UNIVERSITY OF SOUTHAMPTON

Relative Ends and Splittings of Groups

by

Ana Claudia Lopes Onorio

A thesis submitted in partial fulfillment for the
degree of Doctor of Philosophy

in the

Faculty of Social, Human and Mathematical Sciences
Department of Mathematics

August 2018

UNIVERSITY OF SOUTHAMPTON

ABSTRACT

FACULTY OF SOCIAL, HUMAN AND MATHEMATICAL SCIENCES
Department of Mathematics

Doctor of Philosophy

RELATIVE ENDS AND SPLITTINGS OF GROUPS

by Ana Claudia Lopes Onorio

This thesis is motivated by a long-standing conjecture on groups with Bredon cohomological dimension one and their action on trees with stabilisers in a specific family of subgroups. Chapter 1 consists of the first approach to deal with the problem following steps of known results for families of finite and virtually cyclic subgroups. As a consequent of this attempt, we answer a question on the Bredon cohomological and geometric dimension of free abelian groups with finite rank.

The Main Theorem in Chapter 2 provides a partial answer to Kropholler's Conjecture on splittings of groups, which has been thought to be an alternative step for the proof of the conjecture stated in Chapter 1. We define the notion of *relative ends*, *commensurable subgroups*, *almost invariant sets* and the relation between those and splittings of groups, or equivalently, actions on trees with special stabilisers.

Contents

UNIVERSITY OF Southampton

Academic thesis: Declaration of authorship

I, **Ana Claudia Lopes Onorio** , declare that this thesis and the work presented in it are my own and has been generated by me as the result of my own original research.

Relative Ends and Splittings of Groups

I confirm that:

1. This work was done wholly or mainly while in candidature for a research degree at this University;
2. Where any part of this thesis has previously been submitted for a degree or any other qualification at this University or any other institution, this has been clearly stated;
3. Where I have consulted the published work of others, this is always clearly attributed;
4. Where I have quoted from the work of others, the source is always given. With the exception of such quotations, this thesis is entirely my own work;
5. I have acknowledged all main sources of help;
6. Where the thesis is based on work done by myself jointly with others, I have made clear exactly what was done by others and what I have contributed myself;
7. None of this work has been published before submission.

Signed: ....

Dated: ....

Acknowledgements

I would like to express my special thanks to Dr. Nansen Petrosyan, who so thoughtfully guided me throughout the last two years of my PhD; my journey until here has mostly been possible because of his enormous patience, interpersonal skills and encouragement.

Furthermore, I would like to thank Prof. Ian Leary for being ever present during my PhD, reading a draft of my thesis and whose insights in maths inspired me as a mathematician. Also, I would like to thank Prof. Peter Kropholler, whose conjecture motivated my most relevant work in this thesis; thank you for the insightful and passionate conversations about maths and life.

The final stages of writing a thesis can be extremely stressful. Therefore, I have to thank Netflix for providing me with amazing TV shows which, together with the pomodoro technique, kept me going and, most importantly, prevented me from tantrums and kept me sane (or something close to it).

Finally, I would like to express my appreciation to CNPq who, as part of the programme Ciencias sem Fronteiras (Science Without Borders), funded me during my whole PhD. I would not have been able to live these amazing four years without it.

My heartfelt thanks goes to all my friends and colleagues for the constant and non-judgemental support, in particular, Hector, who remained close and helped me to stay grounded. More specially, I thank my parents, for the unconditional support and for always watching me with huge cheering banners from the front row.

Introduction

In the late 60's, Stallings [30] proved that if a finitely generated group G has cohomological dimension one, then G acts freely on a tree, namely, the geometric dimension of G is also one. Later, Swan [31] extended the result for any group. These dimensions are denoted, respectively, by $\mathrm{cd}G$ and $\mathrm{gd}G$. Both definitions are extended to a more general version based on the work of Bredon [3] and further formalised by Lück [23]. In this setting, we consider a family $\mathcal{F}$ of subgroups of G and we refer to the Bredon cohomological and geometric dimensions of G over the family $\mathcal{F}$, respectively denoted by $\mathrm{cd}_{\mathcal{F}}G$ and $\mathrm{gd}_{\mathcal{F}}G$. When $\mathcal{F}$ contains only the trivial subgroup, these dimensions become the classical ones. An exciting open problem known as a general version of Stallings' Theorem conjectures the following:

Conjecture 1. *If G is a group and $\mathcal{F}$ is a family of subgroups of G such that $\mathrm{cd}_{\mathcal{F}}G \leq 1$, then $\mathrm{gd}_{\mathcal{F}}G \leq 1$; that is, G acts on a tree with vertex stabilisers in $\mathcal{F}$.*

This question has been positively answered by Dunwoody [10] when $\mathcal{F}$ is family of the finite subgroups of G and, recently by Degrijse [7] when $\mathcal{F}$ consists of all virtually cyclic subgroups of G, that is, all subgroups of G containing a subgroup of finite index isomorphic to a cyclic group ($\mathbb{Z}$ or $\{1\}$). A special notation is given for both families, respectively $\mathcal{F}_{fin}$ and $\mathcal{F}_{vc}$.

Now, consider a group G and the family $\mathcal{F}_2$ of all virtually cyclic and virtually $\mathbb{Z}^2$ subgroups of G (that is, all subgroups of G containing a subgroup of finite index isomorphic to $\mathbb{Z}^n,\ n = 0, 1, 2$). A natural question to ask is: *if* $\mathrm{cd}_{\mathcal{F}_2}G \leq 1$ *then is* $\mathrm{gd}_{\mathcal{F}_2}G \leq 1$*?* One of the first steps given by Degrijse in his proof is to show that, if $\mathrm{cd}_{\mathcal{F}_{vc}}G = 1$, then G does not contain a copy of the free abelian group on two generators $\mathbb{Z}^2$. This fact rises from the property of Bredon cohomological dimension of groups which states that, if H is a subgroup of G and $\mathcal{F}$ is family of subgroups of G satifying $\mathcal{F} \cap H \subseteq \mathcal{F}$, then

$$\mathrm{cd}_{\mathcal{F}\cap H}H \leq \mathrm{cd}_{\mathcal{F}}G$$

If $H = \mathbb{Z}^2$, it is known that $\mathrm{cd}_{\mathcal{F}\cap\mathbb{Z}^2}\mathbb{Z}^2 = 3$.
In an attempt to follow the same approach, I prove that, if $\mathrm{cd}_{\mathcal{F}_2}G \leq 1$, then G does not contain $\mathbb{Z}^3$. More generally, I show that the following is true:

Proposition A. *For any $n \geq 3$,*

$$\mathrm{cd}_{\mathcal{F}_2}\mathbb{Z}^n = \mathrm{gd}_{\mathcal{F}_2}\mathbb{Z}^n = n + 2$$

The Conjecture above on groups with Bredon cohomological dimension one is very difficult to answer given any family of subgroups of G. For the families $\mathcal{F}_{fin}$ and $\mathcal{F}_{vc}$, the proofs depend on specific results related to each of those families. Therefore, Proposition A concludes the first chapter of this thesis.
At that time during my PhD, there was a discussion on how to approach the solution for groups with Bredon cohomological dimension one for any family of subgroups of G. It was thought to be possible to follow Dunwoody's proof for the case $\mathcal{F}_{fin}$ [10]. Given a group G and a family $\mathcal{F}$ of subgroups of G satisfying $\mathrm{cd}_{\mathcal{F}}G \leq 1$, the idea would be to find a Bredon cohomological condition for G implying an action of G on a tree with stabilisers on $\mathcal{F}$. The best immediate step seemed to be finding a proof for Kropholler's Conjecture on splittings of groups:

The Kropholler Conjecture. *Let G be a finitely generated group and H a subgroup of G. If G contains an H-proper almost invariant subset A such that $AH = A$, then G admits a non-trivial splitting over a subgroup C which is commensurable with a subgroup of H.*

A group G is said to *split* over a subgroup H if G acts without global fixed points and transitively on a tree with an edge stabiliser H.
In the second chapter of this thesis is presented my most relevant work, in which I answer the conjecture for a special case. The following result is my contribution to the proof of Kropholler's Conjecture:

Main Theorem. *Let $H \leq G$ be finitely generated groups satisfying:*

- $2 \leq \widetilde{e}(G, H) < \infty$*;*
- $H \leq_f \mathrm{Comm}_G(H)$*;*
- $\forall g \in G$*, if* gH *is* H*-finite, then* $g \in \mathrm{Comm}_G(H)$.

If G contains an H-proper almost invariant subset A such that $AH = A$, then G admits a non-trivial splitting over a subgroup commensurable with H.

The theorem is a generalisation of [19], Lem. 4.4:

Lemma. *Let G be a finitely generated group and H a Poincaré duality subgroup of G satisfying $H = \mathrm{Comm}_G(H)$. If $\widetilde{e}(G, H) = 2$ and G contains an H-proper almost invariant subset A such that $AH = A$, then G splits over H.*

Given groups $H \leq G$, Kropholler and Roller [21] defined a new invariant of the pair (G, H) denoted by $\widetilde{e}(G, H)$. If H is the trivial subgroup, then the invariant becomes the number of ends of G, known as $e(G)$. The latter is used in Stallings' Theorem [30] and Dunwoody's Theorem [10] on groups with cohomological dimension one. In Section 2.1, we show how $e(G)$ and splittings of groups are directly related.
The motivation for my theorem was given by Peter Kropholler himself. He suggested I should look at the results in [19] for groups satisfying $\widetilde{e}(G, H) = 2$ and ask what happens when $\widetilde{e}(G, H) = 3$. It turns out more general results can be proven when $\widetilde{e}(G, H) = n$, for any $2 \leq n < \infty$, as we will see in Sections 2.8, 2.8.1 and 2.9.
It is possible that the Main Theorem can be extended to a more general version involving the splitting of a group G over a finite family of subgroups of G, as discussed in [20].

Structure of Thesis

This thesis was written with the expectation that the reader is familiar with the following:

- Category Theory: categories, functors, abelian functors, natural transformations, adjoints, universal properties, natural isomorphisms.
- Algebraic Topology: universal cover of a space, CW-complexes, cellular chain complex, Mayer-Vietoris homology sequence for spaces.
- Algebra: basic group theory, Bass-Serre theory, homology and cohomology of groups with coefficients, $\mathbb{Z}G$-modules, free, projective and flat modules.

Although this work consists of many formal statements, I tried to keep a certain informal tone to my writing by creating questions and answering them instead of writing full formal paragraphs linking the topics. I chose to do that for three reasons:

1) Mainly, I believe that, when we read, we automatically ask questions in our minds and we expect the author to fill these gaps as she/he explains the subject. Therefore, why not put those questions in the paper? It will (hopefully) create a dynamical reading and make the text flow better.

2) It helps to compensate a non-native English speaker some lack of proper writing skills using formal English;

3) And, of course, it is less boring this way.

This thesis is divided in two chapters written according to the chronological development of my work during my PhD. Chapter 2 contains my most relevant result.

Chapter 1

Bredon geometric dimension of $\mathbb{Z}^n$

In this chapter, we will work with a given group G and a family of subgroups of G, consisting of virtually $\mathbb{Z}^r$ subgroups, for some $r \geq 0$.

Definition 1.0.1. *Let G be a group and H a subgroup of G. We say H is a **virtually** $\mathbb{Z}^r$ subgroup of G, $r \geq 0$, if H contains a subgroup K of finite index such that $K \cong \mathbb{Z}^r$. If $r = 0$, then H is finite. If $r = 1$, then H is virtually infinite cyclic.*

Let $G = \mathbb{Z}^n$ and $H \leq G$. For $0 \leq r \leq n$, we say H is virtually $\mathbb{Z}^r$ if $H \cong \mathbb{Z}^r$.

Now, given a group G, let $\mathcal{F}_2$ denote the family of all virtually $\mathbb{Z}^r$ subgroups of G, for $r = 0, 1, 2$. In this chapter, we define the Bredon cohomological and geometric dimensions of a group G over a family of subgroups of G and prove the following:

Proposition A. *For any $n \geq 3$,*

$$\mathrm{cd}_{\mathcal{F}_2}\mathbb{Z}^n = \mathrm{gd}_{\mathcal{F}_2}\mathbb{Z}^n = n + 2.$$

This proposition is a partial solution of the following question:

Question 1 ([5], Quest. 2.6)**.** *Let G be a finitely generated abelian group of finite torsion-free rank $n \geq 1$, and denote by $\mathcal{F}_r$ the family of subgroups of G of torsion-free rank less than or equal to $r \geq 0$. Then*

$$\mathrm{gd}_{\mathcal{F}_r}G = n + r.$$

When $r = 0$, $\mathcal{F}_0$ is the family of all finite subgroups and has the special notation $\mathcal{F}_{fin}$. Moreover,

$$\mathrm{cd}_{\mathcal{F}_{fin}}G := \underline{\mathrm{cd}}G \quad \text{and} \quad \mathrm{gd}_{\mathcal{F}_{fin}}G := \underline{\mathrm{gd}}G$$

When $r = 1$, $\mathcal{F}_1$ is the family of all virtually cyclic subgroups of G and has the special notation $\mathcal{F}_{vc}$. Also,

$$\mathrm{cd}_{\mathcal{F}_{vc}} G := \underline{\underline{\mathrm{cd}}} G \quad \text{and} \quad \mathrm{cd}_{\mathcal{F}_{vc}} G := \underline{\underline{\mathrm{cd}}} G$$

It is already known that

$$\underline{\mathrm{cd}}\mathbb{Z}^n = \underline{\mathrm{gd}}\mathbb{Z}^n = n$$

and, for $n > 1$,

$$\underline{\underline{\mathrm{cd}}}\mathbb{Z}^n = \underline{\underline{\mathrm{gd}}}\mathbb{Z}^n = n + 1$$

The first equalities are due to the fact that $\underline{\mathrm{cd}}\mathbb{Z}^n = \mathrm{cd}\mathbb{Z}^n$ and $\underline{\mathrm{gd}}\mathbb{Z}^n = \mathrm{gd}\mathbb{Z}^n$. Hence, the result follows from classical cohomology theory ([4], Ch.VIII, Sec.2, Exm. 5). The second pair of equalities are due to [26], Exm. 5.21.

In order to prove Propostition A, we will construct a *model for a classifying space of* $\mathbb{Z}^n$ *over* $\mathcal{F}_2$, denoted by $E_{\mathcal{F}_2}\mathbb{Z}^n$. In Section 1.10 we conclude the proof by calculating the geometric dimension of such space and applying known results in Bredon cohomology.

Remark 1.0.2. *In Section 1.6, we will mention a construction of models for classifying spaces given by Lück and Weiermann [26]. This construction is used in [5] to obtain the inequality* $\mathrm{gd}_{\mathcal{F}_r} G \leq n + r$ *for a finitely generated abelian group* G *of finite torsion-free rank* $n \geq 1$*. In Section 1.9, we will compare this method to the one used to obtain our main result of this chapter.*

1.1 Bredon modules

Q. *Who is Bredon?*

Glen Eugene Bredon (1932 - 2000) was an American mathematician. In [3], he introduced a homology theory for finite groups involving CW-complexes with a cellular action of a group G (known as G-CW-complexes) with stabilisers in a given family $\mathcal{F}$ of subgroups of G. The theory was later studied, developed and formalised by Lück [23] for arbitrary groups. In classical algebraic topology, we have a classifying space X of a fundamental group G which acts freely on the universal cover of X denoted by $\widetilde{X}$. This action gives a chain complex of $\widetilde{X}$ of free $\mathbb{Z}G$-modules over the trivial $\mathbb{Z}G$-module $\mathbb{Z}$. The Bredon homology theory generalizes the classical homology theory by introducing a family $\mathcal{F}$ of subgroups of the group G which are the only stabilisers of the action of G on a space called *the classifying space of* G *over* $\mathcal{F}$. Similarly, we build a chain complex of what we call *free Bredon modules.*

Q. *What are Bredon modules?*

From now on, we will take G to be a discrete group and, by a family $\mathcal{F}$ of subgroups of G, we mean a set of subgroups of G closed under conjugation and taking subgroups. For any $H \in \mathcal{F}$, the set G/H consists of all the left H-cosets of G. We consider the natural action of G on G/H given by the translation of the left H-cosets by $g \in G$, that is,

$$g \cdot g'H := gg'H$$

Given any subgroups $H, K \in \mathcal{F}$, a G-map between G/H and G/K is a G-equivariant map $f_{H,K} \colon G/H \longrightarrow G/K$ which, by definition, satisfies, $\forall g \in G$,

$$f_{H,K}(gyH) = gf_{H,K}(yH)$$

where $yH \in G/H$. Thus, one can see that $f_{H,K}$ is completely defined by its evaluation in H. Indeed, if $f_{H,K}(H) = xK$, then, for any $yH \in G/H$, we have that

$$f_{H,K}(yH) = yf_{H,K}(H) = yxK$$

We denote this G-map as $f_{H,K,x}$. Observe that if $f_{H,K,x}$ is a G-map, then, $\forall h \in H$,

$$xK = f_{H,K,x}(H) = f_{H,K,x}(hH) = hf_{H,K,x}(H) = hxK$$

Hence, $x^{-1}Hx \subseteq K$.

Definition 1.1.1. *The orbit category of G over $\mathcal{F}$, denoted by $\mathcal{O}_{\mathcal{F}}G$, consists of:*

- *Objects: G/H, for every $H \in \mathcal{F}$;*
- *Morphisms: G-maps $f_{H,K,x} \colon G/H \longrightarrow G/K$, for every $H, K \in \mathcal{F}, x \in G$.*

The abelian category of functors from $\mathcal{O}_{\mathcal{F}}G$ to the category **Ab** of abelian groups is called the *category of Bredon modules over $\mathcal{O}_{\mathcal{F}}G$*. A contravariant (covariant) functor of this category is called a *right (left) Bredon module over $\mathcal{O}_{\mathcal{F}}G$*. Let $\mathcal{M}$ be a contravariant Bredon module over $\mathcal{O}_{\mathcal{F}}G$ and $\phi \colon G/H \longrightarrow G/K$ a morphism in $\mathcal{O}_{\mathcal{F}}G$, $H, K \in \mathcal{F}$. Then, $\mathcal{M}(\phi)$ is a morphism $\mathcal{M}(G/K) \longrightarrow \mathcal{M}(G/H)$, sometimes denoted by ϕ^*. If $\mathcal{M}$ is a covariant Bredon module over $\mathcal{O}_{\mathcal{F}}G$, then $\mathcal{M}(\phi)$ is a morphism $\mathcal{M}(G/H) \longrightarrow \mathcal{M}(G/K)$ and can be denoted by ϕ_*.

Example 1.1.2. *The trivial Bredon module $\mathbb{Z}_{\mathcal{F}} \colon \mathcal{O}_{\mathcal{F}}G \longrightarrow$ **Ab** is defined by:*

- $\mathbb{Z}_{\mathcal{F}}(G/H) = \mathbb{Z}$, $\forall H \in \mathcal{F}$;
- *For any G-map* ϕ *in* $\mathcal{O}_{\mathcal{F}}G$, $\mathbb{Z}_{\mathcal{F}}(\phi)$ *is the identity map on* $\mathbb{Z}$.

The category of right Bredon modules is denoted by $\mathcal{O}_{\mathcal{F}}G$-Mod and the category of left Bredon modules is denoted by Mod-$\mathcal{O}_{\mathcal{F}}G$. The morphisms are maps of functors, which are what we know as natural transformations.

1.2 $\mathbb{Z}G$-modules

Q. *In which sense does the Bredon homology theory generalize the classical homology theory?*

When $\mathcal{F}$ consists of only the trivial subgroup of G, the Bredon modules become $\mathbb{Z}G$-modules.

Q. *Why is that true?*

In this case, the category $\mathcal{O}_{\mathcal{F}}G$ has only the object $G/\{1\}$ and the G-maps in $\mathcal{O}_{\mathcal{F}}G$ are G-equivariant automorphisms of $G/\{1\}$, which are the elements of the set denoted by $\mathrm{Aut}(G/\{1\})$. As seen before, a G-map in $\mathrm{Aut}(G/\{1\})$ will be uniquely defined by its evaluation on $\{1\}$. Therefore, given any $g \in G$, the G-map $\gamma_g \colon G/\{1\} \longrightarrow G/\{1\}$ is defined by $\gamma_g(\{1\}) = g\{1\}$.

Lemma 1.2.1 ([12], p.13)**.** $\mathrm{Aut}(G/\{1\})$ *is a group isomorphic to* G.

Proof. Let $\gamma_g, \gamma_h \in \mathrm{Aut}(G/\{1\})$. Then,

$$\gamma_g \circ \gamma_h(\{1\}) = \gamma_g(h\{1\}) = h\gamma_g(\{1\}) = hg\{1\} = \gamma_{hg}(\{1\})$$

Let $\phi\colon \mathrm{Aut}(G/\{1\}) \longrightarrow G$ be the map defined by $\phi(\gamma_g) = g^{-1}$. Clearly, ϕ is a bijection. Moreover, $\phi(\gamma_1) = 1$ and

$$\phi(\gamma_g \circ \gamma_h) = \phi(\gamma_{hg}) = (hg)^{-1} = g^{-1}h^{-1} = \phi(\gamma_g)\phi(\gamma_h).$$

□

Hence, a right Bredon module $\mathcal{M}$ over $\mathcal{O}_{\mathcal{F}}G$ defines a right action of G on the abelian group $\mathcal{M}(G/\{1\})$ given by

$$x \cdot g := \mathcal{M}(\gamma_{g^{-1}})(x),$$

for any $g \in G, x \in \mathcal{M}(G/\{1\})$. Indeed, for any $g, h \in G$,

$$\begin{aligned} x \cdot (gh) &= \mathcal{M}(\gamma_{(gh)^{-1}})(x) = \mathcal{M}(\gamma_{h^{-1}g^{-1}})(x) \\ &= \mathcal{M}(\gamma_{g^{-1}} \circ \gamma_{h^{-1}})(x) = \mathcal{M}(\gamma_{h^{-1}}) \circ \mathcal{M}(\gamma_{g^{-1}})(x) = (x \cdot g) \cdot h \end{aligned}$$

Also, clearly $x \cdot 1 = \mathcal{M}(\gamma_1)(x) = \mathcal{M}(\mathrm{id}_{G/\{1\}})(x) = x$.
On the other hand, if N is a right $\mathbb{Z}G$-module, let $\mathcal{N} \colon \mathcal{O}_{\mathcal{F}}G \longrightarrow \mathbf{Ab}$ be the functor defined as:

$$\mathcal{N}(G/\{1\}) = N \text{ and, for } x \in N,\ \mathcal{N}(\gamma_g)(x) := x \cdot g^{-1}$$

Then, clearly, $\mathcal{N}(\gamma_1)(x) = \mathcal{N}(id)(x) = x$ and

$$\mathcal{N}(\gamma_g \circ \gamma_h)(x) = \mathcal{N}(\gamma_{hg}) = x \cdot (hg)^{-1} = (x \cdot g^{-1}) \cdot h^{-1} = \mathcal{N}(\gamma_h) \circ \mathcal{N}(\gamma_g)(x)$$

Hence, $\mathcal{N}$ is a right Bredon module over $\mathcal{O}_{\mathcal{F}}G$.

Therefore, when $\mathcal{F} = \{\{1\}\}$, there exists a one-to-one correspondence between

$$\{\mathcal{O}_{\mathcal{F}}G\text{-Mod}\} \longrightarrow \{\text{right } \mathbb{Z}G\text{-modules}\}$$

Now, let $\mathcal{M}$ and $\mathcal{N}$ be right Bredon modules over $\mathcal{O}_{\mathcal{F}}G$ and $\alpha \colon \mathcal{M} \longrightarrow \mathcal{N}$ a natural transformation, defined by a unique homomorphism of abelian groups $\alpha_{\{1\}} \colon \mathcal{M}(G/\{1\}) \longrightarrow \mathcal{N}(G/\{1\})$. Since α is a natural transformation, the following diagram must commute for every $\gamma_g \in \mathrm{Aut}(G/\{1\})$:

$$\begin{array}{ccc} \mathcal{M}(G/\{1\}) & \xrightarrow{\alpha_{\{1\}}} & \mathcal{N}(G/\{1\}) \\ \downarrow{\scriptstyle \mathcal{M}(\gamma_g)} & & \downarrow{\scriptstyle \mathcal{N}(\gamma_g)} \\ \mathcal{M}(G/\{1\}) & \xrightarrow[\alpha_{\{1\}}]{} & \mathcal{N}(G/\{1\}) \end{array}$$

Note that $\mathcal{N}(\gamma_g)$ is an isomorphism of abelian groups whose inverse is $\mathcal{N}(\gamma_{g^{-1}})$. Then, for every $x \in \mathcal{M}(G/\{1\}),\ g \in G$,

$$\begin{aligned} \alpha_{\{1\}}(x \cdot g) &= \mathcal{N}(\gamma_{g^{-1}}) \circ \alpha_{\{1\}} \circ \mathcal{M}(\gamma_g)(x \cdot g) \\ &= \mathcal{N}(\gamma_{g^{-1}}) \circ \alpha_{\{1\}}((x \cdot g) \cdot g^{-1}) \\ &= \mathcal{N}(\gamma_{g^{-1}}) \circ \alpha_{\{1\}}(x) \\ &= \alpha_{\{1\}}(x) \cdot g \end{aligned} \tag{1.1}$$

Hence, $\alpha_{\{1\}}$ is a homomorphism of right $\mathbb{Z}G$-modules. On the other hand, if β is a homomorphism between right $\mathbb{Z}G$-modules M, N, we follow the steps above to construct right Bredon modules $\mathcal{M}$, $\mathcal{N}$ respectively correspondent to M, N and define, in this case, $\alpha_{\{1\}} := \beta$. From the calculation above, we easily see that there exists a natural transformation $\alpha\colon \mathcal{M} \longrightarrow \mathcal{N}$ uniquely defined by $\alpha_{\{1\}}$.
Therefore, one can see that the correspondence above is actually a natural isomorphism of the categories of right Bredon modules over $\mathcal{O}_{\mathcal{F}}G$ and right $\mathbb{Z}G$-modules when $\mathcal{F}$ contains only the trivial subgroup of G. Similarly, one can prove there exists a natural isomorphism between Mod-$\mathcal{O}_{\mathcal{F}}G$ and the category of left $\mathbb{Z}G$-modules.
In this sense, Bredon modules generalise the concept of $\mathbb{Z}G$-modules, and most of the results in classical homology and cohomology theory apply to the Bredon homology theory.

Unless stated otherwise, any further reference to a Bredon module should be considered as a right Bredon module. Moreover, every result and definition concerning such modules can also be similarly stated for left Bredon modules, with the proper adjustments.

1.3 Free Bredon modules

Q. *What does "free" mean in this context?*

Bredon modules are functors. Thus, a *free* Bredon module is a free functor in the categorical sense, that is, a left adjoint to a forgetful functor. A reminder to the reader: a forgetful functor is a functor that, as the name suggests, "forgets" properties of the category in its domain. The reader is probably familiar with the forgetful functor $U\colon \textbf{Grp} \longrightarrow \textbf{Set}$ from the category of groups to the category of sets, which assigns any group G to the group itself without its group properties, and such that any map is just a map of sets and not morphisms of groups. If the definition of left adjoint is also not so fresh, here it goes:

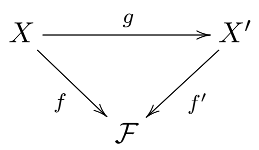
Definition 1.3.1 (D. M. Kan)**.** *Let $F\colon \mathcal{C} \longrightarrow \mathcal{D}$ and $G\colon \mathcal{D} \longrightarrow \mathcal{C}$ be two functors. An* ***adjunction*** *between F and G is a specification, for each pair $A,\ B$ of objects respectively in $\mathcal{C},\ \mathcal{D}$, of a bijection between morphisms $F(A) \longrightarrow B$ in $\mathcal{D}$ and morphisms $A \longrightarrow G(B)$ in $\mathcal{C}$, which is natural in A and B. We say that F is* ***left adjoint*** *to G and G is* ***right adjoint*** *to F.*

In the familiar setting of groups, we say a group G is *free with a generating set* X and we denote $G := F(X)$. In this case, G is the image of the functor $F\colon$ **Set** $\longrightarrow$ **Grp** which assigns a set X to the group $F(X)$ freely generated by X and a map of sets $\phi\colon X \longrightarrow Y$ in **Set** to a homomorphism of groups $F(\phi)\colon F(X) \longrightarrow F(Y)$ satisfying $F(\phi)|_X = \phi$. The attribution *free with generating set* X given to G comes from the fact that F is left adjoint to U. Alternatively, G is free with generating set X if G satisfies the following *universal property*: for any group H and map of sets $f\colon X \longrightarrow U(H)$, there exists a unique homomorphism of groups $\widetilde{f}\colon F(X) \longrightarrow H$ that extends f, that is, $U(\widetilde{f}) \circ i = f$, where, $i\colon X \hookrightarrow U(G)$ is the inclusion map of sets.

We will define a free Bredon module $\mathcal{M}$ in terms of the same universal property. However, we first need to define a *generator* for $\mathcal{M}$. Therefore, we introduce $\mathcal{F}$*-sets*:

Definition 1.3.2. *Given a group G and a family $\mathcal{F}$ of subgroups of G, an* $\mathcal{F}$***-set*** *is a pair (X, f) consisting of a set X and a function $f\colon X \longrightarrow \mathcal{F}$. For $H \in \mathcal{F}$, the H-component of X is the pre-image $f^{-1}(H)$, denoted by X_H.*

Note that (X, f) is defined by its components.
Given two $\mathcal{F}$-sets $(X, f),\ (X', f')$, a map of $\mathcal{F}$-sets $g\colon (X, f) \longrightarrow (X', f')$ is a map of sets $g\colon X \longrightarrow X'$ such that the diagram below commutes:



The commutativity of this diagram means that, for $H \in \mathcal{F}$, g assigns elements from the component X_H to elements in the component X'_H.

Definition 1.3.3. *We say that (X, f) is an* $\mathcal{F}$***-subset*** *of (X', f') if $X_H \subseteq X'_H$ for every $H \in \mathcal{F}$.*

A Bredon module $\mathcal{M}$ can be seen as an $\mathcal{F}$-set, where its components are $\mathcal{M}_H := \mathcal{M}(G/H)$, $H \in \mathcal{F}$, where each $\mathcal{M}(G/H)$ is seen as a set, not as an abelian group. If $\alpha\colon \mathcal{M} \longrightarrow \mathcal{N}$ is a natural transformation of Bredon modules, then α can also be seen as a map of $\mathcal{F}$-sets. Indeed, for each $H \in \mathcal{F}$, there exists $\alpha_H\colon \mathcal{M}_H \longrightarrow \mathcal{N}_H$.

Definition 1.3.4. *Let $\mathcal{M}$ be a Bredon module over $\mathcal{O}_{\mathcal{F}}G$ and (X,f) an $\mathcal{F}$-subset of $\mathcal{M}$. Then, the smallest Bredon module in $\mathcal{M}$ containing (X,f) is called* ***the Bredon submodule of $\mathcal{M}$ generated by the $\mathcal{F}$-set*** *(X,f) and is denoted by $\langle (X,f)\rangle$. If $\mathcal{M} = \langle (X,f)\rangle$, then we say that $\mathcal{M}$ is* ***generated*** *by (X,f).*

Let G be a group and $\mathcal{F}$ a family of subgroups of G. Fix $H \in \mathcal{F}$. The (contravariant) functor

$$\mathbb{Z}[*, G/H]_G\colon \mathcal{O}_{\mathcal{F}}G \longrightarrow \mathbf{Ab}$$

is a Bredon module over $\mathcal{O}_{\mathcal{F}}G$, which sends the object G/K from $\mathcal{O}_{\mathcal{F}}G$ to the free abelian group with basis the set $[G/K, G/H]_G$ of all G-maps from G/K to G/H. If $\phi\colon G/K \longrightarrow G/L$ is a morphism in $\mathcal{O}_{\mathcal{F}}G$, then

$$\phi^*\colon \mathbb{Z}[G/L, G/H]_G \longrightarrow \mathbb{Z}[G/K, G/H]_G$$

is the homomorphism of abelian groups defined as $\phi^*(\gamma) = \gamma \circ \phi$, for $\gamma \in [G/L, G/H]_G$.

Lemma 1.3.5 ([12], Lem. 1.12)**.** *The Bredon module $\mathbb{Z}[*, G/H]_G$ is generated by the $\mathcal{F}$-set (X,f) given by*

$$X_K = \begin{cases} \mathrm{id}_{G/K} & \text{if } K = H, \\ \emptyset & \text{otherwise.} \end{cases}$$

Let $\mathcal{F}$-**Set** be the category whose objects are all the $\mathcal{F}$-sets and the morphisms are all the maps of $\mathcal{F}$-sets. Now, we define the functor

$$F\colon \mathcal{F}\text{-}\mathbf{Set} \longrightarrow \mathcal{O}_{\mathcal{F}}G$$

which sends $\mathcal{F}$-sets (X,f) to $\langle (X,f)\rangle$ and maps of $\mathcal{F}$-sets to the natural transformations corresponding to these maps. Moreover, let

$$U\colon \mathcal{O}_{\mathcal{F}}G \longrightarrow \mathcal{F}\text{-}\mathbf{Set}$$

be the forgetful functor which assigns a Bredon module $\mathcal{M}$ over $\mathcal{O}_{\mathcal{F}}G$ to an $\mathcal{F}$-set also denoted by $\mathcal{M}$, as seen before. The functor F is left adjoint to U.

Definition 1.3.6. *A Bredon module M is* ***free with generating $\mathcal{F}$-set*** *(X, f) if it satisfies the following* ***universal property****: for any Bredon module $\mathcal{N}$ over $\mathcal{O}_{\mathcal{F}}G$ and any map of $\mathcal{F}$-sets $g\colon (X, f) \longrightarrow \mathcal{N}$, there exists a unique map of Bredon modules $\widetilde{g}\colon \mathcal{M} \longrightarrow \mathcal{N}$ that extends g, that is, $U(\widetilde{g}) \circ i = g$, where $i\colon (X, f) \hookrightarrow \mathcal{M}$ is the inclusion map of $\mathcal{F}$-sets.*

$$\begin{array}{ccc} \mathcal{M} & \overset{U(\widetilde{g})}{\dashrightarrow} & \mathcal{N} \\ {\scriptstyle i}\uparrow & \nearrow{\scriptstyle g} & \\ (X, f) & & \end{array}$$

Q. *Is $\mathbb{Z}[*, G/H]_G$ a free Bredon module then?*

Yes! And to answer that we use the following lemma:

Lemma 1.3.7 (Yoneda Type Formula, [12], Lem. 1.14)**.** *Let G be a group and $\mathcal{F}$ a family of subgroups of G. Given $H \in \mathcal{F}$ and a Bredon module $\mathcal{M}$ over $\mathcal{O}_{\mathcal{F}}G$, there exists an isomorphism of abelian groups*

$$e_H\colon \mathrm{Hom}_{\mathcal{F}}(\mathbb{Z}[*, G/H]_G, \mathcal{M}) \longrightarrow \mathcal{M}(G/H)$$

where e_H is the evaluation map given by $e_H(\alpha) := \alpha_H(\mathrm{id}_{G/H})$. This isomorphism is natural in $\mathcal{M}$.

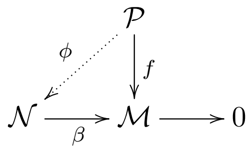
Let $\mathcal{N}$ be a Bredon module over $\mathcal{O}_{\mathcal{F}}G$ and $g\colon (X, f) \longrightarrow \mathcal{N}$ a morphism of $\mathcal{F}$-sets, where (X, f) is the generating set of $\mathbb{Z}[*, G/H]_G$ as given by Lemma 1.3.5. Then, there exists a unique natural transformation $\alpha \in \mathrm{Hom}_{\mathcal{F}}(\mathbb{Z}[*, G/H]_G, \mathcal{N})$ such that $\alpha_H(\mathrm{id}_{G/H}) = g(\mathrm{id}_{G/H})$.

1.4 Projective and flat Bredon modules

Q. *Projective?*

Definition 1.4.1. *A Bredon module $\mathcal{P}$ over $\mathcal{O}_{\mathcal{F}}G$ is called* ***projective*** *if for any morphism $f\colon \mathcal{P} \longrightarrow \mathcal{M}$ to a Bredon module $\mathcal{M}$ over $\mathcal{O}_{\mathcal{F}}G$ and any epimorphism $\beta\colon \mathcal{N} \longrightarrow \mathcal{M}$ of Bredon modules over $\mathcal{O}_{\mathcal{F}}G$ there is a unique morphism $\phi\colon \mathcal{P} \longrightarrow \mathcal{N}$ such that $\beta \circ \phi = f$.*

In cathegory theory, this is the ***universal property of projective functors*** *and is represented by the commutative diagram below:*



Q. *What is the relation between projective and free Bredon modules?*

The answer is given by the next proposition, whose proof can be found in any homological algebra book, as [4], Prop. 8.2. One has to follow the same steps of the proof in the category theory setting:

Proposition 1.4.2 ([12], Prop. 1.23)**.** *Let $\mathcal{P}$ be a Bredon module over $\mathcal{O}_{\mathcal{F}}G$. Then the following statements for $\mathcal{P}$ are equivalent:*

i) *$\mathcal{P}$ is projective;*

ii) *every exact sequence $0 \longrightarrow \mathcal{M} \longrightarrow \mathcal{N} \longrightarrow \mathcal{P} \longrightarrow 0$ splits;*

iii) *$\mathrm{Hom}_{\mathcal{F}}(\mathcal{P}, *)$ is an exact functor;*

iv) *$\mathcal{P}$ is a direct summand of a free Bredon module over $\mathcal{O}_{\mathcal{F}}G$.*

Therefore, free Bredon modules are projective.

Q. *What about flat Bredon modules?*

The *tensor product* of a right Bredon module $\mathcal{M}$ and a left Bredon module $\mathcal{N}$ is defined in [23] by

$$\mathcal{M} \otimes_{\mathcal{F}} \mathcal{N} := P/Q$$

where

$$P = \coprod_{H \in \mathcal{F}} \mathcal{M}(G/H) \otimes_{\mathbb{Z}} \mathcal{N}(G/H)$$

and Q is subgroup of P generated by the elements of the form $\gamma^*(m) \otimes n - m \otimes \gamma_*(n)$, where $\gamma \in [G/K, G/H]_G$, $m \in \mathcal{M}(G/H)$, $n \in \mathcal{N}(G/K)$, $H, K \in \mathcal{F}$, and $\gamma^* := \mathcal{M}(\gamma)$, $\gamma_* := \mathcal{N}(\gamma)$.

Definition 1.4.3. *A right (or left) Bredon module $\mathcal{M}$ is called* ***flat*** *if the functor $\mathcal{M} \otimes_{\mathcal{F}} *$ (or $* \otimes_{\mathcal{F}} \mathcal{M}$) is exact.*

Q. *What is the relation between flat, projective and free Bredon modules?*

Proposition 1.4.4 ([12], Prop. 1.28)**.** *Projective Bredon modules are flat.*

Consequently, free modules are flat.

1.5 Chain Complexes

Q. *Given a topological space X, how does one define a chain complex of Bredon modules?*

Let G be a group and $\mathcal{F}$ a family of subgroups of G. In order to answer that question, we need to introduce a topological space on which G acts cellularly, known as a *G-CW-complex.*

Definition 1.5.1. *A G-CW**-complex** is a CW-complex on which G acts cellularly. Moreover, if $g \in G$ fixes a cell, then g fixes the cell pointwise.*

A formal definition of such space is given in [23], pp. 6f, considering G a topological group.

If H is a subgroup of G and X is a G-CW-complex, then X^H is a CW-subcomplex of X consisting of all points fixed by H.

Let Δ_n denote the set of the n-cells of X. we define the Bredon module over $\mathcal{O}_{\mathcal{F}}G$

$$C_n^{\mathcal{F}}(X) := \mathbb{Z}[*, \Delta_n]_G$$

where, for some $H \in \mathcal{F}$, $\mathbb{Z}[G/H, \Delta_n]_G$ denotes the free abelian group with basis the set $[G/H, \Delta_n]_G$ of all G-maps from G/H to Δ_n.

Q. *Is $\mathbb{Z}[*, \Delta_n]_G$ free?*

Sometimes:

Proposition 1.5.2 ([12], Prop. 1.18)**.** *Let $\mathcal{S}$ be the set of all subgroups of G such that $H \in \mathcal{S}$ if $Hx = x$ for some $x \in \Delta_n$. If $\mathcal{S} \subseteq \mathcal{F}$, then $\mathbb{Z}[*, \Delta_n]_G$ is free.*

Lemma 1.5.3. *Given $H \in \mathcal{F}$,*

$$C_n^{\mathcal{F}}(X)(G/H) \cong C_n(X^H)$$

Notation: $C_n(X^H)$ is the nth element of the classical cellular chain complex $(C_*(X^H), d_{H,*})$, where, for each $n \geq 0$, $C_n(X^H) := \mathbb{Z}[\Delta_n^H]$.

Proof. Take the homomorphisms of abelian groups $\alpha\colon \mathbb{Z}[*, \Delta_n]_G \longrightarrow \mathbb{Z}[\Delta_n^H]$ and $\beta\colon \mathbb{Z}[\Delta_n^H] \longrightarrow \mathbb{Z}[*, \Delta_n]_G$, satisfying $\alpha(f) = f(H)$ and $\beta(\sigma) = g$, such that $f \in [*, \Delta_n]_G$, $\sigma \in \Delta_n^H$ and $g\colon G/H \longrightarrow \Delta_n$ assigns $H \mapsto \sigma$. For all $h \in H$, we have that $g(hH) = h\sigma = \sigma = g(H)$, thus g is well defined. Clearly, $\beta = \alpha^{-1}$. □

Lemma 1.5.4. *For each $n \geq 1$, the functor*

$$d_n\colon C_n^{\mathcal{F}}(X) \longrightarrow C_{n-1}^{\mathcal{F}}(X)$$

given by the set of morphisms $d_{H,n}\colon C_n(X^H) \longrightarrow C_{n-1}(X^H)$, $H \in \mathcal{F}$, is a natural transformation.

Proof. Let $H, K \in \mathcal{F}$ and $\gamma \in [G/H, G/K]_G$, $\gamma(H) = gK$ for some $g \in G$. For each $n \geq 0$,

$$\gamma_n^*\colon C_n^{\mathcal{F}}(X)(G/K) \longrightarrow C_n^{\mathcal{F}}(X)(G/H)$$

is the map assigning $f \longrightarrow f \circ \gamma$, $f \in [*, \Delta_n]_G$. From Lemma 1.5.3, γ_n^* is the morphism induced by the map $X^K \longrightarrow X^H$ which assigns $x \mapsto gx$. Therefore, γ^* defines a chain map, implying the commutativity of the diagram below:

$$\begin{array}{ccc} C_n^{\mathcal{F}}(X)(G/K) & \xrightarrow{\gamma_n^*} & C_n^{\mathcal{F}}(X)(G/H) \\ \downarrow{\scriptstyle d_{K,n}} & & \downarrow{\scriptstyle d_{H,n}} \\ C_{n-1}^{\mathcal{F}}(X)(G/K) & \xrightarrow[\gamma_{n-1}^*]{} & C_{n-1}^{\mathcal{F}}(X)(G/H) \end{array}$$

Hence, d_n is a natural transformation. □

Q. *What about the augmentation map?*

The *augmentation map* is the natural transformation $\epsilon\colon C_0^{\mathcal{F}}(X) \longrightarrow \mathbb{Z}_{\mathcal{F}}$ given by the collection of the augmentation maps $\epsilon_H\colon C_0(X^H) \longrightarrow \mathbb{Z}$, $H \in \mathcal{F}$.

Hence, the chain complex of the G-CW-complex X is

$$\ldots \longrightarrow C_n^{\mathcal{F}}(X) \xrightarrow{d_n} C_{n-1}^{\mathcal{F}}(X) \longrightarrow \ldots \longrightarrow C_1^{\mathcal{F}}(X) \xrightarrow{d_1} C_0^{\mathcal{F}}(X) \xrightarrow{\epsilon} \mathbb{Z}_{\mathcal{F}} \longrightarrow 0$$

Q. *Can this sequence be exact?*

Lemma 1.5.5 ([12], Lem. 2.8)**.** *If X^H is contractible for every $H \in \mathcal{F}$, then the chain complex of X is an exact sequence.*

1.6 Classifying space $E_{\mathcal{F}}G$

In this section, we define a special kind of G-CW-complex:

Definition 1.6.1. *A classifying space of G for the family $\mathcal{F}$ of subgroups of G is a G-CW-complex X, also called a* ***model for*** *$E_{\mathcal{F}}G$, that satisfies:*

- *$X^H = \emptyset$ for every subgroup H of G which is not in $\mathcal{F}$;*
- *X^H is contractible for every $H \in \mathcal{F}$.*

Let X be a model for $E_{\mathcal{F}}G$. Note that, if $H \subseteq K \in \mathcal{F}$ and $x \in X^K$, then $hx = x$, $\forall h \in H \Rightarrow x \in X^H$. Moreover, X is contractible since $\{1\} \in \mathcal{F}$.

Q. *Does a model for $E_{\mathcal{F}}G$ always exist?*

Yes! And details can be found in [25], Thm. 1.9.

Q. *What does it classify?*

The best answer for that, which can be found in [25], p.7, is that $E_{\mathcal{F}}G$ is a terminal object in the G-homotopy category of G-CW-complexes, whose isotropy groups belong to $\mathcal{F}$. In particular, two models for $E_{\mathcal{F}}G$ are G-homotopic equivalent and for two families $\mathcal{F}_o \subseteq \mathcal{F}_1$ there exists, up to G-homotopy, precisely one G-map $E_{\mathcal{F}_0}G \longrightarrow E_{\mathcal{F}_1}G$.

Q. *What would be the equivalent general version of BG?*

Let X be a model for $E_{\mathcal{F}}G$. The quotient X/G of X by the cellular action of G is called a *model for $B_{\mathcal{F}}G$*. When $\mathcal{F} = \{\{1\}\}$, a model for $B_{\mathcal{F}}G$ is a model for BG. Differently from the classical theory where one can build a model for BG given a presentation of G, in the Bredon setting we are interested in building models for $E_{\mathcal{F}}G$.

Definition 1.6.2. *Assume that there exists a finite dimensional model for $E_{\mathcal{F}}G$. Then the least integer $n \geq 0$ for which there exists an n-dimensional model for $E_{\mathcal{F}}G$ is called the* ***Bredon geometric dimension of*** *G* ***for the family*** *$\mathcal{F}$ and we denote this by $\mathrm{gd}_{\mathcal{F}}G := n$. If there exists no finite dimensional model for $E_{\mathcal{F}}G$, then we set $\mathrm{gd}_{\mathcal{F}}G := \infty$.*

Q. *Why $E_{\mathcal{F}}G$?*

When $\mathcal{F}$ contains only the trivial subgroup, then $E_{\{1\}}G = EG$ and a model for $E_{\{1\}}G$ is, for example, the universal cover of a $K(G,1)$. Moreover, if $\mathcal{F}$ contains G, then a model for $E_{\mathcal{F}}G$ is a singleton space. In this case, we have that $\mathrm{gd}_{\mathcal{F}}G = 0$.

Notation: $E_{\mathcal{F}_{fin}}G := \underline{E}G$ and $E_{\mathcal{F}_{vc}}G := \underline{\underline{E}}G$.

Q. *How does one build a model for $E_{\mathcal{F}}G$?*

Building such a classifying space can be difficult. Below, we show a non-trivial example given in [18], Ex.3:

Example 1.6.3. *Let $\{H_i\}_{i\in I}$ be the set of all maximal infinite cyclic subgroups of $\mathbb{Z}^2$, whose index set I can be identified with $\mathbb{Z}$. A model X_i for $E(\mathbb{Z}^2/H_i)$ is a real line on which $\mathbb{Z}^2/H_i$ acts by translation. Let $p\colon \mathbb{Z}^2 \longrightarrow \mathbb{Z}^2/H_i$ be the canonical projection map between both groups. If $g \in \mathbb{Z}^2$ and $x \in X_i$, we define the action*

$$g \cdot x := p(g) \cdot x$$

Note that, if $g \in H_i$ then $g \cdot x = x$.
Now, take

$$X := \left(\coprod_{i\in I} (X_i * X_{i+1}) \right) / \left(\alpha_i(x) \sim \beta_i(x)\right)$$

where $*$ represents the join of the spaces and $\alpha_i\colon X_i \hookrightarrow X_{i-1} * X_i$ and $\beta_i\colon X_i \hookrightarrow X_i * X_{i+1}$ are embedding maps of topological spaces. If H is a (virtually) cyclic subgroup of $\mathbb{Z}^2$, then H is contained in a unique H_i, for some $i \in I$. Moreover, $X_i = X^{H_i} = X^H$ is contractible and, by construction, no subgroup of $\mathbb{Z}^2$ isomorphic to $\mathbb{Z}^2$ stabilises a point in X. In [12], Fig.3, more details and an explicit picture of this 3-dimensional space space can be found. Therefore, X is a model for $\underline{\underline{E}}\mathbb{Z}^2$.

Q. *But is there any method to build these models?*

Yes, and it is given by Lück and Weiermann ([26], Thm. 2.3). Let $\mathcal{F} \subseteq \mathcal{G}$ be two families and $\sim$ an equivalence relation on $\mathcal{G}\backslash\mathcal{F}$ satisfying the following properties:

- if $H, K \in \mathcal{G}\backslash\mathcal{F}$ with $H \subseteq K$, then $H \sim K$;
- if $H, K \in \mathcal{G}\backslash\mathcal{F}$ and $g \in G$, then $H \sim K \Leftrightarrow gHg^{-1} \sim gKg^{-1}$.

Denote by $[\mathcal{G}\backslash\mathcal{F}]$ the set of equivalence classes given by $\sim$. If $H \in \mathcal{G}\backslash\mathcal{F}$, then $[H]$ denotes the equivalence class of H. Now, we define

$$N_G[H] := \left\{g \in G \mid [g^{-1}Hg] = [H]\right\},$$

$$\mathcal{F} \cap N_G[H] := \{K \in \mathcal{F} \mid K \subseteq N_G[H]\},$$

$$\mathcal{G}[H] := \{K \subseteq N_G[H] \mid K \in \mathcal{G}\backslash\mathcal{F},\ [K] = [H]\} \cup (\mathcal{F} \cap N_G[H]).$$

Theorem 1.6.4. *Let I be a complete system of representatives $[H]$ of the G-orbits in $[\mathcal{G}\backslash\mathcal{F}]$ under the G-action coming from conjugation. Choose arbitrary $N_G[H]$-CW-models for $E_{\mathcal{F}\cap N_G[H]}(N_G[H])$ and $E_{\mathcal{G}[H]}(N_G[H])$, and an arbitrary G-CW-model for $E_{\mathcal{F}}G$. Define a G-CW-complex X by the cellular G-push-out*

$$\begin{array}{ccc} \coprod_{[H]\in I} G \times_{N_G[H]} E_{\mathcal{F}\cap N_G[H]}N_G[H] & \xrightarrow{\quad i \quad} & E_{\mathcal{F}}G \\ \Big\downarrow \coprod_{[H]\in I} id_G \times_{N_G[H]} f_{[H]} & & \Big\downarrow \\ \coprod_{[H]\in I} G \times_{N_G[H]} E_{\mathcal{G}[H]}N_G[H] & \xrightarrow{\qquad} & X \end{array}$$

such that $f_{[H]}$ is a cellular $N_G[H]$-map for every $[H] \in I$ and i is an inclusion of G-CW-complexes, or such that every map $f_{[H]}$ is an inclusion of $N_G[H]$-CW-complexes for every $[H] \in I$ and i is a cellular G-map. Then, X is a model for $E_{\mathcal{G}}G$.

In the same paper it is explained that, from the theorem, one can conclude that there exists an n-dimensional model for $E_{\mathcal{G}}G$ if there exists an n-dimensional model for $E_{\mathcal{F}}G$ and, for every $[H] \in I$, an $(n-1)$-dimensional model for $E_{\mathcal{F}\cap N_G[H]}N_G[H]$ and an n-dimensional model for $E_{\mathcal{G}[H]}N_G[H]$.

1.7 Bredon homological and cohomological dimensions

Let G be a group and $\mathcal{F}$ a family of subgroups of G.

Definition 1.7.1. *The **Bredon homological dimension of** G **over** $\mathcal{F}$, denoted by* $\mathrm{hd}_{\mathcal{F}}G$*, is the smallest positive number n such that there exists a flat resolution of Bredon modules over $\mathcal{O}_{\mathcal{F}}G$ of the form*

$$0 \to \mathcal{Q}_n \to \ldots \to \mathcal{Q}_1 \to \mathcal{Q}_0 \to \mathbb{Z}_{\mathcal{F}} \to 0$$

Definition 1.7.2. *The* ***Bredon cohomological dimension of*** G ***over*** $\mathcal{F}$*, denoted by* $\mathrm{cd}_{\mathcal{F}}(G)$*, is the smallest number* n *such that there exists a projetive resolution of Bredon modules over* $\mathcal{O}_{\mathcal{F}}G$ *of the form*

$$0 \to \mathcal{P}_n \to \ldots \to \mathcal{P}_1 \to \mathcal{P}_0 \to \mathbb{Z}_{\mathcal{F}} \to 0$$

Q. *What is the relation between* $\mathrm{hd}_{\mathcal{F}}(G)$ *and* $\mathrm{cd}_{\mathcal{F}}(G)$*?*

Since projective Bredon modules are flat, every projective resolution is a flat resolution. Therefore,

$$\mathrm{hd}_{\mathcal{F}}G \leq \mathrm{cd}_{\mathcal{F}}G$$

These dimensions can also be defined in terms of *Bredon homological and cohomological groups.* Consider a projective resolution of $\mathbb{Z}_{\mathcal{F}}$ over $\mathcal{O}_{\mathcal{F}}G$

$$\mathcal{P}_{\mathcal{F}} : \ldots \longrightarrow \mathcal{P}_n \longrightarrow \mathcal{P}_{n-1} \longrightarrow \ldots \longrightarrow \mathcal{P}_1 \longrightarrow P_0 \longrightarrow \mathbb{Z}_{\mathcal{F}} \longrightarrow 0$$

Definition 1.7.3. *Let* $\mathcal{M}$ *and* $\mathcal{N}$ *be, respectively, right and left Bredon modules over* $\mathcal{O}_{\mathcal{F}}G$*:*

- *The* ***Bredon homological groups with coefficients in*** $\mathcal{N}$ *are defined, for* $n \geq 0$*, by*
$$H_n^{\mathcal{F}}(G, \mathcal{N}) := H_n(\mathcal{P}_{\mathcal{F}} \otimes_{\mathcal{F}} \mathcal{N})$$
- *The* ***Bredon cohomological groups with coefficients in*** $\mathcal{M}$ *are defined, for* $n \geq 0$*, by*
$$H^n_{\mathcal{F}}(G, \mathcal{M}) := H^n(\mathrm{Hom}_{\mathcal{F}}(\mathcal{P}_{\mathcal{F}}, \mathcal{M}))$$

Now, we formulate:

Definition 1.7.4.

$$\mathrm{hd}_{\mathcal{F}}(G) := \sup \left\{ n \mid H_n^{\mathcal{F}}(G, \mathcal{M}) \neq 0 \text{ for some left Bredon module } \mathcal{M} \text{ over } \mathcal{O}_{\mathcal{F}}G \right\}$$

$$\mathrm{cd}_{\mathcal{F}}(G) := \sup \left\{ n \mid H^n_{\mathcal{F}}(G, \mathcal{M}) \neq 0 \text{ for some right Bredon module } \mathcal{M} \text{ over } \mathcal{O}_{\mathcal{F}}G \right\}$$

Q. *As in the classical case, can these Bredon homology groups be defined in terms of the classifying space of* G *over the family* $\mathcal{F}$*?*

Yes!

Theorem 1.7.5 ([12], Thm. 4.2)**.** *For all $n \geq 0$, there exist isomorphisms of abelian groups*

$$H_n^{\mathcal{F}}(G; \mathbb{Z}_{\mathcal{F}}) \cong H_n(B_{\mathcal{F}}G) \quad \text{and} \quad H_{\mathcal{F}}^n(G; \mathbb{Z}_{\mathcal{F}}) \cong H^n(B_{\mathcal{F}}G).$$

Corollary 1.7.6 ([12], Cor. 4.3)**.** *If $H_n(B_{\mathcal{F}}G) \neq 0$, then $\mathrm{hd}_{\mathcal{F}}G \geq n$. Likewise, if $H^m(B_{\mathcal{F}}G) \neq 0$, then $\mathrm{cd}_{\mathcal{F}}G \geq m$.*

Q. *What is the connection between $\mathrm{hd}_{\mathcal{F}}G$, $\mathrm{cd}_{\mathcal{F}}G$ and $\mathrm{gd}_{\mathcal{F}}G$?*

If X is a model for $E_{\mathcal{F}}G$, then by Proposition 1.5.2 and Lemma 1.5.5, the chain complex of X is a free resolution of $\mathbb{Z}_{\mathcal{F}}$ of Bredon modules over $\mathcal{O}_{\mathcal{F}}G$; hence, a projective resolution of $\mathbb{Z}_{\mathcal{F}}$, where each term $C_n^{\mathcal{F}}(X)$ corresponds to the n-cells of X. Therefore,

$$\mathrm{hd}_{\mathcal{F}}G \leq \mathrm{cd}_{\mathcal{F}}G \leq \mathrm{gd}_{\mathcal{F}}G$$

Other results comparing $\mathrm{hd}_{\mathcal{F}}G$ and $\mathrm{cd}_{\mathcal{F}}G$ are known, as for example

Theorem 1.7.7 ([12], Thm. 3.13)**.** *If G and $\mathcal{F}$ are countable, then $\mathrm{cd}_{\mathcal{F}}G \leq \mathrm{hd}_{\mathcal{F}}G + 1$.*

Our main interest though at this part of the thesis are results involving $\mathrm{cd}_{\mathcal{F}}G$ and $\mathrm{gd}_{\mathcal{F}}G$, therefore that will be our focus.

Q. *So when does the equality between $\mathrm{cd}_{\mathcal{F}}G$ and $\mathrm{gd}_{\mathcal{F}}G$ hold?*

Proposition 1.7.8 ([12], Prop. 3.20)**.** *Given any family $\mathcal{F}$ of subgroups of a group G,*

$$\mathrm{cd}_{\mathcal{F}}G = 0 \text{ if and only if } \mathrm{gd}_{\mathcal{F}}G = 0 \text{ if and only if } G \in \mathcal{F}.$$

A complete proof of $\mathrm{cd}_{\mathcal{F}}G = 0 \Leftrightarrow \mathrm{gd}_{\mathcal{F}}G = 0$ can be found in [12], Prop. 3.20. The implication $G \in \mathcal{F} \;\Rightarrow \mathrm{gd}_{\mathcal{F}}G = 0$ was already mentioned before in Section 1.6. Now, if $\mathrm{gd}_{\mathcal{F}}G = 0$, then the singleton space satisfies the condition to be a model for $E_{\mathcal{F}}G$. Clearly, $G \in \mathcal{F}$.

Theorem 1.7.9 ([24], Thm. 0.1)**.** *Given any family $\mathcal{F}$ of subgroups of a group G,*

$$\mathrm{cd}_{\mathcal{F}}G \leq \mathrm{gd}_{\mathcal{F}}G \leq \max\{3, \mathrm{cd}_{\mathcal{F}}G\}$$

If $\mathrm{cd}_{\mathcal{F}}G = 2$, the equality might not always hold. In [2] one can find examples of groups satisfying $\underline{\mathrm{cd}}G = 2$ but $\underline{\mathrm{gd}}G = 3$. In [13], it is shown that for some Coxeter groups one can also have $\underline{\underline{\mathrm{cd}}}G = 2$ but $\underline{\underline{\mathrm{gd}}}G = 3$.

Q. *What about* $\mathrm{cd}_{\mathcal{F}}G = 1$*?*

When $\mathrm{cd}G \leq 1$, it is well known that G satisfies $\mathrm{gd}G \leq 1$, due to Stallings [30] and Swan [31]. Dunwoody [10] later proved that, for any group G, $\underline{\mathrm{cd}}G \leq 1$ implies $\underline{\mathrm{gd}}G \leq 1$. Recently, Degrijse [7] showed that, if $\underline{\underline{\mathrm{cd}}}G \leq 1$, then $\underline{\underline{\mathrm{gd}}}G \leq 1$.

For other families of subgroups of G, the following still holds:

Conjecture 1.7.10. *Given a group G and any family $\mathcal{F}$ of subgroups of G, if* $\mathrm{cd}_{\mathcal{F}}G \leq 1$ *then* $\mathrm{gd}_{\mathcal{F}}G \leq 1$*.*

1.8 The family $\mathcal{F}_2$

The Conjecture 1.7.10 on groups with Bredon cohomological dimension one is a difficult problem which at first sight can not be approached without working with specific families or groups. The proofs given considering each one of the three families for which the conjecture is true rely on particular results related to these families. In an attempt to partially solve this problem, I considered the family that seemed most natural after $\mathcal{F}_{fin}$ and $\mathcal{F}_{vc}$. Given a group G, we take $\mathcal{F}_2$ to denote the family of all subgroups of G that are virtually $\mathbb{Z}^n$, for $n = 0, 1, 2$. It means that $\mathcal{F}_2$ consists of all finite, infinite virtually cyclic and virtually $\mathbb{Z}^2$ subgroups of G. I started by following the same steps of the proof given in [7] for $\mathcal{F}_{vc}$. One of the first challenges is to prove the following:

Lemma 1.8.1 ([7], Lem. 2.3 (i))**.** *If G is a group with $\underline{\underline{\mathrm{cd}}}G \leq 1$, then G does not contain a copy of $\mathbb{Z}^2$.*

The proof of this lemma depends on the next result:

Proposition 1.8.2 ([12], Prop. 3.32)**.** *Let G be a group and $\mathcal{F}$ a family of subgroups of G. If H is a subgroup of G such that $\mathcal{F} \cap H \subseteq \mathcal{F}$, then*

$$\mathrm{cd}_{\mathcal{F}\cap H}H \leq \mathrm{cd}_{\mathcal{F}}G$$

Proof of Lemma 1.8.1. By [26], Exm. 5.21, $\underline{\mathrm{cd}}\mathbb{Z}^2 = 3$. If $\mathbb{Z}^2$ is a subgroup of G and $\mathcal{F}_{vc}$ is the family of all virtually cyclic subgroups of G, then, by Proposition 1.8.2,

$$3 = \underline{\mathrm{cd}}\mathbb{Z}^2 = \mathrm{cd}_{\mathcal{F}_{vc}\cap\mathbb{Z}^2}\mathbb{Z}^2 \leq \underline{\mathrm{cd}}G \leq 1$$

which is clearly a contradiction. □

Similarly, I prove the following:

Lemma 1.8.3. *If G is a group with $\mathrm{cd}_{\mathcal{F}_2} G \leq 1$, then G does not contain a copy of $\mathbb{Z}^3$.*

The proof follows from the next proposition:

Proposition A. *For any $n \geq 3$,*

$$\mathrm{cd}_{\mathcal{F}_2} \mathbb{Z}^n = \mathrm{gd}_{\mathcal{F}_2} \mathbb{Z}^n = n + 2$$

Sections 1.9 and 1.10 will be dedicated to prove Proposition A.

Proof of Lemma 1.8.3. By Proposition A, $\mathrm{cd}_{\mathcal{F}_2 \cap \mathbb{Z}^3} \mathbb{Z}^3 = 5$. Then, by Proposition 1.8.2,

$$5 = \mathrm{cd}_{\mathcal{F}_2 \cap \mathbb{Z}^3} \mathbb{Z}^3 \leq \mathrm{cd}_{\mathcal{F}_2} G \leq 1$$

which is obviously a contradiction. □

1.9 A model for $E_{\mathcal{F}_2}\mathbb{Z}^n$

In order to prove Proposition A, we build a model for $E_{\mathcal{F}_2}\mathbb{Z}^n$, $n \geq 3$, and show that $\mathrm{gd}_{\mathcal{F}_2} \mathbb{Z}^n = n + 2$. Then, the result follows by Theorem 1.7.9.
Before this construction, a few remarks:

Remark 1.9.1. *Let G be an abelian group, H a subgroup of G and $\mathcal{F}$ a family of subgroups of G:*

- *$E_{\mathcal{F}}G$ will be used to denote a model for $E_{\mathcal{F}}G$;*
- *$E_{\leq H}G$ stands for a model of the classifying space of G over the family consisting of all subgroups of H;*
- *Given $i \geq 0$, a $\mathbb{Z}^i$-subgroup of G is a subgroup isomorphic to $\mathbb{Z}^i$;*
- *H is a* maximal *$\mathbb{Z}^i$-subgroup of G if no other $\mathbb{Z}^i$-subgroup of G contains H as a proper subgroup.*

Remark 1.9.2. *Observe that if G is not abelian and H is a subgroup of G, the set of all subgroups of H is a family if and only if H is normal in G.*

Let I be the indexing set of all maximal $\mathbb{Z}$-subgroups of $\mathbb{Z}^n$. Let J_i denote a maximal $\mathbb{Z}$-subgroup, $i \in I$. A standard Bredon categorical argument ([25], p.7) yields a $\mathbb{Z}^n$-map $\sigma_i \colon E\mathbb{Z}^n \longrightarrow E_{\leq J_i}\mathbb{Z}^n$, for each $i \in I$.

Now, let $\sim$ define the equivalence relation in $\mathcal{F}_{vc} \backslash \mathcal{F}_{fin}$ given by, for any $H, S \in \mathcal{F}_{vc} \backslash \mathcal{F}_{fin}$

$$H \sim S \Leftrightarrow \operatorname{rk}(H \cap S) = 1$$

where rk stands for *rank*. Then, $N_{\mathbb{Z}^n}[H] = \mathbb{Z}^n$ and $\mathcal{G}[H]$ is the family of all subgroups of H. By Theorem 1.6.4, the push-out below gives us a model X_0 for $\underline{\underline{E}}\mathbb{Z}^n$:

$$\begin{array}{ccc}
\bigsqcup_{i \in I} E\mathbb{Z}^n & \xrightarrow{\ i\ } & E\mathbb{Z}^n \\
\Big\downarrow {\scriptstyle \bigsqcup_{i\in I} \sigma_i} & & \Big\downarrow \\
\bigsqcup_{i \in I} E_{\leq J_i}\mathbb{Z}^n & \longrightarrow & X_0
\end{array}$$

The next step is where the distinction from the construction for $E_{\mathcal{F}_2}\mathbb{Z}^n$ as in [5] begins. In that case, one consider for any $H, S \in \mathcal{F}_2 \backslash F_{vc}$ the following equivalence relation:

$$H \sim K \Leftrightarrow \operatorname{rk}(H \cap S) = 2$$

Moreover, for any $H \in \mathcal{F}_2 \backslash F_{vc}$, we have that $N_{\mathbb{Z}^n}[H] = \mathbb{Z}^n$ and $\mathcal{G}[H] = \mathcal{F}_{vc} \cup \leq H$.
Let K be the indexing set of all maximal $\mathbb{Z}^2$-subgroups of $\mathbb{Z}^n$. For each $k \in K$, a maximal $\mathbb{Z}^2$-subgroup of $\mathbb{Z}^n$ will be denoted by H_k. For each $k \in K$, let f_k be a $\mathbb{Z}^n$-map $\underline{\underline{E}}\mathbb{Z}^n \longrightarrow E_{\mathcal{F}_{vc} \cup \leq H_k}\mathbb{Z}^n$. By Theorem 1.6.4, the following push-out

$$\begin{array}{ccc}
\bigsqcup_{k \in K} \underline{\underline{E}}\mathbb{Z}^n & \xrightarrow{\ i\ } & \underline{\underline{E}}\mathbb{Z}^n \\
\Big\downarrow {\scriptstyle \bigsqcup_{k\in K} f_k} & & \Big\downarrow \\
\bigsqcup_{k \in K} E_{\mathcal{F}_{vc} \cup \leq H_k}\mathbb{Z}^n & \longrightarrow & X
\end{array}$$

gives us a model X for $E_{\mathcal{F}_2}\mathbb{Z}^n$.

In our case, for each $k \in K$, consider the family $\left\{J_i^{(k)}\right\}_{i\in I_k}$ of all maximal $\mathbb{Z}$-subgroups contained in H_k. We construct the following push-out (1):

$$\begin{array}{ccc}
\coprod\limits_{i\in I_k} E\mathbb{Z}^n & \xrightarrow{\quad i \quad} & E\mathbb{Z}^n \\
\Big\downarrow {\scriptstyle \coprod\limits_{i\in I_k} \sigma_i} & & \Big\downarrow \\
\coprod\limits_{i\in I_k} E_{\leq J_i^{(k)}}\mathbb{Z}^n & \xrightarrow{\qquad} & X_1
\end{array}$$

The inclusion $E_{\mathcal{F}_{vc}\cap H_k}\mathbb{Z}^n \hookrightarrow \underline{\underline{E}}\mathbb{Z}^n$ is a $\mathbb{Z}^n$-map of spaces. Moreover, there exists a $\mathbb{Z}^n$-map $\alpha_k \colon E_{\mathcal{F}_{vc}\cap H_k}\mathbb{Z}^n \longrightarrow E_{\leq H_k}\mathbb{Z}^n$ for each $k \in K$. Then, we build the following push-out (2):

$$\begin{array}{ccc}
\coprod\limits_{k\in K} E_{\mathcal{F}_{vc}\cap H_k}\mathbb{Z}^n & \xrightarrow{\quad i \quad} & \underline{\underline{E}}\mathbb{Z}^n \\
\Big\downarrow {\scriptstyle \coprod\limits_{k\in K} \alpha_k} & & \Big\downarrow \\
\coprod\limits_{i\in I_k} E_{\leq H_k}\mathbb{Z}^n & \xrightarrow{\qquad} & X_2
\end{array}$$

Both push-outs (1) and (2) satisfy enough conditions from Theorem 1.6.4 on the maps in order to conclude that X_1 is a model for $E_{\mathcal{F}_{vc}\cap H_k}$ and X_2 is a model for $E_{\mathcal{F}_2}\mathbb{Z}^n$.

1.10 Bredon geometric dimension of $\mathbb{Z}^n$

In [26], Exm. 5.21, we have that $\underline{\underline{\mathrm{gd}}}\mathbb{Z}^n = n+1$, for $n \geq 2$. In order to find an upper bound to the geometric dimension of $E_{\mathcal{F}_2}\mathbb{Z}^n$, we first see that $\mathrm{gd}_{\mathcal{F}_{vc}\cap H_k}\mathbb{Z}^n \leq n+1$, since $E_{\mathcal{F}_{vc}\cap H_k}\mathbb{Z}^n$ embeds injectively in $\underline{\underline{E}}\mathbb{Z}^n$.

Lemma 1.10.1 ([5], Lem. 2.3)**.** *Let H be a maximal $\mathbb{Z}^r$-subgroup of $\mathbb{Z}^n$, $0 \leq r < n$. Then, $\mathbb{R}^{n-r}$ is a model for $E_{\leq H}\mathbb{Z}^n$ and*

$$\mathrm{gd}_{\leq H}\mathbb{Z}^n = n - r$$

Therefore, from the push-out (2) we conclude that $\mathrm{gd}_{\mathcal{F}_2}\mathbb{Z}^n \leq n+2$.

Knowing that $\mathrm{hd}_{\mathcal{F}_2}\mathbb{Z}^n \leq \mathrm{gd}_{\mathcal{F}_2}\mathbb{Z}^n$, it is sufficient to show that $\mathrm{hd}_{\mathcal{F}_2}\mathbb{Z}^n \geq n+2$. By the definition of Bredon homological dimension, we need to prove that $H_{n+2}^{\mathcal{F}_2}(G, \mathcal{M}) \neq 0$ for some left Bredon module $\mathcal{M}$ over $\mathcal{O}_{\mathcal{F}_2}\mathbb{Z}^n$. By Corollary 1.7.6, it is enough to show that $H_{n+2}\left(B_{\mathcal{F}_2}\mathbb{Z}^n\right) \neq 0$.

Applying the Mayer-Vietoris long exact sequence of homology on the push-out (2), we have:

$$0 \longrightarrow H_{n+2}(B_{\mathcal{F}_2}\mathbb{Z}^n) \longrightarrow \bigoplus_{k\in K} H_{n+1}\left(B_{\mathcal{F}_{vc}\cap H_k}\mathbb{Z}^n\right) \longrightarrow$$

$$\xrightarrow{\phi} H_{n+1}\left(\underline{\underline{B}}\mathbb{Z}^n\right) \oplus H_{n+1}\left(\bigsqcup_{k\in K} B_{\leq H_k}\mathbb{Z}^n\right) \longrightarrow \dots$$

However, note that $H_{n+1}\left(\bigsqcup_{k\in K} B_{\leq H_k}\mathbb{Z}^n\right) = 0$, since $B_{\leq H_k}\mathbb{Z}^n \cong \frac{\mathbb{R}^{n-2}}{\mathbb{Z}^n} = \mathbb{T}^{n-2}$. Therefore, we need to prove that the following map from this sequence is not injective:

$$\bigoplus_{k\in K} H_{n+1}\left(B_{\mathcal{F}_{vc}\cap H_k}\mathbb{Z}^n\right) \xrightarrow{\phi} H_{n+1}\left(\underline{\underline{B}}\mathbb{Z}^n\right)$$

Similarly, applying the Mayer-Vietoris long exact sequence of homology on the push-out (1) gives us

$$0 \longrightarrow \cancelto{0}{H_{n+1}(\mathbb{T}^n)} \oplus \cancelto{0}{H_{n+1}(\bigsqcup_{i\in I}\mathbb{T}^{n-1})} \longrightarrow H_{n+1}\left(\underline{\underline{B}}\mathbb{Z}^n\right) \longrightarrow$$

$$\longrightarrow H_n(\bigsqcup_{i\in I}\mathbb{T}^n) \xrightarrow{\gamma} H_n(\mathbb{T}^n) \oplus \cancelto{0}{H_n(\bigsqcup_{i\in I}\mathbb{T}^{n-1})} \longrightarrow \dots$$

Since the map $\gamma\colon \underset{i\in I}{\oplus}\mathbb{Z} \longrightarrow \mathbb{Z}$ cannot be injective, $\ker\gamma \cong H_{n+1}\left(\underline{\underline{B}}\mathbb{Z}^n\right) \neq 0$.

Now, fix a maximal $\mathbb{Z}$-subgroup J_{i_0} of $\mathbb{Z}^n$. Considering the isomorphism above, $\left\{1^{(i)} - 1^{(i_0)} \mid i \in I\right\}$ is a basis for $H_{n+1}\left(\underline{\underline{B}}\mathbb{Z}^n\right)$, where $1^{(i)}$ corresponds to the identity element of $\mathbb{Z}$ in the ith coordinate of the direct sum $\underset{i\in I}{\oplus}\mathbb{Z}$. Indeed,

$$\ker\gamma = \left\{\sum_{j=1}^{m} z_j^{(i_j)} \in \underset{i\in I}{\oplus}\mathbb{Z} \mid m \in \mathbb{N},\ i_j \in I,\ \gamma\left(\sum_{j=1}^{m} z_j^{(i_j)}\right) = \sum_{j=1}^{m} z_j = 0\right\}.$$

Let $\sum_{j=1}^{m} z_j^{(i_j)} \in \ker\gamma$. Then,

$$\begin{aligned}
\sum_{j=1}^{m} z_j^{(i_j)} &= \sum_{j=1}^{m} z_j^{(i_j)} + \sum_{j=1}^{m} z_j^{(i_0)} - \sum_{j=1}^{m} z_j^{(i_0)} \\
&= \sum_{j=1}^{m} \left(z_j^{(i_j)} - z_j^{(i_0)} \right) + \sum_{j=1}^{m} z_j^{(i_0)} \\
&= \sum_{j=1}^{m} z_j \left(1^{(i_j)} - 1^{(i_0)} \right) + \cancelto{0}{\left(\sum_{j=1}^{m} z_j \right)} \cdot 1^{(i_0)}
\end{aligned}$$

Now, for each $k \in K$ fix an $i_0 \in I$ such that $J_{i_0} \otimes \mathbb{R} \subset H_k \otimes \mathbb{R}$. Then, $H_{n+1}\left(B_{\mathcal{F}_{vc} \cap H_k} \mathbb{Z}^n\right)$ is the group generated by the basis $\left\{1^{(i)} - 1^{(i_0)} \mid i \in I_k\right\}$, where I_k is the subset of I that indexes every maximal $\mathbb{Z}$-subgroup J_i in $\mathbb{Z}^n$ such that $J_i \otimes \mathbb{R} \subset H_k \otimes \mathbb{R}$.

Claim 1.10.2. *ϕ is not injective.*

Proof. Given $x = \sum_{k=1}^{m} \left(\sum_{j=1}^{m_k} z_{k,j}^{(i_{k,j})} \left(1^{(i_{k,j})} - 1^{(k,0)} \right) \right) \in \bigoplus_{k \in K} H_{n+1}\left(B_{\mathcal{F}_{vc} \cap H_k} \mathbb{Z}^n\right)$,

$$\phi(x) = \sum_{k=1}^{m} \left(\sum_{j=1}^{m_k} z_{k,j}^{(i_{k,j})} \left[\left(1^{(i_{k,j})} - 1^{(i_0)} \right) - \left(1^{(i_{k,0})} - 1^{(i_0)} \right) \right] \right)$$

Given $k_1 \in K$, take $i_{k_1,0} \in I$ corresponding to the fixed maximal $\mathbb{Z}$-subgroup $J_{i_{k_1,0}}$ from the basis of $H_{n+1}\left(B_{\mathcal{F}_{vc} \cap H_{k_1}} \mathbb{Z}^n\right)$. We see that

$$J_{i_{k_1,0}} \otimes \mathbb{R} \subset (H_{k_1} \otimes \mathbb{R}) \cap (H_{k_2} \otimes \mathbb{R})$$

for some $k_2 \in K,\ k_1 \neq k_2$. Now, let $J_{i_{k_2,0}}$ be another fixed maximal $\mathbb{Z}$-subgroup from the basis of $H_{n+1}\left(B_{\mathcal{F}_{vc} \cap H_{k_2}} \mathbb{Z}^n\right)$. Then,

$$J_{i_{k_2,0}} \otimes \mathbb{R} \subset (H_{k_2} \otimes \mathbb{R}) \cap (H_{k_3} \otimes \mathbb{R})$$

for some $k_3 \in K,\ k_3 \neq k_1, k_2$. Again, let $J_{i_{k_3,0}}$ be another fixed maximal $\mathbb{Z}$-subgroup in $H_{n+1}\left(B_{\mathcal{F}_{vc} \cap H_{k_3}} \mathbb{Z}^n\right)$. If $J_{i_{k_3,0}} \otimes \mathbb{R} \subset (H_{k_3} \otimes \mathbb{R}) \cap (H_{k_1} \otimes \mathbb{R})$, then take $x \in \bigoplus_{k \in K} H_{n+1}\left(B_{\mathcal{F}_{vc} \cap H_k} \mathbb{Z}^n\right)$, such that

$$x = \left(1^{(i_{k_3,0})} - 1^{(i_{k_1,0})} \right)_{k_1} + \left(1^{(i_{k_1,0})} - 1^{(i_{k_2,0})} \right)_{k_2} + \left(1^{(i_{k_2,0})} - 1^{(i_{k_3,0})} \right)_{k_3}.$$

Then,

$$\phi(x) = \left(1^{(i_{k_3},0)} - 1^{(i_{k_1},0)}\right) + \left(1^{(i_{k_1},0)} - 1^{(i_{k_2},0)}\right) + \left(1^{(i_{k_2},0)} - 1^{(i_{k_3},0)}\right) = 0.$$

Otherwise, let J_i be a maximal $\mathbb{Z}$-subgroup in $\mathbb{Z}^n$ such that $J_i \otimes \mathbb{R} \subset (H_{k_3} \otimes \mathbb{R}) \cap (H_{k_1} \otimes \mathbb{R})$. Then, take $x \in \bigoplus_{k \in K} H_{n+1}\left(B_{\mathcal{F}_{vc} \cap H_k} \mathbb{Z}^n\right)$ where

$$x = \left(1^{(i)} - 1^{(i_{k_1},0)}\right)_{k_1} + \left(1^{(i_{k_1},0)} - 1^{(i_{k_2},0)}\right)_{k_2} + \left[\left(1^{(i_{k_2},0)} - 1^{(i_{k_3},0)}\right) - \left(1^{(i)} - 1^{(i_{k_3},0)}\right)\right]_{k_3}.$$

Hence,

$$\phi(x) = \left(1^{(i)} - 1^{(i_{k_1},0)}\right) + \left(1^{(i_{k_1},0)} - 1^{(i_{k_2},0)}\right) + \left(1^{(i_{k_2},0)} - 1^{(i_{k_3},0)}\right) - \left(1^{(i)} - 1^{(i_{k_3},0)}\right) = 0.$$

□

Chapter 2

Relative ends and splittings of groups

In Section 2.11, we prove the following:

Main Theorem. *Let $H \leq G$ be finitely generated groups satisfying:*

- $2 \leq \widetilde{e}(G, H) < \infty$*;*
- $H \leq_f \mathrm{Comm}_G(H)$*;*
- $\forall g \in G$*, if* gH *is* H*-finite, then* $g \in \mathrm{Comm}_G(H)$*.*

If G contains an H-proper almost invariant subset A such that $AH = A$, then G admits a non-trivial splitting over a subgroup commensurable with H.

This is a particular case of a more general statement conjectured by Peter Kropholler in [21], and formally stated below as in [27]:

The Kropholler Conjecture. *Let G be a finitely generated group and H a subgroup of G. If G contains an H-proper almost invariant subset A such that $AH = A$, then G admits a non-trivial splitting over a subgroup C which is commensurable with a subgroup of H.*

In this chapter, we will explain the terms mentioned in the results above and the motivation that led to the Main Theorem.

The invariant $\widetilde{e}(G,H)$ of the pair of groups (G,H) was first introduced in the paper by Kropholler and Roller [21] in *Relative Ends and Duality Groups.* When $H=\{1\}$, $\widetilde{e}(G,H)$ becomes an invariant of G known as *the number of ends of* G. The latter invariant and splittings of G over finite subgroups are directly related and, being $\widetilde{e}(G,H)$ a more general definition of ends, one would expect similar relation with the splitting of G over H (or over a subgroup "closely related" to H, which we will learn to be a *commensurable* subgroup).

In order to understand the latter relation, we start this chapter by explaining in detail the connection between ends and splittings of a group over finite subgroups.

2.1 Ends and splittings of groups

The *theory of ends* was first introduced in the work of Freudenthal [14] and Hopf [16]. Given a locally finite CW-complex X, it was defined:

Definition 2.1.1. *The* ***number of ends*** *of X, denoted by $e(X)$, is*

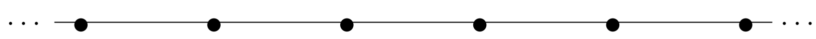
$$e(X)=\varprojlim_{K\subset X}|\text{unbounded components of } X\backslash K|$$

over all compact subsets of X.

Q. *Why "ends"?*

Informally, the number of ends of a space is the *number of unbounded path-connected components of the space at infinity.* Below we show one of the most simple non-trivial examples of a space and its number of ends.

Example 2.1.2. *Let $\mathbb{R}$ be the real line given a CW-complex structure with vertices and edges as below:*



In order to roughly understand the concept of ends of this space, it is not difficult to see that, extracting bigger and bigger compact sets (or finite subcomplexes) from $\mathbb{R}$, as we go to infinity on the right we have one path-connected component. The same happens on the left. Therefore, $e(\mathbb{R})=2$.

Q. *But how does ends of a space relate to groups?*

Given a finitely generated group G, the *Cayley graph of* G over some finite generating set X can be realised as a locally finite CW-complex, and we will denote it by $Cay(G, X)$. Hence,

Definition 2.1.3. *If G is a finitely generated group, then the number of ends of G, denoted by $e(G)$, is defined as*

$$e(G) := e(Cay(G, X))$$

It is important to know that $e(G)$ does not depend on the choice of the finite generating set of G.

If a group G is finite, then its Cayley graph is compact. Consequently, by definition, $e(G) = 0$. The converse is also true ([15], p. 302). More generally, if G acts properly and cocompactly on a path-connected CW-complex X, then $e(G) = e(X)$ ([15], Cor. 13.5.12).

Example 2.1.4. *Here we can see some examples of the number of ends of some finitely generated groups:*

i) $e(\mathbb{Z}) = 2$ *and, if* $n > 1$*, then* $e(\mathbb{Z}^n) = 1$*;*

ii) $e(D_\infty) = 2$*, where* D_∞ *is the infinite dihedral group, which contains an infinite cyclic subgroup of index two;*

iii) the fundamental group of the bitorus $\pi_1(\mathbb{T}^2 \# \mathbb{T}^2)$ *is one-ended, because it acts freely and cocompactly on the hyperbolic plane;*

iv) $e(SL_2(\mathbb{Z})) = \infty$*, because the group of invertible matrices with determinant one and integral entries is isomorphic to a free product with amalgamation of finite cyclic groups*

$$SL_2(\mathbb{Z}) \cong C_4 \underset{C_2}{*} C_6$$

which, from the Bass-Serre theory, acts properly and cocompactly on an infinitely ended tree.

Q. *Does there exist a two-ended group with no infinite cyclic subgroup of finite index?*

No:

Theorem 2.1.5 ([15], Thm. 13.5.9)**.** *A finitely generated group G has two ends if and only if G has an infinite cyclic subgroup of finite index.*

Q. *What about examples of groups with 3 ends?*

That can never happen:

Theorem 2.1.6 ([15], Thm. 13.5.7)**.** *The number of ends of a finitely generated group is 0, 1, 2 or ∞.*

Q. *Can one-ended and infinitely ended groups also be classified?*

One-ended groups are not classified and interesting questions involving hyperbolic groups are still open, as this one due to Gromov: *Does every one-ended hyperbolic group have a surface subgroup?* [1]. On the other hand, to groups with infinitely many ends we attribute to Stallings [30] the following famous result, reworded as in [15], Thm. 13.5.10:

Theorem 2.1.7 (Stallings' Theorem)**.** *A finitely generated group G has infinitely many ends if and only if either (i) $G = G_1 \underset{H}{*} G_2$ where H is finite having index ≥ 2 in G_1 and in G_2, with one of these indices being ≥ 3; or (ii) $G = G_1 *_H$ where H is finite with index ≥ 2 in G_1.*

This decomposition of G is a *splitting* of G. Formally,

Definition 2.1.8. *We say that a group G **splits** non-trivially over a subgroup H if either G can be decomposed as a non-trivial amalgamated free product over H or as a non-trivial HNN-extension over H.*

However, throughout this thesis we will be interested in splittings as actions on trees:

Lemma 2.1.9 ([19], Lem. 1.3)**.** *G splits over a subgroup H if and only if G acts (on the left) on a tree without global fixed points and transitively on the edges with an edge stabiliser H.*

This is a classical result in Bass-Serre theory.

2.2 Almost invariant sets

Q. *Can the notion of end be defined for an infinitely generated group?*

There exists an algebraic definition for the number of ends of an arbitrary group, not necessarily finitely generated, which can be found in [8] and uses the concept of *almost invariant sets.*

Definition 2.2.1. *Let G be a group. A subset A of G is called* ***almost invariant*** *if, $\forall\ g \in G$, the symmetric difference between A and Ag is finite. Moreover, if A and A^c are infinite, then A is called* ***proper****.*

Notation: The symmetric difference between two sets is denoted by $\triangle$.

Let A, B be subsets of G. We say $A \sim_f B$ if $A \triangle B$ is finite. Clearly $\sim_f$ is an equivalence relation and, if A, B belong to the same equivalence class, then $[A] = [B]$. A non-proper subset of G is either in $[G]$ or $[\emptyset]$.

Let $\mathcal{F}(G)$ denote the set of all finite subsets of G and $\mathcal{P}(G)$ the power set of G. The symmetric difference gives to both sets the structure of a vector space over the field $\mathbb{F}_2 = \{0, 1\}$. Indeed, if for any $A \in \mathcal{P}(G)$ we define $1 \cdot A = A$ and $0 \cdot A = \emptyset$, then we see that

$$(1 + 1) \cdots A = A + A = \emptyset = 0 \cdot A$$

and it is not difficult to see that the other properties of vector space apply. Morevover, $\emptyset \in \mathcal{F}(G)$ and, if $A, B \in \mathcal{F}(G)$, then $A + B \in \mathcal{F}(G)$. Therefore, $\mathcal{F}(G)$ is a subspace of $\mathcal{P}(G)$ over $\mathbb{F}_2$.
Now, we define

$$\mathcal{A}(G) := \left(\frac{\mathcal{P}(G)}{\mathcal{F}(G)}\right)^G$$

Definition 2.2.2. *Let G be any group. The* ***number of ends*** *of G is the dimension of the vector space over $\mathbb{F}_2$ of all equivalence classes of almost invariant subsets of G, namely,*

$$e(G) = \dim_{\mathbb{F}_2} \mathcal{A}(G)$$

Q. *Is there an example to see almost invariant sets in the geometric interpretation of ends?*

Take $G = F_2$, the free abelian group on two generators a, b. Let K be a finite subgraph of $Cay(G, \{a, b\})$ containing the vertex corresponding to the trivial element 1. The vertices of the path-connected components of $Cay(G, \{a, b\}) - K$ correspond to proper almost invariant subsets of G.

Q. *Is there some form of Stallings' theorem for infinitely generated groups?*

Stallings' theorem was later extended by any group by Swan [31]. A more general form of the theorem is also given by Dicks and Dunwoody using what they called the Almost Stability Theorem. The statement is as follows:

Theorem 2.2.3 ([8], IV.6.10)**.** *Let G be a group. The following are equivalent:*

i) $e(G) > 1$.

ii) $H^1(G, M) \neq 0$ *for any non-trivial free G-module M.*

iii) There exists a tree on which G acts without global fixed points and finite edge stabilisers.

iv) One of the following holds:

- $G = G_1 \underset{H}{*} G_2$ *where* $G_1 \neq H \neq G_2$ *and H is finite;*
- $G = G_1 \underset{H}{*}$, *where H is finite;*
- *G is countably infinite and locally finite.*

v) $e(G) = 2$ *or* $e(G) = \infty$.

The proof of this theorem relies on the existence of proper almost invariant subsets of G. Note from the algebraic definition of $e(G)$ that $\{[G], [\emptyset]\} \cong \mathbb{F}_2$ in $\mathcal{A}(G)$. Then, $e(G) \geq 2$ if and only if G contains a proper almost invariant subset.

2.3 Ends of pairs of groups

Given a pair of groups (G, H) with $H \leq G$ and G finitely generated, let $Cay_H(G, X)$ be the Cayley graph of G with generating set X quotiented by the left action of H. Houghton [17] introduced an invariant of the pair (G, H), defined as follows:

Definition 2.3.1. *The* ***number of ends of the pair of groups*** (G, H) *is an invariant of the pair denoted by $e(G, H)$ and is defined as*

$$e(G, H) = e(Cay_H(G, X))$$

Again, $e(G, H)$ does not depend on the generating set of G.

The subject was further explored by Scott [29] as an attempt to generalise Stallings' result to groups which split over infinite subgroups. Although Scott concluded that

$e(G, H) \geq 2$ does not necessarily imply G splits over some finite extension of H, his main result showed that

Theorem 2.3.2 ([29], 4.1)**.** *If $H \leq G$ are finitely generated groups and G is H-residually finite, then $e(G, H) \geq 2$ if and only if G has a subgroup G_1 of finite index in G such that G_1 contains H and G_1 splits over H.*

We say a group G is *H-residually finite* if, given $g \in G \backslash H$, there is a subgroup G_1 of finite index in G such that G_1 contains H but not g.

Therefore, if G splits over H then $e(G, H) \geq 2$.

Q. *Does $e(G, H)$ also take only values 0, 1, 2 and ∞?*

Differently from $e(G)$, the number of ends of a pair of groups can take values in any positive integer. One can find examples of finitely generated groups $H \leq G$ with $e(G, H)$ finite and strictly greater than 2 (see [29], 2.1).

Q. *What does it mean when $e(G, H) = 0$?*

Lemma 2.3.3 ([29], Lem. 1.3)**.** *$e(G, H) = 0$ if and only if H has finite index in G.*

Q. *When is $e(G, H) = e(G)$?*

The equality holds when $H = \{1\}$ and that is very easy to see. However, if H is a finite subgroup of G, then the equality does not necessarily hold:

Lemma 2.3.4 ([29], Lem. 1.7)**.** *If H is a subgroup of G and K is a subgroup of finite index n in H, then $e(G, H)$ and $e(G, K)$ are both finite or both infinite. When both are finite, then the following inequality holds:*

$$e(G, H) \leq e(G, K) \leq n \cdot e(G, H)$$

If the reader is interested, the remark in [29] after the lemma provides a counterexample.

Q. *Can $e(G, H)$ also be defined in terms of almost invariant sets, for any group G?*

Yes. Let G a group and H a subgroup of G. Define

$$\mathcal{A}(G/H) := \left(\frac{\mathcal{P}(G/H)}{\mathcal{F}(G/H)} \right)^G$$

Definition 2.3.5. *The **number of ends of the pair** (G, H) is the dimension of the vector space over $\mathbb{F}_2$ of all equivalence classes of almost invariant subsets of G/H, namely,*

$$e(G, H) = \dim_{\mathbb{F}_2} \mathcal{A}(G/H)$$

Lemma 2.3.6 ([29], Lem. 1.6)**.** *$e(G, H) \geq n$ if and only if one can find n disjoint infinite almost invariant subsets of G/H.*

Therefore, $e(G, H) \geq 2$ if and only if G/H contains a proper almost invariant subset.

2.4 H-almost invariant sets

The concept of almost invariant subsets of a group G can be generalised to H-almost invariant subsets of G, where H is a subgroup of G. In this section, we show how they arise and explore several properties through lemmas, remarks and corollaries which will be useful tools further on in this thesis.

Suppose G is finitely generated with generating set X and take the projection map $\pi\colon Cay(G, X) \longrightarrow Cay_H(G, X)$ which restricts to the canonical map $G \longrightarrow G/H$, $g \mapsto Hg$. The following result is a rewording of Lemma 2.3.6:

Theorem 2.4.1 ([28], Thm. 2.3)**.** *$e(G, H) \geq 2$ if and only if there exists $A \subset G$ such that:*

(a) $\pi(A)$ and $\pi(A^c)$ are infinite,

(b) $\forall g \in G$, $\pi(A \triangle Ag)$ is finite,

(c) A is left H-invariant, ie, $\forall h \in H$, $hA = A$.

Note that from (a) A, A^c must contain infinitely many right cosets of H. Moreover, from (b), $\forall g \in G$, $A \triangle Ag$ is contained in a finite union of right cosets of H. This leads us to the next two definitions:

Definition 2.4.2. *Let H be a subgroup of G. A subset of G is said H**-finite** if it is contained in a finite union of right cosets of H. If a subset A of G contains infinitely many right cosets of H, then A is called H**-infinite**. If A and A^c are H-infinite, then we say A is H**-proper**.*

Definition 2.4.3. *A subset A of G that satisfies $Ag \triangle A$ is H-finite, $\forall g \in G$, is called an H-**almost invariant subset of** G.*

Notation 1: An *H-proper almost invariant* subset of G is an H-proper H-almost invariant subset of G and, from now on, will be denoted by *H-p.a.i.*
Notation 2: We have mentioned before that $\mathcal{P}(G)$ equipped with the operation $\triangle$ is a vector space over $\mathbb{F}_2$. From now on, we will represent the binary operation symmetric difference by the symbol $+$.

Remark 2.4.4. *Let A, B be subsets of G. Then,*

- $(A+B)^c = G + A + B = A^c + B = A + B^c$.
- *If $g \in G$, $(Ag)^c = G + Ag = Gg + Ag = (G+A)g = A^c g$. Similarly, $gA^c = (gA)^c$.*

Lemma 2.4.5. *Let A, B, F be subsets of G:*

i) If A, B are H-finite, then $A+B$ is also H-finite.

ii) If A is H-infinite and $A+B$ is H-finite, then B is H-infinite.

iii) If A is H-almost invariant and $A+B$ is H-finite, then B is H-almost invariant.

iv) If A, B are H-almost invariant, then $A+B$ is also H-almost invariant.

v) If $h \in H$ and A is H-p.a.i., then hA, hA^c are H-p.a.i.

vi) If F is finite and A is H-p.a.i., then AF is H-p.a.i.

Proof. i) $A \subseteq HF$ and $B \subseteq HE$, for some finite subsets F, E in G. Thus,

$$A + B \subseteq A \cup B \subseteq HF \cup HE.$$

ii) If B is H-finite, then $A + B + B = A$ is H-finite, contradiction.

iii) Let $g \in G$. Then,

$$\begin{aligned} Bg + B &= Bg + B + A + A + Ag + Ag \\ &= \underbrace{(B+A)g}_{H\text{-finite}} + \underbrace{B+A}_{H\text{-finite}} + \underbrace{A+Ag}_{H\text{-finite}} \end{aligned}$$

iv) $(A+B) + (A+B)g = A + Ag + B + Bg$ is H-finite, for any $g \in G$.

v) $\forall g \in G$, $hA + hAg \subseteq hHF = HF$ for some finite subset F of G. Moreover, hA is H-finite if and only if A is H-finite, so if $(hA)^c$ is H-finite, then hA^c is H-finite and, hence so is A^c.

vi) We first show that $AF + A$ is H-finite. Let $F = \{f_1, \ldots, f_l\}$. Then,

$$\begin{aligned} AF + A &= (Af_1 \cup \ldots \cup Af_l) + A \\ &= [(Af_1 \cup \ldots \cup Af_l) \cap A^c] \cup [A \cap (Af_1 \cup \ldots \cup Af_l)^c] \\ &= (Af_1 \cap A^c) \cup \ldots \cup (Af_l \cap A^c) \cup (A \cap A^c f_1 \cap \ldots \cap A^c f_l) \\ &\subseteq (Af_1 \cap A^c) \cup (A \cap A^c f_1) \cup \ldots \cup (Af_l \cap A^c) \cup (A \cap A^c f_l) \\ &= (Af_1 + A) \cup \ldots \cup (Af_l + A) \end{aligned}$$

which is a finite union of H-finite sets. Now, let $g \in G$. Following similar steps, we have that

$$\begin{aligned} AFg + AF &\subseteq (Af_1 g + AF) \cup \ldots \cup (Af_l g + AF) \\ &= (A + AF(f_1 g)^{-1}) f_1 g \cup \ldots \cup (A + AF(f_l g)^{-1}) f_l g \end{aligned}$$

which is also a finite union of H-finite sets.

□

Lemma 2.4.6 (The Kropholler Corner)**.** *Let A, B be H-almost invariant subsets of G. Then, the same holds for $A \cap B$.*

Proof. Given $g \in G$, we have

$$(A \cap B) + (A \cap B)g = (A \cap B) + (Ag \cap Bg) = ((A + Ag) \cap B) + (Ag \cap (B + Bg))$$

which by hypothesis and Lemma 2.4.5 (i) is H-finite. □

This decomposition was given by Kropholler in the proof of Lemma 4.3 in [19] and is used by Dunwoody in his work on cuts and structure trees, where he defines the term *corner*: each one of the four intersections $A \cap B$, $A^c \cap B$, $A \cap B^c$, $A^c \cap B^c$, where A, B are cuts in a tree. Definitions and further explanations can be found in [9].

If A, B are subsets of G, then $A \sim_H B$ if $A + B$ is H-finite. This defines an equivalence relation. If A, B are in the same equivalence class, then, $[A] = [B]$.

Now, let

$$\mathcal{F}_H(G) := \{\text{all } H\text{-finite subsets of } G\}$$

The vector space over $\mathbb{F}_2$ of all equivalence classes of H-almost invariant subsets of G is defined as

$$\mathcal{A}_H(G) := \left(\frac{\mathcal{P}(G)}{\mathcal{F}_H(G)}\right)^G$$

By Lemma 2.4.5 (iii), $[G]$ is the equivalence class of all subsets of G with H-finite complement and $[\emptyset]$ is the equivalence class of all H-finite subsets of G.

Remark 2.4.7. *(i) Clearly, when we say $\mathcal{A}_H(G)$ is a vector space over $\mathbb{F}_2$, we consider the following binary operation of the vectors: if $[A], [B] \in \mathcal{A}_H(G)$, then*

$$[A] \oplus [B] := [A + B]$$

(ii) If we add to $\mathcal{P}(G)$ a right action of G given by translation of sets (that is, $A \cdot g = Ag\ \forall A \subseteq G,\ g \in G$), then $\mathcal{P}(G)$ becomes a right $\mathbb{F}_2 G$-module. Moreover, $\forall g \in G$, if $A \in \mathcal{F}_H(G)$ then Ag is clearly H-finite, that is, $Ag \in \mathcal{F}_H(G)$. Therefore, $\mathcal{F}_H(G)$ is a right $\mathbb{F}_2 G$-submodule of $\mathcal{P}(G)$.

Definition 2.4.8. *Let $A_1, \ldots, A_k$ be H-almost invariant subsets of G. We say that $A_1, \ldots, A_k$ are **H-linearly independent** if $[A_1], \ldots, [A_k]$ are linearly independent vectors in $\mathcal{A}_H(G)$.*

Notation: If $A_1, \ldots, A_k$ are H-linearly independent H-p.a.i. subsets of G, we will say that $A_1, \ldots, A_k$ are H-*l.i.p.a.i.*

Q. *Given $[A], [B] \in \mathcal{A}_H(G)$, what happens if the following binary operation is defined?*

$$[A] \odot [B] := [A \cap B]$$

Proposition 2.4.9. *$\mathcal{A}_H(G)$ with the binary operations $\oplus, \odot$ is a Boolean ring.*

Proof. From Remark 2.4.4, Lemma 2.4.5 and properties of sets, we see that $\mathcal{A}_H(G)$ with $\oplus$ is an abelian group with identity $[\emptyset]$ and $\mathcal{A}_H(G)$ with $\odot$ is an abelian monoid with identity $[G]$. Every element of $\mathcal{A}_H(G)$ is idempotent over $\odot$. It remains to prove the distributivity law holds. Let $A, B, C \subseteq G$. Clearly $A \cap (B + C) = (B + C) \cap A$. It is enough to show that

$$A \cap (B + C) = (A \cap B) + (A \cap C).$$

We have:

$$\begin{aligned} A \cap (B+C) &= A \cap ((B \cap C^c) \cup (C \cap B^c)) \\ &= (A \cap B \cap C^c) \cup (A \cap C \cap B^c) \\ &= ((A \cap B) \backslash C) \cup ((A \cap C) \backslash B) \\ &= ((A \cap B) \backslash (C \cap A)) \cup ((A \cap C) \backslash (B \cap A)) \\ &= (A \cap B) + (A \cap C) \end{aligned}$$

□

2.5 Commensurability

When working with a subgroup H of G, all the properties of H-almost invariant sets are still preserved if we consider subgroups "closely related" to H, which are known as *commensurable subgroups*:

Definition 2.5.1. *Let H, S be subgroups of G. We say H and S are* ***commensurable*** *if $|H : H \cap S| < \infty$ and $|S : H \cap S| < \infty$. Commensurability is an equivalence relation and we refer to commensurable groups H, S as $H \sim S$.*

If $H \leq_f S$, then $H \sim S$.

Definition 2.5.2. *The* ***commensurator*** *of H in G is the subgroup defined as*

$$\mathrm{Comm}_G(H) = \{g \in G \mid gHg^{-1} \sim H\}$$

Clearly, H is a subgroup of $\mathrm{Comm}_G(H)$.

Lemma 2.5.3. *Let H, S be subgroups of G such that $H \sim S$ and A is a subset of G. Then,*

i) A is H-finite if and only if A is S-finite.

ii) A is H-infinite if and only if A is S-infinite.

Proof. Let F, E be finite subsets of G such that $H = (H \cap S)F$ and $S = (H \cap S)E$. Then, $H \subseteq (H \cap S)EE^{-1}F = SE^{-1}F$ and $S \subseteq (H \cap S)FF^{-1}E = HF^{-1}E$. Thus, i) and ii) follow easily. □

Corollary 2.5.4. *If H, S are subgroups of G such that $H \sim S$, then $\mathcal{A}_H(G)$ and $\mathcal{A}_S(G)$ are isomorphic as right $\mathbb{F}_2 G$-modules.*

Proof. By Lemma 2.5.3, we conclude that a subset A of G is H-almost invariant if and only if A is S-almost invariant. Then, A represents a class in $\mathcal{A}_H(G)$ which we will denote by $[A]_H$ and also a class in $\mathcal{A}_S(G)$ which we will denote by $[A]_S$. Therefore, the map $\phi\colon \mathcal{A}_H(G) \longrightarrow \mathcal{A}_S(G)$ defined by $\phi([A]_H) = [A]_S$ is clearly an isomorphism of vector spaces. Define the right action of G on each space as $[A]_H g := [Ag]_H$ and $[A]_S g := [Ag]_S$. Clearly, the map is right G-invariant. □

2.6 Relative Ends of Pairs of Groups

Now, we would like to introduce an algebraic invariant of the pair (G, H), for an arbitrary group G with a subgroup H, first formally introduced by Kropholler and Roller in [21]. In this section, we discuss its properties and compare it to the other end invariants.

As the reader has probably guessed,

Definition 2.6.1 ([21], p. 200)**.** *Let $H \leq G$ be groups. The* ***number of relative ends*** *is an invariant of the pair (G, H) given by the dimension of the vector space over $\mathbb{F}_2$ of all equivalence classes of H-almost invariant subsets of G, namely,*

$$\widetilde{e}(G, H) = \dim_{\mathbb{F}_2} \mathcal{A}_H(G)$$

Note that, when $H = \{1\}$, $\mathcal{F}_H(G) = \mathcal{F}(G)$ and $G/H = G$. Hence,

$$\widetilde{e}(G, 1) = e(G, 1) = e(G).$$

Q. *What does it mean when $\widetilde{e}(G, H) = 0$?*

Lemma 2.6.2. $\tilde{e}(G, H) = 0$ *if and only if* $|G : H| < \infty$.

Proof. If $[G] \neq [\emptyset]$, $\{[G], [\emptyset]\} \cong \mathbb{F}_2$ is a 1-dimensional subspace of $\mathcal{A}_H(G)$. Then,

$$\tilde{e}(G, H) = 0 \ \Leftrightarrow \ [G] = [\emptyset] \ \Leftrightarrow G \text{ is } H\text{-finite} \Leftrightarrow \ |G : H| < \infty$$

□

Q. *Which values can $\widetilde{e}(G,H)$ take?*

Just as $e(G,H)$, the invariant $\widetilde{e}(G,H)$ can take any positive integer value. For the curious reader, an example can be found in [21], Prop. 4.7.

Q. *What is the relation between $e(G)$, $e(G,H)$ and $\widetilde{e}(G,H)$?*

The next lemma is an extraction of lemmas in [21] whose proofs can be found there.

Lemma 2.6.3 ([21], Lem. 2.4, 2.5)**.** *Let H, S be subgroups of G.*

i) if $S \leq_f H$, then $\tilde{e}(G,H) = \tilde{e}(G,S)$;

ii) if $|G : H| = \infty$ and $K \leq H$, then $\widetilde{e}(G,K) \leq \widetilde{e}(G,H)$;

iii) $e(G,H) \leq \tilde{e}(G,H)$;

iv) if H is finitely generated and $\tilde{e}(G,H)$ is finite, then there exists a subgroup H_0 of finite index in H such that $e(G,H_0) = \tilde{e}(G,H_0) = \tilde{e}(G,H)$.

Corollary 2.6.4. *If $|G : H| = \infty$, then $e(G) \leq \widetilde{e}(G,H)$.*

Proof. In Lemma 2.6.3 (ii), take $K = \{1\}$. □

As we can see by Lemma 2.6.3 (i), the number of relative ends $\widetilde{e}(G,H)$ depends only on the commensurability class of H. Another important result shows how this number can restrict the embedding of H in $\mathrm{Comm}_G(H)$:

Theorem 2.6.5 ([21], Thm. 1.3)**.** *Let $H \leq G$ be finitely generated groups such that H has infinite index in $\mathrm{Comm}_H(G)$. Then, $\tilde{e}(G,H)$ is either 1, 2, or ∞. In the case $\tilde{e}(G,H) = 2$, there are subgroups G_0 and H_0 of finite index in G and H respectively such that H_0 is normal in G_0 and G_0/H_0 is infinite cyclic.*

Therefore, if $3 \leq \tilde{e}(G,H) < \infty$, then $H \leq_f \mathrm{Comm}_G(H)$.

2.7 The Kropholler conjecture

We are now familiar with all the definitions necessary to understand the conjecture that motivated the Main Theorem. The Kropholler Conjecture was first proposed in the joint work of Kropholler and Roller in [21]. They were interested to know whether there is an analogue of Stallings' Theorem on ends of groups [30] for relative ends. They observed that when G splits over H, the kernel of the restriction map

$$Res^G_H \colon H^1(G, \mathcal{F}_H(G)) \longrightarrow H^1(H, \mathcal{F}_H(G))$$

must be non-zero. The conjecture was that, for finitely generated groups $H \leq G$, the non-vanishing of this kernel would imply that G splits over a subgroup related to H. As pointed out in a letter written by Kropholler to Dunwoody in January of 1988, the non-vanishing of the aforementioned kernel is equivalent to the existence of an H-p.a.i. subset A of G satisfying $AH = A$. The conjecture is discussed in [27] and the following formal statement was provided:

The Kropholler Conjecture. *Let G be a finitely generated group and H a subgroup of G. If G contains an H-p.a.i. subset A such that $AH = A$, then G admits a non-trivial splitting over a subgroup C which is commensurable with a subgroup of H.*

The following is true:

Lemma 2.7.1 ([19], Lem. 2.4)**.** *If G splits non-trivially over a subgroup commensurable with H then G contains an H-p.a.i. subset A such that $AH = A$.*

Proof. Because of Lemma 2.5.3, assume G splits non-trivially over H. Then, G acts (left) without global fixed points on a tree Γ transitively on the edges such that there is an egde e with stabiliser H. The edge e separates the tree into two components X_0, X_1. Consider $X_0 \ni e$. Take $A = \{g \in G \mid ge \in X_0\}$. Then, $gHe = ge \Rightarrow AH = A$. Also, let $x \in G$. Then,

$$\begin{aligned} A + Ax &= \left\{g \in G \mid ge \in X_0 \text{ and } gx^{-1}e \in X_1\right\} \cup \left\{g \in G \mid ge \in X_1 \text{ and } gx^{-1}e \in X_0\right\} \\ &= \left\{g \in G \mid e \text{ belongs to the geodesic from } ge \text{ to } gx^{-1}e\right\} \\ &= \left\{g \in G \mid g^{-1}e \text{ belongs to the geodesic from } e \text{ to } x^{-1}e\right\} \end{aligned}$$

which is H-finite, since the geodesic is finite and H stabilises e. Also, A is H-proper since it acts without fixed points. Therefore, A is an H-p.a.i. subset of G satisfying $AH = A$. □

2.8 A cohomological argument for non-splittings of groups

In this section, we explain the relation between the cohomological argument involving the kernel of the restriction map $Res^G_H\colon H^1(G,\mathcal{F}_H(G)) \longrightarrow H^1(H,\mathcal{F}_H(G))$ and the existence of right H-invariant H-p.a.i subsets of G. Moreover, we show properties involving the kernel.

Consider the canonical short exact sequence of right $\mathbb{F}_2G$-modules

$$0 \longrightarrow \mathcal{F}_H(G) \longrightarrow \mathcal{P}(G) \longrightarrow \frac{\mathcal{P}(G)}{\mathcal{F}_H(G)} \longrightarrow 0$$

This sequence induces a long exact sequence in cohomology

$$0 \longrightarrow H^0\left(G,\mathcal{F}_H(G)\right) \longrightarrow H^0\left(G,\mathcal{P}(G)\right) \longrightarrow H^0\left(G,\frac{\mathcal{P}(G)}{\mathcal{F}_H(G)}\right) \longrightarrow$$
$$\longrightarrow H^1\left(G,\mathcal{F}_H(G)\right) \longrightarrow H^1\left(G,\mathcal{P}(G)\right) \longrightarrow \dots$$

We have that:

- $H^0\left(G,\mathcal{F}_H(G)\right) \cong \mathcal{F}_H(G)^G = \{\emptyset\} \cong \{0\} \subset \mathbb{F}_2$
- $H^0\left(G,\mathcal{P}(G)\right) \cong \mathcal{P}(G)^G = \{G,\emptyset\} \cong \mathbb{F}_2$
- $H^0\left(G,\frac{\mathcal{P}(G)}{\mathcal{F}_H(G)}\right) \cong \mathcal{A}_H(G)$
- $H^1\left(G,\mathcal{P}(G)\right) \overset{[[4],p.67]}{=} H^1\left(G,\ \mathrm{Coind}^G_{\{1\}}\mathbb{F}_2\right) \overset{\text{Shapiro's lemma}}{\cong} H^1\left(\{1\},\mathbb{F}_2\right) = \{0\}$

Consequently, we obtain the following short exact sequence of vector spaces over $\mathbb{F}_2$:

$$0 \longrightarrow \mathbb{F}_2 \longrightarrow \mathcal{A}_H(G) \longrightarrow H^1\left(G,\mathcal{F}_H(G)\right) \longrightarrow 0$$

Remark 2.8.1. *In Stallings' paper [30], the number of ends of a finitely generated group G is given in terms of $H^1(G,\mathbb{F}_2G)$ as*

$$e(G) = 1 + \dim_{\mathbb{F}_2} H^1(G,\mathbb{F}_2G)$$

From the short exact sequence above, $\widetilde{e}(G,H)$ can be given a similar definition:

$$\widetilde{e}(G,H) = 1 + \dim_{\mathbb{F}_2} H^1\left(G,\mathcal{F}_H(G)\right)$$

Now, applying restriction maps of cohomology groups on the sequence above, we obtain the following commutative diagram (D) of vector spaces over $\mathbb{F}_2$:

$$\begin{array}{ccccccccccc} & & 0 & \longrightarrow & \mathbb{F}_2 & \longrightarrow & \mathcal{A}_H(G) & \xrightarrow{\alpha} & H^1\left(G, \mathcal{F}_H(G)\right) & \longrightarrow & 0 \\ & & & & \downarrow & & \downarrow r & & \downarrow Res^G_H & & \\ 0 & \longrightarrow & \mathcal{F}_H(G)^H & \longrightarrow & \mathcal{P}(G)^H & \xrightarrow{\beta} & \left(\frac{\mathcal{P}(G)}{\mathcal{F}_H(G)}\right)^H & \xrightarrow{\gamma} & H^1\left(H, \mathcal{F}_H(G)\right) & \longrightarrow & \ldots \end{array}$$

Denote $\mathcal{K} := \alpha^{-1}(\ker Res^G_H)$.

Lemma 2.8.2. $\dim_{\mathbb{F}_2}\mathcal{K} = \dim_{\mathbb{F}_2}\ker Res^G_H + 1.$

Proof. From the exactness of Diagram (D), we know that $\ker\alpha = \{[G], [\emptyset]\} \cong \mathbb{F}_2$. Hence, $\alpha(\mathcal{A}_H(G)) \cong \mathcal{A}_H(G)/\mathbb{F}_2$. Since α is a surjective morphism of vector spaces and $\ker Res^G_H$ is a subspace of $H^1(G, \mathcal{F}_H(G))$, we obtain the result required. □

Proposition 2.8.3. *Suppose $\widetilde{e}(G, H) > 0$. If $[\widetilde{B}] \in \mathcal{K}$, then $[\widetilde{B}] = [B]$, where B is an H-almost invariant subset of G satisfying $BH = B$. Moreover, if B is an H-almost invariant subset of G satisfying $BH = B$, then $[B] \in \mathcal{K}$.*

Proof. If $[\widetilde{B}] \in \{[G], [\emptyset]\}$, then we are done. Therefore, assume $\widetilde{B}$ is H-proper. We look at Diagram (D) and see that

$$\gamma \circ r([\widetilde{B}]) = Res^G_H \circ \alpha([\widetilde{B}]) = 0 \Rightarrow r([\widetilde{B}]) \subseteq \ker\gamma = \mathrm{im}\beta$$

Therefore,

$$r([\widetilde{B}]) = \beta(B) = [B] \in \left(\frac{\mathcal{P}(G)}{\mathcal{F}_H(G)}\right)^H$$

for some $B \in \mathcal{P}(G)^H$, that is, $BH = B$. The image of $r([\widetilde{B}])$ in $\left(\frac{\mathcal{P}(G)}{\mathcal{F}_H(G)}\right)^H$ is an equivalence class of all subsets of G satisfying: $A \in r([\widetilde{B}])$ if and only if $A + \widetilde{B}$ is H-finite and, $\forall h \in H$, $Ah + A$ is H-finite. Therefore, $B + \widetilde{B}$ is H-finite. By Lemma 2.4.5 (iii), we have that $[B] = [\widetilde{B}] \in \mathcal{K} \;\Rightarrow\; B$ is H-p.a.i.

Now, let B be an H-almost invariant subset of G satisfying $BH = B$. We know that $[B] \in \mathcal{A}_H(G)$. If B is not H-proper, then either $[B] = [G]$ or $[B] = [\emptyset]$. For either case, $[B] \in \mathcal{K}$. Then, assume B is H-p.a.i. From the fact that B is right H-invariant, we know that $B \in \mathcal{P}(G)^H$. Moreover, $\beta(B) = r([B])$. Hence,

$$\gamma \circ \beta([B]) = 0 \;\Rightarrow\; \gamma \circ r([B]) = 0 \;\Rightarrow\; Res^G_H \circ \alpha([B]) = 0 \;\Rightarrow\; [B] \in \mathcal{K}.$$

□

Corollary 2.8.4. $\ker Res^G_H \neq \{0\}$ *if and only if* G *contains an* H*-p.a.i. subset* B *satisfying* $BH = B$.

Proof. By Lemma 2.8.2, $\ker Res^G_H \neq \{0\} \Leftrightarrow \dim_{\mathbb{F}_2}\mathcal{K} \geq 2$. Since $\{[G], [\emptyset]\}$ is a 1-dimensional subspace of $\mathcal{K}$, by lemma 2.8.3 there must exist $[B] \in \mathcal{K}$ such that B is an H-p.a.i. subset of G satisfying $BH = B$. □

Q. *What happens if we consider* $\mathcal{K}$ *with the binary operations* $\oplus$, $\odot$ *as defined in Proposition 2.4.9 for* $\mathcal{A}_H(G)$*?*

Corollary 2.8.5. $\mathcal{K}$ *is a subring of* $\mathcal{A}_H(G)$.

Proof. We already know that $[G], [\emptyset] \in \mathcal{K}$. Now, let $[A], [B] \in \mathcal{K}$ such that $AH = A$ and $BH = B$. By Lemmas 2.4.5 and 2.4.6, $A + B$ and $A \cap B$ are H-almost invariant sets. From set theory and the fact that H is a group, the following are also satisfied:

$$(A \cap B)H = A \cap B \quad \text{and} \quad (A + B)H = A + B.$$

By proposition 2.8.3, we conclude that $[A \cap B], [A + B] \in \mathcal{K}$. □

Corollary 2.8.6. *If* $k = \dim_{\mathbb{F}_2}\ker Res^G_H < \infty$*, then there exist* k H*-l.i.p.a.i. subsets* B_i *of* G *such that* $B_iH = B_i$ *for each* $i = 1, \dots, k$.

Proof. We have seen that $\dim_{\mathbb{F}_2}\mathcal{K} = k+1$. We also know that $[G] \in \mathcal{K}$ and $[G] \oplus [G] = [\emptyset]$. Thus, let $\left\{[G], [\widetilde{B}_1], \dots, [\widetilde{B}_k]\right\}$ be a basis for $\mathcal{K}$. Proposition 2.8.3 ensures there exist k H-p.a.i. sets B_i satisfying $B_iH = B_i$, where $[B_i] = [\widetilde{B}_i]$ in $\mathcal{K}$, for each $i = 1, \dots, k$. □

Proposition 2.8.7. *Let* H, S *be subgroups of* G *such that* $H \sim S$*. Then,* $\ker Res^G_H \cong \ker Res^G_S$.

Proof. We assume S is a subgroup of finite index in H and we prove $\ker Res^G_H \cong \ker Res^G_S$. First, we consider the following setting: Let (D_H) be the Diagram (D) with maps α, r, β, γ respectively denoted by $\alpha_H, r_H, \beta_H, \gamma_H$. Then, if S is another subgroup of G, we follow the steps to construct Diagram (D) to build Diagram (D_S), with related maps $\alpha_S, r_S, \beta_S, \gamma_S$.

Let ϕ denote the isomorphism $\mathcal{A}_H(G) \overset{\phi}{\cong} \mathcal{A}_S(G)$ from Corollary 2.5.4.

From these considerations, we have the following diagram:

$$\begin{array}{ccccc} \mathcal{A}_H(G) & \xrightarrow{\pi_H} & \mathcal{A}_H(G)/\mathbb{F}_2 & \xrightarrow{\widetilde{\alpha}_H} & H^1(G, \mathcal{F}_H(G)) \\ \phi \downarrow & & \downarrow \widetilde{\phi} & & \\ \mathcal{A}_S(G) & \xrightarrow{\pi_S} & \mathcal{A}_S(G)/\mathbb{F}_2 & \xrightarrow{\widetilde{\alpha}_S} & H^1(G, \mathcal{F}_S(G)) \end{array}$$

where $\mathbb{F}_2 \cong \{[G], [\emptyset]\}$, the maps π_H, π_S are the canonical projection maps, the map $\widetilde{\phi}$ is the isomorphism induced by ϕ and $\widetilde{\alpha}_H, \widetilde{\alpha}_S$ are the isomorphisms induced respectively by the surjective maps α_H and α_S. Therefore, it suffices to prove that

$$\alpha_H^{-1}(\ker Res_H^G) \overset{\phi}{\cong} \alpha_S^{-1}(\ker Res_S^G)$$

since it implies that

$$\pi_H(\alpha_H^{-1}(\ker Res_H^G)) \overset{\widetilde{\phi}}{\cong} \pi_S(\alpha_S^{-1}(\ker Res_S^G))$$

and hence

$$\widetilde{\alpha}_H^{-1}(\ker Res_H^G) \overset{\widetilde{\phi}}{\cong} \widetilde{\alpha}_S^{-1}(\ker Res_S^G),$$

the result follows.

We now prove the claim above:

Because of the definition of ϕ, we will use $[A]$ to denote a class of H-almost invariant sets in $\mathcal{A}_H(G)$ and, without distinction, the class of S-almost invariant sets $\phi([A])$ in $\mathcal{A}_S(G)$. Let $[A] \in \alpha_H^{-1}(\ker Res_H^G)$. By Proposition 2.8.3, we can choose A to be an H-almost invariant subset of G satisfying $AH = A$, which implies $AS = A$. By Lemma 2.5.3, A is an S-almost invariant subset of G satisfying $AS = A$. Hence, $[A] \in \alpha_S^{-1}(\ker Res_S^G)$.

Now, let $[A] \in \alpha_S^{-1}(\ker Res_S^G)$, such that A is an S-almost invariant subset of G satisfying $AS = A$. By Lemma 2.5.3, A is H-almost invariant. Take F a finite subset of G such that $H = SF$. Then,

$$AFH = ASFH = AH = ASF = AF$$

In additon, Lemma 2.4.5 (vi) gives us that AF is H-almost invariant and $A + AF$ is H-finite. Hence, $[A] \in \alpha_H^{-1}(\ker Res_H^G)$. □

As a conclusion, if the kernel of the restriction map $Res_H^G \colon H^1(G, \mathcal{F}_H(G)) \longrightarrow H^1(H, \mathcal{F}_H(G))$ vanishes, then by Corollary 2.8.4 G does not contain an H-p.a.i. subset B satisfying

$BH = B$ and, by Lemmas 2.7.1 and 2.8.7, G does not split over any subgroup commensurable with H.

2.8.1 The singularity $\mathrm{sing}_G(H)$

In Diagram (D), the image of Res^G_H is a singularity denoted by $\mathrm{sing}_G(H)$ and first defined in [19] in the studies of the splittings of Poincaré duality groups under the assumption that $\widetilde{e}(G,H) = 2$. In this setting, when $\mathrm{sing}_G(H) \neq \{0\}$, the kernel of Res^G_H vanishes and G does not split over any subgroup commensurable with H. When $\widetilde{e}(G,H) > 2$, $\mathrm{sing}_G(H)$ loses its role as an obstruction to the splitting of G over H, but plays a different part in the existence of right H-invariant H-p.a.i. subsets of G. The next results are a generalisation of Lemmas 2.2 and 2.3 in [19] when $\widetilde{e}(G,H) = n$, for $2 \leq n < \infty$.

Proposition 2.8.8. *Let $\widetilde{e}(G,H) = n,\ 2 \leq n < \infty$. The following are equivalent:*

(i) $\mathrm{sing}_G(H) = 0$*;*

(ii) There exist n-1 H-l.i.p.a.i. subsets B_k of G such that, for each $k = 1, \dots, n-1$, $B_kH = B_k$.

Proof. We see that $\mathrm{sing}_G(H) = 0$ if and only if $\dim_{\mathbb{F}_2}\ker Res^G_H = n - 1$.
$(i) \Rightarrow (ii)$: Follows from Corollary 2.8.6.
$(ii) \Rightarrow (i)$: We know that $H^1(G, \mathcal{F}_H(G)) \cong \mathcal{A}_H(G)/\mathbb{F}_2 \;\Rightarrow\; \dim_{\mathbb{F}_2}\ker Res^G_H \leq n-1$. On the other hand, Proposition 2.8.3 gives us that the existence of such subsets B_k implies that $\dim_{\mathbb{F}_2}\ker Res^G_H \geq n-1$. □

Lemma 2.8.9. *Assume $\tilde{e}(G,H) = n,\ 2 \leq n < \infty$. Let H, S be commensurable subgroups of G. Then, $\mathrm{sing}_G(H) = 0$ if and only if $\mathrm{sing}_G(S) = 0$.*

Proof. Follows from Proposition 2.8.7. □

2.9 A special basis for $\mathcal{K}$

In this section, we start to shape the Main Theorem. As mentioned in the introduction, this part of my work started with Peter Kropholler's suggestion to try to generalise the results of his joint paper with Roller [19] for the case when $\widetilde{e}(G,H) = 3$. It turned out that some results can be generalised for $2 \leq \widetilde{e}(G,H) < \infty$ and the hypothesis on H can be weakened. More specifically, let $H \leq G$ be finitely generated such that H satisfies (i) $H \leq_f \mathrm{Comm}_G(H)$ and (ii) a condition called *Property P*. In this section we prove the following:

Proposition 2.9.1. *Suppose* $\dim_{\mathbb{F}_2}\mathcal{K} = k+1$*. Let* $A_1, \ldots, A_k$ *be pairwise disjoint* H*-l.i.p.a.i. subsets of* $G\backslash H$ *satisfying* $A_i H = A_i$*,* $i = 1, \ldots, k$*. For any* $g \in G$*,* $i, j \in \{1, \ldots, k\}$*, the pair* $\{A_i, gA_j\}$ *is nested.*

This result is a general version of

Lemma 2.9.2 ([19], Lem. 4.3)**.** *Let* H *be a Poincaré duality subgroup of a finitely generated group* G *with* $\widetilde{e}(G,H) = 2$ *and* $H = \mathrm{Comm}_H(G)$*. Let* B *be an* H*-p.a.i. subset of* $G\backslash H$ *such that* $BH = B$*. Then,* $\forall g \in G$*, the pair* $\{B, gB\}$ *is nested.*

Q. *What is nested?*

Definition 2.9.3. *We say that a pair* $\{A, B\}$ *of subsets of* G *is* ***nested*** *if one of the following inclusions hold:*

$$\bullet\ A \subseteq B \qquad \bullet\ A^c \subseteq B \qquad \bullet\ A \subseteq B^c \qquad \bullet\ A^c \subseteq B^c$$

Q. *What is Property P?*

Definition 2.9.4. *Let* H *be a subgroup of* G*. The subgroup* H *satisfies* ***Property P*** *if,* $\forall g \in G$*, if* gH *is* H*-finite, then* $g \in \mathrm{Comm}_G(H)$*.*

The Property P is satisfied by Poincaré duality subgroups ([19], Lem. 4.1 (i)).

Note that the converse is always true, that is, if $g \in \mathrm{Comm}_G(H)$, then gH is H-finite. Indeed, $gHg^{-1} \sim H$. Hence,

$$\begin{aligned} gHg^{-1} \cap H \leq_f gHg^{-1} &\Rightarrow gHg^{-1} = (gHg^{-1} \cap H)F, \text{ for some finite subset } F \text{ of } G \\ &\Rightarrow gHg^{-1} \subseteq HF \Rightarrow gH \subseteq HFg \\ &\Rightarrow gH \text{ is } H\text{-finite} \end{aligned}$$

Q. *And why does* H *have finite index in* $\mathrm{Comm}_G(H)$*?*

We recall from Theorem 2.6.5 that, if $2 < \widetilde{e}(G,H) < \infty$, then $H \leq_f \mathrm{Comm}_G(H)$. When $\widetilde{e}(G,H) = 2$, Proposition 2.9.1 is answered by Lemma 2.9.2, in which proof it suffices that H satisfies (i) $H \leq_f \mathrm{Comm}_G(H)$ and (ii) Property P.

Lemma 2.9.5. *If $H \leq_f \mathrm{Comm}_G(H)$ and H satisfies Property P, then $\mathrm{Comm}_G(H)$ satisfies Property P.*

Proof. We know that $H \sim \mathrm{Comm}_G(H)$. Then, for $g \in G$,

$$gHg^{-1} \sim H \;\Leftrightarrow\; g\mathrm{Comm}_G(H)g^{-1} \sim \mathrm{Comm}_G(H)$$

$\Rightarrow\; \mathrm{Comm}_G(\mathrm{Comm}_G(H)) = \mathrm{Comm}_G(H)$.
Let $g \in G$ such that $g\mathrm{Comm}_G(H)$ is $\mathrm{Comm}_G(H)$-finite. Then, $g\mathrm{Comm}_G(H)$ is H-finite $\Rightarrow\; gH$ is H-finite $\;\Rightarrow g \in \mathrm{Comm}_G(H)$. □

The goal of our Main Theorem is to obtain a splitting of G over a subgroup commensurable with H. Therefore, w.l.o.g., for all further results we assume $H = \mathrm{Comm}_G(H)$.

To summarize, in light of the above these are the conditions we will assume from now on:

- $2 \leq \widetilde{e}(G, H) < \infty$,
- $H = \mathrm{Comm}_G(H)$,
- H satisfies Property P.

A subset A of G is an H-p.a.i. subset of $G \backslash H$ if A is an H-p.a.i. subset of G such that $A \cap H = \emptyset$.

Q. *Why do we consider H-p.a.i. subsets of $G \backslash H$?*

This choice is a technicality which will turn out to be useful when picking a basis for $\mathcal{K}$ with pairwise disjoint representatives. Hopefully, it will be made clear later on.

Lemma 2.9.6. *Suppose A, B are subsets of $G \backslash H$ such that $AH = A$ and $BH = B$. If $A + B$ is H-finite, then $A = B$.*

Proof. Let $g \in A + B$. Then, $gH \subseteq (A + B)H = A + B$. Hence, if $A + B$ is H-finite, so is gH. By Property P, we know that $g \in H$, which is a contradiction. Therefore, $A + B = \emptyset \;\Rightarrow\; A = B$. □

We write A^* for the subset $(G \backslash H) \backslash A$.

Remark 2.9.7. *Suppose $A, B \subseteq G \backslash H = H^c$. Then,*

- $A^* = H^c + A$;
- $(A^c)^* = H^c \backslash A^c = H^c \cap A = A$;
- $(A^*)^c = G + A^* = G + (H^c \cap A^c) = G + (H \cup A)^c = A \cup H = A + H$;
- $A^c = (A^c \cap H) + (A^c \cap H^c) = (A^c \cap H) + A^* = H + A^*$;
- *if $h \in H$, then $(Ah)^* = Ah + H^c h = (A + H^c)h = A^* h$ and the same follows with $(hA)^*$;*
- $(A + B)^* = A + B + H^c = A^* + B = A + B^*$.

Suppose $\dim_{\mathbb{F}_2} \mathcal{K} = k + 1$. By Corollary 2.8.6, there exist k H-l.i.p.a.i subsets $A_1, \ldots, A_k$ of G such that $A_i H = A_i$ for each $i = 1, \ldots, k$. Explicitly, we have that

$$\mathcal{K} = \mathrm{Span}_{\mathbb{F}_2} \{[A_1], \ldots, [A_k], [G]\}$$

However, each representative A_i can be chosen as a subset in $G \backslash H$. Indeed,

$$A_i + (A_i \cap H^c) = (A_i \cap G) + (A_i \cap H^c) = A_i \cap (G + H^c) = A_i \cap H \subseteq H$$

which is H-finite. By Lemma 2.4.5 (iii), $[A_i] = [A_i \backslash H]$. Moreover, $(A_i \cap H^c)H = A_i \cap H^c$. Therefore, we can write $\mathcal{K}$ as the spanned set of $\{[A_1 \backslash H], \ldots, [A_k \backslash H], [G \backslash H]\}$. Furthermore, as consequence of Proposition 2.8.3 and Lemma 2.9.6, if A is an H-almost invariant subset of $G \backslash H$ satisfying $AH = A$, then $A \in \mathrm{Span}_{\mathbb{F}_2} \{A_1 \backslash H, \ldots, A_k \backslash H, G \backslash H\}$.

We say $A_1, \ldots, A_k$ are H-l.i.p.a.i. subsets of $G \backslash H$ if $A_1, \ldots, A_k$ are H-p.a.i. subsets of $G \backslash H$ and $[A_1], \ldots, [A_k]$ are linearly independent in $\mathcal{A}_H(G)$.

Given k right H-invariant H-l.i.p.a.i. subsets of G, we can always find H-l.i.p.a.i. subsets $A_1, \ldots, A_k$ of $G \backslash H$ such that $A_i H = A_i$ for each $i = 1, \ldots, k$. We denote

$$\mathcal{K}^* = \mathrm{Span}_{\mathbb{F}_2} \{A_1, \ldots, A_k, G \backslash H\}$$

Clearly, if $\dim_{\mathbb{F}_2} \mathcal{K} = k + 1$, then $\mathcal{K}$ and $\mathcal{K}^*$ are isomorphic vector spaces over $\mathbb{F}_2$ and

$$\dim_{\mathbb{F}_2} \mathcal{K} = \dim_{\mathbb{F}_2} \mathcal{K}^*$$

Note that, by Lemma 2.9.6, $\mathcal{K}^*$ is uniquely defined, that is, if A is an H-almost invariant subset of G satisfying $AH = A$ and $A \cap H = \emptyset$, then $A \in \mathcal{K}^*$.

Proposition 2.9.8. *Suppose* $\dim_{\mathbb{F}_2}\mathcal{K} = k+1$. *There exist pairwise disjoint H-l.i.p.a.i. subsets $A_1, \ldots, A_k$ of $G\backslash H$ such that $A_iH = A_i$, $i = 1, \ldots, k$.*

Proof. By Corollary 2.8.6, there exist H-l.i.p.a.i. subsets $A_1, \ldots, A_k$ of G, which we have seen that can be chosen in $G\backslash H$, such that $A_iH = A_i$, $i = 1, \ldots, k$. When $k = 1$, the proposition clearly holds.
We prove it by induction on k. Suppose $A_1, \ldots, A_{k-1}$ can be chosen to be pairwise disjoint. Let $I = \{1, \ldots, k-1\}$ and I' a subset of I containing every $i \in I$ such that $A_k \cap A_i = A_i$. We can take

$$A'_k := A_k + \sum_{i \in I'} A_i = A_k \backslash \dot{\bigcup_{i \in I'}} A_i \quad \text{(Step S)}$$

Therefore, we may assume the sets $A_1, \ldots, A_k$ satisfy, in addition, the following:

$$A_k \cap A_i \neq A_i, \ \forall i \in I.$$

Below, we study two cases:

1) If $A_k \subset \sum_{i \in I} A_i$, then take

$$A'_k := A_k \cap A_i \text{ and } A'_i := A_i \backslash A'_k = A_i + A'_k,$$

for some $i \in I$ with non-trivial intersection. Note that, if A'_k is a linear combination of $A_1, \ldots, A_{k-1}, G\backslash H$, then $A'_k \cap A_i$ is either A_i or $\emptyset$. In either case, we have a contradiction with our previous assumptions. Therefore, we find new H-l.i.p.a.i. right H-invariant sets

$$A_1, \ldots, A_{i-1}, A'_i, A_{i+1}, \ldots, A_{k-1}, A'_k$$

which are pairwise disjoint.

2) If $A^*_k \subset \sum_{i \in I} A_i$, then take $A'_k := A^*_k$. If necessary, we apply Step S to obtain new H-l.i.p.a.i. sets and we are in case (1) again.

Finally, we may additionally assume that A_k does not satisfy cases (1) and (2).
For each $j \in I$, denote $D_j := A_k \cap A_j$ and let $i \in I$ such that $D_i \neq \emptyset$. Knowing that

$D_i \in \mathcal{K}^*$, we write

$$D_i = B + \lambda_i A_i + \lambda_k A_k + \lambda H^c,$$

where $\lambda_i, \lambda_k, \lambda \in \mathbb{F}_2$ and B is a linear combination of $A_1, \ldots, A_{i-1}, A_{i+1}, \ldots, A_{k-1}$. For the remaining proof, we show that, for any coefficients in $\mathbb{F}_2 = \{0, 1\}$ satisfied by $\lambda_i, \lambda_k, \lambda$, we will end up in a contradiction with our assumptions.

If $B = \emptyset$, then either $D_i = \emptyset$, $D_i = A_i$, $D_i = A_k$ or $D_i = A_k^* + A_i \;\Rightarrow\; A_k^* \subset A_i$. In either case, we have a contradiction with the previous assumptions. Suppose then that $B \neq \emptyset$.

- $\lambda = 0$: We know that $B \cap A_i = \emptyset$ and $D_i \subset A_i$. Hence, $B \subset \lambda_k A_k$. Since $B \neq \emptyset$, we must have $\lambda_k = 1$. But $B \subset A_k \;\Rightarrow\; A_k \cap A_j = A_j$ for some $j \in I$, contradiction.
- $\lambda = 1$:
  - if $\lambda_k = 1$, then

    $$D_i = B + \lambda_i A_i + A_k^* \subset A_k \;\Rightarrow\; A_k^* \subset B + \lambda_i A_i,$$

    contradiction (case (2));
  - otherwise, $\lambda_k = 0$. Then,

    $$D_i = B^* + \lambda_i A_i \subset A_k \;\Rightarrow\; A_k^* \subset B + \lambda_i A_i,$$

    which, again, is a contradiction (case (2)).

Therefore, $D_i = \emptyset, \; \forall i \in I$. □

Lemma 2.9.9. *Suppose X, Y are H-p.a.i. subsets of G satisfying $XH = X$, $YH = Y$ and let $x \in X^c \cap (Y^c)^{-1}$. Then, $X \cap xY \in \mathcal{K}^*$.*

Proof. Let $D = X \cap xY$. The fact that $DH = D$ is clear. Also,

$$D \cap H \neq \emptyset \;\Rightarrow\; xY \cap H \neq \emptyset \;\Rightarrow\; Y \cap x^{-1}H \neq \emptyset \;\Rightarrow\; Y \cap Y^c H = Y \cap Y^c \neq \emptyset$$

which is a contradiction. Thus, $D \subset G \backslash H$. Now, let $g \in G$. We use the Kropholler corner (Lemma 2.4.6) to decompose $D + Dg$ in the following way:

$$\begin{aligned} D + Dg &= (X \cap xY) + (X \cap xY)g = (X \cap xY) + (Xg \cap xYg) \\ &= [(X + Xg) \cap xY] + [(xY + xYg) \cap Xg] \end{aligned}$$

Clearly, $(X+Xg)\cap xY$ is H-finite. Now,

$$\begin{aligned}
Xg\cap(xY+xYg) &= Xg\cap x(Y+Yg)\subseteq Xg\cap xHF,\ \text{for some finite subset } F \text{ of } G\\
&\subseteq Xg\cap X^cHF = Xg\cap X^cF\\
&= \bigcup_{i=1}^{m}(Xg\cap X^cf_i) = \bigcup_{i=1}^{m}(Xg\cap(Xf_i)^c),\ f_i\in F\\
&= \bigcup_{i=1}^{m}(Xg\backslash Xf_i)\subseteq \bigcup_{i=1}^{m}(Xg+Xf_i)\\
&= \bigcup_{i=1}^{m}\underbrace{(Xgf_i^{-1}+X)f_i}_{H\text{-finite}}
\end{aligned}$$

Hence, D is H-almost invariant. By Lemma 2.9.6, $D\in\mathcal{K}^*$. □

Lemma 2.9.10. *If A is an H-p.a.i. subset of $G\backslash H$ satisfying $AH=A$, then the same holds for A^*.*

Proof.

$$A^*H=(H^c\cap A^c)H\subseteq H^cH\cap A^cH=H^c\cap A^c=A^*.$$

Clearly, $A^*\subseteq A^*H$. Also, if $A^*=H^c\cap A^c$ is H-finite, so is $A^c=A^*+(H\cap A^c)\subseteq A^*\cup H$, contradiction. Moreover, $(A^*)^c=A\cup H$ is clearly not H-finite. Furthermore, $\forall\ g\in G$,

$$\begin{aligned}
A^*+A^*g &= (H^c\cap A^c)+(H^cg\cap A^cg)\\
&= (H^c+H^cg)\cap A^c+(A^c+A^cg)\cap H^cg\\
&= (H+Hg)\cap A^c+(A+Ag)\cap H^cg\\
&\subseteq H\cup Hg\cup(A+Ag)
\end{aligned}$$

which is H-finite. □

Lemma 2.9.11. *Let $k=\dim_{\mathbb{F}_2}\mathcal{K}+1$ and define $K=\{h\in H\mid hX=X\}$, where $X\in\mathcal{K}^*$. Then, $|H:K|\leq 2^k$.*

Proof. Clearly, K is a subgroup of H. If $h\in H$, hX is a right H-invariant H-almost invariant subset of $G\backslash H$. By lemma 2.9.6, $hX=Y$ for some $Y\in\mathcal{K}^*$. If $X=\emptyset$ or $X=G\backslash H$, then $K=H$. Suppose then X is H-proper. For each $Y\in\mathcal{K}^*$, fix $h_Y\in H$ such that $h_YX=Y$ (h_Y might not exist for every Y). Then,

$$h=h_Yh_Y^{-1}h \text{ and } h_Y^{-1}hX=h_Y^{-1}Y=X\Rightarrow h\in h_YK.$$

Therefore, K has index at most $|\mathcal{K}| = 2^k$ in H. $\square$

Lemma 2.9.12. *Suppose* $\dim_{\mathbb{F}_2}\mathcal{K} < \infty$. *Let* $x \in G\backslash H$ *and* Y *an* H*-p.a.i subset of* G *satisfying* $YH = Y$. *If* $xY \subseteq Z$ *for some* $Z \in \mathcal{K}^*$, *then* $x \in Z$.

Proof. Let $g \in Y' = Y \cap H^c$. By lemma 2.9.6, we know that $Y' \in \mathcal{K}^*$. Take $K = \{h \in H \mid hY' = Y'\}$. We have that

$$Kg \subseteq Y' \Rightarrow xKg \subseteq xY' \subseteq xY \subseteq Z.$$

If $x \in Z^*$, then

$$xKg = xKg \backslash Z^* \subseteq xHg \backslash Z^* \subseteq Z^* g \backslash Z^* \subseteq Z^* g + Z^*$$

which is H-finite. By lemma 2.9.11, $K \leq_f H$. Then, $xK \subseteq HF_1$ for some finite subset F_1 in G and, for some finite subset F_2 in H, we have that $xH = xKF_2 \subseteq HF_1F_2 \Rightarrow xH$ is H-finite. But, by Property P and the assumption that $H = \mathrm{Comm}_G(H)$, we have that $x \in H$, contradiction. Therefore, $x \in Z$. $\square$

Lemma 2.9.13. *Let* X, Y *be* H*-l.i.p.a.i. subsets of* $G\backslash H$ *satisfying* $XH = X$ *and* $YH = Y$. *For any* $x \in G\backslash H$, *the pair* $\{X, xY\}$ *is nested.*

Proof. Consider the following partition of $G\backslash H$:

$$P_1 = X \cap Y^{-1} \quad P_2 = X \cap (Y^*)^{-1} \quad P_3 = X^* \cap Y^{-1} \quad P_4 = X^* \cap (Y^*)^{-1}$$

(It is not difficult to show that, given any subset M of G, $(M^*)^{-1} = (M^{-1})^*$). Let $x \in P_1$. Then, we define the set $D_1 := X^c \cap xY^c$. For $x \in P_i$, for each $i = 2, 3, 4$, we similarly define a corresponding set D_i. Thus, we have:

$$D_1 := X^c \cap xY^c \quad D_2 := X^c \cap xY \quad D_3 := X \cap xY^c \quad D_4 := X \cap xY$$

Clearly, $D_3 \cap H = \emptyset$ and $D_4 \cap H = \emptyset$. If $D_1 \cap H \neq \emptyset$, then $xY^c \cap H \neq \emptyset \Rightarrow Y^c \cap x^{-1}H \neq \emptyset \Rightarrow Y^c \cap YH = Y^c \cap Y \neq \emptyset$, contradiction. Similarly, we show $D_2 \cap H = \emptyset$. Hence, by Lemmas 2.9.6 and 2.9.9, we see that each $D_i \in \mathcal{K}^*$. Since X is H-proper, we know that $D_i \neq G\backslash H$. For each $i = 1, 2, 3, 4$, if $D_i = \emptyset$, we have the corresponding nesting cases:

$$1)\ xY^c \subseteq X \quad 2)\ xY \subseteq X \quad 3)\ xY^c \subseteq X^c \quad 4)\ xY \subseteq X^c$$

We prove that D_1 and D_4 cannot be any other element in $\mathcal{K}^*$ except $\emptyset$. The same proof will follow for D_2 and D_3.

- Suppose $D_1 \neq \emptyset$. Then, $D_1 \subseteq xY^c = xY^* \dot{\cup} xH$. But

$$D_1 \cap xH \subseteq D_1 \cap X \subseteq X^c \cap X = \emptyset.$$

  Hence, $x^{-1}D_1 \subseteq Y^*$. By Lemma 2.9.12, $x^{-1} \in Y^* \Rightarrow x \in (Y^*)^{-1} = (Y^{-1})^*$, which is a contradiction, because we took $x \in Y^{-1}$.

- Suppose $D_4 \neq \emptyset$. Then, $D_4 \subseteq xY \Rightarrow x^{-1}D_4 \subseteq Y$. By Lemma 2.9.12, $x^{-1} \in Y \Rightarrow x \in Y^{-1}$, which is not possible, since $x \in (Y^*)^{-1} = (Y^{-1})^*$.

□

Proof of Proposition 2.9.1. By Lemma 2.9.13, if $g \notin H$, we know that the pair $\{A_i, gA_j\}$ is nested. Suppose then $g \in H$. We have seen before that gA_j is H-p.a.i. Moreover, $gA_jH = gA_j$ and $gA_j \subset H.H^c = H^c$. Hence, $gA_j \in \mathcal{K}^*$. Thus:

$$gA_j = \lambda_1 A_1 + \ldots + \lambda_k A_k + \lambda H^c$$

where $\lambda_1, \ldots, \lambda_k, \lambda \in \mathbb{F}_2 = \{0, 1\}$. Suppose $\lambda = 0$. Because $A_1, \ldots, A_k$ are pairwise disjoint, any sum of those sets is a disjoint union. Therefore, if $\lambda_i = 1$, then $A_i \subseteq gA_j$. Otherwise, $gA_j \subset A_i^c$.
Now, suppose $\lambda = 1$. If $\lambda_i = 1$, then,

$$gA_j^* = \lambda_1 A_1 + \ldots + \lambda_{i-1} A_{i-1} + A_i + \lambda_{i+1} A_{i+1} + \ldots + \lambda_k A_k$$

which implies $A_i \subset gA_j^* \subset gA_j^c$. Otherwise, $\lambda_i = 0$, and

$$\begin{aligned} gA_j^* &= \lambda_1 A_1 + \ldots + \lambda_{i-1} A_{i-1} + \lambda_{i+1} A_{i+1} + \ldots + \lambda_k A_k \subset A_i^* \\ &\Rightarrow A_i \subset gA_j \end{aligned}$$

□

2.10 Building a cubing

In this section, we describe the construction of a CAT(0)-cube complex, which we will refer as a *cubing*. This construction was given in details by Sageev in [28]. We will be interested in building the 1-skeleton of such complex and studying the conditions for the existence of squares. To the engaged reader curious in learning more about the details of attaching n-cells and preserving the non-positively curved condition of the cubing, we suggest reading [28]. We will leave such details out of the thesis and focus on the results that will help us prove the Main Theorem.

Let $H \leq G$ be groups. In [28], the first step for the construction of a cubing is to assume that $e(G, H) \geq 2$. By Theorem 2.4.1, it implies that G contains an H-p.a.i. subset satisfying $HA = A$. Then, one defines

$$\Sigma = \{gA, gA^c \mid g \in G\},$$

a partially ordered set whose combinatoric properties will be captured in the cubing, which we will denote by X. There exists an action of G on X which is defined as *essential with respect to a hyperplane*. If X is a tree, then G acts essentially on X if G acts without global fixed points on X transitively on the edges such that there is an egde e with stabiliser H, which is exactly what defines a splitting of G over H.

In our setting, we will have a slightly different start. We consider $\dim_{\mathbb{F}_2}\mathcal{K} < \infty$ and we assume that G contains an H-p.a.i. subset satisfying $AH = A$, not necessarily left H-invariant. With this H-p.a.i. subset A we define Σ just as above. The construction of X will follow exactly the same steps of Sageev's construction and such choice of Σ will require only small adjustments further on.

2.10.1 Vertices and Edges

Definition 2.10.1. *A **vertex** V of X is a subset of Σ satisfying the following conditions:*

1. *For all $B \in \Sigma$, exactly one of B, B^c is in V.*

2. *If $B \in V$, $C \in \Sigma$ and $B \subseteq C$, then $C \in V$.*

The set of all vertices is denoted by $\hat{\mathcal{V}}$.

Example 2.10.2. *Given $g \in G$, it is easily seen that the set $V_g = \{B \in \Sigma \mid g \in B\}$ is a vertex of X. Furthermore, if $x \in G$, the action of G on V_g is given by*

$$\begin{aligned} xV_g &= \{xB \in \Sigma \mid g \in B\} \\ &= \{xB \in \Sigma \mid xg \in xB\} \\ &= \{B \in \Sigma \mid xg \in B\} \\ &= V_{xg} \end{aligned} \tag{2.1}$$

Definition 2.10.3. *An* ***edge*** *e is defined by a pair (V, W) of vertices in $\hat{\mathcal{V}}$ such that $|V \backslash W| = |W \backslash V| = 1$. In other words, there exists $B \in V$ such that $W = (V \backslash \{B\}) \cup \{B^c\}$. This set W is denoted by $(V; B)$. The set of all edges is denoted by $\hat{\mathcal{E}}$.*

Lemma 2.10.4. *Suppose V is a vertex of X and $B \in V$. Then, $(V; B)$ is a vertex of X if and only if B is minimal with respect to inclusion in V. Moreover, B^c is minimal in $(V; B)$.*

Proof. If B is not minimal in V, then one can find $C \subset B$, $C \in V$ and, hence, $C \in (V; B)$. By definition of vertex, $B \in (V; B)$, which is an absurd. If B is minimal, it is not difficult to see that $(V; B)$ satisfies the conditions to be a vertex given that V is a vertex. Now, let $(V; B)$ be a vertex and suppose there exists $D \in (V; B)$ such that $D \subset B^c$. Then, $B \subset D^c \Rightarrow D^c \in V$, by definition of vertex. Hence, $|V \backslash (V; B)| = 2$, which is a contradiction with the definition of edge. □

Therefore, if e is an edge with vertices V and $(V; B)$, if we let $(V; B) = W$, then e is an edge with vertices W and $(W; B^c)$.

Q. *Are the edges oriented?*

One can define an orientation to an edge $e \in \hat{\mathcal{E}}$: the starting point of e is $(V; B)$ and its end point is V. We say that e *exits* B. The edge $\bar{e}$ is the edge defined by the same pair of vertices V and $(V; B)$, but with opposite orientation. Then, $\bar{e}$ exits B^c.

Q. *What happens when g acts on an edge, for some $g \in G$?*

If e is an edge defined by a pair of vertices (V, W), then the edge ge is defined by the pair of vertices (gV, gW). Since the action of G on vertices only translates the sets in V, if B is minimal in V then gB is minimal in gV. Therefore, if e exits B, then ge exits gB.

Q. *What does the right H-invariance of A imply in the action of G on X?*

Proposition 2.10.5. *H fixes a point in X.*

Proof. Take $V_1 = \{B \in \Sigma \mid 1 \in B\}$. Clearly, $V_1 \neq \emptyset$, since $1 \in xA$, $\forall x \in A^{-1}$. Since $AH = A$, we have that, $\forall a \in A,\ h \in H$

$$ah \in A \ \Leftrightarrow \ h^{-1}a^{-1} \in A^{-1}$$

which implies $HA^{-1} = A^{-1}$. Similarly, $H(A^c)^{-1} = (A^c)^{-1}$. Hence, if $h \in H$ and $xA \in V_1$, we know that

$$x \in A^{-1} \Rightarrow hx \in A^{-1} \Rightarrow 1 \in hxA \Rightarrow hxA \in V_1.$$

If $h \in H$ and $xA^c \in V_1$, then

$$x \in (A^c)^{-1} \ \Rightarrow \ hx \in (A^c)^{-1} \ \Rightarrow \ 1 \in hxA^c \ \Rightarrow \ hxA^c \in V_1.$$

Therefore, $HV_1 \subseteq V_1$. Clearly, $V_1 \subseteq HV_1$. Therefore, H fixes V_1. □

Notation: Let C be minimal in $(V;B)$. Then, $((V;B)\setminus\{C\}) \cup \{C^c\}$ is denoted by $(V;B,C)$.

Q. *Do $\hat{\mathcal{V}}$ and $\hat{\mathcal{E}}$ compose the 1-skeleton of X then?*

No, because we want a connected 1-skeleton. Thus, only a subset of both will be considered. Given two vertices $V, W \in \hat{\mathcal{V}}$, we say that they are joined by an edge-path if there exists a finite sequence of vertices $V = V_1, \ldots, V_n = W$ such that (V_i, V_{i+1}) are pairs of vertices defining edges $e_i \in \hat{\mathcal{E}}$ for all $i = 1, \ldots, n-1$.
Now, let $\mathcal{V}$ be the subset of $\hat{\mathcal{V}}$ such that $V \in \mathcal{V}$ if there exists an edge-path from V to V_g, for some $g \in G$. Let $\mathcal{E}$ be the set of edges in $\hat{\mathcal{E}}$ that have both endpoints in $\mathcal{V}$. The 1-skeleton $X^{(1)}$ of X is the graph $(\mathcal{V}, \mathcal{E})$.

Theorem 2.10.6 ([28], Thm. 3.3)**.** *$X^{(1)}$ is connected.*

The idea of the proof is to show that, $\forall g_1, g_2 \in G$, the vertices V_{g_1} and V_{g_2} are joined by an edge-path. In order to prove that, a necessary step is to show that $V_{g_1} \triangle V_{g_2}$ is finite.

Lemma 2.10.7. *For any $g_1, g_2 \in G$, the set $V_{g_1} \triangle V_{g_2}$ is finite.*

Proof.

$$\begin{aligned} V_{g_1} \triangle V_{g_2} &= \{B \in \Sigma \mid B \in V_{g_1} \text{ and } B \notin V_{g_2}\} \cup \{B \in \Sigma \mid B \in V_{g_2} \text{ and } B \notin V_{g_1}\} \\ &= \{B \in \Sigma \mid g_1 \in B \text{ and } g_2 \in B^c\} \cup \{B \in \Sigma \mid g_2 \in B \text{ and } g_1 \in B^c\} \\ &= \underbrace{\{gA \mid g_1 \in gA, g_2 \in gA^c\}}_{Y_1} \cup \underbrace{\{gA^c \mid g_1 \in gA^c, g_2 \in gA\}}_{Y_2} \end{aligned}$$

and it suffices to prove that Y_1 is finite.

$$\begin{aligned} gA \in Y_1 \quad &\Leftrightarrow \quad g_1 \in gA, g_2 \in gA^c \Leftrightarrow g^{-1} \in Ag_1^{-1} \cap A^c g_2^{-1} \\ &\Leftrightarrow \quad g^{-1} \in (A \cap A^c g_2^{-1} g_1) g_1^{-1} = A \backslash A g_2^{-1} g_1 \subseteq \underbrace{A + A g_2^{-1} g_1}_{H\text{-finite}} \end{aligned}$$

Therefore, $g^{-1} \in Hx_1 \cup \ldots \cup Hx_n \Rightarrow g \in x_1^{-1} H \cup \ldots \cup x_n^{-1} H$ for some $x_j \in G$. Let

$$K = \{h \in H \mid hA = A\}$$

By lemma 2.9.11, $K \leq_f H$. Since $kA = A$, $\forall k \in K$, and $H = FK$, for some finite set F in H, then

$$x_1^{-1} H \cup \ldots \cup x_n^{-1} H = x_1^{-1} FK \cup \ldots \cup x_n^{-1} FK$$

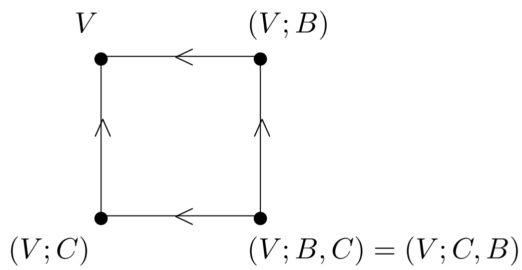
If $g_1, g_2 \in x_i^{-1} fK$ for some $f \in F$, then $g_1 A = x_i^{-1} fA = g_2 A$. Therefore, there is a finite choice of gA in Y_1. □

Q. *How does one build higher dimensional skeletons of X?*

Informally, to build the higher dimensional skeletons of X, an n-cube is glued whenever the boundary of an n-cube appears in $X^{(n-1)}$ and the cube complex is built inductively. The formal details of such construction are left out of this thesis, because our interest will revolve on the existence of squares in X.

2.10.2 Squares

A 2-cube (or a square) is given by



We say $(V, \{B, C\})$ *spans* the square above. As consequence to the existence of such square, the following conditions are satisfied:

1) B, C are minimal in V,

2) B is minimal in $(V;C)$,

3) C is minimal in $(V;B)$.

Lemma 2.10.8. *$(V;\{B,C\})$ spans a square if and only if $B \neq C$ are minimal in V and $B^c \not\subset C$. Moreover, if B, C are nested then $(V;\{B,C\})$ does not spans a square.*

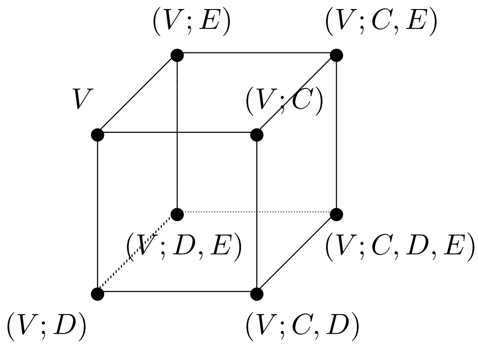
Proof. Given condition 1, note that B is minimal in $(V;C)$ if and only if $C^c \not\subset B$ and C is minimal in $(V;B)$ if and only if $B^c \not\subset C$. If B, C are distinct minimal sets in V, then $(V;B), (V;C)$ exist. Also, $B^c \not\subset C \;\Rightarrow\; B, C$ are minimal in $(V;C)$ and $(V;B)$ respectively. Therefore, $(V;B,C) = (V;C,B)$ exists. Furthermore, the existence of the square implies that

- $B \not\subset C$ and $C \not\subset B$, otherwise it would contradict the minimality of B and C in V (since B, C are distinct);
- $B \not\subset C^c$ (or equivalently $C \not\subset B^c$), otherwise C^c and B^c would be in V, which contradicts the definition of vertex;

Therefore, B and C are not nested. □

Q. *What about n-cubes, $n \geq 3$? How do they look like?*

Each 2-dimensional face of an n-cube, for $n \geq 3$, is a picture similar to the square above spanned by $(V;\{B,C\})$ with distinct spanning sets. We illustrate that with the picture below of a 3-cube (or cube):



Note that, in order to an n-cube to exist, we need each square of this n-cube to exist. Next two lemmas describe n-cubes and the conditions for their existence:

Lemma 2.10.9 ([28], Lem. 3.5)**.** *Suppose $\mathcal{C}$ is an n-cube in X having V as a vertex with neighbouring vertices $(V;B_1),\ldots,(V;B_n)$. Let V' be the vertex diagonally opposite to V in $\mathcal{C}$. Then, $V' = (V;_1,\ldots,B_n)$.*

We say that the n-cube described by this lemma is spanned by $(V;\{B_1,\ldots,B_n\})$.

Lemma 2.10.10 ([28], Lem. 3.6)**.** *Suppose V is a vertex and $S = \{B_1,\ldots,B_n\} \subset V$. Then, $(V;S)$ spans an n-cube in X if and only if $B_i \neq B_j$, each B_i is minimal in V and $B_i^c \not\subset B_j$ for all $i,j \in \{1,\ldots,n\}$.*

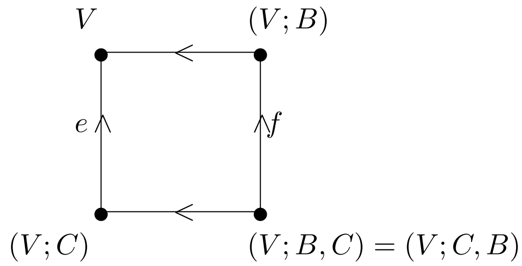
The proof that X is a cubing is found in [28], Thm 3.7.

2.10.3 Hyperplanes

In trees, the geometric concept of a hyperplane is the midpoint of each edge. In this section, we work with combinatorial hyperplanes, defined as an equivalence class of edges. The geometric definition will be provided, but all the results will revolve the first one. We will better understand the action of G on X by studying the action of G on the hyperplanes of X and, finally, show that G acts *essentially* on X *with respect to a hyperplane.*

Q. *Whats is a hyperplane?*

We say two edges e, f are equivalent if there exists a finite sequence of edges $e = e_1,\ldots,e_n = f$ such that for each $i = 1,\ldots,n-1$, e_i, e_{i+1} are opposite sides of the same square, oriented in the same direction.



Definition 2.10.11. *A* ***combinatorial hyperplane*** *J is an equivalence class of edges. If e is an edge in J, then $\bar{J}$ is the hyperplane consisting of edges equivalent to $\bar{e}$.*

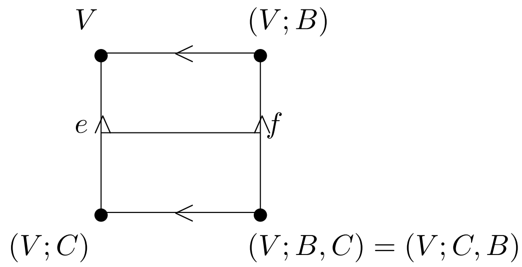
Clearly, each edge defines a unique combinatorial hyperplane.

Q. *What is a geometric hyperplane?*

To each pair of hyperplanes $(J, \bar{J})$ in X there exists a geometric definition of hyperplane equivalent to the pair $(J, \bar{J})$.

Definition 2.10.12. *Given a combinatorial hyperplane J, the* ***geometric hyperplane*** *related to $(J, \bar{J})$ is the collection of the intersection of each n-cube in X containing edges in J with a 1-codimensional hyperplane in $\mathbb{R}^n$ crossing the midpoints of those edges.*

See the line in the middle of the square below to better illustrate the definition:



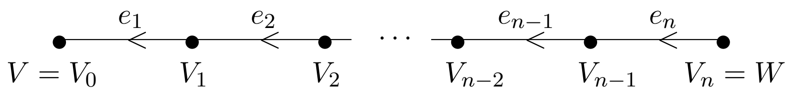
Unless stated otherwise, every further reference of a hyperplane will assume the definition of a combinatorial hyperplane.

Definition 2.10.13. *A hyperplane J* ***crosses*** *an edge-path in X if there exists an edge e in the path such that $e \in J$.*

Definition 2.10.14. *A* ***geodesic*** *in X from vertex V to vertex W is an edge-path from V to W containing the smallest number of edges among all possible edge-paths from V to W. We use the notation* $\text{geod}[V, W]$.

Note that a geodesic is not necessarily unique. Just consider a square spanned by $(V; \{B, C\})$. There are two edge-paths of same length from V to $(V; B, C)$.

Lemma 2.10.15 ([28], 3.8)**.** *Suppose V, W are two vertices of X. Let $\alpha = \mathrm{geod}[V, W]$ consisting of edges $e_1, \ldots, e_n$ arranged as follows:*



where $V_i = (V_{i-1}; B_i)$ for $i = 1, \ldots, n$ and some $B_i \in \Sigma$. Then, the set $\{B_1, \ldots, B_n, B_1^c, \ldots, B_n^c\}$ is a set of distinct elements.

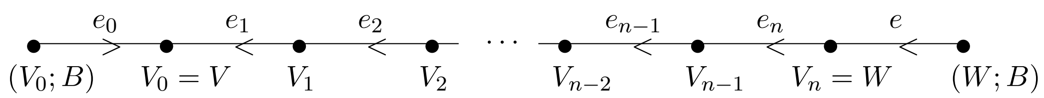
Proof. We use induction on the number of edges (length) of α. If $n = 1$, then the proof is trivial. Now, suppose the lemma is true for $m < n$. The edge-path from V to V_{n-1} is clearly a geodesic. Then, $\left\{B_1, \ldots, B_{n-1}, B_1^c, \ldots, B_{n-1}^c\right\}$ is a set of distinct elements. Suppose $B_n = B_i$ for some $1 \leq i \leq n-1$. We know that B_n is minimal in V_{n-1} and also in V_{i-1}. Since, $V_i = (V_{i-1}; B_i)$, we see that $B_i \notin V_i$. Therefore, if $B_n = B_i$, there must exits $i < j < n$ such that $B_j = B_i^c$, contradiction. Assume then that $B_n = B_i^c$, minimal in V_{n-1}. Suppose B_i^c is not minimal in V_j but minimal in V_{j+1} for some $n - 1 > j > i$. Then, $B_{j+1} \subseteq B_i^c$, which is equivalent to $B_i \subseteq B_{j+1}^c$. But by definition of a vertex in X, $B_{j+1}^c \in V$, contradiction, since $B_{j+1} \in V$. Therefore, B_i^c is minimal in V_j, $i \leq j \leq n-1$. Hence, for $i + 1 \leq l \leq n - 2$, the vertices $(V_l; B_i^c)$ and $(V_l; B_{l+1})$ exist. Moreover, $B_i \not\subset B_{l+1}$ because B_i, B_{l+1} are minimal in V. By Lemma 2.10.8, each $(V_l; \{B_i^c, B_{l+1}\})$ spans a square. Therefore, the sequence of vertices $V_0, V_1, \ldots, V_{i-2}, V_{i-1} = (V_i; B_i^c), (V_{i+1}; B_i^c), \ldots, (V_{n-2}; B_i^c), (V_{n-1}; B_i^c) = V_n$ generates another path from V to W whose length is smaller than the length of α, which is a contradiction since α is a geodesic. $\square$

Lemma 2.10.16 ([28], 3.9)**.** *Suppose J is a hyperplane of X. Then, there exists an $B \in \Sigma$ such that, for every $e \in J$, e exits B. Moreover, every edge which exits B lies in J.*

Proof. Let e_0 be an edge in J and B the set which e_0 exits. Let e be another edge in J. By the definition of hyperplane, there exists a sequence of edges $e_0, e_1, \ldots, e_n = e$ such that for each $i = 0, \ldots, n-1$, e_i, e_{i+1} are opposite sides of the same square in X, with same orientation. Using induction on n, if $n = 1$, then, by the construction of a square seen before, let e_0 be the edge exiting B given by the pair of vertices $((V; B), V)$. Then, for some $B' \in \Sigma$, e_1 is the edge exiting B given by the pair of vertices $((V; B', B), (V; B'))$.

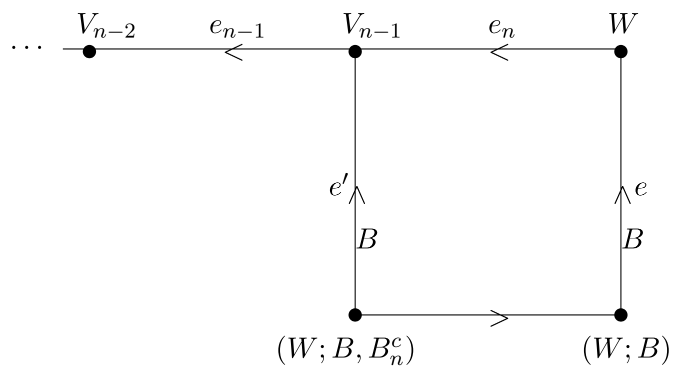
Now, by induction, suppose e_{n-1} exits B. Again, being e_n an edge opposite to e_{n-1} in a same square, with same orientation, by the construction of a square we see that e_n exits B.

Now, let e be an edge in X which exits B and let e_0 be an edge in J. Take a geodesic path α from the tail of e_0 denoted by V_0 to the tail of e denoted by W. Both edges exit B. Thus, we have the following picture:



where $V_i = (V_{i-1}; B_i)$ for $i = 1, \dots, n$ and $B_i \in \Sigma$. (Observe that if $B_1 = B$, then $B_i = B^c$ for some $i = 1, \dots, n-1$, since $B \in W$. By Lemma 2.10.15, $B_i^c \neq B_1$ if α is a geodesic. Similarly, if $V_{n-1} = (W; B)$, then $B_n = B^c$ and $B_i = B$ for some $i = 1, \dots, n-1$. Again, by Lemma 2.10.15, $B_n^c \neq B_i$ if α is a geodesic.)

Suppose $n = 1$. We see that B_1, B are minimal in V and B is minimal in $W = V1 = (V; B_1)$. Therefore, we have sufficient conditions for a square spanned by $(V; \{B, B_1\})$. Hence, e_0 and e are opposite sides of this square $\Rightarrow e \in J$. Now, we can use induction on the length of α. The set B_n^c is minimal in W. If $B_n^c = B$, then $(W; B) = V_{n-1}$ and, by the observation above that cannot happen. Therefore B_n^c and B are distinct minimal sets in W. If B_n^c is not minimal in $(W; B)$, then $B^c \subset B_n^c \Rightarrow B_n \subset B \Rightarrow B$ is not minimal in V, contradiction. Therefore, by Lemma 2.10.8 there exists a square spanned by $(W; \{B, B_n^c\})$. Hence, there exists a path from V to $(W; B, B_n^c)$ of length smaller than n passing by the edge e' exiting B defined by the pair of vertices $((W; B_n^c, B), (W; B_n^c))$ (see picture below). By induction, $e' \in J$. But e' and e are opposite sides of the same square, with same orientation. Therefore, $e \in J$.



□

Remark 2.10.17. *If J_B is the hyperplane consisting of edges exiting B, then gJ_B is the hyperplane whose edges exit gB and is denoted by J_{gB}.*

Recall that, given any G-set Y, the group stabiliser of Y is $\{g \in G \mid gY = Y\}$ denoted by $\mathrm{Stab}(Y)$.

Lemma 2.10.18. *Let $B \in \Sigma$. Then,* $\mathrm{Stab}(J_B) = \mathrm{Stab}(B)$.

Proof. Let $g \in G$.

$$gJ_B = J_{gB} = J_B \underset{\text{Lemma 2.10.16}}{\Leftrightarrow} gB = B \Leftrightarrow g \in \mathrm{Stab}(B).$$

□

Lemma 2.10.19. *If J_1, J_2 are two hyperplanes in X such that $J_1 = xJ_2$ for some $x \in G$, then* $\mathrm{Stab}(J_1) = x\mathrm{Stab}(J_2)x^{-1}$.

Proof.

$$\begin{aligned} \mathrm{Stab}(J_1) &= \{g \in G \mid gJ_1 = J_1\} \\ &= \{g \in G \mid gxJ_2 = xJ_2\} \\ &= \left\{g \in G \mid x^{-1}gxJ_2 = J_2\right\} \\ &= \left\{g \in G \mid x^{-1}gx \in \mathrm{Stab}(J_2)\right\} \\ &= x\mathrm{Stab}(J_2)x^{-1} \end{aligned}$$

□

Corollary 2.10.20. *The action of G on X is transitive on the geometric hyperplanes.*

Proof. Let J_1, J_2 be two geometric hyperplanes respectively related to the combinatorial hyperplanes J_{g_1A}, J_{g_2A}. Then,

$$g_2g_1^{-1}J_{g_1A} = J_{g_2A} \Rightarrow g_2g_1^{-1}J_1 = J_2.$$

□

Lemma 2.10.21. $H \sim \mathrm{Stab}(J_A)$.

Proof. Let $K = \{h \in H \mid hA = A\}$. We have seen that $K \leq_f H$. By Lemma 2.5.3, A is a K-p.a.i. subset of G satisfying $KA = A$. By Theorem 2.4.1, $e(G, K) \geq 2$, which implies by [28], Lem. 2.4, that $|\mathrm{Stab}(A) : K| < \infty$. Since $K = H \cap \mathrm{Stab}(A)$ and $\mathrm{Stab}(J_A) = \mathrm{Stab}(A)$, we obtain the required result. □

2.10.4 Essential action

Let J be a geometric hyperplane of X. Then, J partitions X into two components ([28], Thm. 4.10) denoted by J^+ and J^-. Given a vertex $V \in X$, we define a partition of G into two sets: $A_V = \{g \in G \mid gV \in J^+\}$ and its complement A_V^c.

Definition 2.10.22. *G acts **essentially with respect to a hyperplane** J on the cubing X if there exists a vertex in $V \in X$ such that A_V and A_V^c both contain infinitely many right cosets of* Stab(J).

Theorem 2.10.23. *The action of G on X is essential with respect to the hyperplane J_A.*

Proof. Let e be an edge in J_A with tail V_a, for some $a \in G$. In this case, note that $A \in V_a$. The geometric hyperplane related to J_A separates X into two components J_A^+ and J_A^-. Choose J_A^+ to be the component containing V_a. Define $A_{V_a} = \left\{g \in G \mid gV_a \in J_A^+\right\}$. Then,

$$\begin{aligned} A_{V_a} &= \left\{g \in G \mid V_{ga} \in J_A^+\right\} \\ &= \{g \in G \mid J_A \text{ does not cross an egde-path from } V_a \text{ to } V_{ga}\} \\ &= \{g \in G \mid A \in V_{ga}\} \\ &= \{g \in G \mid ga \in A\} \\ &= \left\{g \in G \mid g \in Aa^{-1}\right\} \\ &= Aa^{-1} \end{aligned}$$

Since A, A^c are H-infinite, we have that $A_{V_a}, A_{V_a}^c$ are also H-infinite. By Lemmas 2.10.21 and 2.5.3, we conclude that $A_{V_a}, A_{V_a}^c$ are Stab(J_A)-infinite. □

Given these settings, we prove the next two lemmas below:

Lemma 2.10.24. *For any vertex W in X, the partition sets A_W, A_W^c of G with respect to J_A are also H-infinite.*

Proof.

$$\begin{aligned} A_W + A_{V_a} &= \left\{g \in G \mid gW \in J_A^+ \text{ and } gV_a \notin J_A^+\right\} \cup \left\{g \in G \mid gW \notin J_A^+ \text{ and } gV_a \in J_A^+\right\} \\ &= \{g \in G \mid \exists e \in J_A \text{ belonging to a geodesic from } gW \text{ to } gV_a\} \\ &= \left\{g \in G \mid \exists e \in J_A \text{ such that } g^{-1}e \text{ belongs to a geodesic from } W \text{ to } V_a\right\} \\ &= \left\{g \in G \mid g^{-1}J_A \text{ is some hyperplane crossing a geodesic } W \text{ to } V_a\right\} \end{aligned}$$

But there is a finite number of hyperplanes containing the edges of any geodesic from W to V_a. Also, if $g_1, g_2 \in G$,

$$g_1 J_A = g_2 J_A \Leftrightarrow g_2^{-1} g_1 \in \mathrm{Stab}(J_A),$$

which implies that $A_W + A_{V_a}$ is contained in a finite union of right cosets of $\mathrm{Stab}(J_A)$. Using the same argument, $A_W^c + A_{V_a}^c$ is also contained in a finite union of right cosets of $\mathrm{Stab}(J_A)$. Therefore, A_W and A_W^c are $\mathrm{Stab}(J_A)$-infinite and hence, H-infinite. □

Hence, for any vertex $W \in X$, A_W and A_W^c give a partition of G into two $\mathrm{Stab}(J_A)$-infinite subsets.

Lemma 2.10.25. *A_W is an H-p.a.i. subset of G.*

Proof. By the previous lemma, we know that A_W is H-proper. Using a similar proof, we show that A_W is H-almost invariant. Let $y \in G$:

$$\begin{aligned} A_W + A_W y &= \left\{g \in G \mid gW \in J_A^+ \text{ and } gy^{-1}W \notin J_A^+\right\} \cup \left\{g \in G \mid gW \notin J_A^+ \text{ and } gy^{-1}W \in J_A^+\right\} \\ &= \left\{g \in G \mid g^{-1}J_A \text{ is some hyperplane crossing an edge-path from } W \text{ to } y^{-1}W\right\}, \end{aligned}$$

which is $\mathrm{Stab}(J_A)$-finite and, hence, H-finite. □

2.11 The Main Theorem

Finally, we have all the necessary definitions and results to understand and prove the Main Theorem of this thesis:

Main Theorem. *Let $H \leq G$ be finitely generated groups satisfying:*

- $2 \leq \widetilde{e}(G, H) < \infty$;
- $H \leq_f \mathrm{Comm}_G(H)$;
- $\forall g \in G$, *if gH is H-finite, then $g \in \mathrm{Comm}_G(H)$.*

If G contains an H-p.a.i. subset A such that $AH = A$, then G admits a non-trivial splitting over a subgroup commensurable with H.

Proof. When $\widetilde{e}(G,H) = 2$, the theorem was mentioned before to be true, see Lemma 2.9.2 and further comments. Then, consider $3 \leq \widetilde{e}(G,H) < \infty$. By Corollary 2.8.4, the fact that G contains an H-p.a.i. subset A satisfying $AH = A$ gives us that

$$1 \leq k = \dim_{\mathbb{F}_2} \ker Res_H^G < \infty$$

which implies that

$$2 \leq \dim_{\mathbb{F}_2} \mathcal{K} = k + 1 < \infty.$$

By Proposition 2.9.8, we can find pairwise disjoint H-l.i.p.a.i. subsets $A_1, \ldots, A_k$ of $G \backslash H$ satisfying $A_i H = A_i$, for $i = 1, \ldots, k$. W.l.o.g., we apply Sageev's construction on $\Sigma = \{gA_1,\ gA_1^c \mid g \in G\}$ to build a cubing X on which G acts essentially with respect to the hyperplane J_{A_1}, as in Theorem 2.10.23.
Let $\{X_1,\ X_2\}$ be a distinct pair of elements in Σ. By Proposition 2.9.1, we have that the pair $\{X_1,\ X_2\}$ is nested. But by Lemma 2.10.8, if V is a vertex in X containing $X_1,\ X_2$ as minimal elements, then $(V, \{X_1,\ X_2\})$ does not span a square. Therefore, X is a tree on which G acts without global fixed points transitively on the edges and, by Lemma 2.10.21, H is commensurable to an edge stabiliser. In other words, G splits non-trivially over a subgroup commensurable with H. □

Bibliography

[1] Bestvina, M. Questions in geometric group theory. http://www.math.utah.edu/ bestvina/eprints/questions-updated.pdf

[2] Brady, N.; Leary, I. J.; Nucinkis, B. E. A. On algebraic and geometric dimensions for groups with torsion, J. London Math. Soc. (2) 64 (2001), no. 2, 4890500

[3] Bredon, G. E. Equivariant cohomology theories. Lecture Notes in Mathematics, No. 34. Springer-Verlag, Berlin, 1967.

[4] Brown, K. S. Cohomology of groups. Bull. Amer. Math. Soc. (N.S.) 11 (1984), no. 1, 240-244.

[5] Cook, G. C.; Moreno, V. G.; Nucinkis, B. E. A.; Pasini, F. W. On the dimension of classifying spaces for families of abelian subgroups (2016), available at arxiv:1604.05889v1

[6] Degrijse, D.; Petrosyan, N. Bredon cohomological dimensions for groups acting on CAT(0)-spaces, Groups, Geometry and Dynamics, accepted (2014).

[7] Degrijse, D. A comohological characterization of locally virtually cyclic groups (2015), available at arXiv:1506.06260 [math.GR]

[8] Dicks, W.; Dunwoody, M. J. Groups acting on Graphs, Cambridge Studies in Advanced Mathematics 17, CUP (1989).

[9] Dunwoody, M. J. Structure trees and networks (2015), available at arXiv:1311.3929v2 [math.CO]

[10] Dunwoody, M. J. Accessibility and groups of cohomological dimension one, Proc. London Math. Soc. (3) 38 (1979), no. 2, 193-215.

[11] Eilenberg, S.; Ganea, T. On the Lusternik-Schnirelmann category of abstract groups, Ann. of Math. (2) 65 (1957), 517-518. MR0085510(19,52d).

[12] Fluch, M. G. On Bredon (Co-)Homological Dimensions of Groups, Ph.D. thesis, University of Southampton (2010), available at arXiv:1009.4633v1.

[13] Fluch, M. G.; Leary, I. J. An Eilenberg-Ganea phenomenon for actions with virtually cyclic stabilisers. Groups Geom. Dyn., 8(1):135-142, 2014.

[14] Freudenthal, H. Uber die Enden topologischer Räume und Gruppen, Math. Z. (33), 1 (1931) 692-713.

[15] Geoghegan, R. Topological methods in group theory, GTM-243 (2008), Springer ISBN 978-0-387-74611-1.

[16] Hopf, H. Enden offener Räume und unendliche diskontinuierliche Gruppen, Comment. Math. Helvetici, 16 (1944), 81-100.

[17] Houghton, C. H. Ends of locally compact groups and their quotient spaces, J. Austral. Math. Soc. 17 (1974) 274-28

[18] Juan-Pineda, D.; Leary, I. J. On classifying spaces for the family of virtually cyclic subgroups, Recent developments in algebraic topology, 2006, pp. 135-145.

[19] Kropholler, P. H.; Roller, M. A. Splittings of Poincare duality groups, Math. Z. 197 (1988) 421-438.

[20] Kropholler, P. H.; Roller, M. A. Splittings of Poincaré duality groups II, J. Lond. Math. Soc., 38 (1988), pp. 410-420

[21] Kropholler, P. H.; Roller, M. A. Relative Ends and Duality Groups, Journal of Pure and Applied Algebra, 61 (1989) 197-210, North Holland.

[22] Leary, I. J.; Petrosyan, N. On dimensions of groups with cocompact classifying spaces for proper actions (2017). Advances in Mathematics, 311, 730-747. DOI: 10.1016/j.aim.2017.03.008

[23] Lück, W. Transformation groups and algebraic K-theory, Lecture Notes in Mathematics, vol. 1408, Springer-Verlag, Berlin, 1989. Mathematica Gottingensis.

[24] Lück, W.; Meintrup, D. On the universal space for group actions with compact isotropy, Geometry and topology: Aarhus (1998), 2000, pp. 293-305.

[25] Lück, W. Survey on classifying spaces for families of subgroups. In Infinite groups: geometric, combinatorial and dynamical aspects, volume 248 of Progr. Math., pages 269-322. Birkhäuser, Basel, 2005.

[26] Lück, W.; Weiermann, M. On the classifying space of the family of virtually cyclic subgroups. Pure App. Math. Q., 8(2):479-555, 2012.

[27] Niblo, G. A.; Sageev, M., The Kropholler conjecture, In Guido's Book of Conjectures, Monographies de L'Enseignement Mathématique, 40. L'Enseignement Mathmatique, Geneva, 2008.

[28] Sageev, M. Ends of group pairs and non-positively curved cube complexes, Proc. London Math. Soc. (3), 71 (1995), 585-617.MR 1347406 (97a:20062)

[29] Scott, G. P. Ends of pairs of groups, J. Pure Appl. Algebra 11 (1977) 179-198.

[30] Stallings, J. R. On torsion-free groups with infinitely many ends. Ann. of Math. (2) 88 (1968) 312-334.

[31] Swan, R. G. Groups of cohomological dimension one. J. Algebra 12 1969 585-610.

[32] Symonds, P. The Bredon cohomology of subgroup complexes, J. Pure Appl. Algebra 199 (2005). 261-298