UNIVERSITY OF SOUTHAMPTON

Homotopy theory of gauge groups over 4-manifolds

by

Tseleung So

A thesis submitted in partial fulfillment for the degree of Doctor of Philosophy

in the

Faculty of Social, Human and Mathematical Sciences
Department of Mathematics

July 2018

UNIVERSITY OF SOUTHAMPTON

ABSTRACT

FACULTY OF SOCIAL, HUMAN AND MATHEMATICAL SCIENCES
Department of Mathematics

Doctor of Philosophy

HOMOTOPY THEORY OF GAUGE GROUPS OVER 4-MANIFOLDS

by Tseleung So

Given a principal G-bundle P over a space X, the gauge group $\mathcal{G}(P)$ of P is the topological group of G-equivariant automorphisms of P which fix X. The study of gauge groups has a deep connection to topics in algebraic geometry and the topology of 4-manifolds. Topologists have been studying the topology of gauge groups of principal G-bundles over 4-manifolds for a long time. In this thesis, we investigate the homotopy types of gauge groups when X is an orientable, connected, closed 4-manifold. In particular, we study the homotopy types of gauge groups when X is a non-simply-connected 4-manifold or a simply-connected non-spin 4-manifold. Furthermore, we calculate the orders of the Samelson products on low rank Lie groups, which help determine the classification of gauge groups over S^4.

Contents

Academic thesis: Declaration of authorship

I, **Tseleung So** , declare that this thesis and the work presented in it are my own and has been generated by me as the result of my own original research.

Homotopy theory of gauge groups over 4-manifolds

I confirm that:

1. This work was done wholly or mainly while in candidature for a research degree at this University;
2. Where any part of this thesis has previously been submitted for a degree or any other qualification at this University or any other institution, this has been clearly stated;
3. Where I have consulted the published work of others, this is always clearly attributed;
4. Where I have quoted from the work of others, the source is always given. With the exception of such quotations, this thesis is entirely my own work;
5. I have acknowledged all main sources of help;
6. Where the thesis is based on work done by myself jointly with others, I have made clear exactly what was done by others and what I have contributed myself;
7. Either none of this work has been published before submission, or parts of this work have been published as:

Signed: ....

Dated: ....

Acknowledgements

First of all, I would like to thank my family who always support my decision and back me up. Without their support I cannot complete this project.

I would like to thank my supervisor Prof Stephen Theriault for his great help in my PhD life. He gave clear guidance on my thesis and papers and spent much time proofreading them, for which he should take most of the credit. Knowing my English is not good, in our meetings he repeated his explanation with patience to ensure I fully understood it and he did a lot to improve my presentation and writing skills. Furthermore, he gave useful career advice and helped me apply for postdoctoral jobs. I was so lucky to have Stephen, who is nice, patient and helpful, as my supervisor.

Thank my examiners Prof Jie Wu and Prof Nansen Petrosyan for reviewing my thesis and giving a lot of useful comments.

Living in Southampton in the last last four years was an amazing and memorable experience to me. There are too many people on my thank-you list to put down in the acknowledgement, including Jelena, Vesna, Bernhard, Shizuo and Mikiko, Mike, Dave, Tom Harris, Joe, Charles, Matt, Ingrid, Ariana, Robin, Hector, Ana, Ashley, Semih, Tyrone, Abi, Xin, Fabio, James, Hollis, Motiejus, Conrad, Naomi, Laurie, Tom Brown, Megan, Simon, Holly, Vlad, Jiawen, Kiko, Mariam, Donya, Jamie, Ilaria, two Martinas, William and Jing, Andy, Jason, Heyi, Ace, Cherry and Louis. In particular, I thank all current and previous people in Room 7007 and 8035 for providing a friendly, supportive and enjoyable atmosphere, thank the algebraic topologists for your company in many happy trips, thank Mike, Matt, Ingrid and Hector for discussing mathematics with me, and thank Mike and Hector for teaching me atypical English culture and knowledge.

I also thank Prof Conan Leung, Prof Haizhong Li and Prof Tom Wan for helping my PhD applications, and my friends in Hong Kong for your friendship. I especially thank Alis, Liu-wong, Cyril, Chara, Man-sai, Sarah and Ryan for your support and encouragement in my difficult times.

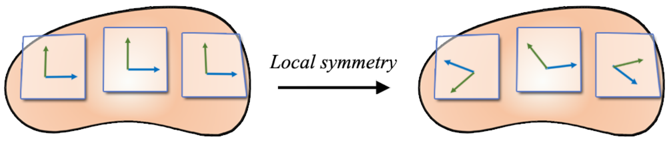
Writing this acknowledgement brings back a lot of memories. Now a verse comes to my head: " 'Not by might nor by power, but by My Spirit', says the Lord of hosts." (Zechariah 4:6) I believe it was my Lord who led me to achieve this. Thank you.

To my parents and to the memory of my grandmother.

Chapter 1

Introduction

Gauge groups arise naturally in physics. In a physical theory, a local coordinate frame at each point is represented by an element of a Lie group G. A gauge group is a collection of local symmetries, which are continuous pointwise transformations of these local bases.



In mathematics, a gauge group $\mathcal{G}(P)$ associated to a principal G-bundle P over X is the topological group of G-equivariant automorphisms of P which fix X. The study of gauge groups has a deep connection with topology and geometry, especially within the study of 4-manifolds, which is one of the most central and difficult topics in modern mathematics. For example, Donaldson applied $SU(2)$-gauge theory to obtain his groundbreaking results on the classification of smooth 4-manifolds, for which he won the Fields Medal.

Suppose X is an orientable, connected, closed 4-manifold and G is a simple, simply-connected, compact Lie group. Every principal G-bundle over X is classified by its second Chern class $k \in \mathbb{Z}$. Denote its associated gauge group by $\mathcal{G}_k(X)$. When X is a simply-connected spin 4-manifold, many cases of the homotopy types of gauge groups of X have been investigated over the last twenty years. For examples, there are 6 distinct homotopy types of $\mathcal{G}_k(X)$'s for $G = SU(2)$ [Kon91], and 8 distinct homotopy types for $G = SU(3)$ [HK06]. When localized rationally or at any odd prime, there are 16 distinct homotopy types for $G = SU(5)$ [The15] and 8 distinct homotopy types for $G = Sp(2)$ [The10a]. However, many problems still remain unsolved. Compared to the extensive work on the spin 4-manifold case, only two cases of $\mathcal{G}_k(X)$ have been studied for non-spin X, which are the $SU(2)$- and $SU(3)$-cases [KT96, The12]. Furthermore, very little is known about $\mathcal{G}_k(X)$ when X is a non-simply-connected 4-manifold due to the complexity of the topology of 4-manifolds.

My Ph.D. research includes three interrelated projects: (1) *the homotopy types of gauge groups over non-simply-connected 4-manifolds*, (2) *calculating the orders of the Samelson products on Lie groups* and (3) *the homotopy types of gauge groups over non-spin 4-manifolds*, of which my findings are summarized in my three articles [So18a, So18c, So18b] in Chapter 6 to 8. In the first project, I developed a homotopy decomposition of gauge groups based on ideas in [Sut92, The10b] that enabled me to decompose the gauge groups over certain non-simply-connected 4-manifolds.

Theorem A. *Let X be an orientable, smooth, connected, closed 4-manifold and let G be a simple, simply-connected, compact Lie group. If $\pi_1(X)=\mathbb{Z}^{*m}$ or $\mathbb{Z}/p^r\mathbb{Z}$ for some odd prime p, then we have*

$$\mathcal{G}_k(X)\simeq\mathcal{G}_k(X')\times\text{“loop spaces” on } G,$$

where X' is S^4 or $\mathbb{C}P^2$. If $\pi_1(X)=(\mathbb{Z}^{*m}*(*_{j=1}^{n}\mathbb{Z}/p_j^{r_j}\mathbb{Z}))$ for odd primes p_j, then we have

$$\mathcal{G}_k(X)\times\prod^{2d}\Omega^2G\simeq\mathcal{G}_k(X')\times\text{“loop spaces” on } G$$

where d is some positive integer.

The space X' is S^4 or $\mathbb{C}P^2$ depending on whether the suspension of the attaching map of the 4-cell in X is null homotopic or not. Also, the term “loop spaces” refer both to iterated based loop spaces $\Omega G,\Omega^2G$ and Ω^3G and modular loop spaces $\Omega G\{p^r\}$ and $\Omega^2G\{p^r\}$, where $G\{p^r\}$ is the homotopy fiber of the p^r-power map on G. Explicit decompositions are stated in Chapter 6.

By Theorem A the homotopy types of $\mathcal{G}_k(X)$ are determined by $\mathcal{G}_k(S^4)$ and $\mathcal{G}_k(\mathbb{C}P^2)$. Combining the known results of $\mathcal{G}_k(S^4)$ and $\mathcal{G}_k(\mathbb{C}P^2)$ [HK06, KK18, KTT17, Kon91, The10a, The15, The17, KT96, The12] gives the following partial classification of $\mathcal{G}_k(X)$.

Corollary B. *If X is an orientable, smooth, connected, closed 4-manifold with $\pi_1(X)=\mathbb{Z}^{*m}$ or $\mathbb{Z}/p^r\mathbb{Z}$, then the following hold:*

- *when $G=SU(2)$, there is a homotopy equivalence $\mathcal{G}_k(X)\simeq\mathcal{G}_l(X)$ if and only if $(12,k)=(12,l)$ for X spin, and $(6,k)=(6,l)$ for X non-spin;*
- *when $G=SU(3)$, there is a homotopy equivalence $\mathcal{G}_k(X)\simeq\mathcal{G}_l(X)$ if and only if $(24,k)=(24,l)$ for X spin; there is a p-local homotopy equivalence $\mathcal{G}_k(X)\simeq\mathcal{G}_l(X)$ if and only if $(12,k)=(12,l)$ for any prime p and X non-spin;*
- *when $G=SU(n)$, there is a p-local homotopy equivalence $\mathcal{G}_k(X)\simeq\mathcal{G}_l(X)$ if and only if $(n(n^2-1),k)=(n(n^2-1),l)$ for any odd prime p such that $n\leq(p-1)^2+1$;*
- *when $G=Sp(2)$, there is a p-local homotopy equivalence $\mathcal{G}_k(X)\simeq\mathcal{G}_l(X)$ if and only if $(40,k)=(40,l)$ for any odd prime p;*

- *when $G = G_2$, there is a p-local homotopy equivalence $\mathcal{G}_k(X) \simeq \mathcal{G}_l(X)$ if and only if $(84, k) = (84, l)$ for any odd prime p.*

The second project was to calculate the orders of the Samelson products on Lie groups G. The commutator map $[\mathbb{1}, \mathbb{1}] : G \times G \to G$ that sends (a, b) to $a^{-1}b^{-1}ab$ descends to a map $\langle \mathbb{1}, \mathbb{1} \rangle$, unique up to homotopy, called the Samelson product. In Chapter 5 we will define the order of $\langle \mathbb{1}, \mathbb{1} \rangle$. It measures the non-commutativity of G and helps determine the homotopy types of gauge groups of principal G-bundles over S^4, which will be explained with more details in Chapter 5. In general, calculating the order of $\langle \mathbb{1}, \mathbb{1} \rangle$ is difficult. To simplify the calculation, I used the homotopy decomposition of low rank Lie groups developed by [MNT77, CN84, The07], which is introduced in Chapter 4, and obtained the following results.

Theorem C. *For a p-localized Lie group G, the order of $\langle \mathbb{1}_G, \mathbb{1}_G \rangle$ is p^r when*

G	$r = 0$	$r = 1$
$SU(n)$	$p > 2n$	$n \leq p < 2n$
$Sp(n)$	$p > 4n$	$2n < p < 4n$
$Spin(2n+1)$	$p > 4n$	$2n < p < 4n$
$Spin(2n)$	$p > 4n-4$	$2n-2 < p < 4n-4$
G_2	$p = 5, p > 11$	$p = 7, 11$
F_4, E_6	$p > 23$	$11 \leq p \leq 23$
E_7	$p > 31$	$17 \leq p \leq 31$
E_8	$p > 59$	$23 \leq p \leq 59$

In the third project, I investigated the homotopy types of $\mathcal{G}_k(\mathbb{CP}^2)$, which determine the homotopy types of $\mathcal{G}_k(X)$ for non-spin 4-manifold X by the homotopy equivalence [So18a]

$$\mathcal{G}_k(X) \simeq \mathcal{G}_k(\mathbb{CP}^2) \times \prod^{d-1} \Omega^2 G.$$

In particular, I studied the boundary map $\partial_1' : G \to \mathrm{Map}^*(\mathbb{CP}^2, BG)$ of the evaluation fibration $\mathrm{Map}^*(\mathbb{CP}^2, BG) \to B\mathcal{G}_k(\mathbb{CP}^2) \to BG$ and showed that its order helps determine the homotopy types of $\mathcal{G}_k(\mathbb{CP}^2)$.

Theorem D. *Let m be the order of ∂_1'. If $(m, k) = (m, l)$, then $\mathcal{G}_k(\mathbb{CP}^2)$ is homotopy equivalent to $\mathcal{G}_l(\mathbb{CP}^2)$ when localized rationally or at any odd prime.*

Furthermore, I used unstable K-theory as in [HK06] to give a lower bound on the order of ∂_1' for $G = SU(n)$.

Theorem E. *When $G = SU(n)$, the order of ∂_1' is at least $\frac{1}{2}n(n^2-1)$ for n odd and $n(n^2-1)$ for n even.*

This thesis is a hybrid of the traditional style and the three-paper style. The structure is as follows. Chapter 2 to 4 give a preliminary introduction to homotopy theory, gauge groups and topological localization of Lie groups. Chapter 5 explains the relation among my three articles [So18a, So18c, So18b], which appear as individual chapters in Chapter 6 to 8.

Chapter 2

A review of homotopy theory

The purpose of this chapter is to prepare background material and give a quick review of basic notions in homotopy theory which are the main tools for calculations in later chapters. Most statements are mentioned without proof. Readers are advised to refer to [Ark11, Sel08, Hat02] for more details.

2.1 Basic notions

Throughout the thesis, we assume that all spaces are locally compact Hausdorff and all maps are continuous.

Homotopy can be thought as a time evolution of maps. It is worth keeping track of where points are going during the evolution. Therefore unless specified, we always equip a space X with a point $x_0 \in X$, called a *basepoint*. The pair (X, x_0) is called a *pointed space* and denoted by X without ambiguity. We also require any map $f : X \to Y$ to be *pointed*, that is f sends the basepoint $x_0 \in X$ to the basepoint $y_0 \in Y$.

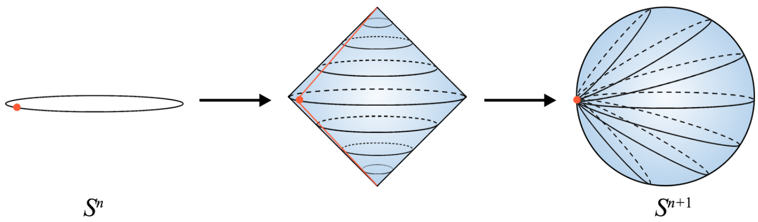
Here we give some basic constructions of pointed spaces. For two pointed spaces (X, x_0) and (Y, y_0), the *wedge* $X \vee Y$ is the quotient $X \sqcup Y / x_0 \sim y_0$ with the basepoint $\langle x_0 \rangle = \langle y_0 \rangle$. It can be regarded as a subspace of the Cartesian product $(X \times Y, (x_0, y_0))$ through an inclusion map

$$i : X \vee Y \to X \times Y, \quad a \mapsto \begin{cases} (a, y_0), & a \in X \\ (x_0, a), & a \in Y. \end{cases}$$

The smash product $X \wedge Y$ is defined to be the quotient $X \times Y / X \vee Y$. Its basepoint is the image $i(X \vee Y)$. The wedge and smash product constructions satisfy associativity and distributivity

$$(X \wedge Y) \wedge Z \cong X \wedge (Y \wedge Z),$$
$$X \wedge (Y \vee Z) \cong (X \wedge Y) \vee (X \wedge Z),$$
$$(X \vee Y) \wedge Z \cong (X \wedge Z) \vee (Y \wedge Z).$$

When $Y = S^1$, the smash product $S^1 \wedge X$ is called the *suspension* of X and denoted by ΣX. For example, ΣS^n is homeomorphic to S^{n+1} for $n \geq 0$.



A point in ΣX can be represented by a quotient class $\langle t, x\rangle$ where $t \in I$ and $x \in X$. The basepoint is $* = \langle 0, x'\rangle = \langle 1, x'\rangle = \langle t', x_0\rangle$ for all $t' \in I$ and $x' \in X$. Given a map $f : X \to Y$, the suspension of f is a map $\Sigma f : \Sigma X \to \Sigma Y$ defined by $\Sigma f\langle t, x\rangle = \langle t, f(x)\rangle$.

Another important construction is the *mapping space*

$$\mathrm{Map}(X, Y) = \{f : X \to Y\},$$

which contains all continuous (but not necessarily pointed) maps from X to Y. It is equipped with the compact-open topology, that is open subsets are generated by the subsets of the form

$$U_{K,V} = \{f : X \to Y | f(K) \subset V\},$$

where $K \subset X$ is compact and $V \subset Y$ is open. Its basepoint is the constant map $* : X \to Y$ which sends all $x \in X$ to $y_0 \in Y$. Similarly we define the *pointed mapping space* $\mathrm{Map}^*(X, Y)$ to be the subspace of $\mathrm{Map}(X, Y)$ containing all pointed continuous maps from X to Y. For any spaces X, Y and Z, their mapping spaces satisfy the *exponential laws*, that is homeomorphism relations

$$\mathrm{Map}(X \times Y, Z) \cong \mathrm{Map}(X, \mathrm{Map}(Y, Z)) \quad \text{and} \quad \mathrm{Map}^*(X \wedge Y, Z) \cong \mathrm{Map}^*(X, \mathrm{Map}^*(Y, Z)).$$

The *path space* PX and the *loop space* ΩX of X are defined to be the pointed mapping spaces $\mathrm{Map}^*(I, X)$ and $\mathrm{Map}^*(S^1, X)$. They contain paths $\gamma \in PX$ and loops $\omega \in \Omega X$ such that their initial values $\gamma(0) = \omega(0) = x_0$. Given a map $g : Y \to X$, the looping of g is a map $\Omega g : \Omega X \to \Omega Y$ defined by $\omega(t) \mapsto f \circ \omega(t)$.

Now we describe the notions of homotopy and homotopy equivalence.

Definition 2.1. Two maps $f, g : X \to Y$ are *homotopic* if there exists a map $H : X \times I \to Y$, called a *homotopy*, such that

$$H(x,t) = \begin{cases} f(x), & t = 0 \\ g(x), & t = 1 \\ y_0, & x = x_0. \end{cases}$$

We denote it by $f \simeq_H g$ or simply $f \simeq g$. If f is homotopic to the constant map $*$, then it is called *null homotopic*.

Definition 2.2. A map $f : X \to Y$ is a *homotopy equivalence* if there exists a map $g : Y \to X$ such that $g \circ f \simeq \mathbb{1}_X$ and $f \circ g \simeq \mathbb{1}_Y$. Then we say that X and Y are *homotopy equivalent*. If X is homotopy equivalent to a point, then it is called a *contractible* space.

Homotopy forms an equivalence relation in the set of maps. The equivalence class of a map f is denoted by $[f]$ and the set of equivalence classes $[X, Y]$ is called a *homotopy set*. Also, homotopy equivalence is an equivalence relation in the set of spaces and the homotopy equivalence class of a space X is called the *homotopy type* of X. For spaces X, Y and Z, by the exponential law there is a bijection

$$[X \wedge Y, Z] \cong [X, \mathrm{Map}^*(Y, Z)].$$

This relation is called the *adjunction*. When $Y = S^1$, it becomes $[\Sigma X, Z] \cong [X, \Omega Z]$.

For $n \geq 0$, the n^{th} *homotopy group* $\pi_n(X)$ is defined to be $[S^n, X]$. The adjunction implies that $\pi_{n+1}(X) \cong \pi_n(\Omega X)$. A space X is n-*connected* if $\pi_i(X)$ is trivial for $i \leq n$, and is *weakly contractible* if $\pi_i(X)$ is trivial for all $i \geq 0$. A map $f : X \to Y$ is n-*equivalence* if the induced homomorphism $f_* : \pi_i(X) \to \pi_i(Y)$ is an isomorphism for $i \leq n$, and is *weak equivalence* if $f_* : \pi_i(X) \to \pi_i(Y)$ is isomorphism for all i. Obviously all homotopy equivalences are weak equivalences. The converse is true when the spaces are CW-complexes.

Theorem 2.3 (Whitehead)**.** *Any weak equivalence $f : X \to Y$ between CW-complexes X and Y is also a homotopy equivalence.*

2.2 H-spaces and co-H-spaces

A topological group G is a topological space having continuous maps $m : G \times G \to G$ and $\imath : G \to G$ satisfying abstract group axioms. From the point of view of homotopy theory, these axioms are too strong and limit the existence of a topological group structure on a space. Instead, we can weaken the axioms so that they are satisfied up to homotopy and generalize the notion of a topological group.

Definition 2.4. A space Y is an *H-space* if there is a map $m : Y \times Y \to Y$, called a *multiplication*, satisfying the homotopy commutative diagram

$$\begin{tikzcd} Y \arrow[r, "i_1"] \arrow[dr, "\mathbb{1}_Y"'] & Y \times Y \arrow[d, "m"] & Y \arrow[l, "i_2"'] \arrow[dl, "\mathbb{1}_Y"] \\ & Y & \end{tikzcd}$$

where i_1 and i_2 are inclusions sending $y \in Y$ to (y, y_0) and (y_0, y).

Definition 2.5. Let Y be an H-space and m be its multiplication. Then we say that:

1. Y is *homotopy associative* if m satisfies the homotopy commutative diagram

$$\begin{tikzcd} Y \times Y \times Y \arrow[r, "\mathbb{1}\times m"] \arrow[d, "m\times\mathbb{1}"'] & Y \times Y \arrow[d, "m"] \\ Y \times Y \arrow[r, "m"] & Y; \end{tikzcd}$$

2. Y is *homotopy commutative* if m satisfies the homotopy commutative diagram

$$\begin{tikzcd} Y \times Y \arrow[r, "T"] \arrow[d, "m"'] & Y \times Y \arrow[d, "m"] \\ Y \arrow[r, equal] & Y \end{tikzcd}$$

   where $T : Y \times Y \to Y \times Y$ swaps (y_1, y_2) to (y_2, y_1);

3. Y has a *homotopy inverse* if there is a map $\iota : Y \to Y$ satisfies the homotopy commutative diagram

$$\begin{tikzcd} Y \arrow[r, "\triangle"] \arrow[drr, "*"'] & Y \times Y \arrow[r, "\iota\times\mathbb{1}"] & Y \times Y \arrow[d, "m"] & Y \times Y \arrow[l, "\mathbb{1}\times\iota"'] & Y \arrow[l, "\triangle"'] \arrow[dll, "*"] \\ & & Y & & \end{tikzcd}$$

   where $\triangle$ is the diagonal map and $*$ is the constant map;

4. Y is an *H-group* if it is homotopy associative and has a homotopy inverse.

Example 2.6. All topological groups are H-groups. In particular, S^0, S^1 and S^3 are Lie groups and hence are H-groups. They correspond to the unit spheres in $\mathbb{R}$, $\mathbb{C}$ and $\mathbb{H}$. While S^7 corresponds to the unit sphere in $\mathbb{O}$ which is not associative, it is an H-space but not a topological group. By a famous theorem of Adams [Ada60], these are the only spheres which are H-spaces.

Example 2.7. For $\omega, \mu \in \Omega X$, define a loop multiplication by

$$(\omega * \mu)(t) = \begin{cases} \omega(2t), & 0 \leq t \leq 1/2 \\ \mu(2t-1), & 1/2 \leq t \leq 1. \end{cases}$$

One can check that ΩX is an H-group with this loop multiplication.

Suppose Y is an H-space with multiplication m. For $f, g : X \to Y$, define $f \times_m g$ to be the composition

$$f \times_m g : X \xrightarrow{\triangle} X \times X \xrightarrow{f \times g} Y \times Y \xrightarrow{m} Y,$$

where $\triangle$ is the diagonal map. The homotopy class $[f \times_m g]$ does not depend on the choice of representative maps of $[f]$ and $[g]$, so $\times_m$ induces a strict multiplication on $[X, Y]$. When Y is an H-group, $[X, Y]$ is a group with $\times_m$ as its operation.

There is a dual notion of an H-space.

Definition 2.8. A space X is a *co-H-space* if there is a map $\sigma : X \to X \vee X$, called a *comultiplication*, satisfying the homotopy diagram

$$\begin{tikzcd} & Y \arrow[dl, "\mathbb{1}_Y"'] \arrow[d, "\sigma"] \arrow[dr, "\mathbb{1}_Y"] & \\ Y & Y \vee Y \arrow[l, "p_1"'] \arrow[r, "p_2"] & Y \end{tikzcd}$$

where p_1 and p_2 pinch the second and the first copy of Y respectively.

Definition 2.9. Let X be a co-H-space and σ be its comultiplication. Then we say that:

1. X is *homotopy coassociative* if σ satisfies the homotopy commutative diagram

$$\begin{tikzcd} X \arrow[r, "\sigma"] \arrow[d, "\sigma"] & X \vee X \arrow[d, "\sigma \vee \mathbb{1}"] \\ X \vee X \arrow[r, "\mathbb{1} \vee \sigma"] & X \vee X \vee X; \end{tikzcd}$$

2. X is *homotopy cocommutative* if σ satisfies the homotopy commutative diagram

$$\begin{tikzcd} X \arrow[r, equal] \arrow[d, "\sigma"] & X \arrow[d, "\sigma"] \\ X \vee X \arrow[r, "T"] & X \vee X \end{tikzcd}$$

   where $T : X \vee X \to X \vee X$ swaps the first and second copy of X;

3. X has a *homotopy coinverse* if there is a map $\iota' : X \to X$ satisfies the homotopy commutative diagram

$$\begin{tikzcd} & & X \arrow[dll, "*"'] \arrow[d, "\sigma"] \arrow[drr, "*"] & & \\ X & X \vee X \arrow[l, "\triangledown"] & X \vee X \arrow[l, "\iota' \vee \mathbb{1}"] \arrow[r, "\mathbb{1} \vee \iota'"'] & X \vee X \arrow[r, "\triangledown"'] & X \end{tikzcd}$$

   where $\triangledown$ is the folding map;

4. X is a *co-H-group* if it is homotopy coassociative and has a homotopy coinverse.

Example 2.10. Let X be a co-H-space and let Y be a space. Then $X \wedge Y$ is a co-H-space and the comultiplication properties of $X \wedge Y$ are inherited from those of X. Since S^1 is a co-H-group, so are suspensions ΣY and all spheres S^n for $n \geq 1$.

Suppose X is a co-H-space with comultiplication σ. For $f, g : X \to Y$, define $f +_\sigma g$ to be the composition

$$f +_\sigma g : X \xrightarrow{\sigma} X \vee X \xrightarrow{f \vee g} Y \vee Y \xrightarrow{\triangledown} Y,$$

where $\triangledown$ is the folding map. The homotopy class $[f +_\sigma g]$ does not depend on the choice of representative maps of $[f]$ and $[g]$, so $+_\sigma$ induces a strict multiplication on $[X, Y]$. When X is a co-H-group, $[X, Y]$ is a group with $+_\sigma$.

There are two remarks on the group structure of $[X, Y]$. First, if Y is an H-space, then m induces another group multiplication $\times_m$ on $[X, Y]$. The two multiplications $\times_m$ and $+_\sigma$ coincide and commute, so $[X, Y]$ is an abelian group (see [Sel08, Theorem 7.2.3]). For example, $\pi_n(X)$ is abelian when $n \geq 2$ or when $n \geq 1$ and X is an H-space.

Second, if X is a suspension $\Sigma X'$, then X is a co-H-group and $[X, Y]$ is a group. Its adjoint $[X', \Omega Y]$ is also a group since ΩY is an H-space. Furthermore, the adjunction $[\Sigma X', Y] \to [X', \Omega Y]$ is a group isomorphism [Ark11, Proposition 2.3.5].

2.3 Fibrations and cofibrations

Fibrations and cofibrations are dual notions having important theoretical and practical value in homotopy theory. Usually fibrations work better with homotopy groups and cofibrations work better with cohomology groups. They generate long exact sequences of homotopy groups or cohomolgoy groups which help calculations.

Definition 2.11. A map $p : X \to B$ has the *homotopy lifting property* with respect to a space Y if given a map $H_0 : Y \to X$ and a homotopy $h : Y \times I \to B$ such that $h(y, 0) = p \circ H_0(y)$ for all $y \in Y$, then there exists a lifting $H : Y \times I \to X$ satisfying the commutative diagram

$$\begin{tikzcd} Y \arrow[r, "H_0"] \arrow[d, "i"'] & X \arrow[d, "p"] \\ Y \times I \arrow[r, "h"] \arrow[ur, dotted, "H"] & B \end{tikzcd}$$

where $i : Y \hookrightarrow Y \times I$ sends $y \in Y$ to $(y, 0) \in Y \times I$.

Definition 2.12. A map $p : X \to B$ is called a *Hurewicz fibration* or a *fibration* if it has the homotopy lifting property with respect to all spaces, and is called a *Serre fibration* or a *weak fibration* if it has the homotopy lifting property with respect to all CW-complexes. The space X is called the *total space* and B the *base space*. We define the *fiber* to be the preimage of the basepoint $F = p^{-1}(b_0)$. The sequence $F \hookrightarrow X \xrightarrow{p} B$ is called a *fibration sequence*. Sometime we refer to a fibration as the map p or the sequence $X \to B$.

Given a fibration $p : X \to B$, the preimages $p^{-1}(b)$ and $p^{-1}(b')$ are homotopy equivalent for any points b and b' in the same connected component of B (see [Sel08, Proposition 7.1.3]). Therefore the homotopy type of the fiber F depends on the connected component of the base B and not the chosen base point.

Example 2.13. A simple example of a fibration is a projection $p : X = F \times B \to B$. The total space X is the Cartesian product of the fiber F and the base space B.

Example 2.14. Let A and B be path-connected subspaces of X. Define a space of paths by

$$P = \{\gamma \in X^I | \gamma(0) \in A, \gamma(1) \in B\}$$

and the evaluation map $e : P \to A \times B$ by $e(\gamma) = (\gamma(0), \gamma(1))$. Then $e : P \to A \times B$ is a fibration. When $X = Y$ and $A = B = Y$, P is the *free path space* of Y and denoted by $\tilde{P}Y$. The fibration sequence becomes

$$\Omega Y \longrightarrow \tilde{P}Y \xrightarrow{e} Y \times Y.$$

When $A = \{y_0\}$ and $B = Y$, all paths are pointed and P becomes the path space PY. Identify $\{y_0\} \times Y$ with Y and the fibration sequence becomes

$$\Omega Y \longrightarrow PY \xrightarrow{e'} Y,$$

where $e' : PY \to Y$ sends $\gamma \in PX$ to $\gamma(1) \in Y$.

Example 2.15. The evaluation map $ev : \mathrm{Map}(X, Y) \to Y$, which sends f to $f(x_0)$, is a fibration with fiber $\mathrm{Map}^*(X, Y)$. That is

$$\mathrm{Map}^*(X, Y) \longrightarrow \mathrm{Map}(X, Y) \xrightarrow{ev} Y.$$

is a fibration sequence.

Example 2.16. A *fiber bundle* $F \to X \xrightarrow{p} B$ is a fibration whose total space X has a collection of coordinate charts $\{(U_\alpha, \phi_\alpha)\}$ such that $\{U_\alpha\}$ is an open cover of B and for each U_α, $\phi_\alpha : U_\alpha \times F \to p^{-1}(U_\alpha)$ is a homeomorphism satisfying the commutative diagram

$$\begin{tikzcd}
U_\alpha \times F \arrow[r, "\phi_\alpha"] \arrow[d, "pr"] & p^{-1}(U_\alpha) \arrow[d, "p"] \\
U_\alpha \arrow[r, equal] & U_\alpha
\end{tikzcd}$$

where $pr : U_\alpha \times F \to U_\alpha$ is the projection. The collection $\{(U_\alpha, \phi_\alpha)\}$ is called a *local trivialization*. Fiber bundles include many important constructions like tangent bundles, vector bundles, principal bundles, and so on. We will give more details in Chapter 2.

A fibration usually works together with another tool called a pullback.

Definition 2.17. Given $f : X \to Z$ and $g : Y \to Z$, a *pullback* is a triple (P, p_1, p_2) where

$$P = \{(x, y) \in X \times Y | f(x) = g(y)\}$$

and $p_1 : P \to X$ and $p_2 : P \to Y$ are projections. It is represented by a pullback square

$$\begin{tikzcd} P \arrow[r, "p_2"] \arrow[d, "p_1"] & Y \arrow[d, "g"] \\ X \arrow[r, "f"] & Z. \end{tikzcd}$$

A pullback has a universal property as follows. For any space W and maps $h : W \to X$ and $k : W \to Y$ such that $f \circ h = g \circ k$, there is a map $\theta : W \to P$ which sends $w \in W$ to $(h(w), k(w)) \in P$. This property is universal because θ is the unique map from W to P (up to homeomorphism) satisfying the commutative diagram

$$\begin{tikzcd} W \arrow[dr, dotted, "\theta"] \arrow[drr, bend left, "k"] \arrow[ddr, bend right, "h"'] & & \\ & P \arrow[r, "p_2"] \arrow[d, "p_1"] & Y \arrow[d, "g"] \\ & X \arrow[r, "f"] & Z. \end{tikzcd}$$

By the universal property, the pullback triple (P, p_1, p_2) is unique up to homeomorphism.

Proposition 2.18. *Let $F \to Y \xrightarrow{g} Z$ be a fibration sequence and let $f : X \to Z$ be a map. If there is a pullback square*

$$\begin{tikzcd} P \arrow[r, "p_2"] \arrow[d, "p_1"] & Y \arrow[d, "g"] \\ X \arrow[r, "f"] & Z, \end{tikzcd}$$

then $p_1 : P \to X$ is a fibration and its fiber is homeomorphic to F.

Apply Proposition 2.18 to the evaluation fibration $e : PZ \to Z$ in Example 2.14. The new fibrations $f_1 : P_f \to X$ is called the *principal fibration* induced by f and its fiber is ΩZ.

$$\begin{tikzcd} P_f \arrow[r] \arrow[d, "f_1"] & PZ \arrow[d, "e"] \\ X \arrow[r, "f"] & Z \end{tikzcd}$$

Proposition 2.19. *If there is a homotopy commutative diagram*

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "\alpha"] & Y \arrow[d, "\beta"] \\ X' \arrow[r, "f'"] & Y', \end{tikzcd}$$

then the homotopy $H : X \times I \to Y'$ between $f' \circ \alpha$ and $\beta \circ f$ induces a map $\theta_H : P_f \to P_{f'}$ between the principal fibrations of f and f' and there is a homotopy commutative diagram

$$\begin{tikzcd} \Omega Y \arrow[r] \arrow[d, "\Omega\beta"] & P_f \arrow[r, "f_1"] \arrow[d, "\theta_H"] & X \arrow[d, "\alpha"] \\ \Omega Y' \arrow[r] & P_{f'} \arrow[r, "f_1'"] & X'. \end{tikzcd}$$

Proposition 2.19 implies that the homotopy type of P_f depends on the homotopy class $[f]$ only. Note that the induced map θ_H is not unique and depends on the choice of H.

For a principal fibration $\Omega Y \to P_f \to X$, the fiber ΩY has an action $\rho : \Omega Y \times P_f \to P_f$ defined by

$$\forall \omega \in \Omega Y \text{ and } (x, \gamma) \in P_f, \quad \rho : (\omega; (x, \gamma)) \mapsto \rho_\omega(x, \gamma) = (x, \omega * \gamma)$$

where

$$(\omega * \gamma)(t) = \begin{cases} \omega(2t), & 0 \le t \le 1/2 \\ \gamma(2t-1), & 1/2 \le t \le 1. \end{cases}$$

One can check that ρ satisfies the following homotopy commutative diagrams

$$(1)\ \begin{tikzcd} \Omega Y \times P_f \arrow[r, "\rho"] \arrow[d, "pr"] & P_f \arrow[d, "f_1"] \\ P_f \arrow[r, "f_1"] & X \end{tikzcd} \quad (2)\ \begin{tikzcd} \Omega Y \times \Omega Y \arrow[r, "m"] \arrow[d, "\mathbb{1} \times i"] & \Omega Y \arrow[d, "i"] \\ \Omega Y \times P_f \arrow[r, "\rho"] & P_f \end{tikzcd} \quad (3)\ \begin{tikzcd} \Omega Y \arrow[r, "i"] \arrow[d, "j"] & P_f \\ \Omega Y \times P_f \arrow[ur, "\rho"'] & \end{tikzcd}$$

where $pr : \Omega Y \times P_f \to P_f$ is the projection, $i : \Omega Y \to P_f$ is the boundary map and $j : \Omega Y \to \Omega Y \times P_f$ is the inclusion. Moreover, if there is a homotopy commutative diagram

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "\alpha"] & Y \arrow[d, "\beta"] \\ X' \arrow[r, "f'"] & Y', \end{tikzcd}$$

then the induced map $\theta_H : P_f \to P_{f'}$ in Proposition 2.19 satisfies the homotopy commutative diagram

$$\begin{tikzcd} \Omega Y \times P_f \arrow[r, "\rho"] \arrow[d, "\Omega\beta\times\theta_H"] & P_f \arrow[d, "\theta_H"] \\ \Omega Y' \times P_{f'} \arrow[r, "\rho'"] & P_{f'} \end{tikzcd}$$

where ρ' is the action of $\Omega Y'$ on $P_{f'}$.

Proposition 2.20. *Let $f_1 : P_f \to X$ be the principal fibration induced by $f : X \to Y$ and let $g : Z \to X$ be a map. Then g has a lifting $\tilde{g} : Y \to P_f$ through f_1 if and only if the composition $f \circ g$ is null homotopic.*

$$\begin{tikzcd} & P_f \arrow[r] \arrow[d, "f_1"] & PY \arrow[d, "e"] \\ Z \arrow[r, "g"] \arrow[ur, dotted, "\tilde{g}"] & X \arrow[r, "f"] & Y \end{tikzcd}$$

In general, a map $f : X \to Y$ is not a fibration, but we can modify it to become a fibration as follows. Define the *mapping path* $\tilde{P}_f$ of f by the pullback square

$$\begin{tikzcd} \tilde{P}_f \arrow[r] \arrow[d, "p"] & \tilde{P}Y \arrow[d, "e"] \\ X \arrow[r, "f"] & Y \end{tikzcd}$$

where $e : \tilde{P}Y \to Y$ sends a free path γ to $\gamma(0)$ and is a fibration. As a subspace of $X \times \tilde{P}Y$, $\tilde{P}_f$ consists of pairs (x, γ) such that $f(x) = \gamma(0)$. There is an inclusion $s : X \to \tilde{P}_f$ sending x to $(x, \gamma_{f(x)})$, where $\gamma_{f(x)}$ is a constant path valued at $f(x)$. One can show that s is a homotopy equivalence and $\tilde{P}_f \simeq X$. Let $\tilde{f} : \tilde{P}_f \to Y$ be a map sending (x, γ) to $\gamma(1)$. Then $f = \tilde{f} \circ s$ and we show that $\tilde{f}$ is a fibration in the following.

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "s"] & Y \\ \tilde{P}_f \arrow[ur, "\tilde{f}"'] & \end{tikzcd}$$

Suppose there is a commutative diagram

$$\begin{tikzcd} Z \arrow[r, "H_0"] \arrow[d, "i"] & \tilde{P}_f \arrow[d, "\tilde{f}"] \\ Z \times I \arrow[r, "h"] \arrow[ur, dotted, "H"] & Y \end{tikzcd}$$

where i sends $z \in Z$ to $(z, 0) \in Z \times I$. We want to find a map $H : Z \times I \to \tilde{P}_f$ satisfying the diagram. Let $H_0(z) = (x_z, \gamma_z) \in \tilde{P}_f$ where γ_z is a free path of Y such that $\gamma_z(0) = f(x_z)$. Then define a map $H : Z \times I \to \tilde{P}_f$ by $H(z,t) = (x_z, \gamma_{z,t})$ where

$$\gamma_{z,t}(\tau) = \begin{cases} \gamma_z(2\tau) & 0 \le \tau \le \frac{1}{2}; \\ h(z,(2\tau-1)t) & \frac{1}{2} \le \tau \le 1. \end{cases}$$

At $\tau = 0$, $\gamma_{z,t}(0) = x_z$ so $(x_z, \gamma_{z,t})$ is in $\tilde{P}_f$. At $\tau = 1$, $\gamma_{z,t}(1) = h(z,t)$, so $H = i \circ H_0$ and $H = h \circ \tilde{f}$. Therefore H is a lifting of h satisfying the commutative diagram and $\tilde{f}$ is a fibration.

Definition 2.21. For $f : X \to Y$, the space P_f is called the *homotopy fiber* of f.

Note that if $f : X \to Y$ is a fibration, then its fiber F is homotopy equivalent to its homotopy fiber P_f. So the two notions coincide.

A sequence $A \xrightarrow{f} B \xrightarrow{g} C$ is called a *homotopy fibration sequence* if there is a fibration sequence $A' \xrightarrow{f'} B' \xrightarrow{g'} C'$ and homotopy equivalences $\alpha : A \to A'$, $\beta : B \to B'$ and $\gamma : C \to C'$ satisfying the homotopy commutative diagram

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "\alpha"] & B \arrow[r, "g"] \arrow[d, "\beta"] & C \arrow[d, "\gamma"] \\ A' \arrow[r, "f'"] & B' \arrow[r, "g'"] & C'. \end{tikzcd}$$

Start with a map $f : X \to Y$. We have seen that the sequence $P_f \xhookrightarrow{i} \tilde{P}_f \xrightarrow{\tilde{f}} Y$ is a fibration, where $\tilde{P}_f$ is homotopy equivalent to X and $\tilde{f}$ is homotopic to f, so $P_f \xrightarrow{f_1} X \xrightarrow{f} Y$ is a homotopy fibration sequence. Extend this sequence to the left by adding the principal fibration P_{f_1} induced by $f_1 : P_f \to X$

$$\begin{tikzcd} P_{f_1} \arrow[r] \arrow[d, "f_2"] & PX \arrow[d, "e"] \\ P_f \arrow[r, "f_1"] & X \end{tikzcd}$$

The homotopy fiber P_{f_1} is homotopy equivalent to the fiber of $f : P_f \to X$, which is ΩY by Proposition 2.19. The extended sequence becomes

$$\Omega Y \xrightarrow{\partial} P_f \xrightarrow{i} \tilde{P}_f \xrightarrow{\tilde{f}} Y$$

where $\partial : \Omega Y \to P_f$ is called the *connecting map*. Iterating this process, we get a long homotopy fibration sequence

$$\cdots \longrightarrow \Omega^2 Y \xrightarrow{\Omega\partial} \Omega P_f \xrightarrow{\Omega(f_1)} \Omega X \xrightarrow{\Omega f} \Omega Y \xrightarrow{\partial} P_f \xrightarrow{i} X \xrightarrow{f} Y.$$

Proposition 2.22. *For any space Z and map $f : X \to Y$, the following sequence is exact.*

$$\cdots \longrightarrow [Z, \Omega^2 Y] \xrightarrow{\Omega\partial_*} [Z, \Omega P_f] \xrightarrow{\Omega(f_1)_*} [Z, \Omega X] \xrightarrow{\Omega f_*} [Z, \Omega Y] \xrightarrow{\partial_*} [Z, P_f] \xrightarrow{(f_1)_*} [Z, X] \xrightarrow{f_*} [Z, Y]$$

This exact sequence is called the *Puppe sequence* and is very useful in calculating homotopy groups. When $Z = S^0$, it becomes a long exact sequence of homotopy groups

$$\cdots \to \pi_{n+1}(B) \xrightarrow{\partial_*} \pi_n(F) \xrightarrow{i_*} \pi_n(X) \xrightarrow{f_*} \pi_n(B) \to \cdots \to \pi_0(F) \xrightarrow{i_*} \pi_0(X) \xrightarrow{f_*} \pi_0(B).$$

For example, apply it to the Hopf fibration $S^1 \to S^3 \to S^2$ to get an exact sequence

$$\cdots \longrightarrow \pi_2(S^2) \longrightarrow \pi_1(S^1) \longrightarrow \pi_1(S^3) \longrightarrow \pi_1(S^2).$$

Then we can work out $\pi_3(S^2) \cong \mathbb{Z}$ and $\pi_n(S^2) \cong \pi_{n-1}(S^3)$ for $n \geq 4$.

A cofibration is a dual notion of a fibration. Most of its properties can be obtained by dualizing the corresponding statements of a fibration.

Definition 2.23. A map $f : X \to Y$ has the *homotopy extension property* with respect to a space Z if given a map $H_0 : Y \to Z$ and a homotopy $h : X \times I \to Z$ such that $h(x, 0) = H_0 \circ f(x)$ for all $x \in X$, then there exists an extension $H : Y \times I \to Z$ satisfying the commutative diagram

$$\begin{tikzcd}
X \arrow[r, "i"] \arrow[d, "f"] & X \times I \arrow[d, "f\times \mathbb{1}"'] \arrow[ddr, bend left, "h"] & \\
Y \arrow[r, "j"] \arrow[drr, bend right, "H_0"] & Y \times I \arrow[dr, dotted, "H"] & \\
& & Z
\end{tikzcd}$$

where $i : X \hookrightarrow X \times I$ sends x to $(x, 0)$ and $j : Y \hookrightarrow Y \times I$ sends y to $(y, 0)$.

Definition 2.24. A map $f : X \to Y$ is called a *cofibration* if it has the homotopy extension property with respect to all spaces. We define the *cofiber* to be the quotient $C = Y/f(X)$. The sequence $X \xrightarrow{f} Y \xrightarrow{q} C$ is called a *cofibration sequence*, where $q : Y \to C$ is the quotient map. Sometime we refer to a cofibration as the map f or the sequence $X \to Y$.

Example 2.25. If a space K is formed by attaching cells onto its subspace L, then (K, L) is called a *relative CW-pair* and the inclusion $i : L \hookrightarrow K$ is a cofibration. In particular, the inclusion $i : A \hookrightarrow X$ of a subcomplex in a CW-complex is a cofibration.

Example 2.26. The *reduced cone* CX of a space X is the quotient

$$X \times I/\sim, \text{ where } (x, 1) \sim (x_0, t) \text{ for all } x \in X \text{ and } t \in I.$$

The inclusion $i : X \hookrightarrow CX$ sending $x \in X$ to $(x, 0) \in CX$ is a cofibration and the cofiber is ΣX.

Definition 2.27. Given $f : X \to Y$ and $g : X \to Z$, a *pushout* is a triple (Q, q_1, q_2) where

$$Q = Y \sqcup Z/\sim, \text{ where } y \sim z \text{ if and only if } y = f(x) \text{ and } z = g(x) \text{ for some } x \in X,$$

and $q_1 : Y \to Q$ and $q_2 : Z \to Q$ are inclusions. It is represented by a pushout square

$$\begin{tikzcd} X \arrow[r, "g"] \arrow[d, "f"] & Z \arrow[d, "q_2"] \\ Y \arrow[r, "q_1"] & Q. \end{tikzcd}$$

For any space W and maps $h : Y \to W$ and $k : Z \to W$ such that $h \circ f = k \circ g$, there is a map $\theta : Q \to W$ sending $\langle y \rangle \in Q$ to $h(y) \in W$ and $\langle z \rangle \in Q$ to $k(z) \in W$. This property is universal because θ is the unique map from Q to W (up to homeomorphism) satisfying the commutative diagram

$$\begin{tikzcd} X \arrow[r, "g"] \arrow[d, "f"] & Z \arrow[d, "q_2"] \arrow[ddr, bend left, "k"] & \\ Y \arrow[r, "q_1"] \arrow[drr, bend right, "h"] & Q \arrow[dr, dotted, "\theta"] & \\ & & W. \end{tikzcd}$$

By the universal property, the pushout triple (Q, q_1, q_2) is unique up to homeomorphism.

Proposition 2.28. *Let $X \xrightarrow{g} Z \to C$ be a cofibration sequence and let $f : X \to Y$ be a map. If there is a pushout square*

$$\begin{tikzcd} X \arrow[r, "g"] \arrow[d, "f"] & Z \arrow[d, "q_2"] \\ Y \arrow[r, "q_1"] & Q, \end{tikzcd}$$

then $q_1 : Y \to Q$ is a cofibration and its cofiber is homeomorphic to C.

Apply Proposition 2.28 to the cofibration $i : X \to CX$ in Example 2.26. The new cofibrations $q_1 : Y \to C_f$ is called the *principal cofibration* induced by f and its cofiber is ΣX.

$$\begin{tikzcd} X \arrow[r, "i"] \arrow[d, "f"] & CX \arrow[d] \\ Y \arrow[r, "q_1"] & C_f \end{tikzcd}$$

Proposition 2.29. *If there is a homotopy commutative diagram*

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "\alpha"] & Y \arrow[d, "\beta"] \\ X' \arrow[r, "f'"] & Y', \end{tikzcd}$$

then the homotopy $H : X \times I \to Y'$ between $f' \circ \alpha$ and $\beta \circ f$ induces a map $\theta_H : C_f \to C_{f'}$ between the principal cofibrations of f and f' and there is a homotopy commutative diagram

$$\begin{tikzcd} Y \arrow[r, "q_1"] \arrow[d, "\beta"] & C_f \arrow[r] \arrow[d, "\theta_H"] & \Sigma X \arrow[d, "\Sigma\alpha"] \\ Y' \arrow[r, "q_1'"] & C_{f'} \arrow[r] & \Sigma X'. \end{tikzcd}$$

Denote a point in C_f by $\overline{(t,x)}$ and a point in ΣX by $\langle t, x\rangle$. For a principal cofibration $X \xrightarrow{i} C_f \xrightarrow{q} \Sigma X$, the cofiber ΣX has a coaction $\psi : C_f \to C_f \vee \Sigma X$ defined by

$$\psi : \overline{(t,x)} \mapsto \begin{cases} \overline{(2t,x)} \vee * & 0 \le t \le 1/2 \\ * \vee \langle 2t-1, x\rangle & 1/2 \le t \le 1. \end{cases}$$

One can check that ψ satisfies the following homotopy commutative diagrams

$$(1)\ \begin{tikzcd} X \arrow[r, "i"] \arrow[d, "i"] & C_f \arrow[d, "j"] \\ C_f \arrow[r, "\psi"] & C_f \vee \Sigma X \end{tikzcd} \qquad (2)\ \begin{tikzcd} C_f \arrow[r, "\psi"] \arrow[d, "q"] & C_f \vee \Sigma X \arrow[d, "q \vee \mathbb{1}"] \\ \Sigma X \arrow[r, "\sigma"] & \Sigma X \vee \Sigma X \end{tikzcd} \qquad (3)\ \begin{tikzcd} C_f \arrow[r, "q"] \arrow[d, "\psi"] & \Sigma X \\ C_f \vee \Sigma X \arrow[ur, "p"'] & \end{tikzcd}$$

where $j : C_f \to C_f \vee \Sigma X$ is the inclusion and $p : C_f \vee \Sigma X \to \Sigma X$ is the pinch map. Moreover, if there is a homotopy commutative diagram

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "\alpha"] & Y \arrow[d, "\beta"] \\ X' \arrow[r, "f'"] & Y', \end{tikzcd}$$

then the induced map $\theta_H : C_f \to C_{f'}$ in Proposition 2.29 satisfies the homotopy commutative diagram

$$\begin{tikzcd} C_f \arrow[r, "\psi"] \arrow[d, "\theta_H"] & C_f \vee \Sigma X \arrow[d, "\theta_H \vee \Sigma\alpha"] \\ C_{f'} \arrow[r, "\psi'"] & C_{f'} \vee \Sigma X' \end{tikzcd}$$

where ψ' is the coaction of $C_{f'}$.

Proposition 2.30. *Let $q_1 : Y \to C_f$ be the principal cofibration induced by $f : X \to Y$ and let $g : Y \to Z$ be a map. Then g has an extension $\tilde{g} : C_f \to Z$ through q_1 if and only* *if the composition $g \circ f$ is null homotopic.*

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "i"] & Y \arrow[r, "g"] \arrow[d, "q_1"] & Z \\ CX \arrow[r] & C_f \arrow[ur, dotted, "\tilde{g}"'] & \end{tikzcd}$$

We can modify a map $f : X \to Y$ to become a cofibration as follows. Define the *mapping cylinder* M_f of f by the pushout square

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "i"] & Y \arrow[d, "j"] \\ X \times I \arrow[r] & M_f \end{tikzcd}$$

where $i : X \to X \times I$ sends $x \in X$ to $(x, 0)$. There is a retract $r : M_f \to Y$ collapsing $\langle x, t\rangle \in M_f$ to $f(x) \in Y$. One can show that r is a homotopy equivalence and $M_f \simeq Y$. Let $\tilde{f} : X \to M_f$ be a map sending x to $(x, 1)$. Then $\tilde{f}$ is a cofibration and $f = r \circ \tilde{f}$.

$$\begin{tikzcd} & M_f \arrow[d, "r"] \\ X \arrow[ur, "\tilde{f}"] \arrow[r, "f"] & Y \end{tikzcd}$$

Definition 2.31. For a map $f : X \to Y$, the space C_f is called the *homotopy cofiber* of f.

If $f : X \to Y$ is a cofibration, then its cofiber C is homotopy equivalent to its homotopy cofiber C_f.

Proposition 2.32. *For any space Z and map $f : X \to Y$, the following sequence is exact.*

$$\cdots \longrightarrow [\Sigma^2 X, Z] \xrightarrow{\Sigma\partial^*} [\Sigma C_f, Z] \xrightarrow{\Sigma(q_1)^*} [\Sigma Y, Z] \xrightarrow{\Sigma f^*} [\Sigma X, Z] \xrightarrow{\partial^*} [C_f, Z] \xrightarrow{(q_1)^*} [Y, Z] \xrightarrow{f^*} [X, Z]$$

where $q_1 : Y \to C_f$ is the inclusion.

For any abelian group G and positive integer n, there is a CW-complex $K(G, n)$, unique up to homotopy, that has $\pi_n(K(G, n))$ being G and other homotopy groups being trivial. The space $K(G, n)$ is called the Eilenberg-MacLane space and its construction is described in [Hat02, Chapter4] . Take Z to be $K(G, n)$ in Proposition 2.32 and use the identification $\tilde{H}^n(X, G) \cong [X, K(G, n)]$ to obtain the usual long exact sequence of reduced cohomology groups

$$
\begin{tikzcd}
\cdots \arrow[r] & \tilde{H}^1(X,G) \arrow[r, "i^*"] & \tilde{H}^1(Y,G) \arrow[r, "q^*"] & \tilde{H}^1(X/Y,G) \arrow[dll, "\partial^*"] \\
& \tilde{H}^2(X,G) \arrow[r, "i^*"] & \tilde{H}^2(Y,G) \arrow[r, "q^*"] & \tilde{H}^2(X/Y,G) \arrow[r] & \cdots
\end{tikzcd}
$$

Chapter 3

Principal G-bundles and gauge groups

In this chapter G is a topological group. We will revise the notion of a principal G-bundle and define its associated gauge groups. Material in this chapter is based on [AB83, Sel08, Sut92].

3.1 Fiber bundles, principal G-bundles and classifying spaces

Let $F \to X \xrightarrow{p} B$ be a fiber bundle. Then X has a local trivialization $\{(U_\alpha, \phi_\alpha)\}$ consisting of pairs of open sets $U_\alpha \subset B$ and maps $\phi_\alpha : U_\alpha \times F \to p^{-1}(U_\alpha)$ such that $\{U_\alpha\}$ forms an open cover of B, and each ϕ_α is a homeomorphism satisfying the commutative diagram

$$\begin{tikzcd} U_\alpha \times F \arrow[r, "\phi_\alpha"] \arrow[d, "pr"] & p^{-1}(U_\alpha) \arrow[d, "p"] \\ U_\alpha \arrow[r, equal] & U_\alpha \end{tikzcd}$$

where $pr : U_\alpha \times F \to U_\alpha$ is the projection. Let $U_{\alpha\beta}$ be the intersection $U_\alpha \cap U_\beta$. For $b \in U_{\alpha\beta}$, any $\xi \in p^{-1}(b)$ has two local trivializations: $\phi_\alpha(b, f)$ on U_α and $\phi_\beta(b, f')$ on U_β. The two trivializations are related by a *transition function* $g_{\alpha\beta} : U_{\alpha\beta} \to Homeo(F)$ given by

$$g_{\alpha\beta}(b) : F \xrightarrow{i} U_{\alpha\beta} \times F \xrightarrow{\phi_\alpha} p^{-1}(U_{\alpha\beta}) \xrightarrow{\phi_\beta^{-1}} U_{\alpha\beta} \times F \xrightarrow{j} F,$$

where the inclusion $i : F \to U_{\alpha\beta} \times F$ sends f to (b, f) and the projection $j : U_{\alpha\beta} \times F \to F$ sends (b, f') to f'. The *structural group* of X is a subgroup of $Homeo(F)$ that contains the ranges of all transition functions.

Example 3.1. A common example of fiber bundles is a *vector bundle*. A rank-k real vector bundle $\mathbb{R}^k \to V \xrightarrow{p'} B$ has fiber being a k-dimensional vector space. Let $\{(U_\alpha, \phi_\alpha)\}$

be a local trivialization. Then for $b \in U_{\alpha\beta}$, we have

$$\phi_\alpha(b, \vec{v}) = \phi_\beta(b, \vec{v}') \quad \text{if and only if} \quad \vec{v}' = A_b \cdot \vec{v},$$

where $A_b \in \mathrm{GL}_k(\mathbb{R})$. The transition function $g_{\alpha\beta} : U_{\alpha\beta} \to \mathrm{GL}_k(\mathbb{R})$ at b sends $\vec{v}$ to $A_b\vec{v}$ and the structural group of V is $\mathrm{GL}_k(\mathbb{R})$.

Example 3.2. Given a vector bundle $\mathbb{R}^k \to V \xrightarrow{p} B$, we can construct a *frame bundle*

$$\mathrm{GL}_k(\mathbb{R}) \longrightarrow F \xrightarrow{\pi} B$$

whose fibers are coordinate frames of fibers in V, where a coordinate frame is an oriented basis in an Euclidean space $\mathbb{R}^k$. Let $\{(U_\alpha, \phi_\alpha)\}$ be a local trivialization of V and let $\{\vec{e}_i\}_{i=1}^k$ be the standard basis of $\mathbb{R}^k$. For $1 \leq i \leq k$, define a local section

$$s_\alpha^i : U_\alpha \to p^{-1}(U_\alpha), \quad s_\alpha^i(b) = \phi_\alpha'(b, \vec{e}_i).$$

Then $\{s^i(b)_\alpha\}_{i=1}^k$ forms an ordered basis in fiber $p^{-1}(b)$ for each $b \in U_\alpha$. By linear algebra, any basis in $p^{-1}(b)$ is uniquely represented by $(s_\alpha^1(b), \cdots, s_\alpha^k(b)) \cdot A_\alpha(b)$ for some invertible matrix $A_\alpha(b) \in \mathrm{GL}_k(\mathbb{R})$. Let F_b be the set of bases in $p^{-1}(b)$. Then F_b is in one-one correspondence with $\mathrm{GL}_k(\mathbb{R})$.

To construct $\mathrm{GL}_k(\mathbb{R}) \to F \xrightarrow{\pi} B$, define F as the disjoint union $F = \sqcup_{b\in B} F_b$, and $\pi : F \to B$ by $\pi(\xi) = b$ for $\xi \in F_b$. A local trivialization $\{(U_\alpha, \tilde{\phi}_\alpha)\}$ is given by

$$\tilde{\phi}_\alpha : U_\alpha \times \mathrm{GL}_k(\mathbb{R}) \quad \tilde{\phi}_\alpha(b, A) = (s_\alpha^1(b), \cdots, s_\alpha^k(b)) \cdot A.$$

If $g_{\alpha\beta}(b)$ is the transition function of V at $b \in U_{\alpha\beta}$, then

$$s_\alpha^j(b) = \phi_\alpha(b, \vec{e}_j) = \phi_\alpha(b, g_{\alpha\beta}(b)\vec{e}_j) = \sum_{i=1}^k g_{ij}\phi_\beta(b, \vec{e}_i) = \sum_{i=1}^k g_{ij} s_\beta^i(b),$$

where $(g_{ij}) = g_{\alpha\beta}(b)$. This implies $(s_\alpha^1(b), \cdots, s_\alpha^k(b)) = (s_\beta^1(b), \cdots, s_\beta^k(b)) \cdot g_{\alpha\beta}^{-1}(b)$ and

$$\tilde{\phi}_\alpha(b, A) = \tilde{\phi}_\beta(b, A') \quad \text{if and only if} \quad A' = g_{\alpha\beta}(b) \cdot A.$$

Therefore the transition function $\tilde{g}_{\alpha\beta} : U_{\alpha\beta} \to \mathrm{GL}_k(\mathbb{R})$ is given by $\tilde{g}_{\alpha\beta}(b)(A) = g_{\alpha\beta}(b) \cdot A$ and the structural group of F is $\mathrm{GL}_k(\mathbb{R})$.

Moreover, there is right $\mathrm{GL}_k(\mathbb{R})$-action $\rho : F \times \mathrm{GL}_k(\mathbb{R}) \to F$. On $\pi^{-1}(U_\alpha)$ it is given by

$$\rho(\Phi_\alpha(b, A), B) = \rho_B(\Phi_\alpha(b, A)) = \Phi_\alpha(b, AB).$$

It is easy to check that the definition is compatible on $U_{\alpha\beta}$ and hence ρ can extend throughout the whole F.

In Example 3.2 a frame bundle F associated to a vector bundle V has fiber $\mathrm{GL}_k(\mathbb{R})$, which is a topological group. In fact, we can generalize the construction of a frame bundle by replacing the structural group $\mathrm{GL}_k(\mathbb{R})$ by any topological group G, resulting in a principal G-bundle.

Definition 3.3. A *principal G-bundle* $G \to P \xrightarrow{p} B$ is a fiber bundle with a right G-action $\rho : G \times P \to P$ such that:

1. the restriction of ρ on each fiber $p^{-1}(b)$ is free and transitive;
2. given a local trivialization $\{(U_\alpha, \phi_\alpha)\}$, the action ρ_g of $g \in G$ sends $\phi_\alpha(b, h)$ to $\phi_\alpha(b, hg)$ for $b \in U_\alpha$;
3. the base space B is homeomorphic to the homogeneous space P/G through p.

For $b \in U_{\alpha\beta}$, $\phi_\alpha(b, h) = \phi_\beta(b, h')$ if and only if $h' = g_b h$, where $g_b \in G$. Therefore the transition function $g_{\alpha\beta} : U_{\alpha\beta} \to G$ is given by $g_{\alpha\beta}(b) = g_b$ and the structural group is G.

Given an effective left G-action $\tau : G \times F \to F$ on a space F, we can construct an *associated bundle* $F \to P \times_\tau F \xrightarrow{p'} B$ as follows. Set the total space to be the quotient

$$P \times_\tau F = P \times F / \sim,$$

where $(\xi, f) \sim (\xi', f')$ if and only if $\xi' = \xi g$ and $f' = \tau_g f$ for some $g \in G$. Denote the equivalence class of (ξ, f) by $\langle \xi, f \rangle \in P \times_\tau F$. The projection p' is defined to send $\langle \xi, f \rangle$ to $p'(\xi)$. One can check that $p' : P \times_\tau F \to B$ is a fiber bundle with fiber F. Let $\{(U_\alpha, \phi_\alpha)\}$ be the local trivialization of P. Then on the intersection $U_{\alpha\beta}$,

$$\phi'_\alpha(b, f) = \phi'_\beta(b, f') \quad \text{if and only if} \quad f' = \tau_{g(b)} f.$$

Therefore the transition function is given by τ and the structural group is G.

Suppose that there is a pullback square

$$\begin{tikzcd} P \arrow[r, "p_2"] \arrow[d, "p_1"] & Y \arrow[d, "g"] \\ X \arrow[r, "f"] & Z \end{tikzcd}$$

and $g : Y \to Z$ is a fiber bundle. Then $p_1 : P \to X$ is a fiber bundle, called the *pullback bundle* of Y and denoted by $f^*(Y)$. Note that if f and f' are homotopic, their pullback bundles $f^*(Y)$ and $f'^*(Y)$ are isomorphic. This implies there exists an automorphism φ satisfying the commutative diagram

$$\begin{tikzcd} f^*(Y) \arrow[r, "\varphi"] \arrow[d, "p"] & f'^*(Y) \arrow[d, "p'"] \\ X \arrow[r, equal] & X \end{tikzcd}$$

where p' is the projection map of $f'^*(Y)$.

Definition 3.4. A principal G-bundle $\pi : EG \to BG$ is called a *universal G-bundle* if for any principal G-bundle $p : P \to B$ there is a map $f : B \to BG$, unique up to homotopy, such that P is the pullback bundle $P = f^*(EG)$.

$$\begin{tikzcd} P = f^*(EG) \arrow[r] \arrow[d, "p"] & EG \arrow[d, "\pi"] \\ B \arrow[r, "f"] & BG \end{tikzcd}$$

We call f the *inducing map* of P and BG the *classifying space* of G.

The existence of a universal bundle $\pi : EG \to BG$ is guaranteed by a theorem of Milnor [Mil56]. The isomorphism classes of principal G-bundles over B is therefore classified by the homotopy set $[B, BG]$. Moreover, EG is weakly contractible, so ΩBG is weakly homotopy equivalent to G and is strictly homotopy equivalent to G when G is a CW-complex.

Example 3.5. The antipodal map $S^\infty \to \mathbb{RP}^\infty$ is the universal $\mathbb{Z}_2$-bundle, so $\mathbb{RP}^\infty$ is the classifying space of $\mathbb{Z}_2$. Similarly $S^\infty \to \mathbb{CP}^\infty$ is the universal bundle for S^1, so $\mathbb{CP}^\infty$ is the classifying space of S^1.

Example 3.6. For $n \leq k$, the Grassmannian $Gr(n, \mathbb{R}^k)$ consists of n-planes in $\mathbb{R}^k$ and the Stiefel manifold $V(n, \mathbb{R}^k)$ consists of n-orthonormal frames in $\mathbb{R}^k$. Let $V(n, \mathbb{R}^\infty)$ be the colimit $\varinjlim_k V(n, \mathbb{R}^k)$ and let $Gr(n, \mathbb{R}^\infty)$ be $\varinjlim_k Gr(n, \mathbb{R}^k)$. Then the principal $O(n)$-bundle $O(n) \to V(n, \mathbb{R}^\infty) \to Gr(n, \mathbb{R}^\infty)$ is a universal bundle, so $Gr(n, \mathbb{R}^\infty)$ is the classifying space of $O(n)$. Similarly, the complex Grassmannian $Gr(n, \mathbb{C}^\infty)$ consisting of n-complex planes in $\mathbb{C}^\infty$ is the classifying space of $U(n)$.

3.2 Gauge groups

Definition 3.7. Given a principal G-bundle $G \to P \xrightarrow{p} X$, the associated *gauge group* $\mathcal{G}(P)$ of P is the topological group $Aut_G(P)$ of G-equivariant automorphisms of P which fix X. That is every $\varphi \in \mathcal{G}(P)$ is a bundle automorphism of P satisfying the commutative diagrams

$$\begin{tikzcd} P \arrow[r, "\varphi"] \arrow[d, "p"] & P \arrow[d, "p"] \\ X \arrow[r, equal] & X \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} P \arrow[r, "\varphi"] \arrow[d, "g"] & P \arrow[d, "g"] \\ P \arrow[r, "\varphi"] & P \end{tikzcd}$$

where $g : P \to P$ is the right action of $g \in G$ on P. Similarly the *pointed gauge group* $\mathcal{G}^*(P)$ is defined to be the subgroup of $\mathcal{G}(P)$ consisting of automorphisms φ such that the restrictions $\varphi|_{p^{-1}(x_0)}$ to the fiber over x_0 are identity.

In general, a manifold X has infinitely many isomorphism classes of principal G-bundles P, so there are infinitely many gauge groups $\mathcal{G}(P)$. However, when X is of finite dimension and G is a compact connected Lie group, Crabb and Sutherland [CS00] showed that there are finitely many distinct homotopy types of gauge groups over X. The following homotopy equivalence proved by Gottlieb, Atiyah and Bott plays an important role in studying the homotopy types of gauge groups in later chapters.

Theorem 3.8 (Gottlieb, [Got72]; Atiyah and Bott, [AB83])**.** *Let $G \to P \xrightarrow{\pi} X$ be a principal G-bundle. Then $B\mathcal{G}(P)$ is weakly homotopy equivalent to $Map_P(X, BG)$, the connected component which contains the inducing map of P. Similarly $B\mathcal{G}^*(P)$ is weakly homotopy equivalent to the corresponding component $Map^*_P(X, BG)$.*

Proof. This sketch of proof is based on [AB83, Got72]. Let $G \to EG \xrightarrow{\pi} BG$ be the universal G-bundle and let $\mathrm{Map}_G(P, EG)$ be the space of G-equivariant maps $f : P \to EG$ satisfying the commutative diagram

$$\begin{array}{ccc} P & \xrightarrow{f} & EG \\ \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle g} \\ P & \xrightarrow{f} & EG \end{array}$$

where the columns are the right G-actions on P and EG. Then $p : P \to X$ and $\pi : EG \to BG$ induce a map $\tilde{\pi} : \mathrm{Map}_G(P, EG) \to \mathrm{Map}_P(X, BG)$ as follows. Given $f \in \mathrm{Map}_G(P, EG)$, define $\tilde{\pi}(f) : X \to BG$ by $\tilde{\pi}(f)(x) = \pi \circ f(\tilde{x})$ for $x \in X$ and $\tilde{x} \in p^{-1}(x)$. It is easy to check that $\tilde{\pi}(f)$ is independent of the choice of $\tilde{x}$ and satisfies the commutative diagram

$$\begin{array}{ccc} P & \xrightarrow{f} & EG \\ \downarrow{\scriptstyle p} & & \downarrow{\scriptstyle \pi} \\ X & \xrightarrow{\tilde{\pi}(f)} & BG \end{array}$$

In fact, $\tilde{\pi} : \mathrm{Map}_G(P, EG) \to \mathrm{Map}_P(X, BG)$ is a fibration. We claim that the fiber is $\mathcal{G}(P)$.

Let $f_0 : X \to BG$ be the map inducing P. Then by definition $P = f_0^*(EG)$ is the pullback bundle of EG and $f_0 = \tilde{\pi}(f_0^*)$:

$$\begin{array}{ccc} P & \xrightarrow{f_0^*} & EG \\ \downarrow{\scriptstyle p} & & \downarrow{\scriptstyle \pi} \\ X & \xrightarrow{f_0} & BG. \end{array}$$

Let $F = \tilde{\pi}^{-1}(f_0) \subset \mathrm{Map}_G(P, EG)$. On the one hand, $\mathcal{G}(P)$ acts freely on F by composition, so $\mathcal{G}(P) \subset F$. On the other hand, for any $\alpha \in F$ we have $\tilde{\pi}(\alpha) = f_0$ by definition, so $\pi \circ \alpha = f_0 \circ p$ and there is a commutative diagram

$$\begin{tikzcd}
P \arrow[drr, bend left, "\alpha"] \arrow[dr, dotted, "\varphi"] \arrow[ddr, bend right, "p"'] & & \\
& P \arrow[r, "f_0^*"] \arrow[d, "p"] & EG \arrow[d, "\pi"] \\
& X \arrow[r, "f_0"] & BG
\end{tikzcd}$$

where the universal property of P induces a unique map $\varphi : P \to P$. Since f_0^* and α are G-equivariant, φ is a G-equivariant automorphism. Therefore $\varphi \in \mathcal{G}(P)$ and $F = \mathcal{G}(P)$.

Furthermore, there is a G-equivariant contraction $EG \to *$, so $\mathrm{Map}_G(P, EG)$ is contractible (see [Got72, Theorem 5.2]). This implies that

$$\mathcal{G}(P) \longrightarrow \mathrm{Map}_G(P, EG) \xrightarrow{\tilde{\pi}} \mathrm{Map}_P(X, BG)$$

is a universal bundle and $B\mathcal{G}(P)$ is weakly equivalent to $\mathrm{Map}_P(X, BG)$. The weak equivalence $\mathrm{Map}_P^*(X, BG) \simeq B\mathcal{G}^*(P)$ can be proved similarly. □

By Theorem 3.8, the gauge group $\mathcal{G}(P)$ is homotopy equivalent to the loop space of the connected component $\mathrm{Map}_P(X, BG)$. Therefore we want to understand the homotopy types of connected components of $\mathrm{Map}(X, BG)$, where X is a 4-dimensional manifold. On the one hand, the connected components of $\mathrm{Map}(X, BG)$ usually have different homotopy types. In [Tsu01, Tsu12] Tsukuda and Tsutaya showed that

$$\mathrm{Map}_P(S^4, BSU(2)) \simeq \mathrm{Map}_{P'}(S^4, BSU(2))$$

if and only if $|k| = |l|$, where k and l are the second Chern classes of P and P' respectively. On the other hand, the following proposition given in [Sut92] shows that all connected components of $\mathrm{Map}^*(X, BG)$ have the same homotopy type.

Proposition 3.9. *Suppose that X is an n-dimensional CW-complex with one n-cell. If G is $(n-2)$-connected and $\pi_{n-1}(G)$ is $\mathbb{Z}$, then all connected components of $\mathrm{Map}^*(X, BG)$ are homotopy equivalent.*

Proof. The proof is based on [Sut92]. Let X_{n-1} be the $(n-1)$-skeleton of X and let $f : S^{n-1} \to X_{n-1}$ be the attaching map. Then the cofibration sequence

$$S^{n-1} \xrightarrow{f} X_{n-1} \xrightarrow{\imath} X \xrightarrow{\rho} S^n \xrightarrow{\Sigma f} \Sigma(X_{n-1}) \longrightarrow \cdots,$$

where ρ is the connecting map, induces an exact sequence

$$\cdots \longrightarrow \pi_{n-1}(G) \xrightarrow{\rho^*} [X, BG] \xrightarrow{\imath^*} [X_{n-1}, BG].$$

By hypothesis $\pi_{n-1}(G)$ is $\mathbb{Z}$ and $[X_{n-1}, BG]$ is zero, so ρ^* is a reduction map. This implies $[X, BG]$ is either 0, $\mathbb{Z}$ or $\mathbb{Z}/k\mathbb{Z}$ for some $k \geq 2$.

If $[X, BG]$ is zero, then $\mathrm{Map}^*(X, BG)$ is connected and the statement is true. If $[X, BG]$ is $\mathbb{Z}/k\mathbb{Z}$ or $\mathbb{Z}$, then the following argument is the same. Assume $[X, BG]$ is $\mathbb{Z}/k\mathbb{Z}$. We claim that $\mathrm{Map}^*_{\bar{l}}(X, BG)$ is homotopy equivalent to $\mathrm{Map}^*_{\overline{l+1}}(X, BG)$ for $\bar{l} \in \mathbb{Z}/k\mathbb{Z}$. Since X is the cofiber of f, there is a coaction $\psi : X \to X \vee S^n$. Let g be a map in $\mathrm{Map}^*_{\bar{l}}(X, BG)$ and let α represent a generator of $\pi_n(BG) \cong \mathbb{Z}$. Define a map $\Theta : \mathrm{Map}^*_{\bar{l}}(X, G) \to \mathrm{Map}^*_{\overline{l+1}}(X, G)$ by

$$\Theta(g) : X \xrightarrow{\psi} X \vee S^n \xrightarrow{g \vee \alpha} BG \vee BG \xrightarrow{\triangledown} BG$$

where $\triangledown$ is the folding map. Since ρ^* is surjective, there exists $\tilde{g} \in \mathrm{Map}^*_l(S^n, BG)$ for some integer l such that $l \equiv \bar{l} \pmod{k}$ and $g = \rho^*(\tilde{g})$. Then we have a homotopy commutative diagram

$$\begin{tikzcd}
X \arrow[r, "\psi"] \arrow[d, "\rho"] \arrow[rrr, bend left, "\Theta(f)"] & X \vee S^4 \arrow[r, "f \vee \alpha"] \arrow[d, "\rho \vee \mathbb{1}"] & BG \vee BG \arrow[r, "\triangledown"] \arrow[d, equal] & BG \arrow[d, equal] \\
S^4 \arrow[r, "\sigma"] \arrow[rrr, bend right, "\tilde{f} + \alpha"'] & S^4 \vee S^4 \arrow[r, "\tilde{f} \vee \alpha"] & BG \vee BG \arrow[r, "\triangledown"] & BG
\end{tikzcd}$$

implying that $\Theta(g) \simeq \rho^*(\tilde{g}+\alpha) \in \mathrm{Map}^*_{\overline{l+1}}(X, BG)$. Define $\Theta' : \mathrm{Map}^*_{\overline{l+1}}(X, BG) \to \mathrm{Map}^*_{\bar{l}}(X, BG)$ by replacing α by $-\alpha$. Obviously Θ' is a homotopy inverse of Θ. Therefore $\mathrm{Map}^*_{\bar{l}}(X, BG)$ and $\mathrm{Map}^*_{\overline{l+1}}(X, BG)$ are homotopy equivalent and the statement follows. $\square$

Chapter 4

Homotopy decomposition of Lie groups

Shown by Cohen, Neisendorfer, Mimura, Nishida, Toda and Theriault [CN84, MNT77, The07], a Lie group can decompose into a product of H-spaces with good properties and simpler structures after localization. The homotopy decomposition of Lie groups has many applications to problems in homotopy theory and has been proved to be a powerful tool. In particular, it is used to study the homotopy types of gauge groups in many cases. In this chapter, we give a preliminary introduction to topological localization in the first section and talk about the homotopy decomposition of Lie groups in the second section.

4.1 Topological localization

Recall the localization of a group in algebra. Let $\mathcal{P}$ be a set of prime numbers and let $\mathbb{Z}_{\mathcal{P}}$ be the ring consisting of all fractions with denominators not divisible by the prime numbers in $\mathcal{P}$, that is

$$\mathbb{Z}_{\mathcal{P}} = \left\{ \frac{a}{b}, \text{ where } gcd(b,p) = 1 \text{ for all } p \in \mathcal{P} \right\}.$$

Given an abelian group G we tensor it by $\mathbb{Z}_{\mathcal{P}}$ to form a new abelian group

$$G_{\mathcal{P}} = G \otimes \mathbb{Z}_{\mathcal{P}} = \{g \otimes \frac{a}{b}, \text{ where } g \in G, \frac{a}{b} \in \mathbb{Z}_{\mathcal{P}}\}.$$

The canonical map $G \to G_{\mathcal{P}}$ sending $g \in G$ to $g \otimes 1 \in G_{\mathcal{P}}$ is called the *$\mathcal{P}$-localization of G*. Sometimes we may refer to $G_{\mathcal{P}}$ as the $\mathcal{P}$-localization as well.

Example 4.1. Let $\mathcal{P}$ be the empty set. Then $\mathbb{Z}_{\mathcal{P}}$ is the rational numbers $\mathbb{Q}$. We denote $G_{\mathcal{P}}$ by $G_{\mathbb{Q}}$ and call it the rational localization of G.

Example 4.2. Let $\mathcal{P}$ be $\{p\}$ for some prime number p. Then $\mathbb{Z}_{\mathcal{P}}$ is $\mathbb{Z}_{(p)}$ which is the ring consisting all fractions with denominator not divisible by p. We denote $G_{\mathcal{P}}$ by $G_{(p)}$ and call it the p-localization of G.

Example 4.3. Let $\mathcal{P}$ be the set of all prime numbers except p, then $\mathbb{Z}_{\mathcal{P}}$ is $\mathbb{Z}_p$ which is the ring consisting of all fractions of the form $\frac{a}{p^r}$. We denote $G_{\mathcal{P}}$ by G_p and call it the localization away from p.

Example 4.4. Let p and q be distinct prime numbers and let G be $\mathbb{Z} \oplus \mathbb{Z}/p\mathbb{Z} \oplus \mathbb{Z}/q\mathbb{Z}$. Then G becomes $\mathbb{Z}_{(p)} \oplus \mathbb{Z}/p\mathbb{Z}$ after p-localization. Denote the generators of $\mathbb{Z}, \mathbb{Z}/p\mathbb{Z}$ and $\mathbb{Z}/q\mathbb{Z}$ by a, b and c respectively. Tensoring with $1 \in \mathbb{Z}_{(p)}$, a and b have non-zero image but not c since $c \otimes 1 = c \otimes \frac{q}{q} = qc \otimes \frac{1}{q} = 0$.

In Example 4.4, the p-localization filters out all free elements and p-torsion and erases other torsion. In general, applying $\mathcal{P}$-localization to a group G removes torsion components with orders coprime to primes in $\mathcal{P}$. Therefore if we are only interested in elements whose orders are products of primes in $\mathcal{P}$, then it suffices to study $G_{\mathcal{P}}$ which has a simpler structure than G. This idea motivates topological localization. Given a space X, we construct a new space $X_{\mathcal{P}}$ such that $\pi_i(X_{\mathcal{P}})$ is the $\mathcal{P}$-localization of $\pi_i(X)$ for all i. Usually the homotopy groups of a space are very complicated. Topological localization allows us to understand the $\mathcal{P}$-local information of $\pi_i(X)$ by studying $X_{\mathcal{P}}$, which is much simpler. There are several methods to construct $X_{\mathcal{P}}$ having different assumptions on X. One commonly used method is as follows.

Let $f : (D^n, S^{n-1}) \to (X, x_0)$ be a representation of $[f] \in \pi_n(X)$ and let $\gamma : ([0,1], \{0,1\}) \to (X, x_0)$ be a representation of $[\gamma] \in \pi_1(X)$. Then γ acts on f by sending f to the map

$$f_\gamma(x) = \begin{cases} f(2x) & 0 \leq |x| \leq \frac{1}{2} \\ \gamma(2|x| - 1) & \frac{1}{2} \leq |x| \leq 1. \end{cases}$$

This defines an action of $\pi_1(X)$ on $\pi_n(X)$.

Definition 4.5. A space X is *simple* (also known as *abelian*) if $\pi_1(X)$ acts trivially on $\pi_n(X)$ for all n.

Example 4.6. A simply-connected space is obviously a simple space.

Example 4.7. Every H-space is simple. Let X be an H-space with multiplication $m : X \times X \to X$. Since $m(x_0, x)$ and $m(x, x_0)$ are homotopic to the identity map $\mathbb{1}_X$ on X, we may modify and assume m is $\mathbb{1}_X$ when restricted to $X \vee X$. Let $\alpha : \mathbb{R}^n \to D^n$ and $\beta : \mathbb{R} \to [0,1]$ be maps

$$\alpha(x) = \begin{cases} x & |x| \leq 1 \\ \frac{x}{|x|} & |x| \geq 1 \end{cases} \quad \text{and} \quad \beta(t) = \begin{cases} 0 & t \leq 0 \\ t & 0 \leq t \leq 1 \\ 1 & t \geq 1. \end{cases}$$

Given any $f \in \pi_n(X)$ and $\gamma \in \pi_1(X)$, f_γ is $m(f\circ\alpha(2x), \gamma\circ\beta(2|x|-1))$. For $x \in D^n$ with norm $|x| \leq \frac{1}{2}$, $f\circ\alpha(2x) = f(2x)$ and $\gamma\circ\beta(2|x|-1) = x_0$, so $m(f\circ\alpha(2x), \gamma\circ\beta(2|x|-1)) = f(2x)$. For $|x| \geq \frac{1}{2}$, $f\circ\alpha(2x) = x_0$ and $\gamma\circ\beta(2|x|-1) = \gamma(2|x|-1)$, so $m(f\circ\alpha(2x), \gamma\circ\beta(2|x|-1)) = \gamma(2|x|-1)$. There is a homotopy $H : (D^n, S^{n-1}) \times I \to (X, x_0)$ given by

$$H(x,t) = m(f \circ \alpha(\frac{2}{1+t}x), \gamma \circ \beta(2|x| + 2t - 1)).$$

At $t = 0$, $H(x,0) = m(f \circ \alpha(2x), \gamma \circ \beta(2|x| - 1)) = f_\gamma(x)$. At $t = 1$, since $2|x| + 1$ is greater than or equal to 1, we have $\gamma \circ \beta(2|x| + 1) = \gamma(1) = x_0$ and

$$H(x,1) = m(f(x), \gamma \circ \beta(2|x| + 1)) = m(f(x), x_0) = f(x).$$

Therefore f_γ is homotopic to f for any $f \in \pi_n(X)$ and $\gamma \in \pi_1(X)$. The action of $\pi_1(X)$ on $\pi_n(X)$ is trivial and X is simple.

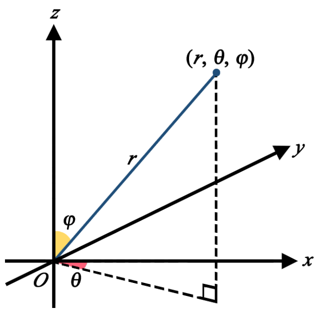
Example 4.8. If f is a loop in $\pi_1(X)$, then by definition $f_\gamma = \gamma f \gamma^{-1}$. When X is simple, the action of γ on any loop $\omega \in \pi_1(X)$ is trivial, so $\gamma\omega\gamma^{-1}$ is homotopic to ω and $\pi_1(X)$ is abelian. This could be the reason why Hatcher calls a simple space an abelian space.

Example 4.9. A space X with abelian $\pi_1(X)$ may not be simple. The projective space $\mathbb{RP}^2$ is constructed by identifying every point x on S^2 with its antipodal point $-x$. There is a fibration sequence

$$S^0 \longrightarrow S^2 \xrightarrow{q} \mathbb{RP}^2, \tag{4.1.1}$$

where q is the identifying map. From (4.1.1) we know $\pi_1(\mathbb{RP}^2) \cong \mathbb{Z}/2\mathbb{Z}$ and $\pi_2(\mathbb{RP}^2) \cong \mathbb{Z}$. We claim that $\pi_1(\mathbb{RP}^2)$ has a non-trivial action on $\pi_2(\mathbb{RP}^2)$.

To describe the action, it is more convenient to embed S^2 into $\mathbb{R}^3$ and use spherical coordinates (r, θ, ϕ) to represent a point $P \in \mathbb{R}^3$, where r is the distance between P and the origin O, θ is the angle between the x-axis and the projection of the line OP on the xy-plane, and ϕ is the angle between OP and the z-axis.



Let the north pole $(1,0,0)$ be the basepoint of S^2. Then $q(1,0,0)$ is the basepoint of $\mathbb{RP}^2$ and the arc $\gamma(t) = (1, 0, \pi t)$ on S^2 between the north pole and the south pole $(1, 0, \pi)$ descends to the generator $\bar{\gamma} = q(\gamma)$ of $\pi_1(\mathbb{RP}^2) \cong \mathbb{Z}/2\mathbb{Z}$. Moreover, q induces an isomorphism $q_* : \pi_2(S^2) \to \pi_2(\mathbb{RP}^2)$, that is q generates $\pi_2(\mathbb{RP}^2)$.

To show that the action is non-trivial, we need to show that $q_{\bar{\gamma}}$ is $-q$. Since $q_* : \pi_2(S^2) \to \pi_2(\mathbb{RP}^2)$ is an isomorphism, it suffices to show that its lift

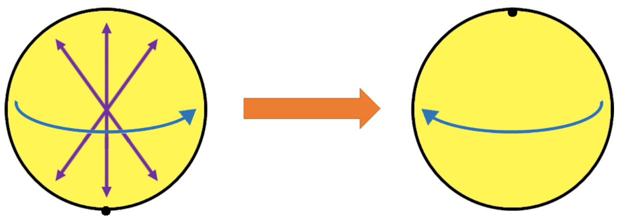
$$\mathbb{1}_\gamma(1,\theta,\phi) = \begin{cases} (1,0,\pi-2\phi) & 0 \le \phi \le \frac{\pi}{2} \\ (1,\theta,2\phi-\pi) & \frac{\pi}{2} \le \phi \le \pi. \end{cases}$$

of $a_{\bar{\gamma}}$ is homotopic to $-\mathbb{1}$. The formula is well-defined since $(1,0,\pi-2\phi)$ and $(1,\theta,2\phi-\pi)$ refer to the north pole when $\phi = \frac{\pi}{2}$. The upper hemisphere is enlarged to become S^2 and the lower hemisphere is merged latitudinally to become a path joining the north and the south pole. Define a homotopy $H : S^2 \times I \to S^2$ between $\mathbb{1}_\gamma$ and -1 by

$$H(1,\theta,\phi;t) = \begin{cases} (1,0,\pi-2\phi) & 0 \le \phi \le (1-t)\frac{\pi}{2} \\ \rho_{t\pi}(1,\theta,\frac{2}{t+1}\phi + \frac{t-1}{t+1}\pi) & (1-t)\frac{\pi}{2} \le \phi \le \pi, \end{cases}$$

where $\rho_{t\pi}$ is the rotation around the y-axis for $t\pi$. The formula is well-defined since $(1,0,\pi-2\phi)$ and $\rho_{t\pi}(1,\theta,\frac{2}{t+1}\phi+\frac{t-1}{t+1}\pi)$ refer to $(1,0,t\pi)$ when $\phi = (1-t)\frac{\pi}{2}$. At time t, we divide S^2 into two parts: the spherical cap above and below the level $\phi = (1-t)\frac{\pi}{2}$. The lower part becomes S^2 and is rotated around the y-axis so that the base point moves to $(1,0,t\pi)$. The upper part becomes a path joining $(1,0,t\pi)$ and the south pole.

At $t = 1$, the sphere is rotated upside down and its base point is at the south pole. The antipodal map $x \mapsto -x$ reverses this upside down sphere so that its basepoint is at the north pole $(1,0,0)$ and its orientation is reversed.



Therefore $\mathbb{1}_\gamma$ is homotopic to $-\mathbb{1}$ in $\pi_2(S^2)$, so $q_{\bar{\gamma}}$ is homotopic to $-q$ in $\pi_2(\mathbb{RP}^2)$. This implies that the action of $\bar{\gamma}$ is non-trivial and $\mathbb{RP}^2$ is not simple.

Proposition 4.10. *Let X be a simple space and let $\mathcal{P}$ be a set of prime numbers. Then there exists a space $X_{\mathcal{P}}$ and a map $\imath : X \to X_{\mathcal{P}}$ such that $\pi_i(X_{\mathcal{P}})$ is a $\mathbb{Z}_{\mathcal{P}}$-module and $\imath$ induces isomorphisms $\imath_* : \pi_i(X) \otimes \mathbb{Z}_{\mathcal{P}} \to \pi_i(X_{\mathcal{P}})$ and $\imath_* : H_i(X) \otimes \mathbb{Z}_{\mathcal{P}} \to H_i(X_{\mathcal{P}})$ for all i. Moreover, let Y be another simple space and let $f : X \to Y$ be a map. If $Y_{\mathcal{P}}$*

and $\imath' : Y \to Y_{\mathcal{P}}$ satisfy the above conditions, then there exists a map $f_{\mathcal{P}} : X_{\mathcal{P}} \to Y_{\mathcal{P}}$, unique up to homotopy, satisfying the commutative diagram

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "\imath"] & Y \arrow[d, "\imath'"] \\ X_{\mathcal{P}} \arrow[r, "f_{\mathcal{P}}"] & Y_{\mathcal{P}}. \end{tikzcd}$$

Proof. The construction of $X_{\mathcal{P}}$ and $\imath$ is long and technical, so here we give a sketch proof. Readers who want more details can refer to [Hat04, Nei10].

Every space X has a Postnikov tower $\cdots X_3 \xrightarrow{j_2} X_2 \xrightarrow{j_1} X_1$, which contains sequences of spaces $\{X_n\}_{n>0}$ and maps $\{j_n : X_{n+1} \to X_n\}_{n>0}$ such that $\pi_i(X_n)$ equals $\pi_i(X)$ for $i \le n$ and zero for $i > n$, and there is a fibration sequence

$$K(\pi_{n+1}(X), n+1) \longrightarrow X_{n+1} \xrightarrow{j_n} X_n.$$

If X is simple, then this fibration sequence is a principal fibration

$$K(\pi_{n+1}(X), n+1) \longrightarrow X_{n+1} \xrightarrow{j_n} X_n \xrightarrow{k_n} K(\pi_{n+1}(X), n+2),$$

where k_n is called the n^{th} k-invariant. The idea is to modify each X_n to produce a new Postnikov tower $\cdots X_3' \xrightarrow{j_2'} X_2' \xrightarrow{j_1'} X_1'$ and maps $\imath_n : X_n \to X_n'$ such that $\pi_i(X_n')$ is the $\mathcal{P}$-localization of $\pi_i(X_n)$ and $\imath_n$ induces isomorphisms

$$\pi_i(X_n) \otimes \mathbb{Z}_{\mathcal{P}} \to \pi_i(X_n') \quad \text{and} \quad H_i(X_n) \otimes \mathbb{Z}_{\mathcal{P}} \to H_i(X_n').$$

For short we denote $\pi_i(X)$ and $\pi_i(X) \otimes \mathbb{Z}_{\mathcal{P}}$ by π_i and π_i'. Notice that X_1 is $K(\pi_1, 1)$ and take X_1' to be $K(\pi_1', 1)$. Let $\imath_1 : X_1 \to X_1'$ be the map induced by localization $\pi_1 \to \pi_1'$. Then $\imath_1$ induces isomorphisms

$$\pi_i(X_1) \otimes \mathbb{Z}_{\mathcal{P}} \to \pi_i(X_1') \quad \text{and} \quad H_i(X_1) \otimes \mathbb{Z}_{\mathcal{P}} \to H_i(X_1').$$

Assume we have X_n' and $\imath_n : X_n \to X_n'$ such that $\pi_i(X_n') = \pi_i(X_n) \otimes \mathbb{Z}_{\mathcal{P}}$ and $\imath_n$ induces isomorphisms

$$\pi_i \otimes \mathbb{Z}_{\mathcal{P}} \to \pi_i(X_n') \quad \text{and} \quad H_i(X_n) \otimes \mathbb{Z}_{\mathcal{P}} \to H_i(X_n').$$

Now we construct X_{n+1}' and $\imath_{n+1} : X_{n+1} \to X_{n+1}'$ as follows. Let $\tilde{\imath} : K(\pi_{n+1}, n+2) \to K(\pi_{n+1}', n+2)$ be the map induced by localization $\pi_{n+1} \to \pi_{n+1}'$. Since $(\imath_n)_* : H_i(X_n) \otimes \mathbb{Z}_{\mathcal{P}} \to H_i(X_n')$

is an isomorphism, there is no obstruction to extend $\imath_n$ to complete the diagram

$$\begin{tikzcd} X_n \arrow[r, "k_n"] \arrow[d, "\imath_n"] & K(\pi_{n+1}, n+2) \arrow[d, "\tilde{\imath}"] \\ X'_n \arrow[r, dotted, "k'_n"] & K(\pi'_{n+1}, n+2) \end{tikzcd}$$

Define X'_{n+1} to be the homotopy fiber of k'_n and $j'_n : X'_{n+1} \to X'_n$ and $\imath_{n+1} : X_{n+1} \to X'_{n+1}$ to be the induced maps

$$\begin{tikzcd} X_{n+1} \arrow[r, "j_n"] \arrow[d, "\imath_{n+1}"] & X_n \arrow[r, "k_n"] \arrow[d, "\imath_n"] & K(\pi_{n+1}, n+2) \arrow[d, "\tilde{\imath}"] \\ X'_{n+1} \arrow[r, "j'_n"] & X'_n \arrow[r, "k'_n"] & K(\pi'_{n+1}, n+2). \end{tikzcd}$$

We can show that the homotopy groups of X'_{n+1} are $\mathbb{Z}_{\mathcal{P}}$-modules and $\imath_{n+1}$ induces isomorphisms

$$\pi_i(X_{n+1}) \otimes \mathbb{Z}_{\mathcal{P}} \to \pi_i(X'_{n+1}) \quad \text{and} \quad H_i(X_{n+1}) \otimes \mathbb{Z}_{\mathcal{P}} \to H_i(X'_{n+1}).$$

Repeat the procedure and construct X'_n and $\imath_n$ for higher n inductively. In the end we can get a Postnikov tower $\cdots \to X'_3 \xrightarrow{j'_2} X'_2 \xrightarrow{j'_1} X'_1$ such that the homotopy groups X'_i are $\mathbb{Z}_{\mathcal{P}}$-local. Take $X_{\mathcal{P}}$ to be the inverse limit $\varprojlim X'_i$ and $\imath$ to be the composition

$$\imath : X \longrightarrow \varprojlim X_i \xrightarrow{\lim \imath_i} \varprojlim X'_i = X_{\mathcal{P}}.$$

Then $X_{\mathcal{P}}$ and $\imath : X \to X_{\mathcal{P}}$ satisfy the asserted properties. □

Definition 4.11. Let X be a simple space. Then the $\mathcal{P}$-localization refers to either the space $X_{\mathcal{P}}$ or the map $\imath : X \to X_{\mathcal{P}}$ stated in Proposition 4.10.

Example 4.12. Let p and q be distinct prime numbers and let $P^n(q)$ be the n-dimensional Moore space of $\mathbb{Z}/q\mathbb{Z}$ for $n \geq 3$. Recall that $P^n(q)$ is the cofiber of the degree map $q : S^{n-1} \to S^{n-1}$. Since $q : S^{n-1} \to S^{n-1}$ induces a homomorphism sending $\alpha \in H^*(S^{n-1})$ to $q\alpha$ that becomes an isomorphism after p-localization, $q : S^{n-1} \to S^{n-1}$ is a homotopy equivalence and $P^n(q)$ is contractible when localized at p.

Example 4.13. It is known that S^{2n+1} is homotopy equivalent to Eilenberg-MacLane space $K(\mathbb{Q}, 2n+1)$ when localized rationally (for example, see [Hat04, Proposition 1.27]). Since $K(\mathbb{Q}, 2n+1) \simeq \Omega K(\mathbb{Q}, 2n+2)$ is a loop space, $S^{2n+1}_{\mathbb{Q}}$ is an H-space.

Example 4.14. After localization spaces may exhibit properties that do not appear originally. On the one hand, S^{2n+1} is an H-space if and only if $n = 0, 1$ or 3 by a famous result of Adams [Ada60]. On the other hand, after localization at any odd prime p there is a homotopy equivalence $\Omega S^{2n+2} \simeq_p S^{2n+1} \times \Omega S^{4n+3}$ [Ser53]. Define a multiplication on S^{2n+1} by

$$S^{2n+1} \times S^{2n+1} \hookrightarrow \Omega S^{2n+2} \times \Omega S^{2n+2} \xrightarrow{m} \Omega S^{2n+2} \xrightarrow{proj} S^{2n+1},$$

where m is the loop multiplication on ΩS^{2n+2}. Therefore S^{2n+1} an H-space for any n.

4.2 Localization of Lie groups

As mentioned in the introduction of this chapter, a Lie group decomposes into a product of simpler spaces after localization. This section is a short survey of homotopy decompositions of localized Lie groups based on the work of Cohen, Neisendorfer, Mimura, Nishida, Toda and Theriault [CN84, MNT77, The07].

Let G be a simple, simply-connected, compact Lie group. By the classification of simple compact Lie groups, G is either $SU(n), Sp(n), Spin(n)$, G_2, F_4, E_6, E_7 or E_8. Since G is a simple space by Example 4.7, Proposition 4.10 implies that there is a $\mathcal{P}$-localization of G for any set $\mathcal{P}$ of prime numbers. When localized rationally, a theorem of Hopf says that G decomposes into a product $\prod_{i=1}^{l} S^{2n_i+1}$ of odd dimensional spheres. When localized at an odd prime p, Mimura, Nishida and Toda [MNT77] showed that any Lie groups whose homology has no p-torsion decomposes into products of H-spaces, and the cells in these H-spaces are joined together by Steenrod powers.

There are maps $\imath : \Sigma\mathbb{CP}^{n-1} \to SU(n)$ and $\imath' : Q_n \to Sp(n)$, where Q_n is the quasi-projective space, such that they induce the inclusions of the generating sets in homology. Let $A(G)$ be $\Sigma\mathbb{CP}^{n-1}$ for $G = SU(n)$ and be Q_n for $G = Sp(n)$. Then the decomposition $G \simeq \prod B_i$ constructed in [MNT77] is compatible with a decomposition $A(G) \simeq \bigvee A_i$ such that $H_*(A_i)$ is the generating set of $H_*(B_i)$. Later, Theriault constructed a corresponding space $A(G)$ for each G and reformulated the Lie group decomposition as follows.

Theorem 4.15 (Mimura, Nishida, Toda [MNT77]; Theriault [The07])**.** *Let p be an odd prime and let G be a compact, simply-connected, simple Lie group whose homology groups $H_*(G)$ have no p-torsion. Then there exists a space A and a map $\imath : A \to G$ such that $\imath_* : H_*(A) \to H_*(G) \cong \Lambda(\tilde{H}_*(A))$ is the inclusion of the generating set in homology. After p-localization, there are homotopy equivalences*

$$A \simeq \bigvee_{i=1}^{p-1} A_i \quad \textit{and} \quad G \simeq \prod_{i=1}^{p-1} B_i,$$

where A_i is a co-H-space and B_i is an H-space for all i. The homology group $H_j(A_i)$ equals $H_j(A)$ for j in the form $2i+2(p-1)k+1$ and equals zero otherwise, and $H_*(B_i) \cong \Lambda(\tilde{H}_*(A_i))$.

Moreover, the composition

$$\imath_i : A_i \longrightarrow A \xrightarrow{\imath} G \longrightarrow B_i$$

induces the inclusion of the generating set in homology.

Example 4.16. The integral homology of $SU(n)$ is $H_*(SU(n)) \cong \Lambda(x_3, x_5, \cdots, x_{2n-1})$, where x_{2i+1} has degree $2i+1$. Localized at an odd prime p, $SU(n)$ decomposes into $\prod_{i=1}^{p-1} B_i$ and B_i has p-local homology groups

$$H_*(B_i; \mathbb{Z}_{(p)}) \cong \Lambda(x_{2i+1}, x_{2i+2p-1}, \cdots, x_{2i+2(p-1)\cdot\lfloor\frac{n-i-1}{p-1}\rfloor+1}),$$

where $\lfloor m \rfloor$ is the greatest integer less than or equal to m. There is an action of the dual Steenrod algebra on $H_*(B_i; \mathbb{Z}_{(p)})$ given by $\mathcal{P}^j_*(x_{2r+1}) = \binom{r}{j} x_{2r+2j(p-1)+1}$. The space A in Theorem 4.15 is $\Sigma\mathbb{CP}^{n-1}$ and it has integral homology groups $H_*(\Sigma\mathbb{CP}^{n-1}) \cong \mathbb{Z}\langle x_3, x_5, \cdots, x_{2n-1}\rangle$. The canonical embedding $\imath : \Sigma\mathbb{CP}^{n-1} \to SU(n)$ induces the inclusion of the generating set in homology. After p-localization, $\Sigma\mathbb{CP}^{n-1}$ is homotopy equivalent to $\bigvee_{i=1}^{p-1} A_i$ and A_i has p-local homology groups

$$H_*(A_i; \mathbb{Z}_{(p)}) \cong \mathbb{Z}_{(p)}\langle x_{2i+1}, x_{2i+2p-1}, \cdots, x_{2i+2(p-1)\cdot\lfloor\frac{n-i-1}{p-1}\rfloor+1}\rangle.$$

On the one hand, the homotopy decomposition stated in Theorem 4.15 holds for a wide range of Lie groups. However, the construction of the H-spaces B_i given in [MNT77] is difficult to use to prove any properties of the B_i's. On the other hand, Cohen and Neisendorfer studied a p-localized H-space X built from a low dimensional CW-complex Y [CN84] and showed that X possesses good properties when the dimension of Y is small with respect to p. Based on their work, Theriault gave an alternative construction of the H-spaces B_i's and showed that they are homotopy associative and homotopy commutative under certain conditions [The07]. Moreover, B_i satisfies the universal property as follows.

Definition 4.17. Given a map $\imath : A \to B$ for a space A and a homotopy associative and homotopy commutative H-space B. We say B is *universal* for A if for any map $f : A \to Z$ to a homotopy associative and homotopy commutative H-space Z there exists an H-map $\bar{f} : B \to Z$, unique up to homotopy, such that $f \simeq \bar{f} \circ \imath$.

$$\begin{tikzcd} A \arrow[rr, "f"] \arrow[dr, "\imath"'] & & Z \\ & B \arrow[ur, "\bar{f}"'] & \end{tikzcd}$$

Theorem 4.18 (Theriault [The07]). *If G and p satisfy one of the following*

$$\begin{array}{ll} SU(n) & n \leq (p-1)(p-2)+1 \\ Sp(n) & 2n \leq (p-1)(p-2) \\ Spin(2n+1) & 2n \leq (p-1)(p-2) \\ Spin(2n) & 2n-2 \leq (p-1)(p-2) \\ G_2, F_4, E_6 & p \geq 5 \\ E_7, E_8 & p \geq 7, \end{array}$$

then B_i is homotopy associative and homotopy commutative for all i. Moreover, ΣA is a retract of ΣG, B_i is universal for A_i and $\prod_{i=1}^{p-1} B_i$ is universal for A.

Chapter 5

Gauge groups and Samelson products

In this chapter I will try to explain the relation of my three papers which are in Chapters 6 to 8.

Let M be a connected, orientable, smooth, compact 4-manifold and let G be a simple, simply-connected, compact Lie group. Then isomorphism classes of principal G-bundles over M are classified by elements in $[M, BG]$, which are integers in this case. Denote the gauge group of the principal bundle corresponding to $k \in \mathbb{Z}$ by $\mathcal{G}_k(M)$. In my first paper "Homotopy types of gauge groups of non-simply-connected closed 4-manifolds" (Chapter 6), I developed a homotopy decomposition method for gauge groups over 4-manifolds and showed that for certain 4-manifolds M its gauge group $\mathcal{G}_k(M)$ decomposes into a product of $\mathcal{G}_k(Y)$, where Y is either S^4 or $\mathbb{CP}^2$, and some complementary factors that do not depend on k. Therefore the homotopy type of $\mathcal{G}_k(M)$ is determined by the homotopy type of $\mathcal{G}_k(S^4)$ and $\mathcal{G}_k(\mathbb{CP}^2)$.

In order to classify $\mathcal{G}_k(Y)$ for $Y = S^4$ or $\mathbb{CP}^2$, we need to study the boundary map

$$\partial_k : G \to \mathrm{Map}^*(Y, BG)$$

of the evaluation fibration

$$\mathcal{G}_k(Y) \longrightarrow G \xrightarrow{\partial_k} \mathrm{Map}_0^*(Y, BG) \longrightarrow B\mathcal{G}_k(Y) \xrightarrow{ev} BG.$$

When Y is S^4, $\mathrm{Map}_0^*(S^4, BG) \cong \Omega_0^3 G$ is an H-space. The order of ∂_k in the group $[G, \Omega_0^3 G]$ is the smallest number m such that the composition

$$G \xrightarrow{\partial_k} \Omega_0^3 G \xrightarrow{m} \Omega_0^3 G$$

is null homotopic, where $m : \Omega_0^3 G \to \Omega_0^3 G$ is the m-power map of $\Omega_0^3 G$. The order of ∂_1 helps determine the homotopy type of $\mathcal{G}_k(S^4)$.

Theorem 5.1 (Theriault, [The10a]; Kishimoto, Kono, Tsutaya, [KKT14])**.** *Let m be the order of $\partial_1 : G \to \Omega_0^3 G$. Denote the greatest common divisor of a and b by (a,b) and the p-component of a by $\nu_p(a)$.*

1. *If $(m,k) = (m,l)$, then $\mathcal{G}_k(S^4)$ is homotopy equivalent to $\mathcal{G}_l(S^4)$ when localized rationally or at any prime.*
2. *If G and p satisfy the conditions in Theorem 4.18 and $\mathcal{G}_k(S^4)$ is homotopy equivalent to $\mathcal{G}_l(S^4)$, then $\nu_p(m,k) = \nu_p(m,l)$ for any odd prime p.*

When Y is $\mathbb{CP}^2$, denote the boundary map by $\partial'_k : G \to \mathrm{Map}_0^*(\mathbb{CP}^2, BG)$. Since $\mathrm{Map}_0^*(\mathbb{CP}^2, BG)$ is not an H-space, so ∂'_k has no order. In [The12], Theriault defined the "order" of ∂'_k to be the smallest number m' such that the composition

$$G \xrightarrow{\partial_k} \Omega_0^3 G \xrightarrow{m'} \Omega_0^3 G \xrightarrow{q^*} \mathrm{Map}_0^*(\mathbb{CP}^2, BG)$$

is null homotopic. When $m' = 1$, this composition gives $\partial'_k = q^*\partial_k$ and it implies ∂'_k is null homotopic. In my third paper "Homotopy types of $SU(n)$-gauge groups over non-spin 4-manifolds" (Chapter 8), I studied the homotopy types of $\mathcal{G}_k(\mathbb{CP}^2)$. In particular, I proved a statement similar to part (1) of Theorem 5.1 and gave a lower bound on the "order" of ∂'_1 for the $SU(n)$ case.

Theorem 5.2. *Let m be the "order" of $\partial'_1 : G \to Map^*(\mathbb{CP}^2, BG)$.*

1. *If $(m,k) = (m,l)$, then $\mathcal{G}_k(\mathbb{CP}^2)$ is homotopy equivalent to $\mathcal{G}_l(\mathbb{CP}^2)$ when localized rationally or at any prime.*
2. *If $G = SU(n)$ and $\mathcal{G}_k(\mathbb{CP}^2)$ is homotopy equivalent to $\mathcal{G}_l(\mathbb{CP}^2)$, then $(\frac{1}{2}n(n^2-1), k) = (\frac{1}{2}n(n^2-1), l)$ for n odd and $(n(n^2-1), k) = (n(n^2-1), l)$ for n even.*

Therefore to classify $\mathcal{G}_k(Y)$ it is important to calculate the order of the boundary maps ∂_1 and ∂'_1. When Y is S^4, Lang showed that [Lan73] ∂_k is the triple adjoint to the composition

$$\langle k\imath, \mathbb{1}\rangle : S^3 \wedge G \xrightarrow{k\imath \wedge \mathbb{1}} G \wedge G \xrightarrow{\langle \mathbb{1}, \mathbb{1}\rangle} G,$$

where $\imath : S^3 \to G$ is the inclusion of the bottom cell and $\langle \mathbb{1}, \mathbb{1}\rangle : G \wedge G \to G$ is the Samelson product defined as follows.

Definition 5.3. Let G be a Lie group and let $[\mathbb{1}, \mathbb{1}] : G \times G \to G$ be the map sending $(x,y) \in G \times G$ to $x^{-1}y^{-1}xy \in G$. Then $[\mathbb{1}, \mathbb{1}]$ descends to a map $\langle \mathbb{1}, \mathbb{1}\rangle$ in $[G \wedge G, G]$, unique up to homotopy. The map $\langle \mathbb{1}, \mathbb{1} \rangle$ is called the *Samelson product*. Furthermore, the *order* of $\langle \mathbb{1}, \mathbb{1} \rangle$ is the minimum number m such that the composition

$$G \wedge G \xrightarrow{\langle \mathbb{1}, \mathbb{1} \rangle} G \xrightarrow{m} G$$

is null-homotopic, where $m : G \to G$ is the m^{th}-power map.

Intuitively, the Samelson product $\langle \mathbb{1}, \mathbb{1} \rangle$ resembles the commutator on G. We are interested in the order of $\langle \mathbb{1}, \mathbb{1} \rangle$ since it measures the non-commutativity of G. Moreover, in our case the order of $\langle \mathbb{1}, \mathbb{1} \rangle$ also gives information about the order of ∂_1, which helps determine the classification of gauge groups of G. When G is $SU(n)$ or $Sp(n)$, the order of $\langle \mathbb{1}, \mathbb{1} \rangle$ can be calculated using Bott's formula in [Bot60]. However, in general calculating the order of $\langle \mathbb{1}, \mathbb{1} \rangle$ is very difficult since it involves analyzing the attaching maps in the cell structure of G, which rapidly increases when the rank of G increases.

In my second paper "The odd primary order of the commutator on low rank Lie groups" (Chapter 7), I applied homotopy decomposition of Lie groups and calculated the order of $\langle \mathbb{1}, \mathbb{1} \rangle$ for certain range of Lie groups G.

HOMOTOPY TYPES OF GAUGE GROUPS OVER NON-SIMPLY-CONNECTED CLOSED 4-MANIFOLDS

TSELEUNG SO

Abstract. Let G be a simple, simply-connected, compact Lie group and let M be an orientable, smooth, connected, closed 4-manifold. In this paper we calculate the homotopy type of the suspension of M and the homotopy types of the gauge groups of principal G-bundles over M when $\pi_1(M)$ is (1) $\mathbb{Z}^{*m}$, (2) $\mathbb{Z}/p^r\mathbb{Z}$, or (3) $\mathbb{Z}^{*m} * (*_{j=1}^{n}\mathbb{Z}/p_j^{r_j}\mathbb{Z})$, where p and the p_j's are odd primes.

1. Introduction

Let G be a topological group and let M be a topological space. Given a principal G-bundle P over M, the associated gauge group $\mathcal{G}(P)$ is the topological group of G-equivariant automorphisms of P which fix M. Atiyah, Bott and Gottlieb [1, 4] showed that its classifying space $B\mathcal{G}(P)$ is homotopy equivalent to the connected component $\mathrm{Map}_P(M, BG)$ of the mapping space $\mathrm{Map}(M, BG)$ which contains the map inducing P. When G is a simple, simply-connected, compact Lie group and M is an orientable, smooth, connected, closed 4-manifold, it can be shown that the set of isomorphism classes of principal G-bundles over M is in one-to-one correspondence with the homotopy set $[M, BG] \cong \mathbb{Z}$. If a principal G-bundle corresponds to an integer t, then we denote its associated gauge group by $\mathcal{G}_t(M)$. In [18] Theriault shows that when M is a simply-connected spin 4-manifold, there is a homotopy equivalence

$$\mathcal{G}_t(M) \simeq \mathcal{G}_t(S^4) \times \prod_{i=1}^{n} \Omega^2 G, \tag{1}$$

where n is the rank of $H^2(M;\mathbb{Z})$. For the non-spin case this homotopy equivalence still holds when localized away from 2. As a result, the study of $\mathcal{G}_t(M)$ can be reduced to that of $\mathcal{G}_t(S^4)$, which has been investigated over the last twenty years. Kishimoto, Kono and Tsutaya gave bounds on the numbers of distinct homotopy types of $\mathcal{G}_t(S^4)$ for all G using localization at odd primes [9]. Moreover, the homotopy types of $\mathcal{G}_t(S^4)$ are classified for many cases. Let (a, b) be the greatest common divisor of a and b. Then [3, 8, 10, 11, 19, 21, 22]

- when $G = SU(2)$, there is a homotopy equivalence $\mathcal{G}_t(S^4) \simeq \mathcal{G}_s(S^4)$ if and only if $(12, t) = (12, s)$;
- when $G = SU(3)$, there is a homotopy equivalence $\mathcal{G}_t(S^4) \simeq \mathcal{G}_s(S^4)$ if and only if $(24, t) = (24, s)$;
- when $G = SU(5)$, there is a p-homotopy equivalence $\mathcal{G}_t(S^4) \simeq \mathcal{G}_s(S^4)$ if and only if $(120, t) = (120, s)$ for any prime p;

2010 *Mathematics Subject Classification.* Primary 54C35, 55P40; Secondary 55R10.
Key words and phrases. homotopy decomposition, gauge groups, 4-manifolds.

- when $G = Sp(2)$, there is a p-local homotopy equivalence $\mathcal{G}_t(S^4) \simeq \mathcal{G}_s(S^4)$ if and only if $(40,t) = (40,s)$ for any prime p;
- when $G = SU(n)$, there is a p-local homotopy equivalence $\mathcal{G}_t(S^4) \simeq \mathcal{G}_s(S^4)$ if and only if $(n(n^2-1),t) = (n(n^2-1),s)$ for any odd prime p such that $n \leq (p-1)^2+1$;
- when $G = Sp(n)$, there is a p-local homotopy equivalence $\mathcal{G}_t(S^4) \simeq \mathcal{G}_s(S^4)$ if and only if $(4n(2n+1),t) = (4n(2n+1),s)$ for any odd prime p such that $2n \leq (p-1)^2+1$;
- when $G = G_2$, there is a p-local homotopy equivalence $\mathcal{G}_t(S^4) \simeq \mathcal{G}_s(S^4)$ if and only if $(84,t) = (84,s)$ for any odd prime p,

In addition, a few cases of $\mathcal{G}_t(\mathbb{CP}^2)$ are worked out [12, 20]:

- when $G = SU(2)$, there is a homotopy equivalence $\mathcal{G}_t(\mathbb{CP}^2) \simeq \mathcal{G}_s(\mathbb{CP}^2)$ if and only if $(6,t) = (6,s)$;
- when $G = SU(3)$, there is a p-local homotopy equivalence $\mathcal{G}_t(\mathbb{CP}^2) \simeq \mathcal{G}_s(\mathbb{CP}^2)$ if and only if $(12,t) = (12,s)$ for any prime p.

On the other hand, very little is known about $\mathcal{G}_t(M)$ when M is non-simply-connected. The goal of this paper is to study the homotopy types of $\mathcal{G}_t(M)$ for certain non-simply-connected 4-manifolds. To achieve this we need a homotopy decomposition statement.

Theorem 1.1. *Suppose that G is a simple, simply-connected, compact Lie group and Y is a CW-complex of dimension at most 3. Let $\phi : Y \to M$ be a map such that $\Sigma\phi$ has a left homotopy inverse. Then we have*

$$\mathcal{G}_t(M) \simeq \mathcal{G}_t(C_\phi) \times Map^*(Y, G),$$

where C_ϕ is the cofiber of ϕ.

Using Theorem 1.1 we calculate the homotopy types of $\mathcal{G}_t(M)$ when $\pi_1(M)$ is (1) $\mathbb{Z}^{*m}$, (2) $\mathbb{Z}/p^r\mathbb{Z}$, or (3) $\mathbb{Z}^{*m} * (*_{j=1}^n \mathbb{Z}/p_j^{r_j}\mathbb{Z})$, where p and the p_j's are odd primes.

Theorem 1.2. *Suppose that G is a simple, simply-connected, compact Lie group and M is an orientable, smooth, connected, closed 4-manifold.*

- *If $\pi_1(M) = \mathbb{Z}^{*m}$ or $\mathbb{Z}/p^r\mathbb{Z}$, then $\mathcal{G}_t(M)$ is homotopy equivalent to a product of $\mathcal{G}_t(S^4)$ or $\mathcal{G}_t(\mathbb{CP}^2)$ and "loop spaces" on G.*
- *If $\pi_1(M) = \mathbb{Z}^{*m} * (*_{j=1}^n \mathbb{Z}/p_j^{r_j}\mathbb{Z})$, then $\mathcal{G}_t(M) \times \prod^{2d} \Omega^2 G$ is homotopy equivalent to a product of $\mathcal{G}_t(S^4)$ or $\mathcal{G}_t(\mathbb{CP}^2)$ and "loop spaces" on G for some number d.*

The term "loop spaces" refers both to iterated based loop spaces ΩG, $\Omega^2 G$ and $\Omega^3 G$ and modular loop spaces $\Omega G\{p^r\}$ and $\Omega^2 G\{p^r\}$, where $G\{p^r\}$ is the homotopy fiber of the p^r-power map on G. Explicit decompositions are stated in Section 3.

Theorem 1.2 shows that the homotopy type of $\mathcal{G}_t(M)$ is related to that of $\mathcal{G}_t(S^4)$ or $\mathcal{G}_t(\mathbb{CP}^2)$ in these three cases. Combining Theorem 1.2 and the known results in [3, 11, 10, 19, 22], we have the following classification.

Corollary 1.3. *If M is an orientable, smooth, connected, closed 4-manifold with $\pi_1(M) = \mathbb{Z}^{*m}$ or $\mathbb{Z}/p^r\mathbb{Z}$, then the followings hold:*

- *when $G = SU(2)$, there is a homotopy equivalence $\mathcal{G}_t(M) \simeq \mathcal{G}_s(M)$ if and only if $(12,t) = (12,s)$ for M spin, and $(6,t) = (6,s)$ for M non-spin;*

- *when* $G = SU(3)$*, there is a homotopy equivalence* $\mathcal{G}_t(M) \simeq \mathcal{G}_s(M)$ *if and only if* $(24, t) = (24, s)$ *for* M *spin; there is a* p*-local homotopy equivalence* $\mathcal{G}_t(M) \simeq \mathcal{G}_s(M)$ *if and only if* $(12, t) = (12, s)$ *for any prime* p *and* M *non-spin;*

- *when* $G = SU(n)$*, there is a* p*-local homotopy equivalence* $\mathcal{G}_t(M) \simeq \mathcal{G}_s(M)$ *if and only if* $(n(n^2-1), t) = (n(n^2-1), s)$ *for any odd prime* p *such that* $n \leq (p-1)^2 + 1$*;*

- *when* $G = Sp(2)$*, there is a* p*-local homotopy equivalence* $\mathcal{G}_t(M) \simeq \mathcal{G}_s(M)$ *if and only if* $(40, t) = (40, s)$ *for any odd prime* p*;*

- *when* $G = G_2$*, there is a* p*-local homotopy equivalence* $\mathcal{G}_t(M) \simeq \mathcal{G}_s(M)$ *if and only if* $(84, t) = (84, s)$ *for any odd prime* p*.*

There is an analogous statement to Corollary 1.3.

Corollary 1.4. *If* M *is an orientable, smooth, connected, closed 4-manifold with* $\pi_1(M) = \mathbb{Z}^{*m} * (*_{j=1}^{n} \mathbb{Z}/p_j^{r_j}\mathbb{Z})$*, then the integral and* p*-local homotopy equivalences in Corollary 1.3 hold for* $\mathcal{G}_t(M) \times \prod^{2d} \Omega^2 G$ *for some number* d*.*

The structure of this paper is as follows. In Section 2 we prove the homotopy decomposition Theorem 1.1 and develop some useful lemmas. In particular, Theorem 1.1 is used to revise homotopy equivalence (1) which is often referred to during the calculations in Section 3. In Section 3 we give the homotopy types of ΣM and $\mathcal{G}_t(M)$ when $\pi_1(M)$ is either $\mathbb{Z}^{*m}$, $\mathbb{Z}/p^r\mathbb{Z}$ or $\mathbb{Z}^{*m} * (*_{j=1}^{n} \mathbb{Z}/p_j^{r_j}\mathbb{Z})$, where p and the p_j's are odd primes.

2. A homotopy decomposition of gauge groups

2.1. **A homotopy decomposition and some useful lemmas.** We are going to extend the homotopy decomposition (1) to a more general situation. Suppose that M is an orientable 4-dimensional CW-complex that is constructed by attaching a 4-cell onto a 3-dimensional CW-complex M_3 by an attaching map $f : S^3 \to M_3$. Let $u : BG \to K(\mathbb{Z}, 4)$ be a generator of $H^4(BG) \cong \mathbb{Z}$. Since it is a 5-equivalence, $u_* : [M, BG] \to [M, K(\mathbb{Z}, 4)] = \mathbb{Z}$ is a bijection. If a principal G-bundle over M corresponds to some integer t, then we denote its associated gauge group by $\mathcal{G}_t(M)$.

In [18], the homotopy decomposition (1) is obtained as a consequence of the attaching map f of the 4-cell in M having the property that Σf is null-homotopic. Observe that Σf is the connecting map of the cofibration sequence

$$M_3 \longrightarrow M \longrightarrow S^4,$$

where M_3 is the 3-skeleton of M. From the point of view of homotopy theory, it can be replaced by a cofibration sequence

$$Y \xrightarrow{\phi} M \xrightarrow{q} C_\phi \tag{2}$$

for some space Y and some map $\phi : Y \to M$, with a connecting map that is null-homotopic. Here q is the quotient map and C_ϕ is the cofiber of ϕ. The nullity condition is equivalent to $\Sigma\phi$ having a left homotopy inverse. If we further restrict the dimension of Y to be at most 3, then by Cellular Approximation Theorem ϕ is homotopic to $\imath \circ \varphi$, where $\varphi : Y \to M_3$ is a map on M_3 and $\imath : M_3 \to M$ is the inclusion. The existence of a left homotopy inverse of $\Sigma\phi$ imposes a strong condition on ΣM.

Lemma 2.1. *Let Y be a CW-complex of dimension at most 3 and let $\phi : Y \to M$ be a map. If ϕ is homotopic to $\imath \circ \varphi$ for some map $\varphi : Y \to M_3$, then the following are equivalent:*

1. *$\Sigma\phi : \Sigma Y \to \Sigma M$ has a left homotopy inverse ψ;*
2. *$\Sigma\varphi$ has a left homotopy inverse ψ' and $\psi' \circ \Sigma f$ is null-homotopic;*
3. *ΣM is homotopy equivalent to $\Sigma Y \vee \Sigma C_\phi$ and $\Sigma\phi$ is homotopic to the inclusion;*
4. *ΣM_3 is homotopy equivalent to $\Sigma Y \vee \Sigma C_\varphi$ where C_φ is the cofiber of φ, $\Sigma\varphi$ is homotopic to the inclusion and $\psi' \circ \Sigma f$ is null-homotopic.*

Proof. First we show that Condition (1) and (2) are equivalent. If $\Sigma\phi$ has a left homotopy inverse ψ, then $\psi' = \psi \circ \Sigma\imath$ is a left homotopy inverse of $\Sigma\varphi$ and $\psi' \circ \Sigma f = \psi \circ \Sigma\imath \circ \Sigma f$ is null-homotopic since $S^4 \xrightarrow{\Sigma f} \Sigma M_3 \xrightarrow{\Sigma\imath} \Sigma M$ is a cofibration sequence. Conversely, assume Condition (2). Consider the homotopy commutative diagram

$$\begin{tikzcd} S^4 \arrow[r, "\Sigma f"] & \Sigma M_3 \arrow[r, "\Sigma\imath"] \arrow[d, "\psi'"] & \Sigma M \arrow[dl, dashed, "\psi"] \\ & \Sigma Y & \end{tikzcd}$$

By hypothesis $\psi' \circ \Sigma f$ is null-homotopic, so ψ' has an extension ψ. Then we have

$$\begin{aligned} \psi \circ \Sigma\phi &\simeq \psi \circ \Sigma(\imath \circ \varphi) \\ &\simeq (\psi \circ \Sigma\imath) \circ \Sigma\varphi \\ &\simeq \psi' \circ \Sigma\varphi \\ &\simeq \mathbb{1}_{\Sigma Y}, \end{aligned}$$

where $\mathbb{1}_{\Sigma Y}$ is the identity map on ΣY. Therefore ψ is a left homotopy inverse of $\Sigma\phi$.

Now we show that Conditions (1) and (3) are equivalent. If $\Sigma\phi$ has a left homotopy inverse ψ, then let h be the composition

$$h : \Sigma M \xrightarrow{\sigma} \Sigma M \vee \Sigma M \xrightarrow{\psi \vee \Sigma q} \Sigma Y \vee \Sigma C_\phi,$$

where σ is the comultiplication. Observe that h induces an isomorphism $H^*(\Sigma Y \vee \Sigma C_\phi) \to H^*(\Sigma M)$. Since these spaces are suspensions and are simply-connected, h is a homotopy equivalence by Whitehead Theorem. Therefore ΣM is homotopy equivalent to $\Sigma Y \vee \Sigma C_\phi$. Moreover, since $\Sigma\phi$ is a co-H-map,

$$\begin{aligned} h \circ \Sigma\phi &\simeq (\psi \vee \Sigma q) \circ \sigma \circ \Sigma\phi \\ &\simeq (\psi \vee \Sigma q) \circ (\Sigma\phi \vee \Sigma\phi) \circ \sigma \\ &\simeq (\psi \circ \Sigma\phi \vee \Sigma q \circ \Sigma\phi) \circ \sigma \\ &\simeq (\mathbb{1}_{\Sigma Y} \vee *) \circ \sigma \end{aligned}$$

is the inclusion $\Sigma Y \hookrightarrow \Sigma Y \vee \Sigma C_\phi$. Conversely, assume Condition (3). Let h be a homotopy equivalence from ΣM to $\Sigma Y \vee \Sigma C_\phi$ and let ψ be the composition

$$\psi : \Sigma M \xrightarrow{h} \Sigma Y \vee \Sigma C_\phi \xrightarrow{p} \Sigma Y.$$

where p is the pinch map. By hypothesis, $h \circ \Sigma\phi$ is homotopic to the inclusion $\Sigma Y \to \Sigma Y \vee \Sigma C_\phi$, so we have

$$\psi \circ \Sigma\phi \simeq p \circ h \circ \Sigma\phi \simeq \mathbb{1}_{\Sigma Y}$$

and ψ is a left homotopy inverse of $\Sigma\phi$.

The equivalence between (2) and (4) can be shown similarly. $\square$

We can extend the cofibration (2) to the homotopy commutative diagram whose rows and columns are cofibration sequences

(3)
$$\begin{tikzcd}
* \arrow[r] \arrow[d] & Y \arrow[r, equal] \arrow[d, "\varphi"] & Y \arrow[r] \arrow[d, "\phi"] & * \arrow[d] \\
S^3 \arrow[r, "f"] \arrow[d, equal] & M_3 \arrow[r, "\imath"] \arrow[d, "q'"] & M \arrow[r, "p"] \arrow[d, "q"] & S^4 \arrow[d, equal] \\
S^3 \arrow[r, "f'"] & C_\varphi \arrow[r] & C_\phi \arrow[r, "p'"] & S^4
\end{tikzcd}$$

where q', p and p' are the quotient maps and $f' = q' \circ f$. The bottom row implies that C_ϕ is constructed by attaching a 4-cell onto C_φ via f'. The generator u of $H^4(BG)$ induces a bijection between $[C_\phi, BG]$ and $[C_\phi, K(\mathbb{Z}, 4)] \cong \mathbb{Z}$, so any principal G-bundle over C_ϕ corresponds to some integer t. Denote the associated gauge group by $\mathcal{G}_t(C_\phi)$. We want to compare $\mathcal{G}_t(M)$ and $\mathcal{G}_t(C_\phi)$ via the pullback q^*.

Lemma 2.2. *Assume the conditions in Lemma 2.1. Then $q^* : [C_\phi, BG] \to [M, BG]$ is a group isomorphism.*

Proof. The naturality of u^* implies the commutative diagram

$$\begin{tikzcd}
{[C_\phi, BG]} \arrow[r, "q^*"] \arrow[d, "u^*"] & {[M, BG]} \arrow[d, "u^*"] \\
H^4(C_\phi) \arrow[r, "q^*"] & H^4(M).
\end{tikzcd}$$

Since the group structures on $[M, BG]$ and $[C_\phi, BG]$ are induced by bijections

$$u^* : [M, BG] \to H^4(M) \cong \mathbb{Z} \quad \text{and} \quad u^* : [C_\phi, BG] \to H^4(C_\phi) \cong \mathbb{Z},$$

it suffices to show that $q^* : H^4(C_\phi) \to H^4(M)$ is an isomorphism.

The cofibration (2) induces an exact sequence

$$H^4(\Sigma M) \xrightarrow{(\Sigma\phi)^*} H^4(\Sigma Y) \xrightarrow{\epsilon^*} H^4(C_\phi) \xrightarrow{q^*} H^4(M) \xrightarrow{\phi^*} H^4(Y),$$

where ϵ^* is induced by the connecting map $\epsilon : C_\phi \to \Sigma Y$. Since $\Sigma\phi$ has a left homotopy inverse, ϵ is null-homotopic and ϵ^* is trivial. Also, the dimension of Y is at most 3, so $H^4(Y)$ is trivial. By exactness $q^* : H^4(C_\phi) \to H^4(M)$ is an isomorphism. $\square$

Lemma 2.3. *For any integer t, let $Map^*_t(M, BG)$ and $Map^*_t(C_\phi, BG)$ be the connected components of $Map^*(M, BG)$ and $Map^*(C_\phi, BG)$ containing $t \in [M, BG] \cong [C_\phi, BG]$ respectively. Then*

$$Map^*(\Sigma Y, BG) \xrightarrow{\epsilon^*} Map^*_t(C_\phi, BG) \xrightarrow{q^*} Map^*_t(M, BG)$$

is a fibre sequence.

Proof. First, $q^* : \mathrm{Map}^*_t(C_\phi, BG) \to \mathrm{Map}^*_t(M, BG)$ is well defined by Lemma 2.2. Second, the cofibration (2) induces a fibre sequence

$$\mathrm{Map}^*(\Sigma Y, BG) \xrightarrow{\epsilon^*} \mathrm{Map}^*(M, BG) \xrightarrow{q^*} \mathrm{Map}^*(C_\phi, BG).$$

Since ϵ^* is null-homotopic, $\mathrm{Map}^*(\Sigma Y, BG)$ is mapped into $\mathrm{Map}^*_0(M, BG)$ and hence is the homotopy fiber of $q^* : \mathrm{Map}^*_0(M, BG) \to \mathrm{Map}^*_0(C_\phi, BG)$. To show that it is true for any $t \in \mathbb{Z}$, let $\alpha_t : \mathrm{Map}^*_0(M, BG) \to \mathrm{Map}^*_t(M, BG)$ be a map sending a pointed map $g : M \to BG$ to the composition

$$\alpha_t(g) : M \xrightarrow{c} M \vee S^4 \xrightarrow{g \vee t} BG \vee BG \xrightarrow{\nabla} BG,$$

where c is the coaction map of M and ∇ is the folding map, and let $\beta_t : \mathrm{Map}^*_0(C_\phi, BG) \to \mathrm{Map}^*_t(C_\phi, BG)$ be a map defined similarly. Since α_t and β_t are homotopy equivalences, it suffices to prove the commutate diagram

$$\begin{tikzcd}
\mathrm{Map}^*_0(C_\phi, BG) \arrow[r, "q^*"] \arrow[d, "\alpha_t"] & \mathrm{Map}^*_0(M, BG) \arrow[d, "\beta_t"] \\
\mathrm{Map}^*_t(C_\phi, BG) \arrow[r, "q^*"] & \mathrm{Map}^*_t(M, BG).
\end{tikzcd}$$

Consider the homotopy commutative diagram

$$\begin{tikzcd}
M \arrow[rrr, "c"] \arrow[dr, "p"] \arrow[dd, "q"'] & & & M \vee S^4 \arrow[dl, "p \vee \mathbb{1}"'] \arrow[dd, "q \vee \mathbb{1}"] \\
& S^4 \arrow[r, "\sigma"] & S^4 \vee S^4 & \\
C_\phi \arrow[rrr, "c'"'] \arrow[ur, "p'"'] & & & C_\phi \vee S^4 \arrow[ul, "p' \vee \mathbb{1}"]
\end{tikzcd}$$

where c' is the coaction map, and p and p' are the quotient maps, and σ is the comultiplication of S^4. The left and the right triangles are due to the bottom right square in diagram (3), the top and the bottom quadrangles are due to the property of coaction maps. Extend it to get the cofibration diagram

$$\begin{tikzcd}
M \arrow[r, "c"] \arrow[d, "q"] & M \vee S^4 \arrow[r, "q^* g \vee t"] \arrow[d, "q \vee \mathbb{1}"] & BG \vee BG \arrow[r, "\nabla"] \arrow[d, equal] & BG \arrow[d, equal] \\
C_\phi \arrow[r, "c'"] & C_\phi \vee S^4 \arrow[r, "g \vee t"] & BG \vee BG \arrow[r, "\nabla"] & BG
\end{tikzcd}$$

The upper row around the diagram is $\beta_t(q^* g)$, while the lower row around the diagram is $q^* \alpha_t(g)$. Therefore q^* commutes with α_t and β_t and the asserted statement follows. $\square$

Theorem 2.4. *Let Y be a CW-complex of dimension at most 3 and let $\phi : Y \to M$ be a map. If ϕ satisfies one of the four conditions in Lemma 2.1, then there are homotopy equivalences*

$$\Sigma M \simeq \Sigma C_\phi \vee \Sigma Y \quad \text{and} \quad \mathcal{G}_t(M) \simeq \mathcal{G}_t(C_\phi) \times \mathit{Map}^*(Y, G).$$

Proof. Let $\mathrm{Map}_t(M, BG)$ and $\mathrm{Map}_t(C_\phi, BG)$ be the connected components of $\mathrm{Map}(M, BG)$ and $\mathrm{Map}(C_\phi, BG)$ containing $t \in [M, BG] \cong [C_\phi, BG]$. There are evaluation fibrations

$$\mathrm{Map}^*_t(M, BG) \to \mathrm{Map}_t(M, BG) \to BG \quad \text{and} \quad \mathrm{Map}^*_t(C_\phi, BG) \to \mathrm{Map}_t(C_\phi, BG) \to BG.$$

By Lemma 2.2 $q^* : [C_\phi, BG] \to [M, BG]$ is an isomorphism, so there is a homotopy commutative diagram whose rows are fibration sequences

$$\begin{tikzcd} G \arrow[r] \arrow[d, equal] & \mathrm{Map}_t^*(C_\phi, BG) \arrow[r] \arrow[d, "q^*"] & \mathrm{Map}_t(C_\phi, BG) \arrow[r] \arrow[d, "q^*"] & BG \arrow[d, equal] \\ G \arrow[r] & \mathrm{Map}_t^*(M, BG) \arrow[r] & \mathrm{Map}_t(M, BG) \arrow[r] & BG \end{tikzcd} \tag{4}$$

As in [18], from the left square in (4) we obtain a homotopy fibration diagram

$$\begin{tikzcd} * \arrow[r] \arrow[d] & \Omega\mathrm{Map}_t^*(M, BG) \arrow[r, equal] \arrow[d] & \Omega\mathrm{Map}_t^*(M, BG) \arrow[d] \\ \mathcal{G}_t(C_\phi) \arrow[r, "q^*"] \arrow[d, equal] & \mathcal{G}_t(M) \arrow[r, "h"] \arrow[d] & \mathrm{Map}^*(\Sigma Y, BG) \arrow[d, "\epsilon^*"] \\ \mathcal{G}_t(C_\phi) \arrow[r] \arrow[d] & G \arrow[r] \arrow[d] & \mathrm{Map}_t^*(C_\phi, BG) \arrow[d] \\ * \arrow[r] & \mathrm{Map}_t^*(M, BG) \arrow[r, equal] & \mathrm{Map}_t^*(M, BG) \end{tikzcd} \tag{5}$$

The right column is due to Lemma 2.3. The nullity of ϵ^* implies that h has a right homotopy inverse. The group multiplication in $\mathcal{G}_t(M)$ then gives a homotopy equivalence

$$\Phi : \mathcal{G}_t(C_\phi) \times \mathrm{Map}^*(\Sigma Y, BG) \longrightarrow \mathcal{G}_t(M) \times \mathcal{G}_t(M) \longrightarrow \mathcal{G}_t(M),$$

so $\mathcal{G}_t(M)$ is homotopy equivalent to $\mathcal{G}_t(C_\phi) \times \mathrm{Map}^*(Y, G)$. □

When calculating $\mathcal{G}_t(M)$, we will use Theorem 2.4 several times and apply it to M and C_ϕ in some cases. So here we establish some facts about C_ϕ.

Lemma 2.5. *Let $q' : M \to C_\phi$ be the quotient map and let f' be the composition*

$$S^3 \xrightarrow{f} M_3 \xrightarrow{q'} C_\varphi.$$

Then $\Sigma f'$ is null-homotopic if and only if Σf is null-homotopic.

Proof. The necessity is obvious. Assume that $\Sigma f'$ is null-homotopic. Since the composition

$$h : \Sigma M_3 \xrightarrow{\sigma} \Sigma M_3 \vee \Sigma M_3 \xrightarrow{\psi \vee \Sigma q'} \Sigma Y \vee \Sigma C_\varphi$$

is a homotopy equivalence, it suffices to show that $h \circ \Sigma f$ is null-homotopic. Consider the homotopy commutative diagram

$$\begin{tikzcd} S^4 \arrow[r, "\sigma"] \arrow[d, "\Sigma f"] & S^4 \vee S^4 \arrow[d, "\Sigma f \vee \Sigma f"] & \\ \Sigma M_3 \arrow[r, "\sigma"] & \Sigma M_3 \vee \Sigma M_3 \arrow[r, "\psi \vee \Sigma q'"] & \Sigma Y \vee \Sigma C_\varphi \end{tikzcd}$$

where the two σ's are the comultiplications of S^4 and ΣM_3. The lower direction around the diagram is $h \circ \Sigma f$, and the upper direction around the diagram is

$$(\psi \circ \Sigma f \vee \Sigma q' \circ \Sigma f) \circ \sigma \simeq (\psi \circ \Sigma f \vee \Sigma f') \circ \sigma$$

By hypothesis $\Sigma f'$ is null-homotopic, and by Lemma 2.1(4) $\psi\circ\Sigma f$ is null-homotopic. Therefore $h\circ\Sigma f$ is null-homotopic and hence so is Σf. $\square$

Lemma 2.6. *Let $H^2_{free}(X)$ be the free part of $H^2(X)$ for any space X. If $H^2(Y)$ is torsion, then $q^*: H^2_{free}(C_\phi)\to H^2_{free}(M)$ is an isomorphism. Moreover, if for any $\alpha'\in H^2_{free}(M)$ there exists $\beta'\in H^2_{free}(M)$ such that $\alpha'\cup\beta'\in H^4(M)$ is a generator, then for any $\alpha\in H^2_{free}(C_\phi)$ there exists $\beta\in H^2_{free}(C_\phi)$ such that $\alpha\cup\beta\in H^4(C_\phi)$ is a generator.*

Proof. Cofibration (2) induces the long exact sequence of cohomology groups

$$\cdots\longrightarrow H^k(C_\phi)\xrightarrow{(\Sigma q)^*}H^k(M)\xrightarrow{(\Sigma\phi)^*}H^k(Y)\longrightarrow\cdots$$

Since $\Sigma\phi$ has a left homotopy inverse, the sequence splits for $k\geq 1$ and we have

$$H^k(M)\cong H^k(Y)\oplus H^k(C_\phi).$$

By hypothesis $H^2(Y)$ is torsion, so $H^2_{\text{free}}(C_\phi)$ is isomorphic to $H^2_{\text{free}}(M)$.

For any $\alpha\in H^2_{\text{free}}(C_\phi)$, $q^*(\alpha)$ is in $H^2_{\text{free}}(M)$. By hypothesis there exists $\beta'\in H^2_{\text{free}}(M)$ such that $q^*(\alpha)\cup\beta'\in H^4(M)$ is a generator. Since $q^*: H^2_{\text{free}}(C_\phi)\to H^2_{\text{free}}(M)$ is an isomorphism, there exists $\beta\in H^2_{\text{free}}(C_\phi)$ such that $q^*(\beta)=\beta'$. Therefore we have $q^*(\alpha)\cup\beta'=q^*(\alpha\cup\beta)$. Observe that $q^*: H^4(C_\phi)\to H^4(M)$ is an isomorphism since Y has dimension at most 3. It follows that $\alpha\cup\beta$ is a generator of $H^4(C_\phi)$. $\square$

The second part of Lemma 2.6 says that the cup product on $H^2_{\text{free}}(C_\phi)$ is unimodular if the cup product on $H^2_{\text{free}(M)}$ is unimodular, which follows from Poincaré Duality when M is an orientable compact manifold. Furthermore, if C_ϕ has a subcomplex Y' satisfying Theorem 2.4, then the cup product on $H^2_{\text{free}}(C_\phi/Y')$ is still unimodular.

Next we consider two variations of Theorem 2.4 when M has a special structure.

Lemma 2.7. *Suppose that M_3 is homotopy equivalent to $Z\vee Z'$. Let Y and Y' be CW-complexes of dimension at most 3, and let $\varphi: Y\to Z$ and $\varphi': Y'\to Z'$ be maps. If $\Sigma\varphi$ and $\Sigma\varphi'$ have left homotopy inverses ψ and ψ' respectively and the compositions*

$$S^4\xrightarrow{\Sigma f}\Sigma M_3\xrightarrow{pinch}\Sigma Z\xrightarrow{\psi}\Sigma Y\quad\textit{and}\quad S^4\xrightarrow{\Sigma f}\Sigma M_3\xrightarrow{pinch}\Sigma Z'\xrightarrow{\psi'}\Sigma Y',$$

are null-homotopic, then we have

$$\Sigma M\simeq\Sigma M'\vee\Sigma Y\vee\Sigma Y'\quad\textit{and}\quad\mathcal{G}_t(M)\simeq\mathcal{G}_t(M')\times Map^*(\Sigma Y,BG)\times Map^*(\Sigma Y',BG)$$

where M' is the cofiber of the map $Y\vee Y'\xrightarrow{\varphi\vee\varphi'}Z\vee Z'\hookrightarrow M$.

Proof. Let Φ be the composition

$$\Phi: Y\vee Y'\xrightarrow{\varphi\vee\varphi'}Z\vee Z'\simeq M_3.$$

The map $\psi\vee\psi':\Sigma Z\vee\Sigma Z'\to\Sigma Y\vee\Sigma Y'$ is a left homotopy inverse of $\Sigma\Phi$. We show that $(\psi\vee\psi')\circ\Sigma f$ is null-homotopic, implying that Φ satisfies the hypothesis of Theorem 2.4.

Notice that the composition

$$h:\Sigma M_3\xrightarrow{\sigma}\Sigma M_3\vee\Sigma M_3\xrightarrow{p_1\vee p_2}\Sigma Z\vee\Sigma Z'$$

is a homotopy equivalence, where $p_1 : \Sigma M_3 \to \Sigma Z$ and $p_2 : \Sigma M_3 \to \Sigma Z'$ are the pinch maps. Since Σf is a co-H-map, we have

$$\begin{aligned}(\psi \vee \psi') \circ h \circ \Sigma f &\simeq (\psi \vee \psi') \circ (p_1 \vee p_2) \circ \sigma \circ \Sigma f \\ &\simeq (\psi \vee \psi') \circ (p_1 \vee p_2) \circ (\Sigma f \vee \Sigma f) \circ \sigma \\ &\simeq (\psi \circ p_1 \circ \Sigma f \vee \psi' \circ p_2 \circ \Sigma f) \circ \sigma\end{aligned}$$

which is null-homotopic by assumption. Therefore $(\psi \vee \psi') \circ \Sigma f$ is null-homotopic and Theorem 2.4 implies the asserted statement. □

Lemma 2.8. *Suppose $M \cong X \# X'$ where X and X' are orientable, smooth, connected, closed 4-manifolds. Let Y and Y' be CW-complexes of dimensions at most 3 and let $\varphi : Y \to X_3$ and $\varphi' : Y' \to X'_3$ be maps satisfying the hypothesis of Theorem 2.4. Then we have*

$$\Sigma M \simeq \Sigma M' \vee \Sigma Y \vee \Sigma Y' \quad \text{and} \quad \mathcal{G}_t(M) \simeq \mathcal{G}_t(M') \times Map^*(\Sigma Y, BG) \times Map^*(\Sigma Y', BG)$$

where M' is the cofiber of the inclusion $Y \vee Y' \xrightarrow{\varphi \vee \varphi'} X_3 \vee X'_3 \hookrightarrow M$.

Proof. Let $f : S^3 \to X_3$ and $f' : S^3 \to X'_3$ be the attaching maps of the 4-cells in X and X' respectively. By Lemma 2.1, $\Sigma\varphi$ and $\Sigma\varphi'$ have left homotopy inverses ψ and ψ' and $\psi \circ \Sigma f$ and $\psi' \circ \Sigma f'$ are null-homotopic. We show that φ and φ' satisfy the hypothesis in Lemma 2.7.

The 3-skeleton of M is $X_3 \vee X'_3$ and the attaching map of the 4-cell is

$$f_\# : S^3 \xrightarrow{\sigma} S^3 \vee S^3 \xrightarrow{f \vee f'} X_3 \vee X'_3$$

Let $p_1 : \Sigma X_3 \vee \Sigma X'_3 \to \Sigma X_3$ and $p_2 : \Sigma X_3 \vee \Sigma X'_3 \to \Sigma X'_3$ be the pinch maps. Consider the homotopy commutative diagram

$$\begin{tikzcd} S^4 \arrow[r, "\Sigma f_\#"] \arrow[d, equal] & \Sigma X_3 \vee \Sigma X'_3 \arrow[d, "p_1"] & \\ S^4 \arrow[r, "\Sigma f"] & \Sigma X_3 \arrow[r, "\psi"] & \Sigma Y \end{tikzcd}$$

Since $\psi \circ \Sigma f$ is null-homotopic, the composition $\psi \circ p_1 \circ \Sigma f_\#$ is null-homotopic. Similarly $\psi' \circ p_2 \circ \Sigma f_\#$ is null-homotopic. Then Lemma 2.7 implies the statement. □

2.2. **Gauge groups over simply-connected 4-manifolds.** Now we revise homotopy equivalence (1) using Theorem 2.4. When M is simply-connected, its 3-skeleton M_3 is homotopy equivalent to $\bigvee_{i=1}^n S^2$. If Σf is null-homotopic, then we can apply Theorem 2.4 by taking Y to be the whole of M_3 and $\varphi : Y \to M_3$ to be the identity map and get the homotopy equivalence (1). In the following we assume the homotopy class of Σf is not trivial.

To distinguish the 2-spheres, denote the i^{th} copy of S^2 in M_3 by S^2_i. Let $\eta : S^3 \to S^2$ be the Hopf map and let η_i be the composition

$$\eta_i : S^3 \xrightarrow{\eta} S^2_i \hookrightarrow \bigvee_{i=1}^n S^2_i.$$

We also denote the suspensions $\Sigma\eta$ and $\Sigma\eta_i$ by $\bar{\eta}$ and $\bar{\eta}_i$ for short.

Lemma 2.9. *If M_3 is homotopy equivalent to $\bigvee_{i=1}^n S^2_i$, then Σf is homotopic to $\sum_{i=1}^n a_i \bar{\eta}_i$ where $a_i \in \mathbb{Z}/2\mathbb{Z}$.*

Proof. The Hilton-Milnor Theorem implies

$$\pi_3(\bigvee_{i=n}^{n} S_i^2) \cong \bigoplus_{i=1}^{n} \pi_3(S_i^2) \oplus \bigoplus_{i\neq j} \pi_3(\Sigma S_i^1 \wedge S_j^1),$$

where $\pi_3(S_i^2)$ is generated by η_i and $\pi_3(\Sigma S_i^1 \wedge S_j^1)$ is generated by Whitehead products of the identity maps on S_i^2 and S_j^2. As an element in $\pi_3(\bigvee_{i=1}^n S_i^2)$, f is homotopic to $\sum_{i=1}^n \tilde{a}_i\eta_i + w$ where $\tilde{a}_i$ is an integer and w is a sum of Whitehead products in $\bigoplus_{i\neq j}\pi_3(S_i^2 \wedge S_j^1)$. After suspension Σf is homotopic to $\sum_{i=1}^n a_i\bar{\eta}_i$, where $a_i \equiv \tilde{a}_i \pmod 2$, since suspensions of Whitehead products are null-homotopic and $\bar{\eta}_i$ has order 2. □

If the homotopy class of Σf is not trivial, then at least one of the a_i's is not zero. Relabelling the spheres if necessary, we may assume that Σf is homotopic to $\sum_{i=1}^m \bar{\eta}_i$ for some integer m such that $1 \leq m \leq n$. Then we can simplify this expression with the following lemma.

Lemma 2.10. *If M_3 is homotopy equivalent to $\bigvee_{i=1}^n S_i^2$ and Σf is homotopic to $\sum_{i=1}^m \bar{\eta}_i$, then there is a map $\tilde{f} : S^3 \to M_3$ such that its cofiber $C_{\tilde{f}}$ is homotopy equivalent to M and $\Sigma\tilde{f}$ is homotopic to $\bar{\eta}_1$. Moreover, $p_1 \circ \tilde{f}$ is homotopic to $p_1 \circ f$ where $p_1 : \bigvee_{i=1}^n S_i^2 \to S_1^2$ is the pinch map.*

Proof. For each $1 < j \leq m$, define a map $\xi_j : M_3 \to M_3$ as follows. On $S_1^2 \vee S_j^2$, ξ_j is the composition

$$S_1^2 \vee S_j^2 \xrightarrow{\sigma\vee\mathbb{1}} S_1^2 \vee S_1^2 \vee S_j^2 \xrightarrow{\mathbb{1}\vee\triangledown} S_1^2 \vee S_j^2,$$

where σ is a comultiplication of S_1^2 and $\triangledown$ is the folding map of S_1^2 and S_j^2. On the remaining spheres, ξ_j is the identity. Consider the homotopy cofibration diagram

$$\begin{tikzcd} S^3 \arrow[r, "f"] \arrow[d, equal] & M_3 \arrow[r] \arrow[d, "\xi_j"] & M \arrow[d, "\tilde{\xi}_j"] \\ S^3 \arrow[r, "\xi_j\circ f"] & M_3 \arrow[r] & C_{\xi_j\circ f} \end{tikzcd}$$

where $\tilde{\xi}_j$ is an induced map and $C_{\xi_j\circ f}$ is the cofiber of $\xi_j \circ f$. Since ξ_j is a homology isomorphism, so is $\tilde{\xi}_j$ by the 5-lemma. Therefore $\tilde{\xi}_j$ is a homotopy equivalence and M is homotopy equivalent to $C_{\xi_j\circ f}$.

By Lemma 2.9, $\Sigma(\xi_j \circ f)$ is homotopic to $\sum_{i=1}^n a_i\bar{\eta}_i$, where $a_i \in \mathbb{Z}/2\mathbb{Z}$. For $1 \leq i \leq n$, let $p_i : \bigvee_{i=1}^n S_i^3 \to S_i^3$ be the pinch map. Then a_i is $p_i \circ \Sigma(\xi_j \circ f)$. By the definition of ξ_j, $a_i = 1$ for $i \neq j$ and $1 \leq i \leq m$, and $a_i = 0$ otherwise, that is

$$\Sigma(\xi_j \circ f) \simeq \bar{\eta}_1 + \cdots + \bar{\eta}_{j-1} + \bar{\eta}_{j+1} + \cdots + \bar{\eta}_m$$

Let $\tilde{f}$ be $\xi_m \circ \cdots \circ \xi_2 \circ f$ and let $C_{\tilde{f}}$ be its cofiber. Then $\Sigma\tilde{f}$ is homotopic to $\bar{\eta}_1$ and $C_{\tilde{f}}$ is homotopy equivalent to M.

Lastly, observe that $p_1 \circ \xi_j$ is homotopic to p_1. It follows that $p_1 \circ \xi_j \circ f$ is homotopic to $p_1 \circ f$ and so is $p_1 \circ \tilde{f}$. □

Lemma 2.11. *Let M be a 4-dimensional simply-connected CW-complex. If Σf is nontrivial, then there are homotopy equivalences*

$$\Sigma M \simeq \Sigma\mathbb{C}\mathbb{P}^2 \vee (\bigvee_{i=1}^{n-1} S^3) \quad \textit{and} \quad \mathcal{G}_t(M) \simeq \mathcal{G}_t(C_{a\eta}) \times \prod_{i=1}^{n-1} \Omega^2 G,$$

where a is an odd integer, n is the rank of $H^2(M)$ and $C_{a\eta}$ is the cofiber of $a\eta$.

Proof. By Lemma 2.9 Σf is homotopic to $\sum_{i=1}^{m} \bar{\eta}_i$ for some m such that $1 \leq m \leq n$. By Lemma 2.10 there exists a map $\tilde{f}$ such that $\Sigma\tilde{f}$ is homotopic to $\bar{\eta}_1$ and its cofiber $\tilde{M}$ is homotopy equivalent to M. Replacing f by $\tilde{f}$ and M by $\tilde{M}$, we can assume $\Sigma f \simeq \bar{\eta}_i$. Use Theorem 2.4 by taking Y to be $\bigvee_{i=2}^{n} S_i^2$ and $\phi : \bigvee_{i=2}^{n} S_i^2 \to M$ to be the inclusion and get

$$\Sigma M \simeq \Sigma C_\phi \vee \left(\bigvee_{i=1}^{n-1} S^3\right) \quad \text{and} \quad \mathcal{G}_t(M) \simeq \mathcal{G}_t(C_\phi) \times \prod_{i=1}^{n-1} \Omega^2 G,$$

where C_ϕ is the cofiber of ϕ. We need to show that C_ϕ is homotopy equivalent to $C_{a\eta}$ for some odd integer a and ΣC_ϕ is homotopy equivalent to $\Sigma\mathbb{C}\mathbb{P}^2$.

Consider the cofibration diagram

$$\begin{tikzcd}
* \arrow[r] \arrow[d] & S^3 \arrow[r, equal] \arrow[d, "f"] & S^3 \arrow[d, "p_1\circ f"] \\
\bigvee_{i=2}^{n} S_i^2 \arrow[r, "\imath"] \arrow[d, equal] & \bigvee_{i=1}^{n} S_i^2 \arrow[r, "p_1"] \arrow[d] & S_1^2 \arrow[d] \\
\bigvee_{i=2}^{n} S_i^2 \arrow[r, "\phi"] & M \arrow[r] & C_\phi
\end{tikzcd}$$

where $\imath : \bigvee_{i=2}^{n} S_i^2 \to \bigvee_{i=1}^{n} S_i^2$ is the inclusion, $p_1 : \bigvee_{i=2}^{n} S_i^3 \to S_1^2$ is the pinch map. Since $p_1 \circ f$ is in $\pi_3(S^2) \cong \mathbb{Z}$, it is homotopic to $a\eta$ for some integer a. The right column implies that C_ϕ is homotopy equivalent to the cofiber $C_{a\eta}$ of $a\eta$. Moreover, Σf is homotopic to $\bar{\eta}_1$, so $\Sigma(p_1 \circ f)$ is homotopic to $\Sigma\eta$ and a is an odd number. It follows that ΣC_ϕ is homotopy equivalent to the cofiber of $\Sigma\eta$, which is $\Sigma\mathbb{C}\mathbb{P}^2$. □

We can modify the result of Lemma 2.11 a bit better.

Lemma 2.12. *Let a be an odd number. Then we have*

$$\mathcal{G}_t(C_{a\eta}) \times \Omega^2 G \simeq \mathcal{G}_t(\mathbb{C}\mathbb{P}^2) \times \Omega^2 G.$$

Proof. Let g be the composition

$$g : S^3 \xrightarrow{\sigma} S^3 \vee S^3 \xrightarrow{\eta\vee a\eta} S^2 \vee S^2$$

and let C_g be its cofiber. To distinguish the 2-spheres in the range, denote the i^{th} copy by S_i^2. Now we calculate $\mathcal{G}_t(C_g)$. By Lemma 2.10, we can assume that Σg is homotopic to $\bar{\eta}_1$ and $p_1 \circ g$ is homotopic to η. Use Theorem 2.4 by taking Y to be S_2^2 and $\varphi : S_2^2 \to S_1^2 \vee S_2^2$ to be the inclusion and obtain $\mathcal{G}_t(C_g) \simeq \mathcal{G}_t(C_g/S_2^2) \times \Omega^2 G$. Consider the homotopy cofibration diagram

$$\begin{tikzcd} * \arrow[r] \arrow[d] & S^2_2 \arrow[r, equal] \arrow[d, "j"] & S^2_2 \arrow[d, "\varphi"] \\ S^3 \arrow[r, "g"] \arrow[d, equal] & S^2_1 \vee S^2_2 \arrow[r] \arrow[d, "p_1"] & C_g \arrow[d] \\ S^3 \arrow[r, "\eta"] & S^2_1 \arrow[r] & \mathbb{CP}^2, \end{tikzcd}$$

where j is the inclusion and p_1 is the pinch map. The right column implies that C_g/S^2_2 is homotopy equivalent to $\mathbb{CP}^2$, so we have

$$\mathcal{G}_t(C_g) \simeq \mathcal{G}_t(\mathbb{CP}^2) \times \Omega^2 G.$$

Similarly, let g' be composition

$$g' : S^3 \xrightarrow{\sigma} S^3 \vee S^3 \xrightarrow{a\eta \vee \eta} S^2_1 \vee S^2_2$$

and let $C_{g'}$ be its cofiber. By Lemma 2.10, $\Sigma g'$ is $\bar{\eta}_1$ and $p_1 \circ g'$ is homotopic to $a\eta$ and $p_1 \circ g'$ is homotopic to $a\eta$. Use Theorem 3.1 by taking Y to be S^2_2 and $\varphi : S^2_2 \to S^2_1 \vee S^2_2$ and obtain $\mathcal{G}_t(M) \simeq \mathcal{G}_t(C_{g'}/S^2_2) \times \Omega^2 G$. Since $C_{g'}/S^2_2$ is homotopy equivalent to $C_{a\eta}$, we have

$$\mathcal{G}_t(C_{g'}) \simeq \mathcal{G}_t(C_{a\eta}) \times \Omega^2 G.$$

Observe that $g = T \circ g'$, where $T : S^2_1 \vee S^2_2 \to S^2_2 \vee S^2_1$ is the swapping map. Since T is a homotopy equivalence, $C_{g'}$ and C_g are homotopy equivalent. Combining the two homotopy equivalences gives the asserted lemma. □

Proposition 2.13. *Suppose that M is a 4-dimensional simply-connected Poincaré-complex. Let the rank of $H^2(M)$ be n. If Σf is null-homotopic, then there are homotopy equivalences*

$$\Sigma M \simeq S^5 \vee (\bigvee_{i=1}^{n} S^3) \quad \textit{and} \quad \mathcal{G}_t(M) \simeq \mathcal{G}_t(S^4) \times \prod_{i=1}^{n} \Omega^2 G.$$

If Σf is nontrivial, then there are homotopy equivalences

$$\Sigma M \simeq \Sigma\mathbb{CP}^2 \vee (\bigvee_{i=1}^{n-1} S^3) \quad \textit{and} \quad \mathcal{G}_t(M) \simeq \mathcal{G}_t(\mathbb{CP}^2) \times \prod_{i=1}^{n-1} \Omega^2 G.$$

Proof. If Σf is null-homotopic, then use Theorem 2.4 by taking Y to be M_3 and $\varphi : Y \to M_3$ to be the identity map and get the first two homotopy equivalences.

If Σf is nontrivial, by Lemma 2.11 there are homotopy equivalences

$$\Sigma M \simeq \Sigma\mathbb{CP}^2 \vee (\bigvee_{i=1}^{n-1} S^3) \quad \text{and} \quad \mathcal{G}_t(M) \simeq \mathcal{G}_t(C_{a\eta}) \times \prod_{i=1}^{n-1} \Omega^2 G,$$

where a is an odd integer. It suffices to show that we can replace $\mathcal{G}_t(C_{a\eta})$ by $\mathcal{G}_t(\mathbb{CP}^2)$.

When $n = 1$, the Poincaré complex condition implies that f has Hopf invariant equal to 1 or -1. Therefore M is homotopy equivalent to $\mathbb{CP}^2$ and the statement holds. When $n \geq 2$, there is at least one copy of $\Omega^2 G$ on the right hand side. By Lemma 2.12, we can replace $\mathcal{G}_t(C_{a\eta})$ by $\mathcal{G}_t(\mathbb{CP}^2)$ to obtain the proposition. □

It would be interesting to know if Lemma 2.12 can be improved to $\mathcal{G}_t(C_{a\eta}) \simeq \mathcal{G}_t(\mathbb{CP}^2)$. Observe that Proposition 2.13 is an improvement on the homotopy equivalence (1) for the non-spin case in [18], which gives a decomposition only after localization away from 2.

3. Gauge groups over non-simply-connected 4-manifolds

From now on we assume that M is an orientable, smooth, connected, closed 4 manifold. By Morse Theory, M admits a CW-structure with one 4-cell [14, Theorem 3.35]. In this section we calculate the homotopy types of ΣM and $\mathcal{G}_t(M)$ when $\pi_1(M)$ is (1) a free group $\mathbb{Z}^{*m}$, (2) a cyclic group $\mathbb{Z}/p^r\mathbb{Z}$, or (3) a free product of types $(\mathbb{Z}^{*m}) * (*_{j=1}^{n}\mathbb{Z}/p_j^{r_j}\mathbb{Z})$, where p and the p_j's are odd primes. Our strategy is to apply Theorem 2.4 and its variations to decompose $\mathcal{G}_t(M)$ into a product of a gauge group of a simply-connected space, whose homotopy type is worked out in Proposition 2.13, and some complementary factors that do not depend on t.

3.1. The case when $\pi_1(M) = \mathbb{Z}^{*m}$.

When $\pi_1(M)$ is a free group, M_3 is homotopy equivalent to a wedge sum of spheres [7]

$$M_3 \simeq (\bigvee_{i=1}^{m} S^3) \vee (\bigvee_{j=1}^{n} S^2) \vee (\bigvee_{k=1}^{m} S^1).$$

Using Theorem 2.4 we can calculate the homotopy types of ΣM and $\mathcal{G}_t(M)$.

Theorem 3.1. *Suppose $\pi_1(M) \cong \mathbb{Z}^{*m}$. Let the rank of $H^2(M)$ be n. If Σf is null-homotopic, then there are homotopy equivalences*

$$\Sigma M \simeq S^5 \vee (\bigvee_{i=1}^{m} S^4) \vee (\bigvee_{j=1}^{n} S^3) \vee (\bigvee_{k=1}^{m} S^2)$$
$$\mathcal{G}_t(M) \simeq \mathcal{G}_t(S^4) \times \prod_{i=1}^{m} \Omega^3 G \times \prod_{j=1}^{n} \Omega^2 G \times \prod_{k=1}^{m} \Omega G.$$

If Σf is nontrivial, then there are homotopy equivalences

$$\Sigma M \simeq \Sigma\mathbb{CP}^2 \vee (\bigvee_{i=1}^{m} S^4) \vee (\bigvee_{j=1}^{n-1} S^3) \vee (\bigvee_{k=1}^{m} S^2)$$
$$\mathcal{G}_t(M) \simeq \mathcal{G}_t(\mathbb{CP}^2) \times \prod_{i=1}^{m} \Omega^3 G \times \prod_{j=1}^{n-1} \Omega^2 G \times \prod_{k=1}^{m} \Omega G.$$

Proof. Denote the i^{th} copy of S^3 in M_3 by S_i^3 and the k^{th} copy of S^1 by S_k^1. We show that the inclusions $\varphi_i^3 : S_i^3 \to M_3$ and $\varphi_k^1 : S_k^1 \to M_3$ satisfy the hypothesis of Lemma 2.7 for all i and k.

Let $p_i^3 : M_3 \to S_i^3$ and $p_k^1 : M_3 \to S_k^1$ be the pinch maps. Then Σp_i^3 and Σp_k^1 are left homotopy inverses of $\Sigma\varphi_i^3$ and $\Sigma\varphi_k^1$. Moreover, $\Sigma p_i^3 \circ \Sigma f \simeq \Sigma(p_i^3 \circ f)$ is null-homotopic since f induces a trivial homomorphism $f^* : H^3(M_3) \to H^3(S^3)$, and $\Sigma p_k^1 \circ \Sigma f \simeq \Sigma(p_k^1 \circ f)$ is null-homotopic since $p_k^1 \circ f$ is null-homotopic by $\pi_3(S^1) = 0$. Apply Lemma 2.7 and get

$$\Sigma M \simeq \Sigma M' \vee (\bigvee_{i=1}^{m} S^4) \vee (\bigvee_{k=1}^{m} S^2) \quad \text{and} \quad \mathcal{G}_t(M) \simeq \mathcal{G}_t(M') \times \prod_{i=1}^{m} \Omega^3 G \times \prod_{k=1}^{m} \Omega G.$$

where M' is the cofiber of the inclusion $(\bigvee_{i=1}^m S^1) \vee (\bigvee_{k=1}^m S^3) \hookrightarrow M$. Observe that the 3-skeleton of M' is homotopy equivalent to $\bigvee_{i=1}^n S^2$. Since $H^2(S_i^1)$ and $H^2(S_k^3)$ are zero, Lemma 2.6 implies that M' satisfies Poincaré Duality. Let f' be the attaching map of the 4-cell in M_3'. By Proposition 2.13, if $\Sigma f'$ is null-homotopic, then we have

$$\Sigma M' \simeq S^5 \vee (\bigvee_{i=1}^n S^3) \quad \text{and} \quad \mathcal{G}_t(M') \simeq \mathcal{G}_t(S^4) \times \prod_{i=1}^n \Omega^2 G.$$

If $\Sigma f'$ is nontrivial, then we have

$$\Sigma M' \simeq \Sigma\mathbb{CP}^2 \vee (\bigvee_{i=1}^{n-1} S^3) \quad \text{and} \quad \mathcal{G}_t(M') \simeq \mathcal{G}_t(\mathbb{CP}^2) \times \prod_{i=1}^{n-1} \Omega^2 G.$$

By Lemma 2.5 $\Sigma f'$ is null-homotopic if and only if Σf is null-homotopic. Combining these homotopy equivalences gives the theorem. $\square$

3.2. **The case when $\pi_1(M) = \mathbb{Z}/p^r\mathbb{Z}$.** Recall that an n-dimensional Moore space $P^n(k)$ is the cofiber of the degree-k map $S^{n-1} \xrightarrow{k} S^{n-1}$ for $n \geq 2$. With integral coefficients, $\tilde{H}_i(P^n(k))$ is $\mathbb{Z}/k\mathbb{Z}$ for $i = n-1$ and is zero otherwise. With mod-k coefficients, $\tilde{H}_i(P^n(k);\mathbb{Z}/k\mathbb{Z})$ is $\mathbb{Z}/k\mathbb{Z}$ for $i = n-1$ and n and is zero otherwise. Let u be a generator of $\tilde{H}_{n-1}(P^n(k))$. By the Universal Coefficient Theorem, $\tilde{H}_{n-1}(P^n(k);\mathbb{Z}/k\mathbb{Z})$ is generated by the mod-k reduction $\bar{u}$ of u, and $\tilde{H}_n(P^n(k);\mathbb{Z}/k\mathbb{Z})$ is generated by $\bar{v} = \beta\bar{u}$, where β is the Bockstein homomorphism.

For any space X, the mod-k homotopy group $\pi_n(X;\mathbb{Z}/k\mathbb{Z})$ is defined to be $[P^n(k), X]$. When $n \geq 3$, $\pi_n(X;\mathbb{Z}/k\mathbb{Z})$ has a group structure induced by the comultiplication of $P^n(k)$ and when $n \geq 4$, $\pi_n(X;\mathbb{Z}/k\mathbb{Z})$ is abelian. There are two associated homomorphisms: the mod-k Hurewicz homomorphism

$$\bar{h} : \pi_n(X;\mathbb{Z}/k\mathbb{Z}) \to H_n(X;\mathbb{Z}/k\mathbb{Z}),$$

which is defined to be $\bar{h}(f) = f_*(\bar{v})$ for $f \in \pi_n(X;\mathbb{Z}/k\mathbb{Z})$, and the homotopy Bockstein homomorphism

$$\bar{\beta}_\pi : \pi_n(X;\mathbb{Z}/k\mathbb{Z}) \to \pi_{n-1}(X),$$

which is defined to be $\bar{\beta}_\pi(f) = \imath^* \circ f$ and $\imath : S^{n-1} \to P^n(k)$ is the inclusion. They are compatible with the standard Hurewicz homomorphism h and Bockstein homomorphisms β in the commutative diagram [15]

$$\begin{tikzcd} \cdots \arrow[r] & \pi_{n+1}(X;\mathbb{Z}/k\mathbb{Z}) \arrow[r, "\beta_\pi"] \arrow[d, "\bar{h}"] & \pi_n(X) \arrow[r, "k"] \arrow[d, "h"] & \pi_n(X) \arrow[r] \arrow[d, "h"] & \pi_n(X;\mathbb{Z}/k\mathbb{Z}) \arrow[r] \arrow[d, "\bar{h}"] & \cdots \\ \cdots \arrow[r] & H_{n+1}(X;\mathbb{Z}/k\mathbb{Z}) \arrow[r, "\beta"] & H_n(X) \arrow[r, "k"] & H_n(X) \arrow[r] & H_n(X;\mathbb{Z}/k\mathbb{Z}) \arrow[r] & \cdots \end{tikzcd} \tag{6}$$

For any map $g : P^3(k) \to P^3(k)$, let C_g be its cofiber, and let a and b be generators of $H^2(C_g;\mathbb{Z}/k\mathbb{Z})$ and $H^4(C_g;\mathbb{Z}/k\mathbb{Z})$. Then the mod-$k$ Hopf invariant $\bar{H}(g) \in \mathbb{Z}/k\mathbb{Z}$ is defined by the formula $a \cup a \equiv \bar{H}(g)b \pmod{k}$.

Lemma 3.2. [16, Corollary 11.12] *Let p be an odd prime and let $g : P^3(p^r) \to P^3(p^r)$ be a map in the kernel of $\bar{h}$. Then g is null-homotopic if and only if its mod-p^r Hopf invariant $\bar{H}(g)$ is zero.*

Back to the calculation of $\mathcal{G}_t(M)$. When $\pi_1(M) = \mathbb{Z}/p^r\mathbb{Z}$, Poincaré Duality and the Universal Coefficient Theorem imply the homology groups of M are as follows:

$$H_i(M;\mathbb{Z}) = \begin{cases} \mathbb{Z} & i = 0, 4 \\ \mathbb{Z}/p^r\mathbb{Z} & i = 1 \\ \mathbb{Z}^{\oplus n} \oplus \mathbb{Z}/p^r\mathbb{Z} & i = 2 \\ 0 & \text{else.} \end{cases}$$

Now we calculate the homotopy type of ΣM to find a possible subcomplex Y satisfying the hypothesis of Theorem 2.4.

Lemma 3.3. *Let p be an odd prime. Then $\pi_4(P^4(p^r))$ and $\pi_4(P^3(p^r))$ are trivial.*

Proof. After localizing away from p, any $P^n(p^r)$ is contractible so $\pi_4(P^4(p^r))$ and $\pi_4(P^3(p^r))$ are p-torsion. Localize at p and consider the long exact sequence of homotopy groups for the pair $(P^4(p^r), S^3)$:

$$\cdots \longrightarrow \pi_4(S^3) \longrightarrow \pi_4(P^4(p^r)) \xrightarrow{j_*} \pi_4(P^4(p^r), S^3) \longrightarrow \cdots$$

The pair $(P^4(p^r), S^3)$ is 3-connected, so $\pi_4(P^4(p^r), S^3)$ is $\mathbb{Z}$ by Hurewicz Theorem. Since $\pi_4(S^3)$ is trivial at odd primes, j_* is an injection. But $\pi_4(P^4(p^r))$ is torsion, so this injection only makes sense if $\pi_4(P^4(p^r))$ is trivial.

Now we calculate $\pi_4(P^3(p^r))$. Let $F^3\{p^r\}$ be the homotopy fiber of the pinch map $P^3(p^r) \longrightarrow S^3$. Then $\pi_4(P^3(p^r))$ equals $\pi_4(F^3\{p^r\})$ since $\pi_4(\Omega S^3)$ and $\pi_4(S^3)$ are trivial at odd primes. By [15, Proposition 11.7.1], there is a p-local homotopy equivalence

$$\Omega F^3\{p^r\} \simeq S^1 \times \Omega\Sigma(\bigvee_{\alpha} P^{n_\alpha}(p^r)) \times \prod_{j} S^{2p^j-1}\{p^{r+1}\},$$

where n_α is either 3 or greater than 4, and $S^n\{p^{r+1}\}$ is the homotopy fiber of the degree map $p^{r+1} : S^n \longrightarrow S^n$. Since $\pi_4(F^3\{p^r\})$ is $\pi_3(\Omega F^3\{p^r\})$, we need to calculate the third homotopy group of each factor on the right hand side. The first factor $\pi_3(S^1)$ and the last factor is trivial since $S^{2p^j-1}\{p^{r+1}\}$ is 3-connected for $j \geq 1$. For the remaining factor of $\Omega F^3\{p^r\}$, consider the string of isomorphisms

$$\begin{aligned} \pi_3(\Omega\Sigma(\bigvee_{\alpha} P^{n_\alpha}(p^r))) &\cong \pi_3(\prod_{\alpha} \Omega\Sigma P^{n_\alpha}(p^r)) \\ &\cong \pi_3(\Omega\Sigma P^3(p^r)) \\ &\cong \pi_4(P^4(p^r)) \\ &\cong 0. \end{aligned}$$

The first isomorphism is obtained from the Hilton-Milnor Theorem using dimension and connectivity considerations. The second isomorphism holds since only one n_α equals 3 while the rest are strictly larger than 4. The third isomorphism holds by adjunction, and the fourth isomorphism holds as we have already seen that $\pi_4(P^4(p^r)) = 0$. Therefore $\pi_4(F^3\{p^r\})$ is trivial and so is $\pi_4(P^3(p^r))$. □

Lemma 3.4. *If $\pi_1(M) \cong \mathbb{Z}/p^r\mathbb{Z}$, then there is a homotopy equivalence*

$$\Sigma M_3 \simeq P^4(p^r) \vee (\bigvee_{i=1}^{n} S^3) \vee P^3(p^r).$$

Proof. Since ΣM_3 is simply-connected, it has a minimal cell structure. By [6, Proposition 4H.3] ΣM_3 is homotopy equivalent to the cofiber of a map

$$g : P^3(p^r) \vee (\bigvee_{i=1}^{n} S^2) \to P^3(p^r)$$

such that g induces a trivial homomorphism $g_* : H_2(P^3(p^r) \vee (\bigvee_{i=1}^{n} S^2)) \to H_2(P^3(p^r))$. We claim that g is null-homotopic. To distinguish the 2-spheres in the wedge sum, denote the i^{th} copy by S^2_i. Since

$$[P^3(p^r) \vee (\bigvee_{i=1}^{n} S^2_i), P^3(p^r)] \cong [P^3(p^r), P^3(p^r)] \oplus \left(\bigoplus_{i=1}^{n}[S^2_i, P^3(p^r)]\right),$$

we write $g = g' \oplus (\bigoplus_{i=1}^{n} g''_i)$, where $g' \in [P^3(p^r), P^3(p^r)]$ and $g''_i \in [S^2_i, P^3(p^r)]$.

Consider the commutative diagram

$$\begin{tikzcd}
\pi_2(S^2_i) \arrow[r, "g''_i"] \arrow[d, "h"'] & \pi_2(P^3(p^r)) \arrow[d, "h"'] \\
H_2(S^2_i) \arrow[r, "(g''_i)_*"] & H_2(P^3(p^r)).
\end{tikzcd}$$

Both Hurewicz homomorphisms h are isomorphisms by Hurewicz Theorem. Since g_* is trivial, so is $(g''_i)_*$. The diagram implies that g''_i is null-homotopic. Therefore ΣM_3 is homotopy equivalent to $C_{g'} \vee (\bigvee_{i=1}^{n} S^3)$, where $C_{g'}$ is the cofiber of g'. It suffices to show that g' is null-homotopic.

Consider the commutative diagram

$$\begin{tikzcd}
\pi_3(P^3(p^r); \mathbb{Z}/p^r\mathbb{Z}) \arrow[r, "\bar{h}"] \arrow[d, "\beta_\pi"] & H_3(P^3(p^r); \mathbb{Z}/p^r\mathbb{Z}) \arrow[d, "\beta"] \\
\pi_2(P^3(p^r)) \arrow[r, "h"] & H_2(P^3(p^r))
\end{tikzcd}$$

from (6). The induced homomorphism g'_* is trivial and we have

$$h \circ \beta_\pi(g') = \beta \circ \bar{h}(g') = \beta \circ ((g')_* v) = 0.$$

Observe that $\beta : H_3(P^3(p^r); \mathbb{Z}/p^r\mathbb{Z}) \to H_2(P^3(p^r))$ is an isomorphism in this case, so g' is in the kernel of $\bar{h}$. Since $C_{g'}$ retracts off the suspension ΣM_3, it is a co-H-space and $H^*(C_{g'}; \mathbb{Z}/p^r\mathbb{Z})$ has trivial cup products. Therefore the mod-p^r Hopf invariant $\bar{H}(g')$ is zero and g' is null-homotopic by Lemma 3.2. □

Lemma 3.4 says that ΣM_3 contains $P^3(p^r) \vee P^4(p^r)$ as its wedge summands. This, however, does not necessarily imply that M_3 contains $P^2(p^r) \vee P^3(p^r)$ since M is not simply-connected.

Lemma 3.5. *If $\pi_1(M) \cong \mathbb{Z}/p^r\mathbb{Z}$, then there exists a map $\epsilon : P^2(p^r) \to M_3$ satisfying the hypothesis of Theorem 2.4 and its cofiber C_ϵ is simply-connected.*

Proof. By the Cellular Approximation Theorem, $\pi_1(M_3)$ equals $\pi_1(M) \cong \mathbb{Z}/p^r\mathbb{Z}$. Let $j : S^1 \to M_3$ represent a generator of $\pi_1(M_3)$. It has order p^r, so there exists an extension $\epsilon : P^2(p^r) \to M_3$.

$$\begin{tikzcd} S^1 \arrow[r, "p^r"] & S^1 \arrow[r] \arrow[d, "j"] & P^2(p^r) \arrow[dl, dashed, "\epsilon"] \\ & M_3 & \end{tikzcd}$$

Since $\pi_1(M_3)$ is abelian, $H_1(M_3)$ is $\pi_1(M_3) \cong \mathbb{Z}/p^r\mathbb{Z}$ by Hurewicz Theorem and the induced map $\epsilon_* : H_1(P^2(p^r)) \to H_1(M_3)$ is an isomorphism. Therefore the cofiber C_ϵ of ϵ has $H_1(C_\epsilon) = 0$, implying that C_ϵ is simply-connected.

Now we show that $\Sigma\epsilon$ has a left homotopy inverse. Let ψ be the composition

$$\psi : \Sigma M_3 \simeq P^4(p^r) \vee (\bigvee_{i=1}^{n} S_i^3) \vee P^3(p^r) \xrightarrow{pinch} P^3(p^r).$$

Observe that

$$\psi_* : H_2(\Sigma M_3; \mathbb{Z}/p^r\mathbb{Z}) \to H_2(P^3(p^r); \mathbb{Z}/p^r\mathbb{Z}) \quad \text{and} \quad (\Sigma\epsilon)_* : H_2(P^3(p^r); \mathbb{Z}/p^r\mathbb{Z}) \to H_2(\Sigma M_3; \mathbb{Z}/p^r\mathbb{Z})$$

are isomorphism, so $(\psi \circ \Sigma\epsilon)_* : H_2(P^3(p^r); \mathbb{Z}/p^r\mathbb{Z}) \to H_2(P^3(p^r); \mathbb{Z}/p^r\mathbb{Z})$ is an isomorphism. Then Bockstein homomorphism implies $(\psi \circ \Sigma\epsilon)_* : H_3(P^3(p^r); \mathbb{Z}/p^r\mathbb{Z}) \to H_3(P^3(p^r); \mathbb{Z}/p^r\mathbb{Z})$ is an isomorphism. Therefore $\psi \circ \Sigma\epsilon$ is a homotopy equivalence and ψ is a left homotopy inverse of $\Sigma\epsilon$.

Moreover, the composition $\psi \circ \Sigma f$ is null-homotopic since $\pi_4(P^3(p^r))$ is trivial by Lemma 3.3. Therefore ϵ satisfies the hypothesis of Theorem 2.4. □

Denote the pointed mapping space $Map^*(P^n(p^r), G)$ by $\Omega^n G\{p^r\}$. This notation is justified since $Map^*(P^n(p^r), G)$ is the homotopy fiber of the power map $p^r : \Omega^n G \to \Omega^n G$. Now we calculate the homotopy type of $\mathcal{G}_t(M)$.

Theorem 3.6. *Suppose $\pi_1(M) \cong \mathbb{Z}/p^r\mathbb{Z}$ where p is an odd prime. Let the rank of $H^2(M)$ be n. If Σf is null-homotopic, then there are homotopy equivalences*

$$\Sigma M \simeq S^5 \vee P^4(p^r) \vee (\bigvee_{i=1}^{n} S^3) \vee P^3(p^r)$$

$$\mathcal{G}_t(M) \simeq \mathcal{G}_t(S^4) \times \Omega^3 G\{p^r\} \times \prod_{i=1}^{n} \Omega^2 G \times \Omega^2 G\{p^r\}.$$

If Σf is nontrivial, then there are homotopy equivalences

$$\Sigma M \simeq \Sigma\mathbb{CP}^2 \vee P^4(p^r) \vee (\bigvee_{i=1}^{n-1} S^3) \vee P^3(p^r)$$

$$\mathcal{G}_t(M) \simeq \mathcal{G}_t(\mathbb{CP}^2) \times \Omega^3 G\{p^r\} \times \prod_{i=1}^{n-1} \Omega^2 G \times \Omega^2 G\{p^r\}.$$

Proof. We decompose $\mathcal{G}_t(M)$ in the following steps.

Step 1: By Lemma 3.5, there exists a map $\epsilon : P^2(p^r) \to M_3$ satisfying the hypothesis of Theorem 2.4. Apply the theorem and get

$$\Sigma M \simeq \Sigma M' \vee P^3(p^r) \quad \text{and} \quad \mathcal{G}_t(M) \simeq \mathcal{G}_t(M') \times \Omega^2 G\{p^r\}.$$

where M' is the cofiber of $P^2(p^r) \xrightarrow{\epsilon} M_3 \hookrightarrow M$.

Step 2: By Lemma 3.5 M' is simply-connected, so its 3-skeleton M_3' is $(\bigvee_{i=1}^n)S^2 \vee P^3(p^r)$. We show that the inclusion $\varphi : P^3(p^r) \hookrightarrow M_3'$ satisfies the condition of Theorem 2.4. The pinch map $\Sigma M_3' \to P^4(p^r)$ is a left homotopy inverse of $\Sigma\varphi$. Let f' be the attaching map of the 4-cell in M'. Then the composition

$$S^4 \xrightarrow{\Sigma f'} \Sigma M_3' \xrightarrow{pinch} P^4(p^r)$$

is null-homotopic since $\pi_4(P^4(p^r))$ is trivial by Lemma 3.3. Apply Theorem 2.4 and get

$$\Sigma M' \simeq \Sigma M'' \vee P^4(p^r) \quad \text{and} \quad \mathcal{G}_t(M') \simeq \mathcal{G}_t(M'') \times \Omega^3 G\{p^r\}.$$

where M'' is $M'/P^3(p^r)$.

Step 3: By Lemma 3.4 the 3-skeleton of M'' is homotopy equivalent to $\bigvee_{i=1}^n S^2$. Since $H^2(P^2(p^r))$ and $H^2(P^3(p^r))$ are torsion, Lemma 2.6 implies that M'' satisfies Poincaré Duality. Let f'' be the attaching map of the 4-cell in M''. By Proposition 2.13, if $\Sigma f''$ is null-homotopic, then

$$\Sigma M'' \simeq S^5 \vee (\bigvee_{i=1}^n S^3) \quad \text{and} \quad \mathcal{G}_t(M'') \simeq \mathcal{G}_t(S^4) \times \prod_{i=1}^n \Omega^2 G$$

If $\Sigma f''$ is nontrivial, then

$$\Sigma M'' \simeq \Sigma\mathbb{CP}^2 \vee (\bigvee_{i=1}^{n-1} S^3) \quad \text{and} \quad \mathcal{G}_t(M'') \simeq \mathcal{G}_t(\mathbb{CP}^2) \times \prod_{i=1}^{n-1} \Omega^2 G$$

Step 4: Combining all the homotopy equivalences from Step 1 to 3 gives the theorem. $\square$

3.3. **The case when $\pi_1 = (\mathbb{Z}^{*m}) * (*_{j=1}^n \mathbb{Z}/p_j^{r_j}\mathbb{Z})$.** Suppose that $\pi_1(M)$ is $(\mathbb{Z}^{*m}) * (*_{j=1}^n \mathbb{Z}/p_j^{r_j}\mathbb{Z})$, where p_j is an odd prime. The Stable Decomposition Theorem [13, Theorem 1.3] implies that for some number d there is a diffeomorphism

$$M\#_d(S^2 \times S^2) \cong N\#(\#_{j=1}^n L_j), \tag{7}$$

where N and L_j's are orientable, smooth, connected, closed 4-manifolds with $\pi_1(N) = \mathbb{Z}^{*m}$ and $\pi_1(L_j) = \mathbb{Z}/p_j^{r_j}\mathbb{Z}$. We will calculate the suspensions and gauge groups for both sides of isomorphism (7).

Lemma 3.7. *There are homotopy equivalences*

$$\Sigma(M\#_d(S^2 \times S^2)) \simeq (\Sigma M) \vee (\bigvee_{s=1}^{2d} S^3) \quad \textit{and} \quad \mathcal{G}_t(M\#_d(S^2 \times S^2)) \simeq \mathcal{G}_t(M) \times \prod_{s=1}^{2d} \Omega^2 G.$$

Proof. By induction it suffices to show the lemma for the $d = 1$ case. We use Lemma 2.8 to prove it. For M, the inclusion $\varphi : \{m_0\} \hookrightarrow M$ of the basepoint m_0 obviously satisfies the hypothesis of Theorem 2.4. For $S^2 \times S^2$, take the inclusion $\varphi' : S^2 \vee S^2 \hookrightarrow S^2 \times S^2$ to be the inclusion of the 3-skeleton. Since $\Sigma(S^2 \times S^2)$ is homotopy equivalent to $S^5 \vee S^3 \vee S^3$, φ' satisfies the hypothesis as well. Apply Lemma 2.8 to get

$$\Sigma(M\#(S^2 \times S^2)) \simeq (\Sigma C_j) \vee (\bigvee_{s=1}^2 S^3) \quad \text{and} \quad \mathcal{G}_t(M\#(S^2 \times S^2)) \simeq \mathcal{G}_t(C_j) \times \prod_{s=1}^2 \Omega^2 G,$$

where C_j is the cofiber of the inclusion $j : \{m_0\} \vee (S^2 \vee S^2) \hookrightarrow M\#(S^2 \times S^2)$. Consider the cofibration diagram

$$\begin{tikzcd}
* \arrow[r] \arrow[d] & S^3 \arrow[r, equal] \arrow[d, "f"] & S^3 \arrow[d, "f_M"] \\
S^2 \vee S^2 \arrow[r] \arrow[d, equal] & M_3 \vee S^2 \vee S^2 \arrow[r] \arrow[d] & M_3 \arrow[d] \\
S^2 \vee S^2 \arrow[r, "j"] & M\#(S^2 \times S^2) \arrow[r] & M
\end{tikzcd}$$

where f_M is the attaching map of the 4-cell in M. The bottom row implies that C_j is homotopy equivalent to M and the asserted homotopy equivalences follow. $\square$

Theorem 3.8. *Let $\pi_1(M) \cong (\mathbb{Z}^{*m}) * (*_{j=1}^{n}\mathbb{Z}/p_j^{r_j}\mathbb{Z})$ where p_j is an odd prime and let l be the rank of $H^2(M)$. Then there exists a number d such that $\Sigma(M\#_d(S^2 \times S^2))$ is homotopy equivalent to either*

$$S^5 \vee (\bigvee_{i=1}^{m} S^4) \vee (\bigvee_{j=1}^{n} P^4(p_j^{r_j})) \vee (\bigvee_{k=1}^{l+2d} S^3) \vee (\bigvee_{j'=1}^{n} P^3(p_{j'}^{r_{j'}})) \vee (\bigvee_{i'=1}^{m} S^2) \qquad \textit{or}$$

$$\Sigma\mathbb{C}\mathbb{P}^2 \vee (\bigvee_{i=1}^{m} S^4) \vee (\bigvee_{j=1}^{n} P^4(p_j^{r_j})) \vee (\bigvee_{k=1}^{l+2d-1} S^3) \vee (\bigvee_{j'=1}^{n} P^3(p_{j'}^{r_{j'}})) \vee (\bigvee_{i'=1}^{m} S^2).$$

In the first case, we have

$$\mathcal{G}_t(M) \times \prod_{s=1}^{2d} \Omega^2 G \simeq \mathcal{G}_t(S^4) \times \prod_{i=1}^{m} \Omega^3 G \times \prod_{j=1}^{n} \Omega^3 G\{p_j^{r_j}\} \times \prod_{k=1}^{l+2d} \Omega^2 G \times \prod_{j'=1}^{n} \Omega^2 G\{p_{j'}^{r_{j'}}\} \times \prod_{i'=1}^{m} \Omega G.$$

In the second case, we have

$$\mathcal{G}_t(M) \times \prod_{s=1}^{2d} \Omega^2 G \simeq \mathcal{G}_t(\mathbb{C}\mathbb{P}^2) \times \prod_{i=1}^{m} \Omega^3 G \times \prod_{j=1}^{n} \Omega^3 G\{p_j^{r_j}\} \times \prod_{k=1}^{l+2d-1} \Omega^2 G \times \prod_{j'=1}^{n} \Omega^2 G\{p_{j'}^{r_{j'}}\} \times \prod_{i'=1}^{m} \Omega G.$$

Proof. Denote $N\#(\#_{j=1}^{n} L_j)$ by X. Using the stable decomposition (7) and Lemma 3.7, we only need to calculate ΣX and $\mathcal{G}_t(X)$. Let N_3 and $(L_j)_3$ be the 3-skeletons of N and L_j. Then the 3-skeleton X_3 of X is the wedge sum $N_3 \vee (\bigvee_{j=1}^{n} (L_j)_3)$.

Step 1: The 3-skeleton N_3 is homotopy equivalent to $(\bigvee_{i=1}^{m} S^3) \vee (\bigvee_{j'=1}^{l'} S^2) \vee (\bigvee_{k=1}^{m} S^1)$. Denote the i^{th} copy of S^3 in N_3 by S_i^3 and the k^{th} copy of S^1 by S_k^1. In the proofs of Theorem 3.1, we show that inclusions $\varphi_i^3 : S_i^3 \to N_3$ and $\varphi_k^1 : S_k^1 \to N_3$ satisfy the hypothesis of Theorem 2.4. For each j, by Lemma 3.5 there exists a map $\epsilon_j : P^2(p_j^{r_j}) \to (L_j)_3$ satisfying the hypothesis as well. Apply Lemma 2.8 and get

$$\Sigma X \simeq \Sigma X' \vee \left(\bigvee_{i=1}^{m} S^4\right) \vee \left(\bigvee_{k=1}^{m} S^2\right) \vee \left(\bigvee_{j=1}^{n} P^3(p_j^{r_j})\right)$$

$$\mathcal{G}_t(X) \simeq \mathcal{G}_t(X') \times \prod_{i=1}^{m} \Omega^3 G \times \prod_{k=1}^{m} \Omega G \times \prod_{j=1}^{n} \Omega G\{p_j^{r_j}\}$$

where X' is the cofiber of the inclusion $(\bigvee_{i=1}^{m} S^3) \vee (\bigvee_{k=1}^{m} S^1) \vee \left(\bigvee_{j=1}^{n} P^2(p_j^{r_j})\right) \hookrightarrow X$.

Step 2: Since X' is simply-connected, its 3-skeleton X'_3 has a minimal cell structure

$$X_3 \simeq \left(\bigvee_{j'=1}^{l''} S^2\right) \vee \left(\bigvee_{j=1}^{n} P^3(p_j^{r_j})\right).$$

We show that inclusions $\varphi_j : P^3(p_j^{r_j}) \to X'_3$ satisfy the hypothesis of Theorem 2.4. For each j, the pinch maps $\Sigma X'_3 \to P^4(p_j^{r_j})$ is a left homotopy inverse of $\Sigma\varphi_j$. Let f' be the attaching map of the 4-cell in X'. Then the composition

$$S^4 \xrightarrow{\Sigma f'} \Sigma X'_3 \xrightarrow{pinch} P^4(p_j^{r_j})$$

is null-homotopic since $\pi_4(P^3(p_j^{r_j}))$ is trivial by Lemma 3.3. Apply Lemma 2.7 and get

$$\Sigma X' \simeq \Sigma X'' \vee \left(\bigvee_{j=1}^{n} P^3(p_j^{r_j})\right) \quad \text{and} \quad \mathcal{G}_t(X') \simeq \mathcal{G}_t(X'') \times \prod_{j=1}^{n} \Omega^2 G\{p_j^{r_j}\},$$

where X'' is $X'/(\bigvee_{j=1}^{n} P^3(p_j^{r_j}))$.

Step 3: The 3-skeleton of X'' is homotopy equivalent to $\bigvee_{j'=1}^{l''} S^2$. Since the subcomplexes $S_i^1, S_k^3, P^2(p_j^{r_j})$ and $P^3(p_j^{r_j})$ in Step 1 and 2 have either zero or torsion second cohomology groups, Lemma 2.6 implies that X'' satisfies Poincaré Duality. By Proposition 2.13, we have

$$\Sigma X'' \simeq S^5 \vee \left(\bigvee_{j'=1}^{l''} S^3\right) \quad \text{and} \quad \mathcal{G}_t(X'') \simeq \mathcal{G}_t(S^4) \times \prod_{j'=1}^{l''} \Omega^2 G$$

or

$$\Sigma X'' \simeq \Sigma\mathbb{CP}^2 \vee \left(\bigvee_{j'=1}^{l''-1} S^3\right) \quad \text{and} \quad \mathcal{G}_t(X'') \simeq \mathcal{G}_t(\mathbb{CP}^2) \times \prod_{j'=1}^{l''-1} \Omega^2 G.$$

Step 4: Combining all homotopy equivalences from Step 1 to 3, the stable decomposition (7) and Lemma 3.7 together imply the theorem. Furthermore, $H_2(M\#_d(S^2\times S^2))$ has rank $l = 2d$, so $l'' = l + 2d$. □

Remark: The proofs of Theorem 3.1 and 3.6 are also valid for any orientable 4-dimensional CW-complex with one 4-cell, while Theorem 3.8 requires the smoothness of M for the stable diffeomorphism splitting in [13]. One can ask under what condition the stabilizing factor $\#_d(S^2 \times S^2)$ can be cancelled so that the factors $\prod_{s=1}^{2d} \Omega^2 G$ can be removed from the equation. If this can be achieved, Theorem 3.1 and 3.6 will be corollaries of Theorem 3.8 when M is smooth.

References

[1] M. Atiyah and R. Bott, *The Yang-Mills equations over Riemann surfaces*, Philosophical Transactions of the Royal Society of London. Series A, Mathematical and Physical Sciences, **308**(1505), (1983), 523 – 615.

[2] F. Cohen, J. Moore, J. Neisendorfer, *Torsion in homotopy groups*, Annals of Mathematics, Second Series, **109**, (1979), 121 – 168.

[3] H. Hamanaka and A. Kono, *Unstable K-group and homotopy type of certain gauge groups*, Proceedings of the Royal Society of Edinburgh Section A: Mathematics, **136**, (2006), 149 – 155.

[4] D. Gottlieb, *Applications of bundle map theory*, Transaction of the American Mathematical Society, **171**, (1972), 23 – 50.

[5] M. Greenberg and J. Harper, *Algebraic Topology: A First Course*, Benjamin/Cummings Publishing Company, 1981.
[6] A. Hatcher, *Algebraic Topology*, Cambridge University Press, 2002.
[7] T. Matumoto and A. Katanaga, *On 4-dimensional closed manifolds with free fundamental groups*, Hiroshima Math J, **25**, (1995), 367 – 370.
[8] D. Kishimoto, A. Kono, *On the homotopy types of $Sp(n)$ gauge groups*, arXiv:1803.06477.
[9] D. Kishimoto, A. Kono, M. Tsutaya, *On p-local homotopy types of gauge groups*, Proceedings of the Royal Society of Edinburgh Section A: Mathematics, **144**, (2014), 149 – 160.
[10] D. Kishimoto, S. Theriault, M. Tsutaya, *The homotopy types of G_2-gauge groups*, Topology and its Applications, **228**, (2017), 92 – 107.
[11] A. Kono, *A note on the homotopy type of certain gauge groups*, Proceedings of the Royal Society of Edinburgh Section A: Mathematics, **117**, (1991), 295 – 297.
[12] A. Kono and S. Tsukuda, *A remark on the homotopy type of certain gauge groups*, Journal of Mathematics of Kyoto University, **36**, (1996), 115 – 121.
[13] M. Kreck, W. Lück and P. Teichner, Stable prime decompositions of 4-manifolds *Prospects in Topology* edited by F. Quinn, Annuals of Mathematics Studies, **138**, Princeton University Press, Princeton, (1995), 251 – 269.
[14] Y. Matsumoto, *An Introduction to Morse Theory (Translations of Mathematical Monographs)*, American Mathematical Society, 2002.
[15] J. Neisendorfer, *Algebraic Methods in Unstable Homotopy Theory*, Cambridge University Press, 2010.
[16] J. Neisendorfer, *Homotopy groups with coefficients*, Journal of Fixed Point Theory and Applications, **8**, (2010), 247 – 338.
[17] W. Sutherland, *Function spaces related to gauge groups*, Proceedings of the Royal Society of Edinburgh, **121A**, (1992), 185 – 190.
[18] S. Theriault, *Odd primary homotopy decompositions of gauge groups*, Algebraic and Geometric Topology, **10**, (2010), 535 – 564.
[19] S. Theriault, *The homotopy types of $Sp(2)$-gauge groups*, Kyoto Journal of Mathematics, **50**, (2010), 591 – 605.
[20] S. Theriault, *Homotopy types of $SU(3)$-gauge groups*, Publ. RIMS Kyoto Univ., **48**, (2012), 543 – 563.
[21] S. Theriault, *The homotopy types of $SU(5)$-gauge groups*, Osaka Journal of Mathematics, **52**, (2015), 15 – 31.
[22] S. Theriault, *Odd primary homotopy types of $SU(n)$-gauge groups*, Algebraic and Geometric Topology, **17**, (2017), 1131 – 1150.

Mathematical Sciences, University of Southampton, SO17 1BJ, UK
E-mail address: `tls1g14@soton.ac.uk`

THE ODD PRIMARY ORDER OF THE COMMUTATOR ON LOW RANK LIE GROUPS

TSELEUNG SO

Abstract. Let G be a simple, simply-connected, compact Lie group of low rank relative to a fixed prime p. After localization at p, there is a space A which "generates" G in a certain sense. Assuming G satisfies a homotopy nilpotency condition relative to p, we show that the Samelson product $\langle \mathbb{1}_G, \mathbb{1}_G\rangle$ of the identity of G equals the order of the Samelson product $\langle \imath, \imath\rangle$ of the inclusion $\imath : A \to G$. Applying this result, we calculate the orders of $\langle \mathbb{1}_G, \mathbb{1}_G\rangle$ for all p-regular Lie groups and give bounds of the orders of $\langle \mathbb{1}_G, \mathbb{1}_G\rangle$ for certain quasi-p-regular Lie groups.

1. Introduction

In this paper, G is a simple, simply-connected, compact Lie group and p is an odd prime. By a theorem of Hopf, G is rationally homotopy equivalent to a product of spheres $\prod_{i=1}^{l} S^{2n_i-1}$, where $n_1 \le \cdots \le n_l$. The sequence $(2n_1 - 1, \cdots, 2n_l - 1)$ is called the *type* of G. Localized at p, it is known [2, 7] that G is homotopy equivalent to a product of H-spaces $\prod_{i=1}^{p-1} B_i$, and there exists a co-H-space A and a map $\imath : A \to G$ such that $H_*(G)$ is the exterior algebra generated by $\imath_*(\tilde{H}_*(A))$. For $1 \le i \le l$, if B_i is S^{2n_i-1}, then we call G *p-regular*. If each B_i is either S^{2n_i-1} or $B(2n_i-1, 2n_i+2p-3)$ that is the S^{2n_i-1}-bundle over S^{2n_i+2p-3} classified by $\frac{1}{2}\alpha \in \pi_{2n_i+2p-4}(S^{2n_i-1})$, then we call G *quasi-p-regular*.

For any maps $f : X \to G$ and $g : Y \to G$, let $c(f, g) : X \times Y \to G$ be a map sending $(x, y) \in X \times Y$ to their commutator $[x, y] = f(x)^{-1}g(y)^{-1}f(x)g(y)$. Then $c(f, g)$ descends to a map $\langle f, g\rangle : X \wedge Y \to G$. The map $\langle f, g\rangle$ is called the *Samelson product* of f and g. The *order* of $\langle f, g\rangle$ is defined to be the minimum number k such that the composition

$$k \circ \langle f, g\rangle : X \wedge Y \xrightarrow{\langle f,g\rangle} G \xrightarrow{k} G$$

is null-homotopic, where $k : G \to G$ is the k^{th}-power map. In particular, when f and g are the identity map $\mathbb{1}_G$ of G, the Samelson product $\langle \mathbb{1}_G, \mathbb{1}_G\rangle$ is universal and we are interested in finding its order.

There is a notion of nilpotency in homotopy theory analogous to that for groups. Let c_1 be the commutator map $c(\mathbb{1}_G, \mathbb{1}_G) : G \times G \to G$, and let $c_n = c_1 \circ (c_{n-1} \times \mathbb{1}_G)$ be the n-iterated commutator for $n > 1$. The *homotopy nilpotence class* of G is the number n such that c_n is null-homotopic but c_{n-1} is not. In certain cases the homotopy nilpotence class of p-localized G is known. Kaji and Kishimoto [3] showed that p-regular Lie groups have homotopy nilpotence class at most 3. When G is quasi-p-regular and $p \ge 7$, Kishimoto [4] showed that $SU(n)$ has homotopy nilpotence class at most 3, and Theriault [10] showed that exceptional Lie groups have homotopy nilpotence class at most 2.

2010 *Mathematics Subject Classification.* 55Q15, 57T20.

Key words and phrases. Lie group, Samelson products, homotopy nilpotence.

Here we restrict G to be a Lie group having low rank with respect to an odd prime p. That is, G and p satisfy:

$$
\begin{array}{ll}
SU(n) & n \le (p-1)(p-2)+1 \\
Sp(n) & 2n \le (p-1)(p-2) \\
Spin(2n+1) & 2n \le (p-1)(p-2) \\
Spin(2n) & 2n-2 \le (p-1)(p-2) \\
G_2, F_4, E_6 & p \ge 5 \\
E_7, E_8 & p \ge 7,
\end{array}
\tag{1}
$$

In these cases, Theriault [9] showed that ΣA is a retract of ΣG. Let $\langle \imath, \imath \rangle$ be the composition

$$\langle \imath, \imath \rangle : A \wedge A \xhookrightarrow{\imath \wedge \imath} G \wedge G \xrightarrow{\langle \mathbb{1}_G, \mathbb{1}_G \rangle} G.$$

Then obviously the order of $\langle \mathbb{1}_G, \mathbb{1}_G \rangle$ is always greater than or equal to the order of $\langle \imath, \imath \rangle$. Conversely, we show that the order of $\langle \imath, \imath \rangle$ restricts the order of $\langle \mathbb{1}_G, \mathbb{1}_G \rangle$ under certain conditions.

Theorem 1.1. *Let G be a compact, simply-connected, simple Lie group of low rank and let p be an odd prime. Localized at p, if the homotopy nilpotence class of G is less than $p^r + 1$, then the order of the Samelson product $\langle \mathbb{1}_G, \mathbb{1}_G \rangle$ is p^r if and only if the order of $\langle \imath, \imath \rangle$ is p^r.*

The strategy for proving Theorem 1.1 is to extend $A \to G$ to an H-map $\Omega\Sigma A \to G$ which has a right homotopy inverse, that is to retract $[G \wedge G, G]$ off $[\Omega\Sigma A \wedge \Omega\Sigma A, G]$, and use commutator calculus to analyze the latter. Combine Theorem 1.1 and the known results in [3, 4, 10] to get the following statement.

Corollary 1.2. *The order of $\langle \mathbb{1}_G, \mathbb{1}_G \rangle$ equals the order of $\langle \imath, \imath \rangle$ when*

- *G is p-regular or;*
- *$p \ge 7$ and G is a quasi-p-regular Lie group which is one of $SU(n), F_4, E_6, E_7$ or E_8.*

On the one hand, there is no good method to calculate the order of $\langle \mathbb{1}_G, \mathbb{1}_G \rangle$ in general. A direct computation is not practical since one has to consider all the cells in $G \wedge G$ and their number grows rapidly when there is a slight increase in the rank of G. On the other hand, Corollary 1.2 says that for all p-regular Lie groups and most of the quasi-p-regular Lie groups, we can determine the order of $\langle \mathbb{1}_G, \mathbb{1}_G \rangle$ by computing the order of $\langle \imath, \imath \rangle$. The latter is easier to work with since A has a much simpler CW-structure than G. To demonstrate the power of Theorem 1.1, we apply this result to compute the order of $\langle \mathbb{1}_G, \mathbb{1}_G \rangle$ for all p-regular cases and some quasi-p-regular cases.

Theorem 1.3. *For a p-localized Lie group G, the order of $\langle \mathbb{1}_G, \mathbb{1}_G\rangle$ is p^r where*

G	$r=0$	$r=1$
$SU(n)$	$p>2n$	$n\leq p<2n$
$Sp(n)$	$p>4n$	$2n<p<4n$
$Spin(2n+1)$	$p>4n$	$2n<p<4n$
$Spin(2n)$	$p>4n-4$	$2n-2<p<4n-4$
G_2	$p=5, p>11$	$p=7,11$
F_4, E_6	$p>23$	$11\leq p\leq 23$
E_7	$p>31$	$17\leq p\leq 31$
E_8	$p>59$	$23\leq p\leq 59$

For many other quasi-p-regular cases, we give rough bounds on the order of $\langle \mathbb{1}_G, \mathbb{1}_G\rangle$ by bounding the order of $\langle \imath, \imath\rangle$.

Here is the structure of this paper. In Section 2 we prove Theorem 1.1 assuming Lemma 2.5, whose proof is given in Section 3 because of its length. Section 3 is divided into two parts. In the first part we consider the algebraic properties of Samelson products and in the second part we use algebraic methods to prove Lemma 2.5. In Section 4 we apply Theorem 1.1 and use other known results to calculate bounds on the order of the Samelson product $\langle \mathbb{1}_G, \mathbb{1}_G\rangle$ for quasi-p-regular Lie groups.

2. Samelson products of low rank Lie groups

Definition 2.1. Let G be a simple, simply-connected, compact Lie group and p be an odd prime. Localized at p, a triple $(A, \imath, G)$ is *retractile* if A is a co-H-space and a subspace of G and $\imath : A \hookrightarrow G$ is an inclusion such that

- there is an algebra isomorphism $H_*(G) \cong \Lambda(\tilde{H}_*(A))$ of homologies with mod-p coefficients;
- the induced homomorphism $\imath_* : H_*(A) \to H_*(G)$ is an inclusion of the generating set;
- the suspension $\Sigma\imath : \Sigma A \to \Sigma G$ has a left homotopy inverse $t : \Sigma G \to \Sigma A$.

We also refer to G as being retractile for short.

From now on, we take p-localization and assume G and p satisfy (1). According to [9], G is retractile. First we want to establish a connection between G and $\Omega\Sigma A$. Consider the homotopy commutative diagram

$$\begin{tikzcd} A \arrow[rr, "\imath"] \arrow[dr, "\Sigma"'] & & G \\ & \Omega\Sigma A \arrow[ur, "\tilde{\imath}"'] & \end{tikzcd} \tag{2}$$

where $\Sigma : A \to \Omega\Sigma A$ is the suspension and $\tilde{\imath} : \Omega\Sigma A \to G$ is an H-map. Since G is retractile, the suspension $\Sigma\imath : \Sigma A \to \Sigma G$ has a left homotopy inverse $t : \Sigma G \to \Sigma A$. Let s be the composition

$$s : G \xrightarrow{\Sigma} \Omega\Sigma G \xrightarrow{\Omega t} \Omega\Sigma A.$$

Lemma 2.1. *The map $\tilde{\imath} \circ s$ is a homotopy equivalence.*

Proof. Denote the composition $\tilde{\imath}\circ s$ by e for convenience. Consider the commutative diagram

$$\begin{tikzcd} A \arrow[r, "\imath"] \arrow[d, "\Sigma"'] & G \arrow[d, "\Sigma"'] \arrow[dr, "s"] & \\ \Omega\Sigma A \arrow[r, "\Omega\Sigma\imath"] & \Omega\Sigma G \arrow[r, "\Omega t"] & \Omega\Sigma A. \end{tikzcd}$$

The commutativity of the left square is due to the naturality of the suspension map, and the commutativity of the right triangle follows from the definition of s. The bottom row is homotopic to the identity since t is a left homotopy inverse for $\Sigma\imath$. Hence we have $s\circ\imath\simeq\Sigma$ and consequently

$$e\circ\imath=\tilde{\imath}\circ s\circ\imath\simeq\tilde{\imath}\circ\Sigma.$$

By Diagram (2) $\tilde{\imath}\circ\Sigma$ is homotopic to $\imath$. This implies that $(e\circ\imath)_*$ sends $H_*(A)$ onto the generating set of $H_*(G)=\Lambda(\tilde{H}_*(A))$ where we consider the mod-p homology. Dually, $(e\circ\imath)^*:H^*(G)\to H^*(A)$ is an epimorphism. The generating set $\imath^*(H^*(A))$ is in $Im(e^*)$. Since $e^*:H^*(G)\to H^*(G)$ is an algebra map, e^* is an epimorphism and hence is an isomorphism. Therefore $e:G\to G$ is a homotopy equivalence. $\square$

We claim that the Samelson product

$$\langle\tilde{\imath},\tilde{\imath}\rangle:\Omega\Sigma A\wedge\Omega\Sigma A\xrightarrow{\tilde{\imath}\wedge\tilde{\imath}}G\wedge G\xrightarrow{\langle\mathbb{1}_G,\mathbb{1}_G\rangle}G$$

has the same order as $\langle\mathbb{1}_G,\mathbb{1}_G\rangle$.

Lemma 2.2. *The map $p^r\circ\langle\mathbb{1}_G,\mathbb{1}_G\rangle$ is null-homotopic if and only if $p^r\circ\langle\tilde{\imath},\tilde{\imath}\rangle$ is null-homotopic.*

Proof. The sufficiency part is obvious. We only show the necessity part. Suppose $p^r\circ\langle\tilde{\imath},\tilde{\imath}\rangle$ is null-homotopic. By Lemma 2.1, $e=\tilde{\imath}\circ s$ is a homotopy equivalence. Composing with its inverse e', the map $\tilde{\imath}\circ s\circ e'$ is homotopic to the identity. Then we obtain

$$p^r\circ\langle\mathbb{1}_G,\mathbb{1}_G\rangle\simeq p^r\circ\langle\tilde{\imath}\circ s\circ e',\tilde{\imath}\circ s\circ e'\rangle=p^r\circ\langle\tilde{\imath},\tilde{\imath}\rangle\circ(s\circ e'\wedge s\circ e')$$

which is null-homotopic since $p^r\circ\langle\tilde{\imath},\tilde{\imath}\rangle$ is null-homotopic. $\square$

Combining Diagram 2 and the fact that $\tilde{\imath}$ is an H-map, we have the commutative diagram

(3)
$$\begin{tikzcd} & (\Omega\Sigma A)^k \arrow[r, "\mu^k"] \arrow[d, "\tilde{\imath}^k"] & \Omega\Sigma A \arrow[d, "\tilde{\imath}"] \\ A^k \arrow[ur, "j^k"] \arrow[r, "\imath^k"] & G^k \arrow[r, "m^k"] & G \end{tikzcd}$$

where μ^k and m^k are the k-fold multiplications in $\Omega\Sigma A$ and G. Let m_k and e_k be the compositions

$$m_k:A^k\xrightarrow{\imath^k}G^k\xrightarrow{m^k}G\quad\text{and}\quad e_k:A^k\xrightarrow{j^k}(\Omega\Sigma A)^k\xrightarrow{\mu^k}\Omega\Sigma A.$$

Then we have the following commutative diagram

$$\begin{tikzcd} A^k\wedge A^l \arrow[d, "e_k\wedge e_l"'] \arrow[dr, "m_k\wedge m_l"] & & & \\ \Omega\Sigma A\wedge\Omega\Sigma A \arrow[r, "\tilde{\imath}\wedge\tilde{\imath}"] & G\wedge G \arrow[r, "\langle\mathbb{1}_G,\mathbb{1}_G\rangle"] & G \arrow[r, "p^r"] & G \end{tikzcd}$$

Observe that there is a string of equalities

$$\begin{aligned}[\Omega\Sigma A\wedge\Omega\Sigma A, G] &= [\Sigma\Omega\Sigma A\wedge\Omega\Sigma A, BG]\\ &= [\bigvee_{k,l=1}^{\infty}\Sigma A^{\wedge k}\wedge A^{\wedge l}, BG]\\ &= \prod_{k,l=1}^{\infty}[A^{\wedge k}\wedge A^{\wedge l}, G].\end{aligned}$$

The first and the third lines are due to adjunction, and the second line is due to James splitting $\Sigma\Omega\Sigma A\simeq\Sigma_{k=1}^{\infty}\Sigma A^{\wedge k}$. It is not hard to see that the nullity of $p^r\circ\langle\tilde{\imath},\tilde{\imath}\rangle$ implies the nullity of the components $p^r\circ\langle m_k, m_l\rangle$. In the following we show that the converse is true.

Lemma 2.3. *Let X be a space and let $f: X\to G$ be a map. If $p^r\circ\langle m_k, f\rangle : A^k\wedge X\to G$ is null-homotopic for all k, then $p^r\circ\langle\tilde{\imath}, f\rangle : \Omega\Sigma A\wedge X\to G$ is null-homotopic. Similarly, if $p^r\circ\langle f, m_l\rangle : X\wedge A^l\to G$ is null-homotopic for all l, then $p^r\circ\langle f,\tilde{\imath}\rangle : X\wedge\Omega\Sigma A\to G$ is null-homotopic.*

Proof. We only prove the first statement since the second statement can be proved similarly. Let $h:\Sigma\Omega\Sigma A\wedge X\to BG$ be the adjoint of $p^r\circ\langle\tilde{\imath}, f\rangle$. It suffices to show that h is null-homotopic.

For any k, choose a right homotopy inverse ψ_k of the suspended quotient map $\Sigma A^k\to\Sigma A^{\wedge k}$, and let Ψ_k be the composition

$$\Psi_k:\Sigma A^{\wedge k}\xrightarrow{\psi_k}\Sigma A^k\xrightarrow{\Sigma e_k}\Sigma\Omega\Sigma A.$$

Observe that e_k is the product of k copies of the suspension j and $j_*: H_*(A)\to H_*(\Omega\Sigma A)$ is the inclusion of the generating set into $H_*(\Omega\Sigma A)\cong T(\tilde{H}_*(A))$. The map $(e_k)_*: H_*(A^k)\to H^*(\Omega\Sigma A)$ sends the submodule $S_k\subset H_*(A^k)\cong H_*(A)^{\otimes k}$ consisting of length k tensor products onto the submodule $M_k\subset T(\tilde{H}_*(A))$ consisting of length k tensor products. Therefore $(\Psi_k)_*$ does the same. Then their sum

$$\Psi=\bigvee_{k=1}^{\infty}\Psi_k:\bigvee_{k=1}^{\infty}\Sigma A^{\wedge k}\to\Sigma\Omega\Sigma A$$

induces a homology isomorphism and hence is a homotopy equivalence.

We claim that $h\circ(\Psi_k\wedge\mathbb{1}_X)$ is null-homotopic for all k, where $\mathbb{1}_X$ is the identity of X. Observe that the adjoint of the composition

$$\Sigma A^k\wedge X\xrightarrow{\Sigma e_k\wedge\mathbb{1}_X}\Sigma\Omega\Sigma A\wedge X\xrightarrow{h}BG$$

is $p^r\circ\langle\tilde{\imath}, f\rangle\circ(e_k\wedge\mathbb{1}_X)\simeq p^r\circ\langle\tilde{\imath}\circ e_k, f\rangle\simeq p^r\circ\langle m_k, f\rangle$ which is null-homotopic by assumption. Therefore $h\circ(\Psi_k\wedge\mathbb{1}_X)=h\circ(\Sigma e_k\wedge\mathbb{1}_X)\circ(\psi_k\wedge\mathbb{1}_X)$ is null-homotopic, and by definition of Ψ, the composition

$$\bigvee_{k=1}^{\infty}\Sigma A^{\wedge k}\wedge X\xrightarrow{\Psi\wedge\mathbb{1}}\Sigma\Omega\Sigma A\wedge X\xrightarrow{h}BG.$$

is null-homotopic. Notice that $(\Psi\wedge\mathbb{1})$ is a homotopy equivalence. It implies that h is null-homotopic and so is $p^r\circ\langle\tilde{\imath}, f\rangle$. ☐

Lemma 2.4. *The map $p^r\circ\langle\tilde{\imath},\tilde{\imath}\rangle$ is null-homotopic if and only if $p^r\circ\langle m_k, m_l\rangle$ is null-homotopic for all k and l.*

Proof. It suffices to prove the necessity part. Suppose $p^r \circ \langle m_k, m_l\rangle$ is null-homotopic for all k and l. Apply the first part of Lemma 2.3 to obtain $p^r \circ \langle \tilde{\imath}, m_l\rangle \simeq *$ for all l and apply the second part of Lemma 2.3 to obtain $p^r \circ \langle \tilde{\imath}, \tilde{\imath}\rangle \simeq *$. □

At this point we have related the order of $\langle \mathbb{1}_G, \mathbb{1}_G\rangle$ to the orders of $\langle m_k, m_l\rangle$ for all k and l. There is one more step to link up with the order of the Samelson product

$$\langle \imath, \imath\rangle : A\wedge A \xrightarrow{\imath\wedge\imath} G\wedge G \xrightarrow{\langle \mathbb{1}_G, \mathbb{1}_G\rangle} G \xrightarrow{p^r} G.$$

Lemma 2.5. *If $p^r \circ \langle \imath, \imath\rangle$ is null-homotopic and G has homotopy nilpotency class less than $p^r + 1$, then $p^r \circ \langle m_k, m_l\rangle$ is null-homotopic for all k and l.*

The proof of Lemma 2.5 is long and we postpone it to the next section so as to avoid interrupting the flow of our discussion. Assuming Lemma 2.5 we can prove our main theorem.

Theorem 2.6. *Suppose that G has homotopy nilpotence class less than $p^r + 1$ after localization at p. Then $\langle \mathbb{1}_G, \mathbb{1}_G\rangle$ has order p^r if and only if $\langle \imath, \imath\rangle$ has order p^r.*

Proof. The order of $\langle \mathbb{1}_G, \mathbb{1}_G\rangle$ is not less than the order of $\langle \imath, \imath\rangle$. Therefore we need to show that the order of $\langle \mathbb{1}_G, \mathbb{1}_G\rangle$ is not greater than the order of $\langle \imath, \imath\rangle$ under the assumption. Assume $\langle \imath, \imath\rangle$ has order p^r, that is $p^r \circ \langle \imath, \imath\rangle$ is null-homotopic. Lemmas 2.2, 2.4 and 2.5 imply that $p^r \circ \langle \mathbb{1}_G, \mathbb{1}_G\rangle$ is null-homotopic, so the order of $\langle \mathbb{1}_G, \mathbb{1}_G\rangle$ is not greater than the order of $\langle \imath, \imath\rangle$. □

3. Proof of Lemma 2.5

In this section we prove Lemma 2.5 by showing $p^r \circ \langle m_k, m_l\rangle$ is null-homotopic assuming the homotopy nilpotence class of G is less than $p^r + 1$. We convert it into an algebraic problem and derive lemmas from group theoretic identities and the topological properties of G. First let us review the algebraic properties of Samelson products.

3.1. **Algebraic properties of Samelson products.** Given two maps $f : X \to G$ and $g : Y \to G$, their Samelson product $\langle f, g\rangle$ sends $(x, y) \in X \wedge Y$ to the commutator of their images $f(x)$ and $g(y)$. It is natural to regard the map $\langle f, g\rangle$ as a commutator in $[X \wedge Y, G]$, but f and g are maps in different homotopy sets and there is no direct multiplication between them. Instead, we can include $[X, G]$, $[Y, G]$ and $[X \wedge Y, G]$ into $[X \times Y, G]$ and identify $\langle f, g\rangle$ as a commutator there.

Lemma 3.1. *For any spaces X and Y, let $\pi_1 : X \times Y \to X$ and $\pi_2 : X \times Y \to Y$ be the projections and let $q : X \times Y \to X \wedge Y$ be the quotient map. Then the images $(\pi_1)^*[X, G]$, $(\pi_2)^*[Y, G]$, $q^*[X \wedge Y, G]$ are subgroups of $[X \times Y, G]$, and $(\pi_1)^* : [X, G] \to [X \times Y, G]$, $(\pi_2)^* : [Y, G] \to [X \times Y, G]$, $q^* : [X \wedge Y, G] \to [X \times Y, G]$ are monomorphisms.*

Proof. Observe that π_1 and π_2 induce group homomorphisms $(\pi_1)^* : [X, G] \to [X \times Y, G]$ and $(\pi_2)^* : [X, G] \to [X \times Y, G]$, so their images $(\pi_1)^*[X, G]$ and $(\pi_2)^*[Y, G]$ are subgroups of $[X \times Y, G]$. Moreover, let $j : X \to X \times Y$ be the inclusion. Since $\pi_1 \circ j$ is the identity, $j^* \circ (\pi_1)^*$ is an isomorphism and $(\pi_1)^*$ is a monomorphism. Therefore $[X, G]$ is isomorphic to $(\pi_1)^*[X, G]$. Similarly $[Y, G]$ is isomorphic to $(\pi_2)^*[Y, G]$.

For $q^*[X \wedge Y, G]$, the cofibration $X \vee Y \xrightarrow{j'} X \times Y \xrightarrow{q} X \wedge Y$ induces an exact sequence

$$\cdots \to [\Sigma X \times Y, G] \xrightarrow{\Sigma j'^*} [\Sigma(X \vee Y), G] \longrightarrow [X \wedge Y, G] \xrightarrow{q^*} [X \times Y, G] \xrightarrow{j'^*} [X \vee Y, G],$$

where j' is the inclusion and q is the quotient map. Since $\Sigma j' : \Sigma(X \vee Y) \hookrightarrow \Sigma(X \times Y)$ has a right homotopy inverse, $q^* : [X \wedge Y, G] \to [X \times Y, G]$ is a monomorphism. Therefore $[X \wedge Y, G]$ is isomorphic to $q^*[X \wedge Y, G]$, which is a subgroup of $[X \times Y, G]$. $\square$

There are two groups in our discussion, namely G and $[X \times Y, G]$. To distinguish their commutators, for any maps $f : X \to G$ and $g : Y \to G$ we use $c(f, g)$ to denote the map which sends $(x, y) \in X \times Y$ to $f(x)^{-1}g(y)^{-1}f(x)g(y) \in G$, and for any maps $a : X \times Y \to G$ and $b : X \times Y \to G$ we use $[a, b]$ to denote the commutator $a^{-1}b^{-1}ab \in [X \times Y, G]$. Lemma 3.1 says that $f : X \to G$ and $g : Y \to G$ can be viewed as being in $[X \times Y, G]$. Their images are the compositions

$$\tilde{f} : X \times Y \xrightarrow{\pi_1} X \xrightarrow{f} G \quad \text{and} \quad \tilde{g} : X \times Y \xrightarrow{\pi_2} Y \xrightarrow{g} G.$$

Consider the diagram

$$\begin{tikzcd}
 & X \times Y \arrow[r, "q"] \arrow[dl, "\triangle"'] \arrow[d, "f\times g"'] & X \wedge Y \arrow[d, "f\wedge g"] \\
(X \times Y) \times (X \times Y) \arrow[r, "\tilde{f}\times\tilde{g}"] & G \times G \arrow[r, "q'"] \arrow[d, "c"'] & G \wedge G \arrow[d, "\langle \mathbb{1}_G, \mathbb{1}_G \rangle"] \\
 & G \arrow[r, "="] & G
\end{tikzcd}$$

where $\triangle$ is the diagonal map, c is the commutator map $c(\mathbb{1}_G, \mathbb{1}_G)$, and q' is the quotient maps. The commutativity of the left triangle is due to the definitions of $\tilde{f}$ and $\tilde{g}$, the commutativity of the top square is due to the naturality of the quotient maps and the commutativity of the bottom square is due to the definition of $\langle \mathbb{1}_G, \mathbb{1}_G \rangle$. The middle column is $c(f, g)$ and the right column is $\langle f, g \rangle$. In order to show that $\langle f, g \rangle$ is null-homotopic, it suffices to consider $c(f, g) \simeq q^*\langle f, g \rangle$ since $q^* : [X \wedge Y, G] \to [X \times Y, G]$ is injective. Observe that $c(f, g)$ is homotopic to the composition

$$X \times Y \xrightarrow{\triangle} (X \times Y) \times (X \times Y) \xrightarrow{\tilde{f}\times\tilde{g}} G \times G \xrightarrow{c} G$$

according to the diagram. That is $c(f, g)$ is the commutator $[\tilde{f}, \tilde{g}] = \tilde{f}^{-1}\tilde{g}^{-1}\tilde{f}\tilde{g}$ in $[X \times Y, G]$.

Let $Ad_b a = b^{-1}ab$ be the conjugation of maps a and b in $[X \times Y, G]$. In group theory, commutators satisfy the following identities:

(4)
(1) $[a, b] = a^{-1} \cdot Ad_b a$;
(2) $[a, b]^{-1} = [b, a]$;
(3) $[a \cdot a', b] = Ad_{a'}[a, b] \cdot [a', b] = [a, b] \cdot [a', [a, b]]^{-1} \cdot [a', b]$;
(4) $[a, b \cdot b'] = [a, b'] \cdot Ad_{b'}[a, b] = [a, b'] \cdot [a, b] \cdot [[a, b], b']$.

In particular, we can substitute $\tilde{f}$ and $\tilde{g}$ to a and b in these identities.

Proposition 3.2. *Let $f, f' : X \to G$ and $g, g' : Y \to G$ be maps. Then in $[X \times Y, G]$,*

(i) $c(f, g) = \tilde{f}^{-1} \cdot Ad_{\tilde{g}}\tilde{f}$;
(ii) $c(f, g)^{-1} = c(g, f) \circ T$;
(iii) $c(f \cdot f', g) = Ad_{\tilde{f}'}c(f, g) \cdot c(f', g) = c(f, g) \cdot [\tilde{f}', [\tilde{f}, \tilde{g}]]^{-1} \cdot c(f', g)$;
(iv) $c(f, g \cdot g') = c(f, g') \cdot Ad_{\tilde{g}'}c(f, g) = c(f, g') \cdot c(f, g) \cdot [[\tilde{f}, \tilde{g}], \tilde{g}']$,

where $T : Y \times X \to X \times Y$ is the swapping map.

Proof. All identities come directly from the identities in (4), while Identity (ii) needs some explanation. Observe there exists a homotopy commutative diagram

$$\begin{tikzcd} X \times Y \arrow[r, "f\times g"] \arrow[d, "T"] & G \times G \arrow[r, "c"] \arrow[d, "T"] & G \arrow[d, "r"] \\ Y \times X \arrow[r, "g\times f"] & G \times G \arrow[r, "c"] & G \end{tikzcd}$$

where $r : G \to G$ is the inversion. The upper direction around the diagram is $c(f,g)^{-1}$, while the lower direction is $c(g,f) \circ T$. So Identity (ii) follows. □

Remark: The iterated commutator $[\tilde{f}'(x), [\tilde{f}(x), \tilde{g}(y)]]$ is the composition

$$X \times Y \xrightarrow{\triangle_X \times \mathbb{1}_Y} X \times X \times Y \xrightarrow{f' \times f \times g} G \times G \times G \xrightarrow{\mathbb{1}_G \times c} G \times G \xrightarrow{c} G$$

where $\triangle_X$ is the diagonal map and $\mathbb{1}_Y$ and $\mathbb{1}_G$ are the identity maps. Let $c_2 = c \circ (\mathbb{1}_G \times c)$ be the 2-iterated commutator on G. Then we can write $[\tilde{f}', [\tilde{f}, \tilde{g}]]$ as $c_2 \circ (f' \times f \times g) \circ (\triangle_X \times \mathbb{1}_Y)$. However, we prefer to stick to the notation $[\tilde{f}', [\tilde{f}, \tilde{g}]]$ because it better indicates it is the commutator of which maps, while $c_2 \circ (f' \times f \times g) \circ (\triangle_X \times \mathbb{1}_Y)$ looks long and confusing.

Since our group $[X \times Y, G]$ has a topological interpretation, the topologies of X and Y add extra algebraic properties to its group structure.

Lemma 3.3. *Let f, g and $h : X \times Y \to G$ be maps. If X is a co-H-space and the restrictions of f and g to $X \vee Y$ are null-homotopic, then in $[X \times Y, G]$ we have*

$$f \cdot g = g \cdot f \quad \textit{and} \quad [f \cdot g, h] = [f, h] \cdot [g, h].$$

Proof. Let $q : X \times Y \to X \wedge Y$ be the quotient map. Observe that there exist f' and g' in $[X \wedge Y, G]$ such that $f = q^* f'$ and $g = q^* g'$. Since $X \wedge Y$ is a co-H-space, $[X \wedge Y, G]$ is an abelian group and f' and g' commute. Therefore f and g commute as q^* is a monomorphism.

To show the linearity, we start with Proposition 3.2 (iii)

$$[f \cdot g, h] = [f, h] \cdot [g, [f, h]] \cdot [g, h].$$

Since $[f, h]$ is also null-homotopic on $X \vee Y$, it commutes with g and their commutator $[g, [f, h]]$ is trivial. Therefore we have $[f \cdot g, h] = [f, h] \cdot [g, h]$. □

3.2. **Main body of the proof.** We go back to the proof of Lemma 2.5. Recall that m_k is the composition $m_k : A^k \xrightarrow{\imath^k} G^k \xrightarrow{m^k} G$. To distinguish the spaces A's, denote the i^{th} copy of A in A^k by A_i. Let a_i and m'_{k-1} be the compositions

$$a_i : A^k \xrightarrow{proj} A_i \xrightarrow{\imath} G \quad \text{and} \quad m'_{k-1} : A^k \xrightarrow{proj} \prod_{i=1}^{k-1} A_i \xrightarrow{m_{k-1}} G$$

respectively. Then we have $m_k = m'_{k-1} \cdot a_i$ in $[A^k, G]$. Include $\langle m_k, m_l \rangle$ in $[A^k \times A^l, G]$ by Lemma 3.1. It becomes the commutator $c(m_k, m_l) = [\tilde{m}_k, \tilde{m}_l]$, where $\tilde{m}_k$ and $\tilde{m}_l$ are compositions

$$\tilde{m}_k : A^k \times A^l \xrightarrow{proj} A^k \xrightarrow{m_k} G \quad \text{and} \quad \tilde{m}_l : A^k \times A^l \xrightarrow{proj} A^l \xrightarrow{m_l} G.$$

Let $\tilde{a}_i$ and $\tilde{m}'_{k-1}$ be compositions

$$\tilde{a}_i : A^k \times A^l \xrightarrow{proj} A_i \xrightarrow{\imath} G \quad \text{and} \quad \tilde{m}'_{k-1} : A^k \times A^l \xrightarrow{proj} \prod_{i=1}^{k-1} A_i \xrightarrow{m_{k-1}} G.$$

Then in $[A^k \times A^l, G]$ we have $\tilde{m}_k = \tilde{m}'_{k-1} \cdot \tilde{a}_k$.

Assume the homotopy nilpotence class of G is less than p^r+1. Now we use induction on k and l show that $c(m_k, m_l)^{p^r}$ is null-homotopic. To start with, we show that this is true for $k=1$ or $l=1$.

Lemma 3.4. *If $c(\imath,\imath)^{p^r}$ is null-homotopic, then $c(m_k,\imath)^{p^r}$ and $c(\imath,m_l)^{p^r}$ are null-homotopic for all k and l.*

Proof. We prove that $c(m_k,\imath)^{p^r}$ is null-homotopic by induction. Since $m_1=\imath$, $c(m_1,\imath)^{p^r}=c(\imath,\imath)^{p^r}$ is null-homotopic by assumption. Suppose $c(m_k,\imath)^{p^r}$ is null-homotopic. We need to show that $c(m_{k+1},\imath)^{p^r}$ is also null-homotopic. Apply Proposition 3.2 (iii) to obtain

$$c(m_{k+1},\imath)=c(m'_k\cdot a_{k+1},\imath)=Ad_{\tilde{a}_{k+1}}c(m'_k,\imath)\cdot c(a_{k+1},\imath).$$

Observe that $c(a_{k+1},\imath)$ and $Ad_{\tilde{a}_{k+1}}c(m'_k,\imath)$ are null-homotopic on $A^{k+1}\vee A$ and $A^{k+1}\wedge A$ is a co-H-space. Lemma 3.3 implies that $c(a_{k+1},\imath)$ and $Ad_{\tilde{a}_{k+1}}c(m'_k,\imath)$ commute and we have

$$\begin{aligned} c(m_{k+1},\imath)^{p^r} &= \left(Ad_{\tilde{a}_{k+1}}c(m'_k,\imath)\cdot c(a_{k+1},\imath)\right)^{p^r}\\ &= \left(Ad_{\tilde{a}_{k+1}}c(m'_k,\imath)\right)^{p^r}\cdot c(a_{k+1},\imath)^{p^r}\\ &= Ad_{\tilde{a}_{k+1}}\left(c(m'_k,\imath)^{p^r}\right)\cdot c(a_{k+1},\imath)^{p^r}. \end{aligned}$$

The last term $c(a_{k+1},\imath)^{p^r}$ is null-homotopic since a_{k+1} is the inclusion $A_{k+1}\xrightarrow{\imath}G$. Also, by the induction hypothesis $c(m'_k,\imath)^{p^r}$ is null-homotopic. Therefore $c(m_{k+1},\imath)^{p^r}$ is null-homotopic and the induction is completed.

Similarly, we can show that $c(\imath,m_l)^{p^r}$ is null-homotopic for all l. $\square$

As a consequence of Lemma 3.4, the following lemma implies that the order of

$$\langle\imath,\mathbb{1}_G\rangle : A\wedge G\xrightarrow{\imath\wedge\mathbb{1}_G}G\wedge G\xrightarrow{\langle\mathbb{1}_G,\mathbb{1}_G\rangle}G$$

equals to the order of its restriction $\langle\imath,\imath\rangle$ without assuming the condition on the homotopy nilpotence of G.

Lemma 3.5. *The map $p^r\circ\langle\imath,\imath\rangle$ is null-homotopic if and only if $p^r\circ\langle\mathbb{1}_G,\imath\rangle$ and $p^r\circ\langle\imath,\mathbb{1}_G\rangle$ are null-homotopic.*

Proof. We only need to prove the sufficient condition. If $p^r\circ\langle\imath,\imath\rangle$ is null-homotopic, then $p^r\circ\langle\imath,m_l\rangle$ is null-homotopic for all l by Lemma 3.4. Lemma 2.3 implies that $p^r\circ\langle\imath,\tilde{\imath}\rangle : A\wedge\Omega\Sigma A\to G$ is null-homotopic. Since $\tilde{\imath}\circ s$ is a homotopy equivalence by Lemma 2.1, $p^r\circ\langle\imath,\mathbb{1}_G\rangle$ is null-homotopic.

The sufficient condition for $p^r\circ\langle\mathbb{1}_G,\imath\rangle$ can be proved similarly. $\square$

Now suppose $c(m_k,m_l)^{p^r}$ is trivial for some fixed k and l in $[A^k\times A^l,G]$. The next step is to show that $c(m_{k+1},m_l)^{p^r}$ is trivial in $[A^{k+1}\times A^l,G]$. At first glance we can follow the proof of Lemma 3.4 and apply Lemmas 3.2 and 3.3 to split $c(m_{k+1},m_l)^{p^r}$ into $c(m'_k,m_l)^{p^r}$ and $c(a_{k+1},m_l)^{p^r}$ which are null-homotopic by the induction hypothesis. However, when $l>1$, A^l is not a co-H-space and we cannot use Lemma 3.3 to argue that $c(m'_k,m_l)$ and $c(a_{k+1},m_l)$ commute. Instead, apply Proposition 3.2 (iii) to obtain

$$\begin{aligned} c(m_{k+1},m_l) &= c(m'_k\cdot a_{k+1},m_l)\\ &= c(m'_k,m_l)\cdot[c(m'_k,m_l),\tilde{a}_{k+1}]\cdot c(a_{k+1},m_l). \end{aligned}$$

Denote $c(m'_k, m_l)$ and $[c(m'_k, m_l), \tilde{a}_{k+1}] \cdot c(a_{k+1}, m_l)$ by α_k and β_k respectively. Observe that the restrictions of any powers and commutators involving β_k to $A_{k+1} \vee (A^k \times A^l)$ are null-homotopic and A_{k+1} is a co-H-space. Therefore they enjoy the conditions of Lemma 3.3.

Lemma 3.6. *For any natural number n, we have*

$$(\alpha_k \cdot \beta_k)^n = \alpha_k^n \cdot \beta_k^n \cdot \left(\prod_{i=1}^{n-1} [\beta_k, \alpha_k^i]\right).$$

Proof. We induct on n. The statement of the lemma is trivial for $n = 1$. Assume the formula holds for an integer n. For the $(n+1)$ case, using the induction hypothesis we have

$$\begin{aligned} (\alpha_k \cdot \beta_k)^{n+1} &= \alpha_k \cdot \beta_k \cdot (\alpha_k \cdot \beta_k)^n \\ &= \alpha_k \cdot \beta_k \cdot \alpha_k^n \cdot \beta_k^n \cdot \left(\prod_{i=1}^{n-1} [\beta_k, \alpha_k^i]\right) \\ &= \alpha_k^{n+1} \cdot \beta_k \cdot [\beta_k, \alpha_k^n] \cdot \beta_k^n \cdot \left(\prod_{i=1}^{n-1} [\beta_k, \alpha_k^i]\right) \end{aligned}$$

In the last line $[\beta_k, \alpha_k^n]$ is formed after we swap β_k and α_k^n. Since the restrictions of β_k, β_k^n and $[\beta_k, \alpha_k^i]$ to $A_{k+1} \vee (A^k \times A^l)$ are null-homotopic, they commute by Lemma 3.3. By commuting the terms, the statement follows. □

In order to prove the triviality of $c(m_{k+1}, m_l)^{p^r}$, by Lemma 3.6 it suffices to show that $\alpha_k^{p^r}$, $\beta_k^{p^r}$ and $\prod_{i=1}^{p^r-1} [\beta_k, \alpha_k^i]$ are null-homotopic. By the induction hypothesis $c(m_k, m_l)^{p^r}$ is null-homotopic, so $\alpha_k^{p^r} = c(m'_k, m_l)^{p^r}$ is null-homotopic. It remains to show that $\beta_k^{p^r}$ and $\prod_{i=1}^{p^r-1} [\beta_k, \alpha_k^i]$ are null-homotopic.

Lemma 3.7. *If $c(\imath, \imath)^{p^r}$ and $\alpha_k^{p^r}$ are null-homotopic, then so is $\beta_k^{p^r}$.*

Proof. By definition, $\beta_k = [c(m'_k, m_l), \tilde{a}_{k+1}] \cdot c(a_{k+1}, m_l)$. Observe that the restrictions of $[c(m'_k, m_l), \tilde{a}_{k+1}]$ and $c(a_{k+1}, m_l)$ to $A_{k+1} \vee (A^k \times A^l)$ are null-homotopic. By Lemma 3.3 they commute and we have

$$\beta_k^{p^r} = ([c(m'_k, m_l), \tilde{a}_{k+1}] \cdot c(a_{k+1}, m_l))^{p^r} = [c(m'_k, m_l), \tilde{a}_{k+1}]^{p^r} \cdot c(a_{k+1}, m_l)^{p^r}.$$

Since $c(\imath, \imath)^{p^r}$ is null-homotopic, so is $c(a_{k+1}, m_l)^{p^r}$ by Lemma 3.4.

On the other hand, recall that $c(m'_k, m_l)$ and $\tilde{a}_{k+1}$ are the compositions

$$c(m'_k, m_l) : A^{k+1} \times A^l \xrightarrow{proj} A^k \times A^l \xrightarrow{c(m_k, m_l)} G \quad \text{and} \quad \tilde{a}_{k+1} : A^{k+1} \times A^l \xrightarrow{proj} A_{k+1} \xrightarrow{\imath} G$$

respectively. Therefore we have

$$\begin{aligned} [c(m'_k, m_l), \tilde{a}_{k+1}]^{p^r} &= p^r \circ c(c(m'_k, m_l), \imath) \\ &= p^r \circ c(\mathbb{1}_G, \imath) \circ (c(m'_k, m_l) \times \mathbb{1}_A) \end{aligned}$$

where $\mathbb{1}_A$ is the identity map of A_{k+1}. Since $p^r \circ c(\mathbb{1}_G, \imath)$ is null-homotopic by Lemma 3.5, $[c(m'_k, m_l), \tilde{a}_{k+1}]^{p^r}$ is null-homotopic and so is $\beta_k^{p^r}$. □

Lemma 3.8. *For any natural number n, we have*

$$\prod_{i=1}^{n-1} [\beta_k, \alpha_k^i] = \prod_{i=1}^{n-1} c_i(\beta_k, \alpha_k, \cdots, \alpha_k)^{\binom{n}{i+1}}$$

where $c_i(\beta_k,\alpha_k,\cdots,\alpha_k)=[[\cdots[[\beta_k,\alpha_k],\alpha_k]\cdots],\alpha_k]$ *is the* i*-iterated commutator.*

Proof. First, by induction we prove

$$[\beta_k,\alpha_k^i]=\prod_{j=1}^{i}c_j(\beta_k,\alpha_k,\cdots,\alpha_k)^{\binom{i}{j}}.$$

It is trivial for $i=1$. Assume the formula holds for $[\beta_k,\alpha_k^i]$. Use the commutator identity in (4) and inductive hypothesis to get

$$\begin{aligned}[\beta_k,\alpha_k^{i+1}] &= [\beta_k,\alpha_k]\cdot[\beta_k,\alpha_k^i]\cdot[[\beta_k,\alpha_k^i],\alpha_k]\\ &= [\beta_k,\alpha_k]\cdot\left(\prod_{j=1}^{i}c_j(\beta_k,\alpha_k,\cdots,\alpha_k)^{\binom{i}{j}}\right)\cdot[\prod_{j=1}^{i}c_j(\beta_k,\alpha_k,\cdots,\alpha_k)^{\binom{i}{j}},\alpha_k]\end{aligned}$$

Since the restriction of $c_j(\beta_k,\alpha_k,\cdots,\alpha_k)$ to $A_{k+1}\vee(A^k\times A^l)$ is null-homotopic for all j, by Lemma 3.3 they commute with each other and

$$\begin{aligned}[\beta_k,\alpha_k^{i+1}] &= [\beta_k,\alpha_k]\cdot\left(\prod_{j=1}^{i}c_j(\beta_k,\alpha_k,\cdots,\alpha_k)^{\binom{i}{j}}\right)\cdot\left(\prod_{j=1}^{i}[c_j(\beta_k,\alpha_k,\cdots,\alpha_k),\alpha_k]^{\binom{i}{j}}\right)\\ &= [\beta_k,\alpha_k]\cdot\left(\prod_{j=1}^{i}c_j(\beta_k,\alpha_k,\cdots,\alpha_k)^{\binom{i}{j}}\right)\cdot\left(\prod_{j=1}^{i}c_{j+1}(\beta_k,\alpha_k,\cdots,\alpha_k)^{\binom{i}{j}}\right)\\ &= [\beta_k,\alpha_k]^{i+1}\cdot\left(\prod_{j=2}^{i}c_j(\beta_k,\alpha_k,\cdots,\alpha_k)^{\binom{i}{j}+\binom{i}{j+1}}\right)\cdot c_{i+1}(\beta_k,\alpha_k,\cdots,\alpha_k)\\ &= \prod_{j=1}^{i+1}c_j(\beta_k,\alpha_k,\cdots,\alpha_k)^{\binom{i+1}{j}}\end{aligned}$$

Therefore the claim is proved.

Now we multiply all $[\beta_k,\alpha_k^i]$'s and use the commutativity of $c_j(\beta_k,\alpha_k,\cdots,\alpha_k)$'s to get

$$\begin{aligned}\prod_{i=1}^{n-1}[\beta_k,\alpha_k^i] &= \prod_{i=1}^{n-1}\prod_{j=1}^{i}c_j(\beta_k,\alpha_k,\cdots,\alpha_k)^{\binom{i}{j}}\\ &= \prod_{j=1}^{n-1}\prod_{i=j}^{n-1}c_j(\beta_k,\alpha_k,\cdots,\alpha_k)^{\binom{i}{j}}\\ &= \prod_{j=1}^{n-1}\left(\prod_{i=j}^{n-1}c_j(\beta_k,\alpha_k,\cdots,\alpha_k)^{\binom{i}{j}}\right)\\ &= \prod_{j=1}^{n-1}c_j(\beta_k,\alpha_k,\cdots,\alpha_k)^{\sum_{i=j}^{n-1}\binom{i}{j}}\end{aligned}$$

The proof will be completed if we can show that $\sum_{i=j}^{n-1}\binom{i}{j}=\binom{n}{j+1}$.

Consider the polynomial

$$\sum_{i=0}^{n-1}(1+x)^i = \sum_{i=0}^{n-1}\sum_{j=0}^{i}\binom{i}{j}x^j = \sum_{j=0}^{n-1}\sum_{i=0}^{n-1}\binom{i}{j}x^j.$$

The coefficient of x^j is $\sum_{i=j}^{n-1}\binom{i}{j}$. On the other hand, it can be written as

$$1+(1+x)+\cdots+(1+x)^{n-1} = \frac{(1+x)^n-1}{x} = \sum_{j=1}^{n}\binom{n}{j}x^{j-1}.$$

The coefficient of x^j is $\binom{n}{j+1}$. By comparing the coefficients of x^j the statement follows. $\square$

Let $nil(G)$ be the homotopy nilpotence class of G, that is, $nil(G)=n$ if and only if the n-iterated commutator c_n is null-homotopic but c_{n-1} is not. In particular, $nil(G)=n$ means all commutators in G of length greater than n are null-homotopic.

Lemma 3.9. *Let $\{f_j : A^k\times A^l \to G\}_{1\leq j\leq n-1}$ be a set of maps. If $nil(G)\leq n$, then $c_{n-1}(\beta_k, f_1,\cdots,f_{n-1})$ is null-homotopic.*

Proof. By definition, $\beta_k = [c(m'_k,m_l),\tilde{a}_{k+1}]\cdot c(a_{k+1},m_l)$. Denote $[c(m'_k,m_l),\tilde{a}_{k+1}]$ and $c(a_{k+1},m_l)$ by γ_0 and γ'_0 respectively. Then we have $\beta_k=\gamma_0\cdot\gamma'_0$. For $1\leq j\leq n-1$, let $\gamma_j=[\gamma_{j-1},f_j]$ and $\gamma'_j=[\gamma'_{j-1},f_j]$. We claim that $c_m(\beta_k,f_1,\cdots,f_m)=\gamma_m\cdot\gamma'_m$ for $1\leq m\leq n-1$.

When $m=1$,

$$c_1(\beta_k,f_1)=[\beta_k,f_1]=[\gamma_0\cdot\gamma'_0,f_1].$$

Since the restrictions of γ_0 and γ'_0 to $A_{k+1}\vee(A^k\times A^l)$ are null-homotopic, by Lemma 3.3 we have

$$[\gamma_0\cdot\gamma'_0,f_1]=[\gamma_0,f_1]\cdot[\gamma'_0,f_1]=\gamma_1\cdot\gamma'_1.$$

Assume the claim is true for $m-1$. By the induction hypothesis,

$$\begin{aligned}
c_m(\beta_k,f_1,\cdots,f_m) &= c_1\circ(c_{m-1}(\beta_k,f_1,\cdots,f_{m-1})\times f_m)\\
&= [c_{m-1}(\beta_k,f_1,\cdots,f_{m-1}),f_m]\\
&= [\gamma_{m-1}\cdot\gamma'_{m-1},f_m]
\end{aligned}$$

Since the restrictions of γ_{m-1} and γ'_{m-1} to $A_{k+1}\vee(A^k\times A^l)$ are null-homotopic, by Lemma 3.3

$$c_m(\beta_k,f_1,\cdots,f_m)=[\gamma_{m-1},f_m]\cdot[\gamma'_{m-1},f_m]=\gamma_m\cdot\gamma'_m.$$

By putting $m=n-1$ we get $c_{n-1}(\beta_k,f_1,\cdots,f_n)=\gamma_{n-1}\cdot\gamma'_{n-1}$. Notice that γ_{n-1} and γ'_{n-1} are commutators of length $n+2$ and $n+1$ respectively, which are null-homotopic due to the condition on the homotopy nilpotency of G. Therefore $c_{n-1}(\beta_k,f_1,\cdots,f_{n-1})$ is null-homotopic. $\square$

Now we have all the ingredients to prove Lemma 2.5.

Proof of Lemma 2.5. Suppose $c(\imath,\imath)^{p^r}$ is null-homotopic and $nil(G)$ is less than p^r+1. We prove that $c(m_k,m_l)^{p^r}$ is null-homotopic for all k and l by induction. By Lemma 3.4, $c(m_k,\imath)^{p^r}$ and $c(\imath,m_l)^{p^r}$ are null-homotopic for all k and l. Assume $c(m_k,m_l)^{p^r}$ is null-homotopic for some fixed k and l. We need to show that $c(m_{k+1},m_l)^{p^r}$ is null-homotopic. By Lemma 3.6, we have

$$c(m_{k+1},m_l)^{p^r}=\alpha_k^{p^r}\cdot\beta_k^{p^r}\cdot\prod_{i=1}^{p^r-1}[\beta_k,\alpha_k^i].$$

The factor $\alpha_k^{p^r}$ is null-homotopic due to the definition of α_k and hypothesis, and $\beta_k^{p^r}$ is null-homotopic by Lemma 3.7. So it remains to show that $\prod_{i=1}^{p^r-1}[\beta_k, \alpha_k^i]$ is null-homotopic.

By Lemma 3.8,

$$\prod_{i=1}^{p^r-1}[\beta_k, \alpha_k^i] = \prod_{j=1}^{p^r-1} c_j(\beta_k, \alpha_k, \cdots, \alpha_k)^{\binom{p^r}{j+1}}$$

Observe that $\binom{p^r}{j+1}$ is divisible by p^r for $1 \leq j \leq p^r - 2$. By Lemma 3.3 we have

$$c_j(\beta_k^n, \alpha_k, \cdots, \alpha_k) = c_j(\beta_k, \alpha_k, \cdots, \alpha_k)^n$$

for all n. In our case we have

$$c_j(\beta_k, \alpha_k, \cdots, \alpha_k)^{\binom{p^r}{j+1}} = c_j(\beta_k^{p^r}, \alpha_k, \cdots, \alpha_k)^{\binom{p^r}{j+1}/p^r}.$$

Also, when $j = p^r - 1$, the term $c_{p^r-1}(\beta_k, \alpha_k, \cdots, \alpha_k)$ is null-homotopic by Lemma 3.9. Putting these together we obtain

$$\prod_{i=1}^{p^r-1}[\beta_k, \alpha_k^i] = \prod_{j=1}^{p^r-2} c_j(\beta_k^{p^r}, \alpha_k, \cdots, \alpha_k)^{\binom{p^r}{j+1}/p^r}.$$

We have shown that $\beta_k^{p^r}$ is null-homotopic in Lemma 3.7, so $\prod_{i=1}^{p^r}[\beta_k, \alpha_k^i]$ is null-homotopic and the induction is completed. □

4. Orders of Samelson products of quasi-p-regular groups

In this section we apply Theorem 2.6 to calculate the orders of $\langle \mathbb{1}_G, \mathbb{1}_G \rangle$ for certain Lie groups G. Recall that G is rationally homotopy equivalent to a product of spheres $\prod_{i=1}^{l} S^{2n_i-1}$, where $n_1 \leq \cdots \leq n_l$. The sequence $(2n_1 - 1, \cdots, 2n_l - 1)$ is called the *type* of G. After localization at p, G is homotopy equivalent to a product of H-spaces $\prod_{i=1}^{p-1} B_i$, and A is homotopy equivalent to a wedge of co-H-spaces $\bigvee_{i=1}^{p-1} A_i$ such that A_i is a subspace of B_i. For $1 \leq i \leq p-1$, let $\imath_i : A_i \to B_i$ be the inclusion. Then $H_*(B_i)$ is the exterior algebra generated by $(\imath_i)_*(\tilde{H}_*(A_i))$. If each B_i is a sphere, then we call G *p-regular*. If each A_i is a sphere or a CW-complex with two cells, then we call G *quasi-p-regular*. When A_i is a CW-complex with two cells, it is homotopy equivalent to the cofibre of α_{2n_i-1}, which is the generator of the homotopy group $\pi_{2n_i+2p-4}(S^{2n_i-1})$, and the corresponding B_i is the S^{2n_i-1}-bundle $B(2n-1, 2n+2p-3)$ over S^{2n_i+2p-3} classified by $\frac{1}{2}\alpha_{2n_i-1}$ [7].

The homotopy nilpotence classes of certain quasi-p-regular Lie groups are known.

Theorem 4.1 (Kaji and Kishimoto [3]). *A p-regular Lie group has homotopy nilpotence class at most 3.*

Theorem 4.2 (Kishimoto [4]). *For $p \geq 7$, a quasi-p-regular $SU(n)$ has homotopy nilpotence class at most 3.*

Theorem 4.3 (Theriault [10]). *For $p \geq 7$, a quasi-p-regular exceptional Lie group has homotopy nilpotence class at most 2.*

For $t = n-p+1$ and $t' = n-\frac{1}{2}p+1$, assume G and p are in the following list:

$$
\begin{array}{llll}
 & SU(n) & \simeq B(3,2p+1)\times\cdots\times B(2t-1,2n-1)\times S^{2t+1}\times\cdots\times S^{2p-1} & p>\frac{n}{2}\\
 & & \simeq S^3\times S^5\times\cdots\times S^{2n-1} & n\leq p\leq\frac{n}{2}\\
 & Sp(n) & \simeq B(3,2p+1)\times\cdots\times B(2t'-1,4n-1)\times S^{2t'+1}\times\cdots\times S^{2p-1} & p>n\\
 & F_4 & \simeq B(3,15)\times B(11,23) & p=7\\
 & & \simeq B(3,23)\times S^{11}\times S^{15} & p=11\\
 & & \simeq S^3\times S^{11}\times S^{15}\times S^{23} & p>11\\
 & E_6 & \simeq F_4\times S^9\times S^{17} & p\geq 7\\
 & E_7 & \simeq B(3,23)\times B(15,35)\times S^{11}\times S^{19}\times S^{27} & p=11\\
(5) & & \simeq B(3,27)\times B(11,35)\times S^{15}\times S^{19}\times S^{23} & p=13\\
 & & \simeq B(3,35)\times S^{11}\times S^{15}\times S^{19}\times S^{23}\times S^{27} & p=17\\
 & & \simeq S^3\times S^{11}\times S^{15}\times S^{19}\times S^{23}\times S^{27}\times S^{35} & p>17\\
 & E_8 & \simeq B(3,23)\times B(15,35)\times B(27,47)\times B(39,59) & p=11\\
 & & \simeq B(3,27)\times B(15,39)\times B(23,47)\times B(35,59) & p=13\\
 & & \simeq B(3,35)\times B(15,47)\times B(27,59)\times S^{23}\times S^{39} & p=17\\
 & & \simeq B(3,39)\times B(23,59)\times S^{15}\times S^{27}\times S^{35}\times S^{47} & p=23\\
 & & \simeq B(3,59)\times S^{15}\times S^{23}\times S^{27}\times S^{35}\times S^{39}\times S^{47} & p=29\\
 & & \simeq S^3\times S^{15}\times S^{23}\times S^{27}\times S^{35}\times S^{39}\times S^{47}\times S^{59} & p>29.
\end{array}
$$

By Theorem 2.6, the order of $\langle \mathbb{1}_G, \mathbb{1}_G\rangle$ equals the order of $\langle\imath,\imath\rangle$ in these groups.

4.1. **Upper bounds on the orders of $\langle \mathbb{1}_G, \mathbb{1}_G\rangle$ for quasi-p-regular Lie groups.** Since $\langle\imath,\imath\rangle \in [A\wedge A, G]$ and

$$
(6)\qquad [A\wedge A,G]\cong[(\bigvee_{i=1}^{p-1}A_i)\wedge(\bigvee_{j=1}^{p-1}A_j),G]\cong\prod_{i,j=1}^{p-1}[A_i\wedge A_j,G]\cong\prod_{i,j,k=1}^{p-1}[A_i\wedge A_j,B_k],
$$

the order of $\langle\imath,\imath\rangle$ cannot exceed the least common multiple of the orders of $[A_i\wedge A_j, B_k]$ for all i,j and k. Let C_{2n_i-1} be the cofiber of the generator α_{2n_i-1} of the homotopy group $\pi_{2n_i+2p-4}(S^{2n_i-1})$. When G is quasi-p-regular, each A_i is a sphere S^{2n_i-1} or C_{2n_i-1}, so $A_i\wedge A_j$ is either $S^{2n_i+2n_j-2}$, $C_{2n_i+2n_j-2}$ or $C_{2i-1}\wedge C_{2n_j-1}$. In the following we consider the orders of $[A_i\wedge A_j, B_k]$ case by case.

If $A_i\wedge A_j$ is $S^{2n_i+2n_j-2}$, then $[A_i\wedge A_j,B_k]$ is $\pi_{2n_i+2n_j-2}(B_k)$. The homotopy groups of B_k are known in a range.

Theorem 4.4 (Toda [11], Mimura and Toda [8], Kishimoto [4])**.** *Localized at p, we have*

$$
\pi_{2n-1+k}(S^{2n-1})\cong\begin{cases}\mathbb{Z}/p\mathbb{Z} & \text{for } k=2i(p-1)-1,\ 1\leq i\leq p-1\\ \mathbb{Z}/p\mathbb{Z} & \text{for } k=2i(p-1)-2,\ n\leq i\leq p-1\\ 0 & \text{other cases for } 1\leq k\leq 2p(p-1)-3,\end{cases}
$$

$$
\pi_{2n-1+k}(B(3,2p+1))\cong\begin{cases}\mathbb{Z}/p\mathbb{Z} & \text{for } k=2i(p-1)-1,\ 2\leq i\leq p-1\\ \mathbb{Z} & \text{for } k=2p-2\\ 0 & \text{other cases for } 1\leq k\leq 2p(p-1)-3,\end{cases}
$$

and

$$\pi_{2n-1+k}(B(2n-1,2n+2p-3)) \cong \begin{cases} \mathbb{Z}/p^2\mathbb{Z} & \text{for } k=2i(p-1)-1,\ 2\leq i\leq p-1 \\ \mathbb{Z}/p\mathbb{Z} & \text{for } k=2i(p-1)-2,\ n\leq i\leq p-1 \\ \mathbb{Z} & \text{for } k=2p-2 \\ 0 & \text{other cases for } 1\leq k\leq 2p(p-1)-3. \end{cases}$$

Since $2n_i+2n_j-2$ is even, $\pi_{2n_i+2n_j-2}(B_k)$ is isomorphic to either 0, $\mathbb{Z}/p\mathbb{Z}$ or $\mathbb{Z}/p^2\mathbb{Z}$. Therefore the order of $[A_i\wedge A_j, B_k]$ is at most p^2.

If $A_i\wedge A_j$ is $C_{2n_i+2n_j-2}$, then the cofibration

$$S^{2n_i+2n_j-2}\to C_{2n_i+2n_j-2}\to S^{2n_i+2n_j+2p-4}$$

induces an exact sequence

$$\pi_{2n_i+2n_j+2p-4}(B_k)\to [C_{2n_i+2n_j-2}, B_k]\to \pi_{2n_i+2n_j-2}(B_k). \tag{7}$$

Since $C_{2n_i+2n_j-2}$ is a suspension and B_k is an H-space, the three groups are abelian. By Theorem 4.4, the first and the last homotopy groups have orders at most p^2, so the order of $[C_{2n_i+2n_j-2}, B_k]$ is at most p^4.

If $A_i\wedge A_j$ is $C_{2n_i-1}\wedge C_{2n_j-1}$, then it is a CW-complex with one cell of dimension $2n_i+2n_j-2$, two cells of dimension $2n_i+2n_j+2p-4$ and one cell of dimension $2n_i+2n_j+4p-6$. Let C' be the $(2n_i+2n_j+4p-7)$-skeleton of $C_{2n_i-1}\wedge C_{2n_j-1}$, that is, $C_{2n_i-1}\wedge C_{2n_j-1}$ minus the top cell. Then the cofibration $C'\to C_{2n_i-1}\wedge C_{2n_j-1}\to S^{2n_i+2n_j+4p-6}$ induces an exact sequence of abelian groups

$$\pi_{2n_i+2n_j+4p-6}(B_k)\longrightarrow [C_{2n_i-1}\wedge C_{2n_j-1}, B_k]\longrightarrow [C', B_k].$$

According to [1], C' is homotopy equivalent to $C_{2n_i+2n_j-2}\vee S^{2n_i+2n_j+2p-4}$, so we have

$$[C', B_k]\cong [C_{2n_i+2n_j-2}, B_k]\oplus \pi_{2n_i+2n_j+2p-4}(B_k).$$

We have shown that $[C_{2n_i+2n_j-2}, B_k]$ has order at most p^4. By Theorem 4.4, $\pi_{2n_i+2n_j+2p-4}(B_k)$ and $\pi_{2n_i+2n_j+4p-6}(B_k)$ have orders at most p^2. Therefore the order of $[C_{2n_i-1}\wedge C_{2n_j-1}, B_k]$ is at most p^6.

Summarizing the above discussion, we have the following proposition.

Proposition 4.5. *Let G and p be in (5). Then the order of $\langle \mathbb{1}_G, \mathbb{1}_G\rangle$ is at most p^6.*

This gives a very rough upper bound on the orders of $\langle \mathbb{1}_G, \mathbb{1}_G\rangle$. We can sharpen the range by refining our calculation according to individual cases of G and p.

Case I: G is p-regular. Suppose G is p-regular. Then $B_i=A_i=S^{2n_i-1}$ and $p\geq n_l$. All summands $[A_i\wedge A_j, B_k]$ in (6) are homotopy groups $\pi_{2n_i+2n_j-2}(S^{2n_k-1})$. According to Theorem 4.4, their orders are at most p since

$$2(n_i+n_j-n_k)-1\leq 2(2n_l-2)-1\leq 2p(p-1)-3$$

for all i, j and k. Therefore the order of $\langle \imath,\imath\rangle$ is at most p and so is the order of $\langle \mathbb{1}_G, \mathbb{1}_G\rangle$ by Theorem 2.6. McGibbon [5] showed that G is homotopy commutative if and only if either $p>2n_l$, or (G,p) is $(Sp(2),3)$ or $(G_2,5)$. Therefore we have the following statement.

Theorem 4.6. *Let G be a p-regular Lie group of type $(2n_1-1,\cdots,2n_l-1)$. Then the order of $\langle \mathbb{1}_G, \mathbb{1}_G\rangle$ is p if $n_l\leq p<2n_l$, and is 1 if $p>2n_l$.*

Case II: G is a quasi-p-regular $SU(n)$ and $p \geq 7$. Suppose $G = SU(n)$ is quasi-p-regular and $p \geq 7$. Then $n_i = i+1$ and $p > \frac{n}{2}$. Let $t = n-p+1$ and $2 \leq t \leq p$. Localized at p, there are homotopy equivalences

$$SU(n) \simeq B(3, 2p+1) \times \cdots \times B(2t-1, 2n-1) \times S^{2t+1} \times \cdots \times S^{2p-1}$$

and

$$A \simeq C_3 \vee \cdots \vee C_{2t-1} \vee S^{2t+1} \vee \cdots \vee S^{2p-1}$$

For $1 \leq j \leq t$ and $t+1 \leq i \leq p$, let ϵ_i and λ_i be the compositions

$$\epsilon_i : S^{2i-1} \hookrightarrow A \xrightarrow{\imath} G \quad \text{and} \quad \lambda_j : C_{2j-1} \hookrightarrow A \xrightarrow{\imath} G.$$

Kishimoto calculated some of their Samelson products in [4].

Theorem 4.7 (Kishimoto [4])**.** *Let G be a quasi-p-regular $SU(n)$. For $2 \leq j, j' \leq t$ and $t+1 \leq i, i' \leq p$,*

(1) the order of $\langle \epsilon_i, \epsilon_{i'} \rangle$ is at most p;
(2) if $i \neq p$ and $j \neq t$, then the order of $\langle \epsilon_i, \lambda_j \rangle$ is at most p;
(3) if $j + j' \leq p$, then $\langle \lambda_j, \lambda_{j'} \rangle$ is null-homotopic;
(4) if $p+1 \leq j+j' \leq 2p-1$, then $\langle \lambda_j, \lambda_{j'} \rangle$ can be compressed into $S^{2(j+j'-p)+1} \subset SU(n)$.

Using these results we can give a bound for the order of $\langle \mathbb{1}_G, \mathbb{1}_G \rangle$.

Theorem 4.8. *For $G = SU(n)$ and $p \geq 7$, let the order of $\langle \mathbb{1}_G, \mathbb{1}_G \rangle$ be p^r.*

- *If $n > 2p$, then $r = 0$;*
- *If $n \leq p < 2p$, then $r = 1$;*
- *If $\frac{2}{3}n + 1 \leq p < n$, then $1 \leq r \leq 2$;*
- *If $\frac{n}{2} < p \leq \frac{2}{3}n$ and $n \neq 2p-1$, then $1 \leq r \leq 3$;*
- *If $n = 2p-1$, then $1 \leq r \leq 6$.*

Proof. When $p \geq n$, G is p-regular and we have shown the first two statements in Theorem 4.6. Assume $\frac{n}{2} < p < n$. By Theorem 2.6, the order of $\langle \mathbb{1}_G, \mathbb{1}_G \rangle$ equals the order of $\langle \imath, \imath \rangle$. Since $\langle \imath, \imath \rangle$ is a wedge of Samelson products of ϵ_i's and λ_j's, we need to consider the orders of $\langle \epsilon_i, \epsilon_{i'} \rangle$, $\langle \epsilon_i, \lambda_j \rangle$ and $\langle \lambda_j, \lambda_{j'} \rangle$.

First, the first two statements of Theorem 4.7 imply that the orders of $\langle \epsilon_i, \epsilon_{i'} \rangle$ and $\langle \epsilon_i, \lambda_j \rangle$ are at most p except for $\langle \epsilon_p, \lambda_t \rangle$. Put $n_i = p$ and $n_j = t$ in (7) to obtain the exact sequence

$$\pi_{4p+2t-4}(B_k) \to [C_{2p+2t-2}, B_k] \to \pi_{2p+2t-2}(B_k)$$

where $2 \leq k \leq p$. According to Theorem 4.4, the two homotopy groups are trivial except for $k = t+1$ or $t = p$ and $k = 2$. In the first case, B_k is S^{2t+1}, and $\pi_{4p+2t-4}(S^{2t+1})$ and $\pi_{2p+2t-2}(S^{2t+1})$ are $\mathbb{Z}/p\mathbb{Z}$. In the second case, B_k is $B(3, 2p+1)$, and $\pi_{6p-4}(B(3, 2p+1))$ and $\pi_{4p-2}(B(3, 2p+1))$ are $\mathbb{Z}/p^2\mathbb{Z}$. By exactness the order of $[C_{2p+2t-2}, B_k]$ is at most p^2 for $2 \leq k \leq p$ and $n \neq 2p-1$, and consequently so is the order of $\langle \epsilon_i, \lambda_j \rangle$.

Second, the third statement of Theorem 4.7 implies that $\langle \lambda_j, \lambda_{j'} \rangle$ is null-homotopic for $j + j' \leq p - 1$. When $p \geq \frac{2}{3}n + 1$, we have

$$n \leq \tfrac{3}{2}(p-1) \quad \text{and} \quad t = n - p + 1 \leq \tfrac{1}{2}(p-1).$$

In this case the order of $\langle \lambda_j, \lambda_{j'} \rangle$ is always 1 since $j + j' \leq 2t \leq p - 1$. When $\frac{n}{2} < p \leq \frac{2}{3}n$, we need to consider the orders of $\langle \lambda_j, \lambda_{j'} \rangle$ for $p + 1 \leq j + j'$. By the last statement of

Theorem 4.7, $\langle \lambda_j, \lambda_{j'} \rangle$ is in $[C_{2j-1} \wedge C_{2j'-1}, S^{2(j+j'-p)+1}]$ if $j + j' \leq 2p - 1$. Since $j, j' \leq t \leq p$, this can always be achieved for $n \neq 2p - 1$. There is a graph of short exact sequences

$$\begin{tikzcd}
 & & \pi_{2j+2j'+2p-4}(S^{2(j+j'-p)+1})^{\oplus 2} \arrow[d] \\
\pi_{2j+2j'+4p-6}(S^{2(j+j'-p)+1}) \arrow[r] & {[C_{2j-1} \wedge C_{2j'-1}, S^{2(j+j'-p)+1}]} \arrow[r] & {[C', S^{2(j+j'-p)+1}]} \arrow[d] \\
 & & \pi_{2j+2j'-2}(S^{2(j+j'-p)+1})
\end{tikzcd}$$

where C' is the subcomplex of $C_{2j-1} \wedge C_{2j'-1}$ without the top cell. By Theorem 4.4, the three homotopy groups are $\mathbb{Z}/p\mathbb{Z}$. The exactness of the column and the row implies that the orders of $[C', S^{2(j+j'-p)+1}]$ and $[C_{2j-1} \wedge C_{2j'-1}, S^{2(j+j'-p)+1}]$ are at most p^2 and p^3. Therefore $\langle \lambda_j, \lambda_{j'} \rangle$ has order at most p^3 when $n \neq 2p - 1$ and $\frac{n}{2} < p \leq \frac{2}{3}n$.

We summarize the above discussion in the following table:

	an upper bound on the order of		
	$\langle \epsilon_i, \epsilon_{i'} \rangle$	$\langle \epsilon_i, \lambda_j \rangle$	$\langle \lambda_j, \lambda_{j'} \rangle$
$\frac{2}{3}n + 1 \leq p \leq n$	p	p^2	1
$\frac{n}{2} < p \leq \frac{2}{3}n$, $n \neq 2p - 1$	p	p^2	p^3
$n = 2p - 1$	p	p^2	p^6

By Theorem 2.6, the order of $\langle \mathbb{1}_G, \mathbb{1}_G \rangle$ equals the order of $\langle \imath, \imath \rangle$ which is the least common multiple of the orders of $\langle \epsilon_i, \epsilon_{i'} \rangle$, $\langle \epsilon_i, \lambda_j \rangle$ and $\langle \lambda_j, \lambda_{j'} \rangle$, so the statement follows. $\square$

Case III: G is a quasi-p-regular exceptional Lie group and $p \geq 7$. Suppose $p \geq 7$ and G is a quasi-p-regular exceptional Lie group. That is

- when $G = F_4$ or E_6, $p = 7$ or 11;
- when $G = E_7$, $p = 11, 13$ or 17;
- when $G = E_8$, $p = 11, 13, 17, 23$ or 29.

For each case, we can calculate bounds on the orders of $[A_i \wedge A_j, B_k]$ for all i, j and k in (6) according to the CW-structure of A. Then we obtain the following statement.

Theorem 4.9. *For $p \geq 7$, suppose G is a quasi-p-regular exceptional Lie group which is not p-regular. Let the order of $\langle \mathbb{1}_G, \mathbb{1}_G \rangle$ be p^r. Then we have the following table*

G	p	*value(s) of* r
F_4	7	$1 \leq r \leq 4$
	11	1
E_6	7	$1 \leq r \leq 4$
	11	1
E_7	11	$1 \leq r \leq 3$
	13	1 *or* 2
	17	1
E_8	11	$1 \leq r \leq 6$
	13	$1 \leq r \leq 4$
	17	$1 \leq r \leq 3$
	19	$1 \leq r \leq 4$
	$23, 29$	1

Remark: It would be interesting if the precise order of $\langle \mathbb{1}_G, \mathbb{1}_G \rangle$ could be obtained in the case of Theorem 4.9.

References

[1] B. Gray, *Associativity in two-cell complexes*, Geometry and Topology: Aarhus (1998), 185-196, Contemp. Math. 258 (2000)

[2] F. Cohen and J. Neisendorfer, *A construction of p-local H-spaces*, Lecture Note in Math. Vol. 1051, Springer, Berlin, (1984), 351 – 359.

[3] S. Kaji, D. Kishimoto, *Homotopy nilpotency in p-regular loop spaces*, Mathematische Zeitschrift, (2010), 264 – 209.

[4] D. Kishimoto, *Homotopy nilpotency in localized $SU(n)$*, Homology, Homotopy and Applications, **11**, (2009), 61 – 79.

[5] C. McGibbon, *Homotopy commutativity in localized groups*, American Journal of Mathematics, **106**(3), (1984), 665 – 687.

[6] M. Mimura, *The homotopy groups of Lie groups of low rank*, Journal of Mathematics of Kyoto University, **6**-2, (1967), 131 – 176.

[7] M. Mimura, G. Nishida, H. Toda *Mod p decomposition of compact Lie groups*, Publ. RIMS, Kyoto Univ, **13**, (1977), 627 – 680.

[8] M. Mimura and H. Toda *Cohomology operations and homotopy of compact Lie groups I*, Topology, **9**, (1970), 317 – 336.

[9] S. Theriault, *Odd primary H-structure of low rank Lie groups and its application to exponents*, Transactions of the American Mathematical Society, **359**, (2007), 4511 – 4535.

[10] S. Theriault, *The dual polyhedral product, cocategory and nilpotence*, arXiv:1506.05998v2, (2016).

[11] H. Toda, *Composition Methods in Homotopy Groups of Spheres*, Annals of Mathematics Studies **49**, Princeton University Press, Princeton N.J., 1962.

Mathematical Sciences, University of Southampton, SO17 1BJ, UK
E-mail address: `tls1g14@soton.ac.uk`

HOMOTOPY TYPES OF $SU(n)$-GAUGE GROUPS OVER NON-SPIN 4-MANIFOLDS

TSELEUNG SO

Abstract. Let M be an orientable, simply-connected, closed, non-spin 4-manifold and let $\mathcal{G}_k(M)$ be the gauge group of the principal G-bundle over M with second Chern class $k \in \mathbb{Z}$. It is known that the homotopy type of $\mathcal{G}_k(M)$ is determined by the homotopy type of $\mathcal{G}_k(\mathbb{CP}^2)$. In this paper we investigate properties of $\mathcal{G}_k(\mathbb{CP}^2)$ when $G = SU(n)$ that partly classify the homotopy types of the gauge groups.

1. Introduction

Let G be a simple, simply-connected, compact Lie group and let M be an orientable, simply-connected, closed 4-manifold. Then a principal G-bundle P over M is classified by its second Chern class $k \in \mathbb{Z}$. The associated gauge group $\mathcal{G}_k(M)$ is the topological group of G-equivariant automorphisms of P which fix M.

When M is a spin 4-manifold, topologists have been studying the homotopy types of gauge groups over M extensively over the last twenty years. On the one hand, Theriault showed that [12] there is a homotopy equivalence

$$\mathcal{G}_k(M) \simeq \mathcal{G}_k(S^4) \times \prod_{i=1}^{d} \Omega^2 G,$$

where d is the second Betti number of M. Therefore to study the homotopy type of $\mathcal{G}_k(M)$ it suffices to study $\mathcal{G}_k(S^4)$. On the other hand, many cases of homotopy types of $\mathcal{G}_k(S^4)$'s are known. For examples, there are 6 distinct homotopy types of $\mathcal{G}_k(S^4)$'s for $G = SU(2)$ [8], and 8 distinct homotopy types for $G = SU(3)$ [4]. When localized rationally or at any prime, there are 16 distinct homotopy types for $G = SU(5)$ [15] and 8 distinct homotopy types for $G = Sp(2)$ [13].

When M is a non-spin 4-manifold, the author in [11] showed that there is a homotopy equivalence

$$\mathcal{G}_k(M) \simeq \mathcal{G}_k(\mathbb{CP}^2) \times \prod_{i=1}^{d-1} \Omega^2 G,$$

so the homotopy type of $\mathcal{G}_k(M)$ depends on the special case $\mathcal{G}_k(\mathbb{CP}^2)$. Compared to the extensive work on $\mathcal{G}_k(S^4)$, only two cases of $\mathcal{G}_k(\mathbb{CP}^2)$ have been studied, which are the $SU(2)$- and $SU(3)$-cases [9, 14]. As a sequel to [11], this paper investigates the homotopy types of $\mathcal{G}_k(\mathbb{CP}^2)$'s in order to explore gauge groups over non-spin 4-manifolds.

A common approach to classifying the homotopy types of gauge groups is as follows. Atiyah, Bott and Gottlieb [1, 2] showed that the classifying space $B\mathcal{G}_k(M)$ is homotopy equivalent to the connected component $\mathrm{Map}_k(M, BG)$ of the mapping space $\mathrm{Map}(M, BG)$

2010 *Mathematics Subject Classification.* Primary 55P15; Secondary 54C35, 81T13.
Key words and phrases. gauge groups, homotopy type, non-spin 4-manifolds.

containing the map $k\alpha \circ q$, where $q : M \to S^4$ is the quotient map and α is a generator of $\pi_4(BG) \cong \mathbb{Z}$. The evaluation map $ev : B\mathcal{G}_k(M) \to BG$ induces a fibration sequence

$$\mathcal{G}_k(M) \longrightarrow G \xrightarrow{\partial_k} \mathrm{Map}_0^*(M, BG) \longrightarrow B\mathcal{G}_k(M) \xrightarrow{ev} BG. \tag{1}$$

For $M = S^4$, the order of $\partial_1 : G \to \Omega_0^3 G$ helps determine the classification of $\mathcal{G}_k(S^4)$'s by the following theorem. The first part is due to [13] and the second is due to [6].

Theorem 1.1 (Theriault, [13]; Kishimoto, Kono, Tsutaya [6]). *Let m be the order of ∂_1. Denote the p-component of a by $\nu_p(a)$ and the greatest common divisor of a and b by (a, b).*

(1) *If $(m, k) = (m, l)$, then $\mathcal{G}_k(S^4)$ is homotopy equivalent to $\mathcal{G}_l(S^4)$ when localized rationally or at any odd prime.*
(2) *If $\mathcal{G}_k(S^4)$ is homotopy equivalent to $\mathcal{G}_l(S^4)$ and G is of low rank (for details please see [6]), then $\nu_p(m, k) = \nu_p(m, l)$ for any odd prime p.*

Therefore the classification problem reduces to calculating the order m of ∂_1. Known examples are $m = 12$ for $G = SU(2)$ [8], $m = 24$ for $G = SU(3)$ [4], $m = 120$ for $G = SU(5)$ [15] and $m = 40$ for $G = Sp(2)$ [13]. For most cases of G, the exact value of m is difficult to compute, but we are still able to obtain partial results. When G is $SU(n)$, the order of ∂_1 and $n(n^2 - 1)$ have the same odd primary components if $n < (p-1)^2 + 1$ [6, 16]. Moreover, Hamanaka and Kono showed a necessary condition $(n(n^2-1), k) = (n(n^2-1), l)$ for a homotopy equivalence $\mathcal{G}_k(S^4) \simeq \mathcal{G}_l(S^4)$ [4].

In this paper we consider gauge groups over $\mathbb{CP}^2$. Take $M = \mathbb{CP}^2$ in (1) and denote the boundary map by $\partial'_k : G \to \mathrm{Map}_0^*(\mathbb{CP}^2, BG)$. Since $\mathrm{Map}_0^*(\mathbb{CP}^2, BG)$ is not an H-space, $[G, \mathrm{Map}_0^*(\mathbb{CP}^2, BG)]$ is not a group so the order of ∂'_k makes no sense. However, we can still define an "order" of ∂'_k [14], which will be mentioned in Section 2. We show that the "order" of ∂'_1 helps determine the homotopy type of $\mathcal{G}_k(\mathbb{CP}^2)$ like part (1) of Theorem 1.1.

Theorem 1.2. *Let m' be the "order" of ∂'_1. If $(m', k) = (m', l)$, then $\mathcal{G}_k(\mathbb{CP}^2)$ is homotopy equivalent to $\mathcal{G}_l(\mathbb{CP}^2)$ when localized rationally or at any prime.*

Theorem 1.2 is Theorem 2.4, which will be proved in Section 2.

In Section 4, we study the $SU(n)$-gauge groups over $\mathbb{CP}^2$ and use unstable K-theory to give a lower bound on the "order" of ∂'_1.

Theorem 1.3. *When G is $SU(n)$, the "order" of ∂'_1 is at least $\frac{1}{2}n(n^2-1)$ for n odd, and $n(n^2-1)$ for n even.*

In Section 5, we prove a necessary condition for the homotopy equivalence $\mathcal{G}_k(\mathbb{CP}^2) \simeq \mathcal{G}_l(\mathbb{CP}^2)$ similar to that in [4].

Theorem 1.4. *Let G be $SU(n)$. If $\mathcal{G}_k(\mathbb{CP}^2)$ is homotopy equivalent to $\mathcal{G}_l(\mathbb{CP}^2)$, then*

$$\begin{cases} (\frac{1}{2}n(n^2-1), k) = (\frac{1}{2}n(n^2-1), l), & \text{for } n \text{ odd;} \\ (n(n^2-1), k) = (n(n^2-1), l), & \text{for } n \text{ even.} \end{cases}$$

2. Some facts about boundary map ∂'_1

Take M to be S^4 and $\mathbb{CP}^2$ respectively in fibration (1) to obtain fibration sequences

$$\mathcal{G}_k(S^4) \longrightarrow G \xrightarrow{\partial_k} \Omega_0^3 G \longrightarrow B\mathcal{G}_k(S^4) \xrightarrow{ev} BG \tag{2}$$

$$\mathcal{G}_k(\mathbb{CP}^2) \longrightarrow G \xrightarrow{\partial'_k} \mathrm{Map}_0^*(\mathbb{CP}^2, BG) \longrightarrow B\mathcal{G}_k(\mathbb{CP}^2) \xrightarrow{ev} BG. \tag{3}$$

There is also a cofibration sequence

$$S^3 \xrightarrow{\eta} S^2 \longrightarrow \mathbb{CP}^2 \xrightarrow{q} S^4, \tag{4}$$

where η is Hopf map and q is the quotient map. Due to the naturality of q^*, we combine fibrations (2) and (3) to obtain a commutative diagram of fibration sequences

$$\begin{tikzcd}
\mathcal{G}_k(S^4) \arrow[r] \arrow[d, "q^*"] & G \arrow[r, "\partial_k"] \arrow[d, equal] & \Omega_0^3 G \arrow[r] \arrow[d, "q^*"] & B\mathcal{G}_k(S^4) \arrow[r] \arrow[d, "q^*"] & BG \arrow[d, equal] \\
\mathcal{G}_k(\mathbb{CP}^2) \arrow[r] & G \arrow[r, "\partial'_k"] & \mathrm{Map}_0^*(\mathbb{CP}^2, BG) \arrow[r] & B\mathcal{G}_k(\mathbb{CP}^2) \arrow[r] & BG
\end{tikzcd} \tag{5}$$

It is known that [10] ∂_k is triple adjoint to Samelson product

$$\langle k\imath, \mathbb{1}\rangle : S^3 \wedge G \xrightarrow{k\imath\wedge\mathbb{1}} G \wedge G \xrightarrow{\langle \mathbb{1},\mathbb{1}\rangle} G,$$

where $\imath : S^3 \to SU(n)$ is the inclusion of the bottom cell and $\langle \mathbb{1}, \mathbb{1}\rangle$ is the Samelson product of the identity on G with itself. The order of ∂_k is its multiplicative order in the group $[G, \Omega_0^3 G]$.

Unlike $\Omega_0^3 G$, $\mathrm{Map}_0^*(\mathbb{CP}^2, BG)$ is not an H-space, so ∂'_k has no order. In [14], Theriault defined the "order" of ∂'_k to be the smallest number m' such that the composition

$$G \xrightarrow{\partial_k} \Omega_0^3 G \xrightarrow{m'} \Omega_0^3 G \xrightarrow{q^*} \mathrm{Map}_0^*(\mathbb{CP}^2, BG)$$

is null homotopic. In the following, we interpret the "order" of ∂'_k as its multiplicative order in a group contained in $[\mathbb{CP}^2 \wedge G, BG]$.

Apply $[- \wedge G, BG]$ to cofibration (4) to obtain an exact sequence of sets

$$[\Sigma^3 G, BG] \xrightarrow{(\Sigma\eta)^*} [\Sigma^4 G, BG] \xrightarrow{q^*} [\mathbb{CP}^2 \wedge G, BG].$$

All terms except $[\mathbb{CP}^2 \wedge G, BG]$ are groups and $(\Sigma\eta)^*$ is a group homomorphism since $\Sigma\eta$ is a suspension. We want to refine this exact sequence so that the last term is replaced by a group. Observe that $\mathbb{CP}^2$ is the cofiber of η and so there is a coaction $\psi : \mathbb{CP}^2 \to \mathbb{CP}^2 \vee S^4$. We show that the coaction gives a group structure on $Im(q^*)$.

Lemma 2.1. *Let Y be a space and let $A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} \Sigma A$ be a cofibration sequence. If ΣA is homotopy cocommutative, then $Im(h^*)$ is an abelian group and*

$$[\Sigma B, Y] \xrightarrow{(\Sigma f)^*} [\Sigma A, Y] \xrightarrow{h^*} Im(h^*) \longrightarrow 0$$

is an exact sequence of groups and group homomorphisms.

Proof. Apply $[-, Y]$ to the cofibration to get an exact sequence of sets

$$[\Sigma B, Y] \xrightarrow{(\Sigma f)^*} [\Sigma A, Y] \xrightarrow{h^*} [C, Y]. \tag{6}$$

Note that $[\Sigma B, Y]$ and $[\Sigma A, Y]$ are groups, and $(\Sigma f)^*$ is a group homomorphism. We will replace $[C, Y]$ by $Im(h^*)$ and define a group structure on it such that $h^* : [\Sigma A, Y] \to Im(h^*)$ is a group homomorphism.

For any α and β in $[\Sigma A, Y]$, we define a binary operator $\boxtimes$ on $Im(h^*)$ by

$$h^*\alpha \boxtimes h^*\beta = h^*(\alpha + \beta).$$

To check this is well-defined we need to show $h^*(\alpha+\beta) \simeq h^*(\alpha'+\beta) \simeq h^*(\alpha+\beta')$ for any $\alpha, \alpha', \beta, \beta'$ satisfying $h^*\alpha \simeq h^*\alpha'$ and $h^*\beta \simeq h^*\beta'$.

First we show $h^*(\alpha+\beta) \simeq h^*(\alpha'+\beta)$. By definition, we have

$$h^*(\alpha+\beta) = (\alpha+\beta)\circ h = \nabla \circ (\alpha \vee \beta) \circ \sigma \circ h,$$

where $\sigma : \Sigma A \to \Sigma A \vee \Sigma A$ is the comultiplication and $\nabla : Y \vee Y \to Y$ is the folding map. Since C is a cofiber, there is a coaction $\psi : C \to C \vee \Sigma A$ such that $\sigma \circ h \simeq (h \vee \mathbb{1}) \circ \psi$.

$$\begin{tikzcd} C \arrow[r, "\psi"] \arrow[d, "h"] & C \vee \Sigma A \arrow[d, "h\vee\mathbb{1}"] \\ \Sigma A \arrow[r, "\sigma"] & \Sigma A \vee \Sigma A \end{tikzcd}$$

Then we obtain a string of equivalences

$$\begin{aligned} h^*(\alpha+\beta) &= \nabla \circ (\alpha \vee \beta) \circ \sigma \circ h \\ &\simeq \nabla \circ (\alpha \vee \beta) \circ (h \vee \mathbb{1}) \circ \psi \\ &\simeq \nabla \circ (\alpha' \vee \beta) \circ (h \vee \mathbb{1}) \circ \psi \\ &\simeq \nabla \circ (\alpha' \vee \beta) \circ \sigma \circ h \\ &= h^*(\alpha'+\beta) \end{aligned}$$

The third line is due to the assumption $h^*\alpha \simeq h^*\alpha'$. Therefore we have $h^*(\alpha+\beta) \simeq h^*(\alpha'+\beta)$. Since ΣA is cocommutative, $[\Sigma A, Y]$ is abelian and $h^*(\alpha+\beta) \simeq h^*(\beta+\alpha)$. Then we have

$$h^*(\alpha+\beta) \simeq h^*(\beta+\alpha) \simeq h^*(\beta'+\alpha) \simeq h^*(\alpha+\beta').$$

This implies $\boxtimes$ is well-defined.

Due to the associativity of $+$ in $[\Sigma A, Y]$, $\boxtimes$ is associative since

$$\begin{aligned} (h^*\alpha \boxtimes h^*\beta) \boxtimes h^*\gamma &= h^*(\alpha+\beta) \boxtimes h^*\gamma \\ &= h^*((\alpha+\beta)+\gamma) \\ &= h^*(\alpha+(\beta+\gamma)) \\ &= h^*\alpha \boxtimes h^*(\beta+\gamma) \\ &= h^*\alpha \boxtimes (h^*\beta \boxtimes h^*\gamma). \end{aligned}$$

Clearly the trivial map $* : C \to Y$ is the identity of $\boxtimes$ and $h^*(-\alpha)$ is the inverse of $h^*\alpha$. Therefore $\boxtimes$ is indeed a group multiplication.

By definition of $\boxtimes$, $h^* : [\Sigma A, Y] \to Im(h^*)$ is a group homomorphism, and hence an epimorphism. Since $[\Sigma A, Y]$ is abelian, so is $Im(h^*)$. We replace $[C, Y]$ by $Im(h^*)$ in (6) to obtain a sequence of groups and group homomorphisms

$$[\Sigma B, Y] \xrightarrow{(\Sigma f)^*} [\Sigma A, Y] \xrightarrow{h^*} Im(h^*) \longrightarrow 0.$$

The exactness of (6) implies $ker(h^*) = Im(\Sigma f)^*$, so the sequence is exact. $\square$

Applying Lemma 2.1 to cofibration $\Sigma^3 G \to \Sigma^2 G \to \mathbb{CP}^2 \wedge G$ and the space $Y = BG$, we obtain an exact sequence of abelian groups

$$[\Sigma^3 G, BG] \xrightarrow{(\Sigma\eta)^*} [\Sigma^4 G, BG] \xrightarrow{q^*} Im(q^*) \longrightarrow 0. \tag{7}$$

In the middle square of (5) $\partial'_k \simeq q^*\partial_k$, so ∂'_k is in $Im(q^*)$. For any number m, $q^*(m\partial_k) = mq^*\partial_k$, so the "order" of ∂'_k defined in [14] coincides with the multiplicative order of ∂'_k in $Im(q^*)$. The exact sequence (7) allows us to compare the orders of ∂_1 and ∂'_1.

Lemma 2.2. *Let m be the order of ∂_1 and let m' be the order of ∂'_1. Then m is m' or $2m'$.*

Proof. By exactness of (7), there is some $f \in [\Sigma^3 G, BG]$ such that $(\Sigma\eta)^* f \simeq m'\partial_1$. Since $\Sigma\eta$ has order 2, $2m'\partial_1$ is null homotopic. It follows that $2m'$ is a multiple of m. Since m is greater than or equal to m', m is either m' or $2m'$. $\square$

When $G = SU(2)$, the order m of ∂_1 is 12 and the order m' of ∂'_1 is 6 [9]. When $G = SU(3)$, $m = 24$ and $m' = 12$ [14]. It is natural to ask whether $m = 2m'$ for all G. However, this is not the case. In a preprint by Theriault and the author, we showed that $m = m' = 40$ for $G = Sp(2)$.

In the S^4 case, part (1) of Theorem 1.1 gives a sufficient condition for $\mathcal{G}_k(S^4) \simeq \mathcal{G}_l(S^4)$ when localized rationally or at any prime. In the $\mathbb{CP}^2$ case, Theriault showed a similar counting statement, in which the sufficient condition depends on the order of ∂_1 instead of ∂'_1.

Theorem 2.3 (Theriault, [14])**.** *Let m be the order of ∂_1. If $(m, k) = (m, l)$, then $\mathcal{G}_k(\mathbb{CP}^2)$ is homotopy equivalent to $\mathcal{G}_l(\mathbb{CP}^2)$ when localized rationally or at any prime.*

Lemma 2.2 can be used to improve the sufficient condition of Theorem 2.3.

Theorem 2.4. *Let m' be the order of ∂'_1. If $(m', k) = (m', l)$, then $\mathcal{G}_k(\mathbb{CP}^2)$ is homotopy equivalent to $\mathcal{G}_l(\mathbb{CP}^2)$ when localized rationally or at any prime.*

Proof. By Lemma 2.2, m is either m' or $2m'$. If $m = m'$, then the statement is same as Theorem 2.3. If we localize rationally or at any odd prime, then $(m, k) = (m', k)$ for any k, so a homotopy equivalence $\mathcal{G}_k(\mathbb{CP}^2) \simeq \mathcal{G}_l(\mathbb{CP}^2)$ follows by Theorem 2.3. It remains to consider the case where $m = 2m'$ when localized at 2.

Assume $m = 2^n$ and $m' = 2^{n-1}$. For any k, $(2^{n-1}, k) = 2^i$ where i an integer such that $0 \le i \le n-1$. If $i \le n-2$, then $k = 2^i t$ for some odd number t and $(2^{n-1}, k) = 2^i$. The sufficient condition $(2^{n-1}, k) = (2^{n-1}, l)$ is equivalent to $(2^n, k) = (2^n, l)$. Again the homotopy equivalence $\mathcal{G}_k(\mathbb{CP}^2) \simeq \mathcal{G}_l(\mathbb{CP}^2)$ follows by Theorem 2.3. If $i = n-1$, then $(2^n, k)$ is either 2^n or 2^{n-1}. We claim that $\mathcal{G}_k(\mathbb{CP}^2)$ has the same homotopy type for both $(2^n, k) = 2^n$ or $(2^n, k) = 2^{n-1}$.

Consider fibration (3)

$$\mathrm{Map}_0^*(\mathbb{CP}^2, G) \longrightarrow \mathcal{G}_k(\mathbb{CP}^2) \longrightarrow G \xrightarrow{\partial'_k} \mathrm{Map}_0^*(\mathbb{CP}^2, BG).$$

If $(2^n, k) = 2^n$, then $k = 2^n t$ for some number t. By linearity of Samelson products, $\partial_k \simeq k\partial_1$. Since $\partial'_k \simeq q^* k\partial_1 \simeq q^* 2^n t \partial_1$ and ∂_1 has order 2^n, ∂'_k is null homotopic and we have

$$\mathcal{G}_k(\mathbb{CP}^2) \simeq G \times \mathrm{Map}_0^*(\mathbb{CP}^2, G).$$

If $(2^n, k) = 2^{n-1}$, then $k = 2^{n-1} t$ for some odd number t. Writing $t = 2s+1$ gives $k = 2^n s + 2^{n-1}$. Since $\partial'_k \simeq q^* k\partial_1 \simeq q^*(2^n s + 2^{n-1})\partial_1 \simeq q^* 2^{n-1}\partial_1$ and ∂'_1 has order 2^{n-1}, ∂'_k is null homotopic and we have

$$\mathcal{G}_k(\mathbb{CP}^2) \simeq G \times \mathrm{Map}_0^*(\mathbb{CP}^2, G).$$

The same is true for $\mathcal{G}_l(\mathbb{CP}^2)$ and hence $\mathcal{G}_k(\mathbb{CP}^2) \simeq \mathcal{G}_l(\mathbb{CP}^2)$. $\square$

3. Plan for the proofs of Theorems 1.3 and 1.4

From this section onward, we will focus on $SU(n)$-gauge groups over $\mathbb{CP}^2$. There is a fibration

$$SU(n) \longrightarrow SU(\infty) \xrightarrow{p} W_n, \tag{8}$$

where $p : SU(\infty) \to W_n$ is the projection and W_n is the symmetric space $SU(\infty)/SU(n)$. Then we have

$$\begin{aligned} \tilde{H}^*(SU(\infty)) &= \Lambda(x_3, \cdots, x_{2n-1}, \cdots), \\ \tilde{H}^*(SU(n)) &= \Lambda(x_3, \cdots, x_{2n-1}), \\ \tilde{H}^*(BSU(n)) &= \mathbb{Z}[c_2, \cdots, c_n], \\ \tilde{H}^*(W_n) &= \Lambda(\bar{x}_{2n+1}, \bar{x}_{2n+3}, \cdots), \end{aligned}$$

where x_{2n+1} has degree $2n+1$, c_i is the i^{th} universal Chern class and $x_{2i+1} = \sigma(c_{i+1})$ is the image of c_{i+1} under the cohomology suspension σ, and $p^*(\bar{x}_{2i+1}) = x_{2i+1}$. Furthermore, $H^{2n}(\Omega W_n) \cong \mathbb{Z}$ and $H^{2n+2}(\Omega W_n) \cong \mathbb{Z}$ are generated by a_{2n} and a_{2n+2}, where a_{2i} is the transgression of x_{2i+1}.

The $(2n+4)$-skeleton of W_n is $\Sigma^{2n-1}\mathbb{CP}^2$ for n odd, and is $S^{2n+3} \vee S^{2n+1}$ for n even, so its homotopy groups are as follows:

(9)

	$\pi_i(W_n)$			
i	$\leq 2n$	$2n+1$	$2n+2$	$2n+3$
n odd	0	$\mathbb{Z}$	0	$\mathbb{Z}$
n even	0	$\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$

The canonical map $\epsilon : \Sigma\mathbb{CP}^{n-1} \to SU(n)$ induces the inclusion $\epsilon_* : H_*(\Sigma\mathbb{CP}^{n-1}) \to H_*(SU(n))$ of the generating set. Let C be the quotient $\mathbb{CP}^{n-1}/\mathbb{CP}^{n-3}$ and let $\bar{q} : \Sigma\mathbb{CP}^{n-1} \to \Sigma C$ be the quotient map. Then there is a diagram

$$\begin{tikzcd} {[\Sigma C, SU(n)]} \arrow[r, "(\partial'_k)_*"] \arrow[d, "\bar{q}^*"] & {[\Sigma C, \mathrm{Map}^*(\mathbb{CP}^2, BSU(n))]} \arrow[r] \arrow[d, "\bar{q}^*"] & {[\Sigma C, B\mathcal{G}_k(\mathbb{CP}^2)]} \arrow[d, "\bar{q}^*"] \\ {[\Sigma\mathbb{CP}^{n-1}, SU(n)]} \arrow[r, "(\partial'_k)_*"] & {[\Sigma\mathbb{CP}^{n-1}, \mathrm{Map}^*(\mathbb{CP}^2, BSU(n))]} \arrow[r] & {[\Sigma\mathbb{CP}^{n-1}, B\mathcal{G}_k(\mathbb{CP}^2)]}, \end{tikzcd}$$

where $(\partial'_k)_*$ sends f to $\partial'_k \circ f$ and the rows are induced by fibration (3). In particular, in the second row the map $\epsilon : \Sigma\mathbb{CP}^{n-1} \to SU(n)$ is sent to $(\partial'_k)_*(\epsilon) = \partial'_k \circ \epsilon$. In Section 4, we use unstable K-theory to calculate the order of $\partial'_1 \circ \epsilon$, giving a lower bound on the order of ∂'_1. Furthermore, in [4] Hamanaka and Kono considered an exact sequence similar to the first row to give a necessary condition for $\mathcal{G}_k(S^4) \simeq \mathcal{G}_l(S^4)$. In Section 5 we follow the same approach and use the first row to give a necessary condition for $\mathcal{G}_k(\mathbb{CP}^2) \simeq \mathcal{G}_l(\mathbb{CP}^2)$.

We remark that it is difficult to use only one of the two rows to prove both Theorems 1.3 and 1.4. On the one hand, $\partial'_1 \circ \epsilon$ factors through a map $\bar{\partial} : \Sigma C \to \mathrm{Map}^*(\mathbb{CP}^2, BSU(n))$. There is no obvious method to show that $\bar{\partial}$ and $\partial'_1 \circ \epsilon$ have the same orders except direct calculation. Therefore we cannot compare the orders of $\bar{\partial}$ and ∂'_1 to prove Theorem 1.3 without calculating the order of $\partial'_1 \circ \epsilon$. On the other hand, applying the method used in Section 5 to the second row gives a much weaker conclusion than Theorem 1.4. This is because $[\Sigma C, B\mathcal{G}_k(\mathbb{CP}^2)]$ is a much smaller group than $[\Sigma\mathbb{CP}^{n-1}, B\mathcal{G}_k(\mathbb{CP}^2)]$ and much information is lost by the map $\bar{q}^*$.

4. A lower bound on the order of ∂_1'

The restriction of ∂_1 to $\Sigma\mathbb{CP}^{n-1}$ is $\partial_1 \circ \epsilon$, which is the triple adjoint of the composition

$$\langle \imath, \epsilon\rangle : S^3 \wedge \Sigma\mathbb{CP}^{n-1} \xrightarrow{\imath\wedge\epsilon} SU(n) \wedge SU(n) \xrightarrow{\langle \mathbb{1},\mathbb{1}\rangle} SU(n).$$

Since $SU(n) \simeq \Omega BSU(n)$, we can further take its adjoint and get

$$\rho : \Sigma S^3 \wedge \Sigma\mathbb{CP}^{n-1} \xrightarrow{\Sigma\imath\wedge\epsilon} \Sigma SU(n) \wedge SU(n) \xrightarrow{[ev,ev]} BSU(n),$$

where $[ev, ev]$ is the Whitehead product of the evaluation map

$$ev : \Sigma SU(n) \simeq \Sigma\Omega BSU(n) \to BSU(n)$$

with itself. Similarly, the restriction $\partial_1' \circ \epsilon$ is adjoint to the composition

$$\rho' : \mathbb{CP}^2 \wedge \Sigma\mathbb{CP}^{n-1} \xrightarrow{q\wedge\mathbb{1}} S^4 \wedge \Sigma\mathbb{CP}^{n-1} \xrightarrow{\Sigma\imath\wedge\epsilon} \Sigma SU(n) \wedge SU(n) \xrightarrow{[ev,ev]} BSU(n).$$

Since we will frequently refer to the facts established in [3, 4], it is easier to follow their setting and consider its adjoint

$$\gamma = \tau(\rho' \circ T) : \mathbb{CP}^2 \wedge \mathbb{CP}^{n-1} \to SU(n),$$

where $T : \Sigma\mathbb{CP}^2\wedge\mathbb{CP}^{n-1} \to \mathbb{CP}^2\wedge\Sigma\mathbb{CP}^{n-1}$ is the swapping map and $\tau : [\Sigma\mathbb{CP}^2\wedge\mathbb{CP}^{n-1}, BSU(n)] \to [\mathbb{CP}^2 \wedge \mathbb{CP}^{n-1}, SU(n)]$ is the adjunction. By adjunction, the orders of $\partial_1' \circ \epsilon$, ρ' and γ are the same. We will calculate the order of γ using unstable K-theory to prove Theorem 1.3.

Apply $[\mathbb{CP}^2 \wedge \mathbb{CP}^{n-1}, -]$ to fibration (8) to obtain the exact sequence

$$\tilde{K}^0(\mathbb{CP}^2 \wedge \mathbb{CP}^{n-1}) \xrightarrow{p_*} [\mathbb{CP}^2 \wedge \mathbb{CP}^{n-1}, \Omega W_n] \longrightarrow [\mathbb{CP}^2 \wedge \mathbb{CP}^{n-1}, SU(n)] \longrightarrow \tilde{K}^1(\mathbb{CP}^2 \wedge \mathbb{CP}^{n-1}).$$

Since $\mathbb{CP}^2 \wedge \mathbb{CP}^{n-1}$ is a CW-complex with even dimensional cells, $\tilde{K}^1(\mathbb{CP}^2 \wedge \mathbb{CP}^{n-1})$ is zero. First we identify the term $[\mathbb{CP}^2 \wedge \mathbb{CP}^{n-1}, \Omega W_n]$.

Lemma 4.1. *We have the following:*

- $[\Sigma^{2n-4}\mathbb{CP}^2, \Omega W_n] \cong \mathbb{Z}$;
- $[\Sigma^{2n-3}\mathbb{CP}^2, \Omega W_n] = 0$ *for n odd;*
- $[\Sigma^{2n-2}\mathbb{CP}^2, \Omega W_n] \cong \mathbb{Z} \oplus \mathbb{Z}$.

Proof. First, apply $[\Sigma^{2n-4}-, \Omega W_n]$ to cofibration (4) to obtain the exact sequence

$$\pi_{2n}(W_n) \longrightarrow \pi_{2n+1}(W_n) \longrightarrow [\Sigma^{2n-4}\mathbb{CP}^2, \Omega W_n] \longrightarrow \pi_{2n-1}(W_n).$$

We refer to Table (9) freely for the homotopy groups of W_n. Since $\pi_{2n-1}(W_n)$ and $\pi_{2n}(W_n)$ are zero, $[\Sigma^{2n-4}\mathbb{CP}^{n-1}, \Omega W_n]$ is isomorphic to $\pi_{2n+1}(W_n) \cong \mathbb{Z}$.

Second, apply $[\Sigma^{2n-3}-, \Omega W_n]$ to (4) to obtain

$$\pi_{2n+2}(W_n) \longrightarrow [\Sigma^{2n-3}\mathbb{CP}^2, \Omega W_n] \longrightarrow \pi_{2n}(W_n).$$

Since $\pi_{2n}(W_n)$ and $\pi_{2n+2}(W_n)$ are zero for n odd, so is $[\Sigma^{2n-3}\mathbb{CP}^2, \Omega W_n]$.

Third, apply $[\Sigma^{2n-2}-, \Omega W_n]$ to (4) to obtain

$$\pi_{2n+2}(W_n) \xrightarrow{\eta_1} \pi_{2n+3}(W_n) \longrightarrow [\Sigma^{2n-2}\mathbb{CP}^2, \Omega W_n] \xrightarrow{j} \pi_{2n+1}(W_n) \xrightarrow{\eta_2} \pi_{2n+2}(W_n),$$

where η_1 and η_2 are induced by Hopf maps $\Sigma^{2n}\eta : S^{2n+3} \to S^{2n+2}$ and $\Sigma^{2n-1}\eta : S^{2n+2} \to S^{2n+1}$, and j is induced by the inclusion $S^{2n+1} \hookrightarrow \Sigma^{2n-2}\mathbb{CP}^2$ of the bottom cell. When n is odd, $\pi_{2n+2}(W_n)$ is zero and $\pi_{2n+1}(W_n)$ and $\pi_{2n+3}(W_n)$ are $\mathbb{Z}$, so $[\Sigma^{2n-2}\mathbb{CP}^{n-1}, \Omega W_n]$ is $\mathbb{Z}\oplus\mathbb{Z}$. When n is even, the $(2n + 4)$-skeleton of W_n is $S^{2n+1} \vee S^{2n+3}$. The inclusions

$$i_1 : S^{2n+1} \to S^{2n+1} \vee S^{2n+3} \quad \text{and} \quad i_2 : S^{2n+3} \to S^{2n+1} \vee S^{2n+3}$$

generate $\pi_{2n+1}(W_n)$ and the $\mathbb{Z}$-summand of $\pi_{2n+3}(W_n)$, and the compositions

$$j_1 : S^{2n+2} \xrightarrow{\Sigma^{2n-1}\eta} S^{2n+1} \xrightarrow{i_1} W_n \quad \text{and} \quad j_2 : S^{2n+3} \xrightarrow{\Sigma^{2n}\eta} S^{2n+2} \xrightarrow{\Sigma^{2n-1}\eta} S^{2n+1} \xrightarrow{i_1} W_n$$

generate $\pi_{2n+2}(W_n)$ and the $\mathbb{Z}/2\mathbb{Z}$-summand of $\pi_{2n+3}(W_n)$ respectively. Since η_1 sends j_1 to j_2, the cokernel of η_1 is $\mathbb{Z}$. Similarly, η_2 sends i_1 to j_1, so $\eta_2 : \mathbb{Z} \to \mathbb{Z}/2\mathbb{Z}$ is surjective. This implies the preimage of j is a $\mathbb{Z}$-summand. Therefore $[\Sigma^{2n-2}\mathbb{CP}^2, \Omega W_n] \cong \mathbb{Z} \oplus \mathbb{Z}$. ☐

Let C be the quotient $\mathbb{CP}^{n-1}/\mathbb{CP}^{n-3}$. Since ΩW_n is $(2n-1)$-connected, $[\mathbb{CP}^2 \wedge \mathbb{CP}^{n-1}, \Omega W_n]$ is isomorphic to $[\mathbb{CP}^2 \wedge C, \Omega W_n]$ which is easier to determine.

Lemma 4.2. *The group* $[\mathbb{CP}^2 \wedge \mathbb{CP}^{n-1}, \Omega W_n] \cong [\mathbb{CP}^2 \wedge C, \Omega W_n]$ *is isomorphic to* $\mathbb{Z}^{\oplus 3}$.

Proof. When n is even, C is $S^{2n-2} \vee S^{2n-4}$. By Lemma 4.1, $[\mathbb{CP}^2 \wedge C, \Omega W_n]$ is $[\Sigma^{2n-2}\mathbb{CP}^2, \Omega W_n] \oplus [\Sigma^{2n-4}\mathbb{CP}^2, \Omega W_n] \cong \mathbb{Z}^{\oplus 3}$.

When n is odd, C is $\Sigma^{2n-6}\mathbb{CP}^2$. Apply $[\Sigma^{2n-6}\mathbb{CP}^2 \wedge -, \Omega W_n]$ to cofibration (4) to obtain the exact sequence

$$[\Sigma^{2n-3}\mathbb{CP}^2, \Omega W_n] \longrightarrow [\Sigma^{2n-2}\mathbb{CP}^2, \Omega W_n] \longrightarrow [\Sigma^{2n-6}\mathbb{CP}^2 \wedge \mathbb{CP}^2, \Omega W_n] \longrightarrow$$
$$\longrightarrow [\Sigma^{2n-4}\mathbb{CP}^2, \Omega W_n] \longrightarrow [\Sigma^{2n-3}\mathbb{CP}^2, \Omega W_n]$$

By Lemma 4.1, the first and the last terms $[\Sigma^{2n-3}\mathbb{CP}^2, \Omega W_n]$ are zero, while the second term $[\Sigma^{2n-2}\mathbb{CP}^2, \Omega W_n]$ is $\mathbb{Z} \oplus \mathbb{Z}$ and the fourth $[\Sigma^{2n-4}\mathbb{CP}^2, \Omega W_n]$ is $\mathbb{Z}$. Therefore $[\mathbb{CP}^2 \wedge C, \Omega W_n]$ is $\mathbb{Z}^{\oplus 3}$. ☐

Define $a : [\mathbb{CP}^2 \wedge \mathbb{CP}^{n-1}, \Omega W_n] \to H^{2n}(\mathbb{CP}^2 \wedge \mathbb{CP}^{n-1}) \oplus H^{2n+2}(\mathbb{CP}^2 \wedge \mathbb{CP}^{n-1})$ to be a map sending $f \in [\mathbb{CP}^2 \wedge \mathbb{CP}^{n-1}, \Omega W_n]$ to $a(f) = f^*(a_{2n}) \oplus f^*(a_{2n+2})$. The cohomology class $\bar{x}_{2n+1}$ represents a map $\bar{x}_{2n+1} : W_n \to K(\mathbb{Z}, 2n+1)$ and $a_{2n} = \sigma(\bar{x}_{2n+1})$ represents its loop $\Omega\bar{x}_{2n+1} : \Omega W_n \to \Omega K(\mathbb{Z}, 2n+1)$. Similarly $a_{2n+2} = \sigma(\bar{x}_{2n+3})$ represents a loop map. This implies a is a group homomorphism. Furthermore, a_{2n} and a_{2n+2} induce isomorphisms between $H^i(\Omega W_n)$ and $H^i(K(2n, \mathbb{Z}) \times K(2n+2, \mathbb{Z}))$ for $i = 2n$ and $2n+2$. Since $[\mathbb{CP}^2 \wedge \mathbb{CP}^{n-1}, \Omega W_n]$ is a free $\mathbb{Z}$-module by Lemma 4.2, a is a monomorphism. Consider the diagram

(10)

$$\begin{tikzcd}
\tilde{K}^0(\mathbb{CP}^2 \wedge \mathbb{CP}^{n-1}) \arrow[r, "p_*"] \arrow[d, equal] & {[\mathbb{CP}^2 \wedge \mathbb{CP}^{n-1}, \Omega W_n]} \arrow[r] \arrow[d, "a"] & {[\mathbb{CP}^2 \wedge \mathbb{CP}^{n-1}, SU(n)]} \arrow[r] \arrow[d, "b"] & 0 \\
\tilde{K}^0(\mathbb{CP}^2 \wedge \mathbb{CP}^{n-1}) \arrow[r, "\Phi"] & H^{2n}(\mathbb{CP}^2 \wedge \mathbb{CP}^{n-1}) \oplus H^{2n+2}(\mathbb{CP}^2 \wedge \mathbb{CP}^{n-1}) \arrow[r, "\psi"] & Coker(\Phi) \arrow[r] & 0
\end{tikzcd}$$

In the left square, Φ is defined to be $a \circ p^*$. In the right square, ψ is the quotient map and b is defined as follows. Any $f \in [\mathbb{CP}^2 \wedge \mathbb{CP}^{n-1}, SU(n)]$ has a preimage $\tilde{f}$ and $b(f)$ is defined to be $\psi(a(\tilde{f}))$. An easy diagram chase shows that b is well-defined and injective. Since b is injective, the order of $\gamma \in [\mathbb{CP}^2 \wedge \mathbb{CP}^{n-1}, SU(n)]$ equals the order of $b(\gamma) \in Coker(\Phi)$. In [3], Hamanaka and Kono gave an explicit formula for Φ.

Theorem 4.3 (Hamanaka, Kono, [3])**.** *For any* $f \in \tilde{K}^0(Y)$, *we have*

$$\Phi(f) = n!ch_{2n}(f) \oplus (n+1)!ch_{2n+2}(f),$$

where $ch_{2i}(f)$ *is the* $2i^{th}$ *part of* $ch(f)$.

Let u and v be the generators of $H^2(\mathbb{CP}^2)$ and $H^2(\mathbb{CP}^{n-1})$. For $1 \le i \le n-1$, denote L_i and L'_i as the generators of $\tilde{K}^0(\mathbb{CP}^2 \wedge \mathbb{CP}^{n-1})$ with Chern characters $ch(L_i) = u^2(e^v-1)^i$ and $ch(L'_i) = (u+\frac{1}{2}u^2)\cdot(e^v-1)^i$. By Theorem 4.3 we have

$$\begin{aligned}\Phi(L_i) &= n(n-1)A_iu^2v^{n-2} + n(n+1)B_iu^2v^{n-1},\\ \Phi(L'_i) &= \frac{n(n-1)}{2}A_iu^2v^{n-2} + nB_iuv^{n-1} + \frac{n(n+1)}{2}B_iu^2v^{n-1},\end{aligned}$$

where

$$A_i = \textstyle\sum_{j=1}^{i}(-1)^{i+j}\binom{i}{j}j^{n-2} \quad\text{and}\quad B_i = \sum_{j=1}^{i}(-1)^{i+j}\binom{i}{j}j^{n-1}.$$

Write an element $xu^2v^{n-2} + yuv^{n-1} + zu^2v^{n-1} \in H^{2n}(\mathbb{CP}^2\wedge\mathbb{CP}^{n-1}) \oplus H^{2n+2}(\mathbb{CP}^2\wedge\mathbb{CP}^{n-1})$ as (x,y,z). Then the coordinates of $\Phi(L_i)$ and $\Phi(L'_i)$ are $(n(n-1)A_i, 0, n(n+1)B_i)$ and $(\frac{n(n-1)}{2}A_i, nB_i, \frac{n(n+1)}{2}B_i)$ respectively.

Lemma 4.4. *For $n \ge 3$, $Im(\Phi)$ is spanned by $(\frac{n(n-1)}{2}, n, \frac{n(n+1)}{2})$, $(n(n-1),0,0)$ and $(0,2n,0)$.*

Proof. By definition, $Im(\Phi) = span\{\Phi(L_i), \Phi(L'_i)\}_{i=1}^{n-1}$. For $i=1$, $A_1 = B_1 = 1$. Then

$$\begin{aligned}\Phi(L_1) &= (n(n-1), 0, n(n+1))\\ &= 2(\frac{1}{2}n(n-1), n, \frac{1}{2}n(n+1)) - (0,2n,0)\\ &= 2\Phi(L'_1) - (0,2n,0)\end{aligned}$$

Equivalently $(0,2n,0) = 2\Phi(L'_1) - \Phi(L_1)$, so $span\{\Phi(L_1), \Phi(L'_1)\} = span\{\Phi(L'_1), (0,2n,0)\}$. For other i's,

$$\begin{aligned}\Phi(L_i) &= (n(n-1)A_i, 0, n(n+1)B_i)\\ &= 2(\frac{1}{2}n(n-1)A_i, nB_i, \frac{1}{2}n(n+1)B_i) - (0, 2nB_i, 0)\\ &= 2\Phi(L'_i) - B_i(0,2n,0)\end{aligned}$$

is a linear combination of $\Phi(L'_i)$ and $(0,2n,0)$, so $Im(\Phi) = span\{\Phi(L'_1), \cdots, \Phi(L'_{n-1}), (0,2n,0)\}$.

We claim that $span\{\Phi(L'_i)\}_{i=1}^{n-1} = span\{\Phi(L'_1), (n(n-1),0,0)\}$. Observe that

$$\begin{aligned}\Phi(L'_i) &= (\frac{n(n-1)}{2}A_i, nB_i, \frac{n(n+1)}{2}B_i)\\ &= (\frac{n(n-1)}{2}B_i, nB_i, \frac{n(n+1)}{2}B_i) + (\frac{n(n-1)}{2}(A_i-B_i), 0, 0)\\ &= B_i\Phi(L'_1) + \frac{A_i-B_i}{2}\cdot(n(n-1),0,0).\end{aligned}$$

The difference

$$\begin{aligned}A_i - B_i &= \sum_{j=1}^{i}(-1)^{i+j}\binom{i}{j}j^{n-2} - \sum_{j=1}^{i}(-1)^{i+j}\binom{i}{j}j^{n-1}\\ &= \sum_{j=1}^{i}(-1)^{i+j+1}\binom{i}{j}(j^{n-1}-j^{n-2})\\ &= \sum_{j=1}^{i}(-1)^{i+j+1}\binom{i}{j}(j-1)j^{n-2}\end{aligned}$$

is even since each term $(j-1)j^{n-2}$ is even and $n \geq 3$. Therefore $\frac{A_i - B_i}{2}$ is an integer and $\Phi(L_i')$ is a linear combination of $\Phi(L_1')$ and $(n(n-1), 0, 0)$.

Furthermore,

$$\begin{aligned}\Phi(L_2') &= B_2\Phi(L_1') + (A_2 - B_2)(\frac{n(n-1)}{2}, 0, 0) \\ &= B_2\Phi(L_1') - 2^{n-3}(n(n-1), 0, 0)\end{aligned}$$

and

$$\begin{aligned}\Phi(L_3') &= B_3\Phi(L_1') + (A_3 - B_3)(\frac{n(n-1)}{2}, 0, 0) \\ &= B_3\Phi(L_1') - (3^{n-2} - 3 \cdot 2^{n-3})(n(n-1), 0, 0).\end{aligned}$$

Since 2^{n-3} and $3^{n-2} - 3 \cdot 2^{n-3}$ are coprime to each other, there exist integers s and t such that $2^{n-3}s + (3^{n-2} - 3 \cdot 2^{n-3})t = 1$ and

$$(n(n-1), 0, 0) = (sB_2 + tB_3)\Phi(L_1') - s\Phi(L_2') - t\Phi(L_3').$$

Therefore $(n(n-1), 0, 0)$ is a linear combination of $\Phi(L_1'), \Phi(L_2')$ and $\Phi(L_3')$. This implies $span\{\Phi(L_1'), (n(n-1), 0, 0)\} = span\{\Phi(L_i')\}_{i=1}^{n-1}$.

Combine all these together to obtain

$$\begin{aligned}Im(\Phi) &= span\{\Phi(L_i), \Phi(L_i')\}_{i=1}^{n-1} \\ &= span\{\Phi(L_1'), (n(n-1), 0, 0), (0, 2n, 0)\} \\ &= span\{(\frac{n(n-1)}{2}, n, \frac{n(n+1)}{2}), (n(n-1), 0, 0), (0, 2n, 0)\}.\end{aligned}$$

□

Back to diagram (10). The map γ has a lift $\tilde{\gamma} : \mathbb{CP}^2 \wedge \mathbb{CP}^{n-1} \to \Omega W_n$. By exactness, the order of γ equals the minimum number m such that $m\tilde{\gamma}$ is contained in $Im(p_*)$. Since a and b are injective, the order of γ equals the minimum number m' such that $m'a(\tilde{\gamma})$ is contained in $Im(\Phi)$.

Lemma 4.5. *Let $\alpha : \Sigma X \to SU(n)$ be a map for some space X. If $\alpha' : \mathbb{CP}^2 \wedge X \to SU(n)$ is the adjoint of the composition*

$$\mathbb{CP}^2 \wedge \Sigma X \xrightarrow{q \wedge \mathbb{1}} \Sigma S^3 \wedge \Sigma X \xrightarrow{\Sigma\imath \wedge \alpha} \Sigma SU(n) \wedge SU(n) \xrightarrow{[ev, ev]} BSU(n),$$

then there is a lift $\tilde{\alpha}$ of α' such that $\tilde{\alpha}^*(a_{2i}) = u^2 \otimes \Sigma^{-1}\alpha^*(x_{2i-3})$, where Σ is the cohomology suspension isomorphism.

$$\begin{tikzcd} & \Omega W_n \arrow[d] \\ \mathbb{CP}^2 \wedge X \arrow[r, "\alpha'"] \arrow[ur, dashed, "\tilde{\alpha}"] & SU(n) \end{tikzcd}$$

Proof. In [3, 4], Hamanaka and Kono constructed a lift $\Gamma : \Sigma SU(n) \wedge SU(n) \to W_n$ of $[ev, ev]$ such that $\Gamma^*(\bar{x}_{2i+1}) = \sum_{j+k=i-1} \Sigma x_{2j+1} \otimes x_{2k+1}$. Let $\tilde{\Gamma}$ be the composition

$$\tilde{\Gamma} : \mathbb{CP}^2 \wedge \Sigma X \xrightarrow{q \wedge \mathbb{1}} \Sigma S^3 \wedge \Sigma X \xrightarrow{\Sigma\imath \wedge \alpha} \Sigma SU(n) \wedge SU(n) \xrightarrow{\Gamma} W_n.$$

Then we have

$$\begin{aligned}
\tilde{\Gamma}^*(\bar{x}_{2i+1}) &= (q\wedge \mathbb{1})^*(\Sigma\imath\wedge\alpha)^*\Gamma^*(\bar{x}_{2i+1})\\
&= (q\wedge \mathbb{1})^*(\Sigma\imath\wedge\alpha)^*\left(\sum_{j+k=i-1}\Sigma x_{2j+1}\otimes x_{2k+1}\right)\\
&= (q\wedge \mathbb{1})^*(\Sigma u_3\otimes\alpha^*(x_{2i-3}))\\
&= u^2\otimes\alpha^*(x_{2i-3}),
\end{aligned}$$

where u_3 is the generator of $H^3(S^3)$.

Let $T:\Sigma\mathbb{CP}^2\wedge X\to\mathbb{CP}^2\wedge\Sigma X$ be the swapping map and let $\tau:[\Sigma\mathbb{CP}^2\wedge X,W_n]\to[\mathbb{CP}^2\wedge X,\Omega W_n]$ be the adjunction. Take $\tilde{\alpha}:\mathbb{CP}^2\wedge X\to\Omega W_n$ to be the adjoint of $\tilde{\Gamma}$, that is $\tilde{\alpha}=\tau(\tilde{\Gamma}\circ T)$. Then $\tilde{\alpha}$ is a lift of α'. Since

$$(\tilde{\Gamma}\circ T)^*(\bar{x}_{2i+1})=T^*\circ\tilde{\Gamma}^*(\bar{x}_{2i+1})=T^*(u^2\otimes\alpha^*(x_{2i-3}))=\Sigma u^2\otimes\Sigma^{-1}\alpha^*(x_{2i-3}),$$

we have $\tilde{\alpha}^*(a_{2i})=u^2\otimes\Sigma^{-1}\alpha^*(x_{2i-3})$. □

Lemma 4.6. *In diagram (10), γ has a lift $\tilde{\gamma}$ such that $a(\tilde{\gamma})=u^2v^{n-2}\oplus u^2v^{n-1}$.*

Proof. Recall that γ is the adjoint of the composition

$$\rho':\mathbb{CP}^2\wedge\Sigma\mathbb{CP}^{n-1}\xrightarrow{q\wedge\mathbb{1}}\Sigma S^3\wedge\mathbb{CP}^{n-1}\xrightarrow{\Sigma\imath\wedge\epsilon}\Sigma SU(n)\wedge SU(n)\xrightarrow{[ev,ev]}BSU(n).$$

Now we use Lemma 4.5 and take α to be $\epsilon:\Sigma\mathbb{CP}^{n-1}\to SU(n)$. Then γ has a lift $\tilde{\gamma}$ such that $\tilde{\gamma}^*(a_{2i})=u^2\otimes\Sigma^{-1}\epsilon^*(x_{2i-3})=u^2\otimes v^{i-2}$. This implies

$$a(\tilde{\gamma})=\tilde{\gamma}^*(a_{2n})\oplus\tilde{\gamma}^*(a_{2n+2})=u^2v^{n-2}\oplus u^2v^{n-1}.$$

□

Now we can calculate the order of $\partial_1'\circ\epsilon$, which gives a lower bound on the order of ∂_1'.

Theorem 4.7. *When $n\geq 3$, the order of $\partial_1'\circ\epsilon$ is $\frac{1}{2}n(n^2-1)$ for n odd and $n(n^2-1)$ for n even.*

Proof. Since $\partial_1'\circ\epsilon$ is adjoint to γ , it suffices to calculate the order of γ. By Lemma 4.4, $Im(\Phi)$ is spanned by $(\frac{1}{2}n(n-1),n,\frac{1}{2}n(n+1)),(n(n-1),0,0)$ and $(0,2n,0)$. By Lemma 4.6, $a(\tilde{\gamma})$ has coordinates $(1,0,1)$. Let m be a number such that $ma(\tilde{\gamma})$ is contained in $Im(\Phi)$. Then

$$m(1,0,1)=s(\frac{1}{2}n(n-1),n,\frac{1}{2}n(n+1))+t(n(n-1),0,0)+r(0,2n,0)$$

for some integers s,t and r. Solve this to get

$$m=\tfrac{1}{2}tn(n^2-1),\quad s=-2r,\quad s=t(n-1).$$

Since $s=-2r$ is even, the smallest positive value of t satisfying $s=t(n-1)$ is 1 for n odd and 2 for n even. Therefore m is $\frac{1}{2}n(n^2-1)$ for n odd and $n(n^2-1)$ for n even. □

For $SU(n)$-gauge groups over S^4, the order m of ∂_1 has the form $m=n(n^2-1)$ for $n=3$ and 5 [4, 15]. If p is an odd prime and $n<(p-1)^2+1$, then m and $n(n^2-1)$ have the same p-components [6, 16]. These facts suggest it may be the case that $m=n(n^2-1)$ for any $n>2$. In fact, one can follow the method Hamanaka and Kono used in [4] and calculate the order of $\partial\circ\epsilon$ to obtain a lower bound $n(n^2-1)$ for n odd. However, it does not work for the n even case since $[S^4\wedge\mathbb{CP}^{n-1},\Omega W_n]$ is not a free $\mathbb{Z}$-module. An interesting corollary of Theorem 4.7 is to give a lower bound on the order of ∂_1 for n even.

Corollary 4.8. *When n is even and greater than 2, the order of ∂_1 is at least $n(n^2-1)$.*

Proof. The order of $\partial_1' \circ \epsilon$ is a lower bound on the order of ∂_1', which is either the same as or half of the order of ∂_1 by Lemma 2.2. The corollary follows from Theorem 4.7. □

5. A necessary condition for $\mathcal{G}_k(\mathbb{CP}^2) \simeq \mathcal{G}_l(\mathbb{CP}^2)$

In this section we follow the approach in [4] to prove Theorem 1.4. The techniques used are similar to that in Section 4, except we are working with the quotient $\Sigma C = \Sigma\mathbb{CP}^{n-1}/\Sigma\mathbb{CP}^{n-1}$ instead of $\Sigma\mathbb{CP}^{n-1}$. When n is odd, C is $\Sigma^{2n-6}\mathbb{CP}^2$, and when n is even, C is $S^{2n-2}\vee S^{2n-4}$. Apply $[\Sigma C, -]$ to fibration (3) to obtain the exact sequence

$$[\Sigma C, SU(n)] \xrightarrow{(\partial_k')_*} [\Sigma C, \mathrm{Map}_0^*(\mathbb{CP}^2, BSU(n))] \longrightarrow [\Sigma C, B\mathcal{G}_k(\mathbb{CP}^2)] \longrightarrow [\Sigma C, BSU(n)],$$

where $(\partial_k')_*$ sends $f \in [\Sigma C, SU(n)]$ to $\partial_k' \circ f \in [\Sigma C, \mathrm{Map}_0^*(\mathbb{CP}^2, BSU(n))]$. Since $BSU(n) \to BSU(\infty)$ is a $2n$-equivalence and ΣC has dimension $2n-1$, $[\Sigma C, BSU(n)]$ is $\tilde{K}^0(\Sigma C)$ which is zero. Similarly, $[\Sigma C, SU(n)] \cong [\Sigma^2 C, BSU(n)]$ is $\tilde{K}^0(\Sigma^2 C) \cong \mathbb{Z}\oplus\mathbb{Z}$. Furthermore, by adjunction we have $[\Sigma C, \mathrm{Map}_0^*(\mathbb{CP}^2, BSU(n))] \cong [\Sigma C\wedge\mathbb{CP}^2, BSU(n)]$. The exact sequence becomes

$$\tilde{K}^0(\Sigma^2 C) \xrightarrow{(\partial_k')_*} [\Sigma C\wedge\mathbb{CP}^2, BSU(n)] \longrightarrow [\Sigma C, B\mathcal{G}_k(\mathbb{CP}^2)] \longrightarrow 0. \tag{11}$$

This implies $[\Sigma C, B\mathcal{G}_k(\mathbb{CP}^2)] \cong [C, \mathcal{G}_k(\mathbb{CP}^2)]$ is $Coker(\partial_k')_*$. Also, apply $[\mathbb{CP}^2\wedge C, -]$ to fibration (8) to obtain the exact sequence

$$[\mathbb{CP}^2\wedge C, \Omega SU(\infty)] \xrightarrow{p_*} [\mathbb{CP}^2\wedge C, \Omega W_n] \longrightarrow [\mathbb{CP}^2\wedge C, SU(n)] \longrightarrow [\mathbb{CP}^2\wedge C, SU(\infty)]. \tag{12}$$

Observe that $[\mathbb{CP}^2\wedge C, \Omega SU(\infty)] \cong \tilde{K}^0(\mathbb{CP}^2\wedge C)$ is $\mathbb{Z}^{\oplus 4}$ and $[\mathbb{CP}^2\wedge C, SU(\infty)] \cong \tilde{K}^1(\mathbb{CP}^2\wedge C)$ is zero. Combine exact sequences (11) and (12) to obtain the diagram

$$\begin{tikzcd}
 & \tilde{K}^0(\mathbb{CP}^2\wedge C) \arrow[d, "p_*"] \arrow[dr, "\Phi"] & & \\
 & {[\mathbb{CP}^2\wedge C, \Omega W_n]} \arrow[r, "a"] \arrow[d] & H^{2n}(\mathbb{CP}^2\wedge C)\oplus H^{2n+2}(\mathbb{CP}^2\wedge C) & \\
\tilde{K}^0(\Sigma^2 C) \arrow[r, "(\partial_k')_*"] & {[\mathbb{CP}^2\wedge C, SU(n)]} \arrow[r] \arrow[d] & {[C, B\mathcal{G}_k(\mathbb{CP}^2)]} \arrow[r] & 0 \\
 & 0 & &
\end{tikzcd}$$

where $a(f) = f^*(a_{2n})\oplus f^*(a_{2n+2})$ for any $f\in[\mathbb{CP}^2\wedge C, \Omega W_n]$, and Φ is defined to be $a\circ p_*$. By Lemma 4.2 $[\mathbb{CP}^2\wedge C, \Omega W_n]$ is free. Following the same argument in Section 4 implies the injectivity of a.

Our strategy to prove Theorem 1.4 is as follows. If $\mathcal{G}_k(\mathbb{CP}^2)$ is homotopy equivalent to $\mathcal{G}_l(\mathbb{CP}^2)$, then $[C, \mathcal{G}_k(\mathbb{CP}^2)] \cong [C, \mathcal{G}_l(\mathbb{CP}^2)]$ and exactness in (12) implies that $Im(\partial_k')_*$ and $Im(\partial_l')_*$ have the same order in $[\mathbb{CP}^2\wedge C, SU(n)]$, resulting in a necessary condition for a homotopy equivalence $\mathcal{G}_k(\mathbb{CP}^2)\simeq\mathcal{G}_l(\mathbb{CP}^2)$. To calculate the order of $Im(\partial_k')_*$, we will find a preimage $\tilde{\partial}_k$ of $Im(\partial_k')_*$ in $[\mathbb{CP}^2\wedge C, \Omega W_n]$. Since a is injective, we can embed $\tilde{\partial}_k$ into $H^{2n}(\mathbb{CP}^2\wedge C)\oplus H^{2n+2}(\mathbb{CP}^2\wedge C)$ and work out the order of $Im(\partial_k')_*$ there.

Let u, v_{2n-4} and v_{2n-2} be generators of $H^2(\mathbb{CP}^2)$, $H^{2n-4}(C)$ and $H^{2n-2}(C)$. Then we write an element $xu^2v_{2n-4} + yuv_{2n-2} + zu^2v_{2n-2} \in H^{2n}(\mathbb{CP}^2 \wedge C) \oplus H^{2n+2}(\mathbb{CP}^2 \wedge C)$ as (x, y, z). First we need to find the submodule $Im(a)$.

Lemma 5.1. *For n odd, $Im(a)$ is $\{(x, y, z)|x + y \equiv z \pmod 2\}$, and for n even, $Im(a)$ is $\{(x, y, z)|y \equiv 0 \pmod 2\}$.*

Proof. When n is odd, C is $\Sigma^{2n-6}\mathbb{CP}^2$ and the $(2n + 3)$-skeleton of ΩW_n is $\Sigma^{2n-2}\mathbb{CP}^2$. To say $(x, y, z) \in Im(a)$ means there exists $f \in [\mathbb{CP}^2 \wedge C, \Omega W_n]$ such that

$$f^*(a_{2n}) = xu^2v_{2n-4} + yuv_{2n-2} \quad \text{and} \quad f^*(a_{2n+2}) = zu^2v_{2n-2}. \tag{13}$$

Reducing to homology with $\mathbb{Z}/2\mathbb{Z}$-coefficients, we have

$$Sq^2(u) = u^2, \quad Sq^2(v_{2n-4}) = v_{2n-2}, \quad Sq^2(a_{2n}) = a_{2n+2}.$$

Apply Sq^2 to (13) to get $x+y \equiv z \pmod 2$. Therefore $Im(a)$ is contained in $\{(x, y, z)|x+y \equiv z \pmod 2\}$. To show that they are equal, we need to show that $(1, 0, 1), (0, 1, 1)$ and $(0, 0, 2)$ are in $Im(a)$. Consider maps

$$\begin{aligned} f_1 &: \mathbb{CP}^2 \wedge C \xrightarrow{q_1} S^4 \wedge C \simeq \Sigma^{2n-2}\mathbb{CP}^2 \hookrightarrow \Omega W_n \\ f_2 &: \mathbb{CP}^2 \wedge C \xrightarrow{q_2} \mathbb{CP}^2 \wedge S^{2n-2} \hookrightarrow \Omega W_n \\ f_3 &: \mathbb{CP}^2 \wedge C \xrightarrow{q_3} S^{2n+2} \xrightarrow{\theta} \Omega W_n \end{aligned}$$

where q_1, q_2 and q_3 are quotient maps and θ is the generator of $\pi_{2n+3}(W_n)$. Their images are

$$a(f_1) = (1, 0, 1) \quad a(f_2) = (0, 1, 1) \quad a(f_3) = (0, 0, 2)$$

respectively, so $Im(a) = \{(x, y, z)|x + y \equiv z \pmod 2\}$.

When n is even, C is $S^{2n-2} \vee S^{2n-4}$ and the $(2n + 3)$-skeleton of ΩW_n is $S^{2n+2} \vee S^{2n}$. Reducing to homology with $\mathbb{Z}/2\mathbb{Z}$-coefficients, $Sq^2(v_{2n-4}) = 0$ and $Sq^2(a_{2n}) = 0$. Apply Sq^2 to (13) to get $y \equiv 0 \pmod 2$. Therefore $Im(a)$ is contained in $\{(x, y, z)|y \equiv 0 \pmod 2\}$. To show that they are equal, we need to show that $(1, 0, 0), (0, 2, 0)$ and $(0, 0, 1)$ are in $Im(a)$. The maps

$$\begin{aligned} f_1' &: \mathbb{CP}^2 \wedge C \xrightarrow{q_1'} S^4 \wedge (S^{2n-2} \vee S^{2n-4}) \xrightarrow{p_1} S^4 \wedge S^{2n-4} \hookrightarrow \Omega W_n \\ f_2' &: \mathbb{CP}^2 \wedge C \xrightarrow{q_2'} S^4 \wedge (S^{2n-2} \vee S^{2n-4}) \xrightarrow{p_2} S^4 \wedge S^{2n-2} \hookrightarrow \Omega W_n \end{aligned}$$

where q_1' and q_2' are quotient maps and p_1 and p_2 are pinch maps, have images $a(f_1') = (1, 0, 0)$ and $a(f_2') = (0, 0, 1)$. To find $(0, 2, 0)$, apply $[- \wedge S^{2n-2}, \Omega W_n]$ to cofibration (4) to obtain the exact sequence

$$\pi_{2n+3}(W_n) \longrightarrow [\mathbb{CP}^2 \wedge S^{2n-2}, \Omega W_n] \xrightarrow{i^*} \pi_{2n+1}(W_n) \xrightarrow{\eta^*} \pi_{2n+2}(W_n)$$

where i^* is induced by the inclusion $i : S^2 \hookrightarrow \mathbb{CP}^2$ and η^* is induced by Hopf map η. The third term $\pi_{2n+1}(W_n) \cong \mathbb{Z}$ is generated by $i' : S^{2n+1} \to W_n$, the inclusion of the bottom cell, and the fourth term $\pi_{2n+2}(W_n) \cong \mathbb{Z}/2\mathbb{Z}$ is generated by $i' \circ \eta$, so $\eta^* : \mathbb{Z} \to \mathbb{Z}/2\mathbb{Z}$ is a surjection. By exactness $[\mathbb{CP}^2 \wedge S^{2n-2}, \Omega W_n]$ has a $\mathbb{Z}$-summand with the property that i^* sends its generator g to $2i'$. Therefore the composition

$$f_3' : \mathbb{CP}^2 \wedge (S^{2n-2} \vee S^{2n-4}) \xrightarrow{pinch} \mathbb{CP}^2 \wedge S^{2n-2} \xrightarrow{g} \Omega W_n$$

has image $(0, 2, 0)$. It follows that $Im(a) = \{(x, y, z)|y \equiv 0 \pmod 2\}$. □

Now we split into the n odd and n even cases to calculate the order of $Im(\partial_k')_*$.

5.1. **The order of $Im(\partial'_k)_*$ for n odd.** When n is odd, C is $\Sigma^{2n-6}\mathbb{CP}^2$. First we find $Im(\Phi)$ in $Im(a)$. For $1 \leq i \leq 4$, let L_i be the generators of $\tilde{K}^0(\mathbb{CP}^2 \wedge C) \cong \mathbb{Z}^{\oplus 4}$ with Chern characters

$$ch(L_1) = (u + \tfrac{1}{2}u^2) \cdot (v_{2n-4} + \tfrac{1}{2}v_{2n-2}) \quad ch(L_2) = (u + \tfrac{1}{2}u^2)v_{2n-2}$$

$$ch(L_3) = u^2(v_{2n-4} + \tfrac{1}{2}v_{2n-2}) \qquad ch(L_4) = u^2 v_{2n-2}.$$

By Theorem 4.3, we have

$$\begin{aligned}
\Phi(L_1) &= \frac{n!}{2}u^2 v_{2n-4} + \frac{n!}{2}uv_{2n-2} + \frac{(n+1)!}{4}u^2 v_{2n-2} \\
\Phi(L_2) &= n!uv_{2n-2} + \frac{(n+1)!}{2}u^2 v_{2n-2} \\
\Phi(L_3) &= n!u^2 v_{2n-4} + \frac{(n+1)!}{2}u^2 v_{2n-2} \\
\Phi(L_4) &= (n+1)!u^2 v_{2n-2}.
\end{aligned}$$

By Lemma 5.1, $Im(a)$ is spanned by $(1,0,1), (0,1,1)$ and $(0,0,2)$. Under this basis, the coordinates of the $\Phi(L_i)$'s are

$$\Phi(L_1) = (\tfrac{n!}{2}, \tfrac{n!}{2}, \tfrac{(n-3)\cdot n!}{8}), \quad \Phi(L_2) = (0, n!, \tfrac{(n-1)\cdot n!}{4}),$$

$$\Phi(L_3) = (n!, 0, \tfrac{(n-1)\cdot n!}{4}), \quad \Phi(L_4) = (0, 0, \tfrac{(n+1)!}{2}).$$

We represent their coordinates by the matrix

$$M_\Phi = L \begin{pmatrix} \frac{n(n-1)}{2} & \frac{n(n-1)}{2} & \frac{n(n-1)(n-3)}{8} \\ 0 & n(n-1) & \frac{n(n-1)^2}{4} \\ n(n-1) & 0 & \frac{n(n-1)^2}{4} \\ 0 & 0 & \frac{n(n^2-1)}{2} \end{pmatrix},$$

where $L = (n-2)!$. Then $Im(\Phi)$ is spanned by the row vectors of M_Φ.

Next, we find a preimage of $Im(\partial'_k)_*$ in $[\mathbb{CP}^2 \wedge C, \Omega W_n]$. In exact sequence (11) $\tilde{K}^0(\Sigma^2 C)$ is $\mathbb{Z} \oplus \mathbb{Z}$. Let α_1 and α_2 be its generators with Chern classes

$$\begin{aligned}
c_{n-1}(\alpha_1) &= (n-2)!\Sigma^2 v_{2n-4} & c_n(\alpha_1) &= \tfrac{(n-1)!}{2}\Sigma^2 v_{2n-2} \\
c_{n-1}(\alpha_2) &= 0 & c_n(\alpha_2) &= (n-1)!\Sigma^2 v_{2n-2}.
\end{aligned}$$

Lemma 5.2. *For $i = 1, 2$, $\xi_k(\alpha_i)$ has a lift $\tilde{\alpha}_{i,k} : \mathbb{CP}^2 \wedge C \to \Omega W_n$ such that*

$$a(\tilde{\alpha}_{i,k}) = ku^2 \otimes \Sigma^{-2}c_{n-1}(\alpha_i) \oplus ku^2 \otimes \Sigma^{-2}c_n(\alpha_i).$$

Proof. For dimension and connectivity reasons, $\alpha_i : \Sigma^2 C \to BSU(\infty)$ lifts through $BSU(n) \to BSU(\infty)$. Label the lift $\Sigma^2 C \to BSU(n)$ by α_i as well. Let $\alpha'_i : \Sigma C \to SU(n)$ be the adjoint of α_i. Then $(\partial'_k)_*(\alpha_i)$ is the adjoint of the composition

$$\mathbb{CP}^2 \wedge \Sigma C \xrightarrow{q \wedge \mathbb{1}} \Sigma S^3 \wedge \Sigma C \xrightarrow{\Sigma k\imath \wedge \alpha'_i} \Sigma SU(n) \wedge SU(n) \xrightarrow{[ev,ev]} BSU(n).$$

By Lemma 4.5, $(\partial'_k)_*(\alpha_i)$ has a lift $\tilde{\alpha}_{i,k}$ such that $\tilde{\alpha}^*_{i,k}(a_{2j}) = ku^2 \otimes \Sigma^{-1}(\alpha')^*(x_{2j-3})$. Since $\sigma(c_{j-1}) = x_{2j-3}$, we have $\tilde{\alpha}^*_{i,k}(a_{2j}) = ku^2 \otimes \Sigma^{-2}c_{j-1}(\alpha_i)$ and

$$a(\tilde{\alpha}_{i,k}) = ku^2 \otimes \Sigma^{-2}c_{n-1}(\alpha_i) \oplus ku^2 \otimes \Sigma^{-2}c_n(\alpha_i).$$

$\square$

By Lemma 5.2, $(\partial'_k)_*(\alpha_1)$ and $(\partial'_k)_*(\alpha_2)$ have lifts

$$\tilde{\alpha}_{1,k} = (n-2)!ku^2v_{2n-4} + \tfrac{(n-1)!}{2}ku^2v_{2n-2} \quad \text{and} \quad \tilde{\alpha}_{2,k} = (n-1)!ku^2v_{2n-2}.$$

We represent their coordinates by the matrix

$$M_\partial = kL\begin{pmatrix} 1 & 0 & \frac{n-3}{4} \\ 0 & 0 & \frac{n-1}{2} \end{pmatrix}.$$

Let $\tilde{\partial}_k = span\{\tilde{\alpha}_{1,k}, \tilde{\alpha}_{2,k}\}$ be the preimage of $Im(\partial'_k)_*$ in $[\mathbb{CP}^2 \wedge C, \Omega W_n]$. Then $\tilde{\partial}_k$ is spanned by the row vectors of M_∂.

Lemma 5.3. *When n is odd, the order of $Im(\partial'_k)_*$ is*

$$|Im(\partial'_k)_*| = \frac{\frac{1}{2}n(n^2-1)}{(\frac{1}{2}n(n^2-1), k)} \cdot \frac{n}{(n,k)}.$$

Proof. Suppose $n = 4m+3$ for some integer m. Then

$$M_\Phi = (4m+3)L\begin{pmatrix} 2m+1 & 2m+1 & 2m^2+m \\ 0 & 4m+2 & 4m^2+4m+1 \\ 4m+2 & 0 & 4m^2+4m+1 \\ 0 & 0 & 8m^2+12m+4 \end{pmatrix}$$

and

$$M_\partial = kL\begin{pmatrix} 1 & 0 & m \\ 0 & 0 & 2m+1 \end{pmatrix}.$$

Transform M_Φ into Smith normal form

$$A \cdot M_\Phi \cdot B = (4m+3)L\begin{pmatrix} (2m+1) & & \\ & (2m+1) & \\ & & (2m+1)(4m+4) \\ & & 0 \end{pmatrix},$$

where

$$A = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -2 & 0 & 1 & 0 \\ 4m+2 & 1 & -(2m+1) & 0 \\ 4 & -2 & -2 & 1 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 1 & -m & -(2m+1) \\ 0 & 0 & 1 \\ 0 & 1 & 2 \end{pmatrix}.$$

The matrix B represents a basis change in $Im(a)$ and A represents a basis change in $Im(\Phi)$. Therefore $[\mathbb{CP}^2 \wedge C, SU(n)]$ is isomorphic to

$$\frac{\mathbb{Z}}{\frac{1}{2}(4m+3)!\mathbb{Z}} \oplus \frac{\mathbb{Z}}{\frac{1}{2}(4m+3)!\mathbb{Z}} \oplus \frac{\mathbb{Z}}{\frac{1}{2}(4m+4)!\mathbb{Z}}.$$

We need to find the representation of $\tilde{\partial}_k$ under the new basis represented by B. The new coordinates of $\tilde{\alpha}_{1,k}$ and $\tilde{\alpha}_{2,k}$ are the row vectors of the matrix

$$M_\partial \cdot \begin{pmatrix} 1 & -m & -(2m+1) \\ 0 & 0 & 1 \\ 0 & 1 & 2 \end{pmatrix} = \begin{pmatrix} kL & 0 & -kL \\ 0 & (2m+1)kL & (4m+2)kL \end{pmatrix}.$$

Apply row operations to get

$$\begin{pmatrix} 1 & 0 \\ 4m+2 & 1 \end{pmatrix} \cdot \begin{pmatrix} kL & 0 & -kL \\ 0 & (2m+1)kL & (4m+2)kL \end{pmatrix} = \begin{pmatrix} kL & 0 & -kL \\ (4m+2)kL & (2m+1)kL & 0 \end{pmatrix}.$$

Let $\mu = (kL, 0, -kL)$ and $\nu = ((4m+2)kL, (2m+1)kL, 0)$. Then

$$\tilde{\partial}_k = \{x\mu + y\nu \in [\mathbb{CP}^2 \wedge C, \Omega W_n] | x, y \in \mathbb{Z}\}.$$

If $x\mu + y\nu$ and $x'\mu + y'\nu$ are the same in $Im(\Phi)$, then we have

$$\begin{cases} xkL + (4m+2)ykL & \equiv & x'kL + (4m+2)y'kL & \pmod{(2m+1)(4m+3)L} \\ (2m+1)ykL & \equiv & (2m+1)y'kL & \pmod{(2m+1)(4m+3)L} \\ xkL & \equiv & x'kL & \pmod{(2m+1)(4m+3)(4m+4)L} \end{cases}$$

These conditions are equivalent to

$$\begin{cases} xk & \equiv & x'k & \pmod{(2m+2)(4m+3)(4m+2)} \\ yk & \equiv & y'k & \pmod{(4m+3)} \end{cases}$$

This implies that there are $\dfrac{(2m+2)(4m+3)(4m+2)}{((2m+2)(4m+3)(4m+2),k)}$ distinct values of x and $\dfrac{4m+3}{(4m+3,k)}$ distinct values of y, so we have

$$|Im(\partial'_k)_*| = \frac{(2m+2)(4m+3)(4m+2)}{((2m+2)(4m+3)(4m+2),k)} \cdot \frac{4m+3}{(4m+3,k)}.$$

When $n = 4m+1$, we can repeat the calculation above to obtain

$$|Im(\partial'_k)_*| = \frac{2m(4m+2)(4m+1)}{(2m(4m+2)(4m+1),k)} \cdot \frac{4m+1}{(4m+1,k)}.$$

□

5.2. **The order of $Im(\partial'_k)_*$ for n even.** When n is even, C is $S^{2n-2} \vee S^{2n-4}$. For $1 \le i \le 4$, let L_i be the generators of $\tilde{K}^0(\mathbb{CP}^2 \wedge C) \cong \mathbb{Z}^{\oplus 4}$ with Chern characters

$$ch(L_1) = (u + \tfrac{1}{2}u^2)v_{2n-4} \quad ch(L_2) = u^2 v_{2n-4}$$

$$ch(L_3) = (u + \tfrac{1}{2}u^2)v_{2n-2} \quad ch(L_4) = u^2 v_{2n-2}.$$

By Theorem 4.3, we have

$$\begin{aligned} \Phi(L_1) &= \frac{n!}{2}u^2 v_{2n-4} \\ \Phi(L_2) &= n!u^2 v_{2n-4} \\ \Phi(L_3) &= n!uv_{2n-2} + \frac{(n+1)!}{2}u^2 v_{2n-2} \\ \Phi(L_4) &= (n+1)!u^2 v_{2n-2}. \end{aligned}$$

By Lemma 5.1, $Im(a)$ is spanned by $(1,0,0), (0,2,0)$ and $(0,0,1)$. Under this basis, the coordinates of the $\Phi(L_i)$'s are

$$\Phi(L_1) = (\tfrac{n!}{2}, 0, 0), \qquad \Phi(L_2) = (n!, 0, 0),$$

$$\Phi(L_3) = (0, \tfrac{n!}{2}, \tfrac{(n+1)!}{2}), \quad \Phi(L_4) = (0, 0, (n+1)!).$$

We represent the coordinates of $\Phi(L_i)$'s by the matrix

$$M_\Phi = \frac{n(n-1)}{2}L\begin{pmatrix}1 & 0 & 0\\ 2 & 0 & 0\\ 0 & 1 & n+1\\ 0 & 0 & 2n+2\end{pmatrix}$$

Then $Im(\Phi)$ is spanned by the row vectors of M_Φ.

In exact sequence (11) $\tilde{K}^0(\Sigma^2 C)$ is $\mathbb{Z}\oplus\mathbb{Z}$. Let α_1 and α_2 be its generators with Chern classes

$$\begin{array}{ll} c_{n-1}(\alpha_1) = (n-2)!\Sigma^2 v_{2n-4} & c_n(\alpha_1) = 0\\ c_{n-1}(\alpha_2) = 0 & c_n(\alpha_2) = (n-1)!\Sigma^2 v_{2n-2}. \end{array}$$

By Lemma 5.2, $(\partial'_k)_*(\alpha_1)$ and $(\partial'_k)_*(\alpha_2)$ have lifts

$$\tilde{\alpha}_{1,k} = (n-2)!ku^2 v_{2n-4} \quad \text{and} \quad \tilde{\alpha}_{2,k} = (n-1)!ku^2 v_{2n-2}.$$

We represent their coordinates by a matrix

$$M_\partial = kL\begin{pmatrix}1 & 0 & 0\\ 0 & 0 & n-1\end{pmatrix}.$$

Then the preimage $\tilde{\partial}_k = span\{\tilde{\alpha}_{1,k},\tilde{\alpha}_{2,k}\}$ of $Im(\partial'_k)_*$ is spanned by the row vectors of M_∂. We calculate as in the proof of Lemma 5.3 to obtain the following lemma.

Lemma 5.4. *When n is even, the order of $Im(\partial'_k)_*$ is*

$$|Im(\partial'_k)_*| = \frac{\frac{1}{2}n(n-1)}{(\frac{1}{2}n(n-1),k)}\cdot\frac{n(n+1)}{(n(n+1),k)}.$$

5.3. **Proof of Theorem 1.4.** Before comparing the orders of $Im(\partial'_k)_*$ and $Im(\partial'_k)_*$, we prove a preliminary lemma.

Lemma 5.5. *Let n be an even number and let p be a prime. Denote the p-component of t by $\nu_p(t)$. If there are integers k and l such that*

$$\nu_p(\frac{1}{2}n,k)\cdot\nu_p(n,k) = \nu_p(\frac{1}{2}n,l)\cdot\nu_p(n,l),$$

then $\nu_p(n,k)=\nu_p(n,l)$.

Proof. Suppose p is odd. If p does not divide n, then $\nu_p(n,k)=\nu_p(n,l)=1$, so the lemma holds. If p divides n, then $\nu_p(\frac{1}{2}n,k)=\nu_p(n,k)$. The hypothesis becomes $\nu_p(n,k)^2=\nu_p(n,l)^2$, implying that $\nu_p(n,k)=\nu_p(n,l)$.

Suppose $p=2$. Let $\nu_2(n)=2^r$, $\nu_2(k)=2^t$ and $\nu_2(l)=2^s$. Then the hypothesis implies

$$min(r-1,t)+min(r,t) = min(r-1,s)+min(r,s). \tag{14}$$

To show $\nu_2(n,k)=\nu_2(n,l)$, we need to show $min(r,t)=min(r,s)$. Consider the following cases: (1) $t,s\geq r$, (2) $t,s\leq r-1$, (3) $t\leq r-1, s\geq r$ and (4) $s\leq r-1, t\geq r$.

Case (1) obviously gives $min(r,t)=min(r,s)$. In case (2), when $t,s\leq r-1$, equation (14) implies $2t=2s$. Therefore $t=s$ and $min(r,t)=min(r,s)$.

It remains to show cases (3) and (4). For case (3) with $t\leq r-1, s\geq r$, equation (14) implies

$$2t = min(r-1,s)+r.$$

Since $s \geq r$, $min(r-1, s) = r-1$ and the right hand side is $2r-1$ which is odd. However, the left hand side is even, leading to a contradiction. This implies that this case does not satisfy the hypothesis. Case (4) is similar. Therefore $\nu_2(n,k) = \nu_2(n,l)$ and the asserted statement follows. $\square$

Proof of Theorem 1.4. In exact sequence (11), $[C, \mathcal{G}_k(\mathbb{CP}^2)]$ is $Coker(\partial'_k)_*$. By hypothesis, $\mathcal{G}_k(\mathbb{CP}^2)$ is homotopy equivalent to $\mathcal{G}_l(\mathbb{CP}^2)$, so $|Im(\partial'_k)_*| = |Im(\partial'_k)_*|$. The n odd and n even cases are proved similarly, but the even case is harder.

When n is even, by Lemma 5.4 the order of $Im(\partial'_k)_*$ is

$$|Im(\partial'_k)_*| = \frac{\frac{1}{2}n(n-1)}{(\frac{1}{2}n(n-1),k)} \cdot \frac{n(n+1)}{(n(n+1),k)},$$

so we have

$$(\frac{1}{2}n(n-1),k) \cdot (n(n+1),k) = (\frac{1}{2}n(n-1),l) \cdot (n(n+1),l). \tag{15}$$

We need to show that

$$\nu_p(n(n^2-1),k) = \nu_p(n(n^2-1),l) \tag{16}$$

for all primes p. Suppose p does not divide $\frac{1}{2}n(n^2-1)$. Equation (16) holds since both sides are 1. Suppose p divides $\frac{1}{2}n(n^2-1)$. Since $n-1$, n and $n+1$ are coprime, p divides only one of them. If p divides $n-1$, then $\nu_p(\frac{1}{2}n,k) = \nu_p(n,k) = \nu_p(n+1,k) = 1$. Equation (15) implies $\nu_p(n-1,k) = \nu_p(n-1,l)$. Since

$$\nu_p(n(n^2-1),k) = \nu_p(n-1,k) \cdot \nu_p(n,k) \cdot \nu_p(n+1,k),$$

this implies equation (16) holds. If p divides $n+1$, then equation (16) follows from a similar argument. If p divides n, then equation (15) implies $\nu_p(\frac{1}{2}n,k) \cdot \nu_p(n,k) = \nu_p(\frac{1}{2}n,l) \cdot \nu_p(n,l)$. By Lemma 5.5 $\nu_p(n,k) = \nu_p(n,l)$, so equation (16) holds.

When n is odd, by Lemma 5.3 the order of $Im(\partial'_k)_*$ is

$$|Im(\partial'_k)_*| = \frac{\frac{1}{2}n(n^2-1)}{(\frac{1}{2}n(n^2-1),k)} \cdot \frac{n}{(n,k)},$$

so we have

$$(\frac{1}{2}n(n^2-1),k) \cdot (n,k) = (\frac{1}{2}n(n^2-1),l) \cdot (n,l).$$

We can argue as above to show that for all primes p,

$$\nu_p(\frac{1}{2}n(n^2-1),k) = \nu_p(\frac{1}{2}n(n^2-1),l).$$

$\square$

References

[1] M. Atiyah and R. Bott, *The Yang-Mills equations over Riemann surfaces*, Philos. Trans. R. Soc. Lond. Ser. A Math. Phys. Eng. Sci., **308**, (1983), 523 – 615.

[2] D. Gottlieb, *Applications of bundle map theory*, Trans. Amer. Math. Soc., **171**, (1972), 23 – 50.

[3] H. Hamanaka and A. Kono, *On* $[X, U(n)]$ *when* $\dim X$ *is* $2n$, J. Math. Kyoto Univ., **43-2**, (2003), 333 – 348.

[4] H. Hamanaka and A. Kono, *Unstable K-group and homotopy type of certain gauge groups*, Proc. Roy. Soc. Edinburgh Sect. A, **136**, (2006), 149 – 155.

[5] D. Kishimoto, A. Kono, S. Theriault, *Refined gauge group decomposition*, J. Math. Kyoto Univ., **54**, (2014), 679 – 691.
[6] D. Kishimoto, A. Kono, M. Tsutaya, *On p-local homotopy types of gauge groups*, Proc. Roy. Soc. Edinburgh Sect. A, **144**, (2014), 149 – 160.
[7] D. Kishimoto, S. Theriault, M. Tsutaya, *The homotopy types of G_2-gauge groups*, Topology Appl., **228**, (2017), 92 – 107.
[8] A. Kono, *A note on the homotopy type of certain gauge groups*, Proc. Roy. Soc. Edinburgh Sect. A, **117**, (1991), 295 – 297.
[9] A. Kono and S. Tsukuda, *A remark on the homotopy type of certain gauge groups*, J. Math. Kyoto Univ., **36**, (1996), 115 – 121.
[10] G. Lang, *The evaluation map and EHP sequences*, Pacific J. Math., **44**, (1973), 201 – 210.
[11] T. So, *Homotopy types of gauge groups over non-simply-connected closed 4-manifolds*, Glasgow Math J., (2018), https://doi.org/10.1017/S0017089518000241.
[12] S. Theriault, *Odd primary homotopy decompositions of gauge groups*, Algebr. Geom. Topol., **10**, (2010), 535 – 564.
[13] S. Theriault, *The homotopy types of $Sp(2)$-gauge groups*, J. Math. Kyoto Univ., **50**, (2010), 591 – 605.
[14] S. Theriault, *Homotopy types of $SU(3)$-gauge groups over simply connected 4-manifolds*, Publ. Res. Inst. Math. Sci., **48**, (2012), 543 – 563.
[15] S. Theriault, *The homotopy types of $SU(5)$-gauge groups*, Osaka J. Math., **1**, (2015), 15 – 31.
[16] S. Theriault, *Odd primary homotopy types of $SU(n)$-gauge groups*, Algebr. Geom. Topol., **17**, (2017), 1131 – 1150.

Mathematical Sciences, University of Southampton, SO17 1BJ, UK
E-mail address: `tls1g14@soton.ac.uk`

Bibliography

[AB83] Michael Atiyah and Raoul Bott, *The yang-mills equations over riemann surfaces*, Philosophical Transactions of the Royal Society of London. Series A, Mathematical and Physical Sciences **308** (1983), no. 1505, 523–615.

[Ada60] Frank Adams, *On the non-existence of elements of hopf invariant one*, Annals of Mathematics **72** (1960), no. 1, 20–104.

[Ark11] Martin Arkowitz, *Introduction to homotopy theory*, Springer Science & Business Media, 2011.

[Bot60] Raoul Bott, *A note on the samelson products in the classical groups*, Commentarii Mathematici Helvetici **34** (1960), no. 1, 249–256.

[CN84] F. Cohen and J. Neisendorfer, *A construction of p-local H-spaces*, Algebraic Topology Aarhus 1982, Springer, 1984, pp. 351–359.

[CS00] M. Crabb and W. Sutherland, *Counting homotopy types of gauge groups*, Proceedings of the London Mathematical Society **81** (2000), no. 3, 747–768.

[Got72] Daniel Gottlieb, *Applications of bundle map theory*, Transactions of the American Mathematical Society **171** (1972), 23–50.

[Hat02] Allen Hatcher, *Algebraic topology*, Cambridge University Press, 2002.

[Hat04] Allen Hatcher, *Spectral sequences in algebraic topology*, Unpublished book project, available on http://www. math. cornell. edu/hatcher/SSAT/SSATpage. html **22** (2004).

[HK06] Hiroaki Hamanaka and Akira Kono, *Unstable K^1-group and homotopy type of certain gauge groups*, Proceedings of the Royal Scoiety of Edinburgh Section A: Mathematics **136** (2006), 149–155.

[KK18] Daisuke Kishimoto and Akira Kono, *On the homotopy types of $Sp(n)$ gauge groups*, arXiv preprint arXiv:1803.06477 (2018).

[KKT14] Daisuke Kishimoto, Akira Kono, and Mitsunobu Tsutaya, *On p-local homotopy types of gauge groups*, Proceedings of the Royal Society of Edinburgh Section A: Mathematics **144** (2014), 149–160.

[Kon91] Akira Kono, *A note on the homotopy type of certain gauge groups*, Proceedings of the Royal Scoiety of Edinburgh Section A: Mathematics **117** (1991), 195–297.

[KT96] Akira Kono and Shuichi Tsukuda, *A remark on the homotopy type of certain gauge groups*, Journal of Mathematics of Kyoto University **36** (1996), no. 1, 115–121.

[KTT17] Daisuke Kishimoto, Stephen Theriault, and Mitsunobu Tsutaya, *The homotopy types of G_2-gauge groups*, Topology and its Applications **228** (2017), 92–107.

[Lan73] George Lang, *The evaluation map and ehp sequences*, Pacific Journal of Mathematics **44** (1973), no. 1, 201–210.

[Mil56] John Milnor, *Construction of universal bundles, ii*, Annals of Mathematics **61** (1956), 430–436.

[MNT77] Mamoru Mimura, Goro Nishida, and Hirosi Toda, *Mod-p decomposition of compact lie groups*, Publications of the Research Institute for Mathematical Sciences **13** (1977), no. 3, 627–680.

[Nei10] Joseph Neisendorfer, *Algebraic methods in unstable homotopy theory*, Cambridge University Press, 2010.

[Sel08] Paul Selick, *Introduction to homotopy theory*, vol. 9, American Mathematical Society, 2008.

[Ser53] Jean-Pierre Serre, *Groupes d'homotopie et classes de groupes abéliens*, Annals of Mathematics **58** (1953), 258–294.

[So18a] Tseleung So, *Homotopy types of gauge groups over non-simply-connected closed 4-manifolds*, Glasgow Mathematical Journal https://doi.org/10.1017/S0017089518000241 (2018).

[So18b] ———, *Homotopy types of $SU(n)$-gauge groups over non-spin 4-manifolds*, arXiv preprint arXiv:1807.01931 (2018).

[So18c] ———, *The odd primary order of the commutator on low rank lie groups*, Topology and its Applications **240** (2018), 210–227.

[Sut92] W. Sutherland, *Function spaces related to gauge groups*, Proceedings of the Royal Society of Edinburgh **121A** (1992), 185–190.

[The07] Stephen Theriault, *The odd primary H-structure of low rank lie groups and its application to exponents*, Transactions of the American Mathematical Society **359** (2007), no. 9, 4511–4535.

[The10a] ______, *The homotopy types of $Sp(2)$-gauge groups*, Kyoto Journal of Mathematics **50** (2010), no. 3, 591–605.

[The10b] ______, *Odd primary homotopy decompositions of gauge groups*, Algebraic and Geometric Topology **10** (2010), 535–564.

[The12] ______, *The homotopy types of $SU(3)$-gauge groups over simply connected 4-manifolds*, Kyoto University. Research Institute for Mathematical Sciences. Publications. **48** (2012), 543–563.

[The15] ______, *The homotopy types of $SU(5)$-gauge groups*, Osaka Journal of Mathematics **52** (2015), 15–31.

[The17] ______, *Odd primary homotopy types of $SU(n)$-gauge groups*, Algebraic and Geometric Topology **17** (2017), 1131–1150.

[Tsu01] Shuichi Tsukuda, *Comparing the homotopy types of the components of Map$(S^4, BSU(2))$*, Journal of Pure and Applied Algebra **161** (2001), no. 1-2, 235–243.

[Tsu12] Mitsunobu Tsutaya, *A note on homotopy types of connected components of Map$(S^4, BSU(2))$*, Journal of Pure and Applied Algebra **216** (2012), 826–832.