UNIVERSITY OF SOUTHAMPTON

FACULTY OF SOCIAL, HUMAN AND MATHEMATICAL SCIENCES

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Generalised conformal structure and its implications

by

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ABSTRACT

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GENERALISED CONFORMAL STRUCTURE AND ITS IMPLICATIONS

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We study the generalised conformal structure of the low-energy dynamics of D-branes in all dimensions and its implications on correlation functions. We begin by considering conformal symmetry and the way it acts on fields. Symmetries and the way they constrain correlation functions are discussed through the derivation of Ward identities. These are used in order to put constraints on CFT n-point functions. The low-energy dynamics of D-branes are discussed and an extensive proof of their generalised conformal symmetry is given. Next, we investigate the analogy of Ward identities for the generalised conformal structure and derive similar equations with an additional term which compensates for the transformation of the coupling constant. Such an equation is used to prove the constraints of generalised dilatations and later in order to study a specific example of this structure - the free massive boson. The thesis ends with a discussion on the implications of this study and possible future steps.

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Declaration of Authorship

I, Stefanos Tyros, declare that the thesis entitled *Generalised conformal structure and its implications* and the work presented in the thesis are both my own, and have been generated by me as the result of my own original research. I confirm that:

- this work was done wholly or mainly while in candidature for a research degree at this University;
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- where I have consulted the published work of others, this is always clearly attributed;
- where I have quoted form the work of others, the source is always given. With the exception of such quotations, this thesis is entirely my own work;
- I have acknowledged all main sources of help;
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Signed: ....

Date: ....

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CHAPTER 1

Introduction

1.1 Solving for the dynamics of a system

The dynamics of a physical system are usually encoded in a Lagrangian. In classical (deterministic) physics the Euler-Lagrange equations are used to produce a set of partial differential equations which describe the evolution of the system in space-time. There exist, therefore, a straightforward algorithm that someone needs to follow in order to apply the general description of the Lagrangian to a particular system (although solving the equations can be rather daunting).

In quantum theory, though, the world is probabilistic. The dynamics are again encoded in a Lagrangian, but the Euler-Lagrange equations only give the classical limit of the system. A general system is described by a quantum state with the Lagrangian (or a Hamiltonian) dictating its time evolution. The end product of this evolution can be a number of potentially infinite states, each accompanied by a certain probability. These probabilities are the main observables in a quantum theory.

The objects that are used in describing those probabilities are the correlation functions. These are in a sense the square root of the probability of an initial state transitioning into another. The difficulty of solving for a specific system given its Lagrangian description depends on two main factors. One is considering all the possible final states given your initial state and the other is actually calculating each correlation function. The later issue is in most cases rather complicated, as exact solutions are most of the times unobtainable. Hence, a lot of techniques have been developed over the years. The most widely used is perturbation theory, where the correlation function is calculated approximately in orders of the coupling constant.

A lot of non-perturbative techniques have been developed as well. One of the most elegant ones is using the symmetries of the system to constrain, if not completely determine, the correlation functions. This technique does not require a specific Lagrangian description for the system. In this report we will investigate the generalised conformal symmetry obeyed by the low energy dynamics of D-branes and the constraints it puts on correlations functions.

1.2 Correlation functions and symmetries

Symmetries are a concept of great significance in physics. They are rather commonly used in classical mechanics in order to simplify descriptions and calculations. In quantum physics they have had a binary role. First, they have been used in order to explain in a simple and concise manner extremely complicated data, such as those arising from particle physics interactions. The foundations of the Standard Model, which is now known to be governed by an $SU(3) \times SU(2) \times U(1)$ symmetry, were laid by various attempts to describe experimental data, starting from the initial nuclear experiments all the way to what are now thought to be the elementary particles. Second, symmetries have more recently been used axiomatically, in order to produce new models, such as SUSY and string theory, that are hoped to answer some more fundamental questions without having available experimental data to start with.

As mentioned earlier, though, beyond using symmetries to describe the dynamics of quantum theories, they can also be used in order to solve for specific systems. In classical physics, Noether's theorem states that for every continuous symmetry of the system there is a corresponding conserved quantity, a current. Similarly, one finds that in quantum physics symmetry generators have a vanishing time derivative.

More importantly symmetries apply direct constraints to correlations functions. Specifically, the Ward identity states that the correlation functions of the conserved currents can be written in term of correlation functions of the fields, as follows [1]

$$\partial_\mu \langle j_a^\mu(x)\Phi(x_1)\dots\Phi(x_n)\rangle = -i\sum_{i=1}^{n}\delta(x-x_i)\langle\Phi(x_1)\dots G_a\Phi(x_i)\dots\Phi(x_n)\rangle. \tag{1.2.1}$$

Here Φ is the dynamical field of the theory, j_a^μ the conserved currents and G_a the generators of the symmetries. By integration of equation (1.2.1) over a region that includes all points of the RHS, x_i, one obtains the following equation

$$\delta\langle\Phi(x_1)\dots\Phi(x_n)\rangle = 0, \tag{1.2.2}$$

where δ indicates an infinitesimal change in the correlation function. This is the cornerstone of this report. Its meaning, though, is rather simple. It says that all the correlation functions of our fields, the observables of the theory, are left intact by a transformation that is a symmetry of the system. This equation, as we will see later on, gives rise to differential equations involving the correlation functions, which after being solved imply constraints on them.

1.3 The generalised conformal symmetry

String theory has emerged over the past years as the best candidate of a theory of quantum gravity. All particle physics theories obey the Poincare symmetry (Lorentz + translations), but string theory obeys a much larger symmetry that includes the later. This symmetry is called conformal symmetry. Conformal field theories have a wide variety of applications in particle physics, condensed matter physics and more. One of the striking features of these kinds of theories is that their correlation functions are highly constrained by the symmetry, as we shall see later on.

D-branes are higher-dimensional dynamical objects that arise in string theory. They are the hypersurfaces on which strings' ends lie on and, therefore, can give rise to interesting field theories. The low energy dynamics of 4-dimensional branes turn out to be conformal, hence, making the investigation of their dynamics a lot more straightforward. This gave rise to a quest of trying to find a possible symmetry that would govern all of D-branes. As we shall see later, such a symmetry is called the generalised conformal symmetry [4] [5] [6], being a generalisation of the former. It is therefore believed that this symmetry shall again put some constraints on the dynamics of D-branes and this will be the goal of this project.

1.4 Outlook

As we have discussed above, the dynamics of a quantum system are encoded in its correlation functions. The later in return can become rather constrained when governed by a symmetry, such as the of D-brane obey. In recent years D-branes, and specifically the conformal 3-branes, have given rise to the so called AdS/CFT correspondence [3]. This is a direct duality between gravitational theories in Anti-de Sitter space and conformal field theories living on its boundary. D-branes of different dimension give rise to more dualities, as a part of the more general Gauge/Gravity duality.

A deeper investigation into the generalised conformal symmetry could give rise to a better understanding of the D-branes. In turn this would further contribute in solidifying the Gauge/Gravity duality by helping us expand the holographic dictionary and give further tests of the correspondence beyond AdS asymptotics. Finally, it could be used to investigate strongly coupled systems via the duality, such as inflationary models [8] or QCD.

1.5 Structure of the thesis

This report is the beginning of an investigation into the generalised conformal symmetry and its consequences on physical systems it governs. Chapter 1 was occupied with providing a short introduction into the area and motivations of the investigation. Chapter 2 provides a thorough investigation of conformal symmetry, CFTs, and the constraints on their correlators as this is the process we shall later try and generalise. Chapter 3 reintroduces generalised conformal symmetry as a generalisation of the conformal symmetry of D3-branes. Chapter 4 provides a detailed proof of the generalised conformal symmetry of the world volume dynamics of Dp-branes and. It also discussed the generalisation of Ward identities for the generalised conformal structure. Chapter 5 investigates the implications of the Ward identities of the generalised dilatations to 2 and 3-point functions. Chapter 6 probes the generalised conformal structure of the Massive Scalar theory. Finally, Chapter 7 discussed the results of future paths of this work.

CHAPTER 2

Conformal field theory (CFT)

A Weyl transformation is defined as preserving the metric up to a constant, namely

$$g_{\mu\nu} \to \Omega^2(x^a) g_{\mu\nu}. \tag{2.0.1}$$

On the other hand, a coordinate transformation $x^\mu \to x'^\mu$ transforms the metric as

$$g'_{\mu\nu} = \frac{\partial x^\alpha}{\partial x'^\mu} \frac{\partial x^\beta}{\partial x'^\nu} g_{\alpha\beta}. \tag{2.0.2}$$

A conformal transformation is defined as a coordinate transformation that acts on the metric as an (inverse) Weyl transformation. In flat space, which will be our focus, this reads as

$$\frac{\partial x^c}{\partial x'^a} \frac{\partial x^d}{\partial x'^b} \eta_{cd} = \Omega^{-2} \eta_{ab}. \tag{2.0.3}$$

2.1 Strings and conformal symmetry

Another way to introduce the conformal symmetry is as the symmetry of the bosonic string worldsheet. A point particle travels through a geodesic determined by extremisation of its length. By analogy a one-dimensional string sweeps out a two-dimensional worldsheet. It's dynamics are given by extremisation of the worldsheet's area, described by the usual Nambu-Goto action, or the physically equivalent Polyakov action

$$S_P = -\frac{1}{4\pi\alpha'}\int d^2x^a \sqrt{-\det(\gamma_{ab})}\gamma^{ab}\partial_a X^\mu \partial_b X_\mu. \quad (2.1.1)$$

Here α' is the Regge slope, γ_{ab} the world sheet metric and X^μ the spacetime coordinates of the worldsheet.

It is clear from equation (2.1.1) that the system is governed by two worldsheet symmetries, namely worldsheet diffeomorphism and Weyl invariance. These two symmetries give us the ability to gauge fix the metric to be flat. A residual symmetry, though, remains. Namely, a combination of the two initial symmetries that leaves the flat metric unchanged

$$\gamma'_{ab} = \frac{\partial x^c}{\partial x'^a}\frac{\partial x^d}{\partial x'^b}\eta_{cd} = \Omega^{-2}\eta_{ab}. \quad (2.1.2)$$

This, as we have seen, is the conformal symmetry in flat space. String theory is, therefore, governed by worldsheet conformal symmetry in two dimensions.

2.2 The conformal group

Equation (2.1.2) defines the conformal symmetry. A conformal transformation is, therefore, a diffeomorphism that changes the flat metric by an amount proportional to itself [1] . Considering an infinitesimal coordinate transformation $x^\mu \to x^\mu + \xi^\mu(x^\mu)$ and $\Omega = 1 + K$ and using equation (2.0.3) we acquire the conformal Killing equation

$$\partial_\mu \xi_\nu + \partial_\nu \xi_\mu = 2K\eta_{\mu\nu}, \quad (2.2.1)$$

where the proportionality constant can be fixed to $K = \frac{1}{d}\partial_\rho \xi^\rho$ by taking the trace on both sides of the equation.

Applying, now, $\partial_\alpha\partial_\beta$ to equation (2.2.1) and contracting with $\eta^{\alpha\mu}\eta^{\beta\nu}$ we have

$$(d-1)\partial^2\partial_\rho\xi^\rho = 0. \quad (2.2.2)$$

Keeping this in mind we repeat the procedure of applying $\partial_\alpha\partial_\beta$ to equation (2.2.1), but this time we contract with $\eta^{\beta\nu}$ only and, therefore, after symmetrisation on α, μ we get

$$\begin{aligned}\frac{d-2}{d}\partial_\alpha\partial_\mu\partial_\beta\xi^\beta &= -\partial^2\partial_{(\alpha}\xi_{\mu)} \\ &= -2\eta_{\alpha\mu}\partial^2 K.\end{aligned} \quad (2.2.3)$$

Hence, using equation (2.2.2) for $d > 1$ (i.e. at least one space dimension) and that $\partial_\beta\xi^\beta =$

dK we conclude that

$$(d-2)\partial_\alpha \partial_\mu K = 0, \tag{2.2.4}$$

i.e. that for $d > 2$ K can be up to a linear function of the coordinates. For $d = 2$ the constraint is automatically satisfied leading to an infinite number of generators that obey the Witt algebra [2]. We shall, from now on, concentrate on the $d > 2$ case.

For $K = 0$ it is clear that equation (2.2.1) becomes the usual Killing equation in flat space time, which is symmetric under Poincare group, and therefore its solutions are translations and Lorentz transformations

$$\delta^T_a x^\mu = a^\mu \qquad \& \qquad \delta^L_\omega x^\mu = \omega^\mu_\nu x^\nu, \qquad \omega_{\mu\nu} = -\omega_{\nu\mu}. \tag{2.2.5}$$

Here, for later convenience, we have introduced the notation δx^μ instead of ξ^μ with a superscript indicating the transformation and a subscript the transformation parameter. For $K = \epsilon$ the solution is a Dilatation (D) with

$$\delta^D_\epsilon x^\mu = \epsilon x^\mu, \tag{2.2.6}$$

as can easily be verified by inserting into (2.2.1). Finally, in the linear level, i.e. for $K = -\epsilon_\mu x^\mu$ the solution is a special conformal transformation (SCT) with

$$\delta^S_\epsilon x^\mu = 2\epsilon \cdot x x^\mu - \epsilon^\mu x^2, \tag{2.2.7}$$

as can again easily be verified.

The Poincare group in d dimensions has the usual generators that form a representation of the $SO(d-1,1)$ group. From equations (2.2.6) and (2.2.7) it is clear that the corresponding generators for dilatations and SCTs are

$$D = -ix^\mu \partial_\mu, \qquad K_\mu = -i(2x_\mu x^\nu \partial_\nu - x^2 \partial_\mu). \tag{2.2.8}$$

These can be put together with the Poincare generators and form an extension, a representation of the $SO(d,2)$ algebra in a d-dimensional space. Finally, the finite transformations for dilations and SCTs are of the form

$$x'^\mu = \lambda x^\mu, \qquad x'^\mu = \frac{x^\mu - b^\mu x^2}{1 - 2b\cdot x + b^2 x^2}, \tag{2.2.9}$$

where λ and b^μ are the finite transformation parameters. The result can easily be verified by inserting to (2.0.3).

2.3 Conformal fields

We shall, now, consider scalar fields under conformal transformations. Under dilatations a scalar field, Φ, will transform as

$$\Phi'(x') = \lambda^{-\Delta_\Phi}\Phi(x), \tag{2.3.1}$$

where Δ_Φ is called the weight of the field. It is already apparent that fields do not transform in the same way as under Lorentz transformations. For a general, now, conformal transformation a scalar field will transform as

$$\Phi'(x') = \left|\frac{\partial x'}{\partial x}\right|^{-\Delta_\Phi/d}\Phi(x). \tag{2.3.2}$$

The infinitesimal change of a field under a dilatations and the special conformal transformation is defined as

$$\delta_\epsilon^{D/S}\Phi(x) = \Phi'(x) - \Phi(x). \tag{2.3.3}$$

Using, also, $\Phi'(x') = \Phi'(x) + \delta_\epsilon x^\mu \partial_\mu \Phi'(x)$ for an infinitesimal transformation we can write out the infinitesimal changes of a scalar field under dilatations and SCT

$$\delta_\epsilon^D\Phi = -\epsilon(\Delta_\Phi + x\cdot\partial)\Phi, \qquad \delta_\epsilon^S\Phi = -(2\Delta_\Phi\epsilon\cdot x + \delta_\epsilon^S x\cdot\partial)\Phi. \tag{2.3.4}$$

2.4 Ward identities

As discussed in Chapter 1, symmetries can put constraints on correlation functions through Ward identities. We will be mostly interested in equation (1.2.2) that shall provide us directly with differential equations of the correlators. As we shall use this procedure later for the generalised conformal symmetry - which is strictly not a symmetry -, we will show explicitly now how it is derived. This is done so that we are later able to appreciate what changes in the Ward identities are caused by using a structure closely related but not identical to a symmetry.

We shall consider, now, correlation functions of multiple operators, $O_i[\Phi](x_i)$, that can be either fundamental fields, which we denote collectively as Φ, or composite operators. They can be thought as a path integral in the following way

$$\langle O_1[\Phi](x_1)\dots O_n[\Phi](x_n)\rangle = \frac{1}{Z}\int [d\Phi]\, O_1[\Phi](x_1)\dots O_n[\Phi](x_n)\exp(-S[\Phi]), \tag{2.4.1}$$

where Z is the vacuum functional, $S[\Phi]$ the action as a functional of all the fundamental fields and $[d\Phi]$ the measure of all the dynamical fields. We now set up the derivation, where we essentially make use of the invariance of the action under transformation of the active fields (the ones appearing in the measure). It follows that [1]

$$\begin{aligned}\langle O_1[\Phi]\dots O_n[\Phi]\rangle &= \frac{1}{Z}\int [d\Phi]\, O_1[\Phi]\dots O_n[\Phi]\exp(-S[\Phi]) \\ &= \frac{1}{Z}\int \left[d\Phi'\right] O_1'[\Phi']\dots O_n'[\Phi']\exp(-S[\Phi']) \\ &= \frac{1}{Z}\int [d\Phi]\, O_1'[\Phi']\dots O_n'[\Phi']\exp(-S[\Phi]) \\ &= \langle O_1'[\Phi']\dots O_n'[\Phi']\rangle.\end{aligned} \tag{2.4.2}$$

Here, from going to the second line we have made a variable change $\Phi \to \Phi'$. Going to the third line, we have used the action invariance under conformal transformations, i.e. $S[\Phi'] = S[\Phi]$, and the measure invariance, $[d\Phi'] = [d\Phi]$, i.e. that there is no anomaly. The final result follows from the definition. Using, finally the definition (2.3.3) we arrive to the desired result

$$\delta\langle O_1[\Phi]\dots O_n[\Phi]\rangle = 0. \tag{2.4.3}$$

This can now be used to produce differential equations that will constrain the correlators.

2.5 CFT correlation functions

We are now in a position to use what we have seen in this chapter, in order to apply and solve the constrains to correlation functions due to conformal symmetry. We shall do this for 2-point scalar field correlation functions, a result long known [9]. 3-point functions give similar results and we will discuss the issues arising when someone tries to consider 4-point functions or higher.

The cornerstone of using the symmetry to constrain the correlators is, of course, equation (2.4.3). To start with, we use the fact that infinitesimal changes as defined by (2.3.3) obey the Leibniz rule and that the δ can be inserted into the correlation function as it leaves the measure and the action unaffected (see equation (2.4.1)). On a 2-point function this reads as

$$\langle \delta_\epsilon^{D/S}\Phi_1(x_1)\Phi_2(x_2)\rangle + \langle \Phi_1(x_1)\delta_\epsilon^{D/S}\Phi_2(x_2)\rangle = 0. \tag{2.5.1}$$

Moreover, given the translational and rotational invariance (part of the conformal symmetry) the correlators are forced to depend on the position coordinates only via $(x_i - x_j)^2$. The 2-point function is, therefore, a function of $(x_1 - x_2)^2$ [10]

$$\langle \Phi_1(x_1)\Phi_2(x_2)\rangle = f\left((x_1 - x_2)^2\right). \tag{2.5.2}$$

Now, we shall impose the restrictions due to dilatation symmetry. Using (2.5.1) and the first equality of (2.3.4) we arrive to a differential equation for the 2-point function

$$(\Delta_{\Phi_1} + \Delta_{\Phi_2} + x_1 \cdot \partial_1 + x_2 \cdot \partial_2)\, f\left((x_1 - x_2)^2\right) = 0. \tag{2.5.3}$$

Using that

$$x_1 \cdot \partial_1 f = x_1 \cdot \partial_1 (x_1 - x_2)^2 \frac{\partial f\left((x_1 - x_2)^2\right)}{\partial (x_1 - x_2)^2} = 2(x_1^2 - x_1 \cdot x_2) \frac{\partial f\left((x_1 - x_2)^2\right)}{\partial (x_1 - x_2)^2} \tag{2.5.4}$$

and similarly for $x_2 \cdot \partial_2 f$, we reach

$$\left(\frac{\Delta_{\Phi_1} + \Delta_{\Phi_2}}{2} + (x_1 - x_2)^2 \frac{\partial}{\partial (x_1 - x_2)^2}\right) f\left((x_1 - x_2)^2\right) = 0. \tag{2.5.5}$$

This simply requires that the function is of order $-\frac{\Delta_{\Phi_1}+\Delta_{\Phi_2}}{2}$, hence we have that

$$\langle \Phi_1(x_1) \Phi_2(x_2) \rangle = \frac{C_{12}}{(x_1 - x_2)^{\Delta_{\Phi_1} + \Delta_{\Phi_2}}}, \tag{2.5.6}$$

where C_{12} is a constant. Similarly, using the second equality of (2.3.4) and the result (2.5.6) we derive the differential equation due to SCT invariance

$$\left(2\Delta_{\Phi_1} \epsilon \cdot x_1 + 2\Delta_{\Phi_2} \epsilon \cdot x_2 + \delta^S_\epsilon x_1 \cdot \partial_1 + \delta^S_\epsilon x_2 \cdot \partial_2\right) \frac{C_{12}}{(x_1 - x_2)^{\Delta_{\Phi_1} + \Delta_{\Phi_2}}} = 0. \tag{2.5.7}$$

Using a similar argument to (2.5.4) and acting with the derivatives we have that

$$\begin{aligned} &\big[(2\Delta_{\Phi_1} \epsilon \cdot x_1 + 2\Delta_{\Phi_2} \epsilon \cdot x_2) \\ &\quad - (\Delta_{\Phi_1} + \Delta_{\Phi_2})(\delta^S_\epsilon x_1 \cdot x_1 - \delta^S_\epsilon x_1 \cdot x_2 - x_1 \cdot \delta^S_\epsilon x_2 + \delta^S_\epsilon x_2 \cdot x_2) \frac{1}{(x_1 - x_2)^2}\big] C_{12} = 0. \end{aligned} \tag{2.5.8}$$

Inserting for the definition of $\delta^S_\epsilon x$ from (2.3.4) in the large factor of the second term we have that

$$(\delta^S_\epsilon x_1 \cdot x_1 - \delta^S_\epsilon x_1 \cdot x_2 - x_1 \cdot \delta^S_\epsilon x_2 + \delta^S_\epsilon x_2 \cdot x_2) = (\epsilon \cdot x_1 + \epsilon \cdot x_2)\,(x_1 - x_2)^2 \tag{2.5.9}$$

Using this and canceling term out we reach

$$\big[(\Delta_{\Phi_1} - \Delta_{\Phi_2})(\epsilon \cdot x_1 + \epsilon \cdot x_2)\big] C_{12} = 0, \tag{2.5.10}$$

meaning that for the equation to be satisfied for all x_1, x_2 we need to have $\Delta_{\Phi_1} - \Delta_{\Phi_2} = 0$, for a non zero 2-point function. Hence, the constraint to the 2-point function of the full conformal symmetry is

$$\langle \Phi_1(x_1)\Phi_2(x_2)\rangle = \begin{cases} \dfrac{C_{12}}{(x_1-x_2)^{2\Delta}}, & \text{if } \Delta_1=\Delta_2=\Delta \\ 0, & \text{otherwise.} \end{cases} \tag{2.5.11}$$

Therefore, remarkably, conformal symmetry completely fixes the 2-point function of scalar field, up to a normalisation constant.

We are able to follow a similar analysis the 3-pt function. Using the same argument as before, due to Lorentz symmetry, the correlator can only have the following dependence

$$\langle \Phi_1(x_1)\Phi_2(x_2)\Phi_3(x_3)\rangle = g\left(x_{12}^2, x_{23}^2, x_{13}^2\right), \tag{2.5.12}$$

where we have introduced the notation $x_{ij} = x_i - x_j$, and as for the 2-point function constraint (2.5.3) due to dilatation invariance we have that

$$(\Delta_{\Phi_1} + \Delta_{\Phi_2} + \Delta_{\Phi_2} + x_1\cdot\partial_1 + x_2\cdot\partial_2 + x_3\cdot\partial_3)\, g = 0 \tag{2.5.13}$$

Following similar step as from equation (2.5.4) we have that

$$\left(\frac{\Delta_{\Phi_1}+\Delta_{\Phi_2}+\Delta_{\Phi_3}}{2} + x_{12}^2\frac{\partial}{\partial x_{12}^2} + x_{23}^2\frac{\partial}{\partial x_{23}^2} + x_{13}^2\frac{\partial}{\partial x_{13}^2}\right) g = 0. \tag{2.5.14}$$

Hence, as for the 2-pt case, the function is or total order $-\frac{\Delta_{\Phi_1}+\Delta_{\Phi_2}+\Delta_{\Phi_3}}{2}$ in the norms, i.e.

$$\langle \Phi_1(x_1)\Phi_2(x_2)\Phi_3(x_3)\rangle = \frac{C_{123}}{x_{12}^a x_{23}^b x_{13}^c}, \qquad a+b+c = \Delta_{\Phi_1}+\Delta_{\Phi_2}+\Delta_{\Phi_3}, \tag{2.5.15}$$

where C_{123} is again a constant. We are, now, ready to use SCT invariance to constraint this further. In the same way as with equation (2.5.7) we have

$$\Big(2\Delta_{\Phi_1}\epsilon\cdot x_1 + 2\Delta_{\Phi_2}\epsilon\cdot x_2 + 2\Delta_{\Phi_3}\epsilon\cdot x_3 \\ + \delta_\epsilon^S x_1\cdot\partial_1 + \delta_\epsilon^S x_2\cdot\partial_2 + \delta_\epsilon^S x_3\cdot\partial_3\Big)\frac{C_{123}}{x_{12}^a x_{23}^b x_{13}^c} = 0. \tag{2.5.16}$$

Following, now, similar steps to those leading to equation (2.5.10), with the difference that this time we need to consider that $x_1\cdot\partial_1 g = x_1\cdot\partial_1 x_{12}^2\frac{\partial g}{\partial_{12}} + x_1\cdot\partial_1 x_{13}^2\frac{\partial g}{\partial_{13}}$, we reach the result

$$[\epsilon\cdot x_1(2\Delta_{\Phi_1}-a-c) + \epsilon\cdot x_2(2\Delta_{\Phi_2}-a-b) + \epsilon\cdot x_3(2\Delta_{\Phi_3}-b-c)] = 0. \tag{2.5.17}$$

For this to hold for all x_1, x_2, x_3 the respective factors in each term must independently be equal to zero. Solving them using also that $a+b+c = \Delta_{\Phi_1} + \Delta_{\Phi_2} + \Delta_{\Phi_3}$ we determine the values of a, b and c, giving the final result due to CFT invariance

$$\langle \Phi_1(x_1)\Phi_2(x_2)\Phi_3(x_3)\rangle = \frac{C_{123}}{x_{12}^{\Delta_1+\Delta_2-\Delta_3} x_{23}^{\Delta_2+\Delta_3-\Delta_1} x_{13}^{\Delta_3+\Delta_1-\Delta_2}}. \tag{2.5.18}$$

Hence, we see that conformal invariance also completely fixes the 3-point function.

The discussion becomes more complicated when one starts considering higher-point correlation functions. This is due to the transformation properties of x_{ij} under conformal transformation. It is clear that ratios of these quantities shall be invariant under dilatations. Under SCTs, though, they transform as

$$x_{ij} \to \frac{x_{ij}}{\left(1 - 2b\cdot x_i + b^2 x_i^2\right)^{1/2} \left(1 - 2b\cdot x_j + b^2 x_j^2\right)^{1/2}}. \tag{2.5.19}$$

Consequently, for four or more distinct coordinates (i.e. four or more fields in the correlator) one can form cross ratios that are invariant under the whole group

$$r_{ijkl} = \frac{x_{ij}x_{kl}}{x_{ik}x_{jl}}. \tag{2.5.20}$$

As these term are invariant, they can appear in the correlators in a way unconstrained by the Ward identities. Hence, these correlators would be of the form

$$\langle \Phi_1(x_1)\Phi_2(x_2)\Phi_3(x_3)\dots\Phi_n(x_n)\rangle = g(r_{ijkl})f(x_{ij}), \tag{2.5.21}$$

with $g(r_{ijkl})$ being an unknown function of the cross ratios and $f(x_{ij})$ a constrained by the Ward identities in a similar way to the one above. Methods that constrain these correlators further have been found, such as conformal bootstrap [11], but this is beyond our scope.

CHAPTER 3

The Generalised Conformal Structure

Within the framework of string theory, D-branes consist of the region where open string end points lie on. On the one hand, their world volume dynamics are described by the gauge fields of the string end points and the Higgs fields, X_m that describe their transverse coordinates. For a (p+1)-dimensional brane the gauge fields dynamics are described by the (p+1)-dimensional super Yang-Mills theory [6]. The bosonic part is described by the following Lagrangian

$$S_{bosonic} = \mathrm{Tr} \int d^{p+1}x \left\{ -\frac{1}{4g^2} F_{\mu\nu} F^{\mu\nu} - \frac{1}{2g^2} D_\mu X_m D^\mu X^m + \frac{1}{4g^2} [X_m, X_n]^2 \right\}, \tag{3.0.1}$$

with $D_\mu = \partial_\mu + A_\mu$. It is well known that for $p = 3$ the branes are conformal.

On the other hand, the mass of the D-branes curves spacetime, giving rise to supergravity theories. In the context of the AdS/CFT correspondence [3], the gauge theory is seen as living in the boundary of the AdS space. In this setup, the conformal symmetry is of central importance. More specifically, both the Yang Mills theory and the $AdS^5 \times S_5$ obey the same conformal group $SO(4, 2)$. This symmetry, along with supersymmetric non-renormalization theorems, was constraining enough to fix the action of the D3-brane. Given that the dual nature of D-branes mentioned above is not a special feature of D3-branes and the constraining power of the conformal symmetry, [4] [5] and [6] have explored its generalisation to arbitrary dimensions.

3.1 D-particles

As thoroughly discussed in the next chapter, it is easy to see from (3.0.1) that the Yang-Mills theory scalar fields of the D3-brane are transformed with a $\Delta_\Phi = +1$, i.e. the scale transformation has the form:

$$X_i'(x_\alpha') = \lambda X_i(x_\alpha) \quad \& \quad x_\alpha' = \lambda^{-1} x_\alpha, \tag{3.1.1}$$

where x_α are the world volume coordinates of the brane and $X_i(x_\alpha)$ the Higgs fields representing the transverse coordinates to the brane. Hence the subscripts run as $i = 1, \ldots, 6$ and $\alpha = 0, 1, 2, 3$. This symmetry is seen in [4] as a symmetry leaving invariant the space-time uncertainty relation

$$\Delta T \Delta X \sim \alpha', \tag{3.1.2}$$

where T represent the longitudinal (world volume) coordinates and X the transverse. Hence, they interpret the full conformal symmetry as the set of general symmetries that leave this space-time uncertainty invariant.

Given that the space-time uncertainty principle is not restricted to the D3-branes, it is natural to extend the symmetry to other dimensions. The first such extension considered was for the D-particle. Its effective super Yang-Mills theory is found to be invariant under the scale transformation, provided that the string coupling constant is simultaneously transformed, i.e.:

$$X_i'(t) = \lambda X_i(t) \quad \& \quad t' = \lambda^{-1} t \quad \& \quad g' = \lambda^3 g. \tag{3.1.3}$$

Here $i = 1, \ldots, 10$ and t is the (only) world volume coordinate. g is the string coupling, which in this case is seen as a dynamical field corresponding to the vacuum expectation value of the dilaton. It was found that the action of the supersymmetric Yang-Mills matrix quantum mechanics is also invariant under the corresponding special conformal transformation as well.

From the supergravity point of view an identical transformation is derived in the near horizon limit. Hence, it is found that, as in the D3-branes case, both sides of the theory obey the same (generalised) conformal symmetry.

In the case of the D3-brane, it has been shown that the conformal symmetry, combined with a supersymmetric non-renormalization theorem, determines the bosonic part of the action. It is shown that the generalised conformal symmetry has the same constraining power in the D-particle case. The scattering of a probe D-particle in the background of the source system with a large number N of coincident D-particles is considered. $U(t)$

is the distance between the probe and the source. Using some simplifying assumptions and the scaling symmetry it is shown in [4] that the action take the form

$$S = \int dt \frac{1}{2g_s} \left(\frac{dU}{dt} \right)^2 F \left(\frac{g_s}{U^3}, \frac{1}{U^4} \left(\frac{dU}{dt} \right)^2 \right). \tag{3.1.4}$$

Expanding the function F as a series, applying the special conformal transformation and the supersymmetric non-renormalization theorem for supersymmetric particle mechanics fixes completely the expression for S. This is found to agree up to at least two loops with other calculations. Hence, it is found that the constraining power of the conformal symmetry is not a feature of only the D3-brane.

3.2 Dp-branes

Following a similar route, the generalisation to all Dp-branes of the generalised conformal symmetry is discussed in [6]. The supergravity metric of N coincident Dp-branes is, first, shown to be invariant under the generalised conformal symmetry, similarly to the D0 case. This holds for all p other than $p = 5$. This symmetry is found to constraint the action of a probe Dp-brane in the field of N coincident Dp-branes placed at the origin, resulting the BDI action calculated elsewhere. Finally, the (p+1)-dimensional super Yang-Mills theory describing the low energy dynamics of near-coincident N Dp-branes is also found to be invariant. An extended proof of the later is presented in the next chapter.

This underlying generalised conformal structure is used in [7], where precision holography is set up for non-conformal branes. This structure is given as the reason that the duality for non-conformal branes can be set up in a similar manner to the D3-brane, which is conformal. It is shown that the generalised dilatation constants two-point functions to a form similar to 2.5.11, with an extra function of an invariant combination of the string coupling and the fields. This is proven and explored further in Chapter 5.

CHAPTER 4

The Generalised Conformal Structure of Dp-branes and their correlation functions

As discussed in the previous chapter, Dp-branes are higher dimensional dynamical objects that arise in string theory. It was mentioned below (3.0.1) that for $p = 3$ the branes are conformal. As we will discuss bellow, there is a certain structure that remains in all dimensions, namely the generalised conformal structure, which in turn constraints their correlation functions. This chapter provides a proof of the former claim for the world volume and explores the latter.

4.1 The generalised conformal structure of D-branes

As we have seen above, the low-energy dynamics of (p+1)-dimensional D-branes are conformal for $p = 3$. As discussed in Chapter 2, this applies major constraints on the correlation functions of such a system and, therefore, allows us to investigate its dynamics in a straightforward manner. Hence, given the fact that D-branes play a central role in string theory and the more recent Gauge/Gravity duality, one would like to investigate the possibility of them obeying a more general symmetry. Such a symmetry could then apply constraints to a much more general class of branes that just 3-branes.

It turns out that such a structure does exist. It was first introduced in [4] and further investigated in [7]. The structure includes a conformal transformation of the dynamical fields, which for $p=3$ leaves the action invariant. For general p, though, in order for the action to be invariant, one also needs to consider the coupling g^2 as a background field transforming like a scalar.

Considering, at first, finite dilatations, $x^\mu \to \lambda x^\mu$, A_μ must transform like ∂_μ due to D_μ, hence $A_\mu \to \lambda^{-1} A_\mu$. Considering then the first and last term of the RHS of (3.0.1) we further conclude the following transformations $X_m \to \lambda^{-1} X_m$ and $g^2 \to \lambda^{p-3} g^2$, and hence $\Delta_X = 1$ and $\Delta_{g^2} = 3-p$ (which is zero for $p=3$, giving rise to a non-transforming coupling constant and, hence, simplifying the structure to the usual conformal symmetry).

Given the above we can now evaluate the infinitesimal changes for the fields under dilatations and special conformal transformation (SCT), as defined in equation (2.3.3). We use (2.3.2), the usual vector transformation and $\Phi'(x') = \Phi'(x) + \delta_\epsilon x^\mu \partial_\mu \Phi'(x)$ for an infinitesimal spatial transformation $x' = x + \delta_\epsilon x$. This leaves us with the following dilatation and SCT of the fields and the coupling constant:

$$\begin{aligned} \delta^D_\epsilon x^\mu &= \epsilon x^\mu, & \delta^D_\epsilon X_m &= -\epsilon(1 + x\cdot\partial)X_m, \\ \delta^D_\epsilon A^\mu &= -\epsilon(1 + x\cdot\partial)A^\mu, & \delta^D_\epsilon g^2 &= -\epsilon((3-p) + x\cdot\partial)g^2 \end{aligned} \tag{4.1.1}$$

and

$$\begin{aligned} &\delta^S_\epsilon x^\mu = 2\epsilon\cdot x x^\mu - \epsilon^\mu x^2, \quad \delta^S_\epsilon X_m = -(2\epsilon\cdot x + \delta^S_\epsilon x\cdot\partial)X_m, \\ &\delta^S_\epsilon A^\mu = -(2\epsilon\cdot x + \delta^S_\epsilon x\cdot\partial)A^\mu - 2(x\cdot A\epsilon^\mu + \epsilon\cdot A x^\mu), \\ &\delta^S_\epsilon g^2 = -(2(3-p)\epsilon\cdot x + \delta^S_\epsilon x\cdot\partial)g^2, \end{aligned} \tag{4.1.2}$$

where for dilatation the parameter is the scalar ϵ and for SCTs the vector ϵ^μ.

It is clear from equation (3.0.1) the action can be written as $S = Tr\int d^{p+1}x \frac{1}{g^2}\mathcal{L}$. For further convenience we shall also extract the factors out of the three terms of $\mathcal{L}$ and define

$$\mathcal{L}_1 = F_{\mu\nu}F^{\mu\nu}, \qquad \mathcal{L}_2 = D_\mu X_m D^\mu X^m, \qquad \mathcal{L}_3 = [X_m, X_n]^2. \tag{4.1.3}$$

We can, therefore, now write the action variations that we are interested in as

$$\delta S_i = \mathrm{Tr}\int d^{p+1}x \left(\delta\left(\frac{1}{g^2}\right)\mathcal{L}_i + \frac{1}{g^2}\delta\mathcal{L}_i\right), \tag{4.1.4}$$

for $i = 1, 2, 3$. We shall now investigate the invariance of each of these terms separately, for both transformations, starting from $\mathcal{L}_3$.

4.1.1 S_3 invariance

We shall first evaluate equation (4.1.4) for the dilation of $\mathcal{L}_3$. We start by concidering

$$\delta\mathcal{L}_3 = 2\delta\left([X_m, X_n]\right)[X^m, X^n] = 2\left([\delta X_m, X_n] + [X_m, \delta X_n]\right)[X^m, X^n], \tag{4.1.5}$$

where we have used that δ acts as a derivation. Substituting, now, for the case of a dilatation, using (4.1.1) one aquires

$$\delta\mathcal{L}_3 = -2\epsilon\left(2[X_m, X_n]^2 + \frac{1}{2}x^k\partial_k\left([X_m, X_n]^2\right)\right). \tag{4.1.6}$$

Inserting this into (4.1.4) and using (4.1.1) to acquire $\delta^D_\epsilon\left(\frac{1}{g^2}\right) = -\epsilon((p-3) + x\cdot\partial)\frac{1}{g^2}$ we get

$$\begin{aligned}
\delta S_3 &= -\epsilon\mathrm{Tr}\int d^{p+1}x\Bigg[(p+1)\frac{1}{g^2}[X_m, X_n]^2 \\
&\quad + x^k\partial_k\left(\frac{1}{g^2}\right)[X_m, X_n]^2 + \frac{1}{g^2}x^k\partial_k\left([X_m, X_n]^2\right)\Bigg] \\
&= -\epsilon\mathrm{Tr}\int d^{p+1}x\left(\partial_k\left(\frac{1}{g^2}x^k[X_m, X_n]^2\right)\right) \\
&= 0.
\end{aligned} \tag{4.1.7}$$

Here, on going to the second line we have used the second and the third term, that respectively come from the $\frac{1}{g^2}$ variation and the $\mathcal{L}_3$ variation, and combined them in one derivative. The reminder term is canceled out using that $\partial_k x^k = p+1$. On going into the final line we have used, as usual, the divergence theorem and integrated over a surface at infinity, assuming the field fall to zero fast enough.

We continue, now, to show invariance of $\mathcal{L}_3$ under SCTs as well. Using equation (4.1.5) and the variation of X_m from (4.1.2) we get

$$\delta\mathcal{L}_3 = -2\left(4\epsilon\cdot x[X_m, X_n]^2 + \frac{1}{2}\delta^S_\epsilon x^k\partial_k\left([X_m, X_n]^2\right)\right). \tag{4.1.8}$$

In parallel to the dilatation discussion, we insert this into (4.1.4) and use (4.1.2) to acquire

$\delta^S_\epsilon\left(\frac{1}{g^2}\right) = -(2(p-3)\epsilon\cdot x + \delta^S_\epsilon x\cdot\partial)\frac{1}{g^2}$ to get

$$\begin{aligned}\delta S_3 &= -\mathrm{Tr}\int d^{p+1}x\Big[2(p+1)\epsilon\cdot x\frac{1}{g^2}[X_m,X_n]^2\\&+\delta^S_\epsilon x^k\partial_k\left(\frac{1}{g^2}\right)[X_m,X_n]^2+\frac{1}{g^2}\delta^S_\epsilon x^k\partial_k\left([X_m,X_n]^2\right)\Big]\\&=-\epsilon\mathrm{Tr}\int d^{p+1}x\left(\partial_k\left(\frac{1}{g^2}\delta^S_\epsilon x^k[X_m,X_n]^2\right)\right)\\&=0.\end{aligned} \tag{4.1.9}$$

Here, the cancelations folowed the dilatation case, with the only difference that the remnant term going to the second line was canceled using $\partial_k\delta^S_\epsilon x^k = 2\epsilon\cdot x(p+1)$.

4.1.2 S_1 invariance

In a similar way to the previous calculation, we shall now prove the invariance of S_1 as well. Using the definition $F^a_{\mu\nu} = 2\partial_{[\mu}A^a_{\nu]} + f^a_{bc}A^b_\mu A^c_\nu$ we have

$$\delta\mathcal{L}_1 = 2\left(\delta F^a_{\mu\nu}\right)F^{\mu\nu}_a = 2F^{\mu\nu}_a\left(2\partial_{[\mu}\delta A^a_{\nu]} + f^a_{bc}\left(\delta A^b_\mu A^c_\nu + A^b_\mu\delta A^c_\nu\right)\right). \tag{4.1.10}$$

Specifying, now, for the dilatation case and using (4.1.1) for the vector field variation, one gets

$$\begin{aligned}\delta\mathcal{L}_1 &= -\epsilon\left[2\left(2\partial_{[\mu}A^a_{\nu]} + x^k\partial_k\partial_{[\mu}A^a_{\nu]}\right) + f^a_{bc}\left(2A^b_\mu A^c_\nu + x^k\partial_k\left(A^b_\mu A^c_\nu\right)\right)\right]\\&= -\epsilon\left(2F^2 + x^k\partial_k\left(F^2\right)\right),\end{aligned} \tag{4.1.11}$$

where in going to the second line we have used the definition of $F^a_{\mu\nu}$ and the abbreviation $F^2 = F^a_{\mu\nu}F^{\mu\nu}_a$. Inserting this into (4.1.4) as before we have

$$\delta S_1 = -\epsilon\mathrm{Tr}\int d^{p+1}x\left((p+1)\frac{1}{g^2}F^2 + x^k\partial_k\left(\frac{1}{g^2}\right)F^2 + \frac{1}{g^2}x^k\partial_k\left(F^2\right)\right) = 0, \tag{4.1.12}$$

having used identical steps to those in (4.1.7). Accordingly, we now move to prove SCT invariance. We use equation (6.2.6) and follow a similar calculation to the dilatation case. The additional terms in $\delta^S_\epsilon A^a_\mu$ and $\delta^S_\epsilon x^k$ in equation (4.1.2) compared to the dilatation case turn out to cancel due to indices symmetry arguments. Hence, we reach the result

$$\delta\mathcal{L}_1 = -2\left(4\epsilon\cdot xF^2 + \frac{1}{2}\delta^S_\epsilon x^k\partial_k\left(F^2\right)\right). \tag{4.1.13}$$

Finally, following the steps of equation (4.1.9) we reach the result

$$\delta S_1 = -\text{Tr}\int d^{p+1}x\left(2(p+1)\epsilon\cdot x\frac{1}{g^2}F^2 + \delta^S_\epsilon x^k\partial_k\left(\frac{1}{g^2}\right)F^2 + \frac{1}{g^2}\delta^S_\epsilon x^k\partial_k\left(F^2\right)\right) = 0. \tag{4.1.14}$$

4.1.3 S_2 invariance

We now continue with the second term in the action. As in the previous calculations, we start with the Lagrangian variation which reads

$$\delta\mathcal{L}_2 = 2\delta\left(D_\mu X_m\right)D^\mu X^m = 2\left(\delta A_\mu X_m D^\mu X^m + \left(D_\mu \delta X_m\right)D^\mu X^m\right). \tag{4.1.15}$$

For a dilatations, using the variations as in the previous case one gets

$$\delta\mathcal{L}_2 = -2\epsilon\left(2D_\mu X_m D^\mu X^m + \frac{1}{2}x\cdot\partial\left(D_\mu X_m D^\mu X^m\right)\right). \tag{4.1.16}$$

Hence, in an identical way as for the other terms we reach $\delta S_2 = 0$. The SCT case becomes a bit more complicated than the rest. Starting again from (4.1.15) and using the SCT variations we reach the result

$$\text{Tr}\left(\delta\mathcal{L}_2\right) = -2\left(4\epsilon\cdot x\text{Tr}\left(D_\mu X_m D^\mu X^m\right) + \frac{1}{2}\delta^S_\epsilon x\cdot\partial\text{Tr}\left(D_\mu X_m D^\mu X^m\right) + \epsilon\cdot\partial\text{Tr}\left(X_m^2\right)\right). \tag{4.1.17}$$

Here, a lot of extra terms have canceled due to symmetry arguments as in the previous SCT case. We have also reintroduced the trace in order to be able to make some manipulations on the non-commuting fields, specifically to use the formula $\partial\left(\text{Tr}\left(MM'\right)\right) = \text{Tr}\left((D_\mu M)M'\right) + \text{Tr}\left(M(D_\mu M')\right)$ for two matrices M and M'. [12]. It is clear that the third term is not one that has appeared in the other calculations. Inserting this to get the action variation and following the steps in the previous calculations we reach

$$\delta S_2 = -\int d^{p+1}x\frac{2}{g^2}\epsilon\cdot\partial\text{Tr}(X_m^2). \tag{4.1.18}$$

It is, therefore, clear that the action in this form is not SCT invariant. Before the transformation, though, where g is not a function of time, one can integrate by parts and get another form of the action, namely

$$S_2' = \text{Tr}\int d^{p+1}x\frac{1}{g^2}X_m D_\mu D^\mu X^m. \tag{4.1.19}$$

Following the same procedure as above one finds that the variation of this action reads

$$\delta S_2' = \int d^{p+1}x\frac{1}{g^2}(p-3)\epsilon\cdot\partial\text{Tr}(X_m^2). \tag{4.1.20}$$

It is possible then to build an invariant action out of (3.0.1), partly integrating the second term by parts. This action is the following

$$S_{invariant} = \mathrm{Tr} \int d^{p+1}x \Big\{ -\frac{1}{4g^2} F_{\mu\nu}F^{\mu\nu} - \frac{1}{2g^2}\frac{1}{p-5}((p-3)D_\mu X_m D^\mu X^m \\ + 2X_m D_\mu D^\mu X^m) + \frac{1}{4g^2}[X_m, X_n]^2 \Big\}. \tag{4.1.21}$$

We did, therefore, prove that the low-energy dynamics of D-branes obey the generalised conformal symmetry in all dimensions. The action above is not well defined for $p = 5$, something that appears in other parts of the brane investigation as well [7].

4.2 Correlation functions

We have, finally, reached the point where we can apply all we have seen in our system of interest. We have talked about symmetries and how they put constraints on systems that obey them. We later saw that D3-branes obey the conformal symmetry, and that there exists a more general structure - the generalised conformal symmetry - which D-branes obey in any dimension.

In Chapter 2 we discussed how conformal symmetry can be used to constrain the dynamics of a system. We shall now follow a similar route in order to acquire the equivalent constraints that generalised conformal symmetry applies on D-branes. In the discussion of conformal symmetry we used the general Ward identity (2.4.3) and then used the transformation properties of scalar field in order to obtain the constraints. This equation, though, was based on the analysis of equation (2.4.2) which used the action invariance. We have mentioned, thought, that the generalised conformal symmetry is not a symmetry in the strict sense. What is meant by that is that the action is not invariant under transformation of the dynamical fields only - i.e. the ones appearing in the path integral measure -, but also needs the coupling constant to transform as well.

Having this in mind, let us repeat the procedure in (2.4.2), but this time for the generalised conformal symmetry. As it is explained under (2.4.2), going from the second to the third line one is using the action invariance. This means that one considers $S[\Phi', g] \to S[\Phi, g] + \delta S$. Under the full generalised conformal transformation we have that $\delta S = 0$. In the path integral, though, only the dynamical fields are transformed, i.e. not the coupling constant g. Hence, defining δ_Φ as the variation of the dynamical fields and δ_g the variation of the coupling constant we have

$$\delta_\Phi S + \delta_g S = 0 \tag{4.2.1}$$

We, therefore, have

$$\begin{aligned}\langle O_1[\Phi]\dots O_n[\Phi]\rangle &= \frac{1}{Z}\int [d\Phi]\, O_1[\Phi]\dots O_n[\Phi]\exp\left(-S[\Phi,g]\right)\\ &= \frac{1}{Z}\int \left[d\Phi'\right] O_1'[\Phi']\dots O_n'[\Phi']\exp\left(-S[\Phi',g]\right)\\ &= \frac{1}{Z}\int [d\Phi]\, O_1'[\Phi']\dots O_n'[\Phi']\exp\left(-S[\Phi,g]+\delta_g S\right)\\ &= \langle O_1'[\Phi']\dots O_n'[\Phi']\rangle + \langle O_1'[\Phi']\dots O_n'[\Phi']\delta_g S\rangle.\end{aligned} \tag{4.2.2}$$

Here, as we said, going from the second to the third line we have used equation (4.2.1) and in going to the last line we have used the fact that the transformation is infinitesimal in order to write the usual exponential series in this limit, $\exp(1+\delta S) = 1+\delta S$. Renaming the variables as $O[\Phi] \leftrightarrow O'[\Phi']$ we have that

$$\langle \delta_\Phi\left(O_1[\Phi]\dots O_n[\Phi]\right)\rangle = \langle O_1[\Phi]\dots O_n[\Phi]\delta_g S\rangle. \tag{4.2.3}$$

We, now, consider the following variation of the correlator

$$\begin{aligned}\delta_g\langle O_1[\Phi]\dots O_n[\Phi]\rangle &= \delta_g\frac{1}{Z}\int [d\Phi]\, O_1[\Phi]\dots O_n[\Phi]\exp\left(-S[\Phi,g]\right)\\ &= \frac{1}{Z}\int [d\Phi]\, O_1[\Phi]\dots O_n[\Phi]\left(\delta_g S\right)\exp\left(-S[\Phi,g]\right)\\ &= -\langle O_1[\Phi]\dots O_n[\Phi]\delta_g S\rangle,\end{aligned} \tag{4.2.4}$$

where in going to the second line we have used that we vary only the coupling constant and not the fields. Therefore, combining equations (4.2.3) and (4.2.4), we arrive to our final result

$$\langle \delta_\Phi\left(O_1[\Phi]\dots O_n[\Phi]\right)\rangle + \delta_g\langle O_1[\Phi]\dots O_n[\Phi]\rangle = 0. \tag{4.2.5}$$

This is the equivalent to the Ward identity (2.4.3) for the generalised conformal structure. One is, therefore, now able to produce equations similar to (2.5.6) and use them to constrain correlation functions. For this example of a 2-point function of a theory obeying the generalised conformal symmetry equation (2.5.2) still holds due to the same argument. Hence, one gets the following differential equation due to the generalised dilatation

$$\left[(\Delta_{\Phi_1} + x_1\cdot\partial_1) + (\Delta_{\Phi_2} + x_2\cdot\partial_2) + \int d^{p+1}x\frac{1}{-\epsilon}\delta g\frac{\partial}{\partial g}\right]\langle\Phi_1(x_1)\Phi_2(x_2)\rangle = 0. \tag{4.2.6}$$

Therefore, we now have similar constraints to those for a CFT. The next step is to try and solve this equation, either directly or perturbatively, both for generalised dilatations and SCTs in order to realise the constraints applied on a system that respects this structure.

CHAPTER 5

Generalised dilatation of 2 & 3-point functions

We shall, now, apply these constrains to 2 & 3-point functions of a theory obeying the generalised conformal symmetry. In this chapter we will analyse the constraints due to the generalised dilatations. and in the next the generalised SCTs. Along with these two new symmetries we also have Poincare symmetry as for CFTs. Hence we can borrow the arguments used in Chapter 2 to state again that the correlation functions must be functions of the x_{ij}^2 variables.

First have a look at equation (4.2.5). What this equation states is that the transformation of the coordinates, as in the CFT Ward constraint (2.4.3), together with the transformation of the coupling constant should leave the correlation function invariant. For dilatations the finite transformation is just multiplication by the parameter to the power of the weight, i.e. $x_{ij} \to \lambda x_{ij}$ and $g^2 \to \lambda^{(3-p)} g^2$. Hence in dimensional grounds we expect the answer to be a function of dimensionless combinations that we shall call the effective coupling constant and for the 2-point function will have the form $g_{eff}^2 = \dfrac{g^2}{x_{12}^{(3-p)}}$. This can, then, of course be multiplied by a function invariant under the usual conformal transformation alone.

Equations (4.1.1) and (4.1.2) give the infinitesimal change of g^2 for generalised dilatations and SCTs. We remember, though, that the coupling constant is considered as a background field only during the transformation and then a constant. As the Ward identity is true in general we can take it to be a constant and hence ignore derivatives of g^2. For dilatations this means that we have

$$\delta g^2 = -\epsilon(3-p)g^2, \tag{5.0.1}$$

therefore independent of x. In this case, we can simplify the third term in equation (4.2.6), by considering

$$\begin{aligned}\delta g^2 \frac{\delta}{\delta g^2}\langle O_1[\Phi]\dots O_n[\Phi]\rangle &= \delta g^2 \frac{\delta}{\delta g^2}\frac{1}{Z}\int [d\Phi]\, O_1[\Phi]\dots O_n[\Phi]\exp\left(-S[\Phi,g]\right)\\ &= -\frac{1}{Z}\int [d\Phi]\, O_1[\Phi]\dots O_n[\Phi]\exp\left(-S[\Phi,g]\right)\delta g^2\frac{\delta}{S}\\ &= -\langle O_1[\Phi]\dots O_n[\Phi]\delta_g S\rangle\\ &= \delta_g\langle O_1[\Phi]\dots O_n[\Phi]\rangle,\end{aligned} \tag{5.0.2}$$

where in going to the last line we have used equation (4.2.4). Using this, then, in the Ward identity we would write $\frac{\partial}{\partial g^2}$ instead of $\frac{\delta}{\delta g^2}$ as g^2 in the correlation is just a constant. Hence, for the 2-point function $\langle \Phi_1(x_1)\Phi_2(x_2)\rangle$ the Ward identity now reads

$$\left[(\Delta_{\Phi_1} + x_1\cdot\partial_1) + (\Delta_{\Phi_2} + x_2\cdot\partial_2) + (3-p)g^2\frac{\partial}{\partial g^2}\right]\langle \Phi_1(x_1)\Phi_2(x_2)\rangle = 0. \tag{5.0.3}$$

5.1 2-point functions

We now focus on the 2-point function $\langle \Phi_1(x_1)\Phi_2(x_2)\rangle$ that shall obey equation (5.0.3). First we use our dimensional argument and prove that it holds true non-perturbatively. Let a function $R\left(g^2_{eff}\right)$, where $g^2_{eff} = \frac{g^2}{x_{12}^{(3-p)}}$, multiplied by the usual dilatation invariant 2-point function (2.5.6)

$$\langle \Phi_1(x_1)\Phi_2(x_2)\rangle = \frac{C_{12}}{x_{12}^{\Delta_{\Phi_1}+\Delta_{\Phi_2}}} R\left(g^2_{eff}\right). \tag{5.1.1}$$

Plugging this into equation (5.0.3) the first two terms acting on the conformal part cancel as in the conformal case and leave an equation for the function R

$$\left[x_1\cdot\partial_1 + x_2\cdot\partial_2 + (3-p)g^2\frac{\partial}{\partial g^2}\right] R\left(g^2_{eff}\right) = 0. \tag{5.1.2}$$

Using, now, $x_1\cdot\partial_1 R\left(g^2_{eff}\right) = g^2(p-3)x_{12}^{p-5}(x_1^2 - x_1x_2)\frac{\partial}{\partial g^2_{eff}}R$, similarly for x_2 and that $\frac{\partial}{\partial g^2}R\left(g^2_{eff}\right) = x_{12}^{p-3}\frac{\partial}{\partial g^2_{eff}}R$ one can easily verify that the equation indeed holds.

The next step is to repeat the calculation perturbatively, without using the initial dimensional argument. This is done in order to see if the function R was over-constrained by the our assumption and dilatations alone can be satisfied by a more general solution. Also this will help us prepare for more complicated cases where we will only be able to find a solution perturbatively. So, we start by doing a convenient expansion in $g^2$of the 2-point function regarded, now, a function of g^2 and x_{12} separately

$$\langle \Phi_1(x_1)\Phi_2(x_2)\rangle = \frac{C_{12}}{x_{12}^{\Delta_{\Phi_1}+\Delta_{\Phi_2}}} \sum_i R_i\left(x_{12}^2\right)(g^2)^i. \tag{5.1.3}$$

Inserting this into equation (5.0.3) the first two terms when acting on the conformal part cancel again, and we are left with an equation for each order in g^2

$$\left[x_1\cdot\partial_1 + x_2\cdot\partial_2 + (3-p)g^2\frac{\partial}{\partial g^2}\right] R_i\left(x_{12}^2\right)(g^2)^i = 0. \tag{5.1.4}$$

Using $x_1\cdot\partial_1 R_i\left(x_{12}^2\right) = 2(x_1^2 - x_1x_2)$, similarly for x_2 and acting with the g^2 derivative as well we reach

$$\left[2x_{12}^2\frac{\partial}{\partial x_{12}^2} + (3-p)i\right] f_i = 0. \tag{5.1.5}$$

This is a simple differential equation with solution $f_i = c_i x_{12}^{(p-3)i}$, with c_i unconstrained. We, therefore, have

$$\begin{aligned} R(g^2, x_{12}) &= \sum_i c_i x_{12}^{(3-p)i}(g^2)^i \\ &= \sum_i c_i (g_{eff}^2)^i \\ &= R\left(g_{eff}^2\right), \end{aligned} \tag{5.1.6}$$

confirming the non-perturbative result.

5.2 3-point functions

Now we turn to 3-point functions. We shall follow the same route as in the 2-point case, by using our dimensional argument to calculate non-perturbatively first and confirming the result in a perturbative way.

As in this case we have three different coordinates one can construct various effective couplings in the following manner

$$g_{eff_i}^2 = g^2 x_{12}^{a_i} x_{23}^{b_i} x_{13}^{c_i}, \qquad \text{where} \quad a_i + b_i + c_i = p-3. \tag{5.2.1}$$

The 3-point function can be a function of any number of them, as they are all invariant under the generalised dilatation. We proceed to prove that as earlier by assuming the following form with the convenient conformally invariant normalisation

$$\langle \Phi_1(x_1)\Phi_2(x_2)\Phi_3(x_3)\rangle = \frac{C_{123}}{x_{12}^a x_{23}^b x_{13}^c} R\left(g^2_{eff_1}, g^2_{eff_2}, \dots\right), \tag{5.2.2}$$

where, as in Chapter 2, $a + b + c = \Delta_{\Phi_1} + \Delta_{\Phi_2} + \Delta_{\Phi_3}$. The Ward identity derived in the same way as equation (5.0.3), but with an extra term arising from the first term of equation (4.2.5) when δ_Φ is acting on Φ_3, is the following

$$\Big[(\Delta_{\Phi_1} + x_1 \cdot \partial_1) + (\Delta_{\Phi_2} + x_2 \cdot \partial_2) + (\Delta_{\Phi_2} + x_3 \cdot \partial_3) + (3-p)g^2 \frac{\partial}{\partial g^2}\Big] \langle \Phi_1(x_1)\Phi_2(x_2)\Phi_3(x_3)\rangle = 0. \tag{5.2.3}$$

Inserting now (5.2.2) the conformal part cancels out as expected and we are left with

$$\left[x_1 \cdot \partial_1 + x_2 \cdot \partial_2 + x_3 \cdot \partial_3 + (3-p)g^2 \frac{\partial}{\partial g^2}\right] R\left(g^2_{eff_1}, g^2_{eff_2}, \dots\right) = 0. \tag{5.2.4}$$

Using, now that

$$x_1 \cdot \partial_1 R\left(g^2_{eff_1}, g^2_{eff_2}, \dots\right) = \sum_i x_1 \cdot \partial_1 g^2_{eff_i} \frac{\partial}{\partial g^2_{eff_i}} R, \tag{5.2.5}$$

and

$$x_1 \cdot \partial_1 g^2_{eff_i} = g^2_{eff_i}\left(a_i\left(x_1^2 - x_1 \cdot x_2\right) x_{12}^{-2} + c_i\left(x_1^2 - x_1 \cdot x_3\right) x_{13}^{-2}\right), \tag{5.2.6}$$

and similarly for x_2 and x_3, we have

$$[x_1 \cdot \partial_1 + x_2 \cdot \partial_2 + x_3 \cdot \partial_3] R\left(g^2_{eff_1}, g^2_{eff_2}, \dots\right) = (p-3)\sum_i g^2_{eff_i} \frac{\partial}{\partial g^2_{eff_i}} R, \tag{5.2.7}$$

where we used the fact that $a_i + b_i + c_i = p - 3$. Working similarly for the g^2 term we have that

$$\frac{\partial}{\partial g^2} R\left(g^2_{eff_1}, g^2_{eff_2}, \dots\right) = \sum_i \frac{\partial}{\partial g^2} g^2_{eff_i} \frac{\partial}{\partial g^2_{eff_i}} R = g^{-2} \sum_i g^2_{eff_i} \frac{\partial}{\partial g^2_{eff_i}} R. \tag{5.2.8}$$

Inserting, now, these two equations into (5.2.4) we see that $R\left(g^2_{eff_1}, g^2_{eff_2}, \cdots\right)$ indeed satisfies the Ward identity.

We are now ready to proceed as in the 2-point function and repeat the calculation perturbatively. We do this by not using the dimensional argument and instead expanding the function R in g^2 in the following way

$$R = \sum_i R_i\left(x_{12}^2, x_{23}^2, x_{13}^2\right)\left(g^2\right)^i, \tag{5.2.9}$$

which is to satisfy equation (5.2.4). Using once again $x_1 \cdot \partial_1 R_i \left(x_{12}^2\right) = 2(x_1^2 - x_1 x_2)$, similarly for x_2 and x_3 acting with the g^2 derivative as well we reach

$$\left[x_{12}^2 \frac{\partial}{\partial x_{12}^2} + x_{23}^2 \frac{\partial}{\partial x_{23}^2} + x_{13}^2 \frac{\partial}{\partial x_{13}^2}\right] R_i = i\frac{p-3}{2} R_i. \tag{5.2.10}$$

It is clear that the solution must be a sum of term of order $i(p-3)$, i.e.

$$R_i = c_i \sum_j n_j x_{12}^{a_j'} x_{23}^{b_j'} x_{13}^{c_j'}, \qquad \text{where} \quad a_j' + b_j' + c_j' = i\left(p-3\right), \tag{5.2.11}$$

where we see that there is an implicit i dependence on each term through the constraint. Inserting this into (5.2.9) we reach

$$\begin{aligned} R_i &= \sum_i c_i \sum_j n_j \left(g^2 x_{12}^{a_j} x_{23}^{b_j} x_{13}^{c_j}\right)^i \\ &= \sum_j n_j \sum_i c_i \left(g_{eff_j}^2\right)^i, \qquad \text{where} \quad a_j + b_j + c_j = \left(p-3\right). \end{aligned} \tag{5.2.12}$$

CHAPTER 6

Generalised Conformal Structure of the Free Massive Scalar

Next, we proceed with solving the SCT Ward identity. As this is a more complicated equation, we begin by examining a how a specific example satisfies the equation. It is well-known that the free massless scalar is a conformal theory. The introduction of the mass, and hence of a scale, brakes this symmetry. It is easy to show, though, that it exhibits generalised conformal symmetry. Starting with the Klein-Gordon action in Euclidean space

$$\mathcal{S} = \int dx^{D+1} \frac{1}{2} \left[(\partial \Phi)^2 + \frac{1}{2} m^2 \Phi^2 \right], \tag{6.0.1}$$

it is easy to see from the first term that, in order for finite generalised dilatations to be a symmetry, we must have $\Delta_\Phi = \frac{D-1}{2}$ and from the second term that $\Delta_m = 1$. Hence, in a similar way to (4.1.1) and (4.1.2) we have that

$$\begin{aligned} &\delta_\epsilon^D x^\mu = \epsilon x^\mu, && \delta_\epsilon^D \Phi = -\epsilon \left(\frac{D-1}{2} + x \cdot \partial \right) \Phi, \\ &\delta_\epsilon^D m = -\epsilon (1 + x \cdot \partial) m \end{aligned} \tag{6.0.2}$$

for the dilatations and

$$\begin{aligned} &\delta_\epsilon^S x^\mu = 2\epsilon \cdot x x^\mu - \epsilon^\mu x^2, \quad \delta_\epsilon^S \Phi = -((D-1)\, \epsilon \cdot x + \delta_\epsilon^S x \cdot \partial) \Phi, \\ &\delta_\epsilon^S m = -(2\epsilon \cdot x + \delta_\epsilon^S x \cdot \partial) m, \end{aligned} \tag{6.0.3}$$

for the SCTs. Working in a similar way to above, we have that for the dilatation

$$\begin{aligned}
\delta S &= \frac{1}{2}\int d^{D+1}x \left[2\partial\Phi\cdot\partial\left(\delta\Phi\right) + 2m\delta m\Phi^2 + 2m^2\Phi\delta\Phi\right] \\
&= -\epsilon\int d^{D+1}x \Bigg[\partial\Phi\cdot\partial\left(\frac{D-1}{2}+x\cdot\partial\right)\Phi \\
&\qquad + m\Phi\left(\Phi\left(1+x\cdot\partial\right)m + m\left(\frac{D-1}{2}+x\cdot\partial\right)\Phi\right)\Bigg] \\
&= -\epsilon\int d^{D+1}x\left[\left(\frac{D+1}{2}+\frac{1}{2}x\cdot\partial\right)\left(\partial\Phi\right)^2 + \left(\left(D+1\right)+x\cdot\partial\right)\left(m^2\Phi^2\right)\right] \\
&= 0,
\end{aligned} \tag{6.0.4}$$

where integration by parts was used in the last equation. And in the same way for SCT

$$\begin{aligned}
\delta S &= \frac{1}{2}\int d^{D+1}x \left[2\partial\Phi\cdot\partial\left(\delta\Phi\right) + 2m\delta m\Phi^2 + 2m^2\Phi\delta\Phi\right] \\
&= -\int d^{D+1}x \Bigg[\partial\Phi\cdot\partial\left(\left(D-1\right)\epsilon\cdot x + \delta x\cdot\partial\right)\Phi \\
&\qquad + m\Phi\left(\Phi\left(2\epsilon\cdot x+\delta x\cdot\partial\right)m + m\left(\left(D-1\right)\epsilon\cdot x+\delta x\cdot\partial\right)\Phi\right)\Bigg] \\
&= -\int d^{D+1}x\left[\left(\left(D+1\right)\epsilon\cdot x+\frac{1}{2}\delta x\cdot\partial\right)\left(\partial\Phi\right)^2 + \left(\left(D+1\right)\epsilon\cdot x+\frac{1}{2}\delta x\cdot\partial\right)\left(m^2\Phi^2\right)\right] \\
&= 0,
\end{aligned} \tag{6.0.5}$$

where again integration by parts was used for the final results and the fact that $\partial\cdot\delta x = 2\left(D+1\right)\epsilon\cdot x$.

6.1 Ward Identities for $\langle\Phi(x)\Phi(0)\rangle$

We shall use this simple example as a guide to solve the general equations. First of all we shall prove that the 2-point function of the free massive scalar satisfies the initial Ward identity (4.2.3), namely

$$L_{CFT}\langle\Phi(x)\Phi(0)\rangle = \langle\Phi(x)\Phi(0)\delta_m S\rangle, \tag{6.1.1}$$

where L_{CFT} is the operator of the CFT Ward identity, i.e. $L_{CFT} = \left(D-1\right)\epsilon\cdot x + \delta x\cdot\partial$. We do this calculation first in momentum space, where the propagator has the same form in all dimensions. The CFT operator in momentum space can be computed in the following way

$$\begin{aligned}
L_{CFT}\langle\Phi\Phi\rangle\,(x) &= (2\Delta\epsilon\cdot x+\delta x\cdot\partial)\,\frac{1}{(2\pi)^{D+1}}\int dp\langle\Phi\Phi\rangle\,(p)\,e^{ip\cdot x}\\
&= \frac{1}{(2\pi)^{D+1}}\int dp\langle\Phi\Phi\rangle\left(2\Delta\epsilon\cdot x+ip_\mu\left(2\epsilon\cdot xx^\mu-\epsilon^\mu x^2\right)\right)e^{ip\cdot x}\\
&= \frac{-i}{(2\pi)^{D+1}}\int dp\langle\Phi\Phi\rangle\left[2\Delta\epsilon\cdot\frac{\partial}{\partial p}+p_\mu\left(2\epsilon\cdot\frac{\partial}{\partial p}\frac{\partial}{\partial p_\mu}-\epsilon^\mu\frac{\partial}{\partial p}\cdot\frac{\partial}{\partial p}\right)\right]e^{ip\cdot x}\\
&= \frac{i}{(2\pi)^{D+1}}\int dp\big[2\Delta\epsilon\cdot\partial_p\langle\Phi\Phi\rangle+\left(p\cdot\partial_p\langle\Phi\Phi\rangle+(D+1/2)\langle\Phi\Phi\rangle\right)2\epsilon\cdot\partial_p\\
&\qquad\qquad-\epsilon\cdot p\left(\partial_{p_\nu}\langle\Phi\Phi\rangle\right)\partial_{p^\nu}\big]e^{ip\cdot x}\\
&= \frac{1}{(2\pi)^{D+1}}\int dpe^{ip\cdot x}i\left[2\left(\Delta-(D+1)\right)\epsilon\cdot\partial_p-2p\cdot\partial_p\epsilon\cdot\partial_p+\epsilon\cdot p\partial_p\cdot\partial_p\right]\langle\Phi\Phi\rangle.
\end{aligned}\tag{6.1.2}$$

Hence, for the free scalar theory where $\Delta_\Phi=\frac{D-1}{2}$ we have that in momentum space

$$L_{CFT}\langle\Phi\Phi\rangle=-i\left[(D+3)\,\epsilon\cdot\partial_p+2p\cdot\partial_p\epsilon\cdot\partial_p-\epsilon\cdot p\partial_p\cdot\partial_p\right]\langle\Phi\Phi\rangle \tag{6.1.3}$$

Using, now, the well known result $\langle\Phi\Phi\rangle=\frac{1}{p^2+m^2}$ and that $\frac{\partial}{\partial p_\mu}=\frac{p^\mu}{p}\frac{\partial}{\partial p}$, $\frac{\partial}{\partial p^\nu}\frac{\partial}{\partial p_\mu}=\left[\frac{\delta^\mu_\nu}{p}-\frac{p^\mu p_\nu}{p^3}\right]\frac{\partial}{\partial p}+\frac{p^\mu p_\nu}{p^2}\frac{\partial^2}{\partial p^2}$, where $p=\sqrt{p_\mu p^\mu}$, we have that

$$\begin{aligned}
L_{CFT}\langle\Phi\Phi\rangle\,(p) &= -i\epsilon\cdot p\left[3\frac{1}{p}\frac{d}{dp}+\frac{d^2}{dp^2}\right]\frac{1}{p^2+m^2}\\
&= \frac{i8m^2\epsilon\cdot p}{(p^2+m^2)^3}.
\end{aligned}\tag{6.1.4}$$

Now, moving on to calculate the right-hand-side of (6.1.1), we have

$$\begin{aligned}
\langle\Phi(x)\Phi(0)\delta_m S\rangle &= \langle\Phi\Phi\int dy^{D+1}m\delta m\Phi^2(y)\rangle\\
&= -\langle\Phi\Phi\int dy^{D+1}m\left(2\epsilon\cdot y+\delta y\cdot\partial\right)m\Phi^2(y)\rangle\\
&= -\langle\Phi\Phi\int dy^{D+1}\left(2\epsilon\cdot y+\frac{1}{2}\delta y\cdot\partial\right)m^2\Phi^2(y)\rangle\\
&= -m^2\langle\Phi\Phi\int dy^{D+1}\left(2\epsilon\cdot y-\frac{1}{2}\partial\cdot\delta y-\frac{1}{2}\delta y\cdot\partial\right)\Phi^2(y)\rangle\\
&= -m^2\langle\Phi\Phi\int dy^{D+1}\left(2\epsilon\cdot y\right)\Phi^2(y)\rangle.
\end{aligned}\tag{6.1.5}$$

Up to this point, we integrated by parts going to the last two lines. In twice so we pulled out m^2 out of the integral, as know that physically m is not a function of y and hence we should be able to take constant at some point of the calculation. It becomes clear, though, by the final result that we can also take m to be constant earlier on the second line. Moving on we have,

$$\begin{aligned}
\langle\Phi(x)\Phi(0)\delta_m S\rangle &= -2m^2\int dy^{D+1}\epsilon\cdot y\langle\Phi(x)\Phi(0)\Phi(y)\Phi(y)\rangle \\
&= -4m^2\int dy^{D+1}\epsilon\cdot yD(x-y)D(y) \\
&= -4m^2\frac{1}{(2\pi)^{2(D+1)}}\int dpdqdy^{D+1}\epsilon\cdot y\frac{e^{ip\cdot(x-y)}}{p^2+m^2}\frac{e^{iq\cdot y}}{q^2+m^2} \\
&= -\frac{4m^2}{(2\pi)^{2(D+1)}}\int dpdqdy^{D+1}\frac{e^{ip\cdot(x-y)}}{(p^2+m^2)(q^2+m^2)}\epsilon\cdot\left(-i\frac{\partial}{\partial q}\right)e^{iq\cdot y} \\
&= -\frac{4im^2}{(2\pi)^{2(D+1)}}\int dpdq\frac{e^{ipx}}{p^2+m^2}\epsilon\cdot\left(\frac{\partial}{\partial q}\frac{1}{q^2+m^2}\right)\int dy^{D+1}e^{iy\cdot(q-p)} \\
&= -\frac{4im^2}{(2\pi)^{D+1}}\int dp\frac{e^{ipx}}{p^2+m^2}\epsilon\cdot\left(\frac{\partial}{\partial p}\frac{1}{p^2+m^2}\right) \\
&= \frac{1}{(2\pi)^{D+1}}\int dpe^{ip\cdot x}\frac{8im^2\epsilon\cdot p}{(p^2+m^2)^3},
\end{aligned} \tag{6.1.6}$$

where in going to the line before the last we have used that $\int dy^{D+1}e^{iy\cdot(q-p)} = (2\pi)^{D+1}\delta(q-p)$. Hence, we see that the Ward identity (6.1.1) is indeed confirmed.

Now, we shall repeat the calculation in position space as well. We first calculate the left-hand side of (6.1.1) using (2.5.7). We shall do so for $D=2$ where the propagator is of the form

$$\langle\Phi(x)\Phi(0)\rangle = D(x) = -\frac{i}{4\pi x}e^{-imx}. \tag{6.1.7}$$

Hence, given that in $D=2$ $\Delta_\Phi = \frac{1}{2}$ we have

$$\begin{aligned}
\langle\Phi(x)\Phi(0)\rangle &= [\epsilon\cdot x+\delta x\cdot\partial]\,D(x) \\
&= \epsilon\cdot x\left[1+x\frac{d}{dx}\right]D(x) \\
&= \epsilon\cdot x\left(-\frac{i}{4\pi}\right)\left[1+x\left(-\frac{1}{x}(1+imx)\right)\right]\frac{1}{x}e^{-imx} \\
&= -im\epsilon\cdot xxD(x),
\end{aligned} \tag{6.1.8}$$

where in going from the first to the second line we have used again that $\frac{\partial}{\partial p_\mu} = \frac{p^\mu}{p}\frac{\partial}{\partial p}$ and that $\delta x\cdot x = \epsilon\cdot xx^2$. Moving on to compute the right-hand side we use the momentum space calculation. Starting from the line before the last of (6.1.6) we have that

$$\begin{aligned}\langle\Phi(x)\Phi(0)\delta_m S\rangle &= -\frac{2im^2}{(2\pi)^{D+1}}\int dpe^{ipx}\epsilon\cdot\frac{\partial}{\partial p}\left(\frac{1}{p^2+m^2}\right)^2 \\ &= -2m^2\epsilon\cdot x\frac{1}{(2\pi)^{D+1}}\int dpe^{ipx}\frac{1}{(p^2+m^2)^2} \\ &= -2m^2\epsilon\cdot x\frac{1}{(2\pi)^{D+1}}\int dpe^{ipx}\left(-\frac{\partial}{\partial m^2}\frac{1}{p^2+m^2}\right) \\ &= 2m^2\epsilon\cdot x\frac{\partial}{\partial m^2}\left[-\frac{i}{4\pi x}e^{-imx}\right] \\ &= -im\epsilon\cdot xxD(x),\end{aligned} \tag{6.1.9}$$

where in moving to the fourth line we again used the $D=2$ result for the position space propagator. Hence, the Ward identity is confirmed in position space as well.

As a last exercise on this two-point function, we return to momentum space and assume we do not know the actual form of the propagator. As in position space, the generalised dilatations constrain the propagator to the conformal propagator (i.e. for m=0) times a function of the weightless factor $y=p^2/m^2$. Hence we have that

$$\langle\Phi(p)\Phi(-p)\rangle = \frac{1}{p^2}f\left(\frac{p^2}{m^2}\right). \tag{6.1.10}$$

We shall, now, see what constraints does the Ward Identity (6.1.1) impose on f. Computing the left-hand side first, we have from equation (6.1.4) that

$$\begin{aligned}L_{CFT}\langle\Phi\Phi\rangle(p) &= -i\epsilon\cdot p\left[3\frac{1}{p}\frac{d}{dp}+\frac{d^2}{dp^2}\right]\left[\frac{1}{p^2}f\left(\frac{p^2}{m^2}\right)\right] \\ &= -i\epsilon\cdot p\frac{4}{m^4}\frac{d^2 f}{dy^2}.\end{aligned} \tag{6.1.11}$$

For the right-hand side we start with the first two lines of equation (6.1.6) and we have that

$$\begin{aligned}
\langle\Phi(x)\Phi(0)\delta_m S\rangle &= -4m^2 \int dy^{D+1}\epsilon\cdot yD(x-y)D(y) \\
&= -4m^2\frac{1}{(2\pi)^{2(D+1)}}\int dpdqdy^{D+1}\epsilon\cdot ye^{ip\cdot(x-y)}\left[\frac{1}{p^2}f\left(\frac{p^2}{m^2}\right)\right]e^{iq\cdot y}\left[\frac{1}{q^2}f\left(\frac{q^2}{m^2}\right)\right] \\
&= -\frac{4m^2}{(2\pi)^{2(D+1)}}\int dpdqdy^{D+1}e^{ip\cdot(x-y)}\left[\frac{1}{p^2}f\left(\frac{p^2}{m^2}\right)\right]\left[\frac{1}{q^2}f\left(\frac{q^2}{m^2}\right)\right]\epsilon\cdot\left(-i\frac{\partial}{\partial q}\right)e^{iq\cdot y} \\
&= -\frac{4im^2}{(2\pi)^{2(D+1)}}\int dpdqe^{ip\cdot x}\left[\frac{1}{p^2}f\left(\frac{p^2}{m^2}\right)\right]\epsilon\cdot\frac{\partial}{\partial q}\left[\frac{1}{q^2}f\left(\frac{q^2}{m^2}\right)\right]\int dy^{D+1}e^{iy\cdot(q-p)} \\
&= -\frac{4im^2}{(2\pi)^{D+1}}\int dpe^{ip\cdot x}\left[\frac{1}{p^2}f\left(\frac{p^2}{m^2}\right)\right]\epsilon\cdot\frac{\partial}{\partial p}\left[\frac{1}{p^2}f\left(\frac{p^2}{m^2}\right)\right] \\
&= -\frac{2im^2}{(2\pi)^{D+1}}\int dpe^{ip\cdot x}\epsilon\cdot\frac{\partial}{\partial p}\left[\frac{1}{p^2}f\left(\frac{p^2}{m^2}\right)\right]^2.
\end{aligned} \tag{6.1.12}$$

Hence, using once more that $\frac{\partial}{\partial p_\mu}=\frac{p^\mu}{p}\frac{\partial}{\partial p}$, we have that

$$\begin{aligned}
\langle\Phi\Phi\delta_m S\rangle &= -2im^2\frac{\epsilon\cdot p}{p}\frac{d}{dp}\left[\frac{1}{p^2}f\left(\frac{p^2}{m^2}\right)\right]^2 \\
&= -4i\frac{\epsilon\cdot p}{m^4}\frac{d}{dy}\left[\frac{f(y)}{y}\right]^2,
\end{aligned} \tag{6.1.13}$$

and equating with the result of (6.1.11) we acquire the ODE

$$\frac{d^2f}{dy^2}=\frac{d}{dy}\left[\frac{f(y)}{y}\right]^2, \tag{6.1.14}$$

which can be integrated once to give

$$\frac{df}{dy}=\left[\frac{f(y)}{y}\right]^2+c. \tag{6.1.15}$$

Making a change of variable $z=\alpha/y$ we get

$$\frac{df}{dz}=-\alpha f^2-\frac{\alpha c}{z^2}, \tag{6.1.16}$$

which, using the substitution $f=\frac{dF/dz}{F}$ and choosing $\alpha=1$, gives the Euler equation

$$z^2\frac{d^2F}{dz^2}+cF=0, \tag{6.1.17}$$

which has a solution

$$F=c_3z^{1/2(1+\sqrt{1-4c})}+c_4z^{1/2(1-\sqrt{1-4c})}. \tag{6.1.18}$$

For $c < 1/4$ $c_1 = \sqrt{1-4c}$ is real. Inverting back $y = 1/z$ and defining $c_2 = c_4/c_3$ we have that

$$f(y) = \frac{y}{2}\left[1 + c_1 \frac{1 - c_2 y^{c_1}}{1 + c_2 y^{c_1}}\right], \tag{6.1.19}$$

which for $c_1 = c_2 = 1$ give $f(y) = \frac{y}{1+y}$, resulting to the propagator $\langle \Phi\Phi\rangle = \frac{1}{p^2} f\left(\frac{p^2}{m^2}\right) = \frac{1}{p^2+m^2}$.

For $c > 1/4$ we can define c_1 as $2ic_1 = \sqrt{1-4c}$. Inverting again $y = 1/z$ and doing some algebra we arrive at the result

$$f(y) = \frac{y}{2}\left[1 + 2c_1 \tan(c_2 + c_1 \ln y)\right], \tag{6.1.20}$$

were the identity $\frac{\tan(\alpha)+\tan(\beta)}{1-\tan(\alpha)\tan(\beta)} = \tan(\alpha+\beta)$ was used, and c_2 is defined as $c_2 = \tan^{-1}\left(\frac{i(c_3-c_4)}{c_1+c_4}\right)$. Since we have used the infinitesimal change of the action due to the mass transformation, these peculiar propagators may correspond to theories with the same infinitesimal change but a different full Lagrangian. It is expected that the General Conformal Symmetry does not fully constraint the propagators, as it is obeyed by very different theories, such as the free boson and D-branes.

These results reveal the implications of the Generalised Conformal Structure, as, given only the infinitesimal change of the action due to the mass transformation (and not the full Lagrangian), we have been able to largely constrain the propagator of the theory, assuming no prior knowledge of it.

6.2 Ward Identities for $\langle \Phi^2(x)\Phi^2(0)\rangle$

In order to probe the constraining power of the generalised conformal structure even further we, now, turn into the composite operator $O(x) = \Phi^2(x)$. We shall first prove that the two-point function of this operator satisfies the corresponding Ward identity

$$L_{CFT}\langle O(x)O(0)\rangle = \langle O(x)O(0)\delta_m S\rangle, \tag{6.2.1}$$

where L_{CFT} has the usual form with $\Delta_O = D - 1$. Using Wick contraction we have that

$$\langle O(x)O(0)\rangle = \langle \Phi^2(x)\Phi^2(0)\rangle = 2\langle \Phi(x)\Phi(0)\rangle^2. \tag{6.2.2}$$

Inserting this result into the LHS of (6.2.1) we get

$$\begin{aligned} L_{CFT}\langle \Phi\Phi\rangle(x) &= \left[2(D-1)\,\epsilon\cdot x + \delta x\cdot\partial\right] 2\langle \Phi(x)\Phi(0)\rangle^2 \\ &= 2\langle \Phi(x)\Phi(0)\rangle \left[2(D-1)\,\epsilon\cdot x + 2\delta x\cdot\partial\right]\langle \Phi(x)\Phi(0)\rangle \\ &= 4\langle \Phi(x)\Phi(0)\rangle L_{CFT}\langle \Phi(x)\Phi(0)\rangle. \end{aligned} \tag{6.2.3}$$

Moving on to the RHS and working as for the Φ case we have

$$\begin{aligned}\langle O(x)O(0)\delta_m S\rangle &= \langle O(x)O(0)\int d^{D+1}y\left(-2m^2\epsilon\cdot y\Phi^2(y)\right)\rangle \\ &= -2m^2\int d^{D+1}y\epsilon\cdot y\langle\Phi^2(x)\Phi^2(0)\Phi^2(y)\rangle \\ &= -16m^2\int d^{D+1}y\epsilon\cdot y\langle\Phi(x)\Phi(0)\rangle\langle\Phi(x)\Phi(y)\rangle\langle\Phi(y)\Phi(0)\rangle \\ &= 4\langle\Phi(x)\Phi(0)\rangle\left[-2m^2\int d^{D+1}y\epsilon\cdot y\langle\Phi(x)\Phi(0)\Phi^2(y)\rangle\right] \\ &= 4\langle\Phi(x)\Phi(0)\rangle\langle\Phi(x)\Phi(0)\delta_m S\rangle,\end{aligned} \tag{6.2.4}$$

where in going to the third and fourth line we used Wick contraction. Compering now the results of (6.2.4) and (6.2.3) we see that the Ward identity for $O(x)$ is satisfied if the Ward identity (6.1.1) is. As this was proven above to be true, we have shown that the Ward identity is indeed satisfied for the composite operator $O(x)$ as well.

The proof above has been heavily based on Wick contraction and did not result in a differential equation that ought to be solved. In a more general scenario one would need to solve such an equation. Hence, we shall proceed to do the same calculation but relying less on Wick contraction, namely by seeing $O(x)$ as an operator of its own right for the LHS of the equation with weight $\Delta_O = D-1$. To start with, we will calculate explicitly the two-point function using the Feynman rules, so that we can later on compare this to the results of the differential equation. We therefore that in momentum space

$$\langle OO\rangle(p) = \frac{2}{(2\pi)^{D+1}}\int d^{D+1}q\frac{1}{q^2+m^2}\frac{1}{(q+p)^2+m^2} = \frac{2}{(2\pi)^{D+1}}L_1. \tag{6.2.5}$$

We now move on to evaluate the integral L_1 using various standard techniques

$$\begin{aligned}L_1 &= \int d^{D+1}q\frac{1}{q^2+m^2}\frac{1}{(q+p)^2+m^2} \\ &= \int d^{D+1}q\int_0^1 dx\frac{1}{\left[(1-x)(q^2+m^2)+x((q+p)^2+m^2)\right]^2} \\ &= \int_0^1 dx\int d^{D+1}q\frac{1}{\left[(q+xp)^2+\Delta\right]^2} \\ &= \int_0^1 dx\int d^{D+1}q\int_0^\infty dtte^{-t(q^2+\Delta)} \\ &= \int_0^1 dx\int_0^\infty dtt^{1-\frac{D+1}{2}}e^{-t\Delta}\int d^{D+1}qe^{-q^2} \\ &= \pi^{\frac{D+1}{2}}\Gamma\left(\frac{3-D}{2}\right)m^{D-3}\int_0^1 dx\Delta_2^{\frac{D-3}{2}},\end{aligned} \tag{6.2.6}$$

where in going to the second line we used Feynman parametrisation and the definition $\Delta = xp^2+m^2-x^2p^2$. In going to the fourth line the change of variable $q\to q+xp$ and

Schwinger parametrisation were used, in going to the next line the change of variable $q \to \sqrt{t}q$ and arriving to the final result the standard integrals $\int d^{D+1}qe^{-q^2} = \pi^{\frac{D+1}{2}}$ and $\int_0^\infty dtt^\alpha e^{-t\Delta} = \Delta^{-1-\alpha}\Gamma(1+\alpha)$ were used along with $\Delta_2 = \Delta/m^2$.

We are now ready to proceed to solve the Ward identity. As we stated above we will do so without using Wick contraction for the LHS and instead calculate it regarding $O(x)$ an operator in its own right. A similar calculation to equations (6.1.3) - (6.1.4) for general operator weight and unknown propagator results to

$$\begin{aligned} L_{CFT}\langle OO\rangle(p) &= i\epsilon\cdot p\left[\frac{1}{p}\left(2\left(\Delta_O-(D+1)\right)+D\right)\frac{d}{dp}-\frac{d^2}{dp^2}\right]\langle OO\rangle(p) \\ &= i\epsilon\cdot p\left[(D-4)\frac{1}{p}\frac{d}{dp}-\frac{d^2}{dp^2}\right]\langle OO\rangle(p), \end{aligned} \tag{6.2.7}$$

where $\Delta_O = D-1$ was used. We shall now use the constraint due to generalised dilatations as in the $\Phi(p)\Phi(-p)\rangle$ case. This means that the correlator $\langle OO\rangle(p)$ will be equal to a conformal part multiplied by a function of the weightless $y = \frac{p^2}{m^2}$. The conformal part ought to be annihilated by L_{CFT}, i.e.

$$L_{CFT}p^\alpha = i\epsilon\cdot p\left[(D-4)\frac{1}{p}\frac{d}{dp}-\frac{d^2}{dp^2}\right]p^\alpha = 0, \tag{6.2.8}$$

which has as non-zero solution $\alpha = D-3$. Hence the $O(x)$ two-point function is constrained to be

$$\langle OO\rangle(p) = p^{D-3}h(y). \tag{6.2.9}$$

Inserting this into the LHS of the Ward identity, i.e. into equation (6.2.7), we arrive (after some algebra) to

$$\begin{aligned} L_{CFT}p^{D-3}h(y) &= -i\frac{\epsilon\cdot p}{m^2}p^{D-3}2\left[2yh'' + (D-1)h'\right] \\ &= -i\frac{\epsilon\cdot p}{m^2}p^{D-3}4\left(y^{\frac{D-1}{2}-1}\right)^{-1}\left[y^{\frac{D-1}{2}}h'' + \frac{D-1}{2}y^{\frac{D-1}{2}-1}h'\right] \\ &= -i\epsilon\cdot p4m^{D-5}\left(y^{\frac{D-1}{2}}h'\right)' \end{aligned} \tag{6.2.10}$$

We now move one to the RHS of the Ward identity, which we have calculated so far in equation (6.2.4). To proceed further we use equation (6.1.12). For this calculation we will assume we are in the usual scalar theory, such that $\frac{1}{p^2}f\left(\frac{p^2}{m^2}\right) = \frac{1}{p^2+m^2}$. Inserting this into equation (6.2.4) we have

$$
\begin{aligned}
&\langle O(x)O(0)\delta_m S\rangle = 4\langle\Phi(x)\Phi(0)\rangle\langle\Phi(x)\Phi(0)\delta_m S\rangle \\
&= -4D(x)\frac{2im^2}{(2\pi)^{D+1}}\int d^{D+1}p e^{ip\cdot x}\epsilon\cdot\frac{\partial}{\partial p}\left[\frac{1}{p^2+m^2}\right]^2 \\
&= -8m^2 i\frac{1}{(2\pi)^{2(D+1)}}\int d^{D+1}q d^{D+1}p e^{iq\cdot x}\frac{1}{q^2+m^2}e^{ip\cdot x}\epsilon\cdot\frac{\partial}{\partial p}\left[\frac{1}{p^2+m^2}\right]^2 \\
&= -8m^2 i\frac{1}{(2\pi)^{2(D+1)}}\int d^{D+1}z d^{D+1}l e^{il\cdot x}\frac{1}{(l-z)^2+m^2}\epsilon\cdot\frac{\partial}{\partial z}\left[\frac{1}{z^2+m^2}\right]^2 \\
&= \frac{1}{(2\pi)^{D+1}}\int d^{D+1}p e^{ip\cdot x}\left[\frac{-8im^2}{(2\pi)^{D+1}}\int d^{D+1}q\frac{1}{(p-q)^2+m^2}\epsilon\cdot\frac{\partial}{\partial q}\left[\frac{1}{q^2+m^2}\right]^2\right],
\end{aligned}
\tag{6.2.11}
$$

where in going to the third line we used the Fourier transform of the $\Phi(x)$ propagator, then used the change of variables $z = p$ and $l = p + q$ which has a Jacobian equal to 1 and in the we renamed $l \to p$ and $z \to q$. Hence we arrive at the result that the RHS of the Ward identity in momentum space is

$$
\langle OO\delta_m S\rangle(p) = -\frac{8im^2}{(2\pi)^{D+1}}L_2, \tag{6.2.12}
$$

where

$$
\begin{aligned}
L_2 &= \int d^{D+1}q\frac{1}{(p-q)^2+m^2}\epsilon\cdot\frac{\partial}{\partial q}\left[\frac{1}{q^2+m^2}\right]^2 \\
&= -\int d^{D+1}q\epsilon\cdot\frac{\partial}{\partial q}\left(\frac{1}{(p-q)^2+m^2}\right)\frac{1}{(q^2+m^2)^2} \\
&= \epsilon\cdot\frac{\partial}{\partial p}\int d^{D+1}q\frac{1}{(p-q)^2+m^2}\frac{1}{(q^2+m^2)^2} \\
&= \epsilon\cdot\frac{\partial}{\partial p}\frac{1}{m^6}\int d^{D+1}q\frac{1}{(\frac{p}{m}-\frac{q}{m})^2+1}\frac{1}{\left((\frac{q}{m})^2+1\right)^2} \\
&= m^{D-7}2\epsilon\cdot p\frac{d}{dy}\int d^{D+1}q\frac{1}{(\sqrt{y}-q)^2+1}\frac{1}{(q^2+1)^2} \\
&= m^{D-7}2\epsilon\cdot p\frac{d}{dy}L_3,
\end{aligned}
\tag{6.2.13}
$$

where in going to the second line we have integrated by parts and then used that $\frac{\partial}{\partial q}f(q-p) = -\frac{\partial}{\partial p}f(q-p)$. In going to the fifth line the change of variables $q \to \frac{q}{m}$ was made, the fact that $\epsilon\cdot\frac{\partial}{\partial p} = 2\frac{\epsilon\cdot p}{m^2}\frac{d}{dy}$ was used and finally the integral L_3 was defined. We now turn to calculate this integral in a similar manner to L_1. We start by making a change of variables $q \to -q$ and we get

$$\begin{aligned}
L_3 &= \int d^{D+1}q \frac{1}{(\sqrt{y}+q)^2+1}\frac{1}{(q^2+1)^2} \\
&= 2\int d^{D+1}q \int_0^1 dxx \left[x\left((q+\sqrt{y})^2+1\right)+(1-x)\left(q^2+1\right)\right]^{-3} \\
&= 2\int d^{D+1}q \int_0^1 dxx \left[\left(q^2+x\sqrt{y}\right)\Delta_2\right]^{-3} \\
&= 2\int_0^1 dxx \int d^{D+1}q \frac{1}{(q^2+\Delta_2)^3} \\
&= 2\int_0^1 dxx \frac{1}{2}\int d^{D+1}q \int_0^\infty dtt^2 e^{-t(q^2+\Delta_2)} \\
&= \int_0^1 dxx \int_0^\infty dtt^{\frac{3-D}{2}} e^{-t\Delta_2^2} \int d^{D+1}q e^{-q^2} \\
&= \Gamma\left(\frac{5-D}{2}\right)\pi^{\frac{D+1}{2}} \int_0^1 dxx\Delta_2^{\frac{D-5}{2}}.
\end{aligned} \tag{6.2.14}$$

Putting everything together we have that the RHS of the Ward identity is

$$\begin{aligned}
\langle OO\delta_m S\rangle(p) &= -\frac{8im^2}{(2\pi)^{D+1}} m^{D-7} 2\epsilon\cdot p \frac{d}{dy}\Gamma\left(\frac{5-D}{2}\right)\pi^{\frac{D+1}{2}}\int_0^1 dxx\Delta_2^{\frac{D-5}{2}}, \\
&= -\frac{16i\pi^{\frac{D+1}{2}}}{(2\pi)^{D+1}}\epsilon\cdot p m^{D-5}\Gamma\left(\frac{5-D}{2}\right)\left(\int_0^1 dxx\Delta_2^{\frac{D-5}{2}}\right)'.
\end{aligned} \tag{6.2.15}$$

Equating this with the LHS (6.2.10) we have

$$\begin{aligned}
-i\epsilon\cdot p4m^{D-5}\left(y^{\frac{D-1}{2}}h'\right)' &= -\frac{16i\pi^{\frac{D+1}{2}}}{(2\pi)^{D+1}}\epsilon\cdot p m^{D-5}\Gamma\left(\frac{5-D}{2}\right)\left(\int_0^1 dxx\Delta_2^{\frac{D-5}{2}}\right)' \\
\Leftrightarrow \left(y^{\frac{D-1}{2}}h'\right)' &= \frac{4\pi^{\frac{D+1}{2}}}{(2\pi)^{D+1}}\Gamma\left(\frac{5-D}{2}\right)\left(\int_0^1 dxx\Delta_2^{\frac{D-5}{2}}\right)' \\
\Leftrightarrow h' &= \frac{4\pi^{\frac{D+1}{2}}}{(2\pi)^{D+1}}\Gamma\left(\frac{5-D}{2}\right)y^{\frac{1-D}{2}}\int_0^1 dxx\Delta_2^{\frac{D-5}{2}} + cy^{\frac{1-D}{2}} \\
\Leftrightarrow h(y) = \frac{4\pi^{\frac{D+1}{2}}}{(2\pi)^{D+1}}\Gamma\left(\frac{5-D}{2}\right) &\int dyy^{\frac{1-D}{2}}\int_0^1 dxx\Delta_2^{\frac{D-5}{2}} + c_1 y^{\frac{3-D}{2}} + c_2,
\end{aligned} \tag{6.2.16}$$

where we defined $c_1 = c\frac{2}{3-D}$. In order to compare this with the result calculated in (6.2.5) - (6.2.6), we shall insert the exact result into the decomposition (6.2.9), so that we get that the exact result for the scalar theory is

$$\begin{aligned}
h(y) &= \frac{\langle OO\rangle(p)}{m^{D-3}y^{\frac{D-3}{2}}} \\
&= \frac{2}{(2\pi)^{D+1}}\pi^{\frac{D+1}{2}}\Gamma\left(\frac{3-D}{2}\right)y^{\frac{3-D}{2}}\int_0^1 dx\Delta_2^{\frac{D-3}{2}}.
\end{aligned} \tag{6.2.17}$$

Hence, in order to make it easier for comparison, we write the exact result as

$$\begin{aligned} g(y) &= \frac{h(y)}{\frac{2\pi^{\frac{D+1}{2}}}{(2\pi)^{D+1}}\Gamma\left(\frac{3-D}{2}\right)} \\ &= y^{\frac{3-D}{2}} \int_0^1 dx \Delta_2^{\frac{D-3}{2}}. \end{aligned} \tag{6.2.18}$$

Similarly for the solution of the differential equation (6.2.16) we can write

$$g(y) = (3-D) \int dy y^{\frac{1-D}{2}} \int_0^1 dx x \Delta_2^{\frac{D-5}{2}} + c_1 y^{\frac{3-D}{2}} + c_2, \tag{6.2.19}$$

where we have absorbed the dividing constants into c_1 and c_2 and used that $\Gamma(x+1) = x\Gamma(x)$. In order, finally, to see the constraining power of the Ward identity we shall present the exact results of equations (6.2.18) and (6.2.19) in various dimensions. We shall start from $D = 2$ as for $D = 1$ the scalar field Φ is not primary [1] and therefore our analysis does not hold. We shall denote the results as g_1 and g_2 respectively (the integrals are calculated with Mathematica). For $D = 2$ we have

$$\begin{aligned} g_1 &= 2\cot^{-1}\left(\frac{2}{\sqrt{y}}\right) \\ g_2 &= 2\cot^{-1}\left(\frac{2}{\sqrt{y}}\right) + c_1 y^{1/2} + c_2, \end{aligned} \tag{6.2.20}$$

which are equal for $c_1 = c_2 = 0$. For $D = 3$

$$\begin{aligned} g_1 &= 1 \\ g_2 &= c_1 + c_2, \end{aligned} \tag{6.2.21}$$

giving the same results for $c_1 + c_2 = 1$ and for $D = 4$

$$\begin{aligned} g_1 &= \frac{2\sqrt{y} + (y+4)\cot^{-1}\left(\frac{2}{\sqrt{y}}\right)}{4y} \\ g_2 &= \frac{2\sqrt{y} + (y+4)\cot^{-1}\left(\frac{2}{\sqrt{y}}\right)}{4y} + c_1 y^{-1/2} + c_2, \end{aligned} \tag{6.2.22}$$

equal for $c_1 = c_2 = 0$. Moving on to $D = 5$

$$\begin{aligned} g_1 &= \frac{1}{y} + \frac{1}{6} \\ g_2 &= \frac{1}{y} + c_1 y^{-1} + c_2, \end{aligned} \tag{6.2.23}$$

being equal for $c_1 = 0$, $c_2 = 1/6$ and for $D = 6$

$$\begin{aligned} g_1 &= \frac{2(3y+20)\sqrt{y} + 3(y+4)^2 \cot^{-1}\left(\frac{2}{\sqrt{y}}\right)}{64y^2} \\ g_2 &= \frac{2(3y+20)\sqrt{y} + 3(y+4)^2 \cot^{-1}\left(\frac{2}{\sqrt{y}}\right)}{64y^2} + c_1 y^{-3/2} + c_2, \end{aligned} \tag{6.2.24}$$

giving the same result once again for $c_1 = c_2 = 0$. And for a final confirmation $D = 7$

$$\begin{aligned} g_1 &= \frac{y+3}{3y^2} + \frac{1}{30} \\ g_2 &= \frac{y+3}{3y^2} + c_1 y^{-2} + c_2, \end{aligned} \tag{6.2.25}$$

being equal for $c_1 = 0$, $c_2 = 1/30$ and for $D = 8$

$$\begin{aligned} g_1 &= \frac{2(5y(3y+32)+528)\sqrt{y} + 15(y+4)^3 \cot^{-1}\left(\frac{2}{\sqrt{y}}\right)}{1536y^3} \\ g_2 &= \frac{2(5y(3y+32)+528)\sqrt{y} + 15(y+4)^3 \cot^{-1}\left(\frac{2}{\sqrt{y}}\right)}{1536y^3} + c_1 y^{-5/2} + c_2, \end{aligned} \tag{6.2.26}$$

giving the same result for $c_1 = c_2 = 0$. A similar pattern seems to hold for all dimensions calculated. It is, therefore, clear that the Ward identity largely constrains correlators of composite operator as well, given that we used as an input only the propagator of the scalar field in our theory.

CHAPTER 7

Discussion

In this thesis we have discussed the generalised conformal symmetry of D-branes and the restrictions it applies to their correlation functions. We have found that, with a small manipulation, the low-energy dynamics of D-branes in all dimensions respect this symmetry. Following a similar procedure to that of the usual Ward identities we have found that even if this is not a symmetry in the strict sense one is still able to derive a differential equation involving correlation functions.

We started by considering conformal symmetry and its generators. Then we investigated the way fields transform in a CFT. We saw how symmetry in a QFT puts constrains to its correlation functions via the Ward identities. When these are applied on CFT correlators it turns out that completely fixes the 2-point and 3-point functions. Higher-point functions are still highly constraint but there is some functional freedom.

We later reviewed D-branes and their low-energy dynamics. These are described a super Yang-Mills Lagrangian. The discussion that followed was concentrated on the bosonic part of the theory; the fermionic part shall be investigated in the work to follow. The bosonic part is conformal in 4 dimensions and it is indeed D3-branes that gave rise to AdS/CFT. Having seen the extend to which conformal symmetry constrains the dynamics of D3-branes we investigated whether an extension of this symmetry is obeyed in general dimensions. It is proved that indeed D-branes obey generalised conformal symmetry in all dimensions, a structure where the coupling constant transforms as a scalar field under conformal transformations along with the dynamical fields. It is noted, thought, that for 6-dimensional branes some issues arise.

We, finally, start the procedure of using this symmetry in order to constrain the dynamics of D-branes. Although the generalised conformal symmetry is not a symmetry in the strict sense, as the coupling constant is not a dynamical field, we are able to derive Ward-like identities. This relies on the fact that g can be thought as a background field and hence the non-invariance under conformal transformations is compensated by adding an extra term to the usual Ward identity. The equation derived, namely (4.2.5), involves the usual differential operator acting on the correlation function as in the typical Ward identities and an additional term due to the transformation of the coupling constant. Once again, as in the case of conformal symmetry, the constraints make no mention of the Lagrangian of the system. It can there become a powerful tool that would allow us to achieve a better understanding of a wide class of theories.

The most obvious of these would be a better understanding of D-brane dynamics as they are. D-branes are fundamental objects in string theory, hence it could provide further investigations with a strong tool. Most importantly, though, one would be able to use it in order to achieve a more clear image of Gauge/Gravity duality and its consequences. D3-branes gave rise to the most famous and better understood example, the AdS/CFT correspondence. The latter provides an image of space-time being emergent from CFT data living on the boundary. It is, therefore, a great case of interest to see wether such a holographic description is realised more generally.

Non-conformal branes [7] give rise to such more general Gauge/Gravity dualities. Hence, finding constraints due to the generalised conformal symmetry they obey could lead to a much better understanding of the holographic dictionary. That in turn would be useful in a number of specific examples other than the better understanding of the duality itself. Such examples can include recent work on early universe models [8] or other strongly coupled systems. Hence, future work could have applications in a variety of problems.

The original goal of this work was to solve the Ward identities of the generalised conformal structure in full generality. That could, furthermore, be combined with a similar generalisation of conformal bootstrap, in order to constraint higher order correlation functions as well. It has been seen, though, that various difficulties arise from the extra term of the Ward identities due to the transformation of the coupling constant, i.e. the second term of (4.2.5). These difficulties have in their core the fact that for the action to be invariant the coupling constant has to be transformed, i.e. the coupling constant depends on some coordinates, but the Ward identity describes the correlation functions of the theory, and hence should be independent of these coordinates. That is become the coupling constant is a constant and, hence, independent of any set of coordinates.

A first attempt was to treat the coupling constant as a field, then in the context of the two-point function the only coordinate it could be allowed to depend on is x_{12}. When this assumption was combined with the transformations (4.1.1) and (4.1.2) the resulting Ward identity had no solution other than the trivial zero two-point function - therefore the assumption turned out to be too restrictive. The second attempt that lead to the concrete results of Chapter 5 did not put any constraints on the coordinated, but used the fact that at some point the coupling constant should be taken as a constant - namely that the second term in the coupling constant transformations in (4.1.1) and (4.1.2) should be taken to be equal to zero. That led to the transformation (5.0.1), which crucially does not depend on the coordinate of the coupling constant. A similar attempt to solve the SCT Ward identity run into trouble, given that the first term of the transformed coupling constant in (4.1.2) does depend on that coordinate.

Hence we focused on solving a specific example - namely the free massive boson - in order to gain a better understanding of the dynamics of the SCT. A more complete understanding of this problematic term would allow to solve the system in general, most likely in a perturbative form.

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