Internal stress calculation

We start form the following equations, based on the equations 56 – 58 in the CICE manual. The revised equations, currently used in CICE, add an extra factor e^2 in the tension and shear equations:

$$\frac{1}{E}\frac{\partial \sigma_1}{\partial t} + \frac{\sigma_1}{2\zeta^{F+C}} + \frac{P^{F+C}}{2\zeta^{F+C}} = D_D, \tag{A.1}$$

$$\frac{e^2}{E}\frac{\partial \sigma_2}{\partial t} + \frac{\sigma_2}{2\eta^{F+C}} = D_T, \tag{A.2}$$

$$\frac{e^2}{E}\frac{\partial \sigma_{12}}{\partial t} + \frac{\sigma_{12}}{2\eta^{F+C}} = \frac{1}{2}D_S. \tag{A.3}$$

The bulk and shear viscosity and pressure term are taken as sums of the collisional (subscript C) and EVP (subscript F) contributions. E is the elastic parameter, $E = \frac{\zeta}{T}$ with T the damping timescale 0.36 times the general time step.

Substitute E and rearrange,

$$\frac{\partial \sigma_1}{\partial t} + \frac{\sigma_1}{2T} + \frac{P^{F+C}}{2T} = \frac{\zeta^{F+C}}{T}D_D, \tag{A.4}$$

$$\frac{\partial \sigma_2}{\partial t} + \frac{\zeta^{F+C}}{e^2\eta^{F+C}}\frac{\sigma_2}{2T} = \frac{\zeta^{F+C}}{e^2T}D_T, \tag{A.5}$$

$$\frac{\partial \sigma_{12}}{\partial t} + \frac{\zeta^{F+C}}{e^2\eta^{F+C}}\frac{\sigma_{12}}{2T} = \frac{\zeta^{F+C}}{2e^2T}D_S. \tag{A.6}$$

Discretise the time derivative with the subcycling time step Δt_e and use the updated values of the stress for other terms to get an explicit formulation:

$$\frac{\partial \sigma}{\partial t} = \frac{\sigma^{n+1} - \sigma^n}{\Delta t_e}, \tag{A.7}$$

$$\frac{\sigma_1^{n+1} - \sigma_1^n}{\Delta t_e} + \frac{\sigma_1^{n+1}}{2T} + \frac{P^{F+C}}{2T} = \frac{\zeta^{F+C}}{T}D_D, \tag{A.8}$$

$$\frac{\sigma_2^{n+1} - \sigma_2^n}{\Delta t_e} + \frac{\zeta^{F+C}}{e^2\eta^{F+C}}\frac{\sigma_2^{n+1}}{2T} = \frac{\zeta^{F+C}}{e^2T}D_T, \tag{A.9}$$

$$\frac{\sigma_{12}^{n+1} - \sigma_{12}^n}{\Delta t_e} + \frac{\zeta^{F+C}}{e^2\eta^{F+C}}\frac{\sigma_{12}^{n+1}}{2T} = \frac{\zeta^{F+C}}{2e^2T}D_S. \tag{A.10}$$

Multiply by Δt_e,

$$\sigma_1^{n+1} - \sigma_1^n + \sigma_1^{n+1}\frac{\Delta t_e}{2T} + P^{F+C}\frac{\Delta t_e}{2T} = \zeta^{F+C}\frac{\Delta t_e}{T}D_D, \tag{A.11}$$

$$\sigma_2^{n+1} - \sigma_2^n + \frac{\zeta^{F+C}}{\eta^{F+C}}\sigma_2^{n+1}\frac{\Delta t_e}{2e^2T} = \zeta^{F+C}\frac{\Delta t_e}{e^2T}D_T, \tag{A.12}$$

$$\sigma_{12}^{n+1} - \sigma_{12}^n + \frac{\zeta^{F+C}}{\eta^{F+C}}\sigma_{12}^{n+1}\frac{\Delta t_e}{2e^2T} = \zeta^{F+C}\frac{\Delta t_e}{2e^2T}D_S, \tag{A.13}$$

rearranging:

$$\sigma_1^{n+1}(1 + \frac{\Delta t_e}{2T}) - \sigma_1^n + P^{F+C}\frac{\Delta t_e}{2T} = \zeta^{F+C}\frac{\Delta t_e}{T}D_D, \tag{A.14}$$

$$\sigma_2^{n+1}(1 + \frac{\zeta^{F+C}}{e^2\eta^{F+C}}\frac{\Delta t_e}{2T}) - \sigma_2^n = \zeta^{F+C}\frac{\Delta t_e}{e^2T}D_T, \tag{A.15}$$

$$\sigma_{12}^{n+1}(1 + \frac{\zeta^{F+C}}{e^2\eta^{F+C}}\frac{\Delta t_e}{2T}) - \sigma_{12}^n = \zeta^{F+C}\frac{\Delta t_e}{2e^2T}D_S. \tag{A.16}$$

Further rearrangement yields:

$$\sigma_1^{n+1}(1 + \frac{\Delta t_e}{2T}) = \sigma_1^n - P^{F+C}\frac{\Delta t_e}{2T} + \zeta^{F+C}\frac{\Delta t_e}{T}D_D, \tag{A.17}$$

$$\sigma_2^{n+1}(1 + \frac{\zeta^{F+C}}{e^2\eta^{F+C}}\frac{\Delta t_e}{2T}) = \sigma_2^n + \zeta^{F+C}\frac{\Delta t_e}{e^2T}D_T, \tag{A.18}$$

$$\sigma_{12}^{n+1}(1 + \frac{\zeta^{F+C}}{e^2\eta^{F+C}}\frac{\Delta t_e}{2T}) = \sigma_{12}^n + \zeta^{F+C}\frac{\Delta t_e}{2e^2T}D_S. \tag{A.19}$$

The same form as the stress equations in the code is now achieved:

$$\sigma_1^{n+1} = (\sigma_1^n - P^{F+C}\frac{\Delta t_e}{2T} + \zeta^{F+C}\frac{\Delta t_e}{T}D_D)\frac{1}{(1 + \frac{\Delta t_e}{2T})}, \tag{A.20}$$

$$\sigma_2^{n+1} = (\sigma_2^n + \zeta^{F+C}\frac{\Delta t_e}{e^2T}D_T)\frac{1}{(1 + \frac{\zeta^{F+C}}{\eta^{F+C}}\frac{\Delta t_e}{2e^2T})}, \tag{A.21}$$

$$\sigma_{12}^{n+1} = (\sigma_{12}^n + \zeta^{F+C}\frac{\Delta t_e}{2e^2T}D_S)\frac{1}{(1 + \frac{\zeta^{F+C}}{\eta^{F+C}}\frac{\Delta t_e}{2e^2T})}. \tag{A.22}$$

Now substitute the new expressions for the viscosities, but keep the compressive strengths to avoid too cumbersome equations:

$$\eta^{F+C} = \frac{\zeta^F}{e^2} + \frac{\pi L_f}{6\sqrt{t}} P^C = \frac{P^F}{2\Delta e^2} + \frac{\pi L_f}{6\sqrt{t}} P^C, \tag{A.23}$$

$$\zeta^{F+C} = \frac{P^F}{2\Delta} + \frac{3\pi L_f}{6\sqrt{t}} P^C = \frac{P^F}{2\Delta} + \frac{\pi L_f}{2\sqrt{t}} P^C, \tag{A.24}$$

$$\sigma_1^{n+1} = (\sigma_1^n - (P^F + P^C)\frac{\Delta t_e}{2T} + (\frac{P^F}{\Delta} + \frac{\pi L_f P^C}{\sqrt{t}})\frac{\Delta t_e}{2T} D_D)\frac{1}{(1 + \frac{\Delta t_e}{2T})}, \tag{A.25}$$

$$\sigma_2^{n+1} = (\sigma_2^n + (\frac{P^F}{\Delta} + \frac{\pi L_f}{\sqrt{t}} P^C)\frac{\Delta t_e}{2e^2 T} D_T)\frac{1}{(1 + \frac{\frac{P^F}{2\Delta} + \frac{\pi L_f P^C}{2\sqrt{t}}}{\frac{P^F}{2\Delta e^2} + \frac{\pi L_f}{6\sqrt{t}} P^C}\frac{\Delta t_e}{2e^2 T})}, \tag{A.26}$$

$$\sigma_{12}^{n+1} = (\sigma_{12}^n + (\frac{P^F}{\Delta} + \frac{\pi L_f}{\sqrt{t}} P^C)\frac{\Delta t_e}{4e^2 T} D_S)\frac{1}{(1 + \frac{\frac{P^F}{2\Delta} + \frac{\pi L_f P^C}{2\sqrt{t}}}{\frac{P^F}{2\Delta e^2} + \frac{\pi L_f}{6\sqrt{t}} P^C}\frac{\Delta t_e}{2e^2 T})}. \tag{A.27}$$

Substitute $arlx1i$, which is used in the CICE code and define extra variables $zetaeta$ and $totpress$:

$$arlx1i = \frac{\delta t_e}{2T}, \tag{A.28}$$

$$zetaeta = \frac{\zeta^{F+C}}{\eta^{F+C}} = \frac{\frac{P^F}{2\Delta} + \frac{\pi L_f}{2\sqrt{t}} P^C}{\frac{P^F}{2\Delta e^2} + \frac{\pi L_f}{6\sqrt{t}} P^C}, \tag{A.29}$$

$$totpress = P^C + P^F, \tag{A.30}$$

$$\sigma_1^{n+1} = (\sigma_1^n - totpress\, arlx1i + (\frac{P^F}{\Delta} + \frac{\pi L_f P^C}{\sqrt{t}}) arlx1i D_D)\frac{1}{(1 + arlx1i)}, \tag{A.31}$$

$$\sigma_2^{n+1} = (\sigma_2^n + (\frac{P^F}{\Delta} + \frac{\pi L_f P^C}{\sqrt{t}})\frac{1}{e^2} arlx1i\, D_T)\frac{1}{(1 + \frac{zetaeta\, arlx1i}{e^2})}, \tag{A.32}$$

$$\sigma_{12}^{n+1} = (\sigma_{12}^n + (\frac{P^F}{\Delta} + \frac{\pi L_f P^C}{\sqrt{t}})\frac{1}{2e^2} arlx1i\, D_S)\frac{1}{(1 + \frac{zetaeta\, arlx1i}{e^2})}. \tag{A.33}$$

Substitute $denom1$, redefine variables $c0$ and $c1$:

$$denom1 = \frac{1}{1 + arlx1i}, \tag{A.34}$$

$$denom2 = \frac{1}{1 + \frac{zetaeta\, arlx1i}{e^2}}, \tag{A.35}$$

$$c1 = (\frac{P^F}{\Delta} + \frac{\pi L_f}{\sqrt{t}} P^C) arlx1i, \tag{A.36}$$

$$c0 = \frac{c1}{e^2} = (\frac{P^F}{e^2 \Delta} + \frac{\pi L_f}{e^2 \sqrt{t}} P^C) arlx1i. \tag{A.37}$$

Finally, the equations to be solved, similar to the EVP equations, become:

$$\sigma_1^{n+1} = (\sigma_1^n - totpress\, arlx1i + c1 D_D) denom1, \tag{A.38}$$

$$\sigma_2^{n+1} = (\sigma_2^n + c0 D_T) denom2, \tag{A.39}$$

$$\sigma_{12}^{n+1} = (\sigma_{12}^n + 0.5\, c0\, D_S) denom2. \tag{A.40}$$