right) &= -G(0,0,0;z)+G(0,0,1;z)+G(0,1,0;z)-G(0,1,1;z) \\ &\quad + i\pi\left[G(0,0;z)-G(0,1;z)\right]+\frac{\pi^2}{2}G(0;z)+\zeta_3-\frac{i\pi^3}{6} \end{aligned} \tag{4.2.66}$$

We can act on both sides with $\mathbf{s}$, and we obtain

$$\mathcal{G}\left(0,1,1;\frac{1}{z}\right) = -\mathcal{G}(0,0,0;z)+\mathcal{G}(0,0,1;z)+\mathcal{G}(0,1,0;z)-\mathcal{G}(0,1,1;z)+2\zeta_3 \tag{4.2.67}$$

This is indeed a valid identity among single-valued hyperlogarithms. We stress the importance of eq. (4.2.52) in order for this to be true.

4.3 MHV amplitudes in MRK

4.3.1 An invitation: the six-point MHV amplitude

In this section we apply the machinery of single-valued iterated integrals on $\mathfrak{M}_{0,N-2}$ of the previous section to the computation of scattering amplitudes in MRK to LLA. We start by reviewing the six-point MHV amplitude in MRK, and we generalise the discussion to more external legs and other helicity configurations in subsequent sections. Most of the techniques introduced in this paper apply also beyond LLA, which we shall explore in coming chapters.

Traditionally, scattering amplitudes in MRK are computed by closing the integration contour in the Fourier-Mellin representation of the amplitude, eq. (4.1.28), and taking residues at the poles of the integrand [100, 102, 110–112, 120]. In the case of the six-point amplitude, the resulting multiple sums can all be performed in terms of polylogarithms using standard techniques [121–125]. For amplitudes with more external legs, performing the multiple sums, however, soon becomes prohibitive.

The goal of this section is to introduce a new way to compute, or rather to circumvent, the Fourier-Mellin transform of eq. (4.1.28). The main idea is to use the convolution theorem (4.1.37) and to perform the computation directly in z-space, rather than evaluating

the Fourier-Mellin transform explicitly. While in itself this idea is not new, performing the convolution integral (4.1.38) requires the evaluation of some integral over the whole complex plane, which seems a daunting task. We show that the fact that amplitudes in MRK are single-valued functions on $\mathfrak{M}_{0,N-2}$ reduces the computation to a simple application of Stokes' theorem.

In order to illustrate our method, we apply it in this section to the six-point MHV amplitude. While the results of this section are not new (see for example ref. [110, 111]), we use them to show all the steps that enter the computation. We start from eq. (4.1.52), and we obtain a recursion for the coefficients to LLA

$$g_{++}^{(l)}(z) = -\frac{1}{2}\,\mathcal{F}\left[\chi^+(\nu,n)\,E_{\nu n}^l\,\chi^-(\nu,n)\right] = g_{++}^{(l-1)}(z) * \mathcal{F}[E_{\nu n}] \tag{4.3.1}$$

We see that increasing the number of loops is equivalent to convoluting the lower loop result with the Fourier-Mellin transform of the BFKL eigenvalue. In order to start the recursion, we need to know $g_{++}^{(l)}(z)$ analytically for some value of l. This can easily be achieved by performing explicitly the Fourier-Mellin transform for $l = 1$ or $l = 2$, cf., e.g., ref. [110]

$$\begin{aligned}\mathcal{F}\left[\chi^+(\nu,n)\chi^-(\nu,n)\right] &= \mathcal{G}_1(z) - \frac{1}{2}\,\mathcal{G}_0(z)\\ \mathcal{F}\left[\chi^+(\nu,n)\,E_{\nu n}\,\chi^-(\nu,n)\right] &= \frac{1}{2}\mathcal{G}_{0,1}(z) + \frac{1}{2}\mathcal{G}_{1,0}(z) - \mathcal{G}_{1,1}(z)\end{aligned} \tag{4.3.2}$$

where we use the notation $\mathcal{G}_{a_1,\ldots,a_w}(z) \equiv \mathcal{G}(a_1,\ldots,a_w;z)$. We also need the Fourier-Mellin transform of the LO BFKL eigenvalue, which can easily be obtained by noting that the functions $\chi^{\pm}(\nu,n)$ have a very simple interpretation in terms of Fourier-Mellin transforms: they are related to derivatives in z-space

$$z\,\partial_z\mathcal{F}\left[\chi^+(\nu,n)\,F(\nu,n)\right] = \mathcal{F}\left[F(\nu,n)\right] \tag{4.3.3}$$

A similar relation holds when replacing z by $\bar{z}$ and χ^+ by χ^-. The Fourier-Mellin transform of the LO BFKL eigenvalue is then given by

$$\mathcal{E}(z) \equiv \mathcal{F}\left[E_{\nu n}\right] = z\,\bar{z}\,\partial_z\,\bar{\partial}_z\mathcal{F}\left[\chi^+(\nu,n)\,E_{\nu n}\,\chi^-(\nu,n)\right] = -\frac{z+\bar{z}}{2\,|1-z|^2} \tag{4.3.4}$$

Next we discuss how we can evaluate the convolution integral. We assume for now that in the multi-Regge limit we can express the amplitude to all loop orders in terms of single-valued hyperlogarithms (This will be proven later in Section 4.5). In ref. [80] it was shown that convolution integrals of this type can be computed using residues. To see how this works, consider a function $f(z)$ that consists of single-valued hyperlogarithms and rational functions with singularities at $z = a_i$ and $z = \infty$. Close to any of these

singularities, f can be expanded into a series of the form

$$\begin{aligned} f(z) &= \sum_{k,m,n} c^{a_i}_{k,m,n} \log^k \left|1 - \frac{z}{a_i}\right|^2 (z-a_i)^m (\bar{z}-\bar{a}_i)^n \quad z \to a_i \\ f(z) &= \sum_{k,m,n} c^{\infty}_{k,m,n} \log^k \frac{1}{|z|^2} \frac{1}{z^m} \frac{1}{\bar{z}^n} \quad z \to \infty \end{aligned} \tag{4.3.5}$$

The *holomorphic residue* of f at the point $z = a$ is then defined as the coefficient of the simple holomorphic pole without logarithmic singularities

$$\text{Res}_{z=a} f(z) \equiv c^a_{0,-1,0} \tag{4.3.6}$$

In ref. [80] it was shown that the integral of f over the whole complex plane, if it exists, can be computed in terms of its holomorphic residues. More precisely, if F is an anti-holomorphic primitive of f, $\bar{\partial}_z F = f$, then

$$\int \frac{d^2z}{\pi} f(z) = \text{Res}_{z=\infty} F(z) - \sum_i \text{Res}_{z=a_i} F(z) \tag{4.3.7}$$

This result is essentially an application of Stokes' theorem to the punctured complex plane. Note that the anti-holomorphic primitive is only defined up to an arbitrary holomorphic function. It was shown in ref. [83] that every single-valued hyperlogarithm has a single-valued primitive, and the sum of residues is independent on the choice of the primitive [80]. It is clear that we can repeat the previous argument by reversing the roles of holomorphic and anti-holomorphic functions.

As a pedagogical example, let us illustrate how this works on the two-loop remainder function in MRK. Using the convolution theorem, we can write

$$\begin{aligned} \mathcal{F}\left[\chi^+(\nu,n)\, E_{\nu n}\, \chi^-(\nu,n)\right] &= \mathcal{F}\left[\chi^+(\nu,n)\chi^-(\nu,n)\right] * \mathcal{E}(z) \\ &= \int \frac{d^2w}{\pi} \underbrace{\left[\frac{1}{2}\mathcal{G}_0(w) - \mathcal{G}_1(w)\right] \frac{\bar{w}z + w\bar{z}}{2\,|w|^2\,|w-z|^2}}_{=f(w)} \end{aligned} \tag{4.3.8}$$

First, we need to compute the anti-holomorphic primitive. Since

$$\mathcal{G}_0(w) = \mathcal{G}_0(\bar{w}) \text{ and } \mathcal{G}_1(w) = \mathcal{G}_1(\bar{w}) \tag{4.3.9}$$

and single-valued hyperlogarithms satisfy the same (holomorphic) differential equa-

tions as their non-single-valued analogues, we obtain

$$\begin{aligned} F(w) &= \int d\bar{w}\, f(w) = \frac{1}{2w\,(w-z)} \int d\bar{w} \left[\frac{1}{2}\,\mathcal{G}_0(\bar{w}) - \mathcal{G}_1(\bar{w})\right] \frac{\bar{w}z + w\bar{z}}{\bar{w}\,(\bar{w}-\bar{z})} \\ &= \frac{1}{4(w-z)} \left[2\mathcal{G}_{0,z}(w) - 4\mathcal{G}_{1,z}(w) - \mathcal{G}_{0,0}(w) + 2\mathcal{G}_{1,0}(w) - 4\mathcal{G}_1(w)\mathcal{G}_0(z)\right. \\ &\qquad \left. +4\mathcal{G}_1(w)\mathcal{G}_1(z) + 2\mathcal{G}_0(z)\mathcal{G}_z(w) - 4\mathcal{G}_1(z)\mathcal{G}_z(w)\right] \\ &+ \frac{1}{4w} \left[-\mathcal{G}_{0,z}(w) + 2\mathcal{G}_{1,z}(w) + 2\mathcal{G}_1(w)\mathcal{G}_0(z) - 2\mathcal{G}_1(w)\mathcal{G}_1(z) - \mathcal{G}_0(z)\mathcal{G}_z(w)\right. \\ &\qquad \left. +2\mathcal{G}_1(z)\mathcal{G}_z(w)\right] \end{aligned} \tag{4.3.10}$$

We anticipate, however, that for higher weights the relation between $\mathcal{G}_{\vec{a}}(w)$ and $\mathcal{G}_{\vec{a}}(\bar{w})$ will not be as easy, but we have

$$\mathcal{G}_{\vec{a}}(w) = \sum_{\vec{b}} c_{\vec{a},\vec{b}}\, \mathcal{G}_{\vec{b}}(\bar{w}) \tag{4.3.11}$$

We see that $F(w)$ has potential poles at $w = 0$, $w = z$ and $w = \infty$. It is easy to check that the residue at $w = 0$ vanishes (because single-valued hyperlogarithms either vanish at $w = 0$, or they have logarithmic singularities). The residue at $w = z$ is easy to obtain

$$\begin{aligned} \text{Res}_{w=z} F(w) &= -\frac{1}{4}\mathcal{G}_{0,0}(z) - \mathcal{G}_{0,1}(z) - \frac{1}{2}\mathcal{G}_{1,0}(z) + 2\mathcal{G}_{1,1}(z) - \mathcal{G}_{1,z}(z) \\ &= -\frac{1}{4}\mathcal{G}_{0,0}(z) - \frac{1}{2}\mathcal{G}_{1,0}(z) + \mathcal{G}_{1,1}(z) \end{aligned} \tag{4.3.12}$$

where the last step follows from the identity

$$\mathcal{G}_{1,z}(z) = \mathcal{G}_{1,1}(z) - \mathcal{G}_{0,1}(z) \tag{4.3.13}$$

Finally, the residue at infinity is obtained by letting $w = 1/u$ (and including the corresponding Jacobian) and expanding the result around $u = 0$. Note that we obtain single-valued hyperlogarithms of the form $\mathcal{G}(\vec{a}; 1/u)$. In order to proceed, we need inversion relations for single-valued hyperlogarithms, which may be obtained from the inversion relations for ordinary hyperlogarithms and then acting with the single-valued map $\mathbf{s}$. We find

$$\text{Res}_{w=\infty} F(w) = \frac{1}{2}\mathcal{G}_{0,1}(z) - \frac{1}{4}\mathcal{G}_{0,0}(z) \tag{4.3.14}$$

Hence

$$\begin{aligned} \mathcal{F}\left[\chi^+(\nu,n)\, E_{\nu n}\, \chi^-(\nu,n)\right] &= \text{Res}_{w=\infty} F(w) - \text{Res}_{w=z} F(w) \\ &= \frac{1}{2}\mathcal{G}_{0,1}(z) + \frac{1}{2}\mathcal{G}_{1,0}(z) - \mathcal{G}_{1,1}(z) \end{aligned} \tag{4.3.15}$$

which is indeed the correct result. This construction is of course not restricted to two loops, but we can now start from the two-loop result we have just computed and obtain

the three, and even higher, loop results by convoluting the two-loop result with the BFKL eigenvalue $\mathcal{E}$.

4.3.2 Higher-point MHV amplitudes and the factorisation theorem

The six-point example from the previous section shows that we can bypass the evaluation of the Fourier-Mellin integrals and the multiple sums, and we can entirely work with convolutions and Stokes' theorem. This procedure can of course be extended to amplitudes with more external legs in a straightforward way. In particular, we obtain the recursion

$$g_{+\ldots+}^{(i_1,\ldots,i_k+1,\ldots,i_{N-5})}(z_1,\ldots,z_{N-5}) = \mathcal{E}(z_k) * g_{+\ldots+}^{(i_1,\ldots,i_{N-5})}(z_1,\ldots,z_{N-5}) \tag{4.3.16}$$

In the previous equation the convolution is carried out only over the variable z_k, even though this is not manifest in the notation. In general, it will always be clear which is the variable that enters the convolution integral. The starting point of the recursion is the two-loop MHV remainder function in MRK, which is known at LLA for an arbitrary number N of external legs [102, 113], cf. eq. (4.2.12). While a direct evaluation of the Fourier-Mellin transform in terms of multiple sums becomes prohibitive because the number of sums increases with the number of external legs, the recursion (4.3.16) requires the evaluation of a single convolution integral at every loop order, independently of the number of external legs. This is one of the key properties why the convolution integral combined with Stokes' theorem gives rise to an efficient algorithm to compute scattering amplitudes in MRK.

In practice, however, if we try to evaluate the convolution integral in terms of residues as we have done for the six-point MHV amplitude, then we have to face a conundrum: The convolution and the BFKL eigenvalue are naturally written in terms of the Fourier-Mellin coordinate z_k. The residues, however, are most easily computed in simplicial coordinates, where the poles in $g_{+\ldots+}^{(i_1,\ldots,i_{N-5})}$ manifest themselves simply as points where simplicial coordinates become equal to $0, 1, \infty$ or to each other. In general, the change of variables from the Fourier-Mellin coordinates to simplicial coordinates is highly non-linear, and will introduce complicated Jacobians. In addition, it will obscure the simple form of the BFKL eigenvalue. This problem arises for the first time for seven points, because for the six-point amplitude the simplicial and Fourier-Mellin coordinate systems coincide.

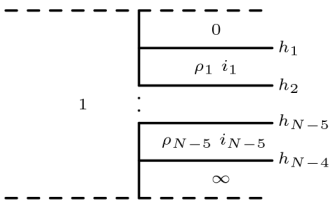
In some cases it is possible to identify a set of coordinates which share the good properties of the simplicial and Fourier-Mellin coordinates even at higher points. We have seen in Section 4.2.2 that there is always a (non unique) system of simplicial coordinates based at z_k with the property that $t_k^{(k)} = z_k$. This system of coordinates has already some of the properties we want: it leaves the BFKL eigenvalue unchanged, because $t_k^{(k)} = z_k$.

However, the change of coordinates may introduce a non-trivial Jacobian, because in general z_{k-2}, z_{k-1} and z_k will depend on the new integration variable $t_k^{(k)}$. There is, however, a special case where the Jacobian is trivial: If we perform a convolution with respect to z_1, and we change variables to simplicial coordinates based at z_1, only z_1 will depend on $t_1^{(1)}$, and so the Jacobian is 1. A similar argument can be made for z_{N-5}, using a slightly different set of simplicial coordinates. Alternatively, we know that we can exchange the roles of z_1 and z_{N-5} using target-projectile symmetry, so it is sufficient to consider z_1. Hence, if we perform a convolution with respect to the first or last cross ratio z_1 or z_{N-5}, we can find a set of simplicial coordinates with the right properties: it leaves the BFKL eigenvalue unchanged, it has a unit Jacobian, and at the same time it exposes all the singularities of $g_{+\ldots+}^{(i_1,\ldots,i_{N-5})}$ in a very simple form. The algorithm to evaluate the recursion (4.3.16) for the first or last cross ratio is then clear: in order to perform the convolution over z_1, we change coordinates to the simplicial coordinates based at z_1, and we evaluate the integral in terms of residues. The change of coordinates requires the use of functional equations among single-valued polylogarithms, which can be obtained using the techniques described in Section 4.2.4.

While the previous considerations answer the question of how to perform convolutions with respect to the first or last cross ratio, we still need to discuss the remaining cases. In the following, we argue that all amplitudes can be constructed by convoluting over the first or last cross ratio only. We only discuss from now on the case of z_1; the case of z_{N-5} is similar by target-projectile symmetry. The proof of this claim relies on a certain factorisation theorem which we present in the following.

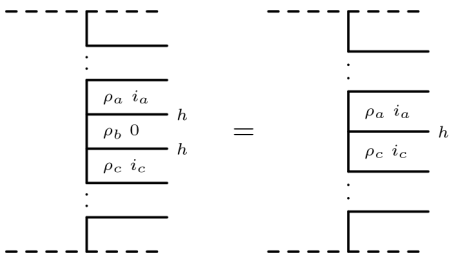
In order to state the factorisation theorem, it is useful to introduce the following graphical representation for the perturbative coefficients,

$$g_{h_1\ldots h_{N-4}}^{(i_1,\ldots,i_{N-5})}(\rho_1,\ldots,\rho_{N-5}) = \tag{4.3.17}$$



We work with the simplicial MRK coordinates ρ_k defined in Section 4.2.2. Every face of the dual graph is associated with a point $\mathbf{x}_k$ (cf. Fig 4.1.1), and we work in a coordinate patch where $(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_{N-2}) = (1, 0, \infty)$. Every outgoing line is labelled by its helicity h_k. In addition, to every face we do not only associate its coordinate ρ_k but also the index i_k. In the following we will not show the points 0, 1 and ∞ explicitly. Using this graphical representation of the perturbative coefficients the factorisation theorem takes

the simple form



(4.3.18)

In other words, whenever the graph representing a perturbative coefficient contains a face with index $i_b = 0$ and the lines adjacent to this face have the same helicity, then this perturbative coefficient is equal to the coefficient where this face has been deleted. We stress that the factorisation theorem holds for arbitrary helicity configurations and is not restricted to MHV amplitudes. The proof of the factorisation theorem for both the MHV and NMHV cases may be found in [38].

In the MHV case, the factorisation theorem implies that we can drop all the faces labelled by a zero,

$$g_{+\ldots+}^{(0,\ldots,0,i_{a_1},0,\ldots,0,i_{a_2},0,\ldots,0,i_{a_k},0,\ldots,0)}(\rho_1,\ldots,\rho_{N-5}) = g_{+\ldots+}^{(i_{a_1},i_{a_2},\ldots,i_{a_k})}(\rho_{i_{a_1}},\rho_{i_{a_2}},\ldots,\rho_{i_{a_k}}) \quad (4.3.19)$$

Let us discuss the implications of this result. First, eq. (4.3.19) implies that we can compute all MHV amplitudes by performing convolutions over the left-most variable z_1. Indeed, assume that we know all MHV amplitude with up to N legs. Then we can write

$$\begin{aligned} g_{+\ldots+}^{(1,i_2,\ldots,i_{N-5})}(\rho_1,\ldots,\rho_{N-5}) &= \mathcal{E}(z_1) * g_{+\ldots+}^{(0,i_2,\ldots,i_{N-5})}(\rho_1,\ldots,\rho_{N-5}) \\ &= \mathcal{E}(z_1) * g_{+\ldots+}^{(i_2,\ldots,i_{N-5})}(\rho_2,\ldots,\rho_{N-5}) \\ g_{+\ldots+}^{(2,i_2,\ldots,i_{N-5})}(\rho_1,\ldots,\rho_{N-5}) &= \mathcal{E}(z_1) * g_{+\ldots+}^{(1,i_2,\ldots,i_{N-5})}(\rho_1,\ldots,\rho_{N-5}) \\ &= \mathcal{E}(z_1) * \mathcal{E}(z_1) * g_{+\ldots+}^{(i_2,\ldots,i_{N-5})}(\rho_2,\ldots,\rho_{N-5}) \end{aligned} \quad (4.3.20)$$

and so on. The amplitude in the right-hand side is a known lower-point amplitude. At the beginning of this section we have argued that we can always easily perform convolutions over z_1 by going to simplicial coordinates based at z_1, because the change of variable has unit Jacobian and leaves the BFKL eigenvalue unchanged. Hence, we conclude that every MHV amplitude can be recursively constructed in this way, and we have thus obtained an efficient algorithm to compute scattering amplitudes in MRK.

Next, let us discuss the implications of the factorisation theorem for the structure of MHV amplitudes. Indeed, since the sum of all indices is related to the loop number, we see that for a fixed number of loops there is a maximal number of non-zero indices, and so there is only a finite number of different perturbative coefficients at every loop order. This generalises the factorisation observed for the two-loop MHV amplitude in MRK to LLA [102, 113, 114]. Indeed, if all indices are zero except for one, say i_a, then eq. (4.3.19)

reduces to

$$g_{+\ldots+}^{(0,\ldots,0,i_a,0,\ldots,0)}(\rho_1,\ldots,\rho_{N-5}) = g_{++}^{(i_a)}(\rho_a) \tag{4.3.21}$$

and so at two loops the amplitude completely factorises, in agreement with ref. [102, 113, 114],

$$\mathcal{R}^{(2)}_{+\ldots+} = \sum_{1\le i\le N-5} \log\tau_i\, g^{(1)}_{++}(\rho_i) \tag{4.3.22}$$

As anticipated in ref. [102], the amplitude does no longer factorise completely beyond two loops. However, we find that at every loop order only a finite number of different functions appear. For example, at three-loop order at most two indices are non-zero, and so we have

$$\mathcal{R}^{(3)}_{+\ldots+} = \frac{1}{2}\sum_{1\le i\le N-5} \log^2\tau_i\, g^{(2)}_{++}(\rho_i) + \sum_{1\le i<j\le N-5} \log\tau_i\,\log\tau_j\, g^{(1,1)}_{+++}(\rho_i,\rho_j) \tag{4.3.23}$$

The only new function that appears at three loops that is not determined by the six-point amplitude is $g^{(1,1)}_{+++}$, which is determined by the three-loop seven-point MHV amplitude. At four loops we have

$$\begin{aligned}\mathcal{R}^{(4)}_{+\ldots+} &= \frac{1}{6}\sum_{1\le i\le N-5} \log^3\tau_i\, g^{(3)}_{++}(\rho_i)\\ &+ \frac{1}{2}\sum_{1\le i<j\le N-5} \Big[\log^2\tau_i\,\log\tau_j\, g^{(2,1)}_{+++}(\rho_i,\rho_j) + \log\tau_i\,\log^2\tau_j\, g^{(1,2)}_{+++}(\rho_i,\rho_j)\Big]\\ &+ \sum_{1\le i<j<k\le N-5} \log\tau_i\,\log\tau_j\,\log\tau_k\, g^{(1,1,1)}_{++++}(\rho_i,\rho_j,\rho_k)\end{aligned} \tag{4.3.24}$$

The four-loop answer is determined for any number of external legs by the six, seven and eight-point amplitudes through four loops. Similar equations can be obtained for higher-loop amplitudes. In general, at L loops $\mathcal{R}^{(L)}_{+\ldots+}$ is determined for any number of legs by the MHV amplitudes involving up to $(L+4)$ external legs.

4.4 Non-MHV amplitudes in MRK

4.4.1 Helicity-flip operations

So far we have only considered MHV amplitudes. In this section we generalise all the results from the previous section to non-MHV amplitudes. In particular, we extend the factorisation theorem (4.3.18) to the non-MHV case. We start by introducing an additional concept before we are ready to prove the factorisation theorem for non-MHV amplitudes.

Let us start by analysing what happens if we start from an MHV amplitude and we flip the helicity on an impact factor. In Fourier-Mellin space, this amounts to replacing $\chi^+(\nu,n)$

by $\chi^-(\nu,n)$,

$$\begin{aligned}\mathcal{F}\left[\chi^+(\nu,n)\,F(\nu,n)\right] &\longrightarrow \mathcal{F}\left[\chi^-(\nu,n)\,F(\nu,n)\right]\\ &= \mathcal{F}\left[\chi^-(\nu,n)/\chi^+(\nu,n)\right] * \mathcal{F}\left[\chi^+(\nu,n)\,F(\nu,n)\right]\\ &= \mathcal{F}\left[\frac{i\nu+\frac{n}{2}}{i\nu-\frac{n}{2}}\right] * \mathcal{F}\left[\chi^+(\nu,n)\,F(\nu,n)\right]\end{aligned} \tag{4.4.1}$$

We see that flipping the helicity on an impact factor amounts to convoluting with the universal *helicity-flip kernel*

$$\mathcal{H}(z) \equiv \mathcal{F}\left[\frac{i\nu+\frac{n}{2}}{i\nu-\frac{n}{2}}\right]\,. \tag{4.4.2}$$

The functional form of $\mathcal{H}(z)$ can easily be obtained by performing explicitly the Fourier-Mellin transform. The integrand has only a simple pole at $i\nu = n/2$, and so we find

$$\mathcal{H}(z) = \mathcal{H}(1/z) = -\frac{z}{(1-z)^2}\,. \tag{4.4.3}$$

Note that helicity-flip kernel is an involution, i.e., flipping the helicity twice on the same impact factor returns the original helicity configuration, and so

$$\mathcal{H}(z) * \mathcal{H}(\bar{z}) = \mathcal{F}[1] = \pi\,\delta^{(2)}(1-z)\,. \tag{4.4.4}$$

Similarly, if we flip the helicity on one of the central emission blocks and use eq. (4.1.49), we obtain

$$\begin{aligned}\mathcal{F}\Big[C^+(\nu,n,\mu,m)F(\nu,n,\mu,m)\Big] &\longrightarrow \mathcal{F}\left[C^-(\nu,n,\mu,m)F(\nu,n,\mu,m)\right]\\ &= \mathcal{F}\left[\frac{C^-(\nu,n,\mu,m)}{C^+(\nu,n,\mu,m)}\right] * \mathcal{F}\left[C^+(\nu,n,\mu,m)F(\nu,n,\mu,m)\right]\\ &= \mathcal{F}\left[\frac{\chi^+(\nu,n)\,\chi^-(\mu,m)}{\chi^-(\nu,n)\,\chi^+(\mu,m)}\right] * \mathcal{F}\left[C^+(\nu,n,\mu,m)F(\nu,n,\mu,m)\right]\\ &= \mathcal{H}(\bar{z}_1) * \mathcal{H}(z_2) * \mathcal{F}\left[C^+(\nu,n,\mu,m)F(\nu,n,\mu,m)\right]\,.\end{aligned} \tag{4.4.5}$$

We see that the flipping of the helicity on a central emission block is controlled by the same kernels as for the impact factor. As a consistency check, the helicity flip kernels

allow us to show that MHV and $\overline{\text{MHV}}$ amplitudes are identical,

$$\begin{aligned}
\mathcal{R}_{-\ldots-}(z_1,\ldots,z_{N-5}) &= \mathcal{H}(z_1) * \mathcal{R}_{+-\ldots-}(z_1,\ldots,z_{N-5}) \\
&= \mathcal{H}(z_1) * \mathcal{H}(\bar{z}_1) * \mathcal{H}(z_2) * \mathcal{R}_{++-\ldots-}(z_1,\ldots,z_{N-5}) \\
&= \mathcal{H}(z_2) * \mathcal{R}_{++-\ldots-}(z_1,\ldots,z_{N-5}) \\
&= \ldots \\
&= \mathcal{H}(z_{N-5}) * \mathcal{R}_{+\ldots+-}(z_1,\ldots,z_{N-5}) \\
&= \mathcal{H}(z_{N-5}) * \mathcal{H}(\bar{z}_{N-5}) * \mathcal{R}_{+\ldots+}(z_1,\ldots,z_{N-5}) \\
&= \mathcal{R}_{+\ldots+}(z_1,\ldots,z_{N-5})\,.
\end{aligned} \tag{4.4.6}$$

Let us conclude this section by making a comment about some classes of non-MHV amplitudes with a special property. In ref. [102] it was argued that flipping the helicity on an impact factor to produce an NMHV amplitude from an MHV amplitude is equivalent to differentiating in the holomorphic variable and integrating in the anti-holomorphic one. Let us see how this arises from the helicity-flip kernel. We have

$$\begin{aligned}
\mathcal{R}_{-+\ldots+}(z_1,\ldots,z_{N-5}) &= \mathcal{H}(z_1) * \mathcal{R}_{+\ldots+}(z_1,\ldots,z_{N-5}) \\
&= -\int \frac{d^2w}{\pi}\,\frac{z_1}{\bar{w}(w-z_1)^2}\,\mathcal{R}_{+\ldots+}(w,z_2\ldots,z_{N-5})\,.
\end{aligned} \tag{4.4.7}$$

We can evaluate eq. (4.4.7) in terms of residues. Let us denote by F the anti-holomorphic primitive,

$$F(w,z_2\ldots,z_{N-5}) \equiv \int \frac{d\bar{w}}{\bar{w}}\,\mathcal{R}_{++\ldots+}(w,z_2\ldots,z_{N-5})\,. \tag{4.4.8}$$

Then $\mathcal{R}_{-+\ldots+}$ is obtained by summing over all the holomorphic residues of F. As MHV amplitudes are a pure functions, they have no poles, and so F has no poles either. Furthermore, it is easy to check that there is no pole at infinity, and so the only residue we need to take into account comes from the double pole at $w = z_1$ in eq. (4.4.7),

$$\begin{aligned}
\mathcal{R}_{-+\ldots+}(z_1,\ldots,z_{N-5}) &= \text{Res}_{w=z_1}\frac{z_1\,F(w,z_2\ldots,z_{N-5})}{(w-z_1)^2} \\
&= z_1\,\partial_{z_1}F(z_1,z_2\ldots,z_{N-5}) \\
&= z_1\partial_{z_1}\int\frac{d\bar{w}}{\bar{w}}\mathcal{R}_{++\ldots+}(w,z_2\ldots,z_{N-5})\,.
\end{aligned} \tag{4.4.9}$$

We see that we recover the rule of ref. [102], but with the differentiation and integration given in the reversed order. While this may look like a minor difference, it is crucial in order to get the complete result. In principle, we need to include a boundary condition when computing the anti-holomorphic primitive. However, if the operations of differentiation and integration are performed in the order shown in eq. (4.4.9), then no boundary condition is needed, because the residue is independent of the choice of the

anti-holomorphic primitive. This is, however, not the case if the two operations are performed in the order given in ref. [102], where one needs to include non-trivial boundary information already for six points.

It is of course tantalising to speculate if this simple rule generalises and all non-MHV amplitudes can be computed by this simple differentiation-integration rule without having to include any boundary information. It turns out that this is not the case, because in general the amplitude in the integrand of the convolution integral (4.4.7) is not a pure function, but may itself have additional poles whose residues need to be taken into account when performing the convolution with the helicity-flip kernel. An explicit counterexample to the simple differentiation-integration rule without boundary terms can be constructed from an eight-point NNMHV amplitude.

Although the simple rule does not hold in general, there are some special cases where it does apply. Besides the case of $\mathcal{R}_{-+\ldots+}$ discussed above, we have identified the following special case in which we can apply the simple differentiation-integration rule without boundary terms: Consider an amplitude whose helicity configuration is given by

$$h_i = \begin{cases} -1 & \text{if } a \leq i \leq b \\ +1 & \text{otherwise} \end{cases} \tag{4.4.10}$$

This amplitude can be written as

$$\mathcal{R}_{+\ldots+-\ldots-+\ldots+} = \mathcal{H}(\bar{z}_{a-1}) * \mathcal{H}(z_b) * \mathcal{R}_{+\ldots+} \tag{4.4.11}$$

Let us first discuss the convolution with $\mathcal{H}(z_b)$. We can repeat exactly the same argument as for $\mathcal{R}_{-+\ldots+}$ and we conclude that

$$\mathcal{H}(z_b) * \mathcal{R}_{+\ldots+} = z_b\, \partial_{z_b} \int \frac{d\bar{z}_b}{\bar{z}_b}\, \mathcal{R}_{+\ldots+} \tag{4.4.12}$$

Next we want to perform the convolution of this function with $\mathcal{H}(\bar{z}_{a-1})$. The function $\mathcal{H}(z_b) * \mathcal{R}_{+\ldots+}$ will have poles, but all of them are holomorphic because they arise from computing the holomorphic derivative with respect to z_b. Hence, they do not give rise to any additional anti-holomorphic poles, and so we have

$$\begin{aligned} \mathcal{R}_{+\ldots+-\ldots-+\ldots+} &= \bar{z}_{a-1}\, \bar{\partial}_{z_{a-1}} \int \frac{dz_{a-1}}{z_{a-1}}\, [\mathcal{H}(z_b) * \mathcal{R}_{+\ldots+}] \\ &= \bar{z}_{a-1}\, \bar{\partial}_{z_{a-1}} \int \frac{dz_{a-1}}{z_{a-1}} z_b\, \partial_{z_b} \int \frac{d\bar{z}_b}{\bar{z}_b}\, \mathcal{R}_{+\ldots+} \end{aligned} \tag{4.4.13}$$

The previous case covers in particular all NMHV amplitudes. Hence, all six and seven-point amplitudes can be computed in this way.

4.4.2 Leading singularities of scattering amplitudes in MRK

In the previous section we have shown how we can compute non-MHV amplitudes via convolution with the universal helicity flip kernel $\mathcal{H}$. Due to the double pole in the helicity flip kernel, non-MHV amplitudes are no longer pure, but the transcendental functions are multiplied by rational prefactors. This is in agreement with the expected structure of scattering amplitudes in full kinematics, where these coefficients are identified with the leading singularities of the amplitudes [126]. In this section we present a way to determine the set of all rational prefactors that can appear in a given non-MHV amplitude in MRK at LLA.

Let us start by defining some concepts that are useful to state the main result. We define *interfaces* of the perturbative coefficients $g^{(i_1,\dots,i_{N-5})}_{h_1\dots h_{N-4}}(\rho_1,\dots,\rho_{N-5})$ as the faces of its graph (see eq. (4.3.17)) that are bounded by two external lines with opposite helicities. In the following we refer to a face of the graph simply by the index of the corresponding dual coordinate (cf. Fig. 4.1.1). We call an interface *holomorphic* if the helicity changes from -1 to $+1$ in the natural order induced by the colour ordering, and *anti-holomorphic* otherwise. We denote by $I = \{a_1,\dots,a_\kappa\}$ the set of all interfaces of the graph (equipped with the natural order induced by the colour ordering) and we let $a_0 = \mathbf{x}_2$ and $a_{\kappa+1} = \mathbf{x}_{N-2}$. For $1 \le k \le \kappa$ we define the sets

$$E^{a_k}_{\uparrow} = \{b \,|\, a_{k-1} \le b < a_k\} \text{ and } E^{a_k}_{\downarrow} = \{b \,|\, a_k < b \le a_{k+1}\} \tag{4.4.14}$$

We also define the cross-ratios

$$R_{bac} = \begin{cases} v_{bac1} & \text{for holomorphic interfaces } a \\ \\ \bar{v}_{bac1} & \text{for anti-holomorphic interfaces } a \end{cases} \tag{4.4.15}$$

with

$$v_{bac1} = \frac{(\mathbf{x}_b - \mathbf{x}_a)(\mathbf{x}_c - \mathbf{x}_1)}{(\mathbf{x}_b - \mathbf{x}_c)(\mathbf{x}_a - \mathbf{x}_1)} \tag{4.4.16}$$

We are now ready to state the main result of this section. We claim that it is possible to write the perturbative coefficients in such a way that all rational prefactors multiplying pure functions take the form

$$\prod_{a\in S} R_{bac} \quad b \in E^a_{\uparrow} \quad c \in \widetilde{E}^a_{\downarrow} \tag{4.4.17}$$

where $S \subseteq I$ is a (possibly empty) subset of interfaces and we have introduced the definition

$$\widetilde{E}^a_{\downarrow} = \{b \,|\, a < b\} \tag{4.4.18}$$

This implies in particular that the building blocks of all rational prefactors in MRK at LLA are contained in the set

$$\mathcal{L} = \{R_{bac} | a \in I\,, b \in E^a_\uparrow\,, c \in \widetilde{E}^a_\downarrow\} \tag{4.4.19}$$

The cross ratios in this set are at the same time the building blocks for all leading singularities in MRK at LLA. We emphasise that this set is an upper bound for the rational prefactors that can appear for a given helicity configuration. In particular, one may wonder whether the asymmetry in eq. (4.4.17) and eq. (4.4.19) between $E^a_\uparrow$ and $\widetilde{E}^a_\downarrow$ could not be lifted, and we could restrict the building blocks to the more symmetric set

$$\mathcal{L}_{\text{sym}} = \{R_{bac} | a \in I\,, b \in E^a_\uparrow\,, c \in E^a_\downarrow\} \tag{4.4.20}$$

Unfortunately, this is incorrect, because the cross ratios R_{bac} are not independent, but they satisfy intricate non-linear relations, e.g.,

$$R_{23c} + R_{234}\, R_{4ac} = R_{23c}\, R_{4ac} + R_{234}\, R_{2ac} \quad a < c \quad a \in I \tag{4.4.21}$$

The apparent asymmetry in the set of building blocks in eq. (4.4.19) can then be lifted through such relations. It would be interesting to have a classification of all the relations among the building blocks R_{bac}. Our building blocks are, however, *linearly* independent, and so we can restrict to the more symmetric set $\mathcal{L}_{\text{sym}}$ in situations where there is at most one interface of a given type (holomorphic or anti-holomorphic). Helicity configurations involving products of building blocks of the same type require at least three interfaces, and the simplest such amplitude is $\mathcal{R}_{-+-+}$. We observe by explicit computation that in this case the restricted set $\mathcal{L}_{\text{sym}}$ is indeed insufficient and a new building block $R_{236} \notin \mathcal{L}_{\text{sym}}$ appears (see Appendix D of [38]).

It is evident from eq. (4.4.17) that every interface contributes at most one factor to the product in eq. (4.4.17), i.e., we never encounter higher powers of R_{bac}.

Second, we see that for a given helicity configuration there is always a finite number of different rational prefactors, independently of the number of loops. The complete set of rational prefactors for a given helicity configuration shows up when all indices are non-zero. In particular, we will see that eq. (4.4.17) is consistent with the factorisation theorem (4.3.18) in the sense that we never need to consider faces b and c bounded by external lines with equal helicities and vanishing index.

Finally, we note that the ratios R_{bac} transform non-trivially under target-projectile symmetry. Target-projectile symmetry obviously maps interfaces to interfaces, and we have

$$R_{bac} \mapsto R_{N-b,N-a,N-c} = 1 - R_{N-c,N-a,N-b} \tag{4.4.22}$$

Let us now illustrate the content of eq. (4.4.17) on some simple examples. MHV and $\overline{\text{MHV}}$ amplitudes do not have any interfaces, so these amplitudes should not contain any non-trivial rational prefactors, in agreement with known results. The simplest amplitudes having a single interface are NMHV amplitudes of the form $\mathcal{R}_{-+\cdots+}$. Since these amplitudes have a single interface, we have $\mathcal{L} = \mathcal{L}_{\text{sym}}$. The amplitude must then take the form

$$\mathcal{R}_{-+\cdots+} = \mathfrak{a} + \sum_{c=4}^{N-2} R_{23c}\, \mathfrak{b}_c \tag{4.4.23}$$

where $\mathfrak{a}$ and $\mathfrak{b}_c$ are pure functions to all loop orders. In the special case $N = 6$ eq. (4.4.23) reduces to the known structure of the six-point NMHV amplitude in MRK [103]

$$\mathcal{R}_{-+} = \mathfrak{a} + R_{234}\, \mathfrak{b} = \mathfrak{a} + \frac{\rho_1}{\rho_1 - 1}\mathfrak{b} \tag{4.4.24}$$

Equation (4.4.17) implies that this structure generalises to an infinite class of N^{k}MHV amplitudes, $k \geq 1$, with a single holomorphic interface

$$\mathcal{R}_{-\cdots-+\cdots+} = \mathfrak{a} + \sum_{b=2}^{a-1} \sum_{c=a+1}^{N-2} R_{bac}\, \mathfrak{b}_{bc} \tag{4.4.25}$$

where a is the holomorphic interface and $\mathfrak{a}$ and $\mathfrak{b}_{bc}$ are pure functions. Products of rational prefactors contribute for the first time for amplitudes with two distinct interfaces, which appear precisely for the helicity configurations considered in eq. (4.4.10). The interfaces are located at $(a_1, a_2) = (a+1, b+2)$. One of them is holomorphic and the other one anti-holomorphic, so we can work with the symmetric set $\mathcal{L}_{\text{sym}}$. We find

$$\begin{aligned}\mathcal{R}_{+\cdots+-\cdots-+\cdots+} = \mathfrak{a} &+ \sum_{c=2}^{a_1-1} \sum_{d=a_1+1}^{a_2} \overline{R}_{ca_1d}\, \mathfrak{b}^1_{cd} + \sum_{c=a_1}^{a_2-1} \sum_{d=a_2+1}^{N-2} R_{ca_2d}\, \mathfrak{b}^2_{cd} \\ &+ \sum_{c_1=2}^{a_1-1} \sum_{d_1=a_1+1}^{a_2} \sum_{c_2=a_1}^{a_2-1} \sum_{d_2=a_2+1}^{N-2} \overline{R}_{c_1a_1d_1}\, R_{c_2a_2d_2}\, \mathfrak{c}^{12}_{c_1d_1c_2d_2}\end{aligned} \tag{4.4.26}$$

where we have indicated the anti-holomorphic rational functions by $\overline{R}_{bac}$ for clarity and $\mathfrak{a}$, $\mathfrak{b}^i_{cd}$ and $\mathfrak{c}^{ij}_{c_1d_1c_2d_2}$ are pure functions

Let us conclude this section by discussing the soft limits of the rational prefactors. First, we can see that R_{bac} has simple poles for $\mathbf{x}_b = \mathbf{x}_c$ and $\mathbf{x}_a = \mathbf{x}_1$. None of these singularities corresponds to a soft limit. This implies in particular that the weight of the pure functions does not drop when taking a soft limit. Next, we see that

$$\lim_{\mathbf{x}_b \to \mathbf{x}_a} R_{bac} = 0 \;\text{ and }\; \lim_{\mathbf{x}_c \to \mathbf{x}_a} R_{bac} = 1 \tag{4.4.27}$$

In order to understand the implication of these relations, let us consider a NMHV am-

plitude, which can be written in the form of eq. (4.4.26) with $a \equiv a_1 = a_2 - 1$

$$\begin{aligned}\mathcal{R}_{+\cdots+-+\cdots+} = \mathfrak{a} + \sum_{c=2}^{a-1} \overline{R}_{ca(a+1)}\, \mathfrak{b}^1_{ca+1} + \sum_{d=a+2}^{N-3} R_{a(a+1)d}\, \mathfrak{b}^2_{cd} \\ + \sum_{c=2}^{a-1} \sum_{d=a+2}^{N-3} \overline{R}_{ca(a+1)}\, R_{a(a+1)d}\, \mathfrak{c}^{12}_{c_1(a+1)ad_2}\end{aligned} \tag{4.4.28}$$

In the limit where the gluon with negative helicity becomes soft, $\mathbf{x}_a \to \mathbf{x}_{a+1}$, the NMHV amplitude reduces to an MHV amplitude, which is a pure function. Equation (4.4.27) guarantees that this is indeed the case, and we find,

$$\lim_{\mathbf{x}_a \to \mathbf{x}_{a+1}} \mathcal{R}_{+\cdots+-+\cdots+} = \mathfrak{a} + \sum_{c=2}^{a-1} \mathfrak{b}^1_{c(a+1)} \tag{4.4.29}$$

4.4.3 Explicit two-loop, seven-point NMHV check

In this section we outline an explicit check of our discussion for the leading logarithmic contribution to the two-loop seven-point NMHV amplitudes in MRK. The symbol of this amplitude was obtained in ref. [57][2]. More precisely, the quantity discussed in ref. [57] the so-called 'BDS-subtracted' amplitude, equivalent to the exponentiated remainder function multiplied by the ratio function. It is given in supersymmetric notation as follows

$$\mathcal{A}^{\text{NMHV}}_{\text{BDS}-\text{subtracted}} = [\tfrac{3}{7}(12) + \tfrac{1}{7}(13) + \tfrac{2}{7}(14) + \text{cyc.}]X + [(67)V_{67} + (47)V_{47} + \text{cyc.}] \tag{4.4.30}$$

In the above formula the quantities X, V_{67} and V_{47} are pure functions based on the heptagon alphabet arising from the cluster algebra structure on $\text{Gr}(4,7)$, as discussed in [10]. The quantities (ij) above represent the R-invariants which encode all the possible NMHV configurations of external states by use of Grassmann odd variables η_i.

We recall that all on-shell states in $\mathcal{N} = 4$ SYM theory can be described by the on-shell supermultiplet written in superspace notation as a function of Grassmann parameters η^A transforming in the fundamental representation of $SU(4)$ (2.2.7). The R-invariants generically depend on five indices $[ijklm]$. In the seven-point case, however, we may simply denote them by the two indices which are absent, e.g.

$$(12) = [34567] \tag{4.4.31}$$

Furthermore, at seven points all R-invariants are of the form $[r\,s-1\,s\,t-1\,t]$ for some

[2] We thank the authors for providing a file with the relevant expressions.

r, s, t and for convenience we recall the functional form (2.2.26)

$$\begin{aligned}[r\,s-1\,s\,t-1\,t] = &\frac{\delta^8(q)}{\langle 12\rangle\dots\langle n1\rangle}\\ &\times\frac{\langle s\,s-1\rangle\langle t\,t-1\rangle\delta^{(4)}(\langle n|x_{ns}x_{st}|\theta^A_{tn}\rangle+\langle n|x_{nt}x_{ts}|\theta^A_{sn}\rangle)}{x^2_{st}\langle n|x_{ns}x_{st}|t\rangle\langle n|x_{ns}x_{st}|t-1\rangle\langle n|x_{nt}x_{ts}|s\rangle\langle n|x_{nt}x_{ts}|s-1\rangle}\end{aligned} \tag{4.4.32}$$

We have included in the above formula the supersymmetric Parke-Taylor-Nair prefactor to exhibit all the relevant η dependence. We recall the explicit form of the delta function $\delta^8(q)$

$$\delta^8(q)=\delta^8\Big(\sum_i\lambda_i\eta_i\Big) \tag{4.4.33}$$

where the λ_i are the spinor-helicity variables introduced in eq. (2.1.14).

Next we calculate the limits of the pure functions X, V_{ij} in MRK and we evaluate the R-invariants in this limit. To perform the second task it is helpful to formulate the passage to multi-Regge kinematics in term of spinor-helicity variables. It is sufficient for us to parametrise our spinors with different powers of ϵ in such a way so as to systematise the strong ordering of the kinematics in the MRK limit, similar to ref. [113]. For example in the $12\to 34567$ kinematics, the λ spinors are parametrised as

$$\lambda_1=\begin{bmatrix}0\\ \Big(-\sum\limits_{i=3}^{7}\frac{|p_i|^2}{p_i^+}\epsilon^{5-i}\Big)^{\frac{1}{2}}\end{bmatrix},\quad \lambda_2=\begin{bmatrix}\Big(-\sum\limits_{i=3}^{7}p_i^+\epsilon^{i-5}\Big)^{\frac{1}{2}}\\ 0\end{bmatrix},\quad \lambda_j=\begin{bmatrix}\sqrt{p_j^+}\,\epsilon^{\frac{j-5}{2}}\\ \frac{p_j}{\sqrt{p_j^+}}\epsilon^{\frac{5-j}{2}}\end{bmatrix} \tag{4.4.34}$$

where $j=3,\dots,7$ and the $\tilde\lambda$ are obtained by conjugation. After calculating the R-invariants using this parametrisation we can recover the MRK value by taking $\epsilon\to 0$.

Projecting out the components of the η's corresponding to the desired helicity configuration and taking the MRK limit we find the following non-vanishing R-invariants for the $(--+)$ configuration,

$$(12)\to 1\,,\quad (23)\to 1-\frac{\rho_1(1-\rho_2)}{\rho_2(1-\rho_1)}\,,\quad (17)\to\frac{\rho_1}{\rho_1-1}$$

$$(27)\to\frac{\rho_1(1-\rho_2)}{\rho_2(1-\rho_1)}\,,\quad (13)\to 1-\frac{\rho_1}{\rho_1-1}\,,\quad (37)\to\frac{\rho_1(1-\rho_2)}{\rho_2(1-\rho_1)}-\frac{\rho_1}{\rho_1-1} \tag{4.4.35}$$

For the $(+-+)$ configuration we find

$$(34)\to\frac{\bar\rho_1(1-\bar\rho_2)}{\bar\rho_2(1-\bar\rho_1)}\Big[1+\frac{(\rho_1-\rho_2)}{(\rho_2-1)}\Big]\,,\quad (24)\to\frac{\bar\rho_1(1-\bar\rho_2)}{\bar\rho_2(1-\bar\rho_1)}\,,\qquad (15)\to 1+\frac{(\rho_1-\rho_2)}{(\rho_2-1)}$$

$$(16) \to \frac{(\rho_1-\rho_2)}{(\rho_2-1)}\Big[\frac{\bar\rho_1(1-\bar\rho_2)}{\bar\rho_2(1-\bar\rho_1)}-1\Big]\,,\quad (25)\to 1\,,\quad (36)\to\Big[1+\frac{(\rho_1-\rho_2)}{(\rho_2-1)}\Big]\Big[1-\frac{\bar\rho_1(1-\bar\rho_2)}{\bar\rho_2(1-\bar\rho_1)}\Big]$$

$$(15) \to \frac{(\rho_2-\rho_1)}{(\rho_2-1)}\,,\quad (26)\to 1-\frac{\bar\rho_1(1-\bar\rho_2)}{\bar\rho_2(1-\bar\rho_1)}\,,\quad (14)\to\frac{(\rho_2-\rho_1)}{(\rho_2-1)}\frac{\bar\rho_1(1-\bar\rho_2)}{\bar\rho_2(1-\bar\rho_1)} \tag{4.4.36}$$

Combining these formulas, we find that the two NMHV helicity configurations become

$$\begin{aligned}\mathcal{R}_{-++} &= \hat X+\hat V_{12}+\hat V_{23}+R_{234}\Big(\hat V_{73}-\hat V_{23}\Big)+R_{235}\Big(\hat V_{71}-\hat V_{73}\Big)\\ \mathcal{R}_{+-+} &= \hat X+\hat V_{25}+\hat V_{36}+\hat V_{62}+\overline{R}_{234}\Big(\hat V_{34}-\hat V_{36}-\hat V_{62}\Big)\\ &\quad+R_{345}\Big(\hat V_{51}-\hat V_{36}\Big)+\overline{R}_{234}R_{345}\Big(\hat V_{14}-\hat V_{34}+\hat V_{36}\Big)\end{aligned} \tag{4.4.37}$$

Here the $\hat V_{ij}$ are the MRK limits of the pure functions V_{ij} of eq. (4.4.30). The explicit forms of $\hat V_{ij}$ at LLA are given in Appendix D of [38] (of course, since we started from just the symbol, these formulas are valid up to terms proportional to multiple zeta values). Note that individually these functions may have beyond-leading log divergences. These extra powers of divergent logarithms cancel in the combinations outlined in eq. (4.4.37). These explicit limits may then be compared to the general structure outlined in eq. (D.11) and the predicted pure functions presented in eqs. (D.20) onwards of reference [38].

4.5 Analytic structure of scattering amplitudes in MRK

It is believed that MHV and NMHV amplitudes are expressible in terms of multiple polylogarithms [21], but it is expected that for more complicated helicity configurations more general classes of special functions may appear [127, 128]. Knowing that in some limit scattering amplitudes can always be expressed in terms of multiple polylogarithms independently of the helicity configuration and the number of external legs can thus provide valuable information and constraints on the analytic structure of scattering amplitudes. A proof of such a property previously only existed for the six-point amplitudes when expanded to leading order around the collinear limit [71] and to LLA in MRK [111,112].

In [38] it was shown that it is possible to construct all amplitudes in MRK to LLA via a sequence of three elementary operations.

1. Flipping the leftmost helicity by convolution with $\mathcal{H}(z_1)$ or $\mathcal{H}(\bar z_1)$ respectively.
2. Increasing the first index by convolution with $\mathcal{E}(z_1)$.
3. Adding more particles to the left with index zero and equal helicities.

In this section we show that this recursive structure of scattering amplitudes in the multi-Regge limit implies that they can always be expressed in terms of single-valued

multiple polylogarithms of maximal and uniform weight, independently of the loop number and the helicity configuration.

Let us start by discussing MHV amplitudes. The algorithm of Section 4.4.2 allows us to construct all MHV amplitudes by adding particles and by convoluting with $\mathcal{E}(z_1)$. We now show that the perturbative MHV coefficients $g_{+\ldots+}^{(i_1,\ldots,i_k)}$ are pure polylogarithmic functions of uniform weight $\omega = 1+i_1+\ldots+i_k$. Obviously, the factorisation theorem (4.3.18) implies that the claim remains true under the elementary operation of adding particles, so it suffices to show that convolution with $\mathcal{E}(z_1)$ has the same property. The proof proceeds by induction. Assume that $g_{+\ldots+}^{(i_1,\ldots,i_k)}$ is a pure function of uniform weight $\omega = 1+i_1+\ldots+i_k$, and let us show that $g_{+\ldots+}^{(i_1+1,\ldots,i_k)} = \mathcal{E}(z_1) * g_{+\ldots+}^{(i_1,\ldots,i_k)}$ is a pure function of uniform weight $\omega+1$. We have

$$\begin{aligned}
&\mathcal{E}(z_1) * g_{+\cdots+}^{(i_1,\ldots,i_{N-5})}(\rho_1,\ldots,\rho_{N-5}) \\
&\quad = -\int \frac{d^2w}{2\pi} g_{+\cdots+}^{(i_1,\ldots,i_{N-5})}(w,t_2,\ldots,t_{N-5}) \frac{\bar{w}t_1 + w\bar{t}_1}{|w|^2|w-t_1|^2} \\
&\quad = -\int \frac{d^2w}{2\pi} g_{+\cdots+}^{(i_1,\ldots,i_{N-5})}(w,t_2,\ldots,t_{N-5}) \frac{1}{w(w-t_1)} \left(\frac{w+t_1}{\bar{w}-\bar{t}_1} - \frac{w}{\bar{w}}\right)
\end{aligned} \tag{4.5.1}$$

We evaluate the integral in terms of residues. As $g_{+\cdots+}^{(i_1,\ldots,i_{N-5})}$ is assumed pure by induction hypothesis and all the denominators are linear in $\bar{w}$, the anti-holomorphic primitive is a pure function (seen as a function of $\bar{w}$) of uniform weight $\omega+1$. The convolution in eq. (4.5.1) can then be written in the form

$$\begin{aligned}
&\mathcal{E}(z_1) * g_{+\cdots+}^{(i_1,\ldots,i_{N-5})}(\rho_1,\ldots,\rho_{N-5}) \\
&\quad = -\int \frac{dw}{2\pi} \left[\frac{1}{w} F_1(w,t_2,\ldots,t_{N-5}) + \frac{1}{w-t_1} F_2(w,t_2,\ldots,t_{N-5})\right]
\end{aligned} \tag{4.5.2}$$

where F_1 and F_2 are pure single-valued polylogarithmic functions of weight $\omega+1$. As all the poles are simple, the holomorphic residues can be computed by simply evaluating the pure functions of weight $\omega+1$ at $w=0$, $w=t_1$ and $w=\infty$ (and dropping all logarithmically divergent terms). Hence, $\mathcal{E}(z_1) * g_{+\cdots+}^{(i_1,\ldots,i_{N-5})}$ is a pure polylogarithmic function of weight $\omega+1$.

While the previous result is not unexpected for MHV amplitudes, we show in the remainder of this section that we can extend the argument to non-MHV amplitudes, independently of the helicity configuration. More precisely, we show that the pure functions multiplying the rational prefactors defined in Section 4.4.2 are always pure polylogarithmic functions of uniform weight $\omega = 1+i_1+\ldots+i_k$. The proof in the MHV case relies crucially on the fact that the anti-holomorphic primitive was a pure function of weight $\omega+1$ and that all the holomorphic poles were simple. Since non-MHV amplitudes are in general not pure but contain rational prefactors, it is not obvious that the same conclusion holds for arbitrary helicity configurations. In addition, for non-MHV we also need

to analyse the effect of the helicity flip operation, which should not change the weight of the function.

We proceed again by induction. Let us start by showing that also in the non-MHV case a convolution with $\mathcal{E}(z_1)$ will increase the weight by one unit. From Section 4.4.2 we know that all poles in z_1 are simple and either holomorphic or anti-holomorphic. In the following we discuss the anti-holomorphic case and the extension to the holomorphic case is trivial. The integrand of the convolution integral in the non-MHV case may have additional poles in $\bar{w}$ at points where R_{2ac} is singular. However none of these additional poles are located at $\bar{w} = 0$ or $\bar{w} = t_1$, and so all the anti-holomorphic poles entering the convolution integral are simple. We can thus repeat the same argument as in the MHV case, and the anti-holomorphic primitive will be a pure polylogarithmic function of weight $\omega+1$. Moreover, there are no additional holomorphic poles in w introduced by the rational prefactors, and so we can compute all the holomorphic residues by evaluating the pure functions of weight $\omega + 1$ at $w \in \{0, t_1, \infty\}$. Hence, a convolution with $\mathcal{E}(z_1)$ produces pure polylogarithmic functions of weight $\omega + 1$ also in the non-MHV case.

To complete the argument, we need to show that flipping the leftmost helicity does not change the weight of the functions. In Section 4.4.2 we have seen that, since all poles in t_1 are simple and either holomorphic or anti-holomorphic, we can always compute the effect of the helicity flip by integrating and differentiating, cf. eq. (4.4.9). Since all poles are simple, the integration will increase the weight by one unit. This effect is compensated by the differentiation, so that the total weight of the functions remains unchanged. Hence, we conclude that non-MHV amplitudes in MRK to LLA are polylogarithmic functions of uniform weight $\omega = 1 + i_1 + \ldots + i_k$ independently of their helicity configuration.

CHAPTER 5

The MRK heptagon at NLLA

In the previous chapter we argued that all amplitudes in MRK at the leading log approximation (LLA) are described by single-valued polylogarithms associated to the moduli space of $(N-2)$ points on a Riemann sphere. Through integrability the impact factor and BFKL eigenvalue, which appear in the hexagon, are known to all orders [109]. Thus in this chapter we turn our attention to the simplest object beyond the solved six-gluon case, the $2 \to 5$ amplitude in MRK. While predictions for the latter to LLA have been worked out in [102,104,105], see also [38] for an extension to N-gluons, a major obstacle for their generalization to arbitrary logarithmic accuracy, is that the dispersion integral yielding the Regge cut contribution diverges when considering the $\mathrm{N}^{\ell-1}$LLA term at ℓ loops. This phenomenon is related to the fact that within the BFKL approach, the dual conformal invariance of the theory is not maintained at the intermediate steps of the calculation, and implies that some terms in the Regge pole contribution develop unphysical poles. As discussed in [102], it is possible to shift these terms from the pole to the cut contribution by modifying their definitions. The necessity of this step suggests that there may be a certain degree of arbitrariness in separating the pole and cut contributions in a conformally invariant theory.

Instead of the BFKL approach, in our analysis we will draw from the eikonal framework of [108], where the two incoming high-energy gluons are approximated by straight Wilson lines. Within this framework, only a Regge cut contribution, respecting high-energy factorization, dual conformal symmetry and consistency with soft limits, is necessary to describe the amplitude in MRK. This procedure also provides a natural regularization

for the Regge cut at finite coupling, and allows for its straightforward expansion at weak coupling. In this manner, we obtain the first significant result of this chapter, namely a dispersion integral describing the $2 \to 5$ amplitude in MRK to arbitrary logarithmic accuracy.

The $2 \to 5$ dispersion integral contains a new building block compared to the six-gluon case, the BFKL central emission vertex, previously only known to leading order [102]. The second significant result of this chapter is the extraction of the NLO correction to the central emission vertex, from the known NLLA contribution to the 2-loop 7-particle MHV amplitude [114], see also refs. [102, 113] for earlier work on the LLA contribution.

More precisely, since the aforementioned NLLA contribution had been previously determined [114] at the level of the symbol [32], in section 5.2 we show how to uniquely promote it to a function, based on information we derive on the leading discontinuity of the amplitude, together with its expected behaviour under soft limits, and single-valuedness properties of the function space in which it "lives". As a bonus, from this result we in fact also obtain the function level 2-loop MHV amplitude for *any* number of gluons in MRK, since the latter has been shown to decompose into 6- and 7-gluon building blocks in momentum space [114].

Then, in section 5.3 we present the final expression for the NLO correction to the BFKL central emission vertex, and detail our approach for obtaining it, by translating the momentum space expression for the amplitude, to the Fourier-Mellin space of the dispersion integral. Using the same approach, we similarly extract the next-to-next-to-leading order (NNLO) correction to the central emission vertex, up to transcendental constants, from the 3-loop MHV heptagon symbol [11].

With the $2 \to 5$ dispersion integral and the NLO BFKL central emission vertex at hand, we then move on to outline how to produce new predictions for the seven-gluon amplitude in MRK to NLLA at higher loops. This is achieved by direct evaluation of the dispersion integral, using a combination of nested sum algorithms and convolution methods, that we describe in section 5.4.

Finally we combine all of the results of this chapter into an all order all multiplicity conjecture for the dispersion integral. Due to the integrability the only quantity that is not known to all orders is the central emission block and it is for this quantity that we wish to propose an Wilson loop OPE inspired expression. Specifically, by considering the NNLO corrections to the central emission block obtained previously we write an ansatz composed of several pieces which we proceed check by considering soft limits and comparing to 4-loop data.

5.1 The BFKL equation at finite coupling

In this section, we will obtain a dispersion integral describing the multi-Regge limit of the $2 \to 5$ amplitude that is well-defined at any logarithmic accuracy, based on the eikonal approach of [108]. We review the basic ingredients of this approach for the $2 \to 4$ amplitude in subsection 5.1.1, before extending it to the $2 \to 5$ MHV amplitude in subsection 5.1.2.

5.1.1 6-points

For $2 \to 4$ scattering in the multi-Regge limit, the six-point remainder function R_6 in the region where we analytically continue the energy components of all produced particles is given by the all-order dispersion relation[1]

$$e^{R_6(z)+i\delta_6(z)} = 2\pi i f_{++} \tag{5.1.1}$$

$$f_{++} = \frac{a}{2}\sum_{n=-\infty}^{\infty}\left(\frac{z}{\bar{z}}\right)^{\frac{n}{2}}\int_{-\infty}^{\infty}\frac{d\nu}{2\pi}\,\tilde{\Phi}(\nu,n)|z|^{2i\nu}e^{-L\omega(\nu,n)} \tag{5.1.2}$$

where

$$L = \log(\sqrt{u_{21}u_{31}}) + i\pi = \log(\tau) + i\pi \tag{5.1.3}$$

contains the logarithms that become large in the limit, whereas $\omega(\nu,n)$ is the BFKL eigenvalue and $\tilde{\Phi}(\nu,n)$ is the product of the impact factors respectively, to all orders in the coupling

$$a = 2g^2 = \frac{\lambda}{8\pi^2}\,. \tag{5.1.4}$$

The phase δ_6 appearing on the left-hand-side is the BDS contribution defined in ref. [129], which is given to all orders in perturbation theory as

$$\delta_6(z) = \pi\,\Gamma\log\frac{u_2u_3}{(1-u_1)^2} = \pi\,\Gamma\log\frac{|z|^2}{|1-z|^4} \qquad \Gamma \equiv \frac{\gamma_K}{8} = \frac{\Gamma_{\text{cusp}}}{4} \tag{5.1.5}$$

where Γ is proportional to the cusp anomalous dimension

$$\gamma_K(a) = 4a-4\zeta_2a^2+22\zeta_4a^3-\Big(\frac{219}{2}\zeta_6+4\zeta_3^2\Big)a^4+\Big(\frac{1774}{3}\zeta_8+8\zeta_2\zeta_3^2+40\zeta_3\zeta_5\Big)a^5+\mathcal{O}(a^6) \tag{5.1.6}$$

known to all loops from integrability (see [63] for a review).

We note that (5.1.2) differs from other results in the literature, in particular from [91]. This can be traced back to the choice of the integration contour to be used in (5.1.2), which we have not specified so far. The two formulations are in fact equivalent via a

[1] Note that we have $(z_i/\bar{z}_i)^{n/2} = (-1)^n(w_i/\bar{w}_i)^{n/2}$ when converting any of the dispersion relations considered in this paper between $z_i \leftrightarrow -w_i$, because any choice of branch on the square roots should also respect complex conjugation. For example, if we choose $\sqrt{z} = i\sqrt{w}$, then we must also have $\sqrt{\bar{z}} = -i\sqrt{\bar{w}}$.

change of integration contour [108], as we explain below. We should also note that one could equally well use Γ as an expansion parameter, rather than a. Based on this choice, and some other considerations we will review in what follows, there exist two different definitions of $\tilde{\Phi}(\nu, n)$ in the literature

$$\frac{a}{2}\tilde{\Phi}(\nu,n) = \frac{a}{2}\frac{\Phi_{\text{reg}}(\nu,n)}{\nu^2+\frac{n^2}{4}} = \frac{\Gamma\Phi(\nu,n)}{\nu^2+\frac{n^2}{4}-\pi^2\Gamma^2} \tag{5.1.7}$$

where the first expression is due to Lipatov, Bartels and collaborators [91, 100], and the second one is due to Caron-Huot [108].

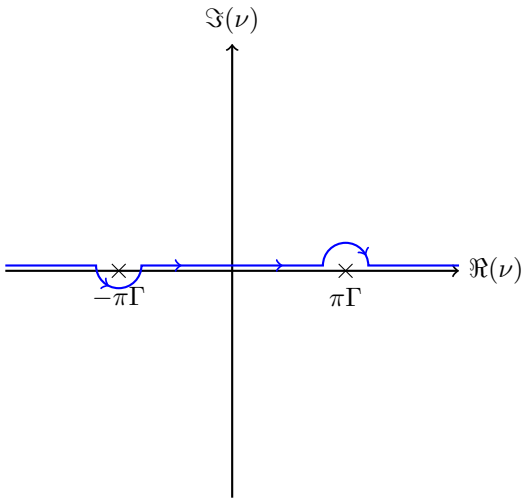


Figure 5.1.1: Integration contour for the six-gluon BFKL integral at finite coupling. Here and in the following figures indicating different integration contours for the six-and seven-gluon BFKL integrals, we only depict the singularities on the integration contour, not the entire ν-plane.

Strong constraints on the analytic structure of the integrand in (5.1.2) at finite coupling can be derived by considering the soft limits $z \to 0$ and $z \to \infty$. The correct soft limit behavior of the BDS ansatz implies that R_6 has to vanish there to all orders in perturbation theory, and thus by virtue of (5.1.5) the left-hand-side of eq. (5.1.1) reduces to

$$\lim_{z\to 0} e^{R_6(z)+i\delta_6(z)} = |z|^{2\pi i\Gamma} \tag{5.1.8}$$

$$\lim_{z\to \infty} e^{R_6(z)+i\delta_6(z)} = |z|^{-2\pi i\Gamma}. \tag{5.1.9}$$

From this we can determine the behavior of the right-hand-side of eq. (5.1.1) for $n = 0$ (terms coming from $n \neq 0$ will be suppressed in the soft limit). It is evident that the integrand should have simple poles at $\nu = \pm\pi\Gamma$, with residues

$$\text{Res}_{\nu=\pm\pi\Gamma}\left(\tilde{\Phi}(\nu,0)e^{-L\omega(\nu,0)}\right) = \pm\frac{1}{\pi a} \tag{5.1.10}$$

in order to capture the all order soft behaviour of the left-hand-side. In fact, a more

detailed analysis of the soft limit reveals that it separately restricts $\omega(\nu,n)$ and $\tilde{\Phi}(\nu,n)$ to obey the *exact bootstrap conditions* [108],

$$\omega(\pm\pi\Gamma,0)=0, \quad \text{and} \quad \text{Res}_{\nu=\pm\pi\Gamma}\tilde{\Phi}(\nu,0)=\pm\frac{1}{\pi a} \tag{5.1.11}$$

where by virtue of (5.1.7), the second relation may also be written as

$$\Phi(\pm\pi\Gamma,0)=1 \tag{5.1.12}$$

Finally, since the integral (5.1.2) diverges if the poles are located right on the real axis, the soft limits (5.1.8)-(5.1.9) also dictate how the contour should be deformed in order to avoid these poles on the real axis: Given that we need to close the contour on the lower (upper) half-plane in ν for z small (large), it is also clear that the integration contour should run above the pole at $\pi\Gamma$ and below the pole at $-\pi\Gamma$, in other words

$$\frac{\Phi(\nu,0)}{\nu^2-\pi^2\Gamma^2}\to\frac{\Phi(\nu,0)}{\nu^2-\pi^2\Gamma^2+i0}=\frac{\Phi(\nu,0)}{(\nu-\pi\Gamma+i0)(\nu+\pi\Gamma-i0)} \tag{5.1.13}$$

as shown in figure 5.1.1.

So far the discussion was restricted to finite coupling. As we will now review, the knowledge of the residues and integration contour that the soft limits provide at finite coupling, is also crucial for appropriately regularizing the divergences that appear in the weak coupling expansion of eq. (5.1.2). In particular, given that at leading order $\Phi(\nu,n)\to 1$, it is evident from (5.1.13) that the integrand becomes ill defined for $n=0$, since evaluating the residue in either the upper or lower half-plane leads to a divergence. In other words, while at finite coupling the contour runs between the poles at $\nu=-\pi\Gamma$ and $\nu=+\pi\Gamma$, in the weak coupling expansion, where $\Gamma=\mathcal{O}(a)$ with $a\to 0$, these two poles will move towards $\nu=0$ and pinch the contour, leading to a divergence.

Consequently, we need to deform the contour at finite coupling before we are allowed to expand in the coupling.

There are two immediate choices for deforming the contour so that it does not pass between the two poles on the real axis any more. We can either take plus the residue at $\nu=-\pi\Gamma$ in order to move the contour above the real line or we can take minus the residue at $\nu=+\pi\Gamma$ to move the contour below the real line. It is possible to preserve the symmetry of the integral by averaging between both choices, in order to find

$$\begin{aligned}e^{R_6+i\delta_6}=&-\frac{\pi a}{2}\operatorname*{Res}_{\nu=-\pi\Gamma}\left(\tilde{\Phi}(\nu,0)|z|^{2i\nu}e^{-L\omega(\nu,0)}\right)+\frac{ia}{4}\sum_{n=-\infty}^{\infty}\left(\frac{z}{\bar{z}}\right)^{\frac{n}{2}}\int_{\uparrow}d\nu\,\tilde{\Phi}(\nu,n)|z|^{2i\nu}e^{-L\omega(\nu,n)}\\&+\frac{\pi a}{2}\operatorname*{Res}_{\nu=\pi\Gamma}\left(\tilde{\Phi}(\nu,0)|z|^{2i\nu}e^{-L\omega(\nu,0)}\right)+\frac{ia}{4}\sum_{n=-\infty}^{\infty}\left(\frac{z}{\bar{z}}\right)^{\frac{n}{2}}\int_{\downarrow}d\nu\,\tilde{\Phi}(\nu,n)|z|^{2i\nu}e^{-L\omega(\nu,n)}\end{aligned}$$

where $\uparrow$ ($\downarrow$) denotes the contour running above (below) the real line, and the contours may be closed in either half-plane.

After we evaluate the residues with the help of the bootstrap conditions (5.1.11)-(5.1.12), combine the two contour integrals by introducing the Cauchy principal value $\mathcal{P}$

$$\mathcal{P}\left(\frac{1}{x}\right) = \frac{1}{2}\left(\frac{1}{x+i0} + \frac{1}{x-i0}\right) \tag{5.1.14}$$

and reexpress the integrand with the help of (5.1.7), we finally obtain the separation into the conformal Regge pole and Regge cut contribution,[2]

$$e^{R_6+i\delta_6} = \cos(\log(|z|^2)\pi\Gamma) + i\frac{a}{2}\sum_{n=-\infty}^{+\infty}\left(\frac{z}{\bar{z}}\right)^{\frac{n}{2}}\mathcal{P}\int_{-\infty}^{\infty}d\nu\,\frac{\Phi_{\text{reg}}(\nu,n)}{\nu^2+\frac{n^2}{4}}|z|^{2i\nu}e^{-L\omega(\nu,n)} \tag{5.1.16}$$

This reproduces the well known expression from ref. [91], see also [99, 100, 129, 130]. We see that the regularization of the integral at $\nu = n = 0$, which in the prescription (5.1.14) amounts to taking half the corresponding residue into account, is intimately connected to the Regge pole contribution. It is worth emphasizing however that not just this symmetric choice, but any contour deformation that avoids a pinching is equally valid for performing the weak coupling expansion. For example, when evaluating R_6 for z small, it is advantageous to pick the contour running below the real axis, so that after closing it from below, the integral will no longer receive any contributions from the poles near the real axis[3]

$$e^{R_6+i\delta_6} = |z|^{2\pi i\Gamma} + i\frac{a}{2}\sum_{n=-\infty}^{+\infty}\left(\frac{z}{\bar{z}}\right)^{\frac{n}{2}}\int_{\downarrow}d\nu\,\tilde{\Phi}(\nu,n)|z|^{2i\nu}e^{-L\omega(\nu,n)} \tag{5.1.17}$$

The generalization to the NMHV case is straightforward. Focusing on the helicity configuration most commonly found in the literature, see e.g. [131], the analogue of (5.1.1)-(5.1.2) is

$$\mathcal{R}_{+-}e^{i\delta_6(z)} = 2\pi i f_{+-} \tag{5.1.18}$$

$$f_{+-} = \frac{a}{2}\sum_{n=-\infty}^{\infty}\left(\frac{z}{\bar{z}}\right)^{\frac{n}{2}}\int_{-\infty}^{\infty}\frac{d\nu}{2\pi}\,\tilde{\Phi}(\nu,n)\overline{H}(\nu,n)|z|^{2i\nu}e^{-L\omega(\nu,n)} \tag{5.1.19}$$

where the helicity flip kernel H will be defined below in eq. (5.1.43). At this point, it is sufficient to note that $H(\nu,0) = 1$, which implies that the MHV and NMHV integrands

[2] In fact, we could have equally well arrived at the expression (5.1.16) from (5.1.1) and (5.1.13) by virtue of the Sokhotski–Plemelj theorem on the real line,

$$\frac{1}{x\pm i0} = \mp\pi\delta(x) + \mathcal{P}\left(\frac{1}{x}\right) \tag{5.1.15}$$

for $x = \nu^2 - \pi^2\Gamma^2$, together with $\delta(x^2-\alpha^2) = (\delta(x-\alpha)+\delta(x+\alpha))/(2|\alpha|)$.

[3] It is worth keeping in mind that there will still be contributions from poles in the interior of the contour for $n = 0$.

become identical for $n = 0$, and so our analysis of the poles on the integration contour as well as the prescription to avoid them generalize straightforwardly to NMHV. Finally, the $\mathcal{R}_{-+}$ helicity configuration may be obtained from (5.1.18)-(5.1.19) by a parity transformation.

5.1.2 7-points MHV

Armed with intuition from the six-point case, we now move on to propose an all-loop dispersion-type formula for the $2 \to 5$ amplitude in MRK, again in the region where we analytically continue the energy components of all produced particles. Our strategy will be as follows:[4]

1. We start with a formula that expresses the remainder plus conformal BDS contribution in the Multi-Regge limit at finite coupling, as a Regge cut (integral) only, i.e. without any explicit Regge poles.

2. We then examine the soft limits of the formula at finite coupling, which reveal to us the positions of the poles of the integral on the real axis, their residues, as well as the prescription for integrating around them[5].

3. In the weak-coupling expansion of the integrand, these poles will pinch the contour, leading to divergences. We may deform the original contour, prior to expanding in the coupling, to any contour that is not pinched in the weak-coupling limit, picking up the residues that are crossed in the process of this deformation at finite coupling, and then expand at weak coupling.

In this manner, for any deformation described in the last step, we will obtain an integral that is well-defined at weak coupling, plus finite-coupling residue contributions, whose values we know from the soft limits.

So let us start with the seven-point analogues of (5.1.1)-(5.1.2) shown in Fig. 5.6.1[6]

$$e^{R_7(z_1,z_2)+i\delta_7(z_1,z_2)} = 2\pi i f_{+++} \tag{5.1.20}$$

$$\begin{aligned} f_{+++} = \frac{a}{2} \sum_{n_1=-\infty}^{\infty} \sum_{n_2=-\infty}^{\infty} \left(\frac{z_1}{\bar{z}_1}\right)^{\frac{n_1}{2}} \left(\frac{z_2}{\bar{z}_2}\right)^{\frac{n_2}{2}} \int \frac{d\nu_1 d\nu_2}{(2\pi)^2} |z_1|^{2i\nu_1} |z_2|^{2i\nu_2} \\ \times e^{-L_1\omega(\nu_1,n_1)} e^{-L_2\omega(\nu_2,n_2)} \chi^+(\nu_1,n_1) C^+(\nu_1,n_1,\nu_2,n_2) \chi^-(\nu_2,n_2) \end{aligned} \tag{5.1.21}$$

[4] A similar strategy for obtaining dispersion integrals of higher-point amplitudes, also in different regions, has also been independently adopted by Basso,Caron-Huot and Sever, see [132].

[5] We assume that no other poles are present on the real axis but the ones dictated by the soft limits.

[6] To make contact with other notations used in the literature, f_{+++} is denoted as $f_{\omega_2\omega_3}$ in [104, 105], and similarly the six-gluon analogue f_{++} of the previous section is denoted as f_{ω_2}.

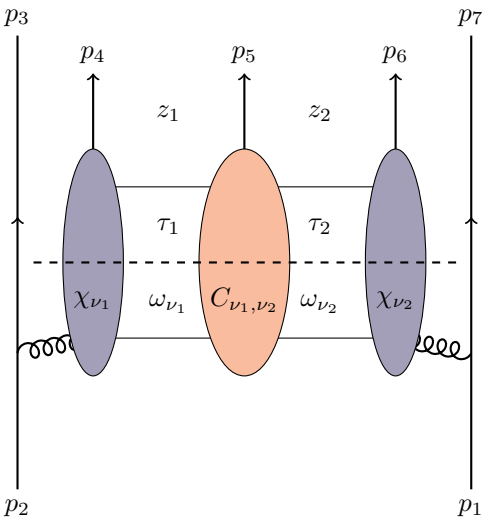


Figure 5.1.2: Structure of the seven point amplitude in MRK.

where

$$L_i = \log\sqrt{u_{2i}u_{3i}} + i\pi = \log\tau_i + i\pi \tag{5.1.22}$$

and the conformally invariant part of the 1-loop Regge cut coming from the BDS ansatz is[7]

$$\delta_7(z_1,z_2) = 2\pi\Gamma\log\frac{\sqrt{u_{21}u_{31}u_{22}u_{32}}}{1-\tilde{u}} = \pi\Gamma\log\frac{|z_1z_2|^2}{|1-z_2+z_1z_2|^4} \tag{5.1.23}$$

In addition, apart from the BFKL eigenvalue encountered in the previous section, $\chi^{\pm}(\nu,n)$ are the two BFKL impact factors at the end of the ladder [100], whose product with equal arguments also appeared in the hexagon case[8]

$$\chi^+(\nu,n)\chi^-(\nu,n) = \tilde{\Phi}(\nu,n) \tag{5.1.24}$$

Finally, $C^+(\nu_1,\nu_2,n_1,n_2)$ is the central emission block, a new ingredient in the BFKL approach to the heptagon compared to the hexagon, first computed to leading order in [102].

Next, we consider the three soft limits where the momentum of one of the produced particles goes to zero, and R_7 reduces to R_6. With the help of (5.1.23) we find that in the

[7] See for example [104], where $\delta_7 \to \delta_{14}$ is expressed in terms of the transverse momenta of the produced gluons, and the momentum transfer between them. It can be recast in terms of (transverse) cross ratios by virtue of the kinematic analysis of [102], as adapted to our conventions in [38].

[8] The fact that the integrand contains impact factors of opposite helicity can be understood by thinking about the momentum flow along the ladder: If we assume that its actual direction is from the χ^+ towards the χ^- impact factor, then in all-outgoing momentum conventions for these impact factors, the helicity assignments of the gluons associated with χ^+ and χ^- will be $(-++)$ and $(--+)$, respectively. Hence the two must be related by parity.

soft limits the left-hand-side of eq. (5.1.20) becomes,

$$\lim_{z_1\to 0} e^{R_7+i\delta_7} = e^{R_6(z_2)+i\delta_6(z_2)}|z_1|^{2\pi i\Gamma} \tag{5.1.25}$$

$$\lim_{z_2\to \infty} e^{R_7+i\delta_7} = e^{R_6(z_1)+i\delta_6(z_1)}|z_2|^{-2\pi i\Gamma} \tag{5.1.26}$$

$$\lim_{z_2\to 0,\, z_1 z_2 \text{ fixed}} e^{R_7+i\delta_7} = e^{R_6(z_1 z_2)+i\delta_6(z_1 z_2)} \tag{5.1.27}$$

Note that the last soft limit is also equivalent to $z_1 \to \infty$ with $z_1 z_2$ fixed. From the behaviour in the soft limits we can determine that the r.h.s of eq. (5.1.20) needs to have a pole $\nu_1 = \pi\Gamma - i0$ for the $n_1 = 0$ term in the sum, a pole at $\nu_2 = -\pi\Gamma + i0$ for $n_2 = 0$, as well as at $\nu_1 = \nu_2 + i0$ (or equivalently at $\nu_2 = \nu_1 - i0$) for $n_1 = n_2$. Furthermore, it is easy to check that the above relations hold if the residues on those poles are equal to

$$\operatorname*{Res}_{\nu_1=\pi\Gamma}\left(\chi^+(\nu_1,0)C^+(\nu_1,0,\nu_2,n_2)\chi^-(\nu_2,n_2)\right) = i\chi^+(\nu_2,n_2)\chi^-(\nu_2,n_2) \tag{5.1.28}$$

$$\operatorname*{Res}_{\nu_2=-\pi\Gamma}\left(\chi^+(\nu_1,n_1)C^+(\nu_1,n_1,\nu_2,0)\chi^-(\nu_2,0)\right) = -i\chi^+(\nu_1,n_1)\chi^-(\nu_1,n_1) \tag{5.1.29}$$

$$\operatorname*{Res}_{\nu_1=\nu_2}\left(\chi^+(\nu_1,n_2)C^+(\nu_1,n_2,\nu_2,n_2)\chi^-(\nu_2,n_2)\right) = -i(-1)^{n_2}e^{i\pi\omega(\nu_2,n_2)}\chi^+(\nu_2,n_2)\chi^-(\nu_2,n_2) \tag{5.1.30}$$

where we already took the condition (5.1.11) into account. The previous equations can be seen as an integrand formulation in Fourier-Mellin space of the soft limits of the amplitude. In particular, if the residues take the values in (5.1.28), (5.1.29) and (5.1.30), then the integrand with respect to the second integration variable becomes identical to the hexagon integrand[9]. At this point we have to make a comment about these relations. Since the soft limits are valid for the integrated amplitude after Fourier-Mellin transformation, any integrand formulation is in principle valid only up to terms that vanish when computing the Fourier-Mellin transform. For example, we could add to (5.1.28) any function of (ν_1, n_1) which is mapped to zero by the Fourier-Mellin transform, without changing the soft limit of the amplitude. Since the Fourier-Mellin transform is invertible, any such function which maps to zero is necessarily trivial, and so the bootstrap conditions (5.1.28), (5.1.29) and (5.1.30) follow indeed from the soft limits of the full amplitude.

Finally, we come to address the necessity for a contour deformation before performing the weak coupling expansion. It is evident that when at least one of the n_i is different from zero, we can have at most one pole on the real axis for either integration variable, and therefore one can safely expand at weak coupling because no pinching can occur. Therefore we only need to consider the $n_1 = n_2 = 0$ case, depicted in figure 5.1.3. There,

[9]The exponential factor in the last relation is present because for $\nu_1 = \nu_2$ the term multiplying $-\omega$ in (5.1.21) becomes $L_1 + L_2$, whereas the corresponding large logarithm in (5.1.27) should be

$$L' = \log(\tau_1\tau_2) + i\pi = L_1 + L_2 - i\pi\,.$$

we see that if after we close the contour at infinity, we receive a contribution from a residue on the real line in any of the integration variables, the integral left to do in the other integration variable will have the same pole structure as the hexagon integral of figure 5.1.1, whereby the poles pinch the contour at weak coupling. We will therefore

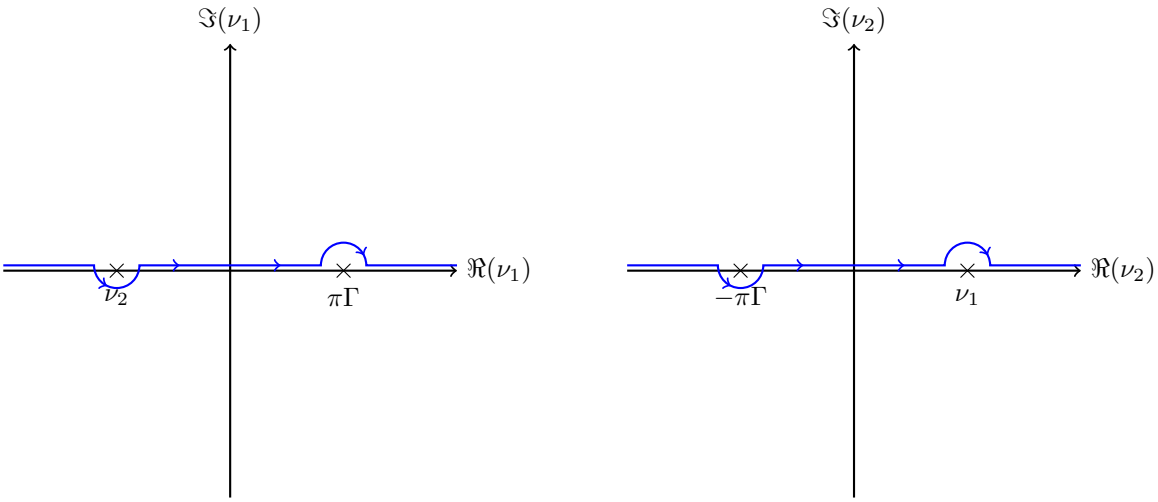


Figure 5.1.3: Integration contour for the seven-gluon BFKL integral.

have to deform the contour before expanding, and the simplest choice will be to pick a new contour such that it does not contain any poles on the real axis. For example, in the region $z_1 \ll 1, z_2 \gg 1$, where we close ν_1 from below and ν_2 from above, we can deform the contour as follows. Let us first schematically rewrite the integrand in a way that exposes the pole structure of the $n_1 = n_2 = 0$ integrand in (5.1.21),

$$\begin{aligned} I &\equiv \frac{a}{2(2\pi)^2}|z_1|^{2i\nu_1}|z_2|^{2i\nu_2}e^{-L_1\omega_1-L_2\omega_2}\chi^+(\nu_1,0)C^+(\nu_1,0,\nu_2,0)\chi^-(\nu_2,0) \\ &= \frac{f(\nu_1,\nu_2)}{(\nu_1-\pi\Gamma)(\nu_1-\nu_2)(\nu_2+\pi\Gamma)} \end{aligned} \tag{5.1.31}$$

Then, we can rewrite our original contour as

$$\begin{aligned} \int \frac{d\nu_1 d\nu_2 f(\nu_1,\nu_2)}{(\nu_1-\pi\Gamma-i0)(\nu_1-\nu_2+i0)(\nu_2+\pi\Gamma+i0)} &= \int \frac{d\nu_1 d\nu_2 f(\nu_1,\nu_2)}{(\nu_1-\pi\Gamma+i0)(\nu_1-\nu_2+i0)(\nu_2+\pi\Gamma+i0)} \\ &\quad - 2\pi i \int \frac{d\nu_2 f(\pi\Gamma,\nu_2)}{(\nu_2-\pi\Gamma-i0)(\nu_2+\pi\Gamma+i0)} \\ &= \int \frac{d\nu_1 d\nu_2 f(\nu_1,\nu_2)}{(\nu_1-\pi\Gamma+i0)(\nu_1-\nu_2+i0)(\nu_2-\pi\Gamma-i0)} \\ &\quad + 2\pi i \int \frac{d\nu_1 f(\nu_1,-\pi\Gamma)}{(\nu_1-\pi\Gamma+i0)(\nu_1+\pi\Gamma+i0)} \\ &\quad - 2\pi i \int \frac{d\nu_2 f(\pi\Gamma,\nu_2)}{(\nu_2-\pi\Gamma-i0)(\nu_2+\pi\Gamma+i0)} \end{aligned} \tag{5.1.32}$$

$$
\begin{aligned}
&= \int \tfrac{d\nu_1 d\nu_2 f(\nu_1,\nu_2)}{(\nu_1-\pi\Gamma+i0)(\nu_1-\nu_2+i0)(\nu_2-\pi\Gamma-i0)} \\
&+ 2\pi i \int \tfrac{d\nu_1 f(\nu_1,-\pi\Gamma)}{(\nu_1-\pi\Gamma-i0)(\nu_1+\pi\Gamma+i0)} \\
&- 2\pi i \int \tfrac{d\nu_2 f(\pi\Gamma,\nu_2)}{(\nu_2-\pi\Gamma-i0)(\nu_2+\pi\Gamma+i0)} + (2\pi i)^2 \frac{f(\pi\Gamma,-\pi\Gamma)}{2\pi\Gamma}
\end{aligned}
$$

where already in the second equality the double integral is free of poles on the real axis, and in going from the second to the third equality, we changed the contour of the single ν_1 integral to make it identical to that of figure 5.1.1. The reason is that due to the bootstrap conditions (5.1.11) and (5.1.28)-(5.1.30), also the integrand of these simple integrals becomes identical to the hexagon integral in (5.1.2), up to factors independent of the integration variable,

$$
\begin{aligned}
2\pi i \int \tfrac{d\nu_1 f(\nu_1,-\pi\Gamma)}{(\nu_1-\pi\Gamma+i0)(\nu_1+\pi\Gamma+i0)} &= |z_2|^{-2\pi i\Gamma} \frac{a}{2} \int \frac{d\nu_1}{2\pi} \tilde{\Phi}(\nu_1,0) |z_1|^{2i\nu_1} e^{-L_1\omega(\nu_1,0)} \\
-2\pi i \int \tfrac{d\nu_2 f(\pi\Gamma,\nu_2)}{(\nu_2-\pi\Gamma-i0)(\nu_2+\pi\Gamma+i0)} &= |z_1|^{2\pi i\Gamma} \frac{a}{2} \int \frac{d\nu_2}{2\pi} \tilde{\Phi}(\nu_2,0) |z_2|^{2i\nu_2} e^{-L_2\omega(\nu_2,0)}
\end{aligned}
\tag{5.1.33}
$$

Similarly, for the double residue in the last line of (5.1.32) we obtain

$$
(2\pi i)^2 \mathrm{Res}_{\nu_1=\pi\Gamma} \mathrm{Res}_{\nu_2=-\pi\Gamma} I = (2\pi i)^2 \frac{f(\pi\Gamma,-\pi\Gamma)}{2\pi\Gamma} = -\frac{1}{2\pi i} \frac{|z_1|^{2\pi i\Gamma}}{|z_2|^{2\pi i\Gamma}} \tag{5.1.34}
$$

Choosing to deform the contour in the same fashion for the case where one of the n_i is zero and the other nonzero, so that they contain no poles on the real axis, we observe that the summands combine nicely to yield

$$
e^{R_7+i\delta_7} = |z_1|^{2\pi i\Gamma} e^{R_6(z_2)+i\delta_6(z_2)} + |z_2|^{-2\pi i\Gamma} e^{R_6(z_1)+i\delta_6(z_1)} - \frac{|z_1|^{2\pi i\Gamma}}{|z_2|^{2\pi i\Gamma}} + 2\pi i \tilde{f}_{+++} \tag{5.1.35}
$$

where $\tilde{f}_{+++}$ is defined precisely as in (5.1.21), but now with the integration contour of eq. (5.1.32), which is also illustrated in figure 5.1.4. Notice that the presence of the second-to-last term from the right, coming from eq.(5.1.34), is necessary for reproducing the soft limits (5.1.25)-(5.1.26). We stress that although the above formula, which is the 7-point analogue of (5.1.17), holds independently of how we choose to close the integration contours, it is only valid in the region $z_1 \ll 1,\ z_2 \gg 1$ which is convenient for expanding at weak coupling. This is because the $\nu_1 = \nu_2$ residue is a simple integral that diverges at weak coupling, and so it cannot be contained in our closed contour. By deforming the contour in (5.1.32) around the $\nu_1 = \nu_2$ pole, we similarly find

$$
e^{R_7+i\delta_7} = e^{R_6(z_1 z_2)+i\delta_6(z_1 z_2)} + 2\pi i \check{f}_{+++} \tag{5.1.36}
$$

where again $\check{f}_{+++}$ is defined as in (5.1.21), but this time the contour is the one that results after we exchange $\nu_1 \leftrightarrow \nu_2$ in figure 5.1.4. The last formula is particularly suited for the

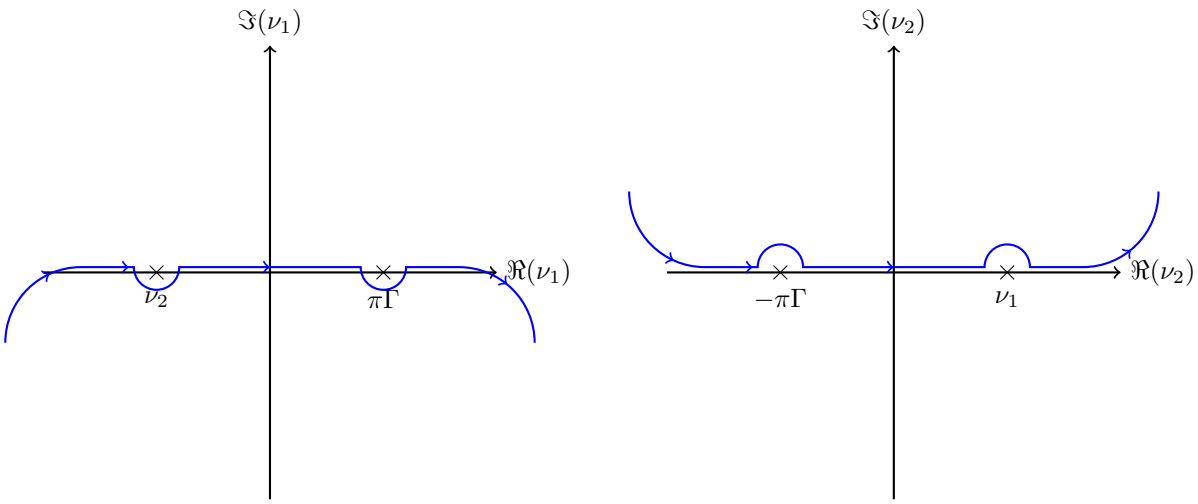


Figure 5.1.4: The deformed integration contour for the seven-gluon BFKL integral, which is convenient for the weak coupling expansion in the region $z_1 \ll 1,\ z_2 \gg 1$.

weak coupling expansion in the $z_1 \gg 1,\ z_2 \ll 1$ region. Finally, one may be tempted to take the average of (5.1.35)-(5.1.36) as the analogue of (5.1.16), however at least currently it seems that it is not convenient for the weak coupling expansion, since it will lead to pinching in both contours.

5.1.3 Summary and extension to any helicity

In the previous section, for simplicity we focused on the MHV $2 \to 5$ amplitude. Here we will present the generalization of the all-loop dispersion integral (5.1.20)-(5.1.23), as well as the exact bootstrap conditions (5.1.28)-(5.1.30) that are obeyed by the building blocks, for arbitrary helicity configurations.

Using definitions of subsection 4.1, the multi-Regge limit of the BDS-normalized $N = 7$ particle amplitude (4.1.27) will be given by

$$\mathcal{R}_{h_1 h_2 h_3} e^{i\delta_7(z_1,z_2)} = 2\pi i f_{h_1 h_2 h_3} \tag{5.1.37}$$

$$\begin{aligned} f_{h_1 h_2 h_3} =& \frac{a}{2} \sum_{n_1,n_2=-\infty}^{\infty} \left(\frac{z_1}{\bar{z}_1}\right)^{\frac{n_1}{2}} \left(\frac{z_2}{\bar{z}_2}\right)^{\frac{n_2}{2}} \int \frac{d\nu_1 d\nu_2}{(2\pi)^2} |z_1|^{2i\nu_1} |z_2|^{2i\nu_2} \tilde{\Phi}(\nu_1, n_1) \tilde{\Phi}(\nu_2, n_2) \\ & \times e^{-L_1 \omega(\nu_1,n_1) - L_2 \omega(\nu_2,n_2)} I^{h_1}(\nu_1, n_1) \tilde{C}^{h_2}(\nu_1, n_1, \nu_2, n_2) \bar{I}^{h_3}(\nu_2, n_2) \end{aligned} \tag{5.1.38}$$

with

$$L_i = \log \tau_i + i\pi \quad \delta_7(z_1, z_2) = \pi\Gamma \log \frac{|z_1 z_2|^2}{|1 - z_2 + z_1 z_2|^4} \tag{5.1.39}$$

as well as

$$\tilde{\Phi}(\nu, n) = \chi^+(\nu, n) \chi^-(\nu, n) \tag{5.1.40}$$

as defined previously, and the contour of integration as depicted in figure 5.1.3. In addition, we have expressed the integrand in terms of the rescaled quantities[10]

$$\tilde{C}^h(\nu_1,n_1,\nu_2,n_2)\equiv\frac{C^h(\nu_1,n_1,\nu_2,n_2)}{\chi^-(\nu_1,n_1)\chi^+(\nu_2,n_2)} \tag{5.1.41}$$

and

$$I^h(\nu,n)\equiv\frac{\chi^h(\nu,n)}{\chi^+(\nu,n)}=\begin{cases}1, & h=+\\ H(\nu,n) & h=-\end{cases} \tag{5.1.42}$$

with $\bar{I}^h$ denoting the complex conjugate of I^h. In the last equation,

$$H(\nu,n)=\frac{x\left(u(\nu)-\frac{in}{2}\right)}{x\left(u(\nu)+\frac{in}{2}\right)} \tag{5.1.43}$$

is the helicity flip kernel, or NMHV form factor, known to all loops from integrability [109], as is also the case for the hexagon impact factor $\tilde{\Phi}$. The precise definition of the Zhukowski variables x and the rapidities u will not be important for our purposes, and we will be explicitly providing the weak-coupling expansion of H in section 5.3. A crucial property that however follows immediately from the above representation, is that

$$H(\nu,0)=1\ \Rightarrow\ I^h(\nu,0)=1 \tag{5.1.44}$$

The most significant advantage of defining a rescaled central emission block as in (5.1.41), is that it allows us to formulate separate exact bootstrap conditions for the latter: Along with the conditions

$$\omega(\pm\pi\Gamma,0)=0,\quad\text{and}\quad \mathrm{Res}_{\nu=\pm\pi\Gamma}\left(\tilde{\Phi}(\nu,0)\right)=\pm\frac{1}{\pi a}, \tag{5.1.45}$$

which as we reviewed in section 5.1.1 follow from the analysis of the six-gluon amplitude, the consistency of soft limits of the seven-point amplitude requires

$$\tilde{C}^h(\pi\Gamma,0,\nu_2,n_2)=i\pi a\, I^h(\nu_2,n_2) \tag{5.1.46}$$

$$\tilde{C}^h(\nu_1,n_1,-\pi\Gamma,0)=-i\pi a\,\bar{I}^h(\nu_1,n_1) \tag{5.1.47}$$

$$\operatorname*{Res}_{\nu_1=\nu_2}\tilde{C}^h(\nu_1,n_2,\nu_2,n_2)=\frac{-i(-1)^{n_2}e^{i\pi\omega(\nu_2,n_2)}}{\tilde{\Phi}(\nu_2,n_2)} \tag{5.1.48}$$

as well as

$$C^h(-\pi\Gamma,0,\nu_2,n_2)=C^h(\nu_1,n_1,\pi\Gamma,0)=0 \tag{5.1.49}$$

In particular, eqs. (5.1.46)-(5.1.48) for $h=+$ follow from (5.1.28)-(5.1.30) and (5.1.45), after

[10] In more detail, the generalization of (5.1.21) to arbitrary helicity follows from $\chi^+C^+\chi^-\to\chi^{h_1}C^{h_2}\chi^{-h3}$, which can then be recast in the form (5.1.38) after we plug in the solution of (5.1.42) and (5.1.41) for χ^h and C^h respectively, and finally use (5.1.40).

also taking into account that $\tilde{C}^h$ must be regular at $n_1 = 0, \nu_1 = \pi\Gamma$ and $n_2 = 0, \nu_2 = -\pi\Gamma$ for the soft limits (5.1.25)-(5.1.26) to hold (e.g. a pole would lead to additional $\log z_i$ dependence that is incompatible with these limits). In a similar spirit, the regularity of the entire integrand at $n_1 = 0, \nu_1 = -\pi\Gamma$ and $n_2 = 0, \nu_1 = \pi\Gamma$ implies (5.1.49), so as to cancel the poles of either of the $\tilde{\Phi}(\nu_i, n_i)$ there.

Then, the extension of these conditions to $\tilde{C}^-$ can be done by recalling that the MHV and $\overline{\text{MHV}}$ amplitudes must be equal to each other, $\mathcal{R}_{+++} = \mathcal{R}_{---}$, as a consequence of their equivalence to the same bosonic Wilson loop under the Wilson loop/amplitude duality. Imposing this on (5.1.37)-(5.1.38) implies

$$\tilde{C}^-(\nu_1, n_1, \nu_2, n_2) = \overline{H}(\nu_1, n_1) C^+(\nu_1, n_1, \nu_2, n_2) H(\nu_2, n_2) \tag{5.1.50}$$

allowing us to obtain (5.1.46)-(5.1.48) for $h = -$ from $h = +$, also with the help of (5.1.44). Note that the last formula implies that we only need consider $\omega, \tilde{\Phi}, H$ and $\tilde{C}^+$ as the independent building blocks of the integrand, and then $\tilde{C}^-$ may be expressed in terms of the last two.

Finally, by deforming the contour, it is possible to equivalently write (5.1.37)-(5.1.38) as

$$\mathcal{R}_{h_1 h_2 h_3} e^{i\delta_7} = |z_1|^{2\pi i\Gamma} \mathcal{R}_{h_2 h_3}(z_2) e^{i\delta_6(z_2)} + |z_2|^{-2\pi i\Gamma} \mathcal{R}_{h_1 h_2}(z_1) e^{i\delta_6(z_1)} - \frac{|z_1|^{2\pi i\Gamma}}{|z_2|^{2\pi i\Gamma}} + 2\pi i \tilde{f}_{h_1 h_2 h_3} \tag{5.1.51}$$

where $\tilde{f}_{h_1 h_2 h_3}$ is defined as in (5.1.41), but with the integration contour illustrated in figure 5.1.4. In what follows, we will almost exclusively be using this form of the dispersion integral, which is particularly convenient for its weak-coupling expansion in the region $z_1 \ll 1,\ z_2 \gg 1$ that we will consider.

5.2 From symbols to functions in MRK

In the previous section, we succeeded in obtaining an all-order dispersion integral describing the multi-Regge limit of the $2 \to 5$ amplitude of any helicity configuration, that is well defined at arbitrary logarithmic accuracy. In order to complete the description we also need to determine the building blocks of the integrand, and while the ones associated to the six-particle amplitude are known to all loops, the (rescaled) central emission block (5.1.41) is only known to leading order [102].

The aim of the next two sections will thus be to extract the central emission block at higher order from the known perturbative data for the amplitude, exploiting the fact that if we know the left-hand side of (5.1.37) at ℓ loops, we can determine all building blocks in (5.1.38) at $\text{N}^{\ell-1}\text{LO}$, since they start at $\mathcal{O}(a^0)$. In particular, in this section we will first promote the known 2-loop symbol of the MHV seven-particle amplitude [114] to a function. From this, we will in fact obtain all 2-loop MHV amplitudes in the multi-Regge

limit, by similarly promoting their interesting factorization property, i.e. their decomposition into building blocks associated with the six- and seven-particle amplitude, that was also discovered in ref. [114].

In subsection 5.2.1, we first establish a necessary result for our subsequent analysis, which is however expected to have other applications as well, since it holds in general kinematics: Based on the framework of the Operator Product Expansion (OPE) for null polygonal Wilson loops [26, 28–31, 115, 133–138], we derive the maximal degree of logarithmic divergences of MHV amplitudes in the Euclidean region for any number of particles N, extending the earlier analysis in ref. [139] of the $N = 6$ case. In subsection 5.2.2, we then use this property, together with information from soft limits, in order to uniquely fix all beyond-the-symbol terms of an ansatz for the 2-loop MHV seven-point amplitude, or more precisely the remainder function $R_7^{(2)}$. Finally, in section 5.2.3 we obtain all 2-loop MHV amplitudes in the multi-Regge limit, by proving that the NLLA factorization of the symbol observed in [114] must necessarily also hold at function level.

5.2.1 Maximal degree of logarithmic divergence from the OPE

Let us start by stating the main result of this subsection: We will prove that the N-point L-loop remainder function $R_N^{(L)}$ in general kinematics may always be written as

$$R_N^{(L)} = \sum_{0\leq j_1+\ldots+j_{N-5}\leq L-1} \log^{j_1} U_1 \ldots \log^{j_{N-5}} U_{N-5} f_{j_1,\ldots,j_{N-5}} \tag{5.2.1}$$

where the functions $f_{j_1,\ldots,j_{N-5}}$ are analytic for any of the cross ratios $U_i \to 0$, and the latter are defined as

$$U_i = U_{N-\lceil \frac{i}{2} \rceil, \lfloor \frac{i}{2} \rfloor + 2} \quad i = 1, \ldots, N-5 \tag{5.2.2}$$

with U_{ij} already defined in (4.1.10), or equivalently any other set obtained by acting on (5.2.2) with dihedral transformations. In the last relation, $\lfloor x \rfloor$ and $\lceil x \rceil$ are the floor and ceiling functions respectively. This way, we find that the only relevant cross ratio for $N = 6$ is U_{52}, for $N = 7$ there is U_{62} and U_{63} and for $N = 8$ we have U_{72}, U_{63} and U_{73} and so on for higher points.

The main content of (5.2.1) is twofold: First, the leading discontinuity of $R_N^{(L)}$, or equivalently its maximal degree of logarithmic divergence is $L - 1$. And second, that one can unshuffle all logarithms in the U_i simultaneously even in general kinematics, which is nontrivial because it cannot be done for general polylogarithmic functions.

We now proceed with the proof, and at the end of this section also mention the generalization of our result beyond the MHV case. We will be relying on the Wilson loop OPE [26] approach and its subsequent refinements [28–31], where the observable of interest is a ratio of bosonic Wilson loops $\mathcal{W}_N$, which has a weak coupling expansion of

the form

$$\mathcal{W}_N = 1 + \sum_{L=1}^{\infty} a^L \mathcal{W}_N^{(L)} \tag{5.2.3}$$

and is related to R_N by

$$R_N = \log \mathcal{W}_N - \frac{\gamma_K}{4} \mathcal{W}_N^{(1)} \tag{5.2.4}$$

where γ_K has already been defined in (5.1.6). The last equation encodes the fact that although $\mathcal{W}_N$ receives tree-level and one-loop contributions, R_N only starts at two loops.

As described in Section 2.2.6 the N-gon Wilson loop dual to the MHV amplitude is tessellated into $N-5$ consecutive squares, where the two segments of each square that are part of the original Wilson loop can be thought of as a quark-antiquark pair sourcing a colour-electric flux tube. We can then decompose the Wilson loop into excitations of this flux tube ψ_i, with energy E_i, momentum p_i and helicity m, corresponding to the three isometries of the square (see Subsection 2.2.6)

$$\mathcal{W}_N = \sum_{\psi_1,\ldots,\psi_{N-5}} e^{\sum_j^{N-5}(-\tau_j E_j + i p_j \sigma_j + i m_j \phi_j)} \mathcal{P}(0|\psi_1)\mathcal{P}(\psi_1|\psi_2)\ldots\mathcal{P}(\psi_{N-6}|\psi_{N-5})\mathcal{P}(\psi_{N-5}|0) \tag{5.2.5}$$

The $3N-15$ algebraically independent variables τ_i, σ_i and ϕ_i parametrize the conformally invariant kinematics, and when expressing the cross-ratios in terms of these variables, it can be shown that the $N-5$ cross ratios of (5.2.2) take the form

$$U_i = \frac{1}{1+e^{2\tau_i}} \tag{5.2.6}$$

This formula can be obtained from the following equivalent parametrization of the exponential factors in terms of so-called 4-brackets $\langle ijkl\rangle$ of momentum twistors

$$\begin{aligned} e^{2\tau_{2j+1}} &\equiv \frac{\langle -j-1, j+1, j+2, j+3\rangle\langle -j-2, -j-1, -j, j+2\rangle}{\langle -j-2, -j-1, j+2, j+3\rangle\langle -j-1, -j, j+1, j+2\rangle} \quad j = 0, \ldots \lfloor \frac{N-6}{2} \rfloor \\ e^{2\tau_{2j}} &\equiv \frac{\langle -j, j+1, j+2, j+3\rangle\langle -j-1, -j, -j+1, j+2\rangle}{\langle -j-1, -j, j+2, j+3\rangle\langle -j, -j+1, j+1, j+2\rangle} \quad j = 1, \ldots \lfloor \frac{N-5}{2} \rfloor \end{aligned} \tag{5.2.7}$$

For our purposes we will require the following identity between six momentum twistors,

$$\langle cdef\rangle\langle abef\rangle - \langle bdef\rangle\langle acef\rangle = -\langle adef\rangle\langle bcef\rangle \tag{5.2.8}$$

The latter is a consequence of the Schouten identity

$$\langle ab\rangle\langle cd\rangle - \langle ac\rangle\langle bd\rangle = -\langle bc\rangle\langle ad\rangle \tag{5.2.9}$$

since six points in $\mathbb{CP}^3$ (the twistors) are equivalent to six points in $\mathbb{CP}^1$ (the spinors), as can be seen by replacing the spinor bracket of two points with a 4-bracket of its com-

plement, $\langle ab\rangle \to \langle cdef\rangle$ etc. With the help of the last two formulas, we can show that indeed

$$\begin{aligned}\frac{1}{1+e^{2\tau_{2j+1}}} &= \frac{\langle -j-2,-j-1,j+2,j+3\rangle\langle -j-1,-j,j+1,j+2\rangle}{\langle -j-2,-j-1,j+1,j+2\rangle\langle -j-1,-j,j+2,j+3\rangle} = U_{-j-1,j+2}\\ \frac{1}{1+e^{2\tau_{2j}}} &= \frac{\langle -j-1,-j,j+2,j+3\rangle\langle -j,-j+1,j+1,j+2\rangle}{\langle -j-1,-j,j+1,j+2\rangle\langle -j,-j+1,j+2,j+3\rangle} = U_{-j,j+2}\end{aligned} \tag{5.2.10}$$

and (5.2.2) neatly combines the separate odd and even cases of the last equation.

At weak coupling, the tree-level term in (5.2.3) comes from the vacuum state, whereas excitations on top of it have energies

$$E_i = M_i + \gamma_i a + \mathcal{O}(a^2) \tag{5.2.11}$$

where M_i is the excitation number. Thus expanding (5.2.5) at weak coupling, and given that

$$\tau_i = -\frac{1}{2}\log U_i + \frac{1}{2}\log(1-U_i) \tag{5.2.12}$$

we see that at L loops the terms that maximize the sum of powers of $\log U_i$ will be

$$\begin{aligned}&\sum_{M_i}\prod_{i=1}^{N-5}\frac{(-a\gamma_i\tau_i)^{j_i}}{j_i!}e^{\sum_i(-\tau_i M_i+ip_i\sigma_i+im_i\phi_i)}a[\mathcal{P}(0|\psi_1)\dots\mathcal{P}(\psi_{N-5}|0)]^{(1)}\\ &= a^L\sum_{l_i=0}^{j_i}\prod_{i=1}^{N-5}\log^{l_i}U_i\times(\text{terms analytic as } U_i=0)\end{aligned} \tag{5.2.13}$$

Where $\sum_i^{N-5} j_i = L-1$. All other terms in the OPE expansion will also have the same general structure as (5.2.13) but with fewer powers of $\log U_i$ (this also includes so-called small fermions which have $\gamma_i = 0$ and in fact start at $\mathcal{O}(a^3)$, see for example [30]), which proves that $\mathcal{W}_N^{(L)}$ has the structure of the right-hand side of eq. (5.2.1). Then by virtue of (5.2.3) and (5.2.4), the same will be true for $R_N^{(L)}$, which thus completes the proof.

A very similar statement also holds true beyond the MHV case, where it is more convenient to consider the entire superamplitude, rather than its gluonic component alone. This is then dual to a super-Wilson loop, whose OPE expansion is equal to the same expansion for MHV, times non-MHV form factors [29,136–138]. These form factors may increase or decrease the order at which OPE excitations begin to contribute, and indeed, some components of the superamplitude, or more precisely ratio function, are nontrivial at tree level. By repeating the arguments presented above, if a component of the dual super-Wilson loop receives its first nontrivial OPE contribution at k loops, then the total degree of its logarithmic divergence at L loops should be $L-k$.

5.2.2 The function-level 7-particle 2-loop MHV amplitude in MRK

In this subsection, we will promote the known 2-loop symbol of the heptagon remainder function in the multi-Regge limit of ref. [114], as defined in (4.1.27) for $N = 7$, and in the region corresponding to the analytic continuation of the cross ratio,

$$\tilde{u} \equiv U_{2,N-1} \to e^{-2\pi i} U_{2,N-1} \tag{5.2.14}$$

to a function $R_7^{(2)}$. Let us start by reviewing the relevant information from the aforementioned paper, where it was shown that

$$\frac{R_7^{(2)}(z_1, z_2)}{2\pi i} = \sum_{i=1}^{2} \Big(2f(\rho_i) \log \tau_i + \tilde{f}(\rho_i) \Big) + g(\rho_1, \rho_2) \tag{5.2.15}$$

where

$$\rho_1 = -\frac{z_1 z_2}{1 - z_2} \quad \rho_2 = (1 - z_1) z_2 \tag{5.2.16}$$

are the coordinates defined in (4.2.9)-(4.2.11) specialized to $N = 7$, and the functions $f, \tilde{f}$ are the LLA and NLLA parts of the hexagon remainder function,

$$\frac{R_6^{(2)}(z_1)}{2\pi i} = 2f(z_1) \log \tau_1 + \tilde{f}(z_1) \tag{5.2.17}$$

this time in the region corresponding to the analytic continuation (5.2.14), but for $N = 6$ (two-particle cut). Explicitly we have [100], in the coupling normalization of (5.1.4)

$$\begin{aligned}
4f(z) = 4f(1/z) &= \frac{1}{2} \log |1 - z|^2 \log \frac{|1 - z|^2}{|z|^2} = -\frac{1}{2} \mathcal{G}_0(z) \mathcal{G}_1(z) + \frac{1}{2} \mathcal{G}_1^2(z) \\
4\tilde{f}(z) = 4\tilde{f}(1/z) &= -4\mathrm{Li}_3(z) - 4\mathrm{Li}_3(\bar{z}) + 2 \log |z|^2 (\mathrm{Li}_2(z) + \mathrm{Li}_2(\bar{z})) \\
&\quad + \frac{1}{3} \log^2 |1 - z|^2 \log \frac{|z|^6}{|1 - z|^4} - \frac{1}{2} \log |1 - z|^2 \log \frac{|1 - z|^2}{|z|^2} \log \frac{|z|^2}{|1 - z|^4} \\
&= -2\mathcal{G}_0(z) \mathcal{G}_{0,1}(z) + 4\mathcal{G}_{0,0,1}(z) + \frac{1}{3} \mathcal{G}_1(z)^3 - \frac{1}{2} \mathcal{G}_0(z) \mathcal{G}_1(z)^2 + \frac{1}{2} \mathcal{G}_0(z)^2 \mathcal{G}_1(z)
\end{aligned} \tag{5.2.18}$$

Finally, the symbol, as well as a 25-parameter functional representative for the genuinely heptagonal NLLA function g, were found in [114]. Here we fix all remaining ambiguity, and

show that

$$\begin{aligned}4g(\rho_1,\rho_2) =& 2\mathcal{G}_{0,1,1/\rho_1}(1/\rho_2) - 2\mathcal{G}_{1,1,1/\rho_1}(1/\rho_2) - \mathcal{G}_{1/\rho_1}(1/\rho_2)\,\mathcal{G}_{0,1}(1/\rho_2)\\ &- \mathcal{G}_0(\rho_1)\,\mathcal{G}_{0,1/\rho_1}(1/\rho_2) + \mathcal{G}_1(\rho_1)\,\mathcal{G}_{0,1/\rho_1}(1/\rho_2)\\ &+ \mathcal{G}_0(\rho_1)\,\mathcal{G}_{1,1/\rho_1}(1/\rho_2) + \mathcal{G}_0(\rho_2)\,\mathcal{G}_{1,1/\rho_1}(1/\rho_2)\\ &- \mathcal{G}_1(\rho_1)\,\mathcal{G}_{1,1/\rho_1}(1/\rho_2) - \mathcal{G}_{1/\rho_1}(1/\rho_2)\,\mathcal{G}_{0,1}(\rho_1)\\ &- \mathcal{G}_1(\rho_1)\,\mathcal{G}_{0,1}(1/\rho_2) + \mathcal{G}_{1,1/\rho_1}(1/\rho_2)\,\mathcal{G}_1(1/\rho_2)\\ &+ \frac{1}{2}\mathcal{G}_0(\rho_1)\,\mathcal{G}_1(\rho_1)\,\mathcal{G}_{1/\rho_1}(1/\rho_2) - \frac{1}{2}\mathcal{G}_0(\rho_1)\,\mathcal{G}_1(\rho_1)\,\mathcal{G}_1(1/\rho_2)\\ &- \frac{1}{2}\mathcal{G}_0(\rho_2)\,\mathcal{G}_1(\rho_1)\,\mathcal{G}_1(1/\rho_2) - \frac{1}{2}\mathcal{G}_0(\rho_2)\,\mathcal{G}_{1/\rho_1}(1/\rho_2)\,\mathcal{G}_1(1/\rho_2)\\ &+ \mathcal{G}_{0,1}(\rho_1)\,\mathcal{G}_1(1/\rho_2)\end{aligned} \tag{5.2.19}$$

Where $\mathcal{G}$ are the single valued polylogarithms defined in section 4.2.4.2, thus fully specifying $R_7^{(2)}$ in MRK at function level.

For the remainder of this subsection, we will describe how we have obtained eq. (5.2.19). Specializing the discussion of subsection 4.2.4.2 to the seven-particle amplitude, we infer that the relevant class of functions for describing it are single-valued A_2 polylogarithms. Then, if we know the symbol of any function in this space (in this case, the symbol of g), we can find a representative function either by matching it against the symbol of an ansatz made of the basis functions of the same weight, which we call the Lyndon basis

$$\mathcal{L} = \left\{G_{\vec{a}}(\rho_1)|a_i \in \{0,1\}\right\} \cup \left\{G_{\vec{a}}(1/\rho_2)|a_i \in \{0,1,1/\rho_1\}\right\} \tag{5.2.20}$$

As was done in [114], or, even better, by directly integrating the symbol along a given contour.

The actual function may differ from the representative function by beyond-the-symbol terms, namely transcendental constants multiplying lower-weight functions of the same type. Assuming that the only transcendental constants appearing here are multiple zeta values (MZV), we thus form an ansatz for the actual function by augmenting the representative function with all products of MZVs with the bases of lower-weight functions, multiplied by yet-to-be-determined coefficients.

In more detail, we may form separate ansätze for the imaginary and real parts of $R_7^{(2)}$, which at two loops will have weight three and two, respectively. However, by virtue of the property (5.2.1), $R_7^{(2)}$ has vanishing real part in the region where all produced particles have a negative energy. The functions f and $\tilde{f}$ are determined from the six-point amplitude and are known to be real. Hence, the function g must also be real. We thus only need form an ansatz for the real part of g, which after taking into account parity and projectile symmetry, will contain just four undetermined coefficients: a constant ζ_3

term, plus ζ_2 times the three parity and flip symmetric weight-one functions

$$\mathcal{G}_0(\rho_1)+\mathcal{G}_0(1/\rho_2) \quad \mathcal{G}_1(\rho_1)+\mathcal{G}_1(1/\rho_2) \quad \mathcal{G}_{\rho_2}(\rho_1) \tag{5.2.21}$$

The final piece of information we will rely on in order to arrive at a unique answer will be the expected behaviour of the amplitude in soft limits. From (5.2.18) we can easily see that

$$f(0)=\tilde{f}(0)=0 \tag{5.2.22}$$

in agreement with (4.1.50) and the fact that $R_5=0$. Similarly, the three soft limits of the heptagon building block g require

$$g(0,\rho_2)=0 \tag{5.2.23}$$

$$g(\rho_1,\infty)=0 \tag{5.2.24}$$

$$g(\rho_1,\rho_1)=-\tilde{f}(\rho_1) \tag{5.2.25}$$

Taking the limits on the left-hand side is straightforward, after expressing it in the Lyndon basis (5.2.20), the first limit sets the coefficients of the constant ζ_3 terms and the first two logarithms in (5.2.21) to zero. The second limit is related to the first one by target-projectile symmetry, which leaves g invariant, $g(\rho_1,\rho_2)=g(1/\rho_2,1/\rho_1)$. Therefore it will not provide any new information, since our ansatz already respects this symmetry. Finally, the third limit $\rho_2\to\rho_1$ also sets the coefficient of the third logarithm in (5.2.21) to zero, since there it is the only term that becomes divergent. We thus arrive at the unique answer (5.2.19) for the function g, or equivalently the heptagon remainder function in MRK.

5.2.3 All function-level 2-loop MHV amplitudes in MRK

Quite interestingly, from the result of the previous section, we can also obtain *all* 2-loop MHV amplitudes, in any region in which the adjacent particles $k+3,k+4,\ldots,l+3$ have their energy signs flipped. In particular, we will show that in the region in question, we have

$$\frac{R^{(2)}_{N[k+3,l+3]}}{2\pi i}=\sum_{i=k}^{l-1}\Big(2f(v_i)\log\tau_i+\tilde{f}(v_i)\Big)+\sum_{i=k}^{l-2}g(v_i,v_{i+1}) \tag{5.2.26}$$

where the hexagon $f,\tilde{f}$ and heptagon g building blocks have already been provided in (5.2.18)-(5.2.19), and the variables v_i are slight generalizations of the ρ_i variables defined in (4.2.9)-(4.2.11), corresponding to simplicial coordinates on the Riemann sphere with

$l-k+3$ marked points.[11] They are related to the usual transverse cross ratios z_i by

$$z_j = \frac{(v_{j-1}-v_j)(1-v_{j+1})}{(v_{j+1}-v_j)(1-v_{j-1})} \quad j \in \{k,\dots,l-1\} \tag{5.2.27}$$

with the boundary conditions $v_{k-1}=0$, $v_l=\infty$.

The above result has already been established at symbol level in [114],[12] so more precisely here we will prove that if it holds at symbol level, then must necessarily also hold at function level.

It can be seen that by virtue of (5.2.22)-(5.2.25), the right-hand side of (5.2.26) has correct soft limit behaviour, namely it reduces to the same function with one leg less. Therefore if this factorization is to break down beyond symbol level, it can only be through terms that vanish in the soft limit.

From the results of subsection (5.2.1) we know that the 2-loop remainder function has a vanishing real part for any N, so the only factorization-violating beyond-the-symbol terms are transcendental constants or weight-1 functions. Given that the latter are single-valued, they can never turn into transcendental constants in the soft limit, and therefore one cannot add transcendental constants to the right-hand side of (5.2.26) without violating soft limits.

With the weight-1 SVMPLs remaining as the only allowed beyond-the symbol terms not captured in (5.2.26), we will now show that there exists no linear combination thereof that vanishes in all soft limits, and therefore they too should be absent. Forgetting target-projectile symmetry momentarily, the weight-1 SVMPLs that can appear in N-point scattering in MRK will be

$$\log|v_i| \quad \log|1-v_i| \quad \log|v_i-v_j| \quad i<j=k,\dots,l-1 \tag{5.2.28}$$

This is a consequence of the fact that in the regions we are considering, the multi-Regge limit is described by a configuration of $l-k+3$ points $\mathbf{x}_i$ (out of a total of $N-2$) in $\mathbb{CP}^1$, which can only have singularities when two of the points coincide, i.e. of the form $\log(\mathbf{x}_i-\mathbf{x}_j)$, plus complex conjugates. Equation (5.2.28) then follows from single-valuedness and the fact that the v_i coordinates are a particular set of simplicial coordinates parametrizing this space, in absolute analogy to (4.2.9) for the $k=1$, $l=N-4$ case.

Soft limits prohibit logarithms of adjacent v_i, $\log|v_{i-1}-v_i|$, to appear, since for each i they will be the only ones that diverge in the $v_{i-1}=v_i$ limit, with all remaining terms being finite. Then $\log|v_{i-2}-v_i|$ are also prohibited, since in the same limit they will be the only

[11] That is, for the long cut with $k=1$, $l=N-4$ the two sets of variables coincide, $v_i=\rho_i$.

[12] Note however that we have modified the conventions slightly: $z_i=-w_{i+3}$ and $v_i^{\text{here}}=-v_{i+3}^{\text{there}}$.

ones that reduce to $\log|v_{i-1}-v_i|$, an independent function whose coefficient should vanish. By extending the argument to differences of v_i with larger and larger separation, we thus sequentially exclude all $\log|v_i-v_j|$ from appearing as extra beyond-the-symbol terms not captured by (5.2.26).

We are thus left with the logarithms of the first two types appearing in eq. (5.2.28). All of them but $\log|1-v_1|$ can be similarly eliminated by the $v_1=0$ limit, namely they will either diverge, or remain independent functions. And going back to any soft limit not involving v_1 will also force $\log|1-v_1|$ to be absent, since it will remain nonzero.

Concluding our analysis, we have shown that there exist no beyond-the-symbol terms respecting soft limits that can be added to the right hand side of eq. (5.2.26) for $N \geq 7$. Therefore the interesting NLLA factorization structure observed at symbol level also holds at function level, and the latter equation accurately describes the MRK of 2-loop MHV amplitudes with any number of points N, in terms of the functions f, $\tilde{f}$ and g.

5.3 Extracting the NLO central emission block

Let us now combine the knowledge of seven-gluon amplitudes in MRK we have gathered so far, namely the function-level 2-loop MHV case of section 5.2, and the dispersion integral governing any helicity configuration to all loops (5.1.37)-(5.1.38). By matching the perturbative two-loop result to the weak coupling expansion of the dispersion integral we determine a main result of this paper: the (rescaled) central emission block (5.1.41) to next-to-leading order. This is a result which cannot be obtained from the well-studied six-gluon amplitudes.

We start by presenting the result, and describe the details of our calculation in the following subsections. If we denote the perturbative expansion of the rescaled central emission block as

$$\tilde{C}^{+}(\nu_1,n_1,\nu_2,n_2)=\tilde{C}^{(0)}(\nu_1,n_1,\nu_2,n_2)+a\tilde{C}^{(1)}(\nu_1,n_1,\nu_2,n_2)+\mathcal{O}(a^2) \tag{5.3.1}$$

we find the result for the $\mathcal{O}(a)$ correction to the central emission block:

$$\begin{aligned}\frac{\tilde{C}^{(1)}(\nu_1,n_1,\nu_2,n_2)}{\tilde{C}^{(0)}(\nu_1,n_1,\nu_2,n_2)}=&\frac{1}{2}\big[DE_1-DE_2+E_1E_2+\tfrac{1}{4}(N_1+N_2)^2+V_1V_2 \\ &+(V_1-V_2)\big(M-E_1-E_2\big)+2\zeta_2+i\pi(V_2-V_1-E_1-E_2)\big]\end{aligned} \tag{5.3.2}$$

Here we normalized to the known leading order result, translated to our conventions [102],

$$\tilde{C}^{(0)}(\nu_1,n_1,\nu_2,n_2)=\frac{\Gamma\big(1-i\nu_1-\frac{n_1}{2}\big)\Gamma\big(1+i\nu_2+\frac{n_2}{2}\big)\Gamma\big(i\nu_1-i\nu_2-\frac{n_1}{2}+\frac{n_2}{2}\big)}{\Gamma\big(i\nu_1-\frac{n_1}{2}\big)\Gamma\big(-i\nu_2+\frac{n_2}{2}\big)\Gamma\big(1-i\nu_1+i\nu_2-\frac{n_1}{2}+\frac{n_2}{2}\big)} \tag{5.3.3}$$

In the above, we have expressed the answer in terms of the hexagon BFKL building blocks [110]

$$\begin{aligned} E(\nu,n) &= -\frac{1}{2}\frac{|n|}{\nu^2+\frac{n^2}{4}} + \psi\left(1+i\nu+\frac{|n|}{2}\right) + \psi\left(1-i\nu+\frac{|n|}{2}\right) - 2\psi(1) \\ V(\nu,n) &\equiv \frac{i\nu}{\nu^2+\frac{n^2}{4}} \qquad N(\nu,n) = \frac{n}{\nu^2+\frac{n^2}{4}} \qquad D_\nu = -i\partial/\partial\nu \end{aligned} \tag{5.3.4}$$

with the shorthand $E_1 = E(\nu_1, n_1)$ etc, as well as a new quantity involving mixed polygamma functions

$$M(\nu_1,n_1,\nu_2,n_2) = \psi(i(\nu_1-\nu_2)-\tfrac{n_1-n_2}{2}) + \psi(1-i(\nu_1-\nu_2)-\tfrac{n_1-n_2}{2}) - 2\psi(1) \tag{5.3.5}$$

Note however that when one changes the integration variables from angular momenta ν_i to integrability-wise more natural rapidities u_i [109] (taking into account that our ν_i differ with the ones used in the latter reference by a factor of $1/2$)

$$\nu_i = u_i + a\frac{iV_i}{2} + O(a^2)\,, \tag{5.3.6}$$

then any dependence on mixed polygamma functions drops out. In other words they only appear when we expand the arguments of the gamma functions in the inverse transformation from the u_i to the ν_i.

We may readily check that our expression (5.3.1)-(5.3.5) indeed obeys the $\mathcal{O}(a)$ expansion of the exact bootstrap conditions (5.1.46)-(5.1.49). For completeness, let us also mention the weak coupling expansion of the BFKL eigenvalue [91, 99], hexagon measure [101, 110, 140] and NMHV helicity flip kernel [103, 109, 131] (see also the last paper for all-loop expressions of these quantities)

$$-\omega(\nu,n) = aE - \frac{a^2}{4}\left(D^2E - 2VDE + 4\zeta_2 E + 12\zeta_3\right) + \mathcal{O}(a^3)\,, \tag{5.3.7}$$

$$\tilde{\Phi}(\nu,n) = \frac{\Phi_{\text{reg}}(\nu,n)}{\nu^2+\frac{n^2}{4}} = \frac{1}{\nu^2+\frac{n^2}{4}}\left[1 - \frac{a}{2}\left(E^2 + \frac{3}{4}N^2 + \frac{\pi^2}{3}\right) + \mathcal{O}(a^2)\right]\,, \tag{5.3.8}$$

$$H(\nu,n) = \frac{\nu-\frac{in}{2}}{\nu+\frac{in}{2}}\left[1 - \frac{a}{2}NV + \mathcal{O}(a^2)\right] \tag{5.3.9}$$

Plugging these formulas back into (5.1.37)-(5.1.38) or (5.1.51), we may obtain predictions for the heptagon to NLLA at higher loop orders. We detail how to evaluate the relevant integrals to obtain explicit expressions in momentum space in section 5.4. Finally, we may invert (5.1.41)-(5.1.42) in order to obtain equivalent perturbative expansions for the

$\chi^{\pm}$ and C^{+} building blocks of the BFKL approach,[13]

$$\begin{aligned}\chi^{+}(\nu,n) &= \sqrt{\frac{\tilde{\Phi}(\nu,n)}{H(\nu,n)}} = \frac{1}{\nu-\frac{in}{2}}\left[1-\frac{a}{4}\left(E^2+\frac{3}{4}N^2-NV+\frac{\pi^2}{3}\right)+\mathcal{O}(a^2)\right]\\ \chi^{-}(\nu,n) &= \chi^{+}(\nu,n)H(\nu,n) = \frac{1}{\nu+\frac{in}{2}}\left[1-\frac{a}{4}\left(E^2+\frac{3}{4}N^2+NV+\frac{\pi^2}{3}\right)+\mathcal{O}(a^2)\right]\end{aligned} \tag{5.3.10}$$

and

$$\begin{aligned}C^{+}(\nu_1,n_1,\nu_2,n_2) = &-\frac{\Gamma\left(1-i\nu_1-\frac{n_1}{2}\right)\Gamma\left(i\nu_2+\frac{n_2}{2}\right)\Gamma\left(i\nu_1-i\nu_2-\frac{n_1}{2}+\frac{n_2}{2}\right)}{\Gamma\left(1+i\nu_1-\frac{n_1}{2}\right)\Gamma\left(-i\nu_2+\frac{n_2}{2}\right)\Gamma\left(1-i\nu_1+i\nu_2-\frac{n_1}{2}+\frac{n_2}{2}\right)}\times\\ \times&\left[1+a\left(\frac{\tilde{C}^{(1)}}{\tilde{C}^{(0)}}-\tfrac{1}{4}(E_1^2+E_2^2+N_1V_1-N_2V_2)-\tfrac{3}{16}(N_1^2+N_2^2)-\zeta_2\right)+\mathcal{O}(a^2)\right]\end{aligned} \tag{5.3.11}$$

where in the first line of (5.3.10) we picked the branch $\sqrt{\left(\nu-\frac{in}{2}\right)^2}=\nu-\frac{in}{2}$.

5.3.1 Building the Fourier-Mellin representation

Here we would like to describe a procedure that can take us from the amplitude in multi-Regge kinematics to its corresponding Fourier-Mellin (FM) representation. As we have seen in previous chapters, in multi-Regge kinematics the amplitude exhibits divergent logarithms which take the form of powers of the $\log\tau_i$, whose coefficients are SVMPLs in the variables z_i. When investigating the heptagon amplitudes we find it useful to define

$$\hat{z}_2 = 1/z_2 \tag{5.3.12}$$

so that the target-projectile symmetry becomes simply $z_1 \leftrightarrow \hat{z}_2$.

The first step in our analysis is to focus solely on the holomorphic part of the heptagon amplitude in MRK, by taking all $\bar{z}_i \to 0$ as well as regularising any logarithms $\log\bar{z}_i \to 0$. That is for a single valued function $F(z,\bar{z})$ we define the holomorphic part by

$$F^{h}(z) = F(z,0)\Big|_{\log\bar{z}\to 0} \tag{5.3.13}$$

We can reconstruct the full kinematic dependence of the latter with the help of the single-valued map, cf. (4.2.50). One is then left with a five-letter A_2 or $\mathcal{M}_{0,5}$ alphabet for the holomorphic part of the form

$$\{z_1,\hat{z}_2,1-z_1,1-\hat{z}_2,1-z_1-\hat{z}_2\} \tag{5.3.14}$$

[13] Note that here we have redefined $\chi^{\pm}$ and C^{+}, compared to e.g. [38], as follows: $\chi^{\pm}_{\text{here}} = i[\chi^{\pm}_{\text{there}}+\mathcal{O}(a)]$, and $C^{+}_{\text{here}} = -[C^{+}_{\text{there}}+\mathcal{O}(a)]$.

In addition to restricting to the holomorphic part we also focus on the terms in the Taylor expansion of the amplitude with strictly positive powers of z_1 and $\hat{z}_2$. In other words we decompose terms in the perturbative expansion of the amplitude in a similar manner to that shown in eq. (5.1.51) and keep only the final term corresponding to $(2\pi i)\tilde{f}_{h_1 h_2 h_3}$. Concretely, this amounts to taking the symbol expression for $A = e^{R_7 + i\delta_7}\Big|_{L \text{ loops}}$ and forming the combination

$$\mathcal{A}^{(L)}(z_1, \hat{z}_2) = A^h(z_1, \hat{z}_2) - A^h(z_1, 0) - A^h(0, \hat{z}_2) + A^h(0, 0) \tag{5.3.15}$$

in order to remove contributions that reduce to lower-point objects. We remind the reader that the superscript h refers to taking the holomorphic part as in eq. (5.3.13). The quantity $\mathcal{A}^{(L)}$ contains all the information necessary to construct the Fourier-Mellin representation we require.

The holomorphic part $\mathcal{A}^{(L)}(z_1, \hat{z}_2)$ will have logarithmic branch cuts around $z_1 = 0$ and $\hat{z}_2 = 0$, in addition to exhibiting the large logarithms $\log \tau_i$ associated with taking the multi-Regge limit. We may render such branch cuts explicit by employing the shuffle relations to obtain each function as a polynomial in $\log z_1$ and $\log \hat{z}_2$ with coefficients which are analytic around $z_1 = 0$ and $\hat{z}_2 = 0$ respectively. Thus we obtain an expression for the holomorphic part of the amplitude in MRK of the form

$$\mathcal{A}^{(L)}(z_1, \hat{z}_2) = \sum_{p,q,r,s} \log^p \tau_1 \log^q \tau_2 \log^r z_1 \log^s \hat{z}_2 f_{pqrs}(z_1, \hat{z}_2)\,, \tag{5.3.16}$$

where $f_{pqrs}(z_1, \hat{z}_2)$ are linear combinations of polylogarithms which are analytic at $z_1 = \hat{z}_2 = 0$.

For each analytic function $f_{p,q,r,s}(z_1, \hat{z}_2)$ we now Taylor expand around the origin $z_1 = \hat{z}_2 = 0$. We may do this simply by employing the following general formula for the Taylor expansion about $z = 0$ of a G-function from e.g. [116]

$$G_{0^{n_r} a_{i_r} \ldots 0^{n_1} a_{i_1}}(z) = \sum \frac{(-1)^r}{m_1^{n_1+1} \ldots m_r^{n_r+1}} \left[\frac{z}{a_{i_1}}\right]^{m_1} \left[\frac{z}{a_{i_2}}\right]^{m_2 - m_1} \ldots \left[\frac{z}{a_{i_r}}\right]^{m_r - m_{r-1}}, \tag{5.3.17}$$

where the nested summation is performed over the region $1 \leq m_1 < \ldots < m_r$ and the $a_{i_r} \neq 0$. Here 0^n denotes a length n sequence of 0. These nested sums actually form their own Hopf algebra, with two closely related representations

$$S(n; m_1, \ldots, m_j; x_1, \ldots, x_j) = \sum_{n \geq i_1 \geq i_2 \geq \ldots \geq i_j \geq 1} \frac{x_1^{i_1}}{i_1^{m_1}} \ldots \frac{x_j^{i_j}}{i_j^{m_j}}\,, \tag{5.3.18}$$

$$Z(n; m_1, \ldots, m_j; x_1, \ldots, x_j) = \sum_{n \geq i_1 > i_2 > \ldots > i_j > 0} \frac{x_1^{i_1}}{i_1^{m_1}} \ldots \frac{x_j^{i_j}}{i_j^{m_j}}\,, \tag{5.3.19}$$

with the generalized harmonic numbers corresponding to the special case $S(k;m;1) = Z(k;m;1)$. Where the multiplication is given by the quasi-shuffle product which may be determined by repeated application of the following identity

$$\sum_{i,j}^{n} a_{ij} = \sum_{i<j}^{n} a_{ij} + \sum_{j<i}^{n} a_{ij} + \sum_{i}^{n} a_{ii} \tag{5.3.20}$$

The coproduct of the coalgebra is given by deconcatenation of the entries

$$\Delta Z(n;m_1,\ldots,m_j;x_1,\ldots,x_j) = \sum_{k=0}^{j} Z(n;m_{k+1},\ldots,m_j;x_{k+1},\ldots,x_j) \otimes Z(n;m_1,\ldots,m_k;x_1,\ldots,x_k) \tag{5.3.21}$$

With a similar expression for the S-sums. The final element of the Hopf algebra is the antipode $\mathcal{S}$ which due to the simple nature of the coproduct together with the quasi-shuffle algebra is given by

$$\mathcal{S}(Z(n;m_1,\ldots,m_j;x_1,\ldots,x_j) = (-1)^j S(n;m_j,\ldots,m_1;x_j,\ldots,x_1) \tag{5.3.22}$$

In the previous chapter we showed that we may build single valued polylogarithms in a purely algebraic way by applying the map (4.2.50) to any holomorphic polylogarithm. Analogously we would like to define a similar map that will generate single-valued Z-sums, in the sense that they provide the integrand for the Fourier-Melin transform of single valued polylogarithms.

$$\boldsymbol{s} = \mu(\tilde{\mathcal{S}} \otimes \mathrm{id})\Delta \tag{5.3.23}$$

However before we may define the modified antipode we need to analytically continue the nested sums to accept complex arguments. That is we will write each nested sum with an integer upper limit as a combination of sums up to infinity.

$$\begin{aligned} Z(n;m_1,\ldots,m_j;x_1,\ldots,x_j) = &\sum_{k=0}^{j}(-1)^k Z(\infty;m_{k+1},\ldots,m_j;x_{k+1},\ldots,x_j) \\ &\times \sum_{i_1,\ldots i_k=0}^{\infty} \frac{x_1^{n+i_1}}{(n+i_1)^{m_1}} \cdots \frac{x_k^{n+i_1+\ldots+i_k}}{(n+i_1+\ldots+i_k)^{m_k}} \end{aligned} \tag{5.3.24}$$

With a similar expression for the S- sums, which apart from the obvious change from Z to S will differ in the lower limit for the indices i.e. in the bottom line the indices would start at 1 instead of 0. Note that for the special case where all the $x_i = 1$, the sums in the first line reduce to the multiple zeta values $Z(\infty;m_1,\ldots,m_j;1,\ldots,1) = \zeta(m_1,\ldots,m_j)$. Having defined the analytically continued sums we may finally write the the action of

the modified antipode appearing in the map (5.3.23)

$$\tilde{\mathcal{S}}\left(Z\left(\frac{n}{2}+i\nu-1;\dots\right)\right)=(-1)^{\sum_i m_i}\mathcal{S}\left(Z\left(\frac{n}{2}-i\nu;\dots\right)\right) \tag{5.3.25}$$

Note that the particular nature of the argument was chosen aposteriori to match the arguments we will be encountering in the Fourier-Mellin integral. By putting all these elements together we may write a self contained expression for the the single-valued version of a Z-sum

$$\begin{aligned}\boldsymbol{s}\left(Z\left(\frac{n}{2}+i\nu-1;\dots\right)\right)=\sum_{k=0}^{j}(-1)^{\sum_{i=k+1}^{j}(m_i+1)}S\left(\frac{n}{2}-i\nu;m_j,\dots,m_{k+1};\bar{x}_j,\dots,\bar{x}_{k+1}\right)\\ \times Z\left(\frac{n}{2}+i\nu-1;m_1,\dots,m_k;x_1,\dots,x_k\right)\end{aligned} \tag{5.3.26}$$

Thus before we can continue we must first write our functions in terms of Z-sums. While the formula (5.3.17) provides the explicit Taylor expansions, we empirically find that f_{pqrs} is always decomposable into sums of the following, much simpler, type involving only simple Z-sums of depth one

$$\begin{aligned}H_{k_1,k_2,\{r_i\},\{s_i\},\{t_i\}}(z_1,z_2)=\sum_{n_1,n_2>0}\left[\frac{z_1^{n_1}\hat{z}_2^{n_2}}{n_1^{k_1}n_2^{k_2}}\frac{\Gamma(n_1+n_2)}{\Gamma(1+n_1)\Gamma(1+n_2)}\right.\\ \left.\left(\prod_i Z(n_1-1;r_i;1)\right)\left(\prod_i Z(n_2-1;s_i;1)\right)\left(\prod_i Z(n_1+n_2-1;t_i;1)\right)\right]\end{aligned} \tag{5.3.27}$$

For the special case of depth one sums the analytic continuation is given by

$$Z(n;r;1)=\sum_{i=1}^{n}\frac{1}{i^r}=\sum_{i=1}^{\infty}\frac{1}{i^r}-\frac{1}{(i+n)^r}=\frac{(-1)^{r-1}}{(r-1)!}\left(\psi^{(r-1)}(n+1)-\psi^{(r-1)}(1)\right) \tag{5.3.28}$$

Where $\psi^{(r)}(z)$ is the polygamma function. The total weight of the representation (5.3.27) is

$$k_1+k_2+\sum_i r_i+\sum_i s_i+\sum_i t_i \tag{5.3.29}$$

We stress that the fact that the functions f_{pqrs} are always expressible as linear combinations of terms of the form (5.3.27) is not at all trivial. For low weights it is possible to use various binomial and harmonic sum identities to go from the general Taylor expansion to the reduced form (5.3.27). Unfortunately this requires cancellations among sums that do not have a simple closed form, and it becomes increasingly intractable at higher weights. However it is simple to explicitly evaluate the Taylor expansions (5.3.17) up to a finite order and thus generate enough terms to reduce the f_{pqrs} to the form (5.3.27) by means of an ansatz and linear algebra. This requires building the vector space spanned by the sums of the form (5.3.27) for each weight required.

The reason why we are interested in solving the linear problem to arrive at the form (5.3.27) is two-fold. Firstly, it gives us a double infinite sum that is reminiscent of the BFKL LLA form. Note that this structure is not automatic just from using the form (5.3.17). Secondly, the reduced sum (5.3.27) is particularly well suited to expressing its single-valued completion through a Fourier-Mellin integral. We would like to stress the last point, since the conjecture for the single valued Z-sums (5.3.26) has been tested only for polylogarithms where all the $x_i = 1$.

To find a Fourier-Mellin representation for the single-valued completion of each of the f_{pqrs} we begin by specifying the following prescription to be applied to the summations in terms of the form (5.3.27)

$$\sum_{n_1,n_2>0} \frac{z_1^{n_1}\hat{z}_2^{n_2}}{n_1^{k_1}n_2^{k_2}} \to \sum_{-\infty<n_1,n_2<\infty} \int_{-\infty}^{\infty} \frac{d\nu_1}{2\pi} \int_{-\infty}^{\infty} \frac{d\nu_2}{2\pi} \frac{z_1^{i\nu_1+\frac{n_1}{2}} \bar{z}_1^{i\nu_1-\frac{n_1}{2}} \hat{z}_2^{i\nu_2+\frac{n_2}{2}} \hat{\bar{z}}_2^{i\nu_2-\frac{n_2}{2}}}{(i\nu_1+\frac{n_1}{2})^{k_1}(i\nu_2+\frac{n_2}{2})^{k_2}} \tag{5.3.30}$$

Here the contours of integration should be taken to be slightly below the real axes in ν_1 and ν_2. Next we specify how to continue the harmonic sums

$$\begin{aligned} &Z(n_j-1;r;1) \to \\ &\frac{(-1)^{r-1}}{(r-1)!}\Big[\psi^{(r-1)}\big(\tfrac{n_j}{2}+i\nu_j\big)+(-1)^{r-1}\psi^{(r-1)}\big(1+\tfrac{n_j}{2}-i\nu_j\big)-2\delta_{r,\mathrm{odd}}\psi^{(r-1)}(1)\Big] \end{aligned} \tag{5.3.31}$$

Finally we provide a prescription for the binomial coefficients

$$\begin{aligned} \frac{\Gamma(n_1+n_2)}{\Gamma(1+n_1)\Gamma(1+n_2)} &\to \frac{\Gamma(\frac{n_1}{2}-i\nu_1)\Gamma(\frac{n_2}{2}-i\nu_2)\Gamma(i\nu_1+i\nu_2+\frac{n_1}{2}+\frac{n_2}{2})}{\Gamma(1+i\nu_1+\frac{n_1}{2})\Gamma(1+i\nu_2+\frac{n_2}{2})\Gamma(1-i\nu_1-i\nu_2+\frac{n_1}{2}+\frac{n_2}{2})} \\ &= \frac{\Gamma(-\frac{n_1}{2}-i\nu_1)\Gamma(-\frac{n_2}{2}-i\nu_2)\Gamma(i\nu_1+i\nu_2-\frac{n_1}{2}-\frac{n_2}{2})}{\Gamma(1+i\nu_1-\frac{n_1}{2})\Gamma(1+i\nu_2-\frac{n_2}{2})\Gamma(1-i\nu_1-i\nu_2-\frac{n_1}{2}-\frac{n_2}{2})} \\ &= \frac{\tilde{C}^{(0)}(\nu_1,n_1,-\nu_2,-n_2)}{\big(i\nu_1+\frac{n_1}{2}\big)\big(i\nu_1-\frac{n_1}{2}\big)\big(i\nu_2+\frac{n_2}{2}\big)\big(i\nu_2-\frac{n_2}{2}\big)} \end{aligned} \tag{5.3.32}$$

The equality of the first and second lines above holds for integer n_1 and n_2 due to the following identity obeyed by the Gamma function

$$\frac{\Gamma\left(x-\frac{k}{2}\right)}{\Gamma\left(1-x-\frac{k}{2}\right)} = (-1)^k \frac{\Gamma\left(x+\frac{k}{2}\right)}{\Gamma\left(1-x+\frac{k}{2}\right)} \tag{5.3.33}$$

The combination of gamma functions provides poles at $\nu_j = -i\left(\frac{n_j}{2}+h\right)$ that recover the holomorphic part for $h = 0$ by closing the contours in the lower half-planes. It is easy to see that for the holomorphic pole the right hand side of (5.3.31) reduces to their initial quantities once we make use of (5.3.28) and similarly for (5.3.32).

Once the prescription is applied one may then evaluate the non-holomorphic residues

and verify that they correctly reproduce the corresponding terms in the expansion of the initial single-valued function. This procedure has been applied and verified for the two-loop heptagon amplitude in MRK as well as many other single-valued polylogarithms.

The final step is to promote the power of $\log z_1$ and $\log \hat{z}_2$ to their single-valued versions $\log|z_1|^2$ and $\log|\hat{z}_2|^2$ and then absorb the log terms into the integrand by writing them as derivatives

$$\log^n|z|^2 \int_{-\infty}^{\infty} \frac{d\nu}{2\pi} z^{i\nu+\frac{n}{2}} \bar{z}^{i\nu-\frac{n}{2}} F(\nu,n) = \int_{-\infty}^{\infty} \frac{d\nu}{2\pi} (-i)^n \frac{\partial^n}{\partial\nu^n} \left(z^{i\nu+\frac{n}{2}} \bar{z}^{i\nu-\frac{n}{2}} \right) F(\nu,n) \quad (5.3.34)$$

Then we may use integration by parts, ignoring surface terms, to shift the derivatives onto the rest of the integrand $F(\nu,n)$. By construction this operation does not spoil the single valuedness of the integrand. However this can also be seen from the fact that the structure on the RHS of (5.3.31) is closed with respect to derivatives. This is immediately evident except for the Gamma functions, for which we have

$$\begin{aligned}(-i)\frac{\partial}{\partial\nu}\frac{\Gamma(\frac{n}{2}-i\nu)}{\Gamma(1+i\nu+\frac{n}{2})} &= -\frac{\Gamma(\frac{n}{2}-i\nu)}{\Gamma(1+i\nu+\frac{n}{2})}\left(\psi(\tfrac{n}{2}-i\nu)+\psi(1+i\nu+\tfrac{n}{2})\right)\\ &= -\frac{\Gamma(\frac{n}{2}-i\nu)}{\Gamma(1+i\nu+\frac{n}{2})}\left(\psi(1+\tfrac{n}{2}-i\nu)+\psi(i\nu+\tfrac{n}{2})+\frac{1}{\frac{n}{2}+i\nu}-\frac{1}{\frac{n}{2}-i\nu}\right)\end{aligned} \quad (5.3.35)$$

The derivative increases the order of the holomorphic pole and is consistent with recovering $\log|z|^2$ from the contour integration. Note that manipulating the arguments of the polygamma functions introduces only rational terms in the integrand and thus does not spoil single-valuedness. Furthermore the derivative of the gamma functions with mixed arguments is already of the single-valued form and requires no rational terms. Altogether this makes it easy to express the integrand in terms of the D, N, V, E, M basis.

It remains to note that to return to the variable z_2 instead of $\hat{z}_2 = 1/z_2$ we simply replace $(\nu_2, n_2) \to (-\nu_2, -n_2)$ in the integrand. In expressions written in the D, N, V, E, M basis this amounts to simply replacing D_2, N_2 and V_2 with $(-D_2)$, $(-N_2)$ and $(-V_2)$.

Applying the above procedure to the finite part of the two-loop heptagon amplitude in MRK yields the correction to the integrand $\tilde{\Phi}\tilde{C}^{+}\tilde{\Phi}$, and by dividing with the known expansion (5.3.8), we arrive at the expression (5.3.2)-(5.3.3). Similarly applying the procedure to the symbol of the amplitude in MRK, obtained from the results of [11], yields the NNLO central emission vertex (up to beyond-the-symbol terms). We will analyse the NNLO results in subsequent setions. Since the intermediate expressions in these calculations can be slightly cumbersome, in the next section we give a worked example of all the steps we have outlined here on a single weight-three polylogarithm.

5.3.2 A worked example

We provide here a demonstration of the entire algorithm on a simple weight-three SVMPL whose holomorphic part admits a representation of the form (5.3.27). We begin with a function with only positive powers of z_1 and $\hat{z}_2$ in its Taylor expansion,

$$G_{1-z_1,1-z_1,0}(\hat{z}_2) - G_{1,1,0}(\hat{z}_2) = \log(\hat{z}_2)[G_{1-z_1,1-z_1}(\hat{z}_2) - G_{1,1}(\hat{z}_2)] - G_{1-z_1,0,1-z_1}(\hat{z}_2) + G_{1,0,1}(\hat{z}_2) - G_{0,1-z_1,1-z_1}(\hat{z}_2) + G_{0,1,1}(\hat{z}_2) \tag{5.3.36}$$

On the RHS of (5.3.36) we have made the logarithmic branch cut at $\hat{z}_2 = 0$ explicit by shuffling out the trailing zeros.

By comparing the explicit sum representation for (5.3.36) against terms of the form (5.3.27) of weight three, we find we can write

$$G_{1-z_1,1-z_1,0}(\hat{z}_2) - G_{1,1,0}(\hat{z}_2) = \sum_{n_1,n_2>0} z_1^{n_1}\hat{z}_2^{n_2}\frac{\Gamma(n_1+n_2)}{\Gamma(1+n_1)\Gamma(1+n_2)}\left[Z(n_2-1;1;1)\Big(\log\hat{z}_2 - \frac{1}{n_2}\Big) - Z(n_2-1;2;1)\right] \tag{5.3.37}$$

Note that the example chosen can be expressed in terms of harmonic polylogarithms and thus the form on the RHS of (5.3.37) is easy to derive. We emphasise again that it is not always simple to derive such a form and in general we have to resort to comparing against an ansatz of terms of the form (5.3.27).

We now pass to the FM representation for the single-valued completion. Following the prescription in (5.3.31) we have

$$\begin{aligned}
&\mathcal{G}_{1-z_1,1-z_1,0}(\hat{z}_2) - \mathcal{G}_{1,1,0}(\hat{z}_2) \\
&\quad= \sum_{-\infty<n_1,n_2<\infty}\int_{-\infty}^{\infty}\frac{d\nu_1}{2\pi}\int_{-\infty}^{\infty}\frac{d\nu_2}{2\pi} z_1^{i\nu_1+\frac{n_1}{2}}\bar{z}_1^{i\nu_1-\frac{n_1}{2}}\hat{z}_2^{i\nu_2+\frac{n_2}{2}}\bar{\hat{z}}_2^{i\nu_2-\frac{n_2}{2}} \\
&\qquad\qquad\times\Big[\log|\hat{z}_2|^2 I_1(\nu_1,\nu_2,n_1,n_2) - I_2(\nu_1,\nu_2,n_1,n_2)\Big] \\
&\quad= \sum_{-\infty<n_1,n_2<\infty}\int_{-\infty}^{\infty}\frac{d\nu_1}{2\pi}\int_{-\infty}^{\infty}\frac{d\nu_2}{2\pi} z_1^{i\nu_1+\frac{n_1}{2}}\bar{z}_1^{i\nu_1-\frac{n_1}{2}}\hat{z}_2^{i\nu_2+\frac{n_2}{2}}\bar{\hat{z}}_2^{i\nu_2-\frac{n_2}{2}} \\
&\qquad\qquad\times\big[i\partial_{\nu_2} I_1(\nu_1,\nu_2,n_1,n_2) - I_2(\nu_1,\nu_2,n_1,n_2)\big]
\end{aligned} \tag{5.3.38}$$

For convenience we have split the integrand into two pieces

$$\begin{aligned} I_1(\nu_1,\nu_2,n_1,n_2) &= \hat{C}^{(0)}(\nu_1,-\nu_2,n_1,-n_2)\Big[\psi\left(\tfrac{n_2}{2}+i\nu_2\right)+\psi\left(1+\tfrac{n_2}{2}-i\nu_2\right)-2\psi(1)\Big] \\ I_2(\nu_1,\nu_2,n_1,n_2) &= \hat{C}^{(0)}(\nu_1,-\nu_2,n_1,-n_2)\Big[-\psi^{(1)}\left(\tfrac{n_2}{2}+i\nu_2\right)+\psi^{(1)}\left(1+\tfrac{n_2}{2}-i\nu_2\right) \\ &\quad + \frac{1}{\frac{n_2}{2}+i\nu_2}\big(\psi\left(\tfrac{n_2}{2}+i\nu_2\right)+\psi\left(1+\tfrac{n_2}{2}-i\nu_2\right)-2\psi(1)\big)\Big] \end{aligned} \tag{5.3.39}$$

where $\hat{C}^{(0)}$ is the quantity which arises from the prescription (5.3.32)

$$\hat{C}^{(0)}(\nu_1,-\nu_2,n_1,-n_2) = \frac{\tilde{C}^{(0)}(\nu_1,-\nu_2,n_1,-n_2)}{\big(i\nu_1+\frac{n_1}{2}\big)\big(i\nu_1-\frac{n_1}{2}\big)\big(i\nu_2+\frac{n_2}{2}\big)\big(i\nu_2-\frac{n_2}{2}\big)} \tag{5.3.40}$$

After performing the differentiation we obtain the desired Fourier-Mellin integrand for the single-valued polylogarithm (5.3.38), here expressed in terms of the N, V, E, M basis.

$$\hat{C}^{(0)}(\nu_1,-\nu_2,n_1,-n_2)\times(E_2+V_2)\left(E_2-M-\frac{1}{2}N_2\right) \tag{5.3.41}$$

Finally, returning to the z_2 variable instead of $\hat{z}_2$ means flipping the sign of ν_2 and n_2. In the above expression this means that $\tilde{C}^{(0)}$ and M acquire arguments with canonical signs and the signs of N_2 and V_2 get flipped.

5.4 Higher-loop NLLA predictions

In the previous section, we used the 2-loop MHV heptagon amplitude in the multi-Regge limit, that we promoted from symbol to function in section 5.2, in order to extract the NLO central emission block (5.3.2)-(5.3.3). Here, we will use this result, together with the analogous weak coupling expansion of the BFKL eigenvalue (5.3.7), hexagon impact factor (5.3.8), and helicity flip kernel (5.3.9), to compute explicit analytic expressions for the heptagon at higher loops in NNLA, from the dispersion integral (5.1.37)-(5.1.38).

Let us start by introducing some useful notation to express the perturbative expansion of the amplitude. At weak coupling, it is evident that the dispersion integral naturally organizes itself into a double expansion in the coupling and in the large logarithms $\log\tau_k$. Separating the coefficients of this expansion into real and imaginary parts, we may define them as

$$\begin{aligned} \mathcal{R}_{h_1,h_2,h_3}(\tau_1,z_1,\tau_2,z_2)\,e^{i\delta_7(z_1,z_2)} &= 1+2\pi i\sum_{\ell=1}^{\infty}\sum_{i_1,i_2=0}^{\ell-1}a^\ell\left(\prod_{k=1}^{2}\frac{1}{i_k!}\log^{i_k}\tau_k\right) \\ &\quad\times\left(\tilde{g}^{(\ell;i_1,i_2)}_{h_1,h_2,h_3}(z_1,z_2)+2\pi i\,\tilde{h}^{(\ell;i_1,i_2)}_{h_1,h_2,h_3}(z_1,z_2)\right) \end{aligned} \tag{5.4.1}$$

Note in particular that we have defined the perturbative coefficients not of $\mathcal{R}_{h_1,h_2,h_3}$

alone, but with its combination with a phase, that is equal to the dispersion integral.

The LLA contribution amounts to the coefficients with $i_1 + i_2 = \ell - 1$, for which it is easy to show that

$$\tilde{h}^{(\ell;i_1,\ell-1-i_1)}_{h_1,h_2,h_3} = 0 \quad i_1 = 0, \ldots, \ell - 1 \tag{5.4.2}$$

In this section, we will be obtaining new results for the coefficients $\tilde{g}^{(\ell;i_1,i_2)}_{h_1,h_2,h_3}$ and $\tilde{h}^{(\ell;i_1,i_2)}_{h_1,h_2,h_3}$ with $i_1 + i_2 = \ell - 2$, or in other words the NLLA contribution.

We will work in the region $z_1 \ll 1$, $z_2 \gg 1$, for which we saw in section 5.1.2 that it is advantageous to deform the contour of the dispersion integral before the weak coupling expansion, so that the latter becomes equal to (5.1.51), with

$$\begin{aligned}\tilde{f}_{h_1 h_2 h_3} =& \frac{a}{2} \sum_{n_1,n_2=-\infty}^{\infty} (-1)^{n_1+n_2} \left(\frac{z_1}{\bar{z}_1}\right)^{\frac{n_1}{2}} \left(\frac{z_2}{\bar{z}_2}\right)^{\frac{n_2}{2}} \int \frac{d\nu_1 d\nu_2}{(2\pi)^2} |z_1|^{2i\nu_1} |z_2|^{2i\nu_2} \tilde{\Phi}(\nu_1, n_1) \tilde{\Phi}(\nu_2, n_2) \\ & \times e^{-L_1 \omega(\nu_1,n_1) - L_2 \omega(\nu_2,n_2)} I^{h_1}(\nu_1, n_1) \tilde{C}^{h_2}(\nu_1, n_1, \nu_2, n_2) \bar{I}^{h_3}(\nu_2, n_2)\end{aligned} \tag{5.4.3}$$

and the integration contour goes below (above) the poles on the real axis for ν_1 (ν_2), as shown in figure 5.1.4. The perturbative coefficients (5.4.1) will be a linear combination of the respective coefficients of all the terms in the right-hand side of (5.1.51). However for the hexagon amplitudes $\mathcal{R}_{h_1 h_2}$ they have already been obtained up to at least 8 loops to NLLA [110, 141], and more generally the holomorphic part may be evaluated in terms of harmonic polylogarithms with the method of [71], see also [112, 120]. So we only need to focus on the last term in (5.4.1), that contains the genuine heptagon contributions.

As we will detail in the next sections, we will compute (5.4.3) with the help of two complementary methods. First, we will use nested sum evaluation algorithms, which are easier to apply for the heptagon to high loop order. Then, we will also rely on Fourier-Mellin convolutions, which are particularly suited for applying to higher-point amplitudes.

Before we proceed with the description of our methods, let us briefly summarize the checks we have performed on our results. First of all, we have confirmed that the two methods yield the same expressions for the $2 \to 5$ amplitude to NLLA, through 3 loops in the $\mathcal{R}_{--++}$ NMHV helicity configuration, and through 4 loops in the MHV case. Up to the same loop orders, we have also checked that under soft limits, the amplitude in any helicity configuration reduces to the known $2 \to 4$ amplitude [110]. Finally, at 3 and 4 loops we have compared the symbol of our expressions for the MHV amplitude with the MRK limit [142] of the known symbol in general kinematics [11, 12], finding perfect agreement.

5.4.1 A nested sum evaluation algorithm

After we expand the integrand in (5.4.3) at weak coupling, we close the integration contour below (above) the real axis for ν_1 (ν_2), and use Cauchy's theorem to express it as a sum over the enclosed residues, with the infinite semicircles giving a vanishing contribution due to $|z_1|^{2|\mathrm{Im}(\nu_1)|}, |z_2|^{-2|\mathrm{Im}(\nu_2)|} \to 0$ in the region $z_1 \ll 1$, $z_2 \gg 1$ we are considering.

In reality, the fact that amplitudes in the multi-Regge limit, and thus also $\tilde{f}_{h_1h_2h_3}$, are expressible in terms of SVMPLs, allows us to compute the latter by only considering the subset of poles $\nu_1 = -in_1/2$ and $\nu_2 = -in_2/2$, with $n_1 > 0$ and $n_2 < 0$, which is equal to its holomorphic part, in the sense of (5.3.13), with respect to the variables $z_1, 1/z_2$ [120][14]

$$\begin{aligned}\tilde{f}^h_{h_1h_2h_3} =& \frac{a}{2}\sum_{n_1=1}^{\infty}\sum_{n_2=-1}^{-\infty}\operatorname*{Res}_{\nu_i=\frac{-in_i}{2}}\Big(z_1^{i\nu_1+\frac{n_1}{2}} z_2^{i\nu_2+\frac{n_2}{2}}\tilde{\Phi}(\nu_1,n_1)\tilde{\Phi}(\nu_2,n_2)\times \\ &\times e^{-L_1\omega(\nu_1,n_1)}e^{-L_2\omega(\nu_2,n_2)}I^{h_1}(\nu_1,n_1)\tilde{C}^{h_2}(\nu_1,n_1,\nu_2,n_2)\bar{I}^{h_3}(\nu_2,n_2)\Big)\end{aligned} \tag{5.4.4}$$

That is, in what follows we will focus on computing $\tilde{f}^h_{h_1h_2h_3}$, and then recover $\tilde{f}_{h_1h_2h_3}$ with the help of the single-valued map (4.2.50) at the very end[15].

After we substitute (5.3.2)-(5.3.3), (5.3.7)-(5.3.9) in (5.4.4), extracting the residues becomes in practice very easy after we also use the symmetry of (5.3.2)-(5.3.3) in order to replace $n_i \to -n_i$ there. In particular, it is manifest that only the rational denominators and $\Gamma(-i\nu_1+\frac{n_1}{2})$ and $\Gamma(i\nu_2-\frac{n_2}{2})$ will have poles, whereas all polygamma functions will have positive arguments. In this manner, and after we set $k=|n_1|, l=|n_2|$, (5.4.4) becomes a sum of terms of the general form

$$\sum_{k,l=1}^{\infty}\frac{z_1^k}{k^{r_1}}\frac{z_2^{-l}}{l^{r_2}}\frac{\Gamma(k+l)}{\Gamma(1+k)\Gamma(1+l)}\prod_{m_i,m_i',m_i''}\psi^{(m_i)}(k+1)\psi^{(m_i')}(l+1)\psi^{(m_i'')}(k+l)\,, \tag{5.4.5}$$

for different choices of integers $r_1, r_2, m_i, m_i', m_i'' \geq 0$, not necessarily different from each other, times factors that do not depend on the summation variables.

Next, we express the polygamma functions in terms of S- or Z-sums [123], via

$$\begin{aligned}\psi(k+1) \equiv \psi^{(0)}(k+1) &= -\gamma_E + S(k;1;1)\\ \psi^{(m-1)}(k+1) &= (-1)^m(m-1)![\zeta_m - S(k;m;1)]\end{aligned} \tag{5.4.6}$$

where ζ_m the Riemann zeta function, $\gamma_E = -\psi(1) \simeq 0.577$ the Euler-Mascheroni constant, and This substitution allows us to use the quasi-shuffle algebra relations of S- or

[14] Note that we need to multiply the integrand with $-(2\pi i)^2$ due to the orientation of our contours.

[15] To be precise, this equality holds if $\tilde{f}_{h_1h_2h_3}$ is a pure function. Subtleties when this is not the case are discussed at the end of this section.

Z-sums, in order to express their products with the same outer summation index, in terms of linear combinations thereof.

As we will see very shortly, it proves advantageous to replace $\psi^{(m_i')}(l+1)$, $\psi^{(m_i)}(k+1)$ by S- and $\psi^{(m_i'')}(k+l)$ by Z-sums respectively. After soaking up the gamma function dependence of (5.4.5) into a rational factor times a binomial coefficient

$$\binom{k+l}{k} = \frac{\Gamma(k+l+1)}{\Gamma(k+1)\Gamma(l+1)} \tag{5.4.7}$$

shifting the summation variable $l \to j = k + l$, and partial fractioning with respect to k, the latter formula splits into terms that look like

$$\sum_{j=1}^{\infty} \frac{(z_2)^{-j}}{j^{n_1}} Z(j-1; n_2, \ldots; 1, \ldots, 1) \times \\ \times \sum_{k=1}^{j-1} \binom{j}{k} \frac{(z_1 z_2)^k}{k^{n_1'}} S(k; n_2', \ldots; 1, \ldots, 1) \frac{1}{(j-k)^{n_1''}} S(j-k; n_2'', \ldots; 1, \ldots, 1) \tag{5.4.8}$$

where we also extended the summation range to include $j = 1$, since $Z(j-1; \ldots)$ vanishes there.

Very crucially, the sum on the second line of (5.4.8) can be evaluated in terms of Z-sums with the help of algorithm D of [123], which has already been implemented in `GiNaC` [143] and `FORM` [144] symbolic computation frameworks, as part of the `nestedsums` library [124] or `XSummer` package [145] respectively. We make use of the former by directly interfacing it to `Mathematica`, in particular by sequentially calling the `transcendental_sum_type_D` and `Ssum_to_Zsum` commands for each sum of this type.

The Z-sums we obtain in this manner may have outer summation index $j - a$ for $a \geq 0$, which from the definition (5.3.19) is equivalent to the statement that in reality the outermost summation range should be $j \geq \max(1, a)$. They may also come with $(z_1 z_2)^j$ or $(1 - z_1 z_2)^j$ prefactors, products/powers of $(j - b)$ denominators with $a - 1 \geq b \geq 0$, as well as factors that do not depend on j. After shifting the summation variable $j \to i = j - \max(1, a) + 1$ for each different a, and partial fractioning in i, we reduce all terms (5.4.8) in our expression for $\tilde{f}^h_{h_1 h_2 h_3}$ into simple sums of the form

$$\sum_{i=1}^{\infty} \frac{x^i}{(i+c)^{n_1}} Z(i+o-1; n_2, \ldots; 1, \ldots, 1) Z(i-1; n_2', \ldots; x_2, \ldots)\,. \tag{5.4.9}$$

We then synchronize the Z-sums, namely remove the offset o of the first of the two, by

recursive definition of the identity[16]

$$Z(i+o-1;m_1,...;x_1,...) \\ = Z(i-1;m_1,...;x_1,...) + \sum_{j=0}^{o-1} x_1^j \frac{x_1^i}{(i+j)^{m_1}} Z(i-1+j;m_2,...;x_2,...) \quad (5.4.11)$$

and once again eliminate any products with quasi-shuffle algebra relations. Similarly, we remove the offset from the denominators with the help of

$$\sum_{i=1}^{\infty} \frac{x^i}{(i+c)^m} Z(i-1,...) = \sum_{i=1}^{\infty} \frac{x^{i-1}}{(i+c-1)^m} Z(i-1,...) - \sum_{i=1}^{\infty} \frac{x^i}{(i+c)^m} \frac{x_1^i}{i^{m_1}} Z(i-1,m_2,...)\,, \quad (5.4.12)$$

or

$$\sum_{i=1}^{\infty} \frac{x^i}{(i+c)^m} = \frac{1}{x} \sum_{i=1}^{\infty} \frac{x^i}{(i+c-1)^m} - \frac{1}{c^m} \quad (5.4.13)$$

if no Z-sums are present. After these steps, the expression (5.4.4) for $\tilde{f}^h_{h_1h_2h_3}$ may be readily evaluated in terms of multiple polylogarithms, thanks to the definition

$$\mathrm{Li}_{m_1,...,m_j}(x_1,\ldots,x_j) = \sum_{i=1}^{\infty} \frac{x_1^i}{i^{m_1}} Z(i-1;m_2,\ldots,m_j;x_2,\ldots,x_j) \quad (5.4.14)$$

The procedure we have described for evaluating the Fourier-Mellin integrand is the same for both the MHV and NMHV case, the only difference being that in the former the powers of the denominators r_1, r_2 are strictly positive, whereas in the latter they can also be zero. This difference is closely related to the fact that the MHV amplitude is a pure function, whereas the NMHV ones also contain some rational factors in the z_i variables.

It is only with respect to these rational factors, that some additional care is needed when considering the projection to the holomorphic part (5.4.4), since this will also set the antiholomorphic rational factors to constants, possibly zero. Particularly for the $\mathcal{R}_{-++}$ NMHV amplitude, it was shown in [38] that the rational factors contain no $\bar{z}_i$ dependence, so similarly to the MHV case, the full $\tilde{f}_{-++}$ may also be obtained with the help of (4.2.50), when the rational factors are considered as constants with respect to the single-valued map.

Using the techniques we have described in this section, we have obtained the MHV $\mathcal{R}_{+++}$ and NMHV $\mathcal{R}_{-++}$ heptagons to NLLA through 5 and 4 loops respectively. The treatment of $\mathcal{R}_{+-+}$ will follow in the next section, with the method of convolutions.

[16] Since in this case the Z-sums with offset have their origin in the terms $\psi^{(m_i'')}(k+l)$ in (5.4.5), we could have alternatively left them in this form, shift their arguments with the identity

$$\psi^{(n)}(z+1) = \psi^{(n)}(z) + (-1)^n n! z^{-n-1} \quad (5.4.10)$$

at this point, and only then use (5.4.6)-(5.3.19) to express them as Z-sums.

5.4.2 Evaluation by Fourier-Mellin convolutions

In this section, we shall use the convolution-based method introduced in the previous chapter to compute amplitudes in MRK, and adapt it for computations beyond LLA.

So as to render the computation by convolutions more transparent, we will define the first few orders of the following building blocks separately,

$$\begin{aligned}\omega(\nu,n) &= -a(E_{\nu,n} + aE^{(1)}_{\nu,n} + \mathcal{O}(a^2))\\ \chi^{\pm}(\nu,n) &= \chi^{\pm}_0(\nu,n)(1 + a\kappa^{\pm}_1(\nu,n) + \mathcal{O}(a^2))\\ C^{\pm}(\nu,n,\mu,m) &= C^{\pm}_0(\nu,n,\mu,m)(1 + ac^{\pm}_1(\nu,n,\mu,m) + \mathcal{O}(a^2))\end{aligned} \tag{5.4.15}$$

and we define

$$\begin{aligned} E_i &\equiv E_{\nu_i n_i}, & \chi^{\pm}_{0,i} &\equiv \chi^{\pm}_0(\nu_i,n_i), & \kappa^{\pm}_{0,i} &\equiv \kappa^{\pm}_0(\nu_i,n_i)\\ C^{\pm}_{0,ij} &\equiv C^{\pm}_0(\nu_i,n_i,\nu_j,n_j), & c^{\pm}_{1,ij} &\equiv c^{\pm}_1(\nu_i,n_i,\nu_j,n_j), \end{aligned} \tag{5.4.16}$$

We also define a shorthand for the product of leading order impact factors and central emission blocks at seven points,

$$\varpi_7 \equiv \varpi_7^{h_1h_2h_3} = \chi^{h_1}_{0,1} C^{h_2}_{0,12} \chi^{-h_3}_{0,2} \tag{5.4.17}$$

where we drop explicit dependence on the helicities.

Then at LLA, (i.e. for $i_1 + i_2 = \ell - 1$) we find

$$\tilde{g}^{(\ell;i_1,i_2)}_{h_1h_2h_3}(z_1,z_2) = \frac{1}{2}\,\mathcal{F}_2\left[\varpi_7 E_1^{i_1} E_2^{i_2}\right] \tag{5.4.18}$$

where

$$\mathcal{F}_2[F] = \sum_{n_1,n_2=-\infty}^{\infty} \left(\frac{z_1}{\bar{z}_1}\right)^{\frac{n_1}{2}} \left(\frac{z_2}{\bar{z}_2}\right)^{\frac{n_2}{2}} \int_{-\infty}^{+\infty} \frac{d\nu_1}{2\pi}\frac{d\nu_2}{2\pi}\ |z_1|^{2i\nu_1}|z_2|^{2i\nu_2} F \tag{5.4.19}$$

denotes the two-fold Fourier-Mellin transform. At NLLA, (i.e. for $i_1+i_2 = \ell-2$), we write the perturbative coefficients as

$$\begin{aligned}\tilde{g}^{(\ell;i_1,i_2)}_{h_1h_2h_3}(z_1,z_2) &= \sum_{j=1}^{2} i_j \tilde{g}^{j;(\ell;i_1,i_2)}_{h_1h_2h_3}(z_1,z_2) + \sum_{j=1}^{3} \tilde{g}^{(\ell;i_1,i_2)}_{j;h_1h_2h_3}(z_1,z_2) + P^{(\ell;i_1,i_2)}_{h_1h_2h_3}(z_1,z_2) \text{ and}\\ \tilde{h}^{(\ell;i_1,i_2)}_{h_1h_2h_3}(z_1,z_2) &= \sum_{j=1}^{2} \tilde{h}^{j;(\ell;i_1,i_2)}_{h_1h_2h_3}(z_1,z_2) + \sum_{j=1}^{3} \tilde{h}^{(\ell;i_1,i_2)}_{j;h_1h_2h_3}(z_1,z_2) + Q^{(\ell;i_1,i_2)}_{h_1h_2h_3}(z_1,z_2)\end{aligned} \tag{5.4.20}$$

where P and Q are due to the contributions from the first three terms of eq. (5.1.51),

explicitly given by

$$P^{(\ell;i_1,i_2)}_{h_1h_2h_3}(z_1,z_2) = \delta_{i_2,0}\,\tilde{g}^{(\ell;i_1)}_{h_1h_2}(z_1) + \delta_{i_1,0}\,\tilde{g}^{(\ell;i_2)}_{h_2h_3}(z_2) \tag{5.4.21}$$

$$\begin{aligned} Q^{(\ell;i_1,i_2)}_{h_1h_2h_3}(z_1,z_2) = {}& \delta_{i_2,0}\left(\tilde{h}^{(\ell;i_1)}_{h_1h_2}(z_1) - \frac{1}{4}\mathcal{G}_0(z_2)\tilde{g}^{(\ell-1;i_1)}_{h_1h_2}(z_1)\right) \\ & + \delta_{i_1,0}\left(\tilde{h}^{(\ell;i_2)}_{h_2h_3}(z_2) + \frac{1}{4}\mathcal{G}_0(z_1)\tilde{g}^{(\ell-1;i_2)}_{h_2h_3}(z_2)\right) \\ & + \delta_{\ell,2}\frac{1}{16}\mathcal{G}_0(z_1)\mathcal{G}_0(z_2) \end{aligned} \tag{5.4.22}$$

and we have also introduced *corrected perturbative coefficients* describing different contributions to the expansion of the purely heptagonal $\tilde{f}_{h_1h_2h_3}$ term of the latter equation. Perturbative coefficients with an additional upper index correspond to insertions of the NLO corrections to the BFKL eigenvalue and perturbative coefficients with an additional lower index correspond to insertions of NLO corrections to the impact factors or central emission blocks. Then these corrected perturbative coefficients are given by

$$\begin{aligned} \tilde{g}^{j;(\ell;i_1,i_2)}_{h_1h_2h_3}(z_1,z_2) &= \frac{1}{2}\,\mathcal{F}_2\left[\varpi_7\,E_1^{i_1-\delta_{1j}}E_2^{i_2-\delta_{2j}}E_j^{(1)}\right] \\ \tilde{g}^{(\ell;i_1,i_2)}_{1;h_1h_2h_3}(z_1,z_2) &= \frac{1}{2}\,\mathcal{F}_2\left[\varpi_7\,E_1^{i_1}E_2^{i_2}\kappa_{1,1}^{h_1}\right] \\ \tilde{g}^{(\ell;i_1,i_2)}_{3;h_1h_2h_3}(z_1,z_2) &= \frac{1}{2}\,\mathcal{F}_2\left[\varpi_7\,E_1^{i_1}E_2^{i_2}\kappa_{1,2}^{-h_3}\right] \\ \tilde{g}^{(\ell;i_1,i_2)}_{2;h_1h_2h_3}(z_1,z_2) &= \frac{1}{2}\,\mathcal{F}_2\left[\varpi_7\,E_1^{i_1}E_2^{i_2}\Re\left(c_{1,12}^{h_2}\right)\right] \end{aligned} \tag{5.4.23}$$

and

$$\begin{aligned} \tilde{h}^{j;(\ell;i_1,i_2)}_{h_1h_2h_3}(z_1,z_2) &= -\frac{1}{4}\,\mathcal{F}_2\left[\varpi_7\,E_1^{i_1}E_2^{i_2}E_j\right] \\ \tilde{h}^{(\ell;i_1,i_2)}_{j;h_1h_2h_3}(z_1,z_2) &= 0, \quad j\in\{1,3\} \\ \tilde{h}^{(\ell;i_1,i_2)}_{2;h_1h_2h_3}(z_1,z_2) &= \frac{1}{4\pi}\,\mathcal{F}_2\left[\varpi_7\,E_1^{i_1}E_2^{i_2}\Im\left(c_{1,12}^{h_2}\right)\right] \end{aligned} \tag{5.4.24}$$

Given the form of the recursion relation

$$g^{(\ell;\ell-1)}_{++} = -\frac{1}{2}\mathcal{F}\left[\chi^{+}_{0,1}E_1^{\ell-1}\chi^{-}_{0,1}\right] = g^{(\ell-1;\ell-2)}_{++} * \mathcal{F}[E] \tag{5.4.25}$$

all we need in addition to the integration kernels $\mathcal{E}$ (4.3.4) and $\mathcal{H}$ (4.4.3), which we may extend to higher orders by using (5.3.9), is a starting point for the recursion. Starting from the two-loop NLLA amplitude (5.2.15) we computed the perturbative coefficients $\tilde{g}$ and $\tilde{h}$ through four loops in the MHV case and through three loops in all other helicity configurations.

We will conclude this section by commenting on some details of this computation. At NLLA, we had to introduce the terms $|z_1|^{2\pi i\Gamma}e^{R_6(z_2)+i\delta_6(z_2)}$ (the same discussion applies

to the corresponding terms with $z_1 \leftrightarrow z_2$) and $|z_1|^{2\pi i\Gamma}/|z_2|^{2\pi i\Gamma}$ in (5.1.35) in order to avoid a pinching of our integration contour. Since only the term $\tilde{f}_{+++}$ corresponds to the two-fold Fourier-Mellin integral, when relating different perturbative coefficients via the recursion (5.4.25), these additional terms should be subtracted before performing convolutions and added back afterwards. At NLLA, for example, the term $|z_1|^{2\pi i\Gamma}/|z_2|^{2\pi i\Gamma}$ only contributes at two loops, as it is independent of the large logarithms $\log \tau_i$. Naively, convoluting over this term will introduce additional terms at three-loop order that should not be there. It turns out, however, that these terms only interfere with our computations when raising the loop order of the real part from two- to three loops. This is easy to see when analysing how the individual parts of these terms behave under convolutions with the $\mathcal{E}$ and $\mathcal{H}$ kernels. Expanding the extra terms in a only yields powers of logarithms $\mathcal{G}_0(z_i)^k$, with $0 \leq k \leq \ell$ and six-point perturbative coefficients $\tilde{g}$ and $\tilde{h}$ at any given order ℓ, and though NLLA, we can limit our analysis to $k \leq 2$. When convoluted with $\mathcal{E}$ and $\mathcal{H}$, these logarithms yield the following results.

$$\begin{aligned} 1 * \mathcal{E}(z) &= 0 \\ \mathcal{G}_0(z) * \mathcal{E}(z) &= 0 \\ \mathcal{G}_0(z)^2 * \mathcal{E}(z) &= -4\zeta_3 \end{aligned} \tag{5.4.26}$$

$$\begin{aligned} 1 * \mathcal{H}^{(0)}(z) &= 1 \\ \mathcal{G}_0(z) * \mathcal{H}^{(0)}(z) &= \mathcal{G}_0(z) \\ \mathcal{G}_0(z)^2 * \mathcal{H}^{(0)}(z) &= \mathcal{G}_0(z)^2 \end{aligned} \tag{5.4.27}$$

$$1 * \mathcal{H}^{(1)}(z) = \mathcal{G}_0(z) * \mathcal{H}^{(1)}(z) = \mathcal{G}_0(z)^2 * \mathcal{H}^{(1)}(z) = 0 \tag{5.4.28}$$

Let us now have a look at the two different kinds of extra terms individually. We will start with the fraction term $|z_1|^{2\pi i\Gamma}/|z_2|^{2\pi i\Gamma}$, which has no dependence on the large logarithms $\log \tau_i$ and therefore, at NLLA, should only affect the 2-loop amplitude. As convolutions with $\mathcal{E}(z_i)$ and $\mathcal{H}^{(1)}(z_i)$ both appear with a factor of a in the Fourier-Mellin integrand, they both result in a higher-order contribution and should evaluate to zero when convoluted with the extra term. Furthermore, it does not depend on the helicity configuration of the amplitude, which suggests that it should be invariant under convolution with leading order helicity flip kernels $\mathcal{H}^{(0)}(z_i)$. Expanding the term in a, we find

$$\begin{aligned} \frac{|z_1|^{2\pi i\Gamma}}{|z_2|^{2\pi i\Gamma}} = 1 &+ \frac{a}{2} i\pi(\mathcal{G}_0(z_1) - \mathcal{G}_0(z_2)) - \frac{a^2}{12} i\pi^3(\mathcal{G}_0(z_1) - \mathcal{G}_0(z_2)) \\ &+ \frac{a^2}{8}\pi^2 \left(\mathcal{G}_0(z_1)^2 - 2\mathcal{G}_0(z_1)\mathcal{G}_0(z_2) + \mathcal{G}_0(z_2)^2\right) + \mathcal{O}(a^3) \end{aligned} \tag{5.4.29}$$

Considering (5.4.26) - (5.4.28), we see immediately that the aforementioned criteria are only violated by the terms $\mathcal{G}_0(z_i)^2$ appearing in the 2-loop real part. We will therefore have to subtract these terms before performing convolutions with $\mathcal{E}(z_i)$.

We will now focus on the extra terms $|z_1|^{2\pi i\Gamma} e^{R_6(z_2)+i\delta_6(z_2)}$ (and the one with $z_1 \leftrightarrow z_2$) containing the 6-point amplitude. At LLA, it contributes only through the exponential $e^{R_6(z_2)+i\delta_6(z_2)}\big|_{LLA}$, which by definition transforms correctly under convolutions with $\mathcal{E}(z_2)$ and $\mathcal{H}(z_2)$. Since the term only comes with large logarithms $\log \tau_2$, it should vanish when convoluting with $\mathcal{E}(z_1)$ so that its presence will not spoil the terms proportional to both large logarithms. Furthermore, it should be invariant under leading order helicity flips $\mathcal{H}^{(0)}(z_1)$ and should vanish under first order helicity flips $\mathcal{H}^{(1)}(z_1)$. This follows again from (5.4.26) - (5.4.28). At NLLA the situation is similar. Here we will again encounter terms that only arise from $e^{R_6(z_2)+i\delta_6(z_2)}\big|_{NLLA}$[17], i.e. the NLLA hexagon contributions. In addition, we will find terms arising from $\log|z_1|^2 e^{R_6(z_2)+i\delta_6(z_2)}\big|_{LLA}$. Since Fourier-Mellin convolutions are also suited for the computation of hexagon NLLA amplitudes, convolutions in z_2 will behave in the desired fashion. Once again, we have to ensure that convolutions in z_1 do not spoil our results, which means that convolutions with $\mathcal{E}(z_1)$ and $\mathcal{H}^{(1)}(z_1)$ should vanish and that leading order helicity flips $\mathcal{H}^{(0)}(z_1)$ should have no impact. This is again given by (5.4.26) - (5.4.28). We see therefore, that these potentially dangerous terms appearing in the amplitude are sufficiently well behaved and we can therefore completely ignore the presence of these terms and perform our convolutions without taking further precautions.

Let us now briefly summarize the previous observations. We have seen that at NLLA, we need to introduce extra terms to our amplitudes that are not part of the two-fold Fourier-Mellin integral (5.1.35) due to the presence of Regge poles. Even though these extra terms contribute to the perturbative coefficients at all orders, we only have to subtract these terms when convoluting $\tilde{h}^{(2;0,0)}_{h_1 h_2}$ with $\mathcal{E}$ in order to raise its loop order. In all other cases, and in particular for all helicity flips, convolutions can be applied directly to the full perturbative coefficients.

5.5 All-order central emission vertex

As mentioned in the previous sections the $N = 6$ case is well studied and all-order expressions are already known for the BFKL eigenvalue ω, measure factor $\tilde{\Phi}$ and helicity-flip kernel H [109]. The only quantity for which the planar hexagon does not provide an all order expression is the central emission vertex $\tilde{C}$. In section 5.3 we determined $\tilde{C}^+$ to NLL accuracy and here we give an all-loop generalisation of that formula, consistent

[17] Note that $e^{R_6(z_2)+i\delta_6(z_2)}$ also contains an extra term that is not part of the Fourier-Mellin transform. This term is the 6-point equivalent of the fractional term we discussed before and all observations apply here, too.

with the known three-loop heptagon data obtained via the cluster bootstrap [11] as well as soft limits on the amplitudes in MRK. In the following we focus on $\tilde{C}^+$ as $\tilde{C}^-$ is related to it by the helicity flip kernels

5.5.1 Central emission vertex at finite coupling

A crucial feature of the following analysis is that at finite coupling it is more appropriate to use the rapidity variable u rather than the variables ν in order to write the integral expression. One may then write a uniform relation for ω and ν both in terms of the rapidity u

$$\omega(u,n) = -4g(\mathbb{Q}\cdot\mathbb{M}\cdot\kappa)_1 \quad \nu = u - 2g(\mathbb{Q}\cdot\mathbb{M}\cdot\tilde{\kappa})_1 \tag{5.5.1}$$

where the subscript $(\dots)_1$ denotes the first element in the enclosed vector and the sources κ and $\tilde{\kappa}$ were obtained in [109] after analytic continuation of the OPE sources to the BFKL regime

$$\kappa_j(u,n) = -\int_0^\infty \frac{dt}{t}\frac{J_j(2gt)}{e^t-1}\phi_j(t;u,n) \tag{5.5.2}$$

and similarly for $\tilde{\kappa}$ with ϕ replaced by $\tilde{\phi}$ where

$$\begin{aligned}\phi_j &= \tfrac{1}{2}\big(e^{t\delta_j^{\text{even}}} - (-1)^j e^{t\delta_j^{\text{odd}}}\big)\cos(ut)e^{-\frac{nt}{2}} - J_0(2gt)\\ \tilde{\phi}_j &= \tfrac{1}{2}\big(e^{t\delta_j^{\text{even}}} + (-1)^j e^{t\delta_j^{\text{odd}}}\big)\sin(ut)e^{-\frac{nt}{2}}\end{aligned} \tag{5.5.3}$$

The matrices $\mathbb{Q}$ and $\mathbb{M}$ are given by

$$\begin{aligned}\mathbb{Q}_{ij} &= \delta_{ij}(-1)^{i+1}i \qquad\qquad \mathbb{M} = (1+\mathbb{K})^{-1}\\ \mathbb{K}_{ij} &= 2j(-1)^{j(i+1)}\int_0^\infty \frac{dt}{t}\frac{J_i(2gt)J_j(2gt)}{e^t-1}\end{aligned} \tag{5.5.4}$$

The impact factor is related to the analytically continued OPE measure μ as follows

$$\frac{d\nu}{2\pi}\tilde{\Phi}(\nu,n) = \frac{du}{2\pi}\mu(u,n) \tag{5.5.5}$$

with the BFKL measure $\mu(u,n)$ given by [109]

$$\mu(u,n) = \frac{g^2(x^+x^- - g^2)e^{A-2f+2\tilde{f}}}{x^+x^-\sqrt{(x^+x^+ - g^2)(x^-x^- - g^2)}} \tag{5.5.6}$$

Here we have

$$f = 2\kappa\cdot\mathbb{Q}\cdot\mathbb{M}\cdot\kappa \qquad \tilde{f} = 2\tilde{\kappa}\cdot\mathbb{Q}\cdot\mathbb{M}\cdot\tilde{\kappa} \tag{5.5.7}$$

and we introduce the Zhukowski variables

$$x^{\pm} = x(u \pm i\tfrac{n}{2}) \qquad x(u) = \tfrac{1}{2}(u + \sqrt{u^2 - 4g^2}) \tag{5.5.8}$$

The quantity A which appears in (5.5.6) is given by

$$A = 2\int_0^\infty \frac{dt}{t}\frac{1 - J_0(2gt)^2}{e^t - 1} - \frac{\pi^2}{4}\Gamma_{\text{cusp}} \tag{5.5.9}$$

Thus we have

$$e^{-A} = 1 + 2g^2\zeta_2 - 7g^4\zeta_4 + O(g^6) \tag{5.5.10}$$

Finally the helicity-flip kernel is simply given by

$$H\big(\nu(u,n), n\big) = \frac{x^-}{x^+} \tag{5.5.11}$$

Here we highlight the factorised structure of the Fourier-Mellin space representation for the heptagon amplitude in multi-Regge kinematics. From the explicit data for the symbol of the MHV heptagon that we have up to three loops we are able to observe patterns which strongly resemble objects that should arise from considering the appropriate analytic continuation of the pentagon transitions of gluon bound states in the OPE of [31]. Specifically we observe that the part of the Mellin integrand (5.1.21)

$$\Psi(\nu_1, n_1, \nu_2, n_2) \equiv \chi_1(\nu_1, n_1) C(\nu_1, n_1, \nu_2, n_2) \chi_2(\nu_2, n_2) \tag{5.5.12}$$

respects a factorisation of the following form

$$\begin{aligned} &\frac{d\nu_1}{2\pi}\frac{d\nu_2}{2\pi}\Psi(\nu_1, n_1, \nu_2, n_2) \\ &= \frac{du_1}{2\pi}\frac{du_2}{2\pi}\mu_1\mu_2 h_1 \hat{h}_2 Z_{12} e^{2(-if_1 + if_2 - f_3 + f_4) - A} C_0 \end{aligned} \tag{5.5.13}$$

The factor C_0 is the leading order Mellin integrand with rapidities u_i as arguments rather than the ν_i. It is expressed purely in terms of gamma functions

$$\begin{aligned} C_0 &= \frac{\Gamma\big(1 - iu_1 - \frac{n_1}{2}\big)\Gamma\big(1 + iu_2 + \frac{n_2}{2}\big)\Gamma\big(iu_1 - iu_2 - \frac{n_1 - n_2}{2}\big)}{\Gamma\big(iu_1 - \frac{n_1}{2}\big)\Gamma\big(-iu_2 + \frac{n_2}{2}\big)\Gamma\big(1 - iu_1 + iu_2 - \frac{n_1 - n_2}{2}\big)} \\ &= \frac{C^+(u_1, n_1, u_2, n_2)}{\chi^-(u_1, n_1)\chi^+(u_2, n_2)} \end{aligned} \tag{5.5.14}$$

Expanding C_0 in terms of the ν_i variables generates terms containing the quantity M, defined in eq. (5.3.5), which encapsulates all dependence on polygammas with $i(\nu_1 - \nu_2)$ in the argument. In particular it correctly generates the term proportional to M in (5.3.2).

The functions f_i are given by

$$\begin{aligned} f_1 &= 2\tilde{\kappa}_1 \cdot \mathbb{Q} \cdot \mathbb{M} \cdot \kappa_2 \qquad & f_2 &= 2\kappa_1 \cdot \mathbb{Q} \cdot \mathbb{M} \cdot \tilde{\kappa}_2 \\ f_3 &= 2\tilde{\kappa}_1 \cdot \mathbb{Q} \cdot \mathbb{M} \cdot \tilde{\kappa}_2 \qquad & f_4 &= 2\kappa_1 \cdot \mathbb{Q} \cdot \mathbb{M} \cdot \kappa_2 \end{aligned} \tag{5.5.15}$$

with $\kappa_i = \kappa(u_i, n_i)$ etc.

With the above definitions we find the perturbative expansion of the exponential factor dependent on the f_i is

$$e^{2(-if_1+if_2-f_3+f_4)} = 1 + g^2(E_1E_2 - E_2V_1 + E_1V_2 + V_1V_2) + O(g^4) \tag{5.5.16}$$

Note that the arguments of the E and V functions are taken as the rapidities u_i, not the ν_i in the above equation. At weak coupling the two variables are related by the expansion of the finite coupling relation (5.5.1)

$$\nu = u + ig^2V(u,n) - \frac{i}{2}g^4(D_u^2V(u,n) + 4\zeta_2V(u,n)) + O(g^6) \tag{5.5.17}$$

The factor Z in (5.5.13) is expressed in terms of the Zhukowski variables $x^{\pm}$

$$Z = \sqrt{\frac{(x_1^-x_2^- - g^2)(x_1^+x_2^+ - g^2)}{(x_1^+x_2^- - g^2)(x_1^-x_2^+ - g^2)}} \tag{5.5.18}$$

The perturbative expansion of Z is as follows,

$$Z = 1 + \frac{1}{2}g^2N_1N_2 - \frac{1}{8}g^4N_1N_2(N_1^2 - N_1N_2 + N_2^2 + 12V_1^2 + 8V_1V_2 + 12V_2^2) + O(g^6) \tag{5.5.19}$$

Once again the arguments of the N_i and V_i in the above equation are taken to be the rapidity variables u_i.

The quantity $\mu(u,n)$ is the measure for the six-point amplitude in multi-Regge kinematics written in rapidity variables. The measure μ has a perturbative expansion of the form

$$\begin{aligned} \mu(u,n) = \frac{1}{u^2 + \frac{n^2}{4}} \Bigg[1 - g^2(E^2 + N^2 + 3V^2 + 2\zeta_2) + g^4 \Bigg(\frac{1}{2}(DE)^2 + ED^2E + \frac{1}{2}E^4 + E^2N^2 \\ + \frac{31}{32}N^4 + 3E^2V^2 + \frac{53}{4}N^2V^2 + 10V^4 + 4E^2\zeta_2 + \frac{3}{2}N^2\zeta_2 + 6V^2\zeta_2 \\ + 12E\zeta_3 + 22\zeta_4 \Bigg) + O(g^6) \Bigg] \end{aligned} \tag{5.5.20}$$

Again, the arguments of the E, N, V appearing above are the rapidity u. Note that

$$\mu(-u,-n) = \mu(u,n) \tag{5.5.21}$$

The remaining factor to be described in (5.5.13) is $h(u,n)$. We write this factor as a product

$$h(u,n) = h_{\text{even}}(u,n)h_{\text{odd}}(u,n) \tag{5.5.22}$$

where

$$h_{\text{even}}(-u,-n) = h_{\text{even}}(u,n) \qquad h_{\text{odd}}(-u,-n) = [h_{\text{odd}}(u,n)]^{-1} \tag{5.5.23}$$

By looking at the symbol level data from the imaginary part of the amplitude up to three loops we determine that

$$h_{\text{even}}(u,n) = 1 + g^2\left(V^2 + \frac{1}{4}N^2\right) - g^4\left(V^4 + 2V^2N^2 + \frac{1}{16}N^4\right) + O(g^6) \tag{5.5.24}$$

while

$$h_{\text{odd}}(u,n) = e^{r(u,n)} \tag{5.5.25}$$

with

$$r(u,n) = g^2DE - \frac{1}{4}g^4D^3E + O(g^6) \tag{5.5.26}$$

In (5.5.24) we have not included various terms proportional to $i\pi$ which come from the real part of the amplitude.

So far we have explored the structure of the expression which we derived from the symbols of the two and three-loop MHV heptagon amplitudes. By inspecting this data we deduced that the Mellin integrand is most naturally expressed in terms of the rapidity variables u_i and furthermore that it decomposes naturally into various factors. Now let us *assume* that the factorisation (5.5.13) found above holds in general and see what we can determine about the form of $h(u,n)$. Consider the residue of $\Psi(\nu_1,n_1,\nu_2,n_2)$ at $\nu_1 = \nu_2$ for $n_1 = n_2$. The residue of Ψ is determined by the residue of the RHS of (5.5.13) for $u_1 = u_2$ when $n_1 = n_2$. Because there are no M type functions in the rapidity variables the only pole comes from the factor $C_0(u_1,n_1,u_2,n_2)$ whose residue is

$$\text{Res}_{u_1=u_2}C_0(u_1,n_2,u_2,n_2) = (-1)^{n_2}i\chi^+(u_2,n_2)\chi^-(u_2,n_2) = -(-1)^{n_2}i\left(u_2^2 + \frac{n_2^2}{4}\right) \tag{5.5.27}$$

We therefore find

$$\begin{aligned}
&\oint_{\nu_1=\nu_2}\frac{d\nu_1}{2\pi}\frac{d\nu_2}{2\pi}\Psi(\nu_1,n_2,\nu_2,n_2)\\
&\quad=\oint_{u_1=u_2}\frac{du_1}{2\pi}\frac{du_2}{2\pi}\Big[\mu(u_1,n_1)\mu(u_2,n_2)h(u_1,n_1)h(-u_2,-n_2)\\
&\qquad\times Z(u_1,n_1,u_2,n_2)e^{-A+2(-if_1+if_2-f_3+f_4)}C_0(u_1,n_1,u_2,n_2)\Big]\\
&\quad=(-1)^{n_2}\frac{du_2}{2\pi}\Big(u_2^2+\frac{n_2^2}{4}\Big)\mu(u_2,n_2)^2h(u_2,n_2)h(-u_2,-n_2)Z(u_2,n_2,u_2,n_2)e^{-A+2(\hat{f}_4-\hat{f}_3)}\\
&\quad=(-1)^{n_2}\frac{du_2}{2\pi}\Big(u_2^2+\frac{n_2^2}{4}\Big)\mu(u_2,n_2)\frac{1}{x_2^+x_2^-}h(u_2,n_2)h(-u_2,-n_2)
\end{aligned}\tag{5.5.28}$$

On the other hand by the general analysis of soft limits described in (5.1.30) we expect that

$$\begin{aligned}
\oint_{\nu_1=\nu_2}\frac{d\nu_1}{2\pi}\frac{d\nu_2}{2\pi}\Psi(\nu_1,n_2,\nu_2,n_2)&=\frac{d\nu_2}{2\pi}(-1)^{n_2}e^{i\pi\omega(\nu_2,n_2)}\tilde{\Phi}(\nu_2,n_2)\\
&=\frac{du_2}{2\pi}\mu(u_2,n_2)(-1)^{n_2}e^{i\pi\omega(\nu_2(u_2),n_2)}
\end{aligned}\tag{5.5.29}$$

Comparing (5.5.28) with (5.5.29) we find

$$h(u_2,n_2)h(-u_2,-n_2)=h_{\text{even}}(u_2,n_2)^2=\frac{x_2^+x_2^-}{u_2^2+\frac{n_2^2}{4}}e^{i\pi\omega(\nu_2(u_2),n_2)}\tag{5.5.30}$$

This then fixes the function $h_{\text{even}}(u,n)$,

$$h_{\text{even}}(u,n)=\sqrt{\frac{x_2^+x_2^-}{u_2^2+\frac{n_2^2}{4}}}e^{\frac{1}{2}[i\pi\omega(\nu_2(u_2),n_2)]}\tag{5.5.31}$$

where now we have included the necessary $i\pi$ terms needed for the soft limit to hold. Note that we may also equivalently write the above formula such that the u-dependence only appears manifestly in the exponent, with the help of the identity

$$\sqrt{\frac{x^+x^-}{u^2+\frac{n^2}{4}}}=e^{\int_0^\infty\frac{dt}{t}[1-J_0(2gt)]\cos(ut)e^{-\frac{n}{2}t}}\tag{5.5.32}$$

The veracity of this identity may be confirmed by e.g. expanding both sides at weak coupling, and most probably it may also be proven at finite coupling along the lines of [27, 146].

Finally, we may resort to the $\nu_1=\pi\Gamma$ exact bootstrap condition in order to constrain the remaining factor $h_{\text{odd}}(u,n)$, which drops out in the above analysis. As explained in appendix C of [109], the only additional complication is that for $n_1=0$, we need two sheets in u_1 in order to cover the entire ν_1 line, and the domain of ν_1 that the finite-

coupling analogue of (5.5.17) covers, does not contain the point $\nu_1 = \pi\Gamma$ we are after. Thus if we wish to enforce the exact bootstrap in question at finite coupling, we need to first analytically continue u_1 to the second sheet.

Since this issue, however, occurs only when expressing the Fourier-Mellin integrand in the u-variables, we can make use of the $\nu_1 = \pi\Gamma$ exact bootstrap condition perturbatively: First of all, in the ν-variables, nothing special happens when taking $n_1 = 0$, e.g. eq.(5.3.7) is still valid for $\omega(\nu_1, 0)$. Furthermore, as we saw in section 5.1.2, taking the residue $\nu_1 = \pi\Gamma$ of the integrand at finite coupling, and then performing the weak coupling expansion, or first expanding the integrand at weak coupling, and then taking the $\nu_1 = 0$ residue, should yield the same result provided $n_2 \neq 0$: The pinching of the contour only occurs for $\nu_1 = \nu_2 = n_1 = n_2 = 0$.

The conclusion is that if we have a candidate, weak coupling expression for the Fourier-Mellin integrand in the u-variables (including $h_{\text{odd}}(u_i, n_i)$), we can translate it to an expression in the ν-variables (now containing $h_{\text{odd}}(\nu_i, n_i)$) with the help of (5.5.17), set $n_1 = 0$, and finally take the $\nu_1 = 0$ residue. This should then be equal to the right-hand side of (5.1.28), expanded at weak coupling.

We make use of this condition by forming a weak-coupling ansatz for $r(u, n)$, consisting of all odd combinations of the single-variable building blocks $V, N, D^i E, i = 0, \ldots$ with undetermined coefficients, and imposing it on the entire Fourier-Mellin integrand. We find that it uniquely fixes the ansatz, up to 3 loops that we tried. More excitingly, we find that the weak-coupling expansion for $h_{\text{odd}}(u, n)$ that we determined, follows from the finite-coupling formula

$$h_{\text{odd}}(u, n) = e^{i \int_0^\infty \frac{dt}{t} \frac{(J_0(2gt)-1)(e^t+1)}{(e^t-1)} \sin(ut) e^{-\frac{n}{2}t} + \pi(u-\nu)} \tag{5.5.33}$$

All in all, and in conventions where we rescale the central emission block so that the Mellin integrand (5.1.21) contains the combination of impact factors of eq.(5.1.24) appearing in the hexagon amplitude,

$$\tilde{\Phi}(\nu_1, n_1)\tilde{C}(\nu_1, n_1, \nu_2, n_2)\tilde{\Phi}(\nu_2, n_2) \equiv \chi_1(\nu_1, n_1) C(\nu_1, n_1, \nu_2, n_2) \chi_2(\nu_2, n_2) = \Psi \tag{5.5.34}$$

the final finite-coupling expression for $\tilde{C}$ is

$$\begin{aligned} \tilde{C}(u_1, n_1, u_2, n_2) =& C_0(u_1, n_1, u_2, n_2) h(u_1, n_1) h(-u_2, -n_2) Z(u_1, n_1, u_2, n_2) \\ & \times e^{2(-if_1 + if_2 - f_3 + f_4) - A} \end{aligned} \tag{5.5.35}$$

Finally, let us summarize the checks of the finite-coupling expression for $\tilde{C}$, or equivalently heptagon Fourier-Mellin integrand, have been performed so far:

- It respects the discrete symmetries $(\nu_1 \leftrightarrow -\nu_2, n_1 \leftrightarrow n_2)$ and $n_i \leftrightarrow -n_i$.

- It agrees with the previously determined Fourier-Mellin transform of $R_7^{(2)}$ at function level, for both the real and the imaginary part.
- It agrees with the imaginary part of the symbol of $R_7^{(3)}$ (full $w_i, \bar{w}_i$ dependence).
- Last but not least, without having used any 4-loop data as input, it indeed matches the holomorphic terms (no $\bar{w}_i$ dependence) for the imaginary part of the symbol of $R_7^{(4)}$.

5.6 Conclusion

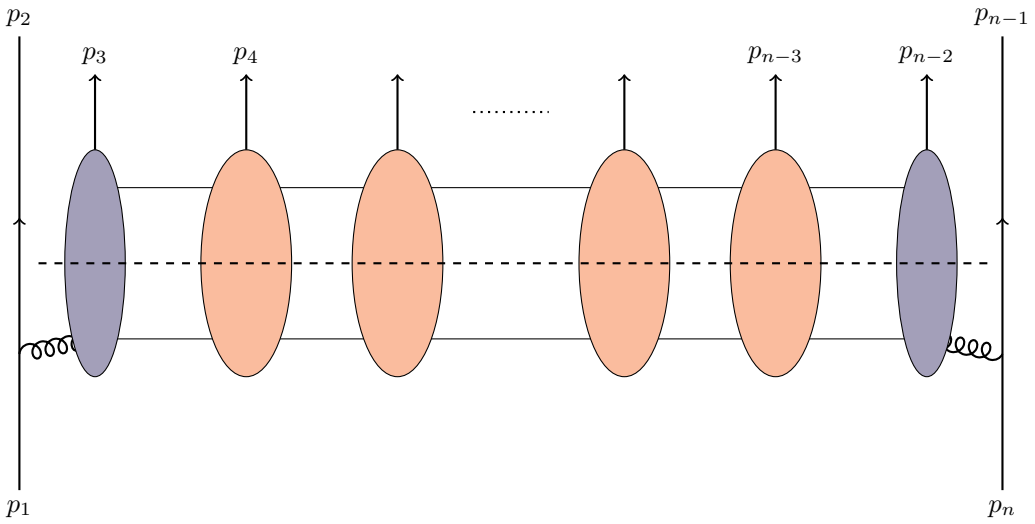


Figure 5.6.1: General n-point factorisation structure of the scattering amplitude in multi-Regge kinematics.

From Chapter 4 onwards we saw how in MRK both perturbative and finite coupling amplitudes in planar $\mathcal{N} = 4$ SYM are simple enough to be determined in all aspects. Because the kinematics are reduced from four to two dimensions the limit is significantly more tractable than general kinematics, both from the point of view of the Wilson loop OPE expansion [28–31, 109] and from the analytic structure of perturbative amplitudes.

This simplification was further reflected in the singularity structure of the amplitudes, and the cluster algebras describing the general kinematics collapsed into two algebras of finite type $A_{n-3} \times A_{n-3}$ which are complex conjugate to each other. It followed that the cluster algebra A_{n-3} was isomorphic to that of $\mathfrak{M}_{0,n}$ the moduli space of Riemann spheres with marked points. Using the well understood structure of iterated integrals on $\mathfrak{M}_{0,n}$ we were able to show that amplitudes in MRK at all orders and all multiplicities are given in terms of single-valued multiple polylogarithms. Furthermore we conjectured that the remainder functions for amplitudes factorise in Fourier-Mellin space and they

take the form

$$\mathcal{R}_N = \prod_{r=1}^{N-5}\left[\sum_{n_r}\left(\frac{z_r}{\bar{z}_r}\right)^{n_r}\int\frac{d\nu_r}{2\pi}|z_r|^{2i\nu_r}\tilde{\Phi}_r e^{-L_r\omega_r}\right] I_1^{h_1}\tilde{C}_{12}^{h_2}\ldots\tilde{C}_{N-6,N-5}^{h_{N-5}}\bar{I}_{N-5}^{h_{N-4}} \tag{5.6.1}$$

By considering soft limits we then showed the validity of the dispersion integral in the region where all the energies of the produced particles have been analytically continued. The Fourier-Mellin transform maps products into convolutions, and it was possible to obtain higher order results and at different helicity configurations by convoluting known results with the BFKL eigenvalue and the helicity flip kernel respectively.

At 6 points we could obtain all order results as all elements appearing in the Fourier-Mellin integral are known at finite coupling due to integrability. At 7-points and beyond however a new quantity called the central emission vertex $\tilde{C}$ appears in the factorisation which was known only to leading order. We were able to use a prescription for deriving Fourier-Mellin integrands from single valued polylogarithms and combine it with existing MHV data at 7-points to obtain higher order corrections to the central emission vertex. Finally by observing the perturbative corrections we were able to conjecture a finite coupling expression for the central emission vertex that is consistent with soft limits and higher order data at 7-points. Thus by incorporating this finite coupling conjecture into (5.6.1) we are able to obtain a dispersion integral valid for all multiplicities, all helicity configurations and at arbitrary perturbative order.

APPENDIX A

2-loop, 5-point integral result

In this appendix we present the solution in MPL form of the differential equation discussed in Section 3. We first define the quantities

$$
\begin{aligned}
p_1 &= \frac{\bar{a}_1 a_2}{a_1} \\
p_2 &= \frac{\bar{a}_1 - a_1}{\bar{a}_1 - 1} \\
p_3 &= \frac{a_1 - a_2 + \bar{a}_1(a_2 - 1)}{a_1 - 1} \\
p_4 &= \frac{a_2 \bar{a}_1 (a_1 - 1)}{a_1(a_2 + \bar{a}_1 - 1) - \bar{a}_1 a_2}
\end{aligned}
\tag{A.0.1}
$$

We give the integrated symbol of the solution along the path beginning at the origin and following $a_1 \to \bar{a_1} \to a_2 \to \bar{a}_2$.

$$
\begin{aligned}
&G(0;a_1)G(1;\bar{a}_1)G(a_1;a_2)G(\bar{a}_1;\bar{a}_2) - G(0;\bar{a}_1)G(1;a_1)G(a_1;a_2)G(\bar{a}_1;\bar{a}_2) + \\
&G(0;a_1)G(0;\bar{a}_1)G(1;a_2)G(a_2;\bar{a}_2) - G(0;\bar{a}_1)G(1;a_1)G(1;a_2)G(a_2;\bar{a}_2) + \\
&G(0;\bar{a}_1)G(1;a_1)G(a_1;a_2)G(p_1;\bar{a}_2) + G(0;\bar{a}_1)G(1;a_1)G(\bar{a}_1;\bar{a}_2)G(p_2;a_2) - \\
&G(0;a_1)G(1;\bar{a}_1)G(\bar{a}_1;\bar{a}_2)G(p_2;a_2) + G(0;\bar{a}_1)G(1;a_1)G(a_2;\bar{a}_2)G(p_2;a_2) - \\
&G(0;a_1)G(1;\bar{a}_1)G(a_2;\bar{a}_2)G(p_2;a_2) - G(0;\bar{a}_1)G(1;a_1)G(p_1;\bar{a}_2)G(p_2;a_2) + \\
&G(0;a_1)G(1;\bar{a}_1)G(p_1;\bar{a}_2)G(p_2;a_2) - G(0;a_1)G(0;\bar{a}_1)G(1;a_2)G(p_3;\bar{a}_2) + \\
&G(0;\bar{a}_1)G(1;a_1)G(1;a_2)G(p_3;\bar{a}_2) - G(0;a_1)G(1;\bar{a}_1)G(a_1;a_2)G(p_3;\bar{a}_2) -
\end{aligned}
$$

$$
\begin{aligned}
&G(0;\bar{a}_1)G(1;a_1)G(p_2;a_2)G(p_3;\bar{a}_2) + G(0;a_1)G(1;\bar{a}_1)G(p_2;a_2)G(p_3;\bar{a}_2) + \\
&G(1;\bar{a}_1)G(\bar{a}_1;\bar{a}_2)G(0,0;a_1) + G(1;a_2)G(a_2;\bar{a}_2)G(0,0;a_1) - G(1;\bar{a}_1)G(p_2;a_2)G(0,0;a_1) - \\
&G(1;\bar{a}_1)G(p_3;\bar{a}_2)G(0,0;a_1) - G(1;a_2)G(p_3;\bar{a}_2)G(0,0;a_1) - G(1;a_1)G(a_1;a_2)G(0,0;\bar{a}_1) + \\
&G(1;a_2)G(a_2;\bar{a}_2)G(0,0;\bar{a}_1) + G(1;a_1)G(p_2;a_2)G(0,0;\bar{a}_1) + G(1;a_1)G(p_3;\bar{a}_2)G(0,0;\bar{a}_1) - \\
&G(1;a_2)G(p_3;\bar{a}_2)G(0,0;\bar{a}_1) + G(0;\bar{a}_1)G(a_1;a_2)G(0,1;a_1) - G(0;\bar{a}_1)G(\bar{a}_1;\bar{a}_2)G(0,1;a_1) + \\
&G(a_1;a_2)G(\bar{a}_1;\bar{a}_2)G(0,1;a_1) - G(\bar{a}_1;\bar{a}_2)G(p_2;a_2)G(0,1;a_1) - G(a_2;\bar{a}_2)G(p_2;a_2)G(0,1;a_1) + \\
&G(p_1;\bar{a}_2)G(p_2;a_2)G(0,1;a_1) - G(a_1;a_2)G(p_3;\bar{a}_2)G(0,1;a_1) + G(p_2;a_2)G(p_3;\bar{a}_2)G(0,1;a_1) - \\
&G(0,0;\bar{a}_1)G(0,1;a_1) + G(0;a_1)G(a_1;a_2)G(0,1;\bar{a}_1) - G(0;a_1)G(\bar{a}_1;\bar{a}_2)G(0,1;\bar{a}_1) - \\
&G(a_1;a_2)G(\bar{a}_1;\bar{a}_2)G(0,1;\bar{a}_1) - G(1;a_2)G(a_2;\bar{a}_2)G(0,1;\bar{a}_1) + G(a_1;a_2)G(p_1;\bar{a}_2)G(0,1;\bar{a}_1) + \\
&G(\bar{a}_1;\bar{a}_2)G(p_2;a_2)G(0,1;\bar{a}_1) + G(a_2;\bar{a}_2)G(p_2;a_2)G(0,1;\bar{a}_1) - G(p_1;\bar{a}_2)G(p_2;a_2)G(0,1;\bar{a}_1) + \\
&G(1;a_2)G(p_3;\bar{a}_2)G(0,1;\bar{a}_1) - G(p_2;a_2)G(p_3;\bar{a}_2)G(0,1;\bar{a}_1) + G(0,0;a_1)G(0,1;\bar{a}_1) + \\
&G(0;a_1)G(0;\bar{a}_1)G(0,1;a_2) - G(0;\bar{a}_1)G(1;a_1)G(0,1;a_2) + G(0;a_1)G(\bar{a}_1;\bar{a}_2)G(0,1;a_2) + \\
&G(0;\bar{a}_1)G(\bar{a}_1;\bar{a}_2)G(0,1;a_2) - G(1;a_1)G(\bar{a}_1;\bar{a}_2)G(0,1;a_2) - G(1;\bar{a}_1)G(\bar{a}_1;\bar{a}_2)G(0,1;a_2) - \\
&G(0;a_1)G(p_3;\bar{a}_2)G(0,1;a_2) - G(0;\bar{a}_1)G(p_3;\bar{a}_2)G(0,1;a_2) + G(1;a_1)G(p_3;\bar{a}_2)G(0,1;a_2) + \\
&G(1;\bar{a}_1)G(p_3;\bar{a}_2)G(0,1;a_2) + G(0,0;a_1)G(0,1;a_2) + G(0,0;\bar{a}_1)G(0,1;a_2) - G(0,1;\bar{a}_1)G(0,1;a_2) + \\
&G(0;\bar{a}_1)G(1;a_1)G(0,a_1;a_2) + G(1;a_1)G(\bar{a}_1;\bar{a}_2)G(0,a_1;a_2) + G(1;\bar{a}_1)G(\bar{a}_1;\bar{a}_2)G(0,a_1;a_2) - \\
&G(1;a_1)G(p_3;\bar{a}_2)G(0,a_1;a_2) - G(1;\bar{a}_1)G(p_3;\bar{a}_2)G(0,a_1;a_2) + G(0,1;\bar{a}_1)G(0,a_1;a_2) - \\
&G(0;\bar{a}_1)G(a_1;a_2)G(1,0;a_1) - G(a_1;a_2)G(\bar{a}_1;\bar{a}_2)G(1,0;a_1) - G(1;a_2)G(a_2;\bar{a}_2)G(1,0;a_1) + \\
&G(a_1;a_2)G(p_1;\bar{a}_2)G(1,0;a_1) + G(0;\bar{a}_1)G(p_2;a_2)G(1,0;a_1) + G(\bar{a}_1;\bar{a}_2)G(p_2;a_2)G(1,0;a_1) + \\
&G(a_2;\bar{a}_2)G(p_2;a_2)G(1,0;a_1) - G(p_1;\bar{a}_2)G(p_2;a_2)G(1,0;a_1) + G(0;\bar{a}_1)G(p_3;\bar{a}_2)G(1,0;a_1) + \\
&G(1;a_2)G(p_3;\bar{a}_2)G(1,0;a_1) - G(p_2;a_2)G(p_3;\bar{a}_2)G(1,0;a_1) - G(0,1;a_2)G(1,0;a_1) + \\
&G(0,a_1;a_2)G(1,0;a_1) + G(0;a_1)G(\bar{a}_1;\bar{a}_2)G(1,0;\bar{a}_1) + G(a_1;a_2)G(\bar{a}_1;\bar{a}_2)G(1,0;\bar{a}_1) - \\
&G(0;a_1)G(p_2;a_2)G(1,0;\bar{a}_1) - G(\bar{a}_1;\bar{a}_2)G(p_2;a_2)G(1,0;\bar{a}_1) - G(a_2;\bar{a}_2)G(p_2;a_2)G(1,0;\bar{a}_1) + \\
&G(p_1;\bar{a}_2)G(p_2;a_2)G(1,0;\bar{a}_1) - G(0;a_1)G(p_3;\bar{a}_2)G(1,0;\bar{a}_1) - G(a_1;a_2)G(p_3;\bar{a}_2)G(1,0;\bar{a}_1) + \\
&G(p_2;a_2)G(p_3;\bar{a}_2)G(1,0;\bar{a}_1) + G(0;a_1)G(a_2;\bar{a}_2)G(1,a_1;a_2) + G(0;\bar{a}_1)G(a_2;\bar{a}_2)G(1,a_1;a_2) - \\
&G(0;a_1)G(p_3;\bar{a}_2)G(1,a_1;a_2) - G(0;\bar{a}_1)G(p_3;\bar{a}_2)G(1,a_1;a_2) + G(0;a_1)G(0;\bar{a}_1)G(a_1,1;a_2) - \\
&G(0;\bar{a}_1)G(1;a_1)G(a_1,1;a_2) + G(0,0;a_1)G(a_1,1;a_2) + G(0,0;\bar{a}_1)G(a_1,1;a_2) - G(0,1;\bar{a}_1)G(a_1,1;a_2) - \\
&G(1,0;a_1)G(a_1,1;a_2) + G(0;\bar{a}_1)G(1;a_1)G(a_1,p_2;a_2) - G(0;a_1)G(1;\bar{a}_1)G(a_1,p_2;a_2) - \\
&G(0,1;a_1)G(a_1,p_2;a_2) + G(0,1;\bar{a}_1)G(a_1,p_2;a_2) + G(1,0;a_1)G(a_1,p_2;a_2) - G(1,0;\bar{a}_1)G(a_1,p_2;a_2) + \\
&G(a_1,1;a_2)G(\bar{a}_1,0;\bar{a}_2) - G(0;a_1)G(0;\bar{a}_1)G(\bar{a}_1,1;\bar{a}_2) + G(0;a_1)G(1;\bar{a}_1)G(\bar{a}_1,1;\bar{a}_2) - \\
&G(0,0;a_1)G(\bar{a}_1,1;\bar{a}_2) - G(0,0;\bar{a}_1)G(\bar{a}_1,1;\bar{a}_2) + G(0,1;a_1)G(\bar{a}_1,1;\bar{a}_2) + G(1,0;\bar{a}_1)G(\bar{a}_1,1;\bar{a}_2) - \\
&G(a_1,0;a_2)G(\bar{a}_1,1;\bar{a}_2) - G(0,1;a_2)G(\bar{a}_1,a_2;\bar{a}_2) + G(1,0;a_2)G(\bar{a}_1,a_2;\bar{a}_2) + \\
&G(0;a_1)G(a_1;a_2)G(\bar{a}_1,p_1;\bar{a}_2) + G(0;\bar{a}_1)G(a_1;a_2)G(\bar{a}_1,p_1;\bar{a}_2) + G(0,a_1;a_2)G(\bar{a}_1,p_1;\bar{a}_2) - \\
&G(a_1,0;a_2)G(\bar{a}_1,p_1;\bar{a}_2) + G(0;\bar{a}_1)G(1;a_1)G(\bar{a}_1,p_3;\bar{a}_2) - G(0;a_1)G(1;\bar{a}_1)G(\bar{a}_1,p_3;\bar{a}_2) - \\
&G(0;a_1)G(1;a_2)G(\bar{a}_1,p_3;\bar{a}_2) - G(0;\bar{a}_1)G(1;a_2)G(\bar{a}_1,p_3;\bar{a}_2) + G(1;a_1)G(1;a_2)G(\bar{a}_1,p_3;\bar{a}_2) + \\
&G(1;\bar{a}_1)G(1;a_2)G(\bar{a}_1,p_3;\bar{a}_2) - G(1;a_1)G(a_1;a_2)G(\bar{a}_1,p_3;\bar{a}_2) - G(1;\bar{a}_1)G(a_1;a_2)G(\bar{a}_1,p_3;\bar{a}_2) - \\
&G(0,1;a_1)G(\bar{a}_1,p_3;\bar{a}_2) + G(0,1;\bar{a}_1)G(\bar{a}_1,p_3;\bar{a}_2) + G(1,0;a_1)G(\bar{a}_1,p_3;\bar{a}_2) - G(1,0;\bar{a}_1)G(\bar{a}_1,p_3;\bar{a}_2) - \\
&G(1,a_1;a_2)G(\bar{a}_1,p_3;\bar{a}_2) + G(a_1,1;a_2)G(\bar{a}_1,p_3;\bar{a}_2) + G(0;a_1)G(1;a_2)G(\bar{a}_1,p_4;\bar{a}_2) + \\
&G(0;\bar{a}_1)G(1;a_2)G(\bar{a}_1,p_4;\bar{a}_2) - G(1;a_1)G(1;a_2)G(\bar{a}_1,p_4;\bar{a}_2) - G(1;\bar{a}_1)G(1;a_2)G(\bar{a}_1,p_4;\bar{a}_2) - \\
&G(0;a_1)G(a_1;a_2)G(\bar{a}_1,p_4;\bar{a}_2) - G(0;\bar{a}_1)G(a_1;a_2)G(\bar{a}_1,p_4;\bar{a}_2) + G(1;a_1)G(a_1;a_2)G(\bar{a}_1,p_4;\bar{a}_2) + \\
&G(1;\bar{a}_1)G(a_1;a_2)G(\bar{a}_1,p_4;\bar{a}_2) + G(0,1;a_2)G(\bar{a}_1,p_4;\bar{a}_2) - G(0,a_1;a_2)G(\bar{a}_1,p_4;\bar{a}_2) -
\end{aligned}
$$

$$G(1,0;a_2)G(\bar{a}_1,p_4;\bar{a}_2)+G(1,a_1;a_2)G(\bar{a}_1,p_4;\bar{a}_2)+G(a_1,0;a_2)G(\bar{a}_1,p_4;\bar{a}_2)-$$
$$G(a_1,1;a_2)G(\bar{a}_1,p_4;\bar{a}_2)+G(a_1,1;a_2)G(a_2,0;\bar{a}_2)-G(0;a_1)G(0;\bar{a}_1)G(a_2,1;\bar{a}_2)+$$
$$G(0;a_1)G(1;\bar{a}_1)G(a_2,1;\bar{a}_2)-G(0,0;a_1)G(a_2,1;\bar{a}_2)-G(0,0;\bar{a}_1)G(a_2,1;\bar{a}_2)+$$
$$G(0,1;a_1)G(a_2,1;\bar{a}_2)+G(1,0;\bar{a}_1)G(a_2,1;\bar{a}_2)-G(a_1,0;a_2)G(a_2,1;\bar{a}_2)-G(0,1;a_2)G(a_2,\bar{a}_1;\bar{a}_2)+$$
$$G(1,0;a_2)G(a_2,\bar{a}_1;\bar{a}_2)-G(0,1;a_2)G(a_2,a_2;\bar{a}_2)+G(1,0;a_2)G(a_2,a_2;\bar{a}_2)+$$
$$G(0;a_1)G(a_1;a_2)G(a_2,p_1;\bar{a}_2)+G(0;\bar{a}_1)G(a_1;a_2)G(a_2,p_1;\bar{a}_2)+G(0,a_1;a_2)G(a_2,p_1;\bar{a}_2)-$$
$$G(a_1,0;a_2)G(a_2,p_1;\bar{a}_2)+G(0;\bar{a}_1)G(1;a_1)G(a_2,p_3;\bar{a}_2)-G(0;a_1)G(1;\bar{a}_1)G(a_2,p_3;\bar{a}_2)-$$
$$G(0;a_1)G(1;a_2)G(a_2,p_3;\bar{a}_2)-G(0;\bar{a}_1)G(1;a_2)G(a_2,p_3;\bar{a}_2)+G(1;a_1)G(1;a_2)G(a_2,p_3;\bar{a}_2)+$$
$$G(1;\bar{a}_1)G(1;a_2)G(a_2,p_3;\bar{a}_2)-G(1;a_1)G(a_1;a_2)G(a_2,p_3;\bar{a}_2)-G(1;\bar{a}_1)G(a_1;a_2)G(a_2,p_3;\bar{a}_2)-$$
$$G(0,1;a_1)G(a_2,p_3;\bar{a}_2)+G(0,1;\bar{a}_1)G(a_2,p_3;\bar{a}_2)+G(1,0;a_1)G(a_2,p_3;\bar{a}_2)-G(1,0;\bar{a}_1)G(a_2,p_3;\bar{a}_2)-$$
$$G(1,a_1;a_2)G(a_2,p_3;\bar{a}_2)+G(a_1,1;a_2)G(a_2,p_3;\bar{a}_2)+G(0;a_1)G(1;a_2)G(a_2,p_4;\bar{a}_2)+$$
$$G(0;\bar{a}_1)G(1;a_2)G(a_2,p_4;\bar{a}_2)-G(1;a_1)G(1;a_2)G(a_2,p_4;\bar{a}_2)-G(1;\bar{a}_1)G(1;a_2)G(a_2,p_4;\bar{a}_2)-$$
$$G(0;a_1)G(a_1;a_2)G(a_2,p_4;\bar{a}_2)-G(0;\bar{a}_1)G(a_1;a_2)G(a_2,p_4;\bar{a}_2)+G(1;a_1)G(a_1;a_2)G(a_2,p_4;\bar{a}_2)+$$
$$G(1;\bar{a}_1)G(a_1;a_2)G(a_2,p_4;\bar{a}_2)+G(0,1;a_2)G(a_2,p_4;\bar{a}_2)-G(0,a_1;a_2)G(a_2,p_4;\bar{a}_2)-$$
$$G(1,0;a_2)G(a_2,p_4;\bar{a}_2)+G(1,a_1;a_2)G(a_2,p_4;\bar{a}_2)+G(a_1,0;a_2)G(a_2,p_4;\bar{a}_2)-$$
$$G(a_1,1;a_2)G(a_2,p_4;\bar{a}_2)-G(a_1,1;a_2)G(p_1,0;\bar{a}_2)-G(0;a_1)G(1;\bar{a}_1)G(p_1,\bar{a}_1;\bar{a}_2)-$$
$$G(0,1;a_1)G(p_1,\bar{a}_1;\bar{a}_2)+G(0,1;a_2)G(p_1,\bar{a}_1;\bar{a}_2)-G(1,0;\bar{a}_1)G(p_1,\bar{a}_1;\bar{a}_2)+G(0,1;a_2)G(p_1,a_2;\bar{a}_2)-$$
$$G(1,0;a_2)G(p_1,a_2;\bar{a}_2)-G(0;a_1)G(a_1;a_2)G(p_1,p_1;\bar{a}_2)-G(0;\bar{a}_1)G(a_1;a_2)G(p_1,p_1;\bar{a}_2)-$$
$$G(0,a_1;a_2)G(p_1,p_1;\bar{a}_2)+G(a_1,0;a_2)G(p_1,p_1;\bar{a}_2)-G(0;\bar{a}_1)G(1;a_1)G(p_1,p_3;\bar{a}_2)+$$
$$G(0;a_1)G(1;\bar{a}_1)G(p_1,p_3;\bar{a}_2)+G(0;a_1)G(1;a_2)G(p_1,p_3;\bar{a}_2)+G(0;\bar{a}_1)G(1;a_2)G(p_1,p_3;\bar{a}_2)-$$
$$G(1;a_1)G(1;a_2)G(p_1,p_3;\bar{a}_2)-G(1;\bar{a}_1)G(1;a_2)G(p_1,p_3;\bar{a}_2)+G(1;a_1)G(a_1;a_2)G(p_1,p_3;\bar{a}_2)+$$
$$G(1;\bar{a}_1)G(a_1;a_2)G(p_1,p_3;\bar{a}_2)+G(0,1;a_1)G(p_1,p_3;\bar{a}_2)-G(0,1;\bar{a}_1)G(p_1,p_3;\bar{a}_2)-$$
$$G(1,0;a_1)G(p_1,p_3;\bar{a}_2)+G(1,0;\bar{a}_1)G(p_1,p_3;\bar{a}_2)+G(1,a_1;a_2)G(p_1,p_3;\bar{a}_2)-G(a_1,1;a_2)G(p_1,p_3;\bar{a}_2)-$$
$$G(0;a_1)G(1;a_2)G(p_1,p_4;\bar{a}_2)-G(0;\bar{a}_1)G(1;a_2)G(p_1,p_4;\bar{a}_2)+G(1;a_1)G(1;a_2)G(p_1,p_4;\bar{a}_2)+$$
$$G(1;\bar{a}_1)G(1;a_2)G(p_1,p_4;\bar{a}_2)+G(0;a_1)G(a_1;a_2)G(p_1,p_4;\bar{a}_2)+G(0;\bar{a}_1)G(a_1;a_2)G(p_1,p_4;\bar{a}_2)-$$
$$G(1;a_1)G(a_1;a_2)G(p_1,p_4;\bar{a}_2)-G(1;\bar{a}_1)G(a_1;a_2)G(p_1,p_4;\bar{a}_2)-G(0,1;a_2)G(p_1,p_4;\bar{a}_2)+$$
$$G(0,a_1;a_2)G(p_1,p_4;\bar{a}_2)+G(1,0;a_2)G(p_1,p_4;\bar{a}_2)-G(1,a_1;a_2)G(p_1,p_4;\bar{a}_2)-$$
$$G(a_1,0;a_2)G(p_1,p_4;\bar{a}_2)+G(a_1,1;a_2)G(p_1,p_4;\bar{a}_2)-G(0;a_1)G(0;\bar{a}_1)G(p_2,1;a_2)+$$
$$G(0;\bar{a}_1)G(1;a_1)G(p_2,1;a_2)-G(0;a_1)G(\bar{a}_1;\bar{a}_2)G(p_2,1;a_2)-G(0;\bar{a}_1)G(\bar{a}_1;\bar{a}_2)G(p_2,1;a_2)+$$
$$G(1;a_1)G(\bar{a}_1;\bar{a}_2)G(p_2,1;a_2)+G(1;\bar{a}_1)G(\bar{a}_1;\bar{a}_2)G(p_2,1;a_2)-G(0;a_1)G(a_2;\bar{a}_2)G(p_2,1;a_2)-$$
$$G(0;\bar{a}_1)G(a_2;\bar{a}_2)G(p_2,1;a_2)+G(1;a_1)G(a_2;\bar{a}_2)G(p_2,1;a_2)+G(1;\bar{a}_1)G(a_2;\bar{a}_2)G(p_2,1;a_2)+$$
$$G(0;a_1)G(p_1;\bar{a}_2)G(p_2,1;a_2)+G(0;\bar{a}_1)G(p_1;\bar{a}_2)G(p_2,1;a_2)-G(1;a_1)G(p_1;\bar{a}_2)G(p_2,1;a_2)-$$
$$G(1;\bar{a}_1)G(p_1;\bar{a}_2)G(p_2,1;a_2)+G(0;a_1)G(p_3;\bar{a}_2)G(p_2,1;a_2)+G(0;\bar{a}_1)G(p_3;\bar{a}_2)G(p_2,1;a_2)-$$
$$G(1;a_1)G(p_3;\bar{a}_2)G(p_2,1;a_2)-G(1;\bar{a}_1)G(p_3;\bar{a}_2)G(p_2,1;a_2)-G(0,0;a_1)G(p_2,1;a_2)-$$
$$G(0,0;\bar{a}_1)G(p_2,1;a_2)+G(0,1;\bar{a}_1)G(p_2,1;a_2)+G(1,0;a_1)G(p_2,1;a_2)-G(0;a_1)G(1;\bar{a}_1)G(p_2,a_1;a_2)-$$
$$G(1;a_1)G(\bar{a}_1;\bar{a}_2)G(p_2,a_1;a_2)-G(1;\bar{a}_1)G(\bar{a}_1;\bar{a}_2)G(p_2,a_1;a_2)-G(1;a_1)G(a_2;\bar{a}_2)G(p_2,a_1;a_2)-$$
$$G(1;\bar{a}_1)G(a_2;\bar{a}_2)G(p_2,a_1;a_2)+G(1;a_1)G(p_1;\bar{a}_2)G(p_2,a_1;a_2)+G(1;\bar{a}_1)G(p_1;\bar{a}_2)G(p_2,a_1;a_2)+$$
$$G(1;a_1)G(p_3;\bar{a}_2)G(p_2,a_1;a_2)+G(1;\bar{a}_1)G(p_3;\bar{a}_2)G(p_2,a_1;a_2)-G(0,1;a_1)G(p_2,a_1;a_2)-$$
$$G(1,0;\bar{a}_1)G(p_2,a_1;a_2)-G(0;\bar{a}_1)G(1;a_1)G(p_2,p_2;a_2)+G(0;a_1)G(1;\bar{a}_1)G(p_2,p_2;a_2)+$$
$$G(0,1;a_1)G(p_2,p_2;a_2)-G(0,1;\bar{a}_1)G(p_2,p_2;a_2)-G(1,0;a_1)G(p_2,p_2;a_2)+G(1,0;\bar{a}_1)G(p_2,p_2;a_2)+$$

$$
G(0;a_1)G(0;\bar{a}_1)G(p_3,1;\bar{a}_2) - G(0;a_1)G(1;\bar{a}_1)G(p_3,1;\bar{a}_2) + G(0,0;a_1)G(p_3,1;\bar{a}_2) +
G(0,0;\bar{a}_1)G(p_3,1;\bar{a}_2) - G(0,1;a_1)G(p_3,1;\bar{a}_2) - G(1,0;\bar{a}_1)G(p_3,1;\bar{a}_2) + G(a_1,0;a_2)G(p_3,1;\bar{a}_2) +
G(0;\bar{a}_1)G(1;a_1)G(p_3,\bar{a}_1;\bar{a}_2) + G(0,1;\bar{a}_1)G(p_3,\bar{a}_1;\bar{a}_2) + G(1,0;a_1)G(p_3,\bar{a}_1;\bar{a}_2) -
G(1,0;a_2)G(p_3,\bar{a}_1;\bar{a}_2) + G(0,1;a_2)G(p_3,a_2;\bar{a}_2) - G(1,0;a_2)G(p_3,a_2;\bar{a}_2) -
G(0;a_1)G(a_1;a_2)G(p_3,p_1;\bar{a}_2) - G(0;\bar{a}_1)G(a_1;a_2)G(p_3,p_1;\bar{a}_2) - G(0,a_1;a_2)G(p_3,p_1;\bar{a}_2) +
G(a_1,0;a_2)G(p_3,p_1;\bar{a}_2) - G(0;\bar{a}_1)G(1;a_1)G(p_3,p_3;\bar{a}_2) + G(0;a_1)G(1;\bar{a}_1)G(p_3,p_3;\bar{a}_2) +
G(0;a_1)G(1;a_2)G(p_3,p_3;\bar{a}_2) + G(0;\bar{a}_1)G(1;a_2)G(p_3,p_3;\bar{a}_2) - G(1;a_1)G(1;a_2)G(p_3,p_3;\bar{a}_2) -
G(1;\bar{a}_1)G(1;a_2)G(p_3,p_3;\bar{a}_2) + G(1;a_1)G(a_1;a_2)G(p_3,p_3;\bar{a}_2) + G(1;\bar{a}_1)G(a_1;a_2)G(p_3,p_3;\bar{a}_2) +
G(0,1;a_1)G(p_3,p_3;\bar{a}_2) - G(0,1;\bar{a}_1)G(p_3,p_3;\bar{a}_2) - G(1,0;a_1)G(p_3,p_3;\bar{a}_2) + G(1,0;\bar{a}_1)G(p_3,p_3;\bar{a}_2) +
G(1,a_1;a_2)G(p_3,p_3;\bar{a}_2) - G(a_1,1;a_2)G(p_3,p_3;\bar{a}_2) - G(0;a_1)G(1;a_2)G(p_3,p_4;\bar{a}_2) -
G(0;\bar{a}_1)G(1;a_2)G(p_3,p_4;\bar{a}_2) + G(1;a_1)G(1;a_2)G(p_3,p_4;\bar{a}_2) + G(1;\bar{a}_1)G(1;a_2)G(p_3,p_4;\bar{a}_2) +
G(0;a_1)G(a_1;a_2)G(p_3,p_4;\bar{a}_2) + G(0;\bar{a}_1)G(a_1;a_2)G(p_3,p_4;\bar{a}_2) - G(1;a_1)G(a_1;a_2)G(p_3,p_4;\bar{a}_2) -
G(1;\bar{a}_1)G(a_1;a_2)G(p_3,p_4;\bar{a}_2) - G(0,1;a_2)G(p_3,p_4;\bar{a}_2) + G(0,a_1;a_2)G(p_3,p_4;\bar{a}_2) + G(1,0;a_2)G(p_3,p_4;\bar{a}_2) -
G(1,a_1;a_2)G(p_3,p_4;\bar{a}_2) - G(a_1,0;a_2)G(p_3,p_4;\bar{a}_2) + G(a_1,1;a_2)G(p_3,p_4;\bar{a}_2) + G(0;\bar{a}_1)G(0,0,1;a_1) +
G(\bar{a}_1;\bar{a}_2)G(0,0,1;a_1) - G(p_2;a_2)G(0,0,1;a_1) - G(p_3;\bar{a}_2)G(0,0,1;a_1) - G(0;a_1)G(0,0,1;\bar{a}_1) -
G(a_1;a_2)G(0,0,1;\bar{a}_1) + G(p_2;a_2)G(0,0,1;\bar{a}_1) + G(p_3;\bar{a}_2)G(0,0,1;\bar{a}_1) - G(0;\bar{a}_1)G(0,1,0;a_1) +
G(a_1;a_2)G(0,1,0;a_1) - G(\bar{a}_1;\bar{a}_2)G(0,1,0;a_1) + G(0;a_1)G(0,1,0;\bar{a}_1) + G(a_1;a_2)G(0,1,0;\bar{a}_1) -
G(\bar{a}_1;\bar{a}_2)G(0,1,0;\bar{a}_1) + G(0;a_1)G(0,1,a_1;a_2) + G(0;\bar{a}_1)G(0,1,a_1;a_2) + G(\bar{a}_1;\bar{a}_2)G(0,1,a_1;a_2) -
G(p_3;\bar{a}_2)G(0,1,a_1;a_2) - G(\bar{a}_1;\bar{a}_2)G(0,a_1,1;a_2) - G(a_2;\bar{a}_2)G(0,a_1,1;a_2) + G(p_1;\bar{a}_2)G(0,a_1,1;a_2) +
G(p_3;\bar{a}_2)G(0,a_1,1;a_2) - G(a_1;a_2)G(1,0,0;a_1) + G(p_2;a_2)G(1,0,0;a_1) + G(p_3;\bar{a}_2)G(1,0,0;a_1) +
G(\bar{a}_1;\bar{a}_2)G(1,0,0;\bar{a}_1) - G(p_2;a_2)G(1,0,0;\bar{a}_1) - G(p_3;\bar{a}_2)G(1,0,0;\bar{a}_1) + G(a_2;\bar{a}_2)G(1,0,a_1;a_2) -
G(p_3;\bar{a}_2)G(1,0,a_1;a_2) - G(\bar{a}_1;\bar{a}_2)G(a_1,0,1;a_2) - G(a_2;\bar{a}_2)G(a_1,0,1;a_2) + G(p_3;\bar{a}_2)G(a_1,0,1;a_2) +
G(\bar{a}_1;\bar{a}_2)G(a_1,1,0;a_2) + G(a_2;\bar{a}_2)G(a_1,1,0;a_2) - G(p_1;\bar{a}_2)G(a_1,1,0;a_2) + G(0;a_1)G(a_1,1,a_1;a_2) +
G(0;\bar{a}_1)G(a_1,1,a_1;a_2) - G(0;a_1)G(a_1,p_2,1;a_2) - G(0;\bar{a}_1)G(a_1,p_2,1;a_2) + G(1;a_1)G(a_1,p_2,1;a_2) +
G(1;\bar{a}_1)G(a_1,p_2,1;a_2) - G(1;a_1)G(a_1,p_2,a_1;a_2) - G(1;\bar{a}_1)G(a_1,p_2,a_1;a_2) + G(0;a_1)G(\bar{a}_1,0,1;\bar{a}_2) +
G(0;\bar{a}_1)G(\bar{a}_1,0,1;\bar{a}_2) - G(1;a_1)G(\bar{a}_1,0,1;\bar{a}_2) - G(1;\bar{a}_1)G(\bar{a}_1,0,1;\bar{a}_2) + G(a_1;a_2)G(\bar{a}_1,0,1;\bar{a}_2) +
G(1;a_1)G(\bar{a}_1,0,\bar{a}_1;\bar{a}_2) + G(1;\bar{a}_1)G(\bar{a}_1,0,\bar{a}_1;\bar{a}_2) - G(a_1;a_2)G(\bar{a}_1,1,0;\bar{a}_2) - G(0;a_1)G(\bar{a}_1,1,\bar{a}_1;\bar{a}_2) -
G(0;\bar{a}_1)G(\bar{a}_1,1,\bar{a}_1;\bar{a}_2) + G(1;a_2)G(\bar{a}_1,a_2,0;\bar{a}_2) - G(0;a_2)G(\bar{a}_1,a_2,1;\bar{a}_2) - G(a_1;a_2)G(\bar{a}_1,p_1,0;\bar{a}_2) -
G(0;a_1)G(\bar{a}_1,p_1,\bar{a}_1;\bar{a}_2) - G(0;\bar{a}_1)G(\bar{a}_1,p_1,\bar{a}_1;\bar{a}_2) + G(0;a_2)G(\bar{a}_1,p_1,\bar{a}_1;\bar{a}_2) + G(0;a_1)G(\bar{a}_1,p_3,1;\bar{a}_2) +
G(0;\bar{a}_1)G(\bar{a}_1,p_3,1;\bar{a}_2) - G(1;a_1)G(\bar{a}_1,p_3,1;\bar{a}_2) - G(1;\bar{a}_1)G(\bar{a}_1,p_3,1;\bar{a}_2) + G(a_1;a_2)G(\bar{a}_1,p_3,1;\bar{a}_2) +
G(1;a_1)G(\bar{a}_1,p_3,\bar{a}_1;\bar{a}_2) + G(1;\bar{a}_1)G(\bar{a}_1,p_3,\bar{a}_1;\bar{a}_2) - G(1;a_2)G(\bar{a}_1,p_3,\bar{a}_1;\bar{a}_2) - G(1;a_2)G(\bar{a}_1,p_4,0;\bar{a}_2) +
G(a_1;a_2)G(\bar{a}_1,p_4,0;\bar{a}_2) - G(0;a_1)G(\bar{a}_1,p_4,1;\bar{a}_2) - G(0;\bar{a}_1)G(\bar{a}_1,p_4,1;\bar{a}_2) + G(0;a_2)G(\bar{a}_1,p_4,1;\bar{a}_2) +
G(1;a_1)G(\bar{a}_1,p_4,1;\bar{a}_2) + G(1;\bar{a}_1)G(\bar{a}_1,p_4,1;\bar{a}_2) - G(a_1;a_2)G(\bar{a}_1,p_4,1;\bar{a}_2) + G(0;a_1)G(\bar{a}_1,p_4,\bar{a}_1;\bar{a}_2) +
G(0;\bar{a}_1)G(\bar{a}_1,p_4,\bar{a}_1;\bar{a}_2) - G(0;a_2)G(\bar{a}_1,p_4,\bar{a}_1;\bar{a}_2) - G(1;a_1)G(\bar{a}_1,p_4,\bar{a}_1;\bar{a}_2) - G(1;\bar{a}_1)G(\bar{a}_1,p_4,\bar{a}_1;\bar{a}_2) +
G(1;a_2)G(\bar{a}_1,p_4,\bar{a}_1;\bar{a}_2) + G(0;a_1)G(a_2,0,1;\bar{a}_2) + G(0;\bar{a}_1)G(a_2,0,1;\bar{a}_2) - G(1;a_1)G(a_2,0,1;\bar{a}_2) -
G(1;\bar{a}_1)G(a_2,0,1;\bar{a}_2) + G(a_1;a_2)G(a_2,0,1;\bar{a}_2) + G(1;a_1)G(a_2,0,\bar{a}_1;\bar{a}_2) + G(1;\bar{a}_1)G(a_2,0,\bar{a}_1;\bar{a}_2) -
G(a_1;a_2)G(a_2,1,0;\bar{a}_2) - G(0;a_1)G(a_2,1,\bar{a}_1;\bar{a}_2) - G(0;\bar{a}_1)G(a_2,1,\bar{a}_1;\bar{a}_2) + G(1;a_2)G(a_2,\bar{a}_1,0;\bar{a}_2) -
G(0;a_2)G(a_2,\bar{a}_1,1;\bar{a}_2) + G(1;a_2)G(a_2,a_2,0;\bar{a}_2) - G(0;a_2)G(a_2,a_2,1;\bar{a}_2) - G(a_1;a_2)G(a_2,p_1,0;\bar{a}_2) -
G(0;a_1)G(a_2,p_1,\bar{a}_1;\bar{a}_2) - G(0;\bar{a}_1)G(a_2,p_1,\bar{a}_1;\bar{a}_2) + G(0;a_2)G(a_2,p_1,\bar{a}_1;\bar{a}_2) + G(0;a_1)G(a_2,p_3,1;\bar{a}_2) +
G(0;\bar{a}_1)G(a_2,p_3,1;\bar{a}_2) - G(1;a_1)G(a_2,p_3,1;\bar{a}_2) - G(1;\bar{a}_1)G(a_2,p_3,1;\bar{a}_2) + G(a_1;a_2)G(a_2,p_3,1;\bar{a}_2) +
G(1;a_1)G(a_2,p_3,\bar{a}_1;\bar{a}_2) + G(1;\bar{a}_1)G(a_2,p_3,\bar{a}_1;\bar{a}_2) - G(1;a_2)G(a_2,p_3,\bar{a}_1;\bar{a}_2) -
$$

$$G(1;a_2)G(a_2,p_4,0;\bar{a}_2)+G(a_1;a_2)G(a_2,p_4,0;\bar{a}_2)-G(0;a_1)G(a_2,p_4,1;\bar{a}_2)-$$
$$G(0;\bar{a}_1)G(a_2,p_4,1;\bar{a}_2)+G(0;a_2)G(a_2,p_4,1;\bar{a}_2)+G(1;a_1)G(a_2,p_4,1;\bar{a}_2)+$$
$$G(1;\bar{a}_1)G(a_2,p_4,1;\bar{a}_2)-G(a_1;a_2)G(a_2,p_4,1;\bar{a}_2)+G(0;a_1)G(a_2,p_4,\bar{a}_1;\bar{a}_2)+$$
$$G(0;\bar{a}_1)G(a_2,p_4,\bar{a}_1;\bar{a}_2)-G(0;a_2)G(a_2,p_4,\bar{a}_1;\bar{a}_2)-G(1;a_1)G(a_2,p_4,\bar{a}_1;\bar{a}_2)-$$
$$G(1;\bar{a}_1)G(a_2,p_4,\bar{a}_1;\bar{a}_2)+G(1;a_2)G(a_2,p_4,\bar{a}_1;\bar{a}_2)-G(0;a_1)G(p_1,0,1;\bar{a}_2)-$$
$$G(0;\bar{a}_1)G(p_1,0,1;\bar{a}_2)+G(1;a_1)G(p_1,0,1;\bar{a}_2)+G(1;\bar{a}_1)G(p_1,0,1;\bar{a}_2)-$$
$$G(a_1;a_2)G(p_1,0,1;\bar{a}_2)-G(1;a_1)G(p_1,0,\bar{a}_1;\bar{a}_2)-G(1;\bar{a}_1)G(p_1,0,\bar{a}_1;\bar{a}_2)+$$
$$G(0;a_2)G(p_1,\bar{a}_1,1;\bar{a}_2)-G(1;a_2)G(p_1,a_2,0;\bar{a}_2)+G(0;a_2)G(p_1,a_2,1;\bar{a}_2)+$$
$$G(a_1;a_2)G(p_1,p_1,0;\bar{a}_2)+G(0;a_1)G(p_1,p_1,\bar{a}_1;\bar{a}_2)+G(0;\bar{a}_1)G(p_1,p_1,\bar{a}_1;\bar{a}_2)-$$
$$G(0;a_2)G(p_1,p_1,\bar{a}_1;\bar{a}_2)-G(0;a_1)G(p_1,p_3,1;\bar{a}_2)-G(0;\bar{a}_1)G(p_1,p_3,1;\bar{a}_2)+$$
$$G(1;a_1)G(p_1,p_3,1;\bar{a}_2)+G(1;\bar{a}_1)G(p_1,p_3,1;\bar{a}_2)-G(a_1;a_2)G(p_1,p_3,1;\bar{a}_2)-$$
$$G(1;a_1)G(p_1,p_3,\bar{a}_1;\bar{a}_2)-G(1;\bar{a}_1)G(p_1,p_3,\bar{a}_1;\bar{a}_2)+G(1;a_2)G(p_1,p_3,\bar{a}_1;\bar{a}_2)+$$
$$G(1;a_2)G(p_1,p_4,0;\bar{a}_2)-G(a_1;a_2)G(p_1,p_4,0;\bar{a}_2)+G(0;a_1)G(p_1,p_4,1;\bar{a}_2)+$$
$$G(0;\bar{a}_1)G(p_1,p_4,1;\bar{a}_2)-G(0;a_2)G(p_1,p_4,1;\bar{a}_2)-G(1;a_1)G(p_1,p_4,1;\bar{a}_2)-$$
$$G(1;\bar{a}_1)G(p_1,p_4,1;\bar{a}_2)+G(a_1;a_2)G(p_1,p_4,1;\bar{a}_2)-G(0;a_1)G(p_1,p_4,\bar{a}_1;\bar{a}_2)-$$
$$G(0;\bar{a}_1)G(p_1,p_4,\bar{a}_1;\bar{a}_2)+G(0;a_2)G(p_1,p_4,\bar{a}_1;\bar{a}_2)+G(1;a_1)G(p_1,p_4,\bar{a}_1;\bar{a}_2)+$$
$$G(1;\bar{a}_1)G(p_1,p_4,\bar{a}_1;\bar{a}_2)-G(1;a_2)G(p_1,p_4,\bar{a}_1;\bar{a}_2)-G(0;a_1)G(p_2,0,1;a_2)-$$
$$G(0;\bar{a}_1)G(p_2,0,1;a_2)+G(1;a_1)G(p_2,0,1;a_2)+G(1;\bar{a}_1)G(p_2,0,1;a_2)-$$
$$G(1;a_1)G(p_2,0,a_1;a_2)-G(1;\bar{a}_1)G(p_2,0,a_1;a_2)-G(0;a_1)G(p_2,1,a_1;a_2)-$$
$$G(0;\bar{a}_1)G(p_2,1,a_1;a_2)-G(\bar{a}_1;\bar{a}_2)G(p_2,1,a_1;a_2)-G(a_2;\bar{a}_2)G(p_2,1,a_1;a_2)+$$
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$$G(1;a_1)G(p_3,p_3,1;\bar{a}_2)+G(1;\bar{a}_1)G(p_3,p_3,1;\bar{a}_2)-G(a_1;a_2)G(p_3,p_3,1;\bar{a}_2)-$$
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$$G(1;\bar{a}_1)G(p_3,p_4,\bar{a}_1;\bar{a}_2)-G(1;a_2)G(p_3,p_4,\bar{a}_1;\bar{a}_2)+G(0,0,1,0;a_1)-G(0,0,1,0;\bar{a}_1)-$$
$$G(0,1,0,0;a_1)+G(0,1,0,0;\bar{a}_1)+G(0,1,0,a_1;a_2)-G(0,a_1,0,1;a_2)-G(a_1,0,a_1,1;a_2)+$$
$$G(a_1,1,0,a_1;a_2)-G(a_1,p_2,1,a_1;a_2)+G(a_1,p_2,a_1,1;a_2)+G(\bar{a}_1,0,1,\bar{a}_1;\bar{a}_2)-G(\bar{a}_1,1,0,\bar{a}_1;\bar{a}_2)+$$
$$G(\bar{a}_1,a_2,0,1;\bar{a}_2)-G(\bar{a}_1,a_2,1,0;\bar{a}_2)-G(\bar{a}_1,p_1,0,\bar{a}_1;\bar{a}_2)+G(\bar{a}_1,p_1,\bar{a}_1,0;\bar{a}_2)+G(\bar{a}_1,p_3,1,\bar{a}_1;\bar{a}_2)-$$
$$G(\bar{a}_1,p_3,\bar{a}_1,1;\bar{a}_2)-G(\bar{a}_1,p_4,0,1;\bar{a}_2)+G(\bar{a}_1,p_4,0,\bar{a}_1;\bar{a}_2)+G(\bar{a}_1,p_4,1,0;\bar{a}_2)-G(\bar{a}_1,p_4,1,\bar{a}_1;\bar{a}_2)-$$
$$G(\bar{a}_1,p_4,\bar{a}_1,0;\bar{a}_2)+G(\bar{a}_1,p_4,\bar{a}_1,1;\bar{a}_2)+G(a_2,0,1,\bar{a}_1;\bar{a}_2)-G(a_2,1,0,\bar{a}_1;\bar{a}_2)+G(a_2,\bar{a}_1,0,1;\bar{a}_2)-$$
$$G(a_2,\bar{a}_1,1,0;\bar{a}_2)+G(a_2,a_2,0,1;\bar{a}_2)-G(a_2,a_2,1,0;\bar{a}_2)-G(a_2,p_1,0,\bar{a}_1;\bar{a}_2)+G(a_2,p_1,\bar{a}_1,0;\bar{a}_2)+$$

$G(a_2,p_3,1,\bar{a}_1;\bar{a}_2)-G(a_2,p_3,\bar{a}_1,1;\bar{a}_2)-G(a_2,p_4,0,1;\bar{a}_2)+G(a_2,p_4,0,\bar{a}_1;\bar{a}_2)+G(a_2,p_4,1,0;\bar{a}_2)-$
$G(a_2,p_4,1,\bar{a}_1;\bar{a}_2)-G(a_2,p_4,\bar{a}_1,0;\bar{a}_2)+G(a_2,p_4,\bar{a}_1,1;\bar{a}_2)-G(p_1,0,1,\bar{a}_1;\bar{a}_2)+G(p_1,\bar{a}_1,1,0;\bar{a}_2)-$
$G(p_1,a_2,0,1;\bar{a}_2)+G(p_1,a_2,1,0;\bar{a}_2)+G(p_1,p_1,0,\bar{a}_1;\bar{a}_2)-G(p_1,p_1,\bar{a}_1,0;\bar{a}_2)-G(p_1,p_3,1,\bar{a}_1;\bar{a}_2)+$
$G(p_1,p_3,\bar{a}_1,1;\bar{a}_2)+G(p_1,p_4,0,1;\bar{a}_2)-G(p_1,p_4,0,\bar{a}_1;\bar{a}_2)-G(p_1,p_4,1,0;\bar{a}_2)+G(p_1,p_4,1,\bar{a}_1;\bar{a}_2)+$
$G(p_1,p_4,\bar{a}_1,0;\bar{a}_2)-G(p_1,p_4,\bar{a}_1,1;\bar{a}_2)-G(p_2,0,1,a_1;a_2)+G(p_2,0,a_1,1;a_2)-G(p_2,1,0,a_1;a_2)+$
$G(p_2,a_1,0,1;a_2)+G(p_2,p_2,1,a_1;a_2)-G(p_2,p_2,a_1,1;a_2)+G(p_3,1,0,\bar{a}_1;\bar{a}_2)-G(p_3,\bar{a}_1,0,1;\bar{a}_2)-$
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$G(p_3,p_3,\bar{a}_1,1;\bar{a}_2)+G(p_3,p_4,0,1;\bar{a}_2)-G(p_3,p_4,0,\bar{a}_1;\bar{a}_2)-G(p_3,p_4,1,0;\bar{a}_2)+G(p_3,p_4,1,\bar{a}_1;\bar{a}_2)+$
$G(p_3,p_4,\bar{a}_1,0;\bar{a}_2)-G(p_3,p_4,\bar{a}_1,1;\bar{a}_2)$

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