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HOW'S THE PERFORMANCE OF THE OPTIMIZED PORTFOLIOS BY SAFETY-FIRST RULES: THEORY WITH EMPIRICAL COMPARISONS

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Abstract. Safety-first (SF) rules have been increasingly useful in particular for construction of optimal portfolios related to pension and other social insurance funds. How's the performance of the optimal portfolios constructed by different SF rules is an interesting practical question but yet less investigated theoretically. In this paper, we therefore analytically investigate the properties of the risky portfolios constructed by the three popular SF rules, denoted by the RSF, TSF and KSF, which are suggested and developed by A. D. Roy, L. G. Telser and S. Kataoka, respectively. Using Sharpe ratio as a measure of portfolio performance, we theoretically derive that the performance of an optimal portfolio constructed by the KSF approach depends on an acceptable level of extreme risk tolerance. The unique solution where the performance of the KSF portfolio is the same as that of the other two SF portfolios is found. By this we interestingly find that except this special case, under the finite optimal portfolios existent, the KSF portfolio always dominates the TSF portfolio in terms of the Sharpe ratio. In addition, in some market scenarios, even when the RSF and TSF portfolios do not exist in finite forms, the KSF rule can still apply to get a finite optimal portfolio. Moreover, in comparison with the RSF rule, a series of finite KSF portfolios can be interestingly constructed with their Sharpe ratios approaching to the maximum Sharpe ratio, which however cannot be reached by any corresponding finite RSF portfolio. Numerical comparisons of these rules by using a set of real data are further empirically demonstrated.

1. **Introduction.**

Beyond Markowitz's (1952)[16] mean-variance methodology, the safety-first (SF) criteria, suggested originally by Roy (1952)[21] and then developed by Telser (1955)[23] and Kataoka (1963)[11], have been well-known in risky assets allocation. Economic implications of using these SF rules as well as comparison with other known criteria, such as expected utility maximization and stochastic dominance, for portfolio optimization can be seen, for example, in Pyle & Turnovsky (1970)[20], Levy & Sarnat (1972)[12], Gressis & Remaley (1974)[8], Bawa (1978)[1] and Ortobelli & Rachev (2001)[19]. However, until the global financial crises have

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the SF rules been in the shadow of the Markowitz's theory. Recently, the SF rules have become increasingly popular in particular for constructing the optimal portfolios related to pension and other social insurance funds (Norkin & Boyko 2012)[17]. See also Ding & Zhang (2009a, b)[3][4], Li et al. (2010)[14], Ding and Lu (2016)[5] and the references therein, for the recent work, among others.

In this paper, our main objective is to investigate the performance of the optimized portfolios by the different SF rules. We will focus on the three most popular SF rules suggested and developed by Roy (1952)[21], Telser (1955)[23] and Kataoka (1963)[11], which are denoted by the RSF, TSF and KSF, respectively, in the sequel. How's the performance of the optimal portfolios constructed by the different SF rules is an interesting practical question but yet less investigated theoretically. We will explore this from the practical perspective of Sharpe ratio as a measure of the performance of a portfolio. The Sharpe ratio has been practically widely used in evaluating the performance of portfolios. For example, a portfolio which maximizes the Sharpe ratio is called the Sharpe portfolio, which is shown to be equivalent to the Roy safety-first (RSF) portfolio (Haley & McGee, 2006)[10] and belongs, in the case of no risk-free asset, to the mean-variance efficient frontier (Bodnar & Zabolotskyy, 2017)[2]. For statistical inference of the weights of the Sharpe portfolio, the reader is referred to Okhrin & Schmid (2006) [18]. Numerical comparison of the performance in terms of the Sharpe measure and the measure of the Telser safety-first (TSF) portfolio can be found in Hagigi & Kluger (1987)[9], who demonstrated that the two performance measures are equivalent only in a special case where the disaster level is equal to the risk-free return rate, and all the assets are unsafe with respect to the acceptable probability. It has also been shown (c.f., Durand & et al (2010)[6]) that the Sharpe portfolio has the smallest VaR (value at risk) relative to the reference return rate. However, to the best of our knowledge, there is still little investigation in the literature into comparing the performance of the portfolios by the different SF rules mentioned above in theory.

We are therefore investigating the properties of the risky portfolios by the Kataoka SF (KSF) rule first and then comparing analytically the KSF portfolio with the Sharpe portfolio as well as the RSF and the TSF portfolios respectively. The unique solution where the performance of the KSF portfolio is the same as that of the other two SF portfolios is found (note that the RSF portfolio is equivalent to the Sharpe portfolio in terms of Sharpe ratio). By this we interestingly find that except this special case, when the finite optimal portfolios exist, the KSF portfolio always dominates the TSF portfolio in terms of the Sharpe ratio. In addition, in some market scenarios, even when both the RSF and the TSF portfolios do not exist in finite forms, the KSF rule can still apply to get a finite optimal portfolio. Moreover, in comparison with the RSF, a series of finite KSF portfolios can be interestingly constructed with their Sharpe ratios approaching the maximum Sharpe ratio, which however cannot be reached by any corresponding finite RSF or Sharpe portfolio.

The structure of this paper is arranged as follows. In Section 2, notation and definition needed will be introduced, including mild assumptions and definitions on the Sharpe ratio and the RSF, the TSF and the KSF portfolios. In Section 3, we theoretically derive some novel characteristics on the Sharpe ratio of the KSF portfolio. In Section 4, analytical comparisons of the KSF portfolio with the RSF and the TSF portfolios in terms of the Sharpe ratio will be explored. Numerical comparisons by using a set of real data will be empirically demonstrated in Section 5. Section 6 concludes.

2. **Notation and definition.** Given a universe of n risky securities with an n-vector of return rates, $R = (R_1, R_2, \cdots, R_n)'$, an individual seeks to allocate on the risky securities to reach his/her objective on the portfolio. Suppose the vector of return rates, R, is distributed with a mean vector $\mu = (\mu_1, \mu_2, \cdots, \mu_n)'$ and a finite positive definite n by n covariance matrix Σ whose (i,j)-th component is $cov(R_i, R_j)$. The allocation of the assets is to find a portfolio of the securities, denoted as an n-vector $X = (X_1, X_2, \cdots, X_n)'$ of the security weights satisfying the budget constraint $X'e = 1$. Here e is an n-vector, each of its elements being 1. If no short sales are allowed in the security market, then the constraints $X \geq 0$, i.e., $X_i \geq 0$ for $i = 1, 2, \cdots, n$, should be satisfied. However in the complete market with marginal trading allowed, the weights of the securities can be negative and we need not consider the non-negativity constraints. Then $R_p = X'R$ is the return rate of the portfolio with weights of X, and R_p follows a distribution with the mean $\mu_p = E(R_p) = X'\mu$ and the variance $\sigma_p^2 = X'\Sigma X$. Similar to Pyle & Turnovsky (1970)[20], it is assumed that the distribution of R_p can be described by two parameters, a location parameter μ_p and a dispersion parameter $\sigma_p = \sqrt{X'\Sigma X}$, so that $(R_p - \mu_p)/\sigma_p$ has a cumulative distribution function F with defined functional form and without unknown parameters. The following notations describing the market situations will be used: $A = \mu'\Sigma^{-1}\mu$, $B = \mu'\Sigma^{-1}e$, $C = e'\Sigma^{-1}e$, $D = AC - B^2$.

Definition 2.1. [Sharpe ratio] Let R_b be the return rate of a benchmark security or portfolio, then the Sharpe ratio of a portfolio with weight of X, proposed by Sharpe (1994) [22], is defined as expected excess return per unit of risk, with the risk measured by the standard deviation of the excess return, that is

$$SR = \frac{E(X'R - R_b)}{\sqrt{E(X'R - X'\mu)^2}}.$$

In practice, either $R_b = 0$ or $R_b = r_f$, the riskless return rate, is typically used to calculate the Sharpe ratio of a portfolio.

Definition 2.2. [RSF portfolio] Based on the Roy safety-first (RSF) rule proposed by Roy (1952)[21], an individual seeks to reduce as far as is possible the chance of a disaster occurring. Let R_b be the return rate of a benchmark security or portfolio, then the RSF portfolio is defined as the optimal portfolio obtained by minimizing $P(X'R \leq R_b)$ subject to $X'e = 1$.

As explained in Roy (1952)[21], the RSF portfolio can be analytically obtained by solving

$$\max_X \quad \frac{X'\mu - R_b}{\sqrt{X'\Sigma X}}, \quad subject \quad to \quad X'e = 1, \tag{1}$$

and geometrically shown at the tangency point S in Figure 1. Hence, the RSF portfolio is the same as the Sharpe portfolio which is defined as that reaches the maximal Sharpe ratio, see Haley & McGee (2006) [10]. In the latter part, we will use X_{sh} or X_{rsf} to denote the RSF porfolio.

Definition 2.3. [KSF portfolio] The Kataoka safety-first (KSF) rule asserts that the best portfolio (KSF portfolio) is the one which maximizes the lower limit return rate R_d, subject to the constraint that the probability that the return rate R_p occurs to be less than or equal to the lower limit R_d is not greater than some predetermined value α (acceptable probability). For a given α (usually $0 < \alpha < 0.5$), the KSF portfolio is defined as the portfolio with weight of vector X_{ksf} obtained by solving:

$$\max_X \quad R_d, \quad subject \quad to \quad P(X'R \leq R_d) \leq \alpha, \quad X'e = 1. \tag{2}$$

Definition 2.4. [TSF portfolio] Based on the Telser safety-first (TSF) rule (Telser 1955)[23], an investor selects a portfolio (TSF portfolio) to maximize the expected return rate under the constraint that the probability for the return rate being less than or equal to some reference return rate R_b is not greater than some predetermined value α. Given R_b and α (usually $0 < \alpha < 0.5$), the TSF portfolio is defined as the portfolio with weight of vector X_{tsf} obtained by solving:

$$\max_X \quad E(X'R), \quad subject \quad to \quad P(X'R \leq R_b) \leq \alpha, \quad X'e = 1. \tag{3}$$

Figure 1 illustrates the RSF portfolio (at tangency point S), the KSF portfolio (at tangency point K), and the TSF portfolio (at intersection point T) on the mean-variance efficient frontier. It was drawn by using $R_b = 0.00210$, $\alpha = 0.41000$, and $A = 0.06703$, $B = 2.40884$, $C = 168.23326$, $D = 5.47400$.

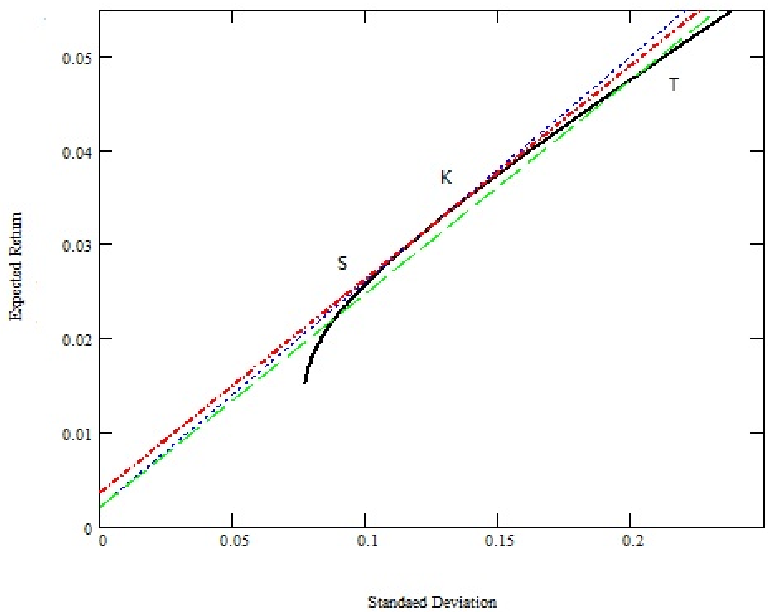

FIGURE 1. RSF portfolio, KSF portfolio and TSF portfolio

3. **Theoretical characteristics of the KSF portfolio.** Since $P(R_p \leq R_d) \leq \alpha$ is equivalent to $\mu_p + z_\alpha \sigma_p \geq R_d$, with $z_\alpha = F^{-1}(\alpha)$, the KSF portfolio by Definition 2.3 is the optimal solution of the following programming:

$$\max_X \quad X'\mu + z_\alpha \sqrt{X'\Sigma X} \quad subject \quad to \quad X'e = 1.$$

According to Ding & Zhang (2009a, b)[3][4], we know that if and only if the acceptable probability α is smaller enough such that $z_\alpha < -\sqrt{\frac{D}{C}}$, the KSF portfolio exists and can be analytically expressed as

$$X_{ksf} = \frac{1}{\sqrt{Cz_\alpha^2 - D}} \left(\Sigma^{-1}\mu + \frac{-B + \sqrt{Cz_\alpha^2 - D}}{C} \Sigma^{-1} e \right). \tag{4}$$

The maximum lower limit level of return rate R_d is

$$R_{ksf} = \frac{B - \sqrt{Cz_\alpha^2 - D}}{C} \tag{5}$$

The expected return rate and the standard deviation of the KSF portfolio are given respectively as:

$$\mu_{ksf} = \frac{B}{C} + \frac{D}{C\sqrt{Cz_\alpha^2 - D}}, \tag{6}$$

$$\sigma_{ksf} = \sqrt{\frac{1}{C}\left(1 + \frac{D}{Cz_\alpha^2 - D}\right)} = \frac{-z_\alpha}{\sqrt{Cz_\alpha^2 - D}}. \tag{7}$$

Therefore, the Sharpe ratio of the KSF portfolio is

$$SR_{ksf} = \frac{\mu_{ksf} - R_b}{\sigma_{ksf}} = \sqrt{\frac{D}{C}}\sqrt{\frac{D}{Cz_\alpha^2}} + \frac{B - CR_b}{\sqrt{C}}\sqrt{1 - \frac{D}{Cz_\alpha^2}}. \tag{8}$$

Eq. (8) describes the performance of the KSF portfolios, which depends on the acceptable probability α for a given benchmark return rate R_b.

Now let $g(y, R_b) = \sqrt{\frac{D}{C}}y + \frac{B-CR_b}{\sqrt{C}}\sqrt{1-y^2}$, with $0 < y < 1$, then $SR_{ksf} = g\left(\sqrt{\frac{D}{Cz_\alpha^2}}, R_b\right)$. In the following, we further describe some Sharpe ratio characteristics of the KSF portfolio.

Proposition 3.1. *Given $R_b > \frac{B}{C}$, the Sharpe ratio SR_{ksf} of the KSF portfolio is a strictly increasing function of the acceptable probability α in the field of $0 < \alpha < F(-\sqrt{\frac{D}{C}})$, and it satisfies*

$$\frac{B - CR_b}{\sqrt{C}} < SR_{ksf} < \sqrt{\frac{D}{C}},$$

$$\lim_{\alpha \to F(-\sqrt{\frac{D}{C}})} SR_{ksf} = \sqrt{\frac{D}{C}},$$

$$\lim_{\alpha \to 0} SR_{ksf} = \frac{B - CR_b}{\sqrt{C}}.$$

Proof. Note that when $R_b > \frac{B}{C}$, we have

$$\frac{\partial g(y, R_b)}{\partial y} = \sqrt{\frac{D}{C}} - \frac{B - CR_b}{\sqrt{C}} \frac{y}{\sqrt{1-y^2}} > 0$$

for any y $(0 < y < 1)$. Hence $g(y, R_b)$ is strictly increasing as a function of y. Now let $y = \sqrt{\frac{D}{Cz_\alpha^2}}$. Then it follows that $0 < y < 1$ for $z_\alpha < -\sqrt{\frac{D}{C}}$ or $0 < \alpha < F(-\sqrt{\frac{D}{C}})$

and that $y = \sqrt{\frac{D}{Cz_\alpha^2}}$ is strictly increasing too as a function of α. Therefore, the Sharpe ratio

$$SR_{ksf} = g\left(\sqrt{\frac{D}{Cz_\alpha^2}}, R_b\right)$$

is strictly increasing as a function of α over the defined field.

Thus SR_{ksf} approaches to $\frac{B-CR_b}{\sqrt{C}}$ as α tends to zero such that y approaches to zero. Also, SR_{ksf} approaches to $\sqrt{\frac{D}{C}}$ as α tends to the value $F(-\sqrt{\frac{D}{C}})$ with the corresponding $z_\alpha = -\sqrt{\frac{D}{C}}$ such that y approaches to 1. □

Proposition 3.1 shows that when $R_b > \frac{B}{C}$, the Sharpe ratio of the KSF portfolio with a smaller α or a higher safety level is always smaller than the Sharpe ratio of the KSF portfolio with a larger α or a lower safety level. Thus according to the KSF criterion, if $R_b > \frac{B}{C}$, then an investor can not get a portfolio with the highest safety level and the best performance.

Proposition 3.2. *Given $R_b = \frac{B}{C}$, the Sharpe ratio SR_{ksf} of the KSF portfolio is a strictly increasing function of the acceptable probability α in the field of $0 < \alpha < F(-\sqrt{\frac{D}{C}})$, and it satisfies*

$$0 < SR_{ksf} < \sqrt{\frac{D}{C}},$$

$$\lim_{\alpha \to F(-\sqrt{\frac{D}{C}})} SR_{ksf} = \sqrt{\frac{D}{C}},$$

and

$$\lim_{\alpha \to 0} SR_{ksf} = 0.$$

Proof. It suffices to notice that as $R = B/C$, we have $SR_{ksf} = \sqrt{\frac{D}{C}}\sqrt{\frac{D}{Cz_\alpha^2}}$ by the Eq. (8). □

Proposition 3.2 shows that as $R_b = \frac{B}{C}$, the Sharpe ratio of the KSF portfolio with a smaller α or a higher safety level is always smaller than the Sharpe ratio of the KSF portfolio with a larger α or a lower safety level. Thus according to the KSF criterion, if $R_b = \frac{B}{C}$, then an investor can not get a portfolio with the highest safety level and the best performance. Obviously, Proposition 3.2 can be combined with Proposition 3.1 in the case where $R_b \geq \frac{B}{C}$.

Proposition 3.3. *Given $R_b < \frac{B}{C}$, the Sharpe ratio SR_{ksf} of the KSF portfolio is a convex function of the acceptable probability α in the field of $0 < \alpha < F(-\sqrt{\frac{D}{C}})$, it attains the maximum value of $\sqrt{A - 2BR_b + CR_b^2}$ at the point of α with $z_\alpha = -\sqrt{A - 2BR_b + CR_b^2}$, and it satisfies that*

$$\lim_{\alpha \to F(-\sqrt{\frac{D}{C}})} SR_{ksf} = \sqrt{\frac{D}{C}},$$

$$\lim_{\alpha \to 0} SR_{ksf} = \frac{B - CR_b}{\sqrt{C}}.$$

Proof. Recall that $\frac{\partial g(y,R_b)}{\partial y} = \sqrt{\frac{D}{C}} - \frac{B-CR_b}{\sqrt{C}}\frac{y}{\sqrt{1-y^2}}$. By letting $\frac{\partial g(y,R_b)}{\partial y} = 0$, we have $\frac{\sqrt{D}}{B-CR_b} = \frac{y}{\sqrt{1-y^2}}$ and thus $y = \sqrt{\frac{D}{(B-CR_b)^2+D}}$.

Since $R_b < \frac{B}{C}$ and $\frac{y}{\sqrt{1-y^2}}$ is strictly increasing in the field of $0 < y < 1$, it follows that if $0 < y < \sqrt{\frac{D}{(B-CR_b)^2+D}}$, then $\frac{y}{\sqrt{1-y^2}} < \frac{\sqrt{D}}{B-CR_b}$ and thus $\frac{\partial g(y,R_b)}{\partial y} > 0$. Similarly, if $\sqrt{\frac{D}{(B-CR_b)^2+D}} < y < 1$, then it follows that $\frac{y}{\sqrt{1-y^2}} > \frac{\sqrt{D}}{B-CR_b}$ and thus $\frac{\partial g(y,R_b)}{\partial y} < 0$. Hence $g(y, R_b)$ is a convex function of y in the field of $0 < y < 1$, and it attains the maximum value at the point $y = \sqrt{\frac{D}{(B-CR_b)^2+D}}$. Also, for the given constant R_b, the maximum value of $g(y, R_b)$ is $\sqrt{\frac{(B-CR_b)^2+D}{C}}$.

For $0 < \alpha < F(-\sqrt{\frac{D}{C}})$, since $z_\alpha < -\sqrt{\frac{D}{C}}$, we have $y = \sqrt{\frac{D}{Cz_\alpha^2}}$ is strictly increasing as a function of α. Thus, by solving $\sqrt{\frac{D}{(B-CR_b)^2+D}} = \sqrt{\frac{D}{Cz_\alpha^2}}$, we get $z_\alpha = -\sqrt{A - 2BR_b + CR_b^2}$. Therefore, the Sharpe ratio SR_{ksf} of the KSF portfolio is a convex function of the acceptable probability α for $0 < \alpha < F(-\sqrt{\frac{D}{C}})$, and at the point with $z_\alpha = -\sqrt{A - 2BR_b + CR_b^2}$, it attains the maximum value of $\sqrt{A - 2BR_b + CR_b^2}$.

Finally, obviously by Eq. (8), $\lim_{\alpha \to F(-\sqrt{D/C})} SR_{ksf} = \sqrt{\frac{D}{C}}$ and $\lim_{\alpha \to 0} SR_{ksf} = \frac{B-CR_b}{\sqrt{C}}$. □

Proposition 3.3 shows that as $R_b < \frac{B}{C}$, among all the efficient KSF portfolios, there is a unique KSF portfolio with the highest Sharpe ratio, and the corresponding extreme risk level $\alpha = F(-\sqrt{A - 2BR_b + CR_b^2})$.

4. **Analytical comparisons.**

On the basis of the theoretical characteristics on the KSF portfolios derived in Section 3, we are analytically comparing the performance of the KSF portfolios with other two SF portfolios.

4.1. **Comparing the KSF portfolio with the Sharpe portfolio.**

Proposition 4.1. *Under the notations above (without further restrictions), we have*

$$\sup_{e'X=1} \frac{\mu' X - R_b}{\sqrt{X'\Sigma X}} = \begin{cases} \sqrt{A - 2BR_b + CR_b^2}, & if \quad R_b < \frac{B}{C}; \\ \sqrt{\frac{D}{C}}, & if \quad R_b \geq \frac{B}{C}. \end{cases} \tag{9}$$

And if and only if $R_b < \frac{B}{C}$, the maximum value is achieved for an unique finite Sharpe portfolio or RSF portfolio, which is given by

$$X_{sh} = \frac{\Sigma^{-1}\mu - R_b\Sigma^{-1}e}{B - CR_b}. \tag{10}$$

Proof. Let $\tilde{\mu} = \mu - R_b e$. Then

$$e'\Sigma^{-1}\tilde{\mu} = e'\Sigma^{-1}\mu - R_d e'\Sigma^{-1}e = B - CR_b,$$

and

$$\sup_{e'X=1} \frac{\mu' X - R_b}{\sqrt{X'\Sigma X}} = \sup_{e'X=1} \frac{\tilde{\mu}' X}{\sqrt{X'\Sigma X}}.$$

Referring to Maller & Turkington (2002)[15], we have

$$\sup_{e'X=1} \frac{\tilde{\mu}'X}{\sqrt{X'\Sigma X}} = \begin{cases} +\sqrt{\tilde{\mu}'\Sigma^{-1}\tilde{\mu}}, & if \quad e'\Sigma^{-1}\tilde{\mu} > 0; \\ +\sqrt{\tilde{\mu}'\Sigma^{-1}\tilde{\mu} - (e'\Sigma^{-1}\tilde{\mu})^2/e'\Sigma^{-1}e}, & if \quad e'\Sigma^{-1}\tilde{\mu} \leq 0. \end{cases}$$

Thus, if and only if $e'\Sigma^{-1}\mu > 0$, the maximum value of the Sharpe ratio is achieved at an unique finite Sharpe portfolio (or RSF portfolio) with $X_{sh} = \frac{\Sigma^{-1}\tilde{\mu}}{e'\Sigma^{-1}\tilde{\mu}} = \frac{\Sigma^{-1}\mu - R_b\Sigma^{-1}e}{B-CR_b}$, which is as given in the Eq. (10).

Now notice that $\tilde{\mu}'\Sigma^{-1}\tilde{\mu} = A - 2BR_b + CR_b^2$ and $\frac{\tilde{\mu}'\Sigma^{-1}\tilde{\mu} - (e'\Sigma^{-1}\tilde{\mu})^2}{e'\Sigma^{-1}e} = \frac{AC-B^2}{C} = \frac{D}{C}$. Then Eq. (9) holds true. □

Proposition 4.2. *If $R_b < \frac{B}{C}$, then if and only if the acceptable probability α satisfies $z_\alpha = -\sqrt{A - 2BR_b + CR_b^2}$ or $\alpha = F\left(-\sqrt{A - 2BR_b + CR_b^2}\right)$, the KSF portfolio is the same as the Sharpe portfolio or the RSF portfolio.*

Proof. From Proposition 3.3, if and only if $z_\alpha = -\sqrt{A - 2BR_b + CR_b^2}$, the KSF portfolio, given by

$$X_{ksf} = \frac{\Sigma^{-1}\mu - R_b\Sigma^{-1}e}{B - R_bC},$$

attains its maximum Shape ratio $\sqrt{A - 2BR_b + CR_b^2}$, which is the same as the Sharpe ratio of the Sharpe portfolio or the RSF portfolio given by Eq. (10). □

Proposition 4.2 implies that the Sharpe portfolio or the RSF portfolio lies on the KSF efficient frontier as $R_b < \frac{B}{C}$.

Proposition 4.3. *If $R_b \geq \frac{B}{C}$, then any finite Sharpe portfolio (or RSF portfolio) does not exist. But a finite KSF portfolio always exists if and only if the acceptable probability α is small enough such that $0 < \alpha < F\left(-\sqrt{\frac{D}{C}}\right)$, and with the acceptable probability increasing, the Sharpe ratios of the KSF portfolios approach to the maximum Sharpe ratio.*

Proof. From Proposition 4.1, any finite Sharpe portfolio does not exist if $R_b \geq \frac{B}{C}$. The other part of Proposition 4.3 comes true directly by combining Propositions 3.1, 3.2, and 4.1. □

Proposition 4.3 implies that the Sharpe ratio optimization rule or the original Roy safety-first rule is not applicable to the allocation of risky assets in the financial markets when $R_b \geq \frac{B}{C}$, but the Kataoka safety-first rule still applies for investors with $0 < \alpha < F(-\sqrt{\frac{D}{c}})$. In such a case, the maximum Sharpe ratio can be achieved by the limit of the Sharpe ratios of a series of KSF portfolios, which we express as follows

$$X_{sh}^{\infty} = \lim_{z_\alpha \uparrow -\sqrt{D/C}} \frac{1}{\sqrt{Cz_\alpha^2 - D}} \left(\Sigma^{-1}\mu + \frac{-B + \sqrt{Cz_\alpha^2 - D}}{C} \Sigma^{-1}e \right).$$

4.2. **Comparing the KSF portfolio with the TSF portfolio.** Since $P(R_p \leq R_b) \leq \alpha$ is equivalent to $\mu_p + z_\alpha\sigma_p \geq R_b$, where $z_\alpha = F^{-1}(\alpha)$, the TSF portfolio by Definition 2.4 is the optimal solution to the following programming:

$$\max_X \quad \mu'X \quad subject \quad to \quad \mu'X + z_\alpha\sqrt{X'\Sigma X} \geq R_b, \quad e'X = 1 \tag{11}$$

Engels (2004) [7] solved the optimized TSF portfolio for the model taking R_b as the negative of the initial wealth. Li & Chen (2005) [13] discussed the solution of the TSF model in form of programming (11) with $z_\alpha = -\frac{1}{\sqrt{\alpha}}$. Here we provide some further discussion on the general TSF model in form of programming (11).

Proposition 4.4. *(i) If and only if $z_\alpha < -\sqrt{\frac{D}{C}}$ and $R_b < \frac{B-\sqrt{Cz_\alpha^2-D}}{C}$, an uniquely finite TSF portfolio exists, which can be expressed as*

$$X_{tsf} = \frac{1}{D}\Sigma^{-1}\left[(Ae - B\mu) + (C\mu - Be)\,\tilde{E}\right], \tag{12}$$

with the expected return rate as

$$\tilde{E} = \frac{Bz_\alpha^2 - DR_b + \sqrt{Dz_\alpha^2\left(A - 2BR_b + CR_b^2 - z_\alpha^2\right)}}{Cz_\alpha^2 - D}. \tag{13}$$

The Sharpe ratio of the TSF portfolio X_{tsf} is $SR_{tsf} = -z_\alpha$.

(ii) If $z_\alpha < -\sqrt{\frac{D}{C}}$ and $R_b = \frac{B-\sqrt{Cz_\alpha^2-D}}{C}$, then

$$X_{tsf} = \frac{\Sigma^{-1}\mu - R_b\Sigma^{-1}e}{B - R_bC}. \tag{14}$$

The optimal expected return rate is

$$\tilde{E} = \frac{B}{C} + \frac{D}{C\,(B - R_bC)}. \tag{15}$$

Proof. Part (i) follows by using the similar argument to that used in [7] and [13]. As shown in Figure 1, it is known from [3] that $z_\alpha < -\sqrt{\frac{D}{C}}$ is the necessary and sufficient condition for existence of the KSF portfolio (illustrated at the tangent point K), and the optimal return rate is $R_d = \frac{B-\sqrt{Cz_\alpha^2-D}}{C}$. Hence, in order to make the finite TSF portfolio exist at the intersection point T, it is necessary that $z_\alpha < -\sqrt{\frac{D}{C}}$ and $R_b \le R_d$.

Let the TSF portfolio have an expected return rate $\tilde{E}$ and standard deviation $\tilde{\sigma}$. Then

$$\tilde{E} + z_\alpha\tilde{\sigma} = R_b, \tag{16}$$

$$\frac{\tilde{\sigma}^2}{1/C} - \frac{(\tilde{E} - B/C)^2}{D/C^2} = 1, \tilde{\sigma} > 0, \tilde{E} \ge \frac{B}{C}, \tag{17}$$

where Eq. (16) shows that the TSF portfolio lies on the line with a slope of $-z_\alpha$ and a vertical intercept of R_b, while Eq. (17) indicates that the TSF portfolio also lies on the mean-variance (MV) efficient frontier.

Further, if $z_\alpha < -\sqrt{\frac{D}{C}}$ and $R_b < \frac{B-\sqrt{Cz_\alpha^2-D}}{C}$, then $z_\alpha^2 \le A - 2BR_b + CR_b^2$. Also, from Eqs. (16) and (17), we have, respectively,

$$\tilde{\sigma}^2 = \left(\frac{R_b - \tilde{E}}{z_\alpha}\right)^2,$$

$$\tilde{\sigma}^2 = \frac{C}{D}\left(\tilde{E} - \frac{B}{C}\right)^2 + \frac{1}{C} = \frac{1}{D}\left(C\tilde{E}^2 - 2B\tilde{E} + A\right),$$

where $D = AC - B^2$. Thus, it easily follows that

$$\left(Cz_\alpha^2 - D\right)\tilde{E}^2 + 2\left(DR_b - Bz_\alpha^2\right)\tilde{E} + Az_\alpha^2 - DR_b^2 = 0.$$

This quadratic equation can be solved by using the abc-formula. Since the discriminator

$$4\left(DR_b - Bz_\alpha^2\right)^2 - 4\left(Cz_\alpha^2 - D\right)\left(Az_\alpha^2 - DR_b^2\right) = 4Dz_\alpha^2\left(A - 2BR_b + CR_b^2 - z_\alpha^2\right),$$

and $Cz_\alpha^2 - D > 0$, applying the abc-formula gives

$$\tilde{E} = \frac{2\left(Bz_\alpha^2 - DR_b\right) + 2\sqrt{Dz_\alpha^2\left(A - 2BR_b + CR_b^2 - z_\alpha^2\right)}}{2\left(Cz_\alpha^2 - D\right)},$$

which is just the desired Eq. (13).

By using the well known expression of the mean-variance efficient portfolio (see Wang & Xia 2002)[24], the portfolio at the intersection point T is obtained by Eq. (12). Also, Eq. (16) implies that the Sharpe ratio of the TSF portfolio is $SR_{tsf} = -z_\alpha$. We thus finish the proof of Part (i).

Part (ii) is a special case of Part (i). If $z_\alpha < -\sqrt{\frac{D}{C}}$ and $R_b = \frac{B-\sqrt{Cz_\alpha^2 - D}}{C}$, then $z_\alpha^2 = A - 2BR_b + CR_b^2$. Thus, Eq. (13) is reduced to Eq. (15), and Eq. (14) is obtained from Eq. (12) by replacing $\tilde{E}$ with Eq. (15). □

Proposition 4.4 implies that the TSF rule, like the RSF rule, is not suitable for portfolio choice in the market situation where $R_b \geq \frac{B}{C}$.

Proposition 4.5. *Given $R_b < \frac{B}{C}$, then the following holds.*
(i) if and only if $-\sqrt{A - 2BR_b + CR_b^2} \leq z_\alpha < -\sqrt{D/C}$, the TSF portfolio exists and $SR_{tsf} \leq SR_{ksf}$.
(ii) if and only if $z_\alpha = -\sqrt{A - 2BR_b + CR_b^2}$, $SR_{tsf} = SR_{ksf}$.

Proof. (i) Notice that $C\left(A - 2BR_b + CR_b^2\right) = D + \left(B - R_bC\right)^2$, and thus

$$-\sqrt{A - 2BR_b + CR_b^2} \leq z_\alpha$$

is equivalent to $\left(B - R_bC\right)^2 \geq Cz_\alpha^2 - D$. Therefore, when $R_b < \frac{B}{C}$,

$$-\sqrt{A - 2BR_b + CR_b^2} \leq z_\alpha < -\sqrt{\frac{D}{C}}$$

is equivalent to $z_\alpha < -\sqrt{\frac{D}{C}}$ and $R_b \leq \frac{B-\sqrt{Cz_\alpha^2 - D}}{C}$ together. Therefore by Proposition 4.4, the TSF portfolio exists and $SR_{tsf} = -z_\alpha$.

By Definition 2.1 and Eq. (5), we have

$$SR_{ksf} = \frac{\mu_{ksf} - R_b}{\sigma_{ksf}} = \frac{\mu_{ksf} - R_{ksf}}{\sigma_{ksf}} + \frac{R_{ksf} - R_b}{\sigma_{ksf}} \geq \frac{\mu_{ksf} - R_{ksf}}{\sigma_{ksf}}, \tag{18}$$

since $R_b \leq \frac{B-\sqrt{Cz_\alpha^2 - D}}{C}$. Recalling Eqs. (5) and (6), we obtain

$$\begin{aligned} \mu_{ksf} - R_{ksf} &= \frac{B}{C} + \frac{D}{C\sqrt{Cz_\alpha^2 - D}} - \frac{B - \sqrt{Cz_\alpha^2 - D}}{C} \\ &= \frac{z_\alpha^2}{\sqrt{Cz_\alpha^2 - D}} = -z_\alpha \frac{-z_\alpha}{\sqrt{Cz_\alpha^2 - D}} = -z_\alpha \sigma_{ksf}. \end{aligned} \tag{19}$$

Thus, combining Eqs. (18) and (19), we get $SR_{ksf} \geq -z_\alpha$. That is $SR_{tsf} \leq SR_{ksf}$.

For the proof of (ii), notice that $z_\alpha = -\sqrt{A - 2BR_b + CR_b^2}$ is equivalent to $R_b = \frac{B - \sqrt{Cz_\alpha^2 - D}}{C} = R_{ksf}$ when $R_b < \frac{B}{C}$. On the other hand, by Eqs. (18) and (19),

$$SR_{ksf} = \frac{\mu_{ksf} - R_b}{\sigma_{ksf}} = -z_\alpha + \frac{R_{ksf} - R_b}{\sigma_{ksf}}.$$

Therefore, $SR_{tsf} = SR_{ksf} = -z_\alpha$ if and only if $z_\alpha = -\sqrt{A - 2BR_b + CR_b^2}$. □

It follows from part (i) of Proposition 4.5 that if $R_b < \frac{B}{C}$, the KSF rule can help investors to find optimal portfolios, but the TSF rule cannot, when $z_\alpha < -\sqrt{A - 2BR_b + CR_b^2}$. Even in the case where the optimal TSF portfolio exists, it cannot perform better than the KSF portfolio in terms of the Sharpe ratio. Part (ii) of Proposition 4.5 implies that there is a unique special case where the KSF and the TSF optimal portfolios are the same. This special case is just the situation that the KSF and the RSF optimal portfolios are the same, implied by Proposition 4.2. In addition, when $R_b < \frac{B}{C}$, in the case that $-\sqrt{A - 2BR_b + CR_b^2} \le z_\alpha < -\sqrt{\frac{D}{C}}$, the optimal KSF portfolio is safer than the optimal TSF portfolio with respect to the referenced rate of return, because $R_b < R_d$ implies $P\left(R'X_{ksf} \le R_b\right) < P\left(R'X_{ksf} \le R_d\right) \le \alpha = P\left(R'X_{tsf} \le R_b\right)$.

Proposition 4.6. *Let R_d be the secured rate of return optimized by the KSF rule with a given α satisfying $0 < \alpha < F(-\sqrt{\frac{D}{C}})$, and set $P\left(R'X_{ksf}\left(\alpha\right) \le R_b\right) = \bar{\alpha}$. If the TSF portfolio exists and $P\left(R'X_{tsf}\left(\bar{\alpha}\right) \le R_b\right) = \bar{\alpha}$, then the Sharpe ratio of the TSF portfolio equals that of the KSF portfolio.*

Proof. According to Proposition 4.4, the Sharpe ratio of the TSF portfolio with $P\left(R'X_{tsf}\left(\bar{\alpha}\right) \le R_b\right) = \bar{\alpha}$ is $SR_{tsf}\left(\bar{\alpha}\right) = -z_{\bar{\alpha}}$. Since $P\left(R'X_{ksf}\left(\alpha\right) \le R_b\right) = \bar{\alpha}$, it follows that

$$P\left(\frac{R'X_{ksf}\left(\alpha\right) - \mu'X_{ksf}\left(\alpha\right)}{\sqrt{X'_{ksf}\left(\alpha\right)\Sigma X'_{ksf}\left(\alpha\right)}} \le \frac{R_b - \mu'X_{ksf}\left(\alpha\right)}{\sqrt{X'_{ksf}\left(\alpha\right)\Sigma X'_{ksf}\left(\alpha\right)}}\right) = \bar{\alpha}.$$

Hence $\frac{R_b - \mu'X_{ksf}(\alpha)}{\sqrt{X'_{ksf}(\alpha)\Sigma X'_{ksf}(\alpha)}} = F^{-1}\left(\bar{\alpha}\right) = z_{\bar{\alpha}}$, which gives that $SR_{ksf}\left(\alpha\right) = -z_{\bar{\alpha}} = SR_{tsf}\left(\bar{\alpha}\right)$. □

5. **Numerical comparisons.** In this section we are demonstrating the numerical comparisons of the outcomes given in the above section using a set of real data. We examine the risky assets market, which consists of stocks, funds, real estate and bonds. According to the Great Intelligence Software Database, we collected the monthly closing data for the SSE 180 Index (1B0010), the Securities Fund Index (1B0011), the Real Estate Index (1B0006), the Treasury Bond Index (1B0012) and the Enterprises Bond Index (1B0013) for the period from June 2003 to December 2011. Since the sample data cannot reject the null hypothesis of the normal distribution, we consider the joint distribution of the returns of assets as a normal distribution. In our computation, the bank deposit and loan interest rates are based on the annual interest rate of financial institutions at the beginning of each year, and the average monthly calculation is adopted. Based on the bank data from June 2003 to December 2011, the monthly deposit and loan rates are $r_l = 0.00210$ and $r_b = 0.00482$, respectively. MathCAD 2000 was used for conducting the calculations and graphs in this section and former sections.

5.1. **Case of $R_b \leq \frac{B}{C}$.** In this case, we take the given reference rate of return $R_b = r_l = 0.00210$ and consider three risky assets including the SSE 180 Index, the Securities Fund Index and the Real Estate Index. Table 1 displays the expected returns and covariance of them, which are estimated from the sample data.

TABLE 1. Expected Returns and Covariance of Three Risky Assets

Covariance	1B0010	1B0011	1B0006	Returns
1B0010	0.00863	0.00657	0.00830	0.00497
1B0011	0.00657	0.00609	0.00648	0.01214
1B0006	0.00830	0.00648	0.01390	0.00613

By computation we have $A = 0.06703$, $B = 2.40884$, $C = 168.23326$, $D = 5.47400$, and thus $\frac{B}{C} = 0.01432$, $\sqrt{\frac{D}{C}} = 0.18038$, $\sqrt{A - 2BR_b + CR_b^2} = 0.24011$. As $R_b < \frac{B}{C}$, the optimal RSF portfolio exists and its weight vector is $X_{rsf} = (-2.51889, 3.51014, 0.00875)'$, where the corresponding $P(X'_{rsf}R \leq R_b)$ is 0.40512, with the Sharpe ratio of 0.24011. Further, the TSF rule is only useful for an investor with a given α satisfying $0.40512 < \alpha < 0.42843$, say for $\alpha = 0.41000$, the optimal TSF portfolio is $X_{tsf} = (-4.95741, 5.94221, 0.01520)'$ with the Sharpe ratio of 0.22754. But the KSF rule is useful for an investor with a larger range of α satisfying $0 < \alpha < 0.42843$, say for $\alpha = 0.10000$, the optimal KSF portfolio is $(-0.58263, 1.57900, 0.00363)'$ with the Sharpe ratio of 0.18229. Figure 2 demonstrates the characteristics of the Sharpe ratio of a KSF portfolio against α, where the KSF portfolio with a predetermined $\alpha = 0.40512$ has the maximum Sharpe ratio of 0.24011.

In Table 2 we list part of the optimized portfolios under the RSF, TSF and KSF rules. In each panel, $R_b \equiv 0.00210$, the KSF rule decides R_d and the KSF portfolio satisfying $P(X'R \leq R_d) = \alpha$, and the TSF rule gives the TSF portfolio satisfying $P(X'R \leq R_b) = \alpha$, for a given α. In the table, the notation N stands for no existence of a finitely optimal portfolio. Different panels in Table 2 are based on different values of α. The portfolios by different rules are based on the same given α in each of Panels 1-7. Panels 8-11 are used to illustrate Proposition 10.

It can be seen from Table 2 that:

(1) There is one special case where the three safety-first rules give the same optimized portfolio, that is $\alpha = \Phi\left(-\sqrt{A - 2BR_b + CR_b^2}\right) \approx 0.40512$, See Panel 1.

(2) By comparing $P(R'X \leq R_b)$ of the portfolios by the three rules in each of Panels 2-4 (with three different values of α), the RSF portfolio is safest among the three SF portfolios, and the KSF portfolio is safer than the TSF portfolio, in view of the extreme risk prevention with respect to the given referenced rate of return R_b. Figure 3 illustrates $tP(\alpha) = P(X'R \leq R_b)$ vs. α, where X is optimized portfolios by RSF rule, TSF rule or KSF rule.

For a given α, let R_d is the secured rate of return optimized by the KSF rule. By comparing $P(R'X \leq R_d)$ of the portfolios by the three rules in each panel from Panel 2 to Panel 4, the RSF portfolio is safer than the TSF portfolio and the KSF portfolio is safest in view of the extreme risk prevention with respect to the optimal secured rate of return R_d at α. Figure 4 compares the $P(X'R \leq R_d)$ between KSF and RSF Portfolios.

(3) From the last columns of Panels 2-4, we can see that with the same given α, the optimized portfolio by the KSF rule always has a higher Sharpe ratio than that

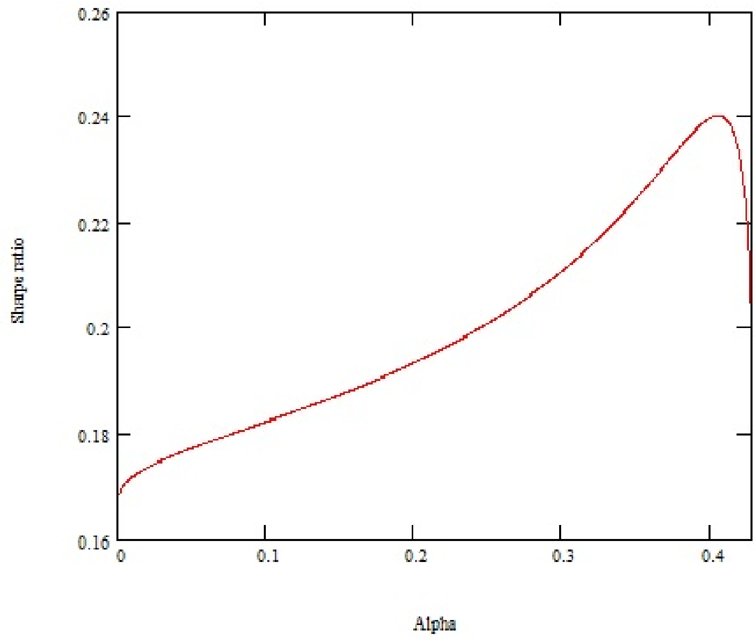

FIGURE 2. Characteristics of Sharpe ratio of KSF portfolio vs. α when $R_b < \frac{B}{C}$

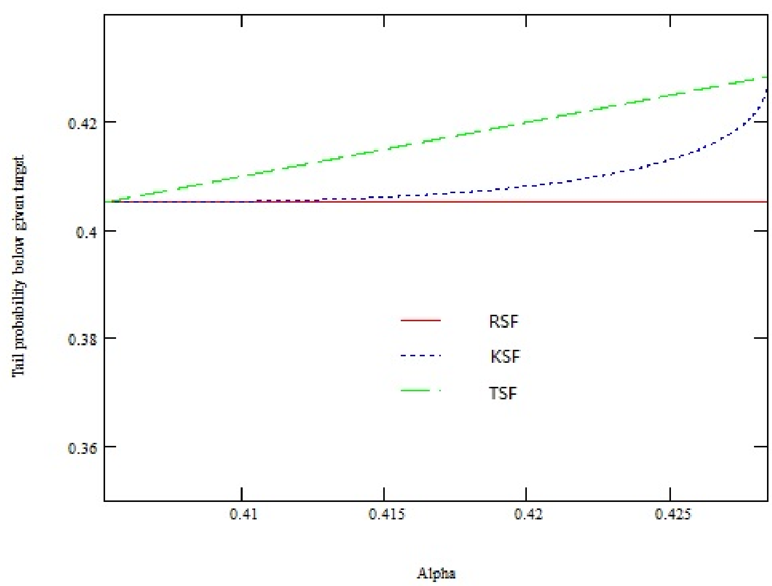

FIGURE 3. Comparison of $P(X'R \leq R_b)$ among the SF Portfolios

TABLE 2. Some results of RSF, TSF and KSF portfolios

Panel	Rule	$P(R'X \le R_d)$	$P(R'X \le R_b)$	X_1	X_2	X_3	SR
1	RSF	0.40512	0.40512	-2.51889	3.51014	0.00875	0.24011
	TSF	0.40512	0.40512	-2.51889	3.51014	0.00875	0.24011
	KSF	0.40512	0.40512	-2.51889	3.51014	0.00875	0.24011
2	RSF	0.41019	0.40512	-2.51889	3.51014	0.00875	0.24011
	TSF	0.41297	0.41000	-4.95741	5.94221	0.01520	0.22754
	KSF	0.41000	0.40532	-2.83436	3.82477	0.00959	0.23960
3	RSF	0.41778	0.40512	-2.51889	3.51014	0.00875	0.24011
	TSF	0.42074	0.41650	-8.91672	9.89105	0.02567	0.21086
	KSF	0.41650	0.40657	-3.51776	4.50637	0.01139	0.23638
4	RSF	0.42255	0.40512	-2.51889	3.51014	0.00875	0.24011
	TSF	0.42393	0.42000	-13.26792	14.23075	0.03718	0.20189
	KSF	0.42000	0.40813	-4.17332	5.16010	0.01313	0.23235
5	RSF	0.11226	0.40512	-2.51889	3.51014	0.00875	0.24011
	TSF	N	0.05000	N	N	N	N
	KSF	0.05000	0.42963	-0.52073	1.51727	0.00347	0.17730
6	RSF	0.20922	0.40512	-2.51889	3.51014	0.00875	0.24011
	TSF	N	0.15000	N	N	N	N
	KSF	0.15000	0.42565	-0.64983	1.64602	0.00381	0.18745
7	RSF	0.40016	0.40512	-2.51889	3.51014	0.00875	0.24011
	TSF	N	0.35000	N	N	N	N
	KSF	0.35000	0.41119	-1.33611	2.33049	0.00562	0.22449
8	RSF	0.43132	0.40512	-2.51889	3.51014	0.00875	0.24011
	TSF	0.42500	0.41307	-6.47843	7.45921	0.01922	0.21966
	KSF	0.42500	0.41307	-6.47843	7.45921	0.01922	0.21966
9	RSF	0.32258	0.40512	-2.51889	3.51014	0.00875	0.24011
	TSF	0.38805	0.41650	-8.91672	9.89105	0.02567	0.21086
	KSF	0.30000	0.41650	-1.01839	2.01361	0.00478	0.21086
10	RSF	0.24886	0.40512	-2.51889	3.51014	0.00875	0.24011
	TSF	0.40088	0.42330	-22.61759	23.55569	0.06190	0.19346
	KSF	0.20000	0.42330	-0.73281	1.72878	0.00403	0.19346
11	RSF	0.16529	0.40512	-2.51889	3.51014	0.00875	0.24011
	TSF	0.42240	0.42768	-160.68449	161.25747	0.42702	0.18229
	KSF	0.10000	0.42768	-0.58263	1.57900	0.00363	0.18229

by the TSF rule. See Figure 5 which displays the Sharpe ratios of the RSF, KSF and TSF portfolios against the given α. Panels 5-7 show that the TSF rule cannot give the optimized portfolios in some situations, and the KSF rule can provide optimized portfolios with less extreme risks than RSF rule.

(4) From Panels 8-11, among the TSF and KSF portfolios with the same level of the Sharpe ratio and the loss probability that the return occurs below R_b, the KSF portfolio is safer than the TSF portfolio (if they are different) in view of the extreme risk prevention with respect to the optimal secured rate of return R_d. See Figure 6 for an illustration, where the y-axis is $rP(a) = P(R'X \le R_d(a))$, with $R_d(a)$ the optimal secured rate of return obtained by the KSF rule for the given $\alpha = a$, $rP(a) = P\left(R'X_{ksf}\left(a\right) \le R_d\left(a\right)\right)$ for KSF portfolio, and $rP\left(a\right) = P\left(R'X_{tsf}\left(\bar{a}\right) \le R_d\left(a\right)\right)$ for TSF portfolio with $P\left(R'X_{ksf}\left(a\right) \le R_b\right) = \bar{a}$. Here, for each $a \ge 0.40512$, the TSF portfolio and the KSF portfolio are the same.

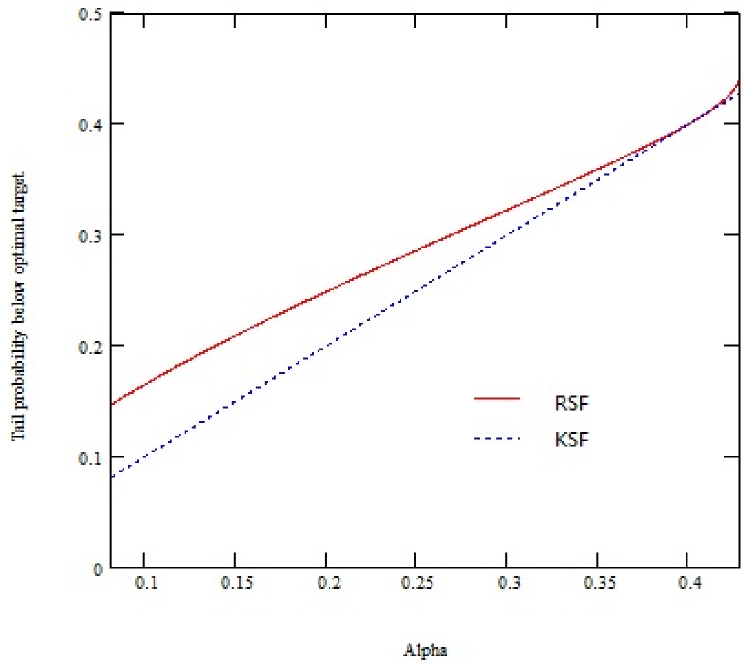

FIGURE 4. Comparison of $P(X'R \leq R_d)$ between KSF and RSF Portfolios

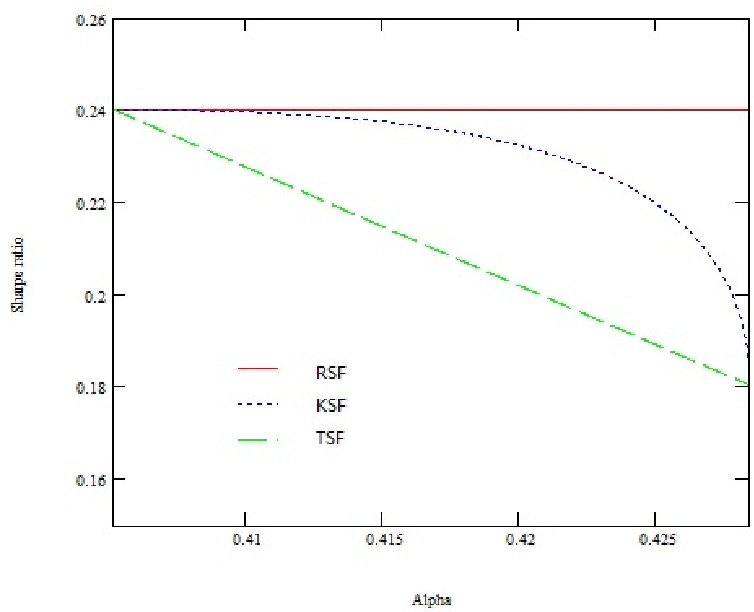

FIGURE 5. Comparison of Sharpe ratios of RSF, KSF and TSF portfolios

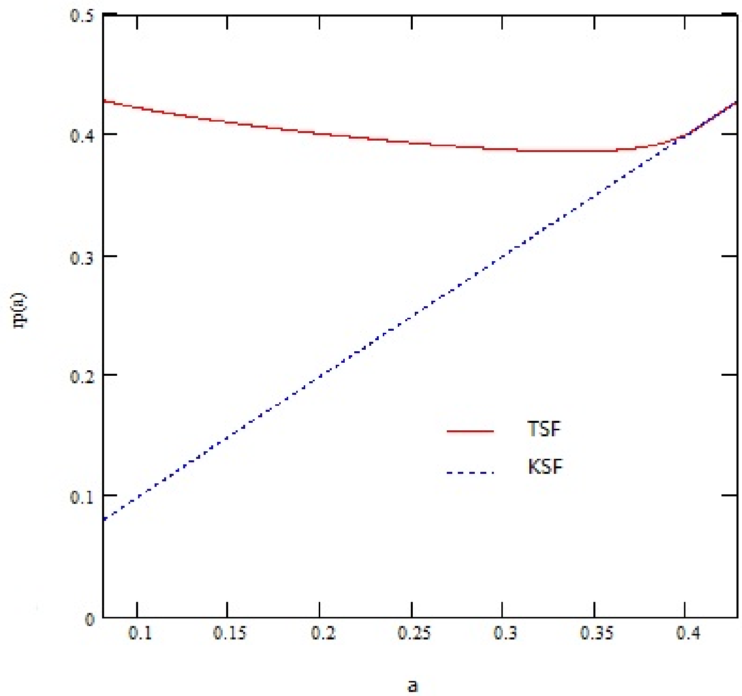

FIGURE 6. Extreme risks of KSF and TSF Portfolios with Equal Sharpe Ratios

5.2. **Case of $R_b > \frac{B}{C}$.** We take $R_b = r_b = 0.00482$ as the given benchmark rate of return. In order to further demonstrate the advantage of the KSF rule over the others, we consider all the five alternative risky assets for portfolio choice including the Treasury Bond Index (1B0012) and the Enterprises Bond Index (1B0013). The covariance matrix and the expected rates of returns are provided in Table 3.

TABLE 3. Expected Returns and Covariance of Five Risky Assets

Covariance	1B0012	1B0013	1B0010	1B0011	1B0006	Returns
1B0012	8.62E-05	1.07E-04	-4.22E-05	-5.38E-05	-5.25E-05	0.00246
1B0013	1.07E-04	1.72E-04	-1.62E-04	-1.61E-04	-5.79E-05	0.00351
1B0010	-4.22E-05	-1.62E-04	0.00863	0.00657	0.00830	0.00497
1B0011	-5.38E-05	-1.61E-04	0.00657	0.00609	0.00648	0.01214
1B0006	-5.25E-05	-5.79E-05	0.00830	0.00648	0.01390	0.00613

By calculations, we get $A = 0.15994$, $B = 29.87016$, $C = 13441.80372$ and $D = 1257.69763$. Thus $\frac{B}{C} = 0.00222$, and $R_b > \frac{B}{C}$. Note that in this case, both the RSF and the TSF rules are not applicable, but the KSF rule can apply, to get the optimized safety-first portfolio for a given α in $0 < \alpha < 0.37985 = \Phi\left(-\sqrt{\frac{D}{C}}\right)$, where $\sqrt{\frac{D}{C}} = 0.30589$ is the maximal Sharpe ratio. Further, by Eq. (10), the constructed portfolio is

$$X_{rsf}^{*} = (2.41222, -1.35322, 0.10655, -0.19240, 0.02685)'$$

with its Sharpe ratio $SR^*_{rsf} = -0.42928 = -\sqrt{A - 2BR_b + CR_b^2}$ and $P(\mu' X^*_{rsf} \leq R_b) = 0.66614$. While by Eqs. (12) and (13) with $\alpha = 0.35$, the constructed portfolio is

$$X^*_{tsf} = (1.77913, -0.76800, 0.02107, -0.04816, 0.01597)'$$

with its Sharpe ratio $SR^*_{tsf} = -0.38532$ and $P\left(\mu' X^*_{tsf} \leq R_b\right) = 0.65000$. Obviously, X^*_{tsf} is not a TSF optimized portfolio since it does not satisfy the safety-first constraint, while X^*_{rsf} is not a RSF optimized portfolio since X^*_{tsf} has a higher Sharpe ratio with a smaller tail probability than X^*_{rsf} has.

TABLE 4. Expected Returns and Covariance of Five Risky Assets

weight	α						
	0.10000	0.20000	0.30000	0.35000	0.36000	0.37000	0.37984
X_1	1.21251	1.07523	0.76327	0.20470	-0.11043	-0.81445	-93.92237
X_2	-0.24424	-0.11734	0.17103	0.68736	0.97865	1.62943	87.69587
X_3	-0.05544	-0.07397	-0.11610	-0.19151	-0.23406	-0.32912	-12.90054
X_4	0.08093	0.11220	0.18327	0.31053	0.38232	0.54272	21.75489
X_5	0.00624	0.00388	-0.00148	-0.01107	-0.01648	-0.02858	-1.62785
SR	-0.21947	-0.16942	-0.06622	0.05969	0.10400	0.16515	0.30287

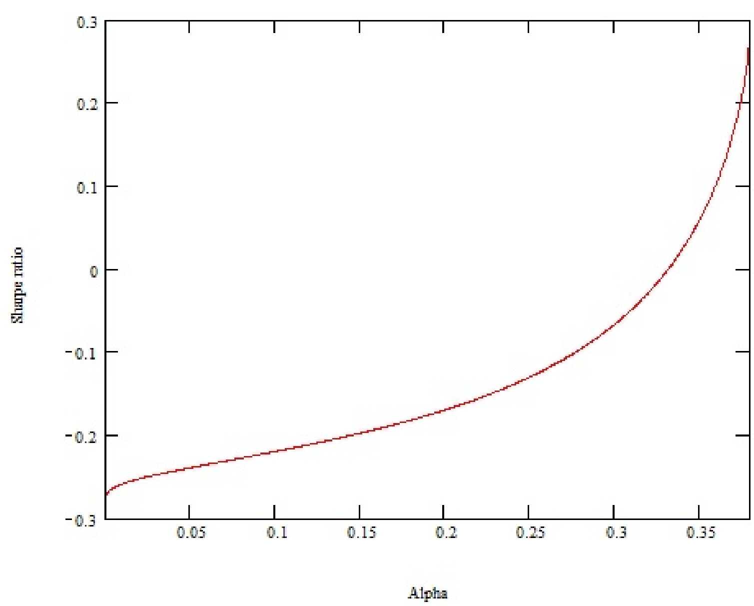

FIGURE 7. Characteristics of Sharpe ratio of KSF portfolio vs. α when $R_b > \frac{B}{C}$

By Proposition 4.3, the RSF portfolio with the maximum Sharpe ratio of 0.30589 cannot be constructed finitely, but the limit of a series of the optimized KSF portfolios can approach the maximum Sharpe portfolio. See Table 4 for a series of the

KSF portfolios as an illustration, depicted in Figure 7. It is clearly noticed that the Sharpe ratio of a KSF portfolio depends increasingly on the predetermined level of tolerance of extreme risk, α.

6. **Conclusion.** It is well know that the Roy safey-first optimization does not depend on the loss probability, while both the Telser safety-first and the Kataoka safety-first optimizations depend on the predetermined loss probability or the acceptable tolerance level to extreme risk. To the best of our knowledge, no literature was given with detailed theoretical comparison on the three classical safety-first rules, especially between the KSF rule and the TSF rule. This paper compares the three safety-first rules and assesses the KSF portfolio in view of the Sharpe ratio, with numerical demonstration by using a set of real data presented. Given A, B, C, and D the market parameters determined only by the financial market, a given reference return rate R_b either for a risk-free asset or the market portfolio, and $z_\alpha = F^{-1}(\alpha)$ with a loss probability α and a cumulative distribution function F without involving parameter, the novel findings in the above sections can be summarized as follows.

Firstly, in the market situation $R_b \geq \frac{B}{C}$, either the RSF rule or the TSF rule cannot be used to find an unique and finite optimal portfolio but the KSF rule can if and only if the investor has a mild tolerance level to the extreme risk such that $z_\alpha < -\sqrt{\frac{D}{C}}$. In this situation, the higher the tolerance level to extreme risk is, the higher the Sharpe ratio of the KSF portfolio is. Also the Sharpe portfolio with the finite maximum Sharpe ratio of $\sqrt{\frac{D}{C}}$ does not exist, but a series of KSF portfolios with different α can be constructed so that the limit of the Sharpe ratios of these KSF portfolios reaches the maximum Sharpe ratio. It implies that the KSF rule can be applied in wider market situations than the others do.

Secondly, in the market situation $R_b < \frac{B}{C}$, the TSF rule can only be used to get the optimal portfolio under the condition that $-\sqrt{A - 2BR_b + CR_b^2} \leq z_\alpha < -\sqrt{\frac{D}{C}}$, but the KSF rule can be used under the condition that $z_\alpha < -\sqrt{\frac{D}{C}}$. It implies that the KSF rule suits for more investors to make decision on portfolio choice than the TSF rule does.

Thirdly, in the market situation $R_b < \frac{B}{C}$, there is a unique special case that $z_\alpha = -\sqrt{A - 2BR_b + CR_b^2}$, where the three safety-first portfolios are the same with the maximum Sharpe ratio $\sqrt{A - 2BR_b + CR_b^2}$.

Fourthly, in the market situation of $R_b < \frac{B}{C}$, when $-\sqrt{A - 2BR_b + CR_b^2} \leq z_\alpha < -\sqrt{\frac{D}{C}}$, both TSF and KSF rules can be used to get optimized portfolio, but the KSF portfolio always has a higher Sharpe ratio than the TSF portfolio. In addition, among the TSF and KSF portfolios with the same level of the Sharpe ratio and the loss probability that the return occurs below R_b, the KSF portfolio has less or equal extreme loss probability than the TSF portfolio. It implies that the KSF rule performs better than the TSF rule in view of either Sharpe ratio or extreme risk prevention.

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