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Homotopy types of $SU(n)$-gauge groups over non-spin 4-manifolds

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Abstract
Let M be an orientable, simply-connected, closed, non-spin 4-manifold and let $\mathcal{G}_k(M)$ be the gauge group of the principal G-bundle over M with second Chern class $k \in \mathbb{Z}$. It is known that the homotopy type of $\mathcal{G}_k(M)$ is determined by the homotopy type of $\mathcal{G}_k(\mathbb{CP}^2)$. In this paper we investigate properties of $\mathcal{G}_k(\mathbb{CP}^2)$ when $G = SU(n)$ that partly classify the homotopy types of the gauge groups.

Keywords Gauge groups · Homotopy type · Non-spin 4-manifolds

Mathematics Subject Classification Primary 55P15; Secondary 54C35 · 81T13

1 Introduction

Let G be a simple, simply-connected, compact Lie group and let M be an orientable, simply-connected, closed 4-manifold. Then the isomorophism class of a principal G-bundle P over M is classified by its second Chern class $k \in \mathbb{Z}$. In particular, if $k = 0$, then P is a trivial G-bundle. The associated gauge group $\mathcal{G}_k(M)$ is the topological group of G-equivariant automorphisms of P which fix M.

A simply-connected 4-manifold is spin if and only if its intersection form is even. In the case of simply-connected 4-manifolds, the spin condition is equivalent to all cup product squares being trivial in mod 2 cohomology. In this paper, we consider the homotopy types of gauge groups $\mathcal{G}_k(M)$, where M is a non-spin 4-manifold such as $\mathbb{CP}^2$. When M is a spin 4-manifold, topologists have been studying the homotopy types of gauge groups over M extensively over the last twenty years. On the one hand, Theriault showed in [16] that there is a homotopy equivalence

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$$\mathcal{G}_k(M) \simeq \mathcal{G}_k(S^4) \times \prod_{i=1}^{d} \Omega^2 G,$$

where d is the second Betti number of M. Therefore to study the homotopy type of $\mathcal{G}_k(M)$ it suffices to study $\mathcal{G}_k(S^4)$. On the other hand, many cases of homotopy types of $\mathcal{G}_k(S^4)$'s are known. For examples, there are 6 distinct homotopy types of $\mathcal{G}_k(S^4)$'s for $G = SU(2)$ [11], and 8 distinct homotopy types for $G = SU(3)$ [5]. When localized rationally or at any prime, there are 16 distinct homotopy types for $G = SU(5)$ [19] and 8 distinct homotopy types for $G = Sp(2)$ [17].

When M is a non-spin 4-manifold, the author in [14] showed that there is a homotopy equivalence

$$\mathcal{G}_k(M) \simeq \mathcal{G}_k(\mathbb{CP}^2) \times \prod_{i=1}^{d-1} \Omega^2 G,$$

so the homotopy type of $\mathcal{G}_k(M)$ depends on the special case $\mathcal{G}_k(\mathbb{CP}^2)$. Compared to the extensive work on $\mathcal{G}_k(S^4)$, only two cases of $\mathcal{G}_k(\mathbb{CP}^2)$ have been studied, which are the $SU(2)$- and $SU(3)$-cases [12,18]. As a sequel to [14], this paper investigates the homotopy types of $\mathcal{G}_k(\mathbb{CP}^2)$'s in order to explore gauge groups over non-spin 4-manifolds.

A common approach to classifying the homotopy types of gauge groups is as follows. Atiyah, Bott and Gottlieb [1,3] showed that the classifying space $B\mathcal{G}_k(M)$ is homotopy equivalent to the connected component $\mathrm{Map}_k(M, BG)$ of the mapping space $\mathrm{Map}(M, BG)$ containing the map $k\alpha \circ q$, where $q : M \to S^4$ is the quotient map and α is a generator of $\pi_4(BG) \cong \mathbb{Z}$. The evaluation map $ev : B\mathcal{G}_k(M) \to BG$ induces a fibration sequence

$$\mathcal{G}_k(M) \longrightarrow G \xrightarrow{\partial_k} \mathrm{Map}_k^*(M, BG) \longrightarrow B\mathcal{G}_k(M) \xrightarrow{ev} BG, \tag{1}$$

where $\partial_k : G \to \mathrm{Map}_k^*(M, BG)$ is the boundary map. The action of $\pi_4(BG) \cong \mathbb{Z}$ on $\mathrm{Map}_k^*(M, BG)$ induces a homotopy equivalence $\mathrm{Map}_k^*(M, BG) \simeq \mathrm{Map}_0^*(M, BG)$. Denote the composition $G \xrightarrow{\partial_k} \mathrm{Map}_k^*(M, BG) \simeq \mathrm{Map}_0^*(M, BG)$ also by ∂_k for convenience. For $M = S^4$, $\mathrm{Map}_0^*(M, BG) \simeq \Omega_0^3 G$ is an H-group so $[G, \Omega_0^3 G]$ is a group. The order of $\partial_1 : G \to \Omega_0^3 G$ is important for distinguishing the homotopy types of $\mathcal{G}_k(S^4)$.

Theorem 1.1 (Theriault, [17]) *Let m be the order of ∂_1. If $(m, k) = (m, l)$, then $\mathcal{G}_k(S^4)$ is homotopy equivalent to $\mathcal{G}_l(S^4)$ when localized rationally or at any prime.*

For most cases of G, the exact value of the order of ∂_1 is difficult to compute. When $G = SU(n)$, the exact value or a partial result of the order of ∂_1 was worked out for certain cases. For any number $a = p^r q$ where q is coprime to p, the p-component of a is p^r and is denoted by $\nu_p(a)$.

Theorem 1.2 ([2,5,9,11,19,20]) *Let G be $SU(n)$ and let m be the order of ∂_1. Then*

- $m = 12$ *for* $n = 2$
- $m = 24$ *for* $n = 3$
- $m = 120$ *for* $n = 5$
- $m = 60$ *or 120 for* $n = 4$
- $\nu_p(m) = \nu_p(n(n^2-1))$ *for* $n < (p-1)^2 + 1$.

In Theorem 1.1, the g.c.d condition $(m, k) = (m, l)$ gives a sufficient condition for the homotopy equivalence $\mathcal{G}_k(S^4) \simeq \mathcal{G}_l(S^4)$. Conversely, there is a partial necessary condition for certain cases of $G = SU(n)$.

Theorem 1.3 (Hamanaka and Kono [5]; Kishimoto, Kono and Tsutaya [9]) *Let G be $SU(n)$ and let p be an odd prime. If $\mathcal{G}_k(S^4)$ is homotopy equivalent to $\mathcal{G}_l(S^4)$, then*

- $(n(n^2-1), k) = (n(n^2-1), l)$ *for n odd,*
- $\nu_p(n(n^2-1), k) = \nu_p(n(n^2-1), l)$ *for n less than* $(p-1)^2 + 1$.

In this paper we consider gauge groups over $\mathbb{CP}^2$. Take $M = \mathbb{CP}^2$ in (1) and denote the boundary map by $\partial'_k : G \to \mathrm{Map}_0^*(\mathbb{CP}^2, BG)$. Since $\mathrm{Map}_0^*(\mathbb{CP}^2, BG)$ is not an H-space, $[G, \mathrm{Map}_0^*(\mathbb{CP}^2, BG)]$ is not a group so the order of ∂'_k makes no sense. However, we can still define an "order" of ∂'_k [18], which will be described in Sect. 2. We show that the "order" of ∂'_1 helps distinguish the homotopy type of $\mathcal{G}_k(\mathbb{CP}^2)$ as in Theorem 1.1.

Theorem 1.4 *Let m' be the "order" of ∂'_1. If $(m', k) = (m', l)$, then $\mathcal{G}_k(\mathbb{CP}^2)$ is homotopy equivalent to $\mathcal{G}_l(\mathbb{CP}^2)$ when localized rationally or at any prime.*

We study the $SU(n)$-gauge groups over $\mathbb{CP}^2$ and use unstable K-theory to give a lower bound on the "order" of ∂'_1 that is in the spirit of Theorem 1.2.

Theorem 1.5 *When G is $SU(n)$, the "order" of ∂'_1 is at least $\frac{1}{2}n(n^2-1)$ for n odd, and $n(n^2-1)$ for n even.*

Localized rationally or at an odd prime, we have $\mathcal{G}_k(\mathbb{CP}^2) \simeq \mathcal{G}_k(S^4) \times \Omega^2 G$ [16]. The homotopy types of $\mathcal{G}_k(\mathbb{CP}^2)$ are then completely determined by that of $\mathcal{G}_k(S^4)$, which have been investigated in many cases when the localizing prime is relatively large [6,7,9,10,20]. A large part of the remaining cases can be understood by studying the 2-localized order of ∂'_1, on which Theorem 1.5 gives bounds for the $SU(n)$ case. For example, combining Theorem 1.5 with Lemma 2.2 implies the order of ∂'_1 is either 120 or 60 for $G = SU(5)$. Furthermore, when $G = SU(4)$ since the order of ∂_1 is either 120 or 60, the order of ∂'_1 is either 60 or 120.

Finally we prove a necessary condition for the homotopy equivalence $\mathcal{G}_k(\mathbb{CP}^2) \simeq \mathcal{G}_l(\mathbb{CP}^2)$ similar to Theorem 1.3.

Theorem 1.6 *Let G be $SU(n)$. If $\mathcal{G}_k(\mathbb{CP}^2)$ is homotopy equivalent to $\mathcal{G}_l(\mathbb{CP}^2)$, then*

- $(\frac{1}{2}n(n^2-1), k) = (\frac{1}{2}n(n^2-1), l)$ *for n odd,*
- $(n(n^2-1), k) = (n(n^2-1), l)$ *for n even.*

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2 Some facts about boundary map ∂'_1

Take M to be S^4 and $\mathbb{CP}^2$ respectively in fibration (1) to obtain fibration sequences

$$\mathcal{G}_k(S^4) \longrightarrow G \xrightarrow{\partial_k} \Omega^3_0 G \longrightarrow B\mathcal{G}_k(S^4) \xrightarrow{ev} BG \tag{2}$$

$$\mathcal{G}_k(\mathbb{CP}^2) \longrightarrow G \xrightarrow{\partial'_k} \mathrm{Map}^*_0(\mathbb{CP}^2, BG) \longrightarrow B\mathcal{G}_k(\mathbb{CP}^2) \xrightarrow{ev} BG. \tag{3}$$

There is also a cofibration sequence

$$S^3 \xrightarrow{\eta} S^2 \longrightarrow \mathbb{CP}^2 \xrightarrow{q} S^4, \tag{4}$$

where η is Hopf map and q is the quotient map. Due to the naturality of q^*, we combine fibrations (2) and (3) to obtain a commutative diagram of fibration sequences

$$\begin{tikzcd}
\mathcal{G}_k(S^4) \arrow[r] \arrow[d, "q^*"] & G \arrow[r, "\partial_k"] \arrow[d, equal] & \Omega^3_0 G \arrow[r] \arrow[d, "q^*"] & B\mathcal{G}_k(S^4) \arrow[r] \arrow[d, "q^*"] & BG \arrow[d, equal] \\
\mathcal{G}_k(\mathbb{CP}^2) \arrow[r] & G \arrow[r, "\partial'_k"] & \mathrm{Map}^*_0(\mathbb{CP}^2, BG) \arrow[r] & B\mathcal{G}_k(\mathbb{CP}^2) \arrow[r] & BG
\end{tikzcd} \tag{5}$$

It is known, [13], that ∂_k is triple adjoint to Samelson product

$$\langle k\imath, \mathbb{1}\rangle : S^3 \wedge G \xrightarrow{k\imath \wedge \mathbb{1}} G \wedge G \xrightarrow{\langle \mathbb{1}, \mathbb{1}\rangle} G,$$

where $\imath : S^3 \to SU(n)$ is the inclusion of the bottom cell and $\langle \mathbb{1}, \mathbb{1}\rangle$ is the Samelson product of the identity on G with itself. The order of ∂_k is its multiplicative order in the group $[G, \Omega^3_0 G]$.

Unlike $\Omega^3_0 G$, $\mathrm{Map}^*_0(\mathbb{CP}^2, BG)$ is not an H-space, so ∂'_k has no order. In [18], Theriault defined the "order" of ∂'_k to be the smallest number m' such that the composition

$$G \xrightarrow{\partial_k} \Omega^3_0 G \xrightarrow{m'} \Omega^3_0 G \xrightarrow{q^*} \mathrm{Map}^*_0(\mathbb{CP}^2, BG)$$

is null homotopic. In the following, we interpret the "order" of ∂'_k as its multiplicative order in a group contained in $[\mathbb{CP}^2 \wedge G, BG]$.

Apply $[- \wedge G, BG]$ to cofibration (4) to obtain an exact sequence of sets

$$[\Sigma^3 G, BG] \xrightarrow{(\Sigma\eta)^*} [\Sigma^4 G, BG] \xrightarrow{q^*} [\mathbb{CP}^2 \wedge G, BG].$$

All terms except $[\mathbb{CP}^2 \wedge G, BG]$ are groups and $(\Sigma\eta)^*$ is a group homomorphism since $\Sigma\eta$ is a suspension. We want to refine this exact sequence so that the last term is replaced by a group. Observe that $\mathbb{CP}^2$ is the cofiber of η and so there is a coaction $\psi : \mathbb{CP}^2 \to \mathbb{CP}^2 \vee S^4$. We show that the coaction gives a group structure on $Im(q^*)$.

Lemma 2.1 *Let Y be a space and let $A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} \Sigma A$ be a cofibration sequence. If ΣA is homotopy cocommutative, then $Im(h^*)$ is an abelian group and*

$$[\Sigma B, Y] \xrightarrow{(\Sigma f)^*} [\Sigma A, Y] \xrightarrow{h^*} Im(h^*) \longrightarrow 0$$

is an exact sequence of groups and group homomorphisms.

Proof Apply $[-, Y]$ to the cofibration to get an exact sequence of sets

$$[\Sigma B, Y] \xrightarrow{(\Sigma f)^*} [\Sigma A, Y] \xrightarrow{h^*} [C, Y]. \tag{6}$$

Note that $[\Sigma B, Y]$ and $[\Sigma A, Y]$ are groups, and $(\Sigma f)^*$ is a group homomorphism. We will replace $[C, Y]$ by $Im(h^*)$ and define a group structure on it such that $h^* : [\Sigma A, Y] \to Im(h^*)$ is a group homomorphism.

For any α and β in $[\Sigma A, Y]$, we define a binary operator $\boxtimes$ on $Im(h^*)$ by

$$h^*\alpha \boxtimes h^*\beta = h^*(\alpha + \beta).$$

To check this is well-defined we need to show $h^*(\alpha + \beta) \simeq h^*(\alpha' + \beta) \simeq h^*(\alpha + \beta')$ for any $\alpha, \alpha', \beta, \beta'$ satisfying $h^*\alpha \simeq h^*\alpha'$ and $h^*\beta \simeq h^*\beta'$.

First we show $h^*(\alpha + \beta) \simeq h^*(\alpha' + \beta)$. By definition, we have

$$h^*(\alpha + \beta) = (\alpha + \beta) \circ h = \nabla \circ (\alpha \vee \beta) \circ \sigma \circ h,$$

where $\sigma : \Sigma A \to \Sigma A \vee \Sigma A$ is the comultiplication and $\nabla : Y \vee Y \to Y$ is the folding map. Since C is a cofiber, there is a coaction $\psi : C \to C \vee \Sigma A$ such that $\sigma \circ h \simeq (h \vee \mathbb{1}) \circ \psi$.

$$\begin{tikzcd} C \arrow[r, "\psi"] \arrow[d, "h"] & C \vee \Sigma A \arrow[d, "h \vee \mathbb{1}"] \\ \Sigma A \arrow[r, "\sigma"] & \Sigma A \vee \Sigma A \end{tikzcd}$$

Then we obtain a string of equivalences

$$\begin{aligned} h^*(\alpha + \beta) &= \nabla \circ (\alpha \vee \beta) \circ \sigma \circ h \\ &\simeq \nabla \circ (\alpha \vee \beta) \circ (h \vee \mathbb{1}) \circ \psi \\ &\simeq \nabla \circ (\alpha' \vee \beta) \circ (h \vee \mathbb{1}) \circ \psi \\ &\simeq \nabla \circ (\alpha' \vee \beta) \circ \sigma \circ h \\ &= h^*(\alpha' + \beta) \end{aligned}$$

The third line is due to the assumption $h^*\alpha \simeq h^*\alpha'$. Therefore we have $h^*(\alpha + \beta) \simeq h^*(\alpha' + \beta)$. Since ΣA is cocommutative, $[\Sigma A, Y]$ is abelian and $h^*(\alpha + \beta) \simeq h^*(\beta + \alpha)$. Then we have

$$h^*(\alpha+\beta) \simeq h^*(\beta+\alpha) \simeq h^*(\beta'+\alpha) \simeq h^*(\alpha+\beta').$$

This implies $\boxtimes$ is well-defined.

Due to the associativity of $+$ in $[\Sigma A, Y]$, $\boxtimes$ is associative since

$$\begin{aligned}(h^*\alpha \boxtimes h^*\beta) \boxtimes h^*\gamma &= h^*(\alpha+\beta) \boxtimes h^*\gamma \\ &= h^*((\alpha+\beta)+\gamma) \\ &= h^*(\alpha+(\beta+\gamma)) \\ &= h^*\alpha \boxtimes h^*(\beta+\gamma) \\ &= h^*\alpha \boxtimes (h^*\beta \boxtimes h^*\gamma).\end{aligned}$$

Clearly the trivial map $* : C \to Y$ is the identity of $\boxtimes$ and $h^*(-\alpha)$ is the inverse of $h^*\alpha$. Therefore $\boxtimes$ is indeed a group multiplication.

By definition of $\boxtimes$, $h^* : [\Sigma A, Y] \to Im(h^*)$ is a group homomorphism, and hence an epimorphism. Since $[\Sigma A, Y]$ is abelian, so is $Im(h^*)$. We replace $[C, Y]$ by $Im(h^*)$ in (6) to obtain a sequence of groups and group homomorphisms

$$[\Sigma B, Y] \xrightarrow{(\Sigma f)^*} [\Sigma A, Y] \xrightarrow{h^*} Im(h^*) \longrightarrow 0.$$

The exactness of (6) implies $ker(h^*) = Im(\Sigma f)^*$, so the sequence is exact. $\square$

Applying Lemma 2.1 to cofibration $\Sigma^3 G \to \Sigma^2 G \to \mathbb{CP}^2 \wedge G$ and the space $Y = BG$, we obtain an exact sequence of abelian groups

$$[\Sigma^3 G, BG] \xrightarrow{(\Sigma\eta)^*} [\Sigma^4 G, BG] \xrightarrow{q^*} Im(q^*) \longrightarrow 0. \tag{7}$$

In the middle square of (5) $\partial'_k \simeq q^*\partial_k$, so ∂'_k is in $Im(q^*)$. For any number m, $q^*(m\partial_k) = mq^*\partial_k$, so the "order" of ∂'_k defined in [18] coincides with the multiplicative order of ∂'_k in $Im(q^*)$. The exact sequence (7) allows us to compare the orders of ∂_1 and ∂'_1.

Lemma 2.2 *Let m be the order of ∂_1 and let m' be the order of ∂'_1. Then m is m' or $2m'$.*

Proof By exactness of (7), there is some $f \in [\Sigma^3 G, BG]$ such that $(\Sigma\eta)^* f \simeq m'\partial_1$. Since $\Sigma\eta$ has order 2, $2m'\partial_1$ is null homotopic. It follows that $2m'$ is a multiple of m. Since m is greater than or equal to m', m is either m' or $2m'$. $\square$

When $G = SU(2)$, the order m of ∂_1 is 12 and the order m' of ∂'_1 is 6 [12]. When $G = SU(3)$, $m = 24$ and $m' = 12$ [18]. When $G = Sp(2)$, $m = 40$ and $m' = 20$ [15]. It is natural to ask whether $m = 2m'$ for all G.

In the S^4 case, Theorem 1.1 gives a sufficient condition for $\mathcal{G}_k(S^4) \simeq \mathcal{G}_l(S^4)$ when localized rationally or at any prime. In the $\mathbb{CP}^2$ case, Theriault showed a similar counting statement, in which the sufficient condition depends on the order of ∂_1 instead of ∂'_1.

Theorem 2.3 (Theriault, [18]) *Let m be the order of ∂_1. If $(m,k)=(m,l)$, then $\mathcal{G}_k(\mathbb{CP}^2)$ is homotopy equivalent to $\mathcal{G}_l(\mathbb{CP}^2)$ when localized rationally or at any prime.*

Lemma 2.2 can be used to improve the sufficient condition of Theorem 2.3.

Theorem 2.4 *Let m' be the order of ∂'_1. If $(m',k)=(m',l)$, then $\mathcal{G}_k(\mathbb{CP}^2)$ is homotopy equivalent to $\mathcal{G}_l(\mathbb{CP}^2)$ when localized rationally or at any prime.*

Proof By Lemma 2.2, m is either m' or $2m'$. If $m=m'$, then the statement is same as Theorem 2.3. Assume $m=2m'$. Localize at an odd prime p. Let p^r be the p-component of m, that is $m=p^r\cdot q$ where q is coprime to p. Observe that $m\circ\partial_1\simeq(p^r\cdot q)\circ\partial_1\simeq p^r\circ\partial_1$ since the power map $q:\Omega_0^3G\to\Omega_0^3G$ is a homotopy equivalence. Therefore p^r is the order of ∂_1 after localization. The hypothesis $(m',k)=(m',l)$ implies $(p^r,k)=(p^r,l)$, so a homotopy equivalence $\mathcal{G}_k(\mathbb{CP}^2)\simeq\mathcal{G}_l(\mathbb{CP}^2)$ follows by Theorem 2.3. A similar argument works for rational localization. Now it remains to consider the case where $m=2m'$ when localized at 2.

Assume $m=2^n$ and $m'=2^{n-1}$. For any k, $(2^{n-1},k)=2^i$ where i an integer such that $0\leq i\leq n-1$. If $i\leq n-2$, then $k=2^it$ for some odd number t and $(2^{n-1},k)=2^i$. The sufficient condition $(2^{n-1},k)=(2^{n-1},l)$ is equivalent to $(2^n,k)=(2^n,l)$. Again the homotopy equivalence $\mathcal{G}_k(\mathbb{CP}^2)\simeq\mathcal{G}_l(\mathbb{CP}^2)$ follows by Theorem 2.3. If $i=n-1$, then $(2^n,k)$ is either 2^n or 2^{n-1}. We claim that $\mathcal{G}_k(\mathbb{CP}^2)$ has the same homotopy type for both $(2^n,k)=2^n$ or $(2^n,k)=2^{n-1}$.

Consider fibration (3)

$$\mathrm{Map}_0^*(\mathbb{CP}^2,G)\longrightarrow\mathcal{G}_k(\mathbb{CP}^2)\longrightarrow G\xrightarrow{\partial'_k}\mathrm{Map}_0^*(\mathbb{CP}^2,BG).$$

If $(2^n,k)=2^n$, then $k=2^nt$ for some number t. By linearity of Samelson products, $\partial_k\simeq k\partial_1$. Since $\partial'_k\simeq q^*k\partial_1\simeq q^*2^nt\partial_1$ and ∂_1 has order 2^n, ∂'_k is null homotopic and we have

$$\mathcal{G}_k(\mathbb{CP}^2)\simeq G\times\mathrm{Map}_0^*(\mathbb{CP}^2,G).$$

If $(2^n,k)=2^{n-1}$, then $k=2^{n-1}t$ for some odd number t. Writing $t=2s+1$ gives $k=2^ns+2^{n-1}$. Since $\partial'_k\simeq q^*k\partial_1\simeq q^*(2^ns+2^{n-1})\partial_1\simeq q^*2^{n-1}\partial_1$ and ∂'_1 has order 2^{n-1}, ∂'_k is null homotopic and we have

$$\mathcal{G}_k(\mathbb{CP}^2)\simeq G\times\mathrm{Map}_0^*(\mathbb{CP}^2,G).$$

The same is true for $\mathcal{G}_l(\mathbb{CP}^2)$ and hence $\mathcal{G}_k(\mathbb{CP}^2)\simeq\mathcal{G}_l(\mathbb{CP}^2)$. $\square$

3 Plan for the proofs of Theorems 1.5 and 1.6

From this section onward, we will focus on $SU(n)$-gauge groups over $\mathbb{CP}^2$. There is a fibration

$$SU(n)\longrightarrow SU(\infty)\xrightarrow{p}W_n, \tag{8}$$

where $p : SU(\infty) \to W_n$ is the projection and W_n is the symmetric space $SU(\infty)/SU(n)$. Then we have

$$\begin{aligned}\tilde{H}^*(SU(\infty)) &= \Lambda(x_3, \ldots, x_{2n-1}, \ldots),\\ \tilde{H}^*(SU(n)) &= \Lambda(x_3, \ldots, x_{2n-1}),\\ \tilde{H}^*(BSU(n)) &= \mathbb{Z}[c_2, \ldots, c_n],\\ \tilde{H}^*(W_n) &= \Lambda(\bar{x}_{2n+1}, \bar{x}_{2n+3}, \ldots),\end{aligned}$$

where x_{2n+1} has degree $2n+1$, c_i is the ith universal Chern class and $x_{2i+1} = \sigma(c_{i+1})$ is the image of c_{i+1} under the cohomology suspension σ, and $p^*(\bar{x}_{2i+1}) = x_{2i+1}$. Furthermore, $H^{2n}(\Omega W_n) \cong \mathbb{Z}$ and $H^{2n+2}(\Omega W_n) \cong \mathbb{Z}$ are generated by a_{2n} and a_{2n+2}, where a_{2i} is the transgression of x_{2i+1}.

The $(2n+4)$-skeleton of W_n is $\Sigma^{2n-1}\mathbb{CP}^2$ for n odd, and is $S^{2n+3} \vee S^{2n+1}$ for n even, so its homotopy groups are as follows:

	$\pi_i(W_n)$			
i	$\leq 2n$	$2n+1$	$2n+2$	$2n+3$
n odd	0	$\mathbb{Z}$	0	$\mathbb{Z}$
n even	0	$\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$

(9)

The canonical map $\epsilon : \Sigma\mathbb{CP}^{n-1} \to SU(n)$ induces the inclusion $\epsilon_* : H_*(\Sigma\mathbb{CP}^{n-1}) \to H_*(SU(n))$ of the generating set. Let C be the quotient $\mathbb{CP}^{n-1}/\mathbb{CP}^{n-3}$ and let $\bar{q} : \Sigma\mathbb{CP}^{n-1} \to \Sigma C$ be the quotient map. Then there is a diagram

$$\begin{tikzcd}
{[\Sigma C, SU(n)]} \arrow[r, "(\partial'_k)_*"] \arrow[d, "\bar{q}^*"] & {[\Sigma C, \mathrm{Map}^*(\mathbb{CP}^2, BSU(n))]} \arrow[r] \arrow[d, "\bar{q}^*"] & {[\Sigma C, B\mathcal{G}_k(\mathbb{CP}^2)]} \arrow[d, "\bar{q}^*"] \\
{[\Sigma\mathbb{CP}^{n-1}, SU(n)]} \arrow[r, "(\partial'_k)_*"] & {[\Sigma\mathbb{CP}^{n-1}, \mathrm{Map}^*(\mathbb{CP}^2, BSU(n))]} \arrow[r] & {[\Sigma\mathbb{CP}^{n-1}, B\mathcal{G}_k(\mathbb{CP}^2)]},
\end{tikzcd}$$

where $(\partial'_k)_*$ sends f to $\partial'_k \circ f$ and the rows are induced by fibration (3). In particular, in the second row the map $\epsilon : \Sigma\mathbb{CP}^{n-1} \to SU(n)$ is sent to $(\partial'_k)_*(\epsilon) = \partial'_k \circ \epsilon$. In Sect. 4, we use unstable K-theory to calculate the order of $\partial'_1 \circ \epsilon$, giving a lower bound on the order of ∂'_1. Furthermore, in [5] Hamanaka and Kono considered an exact sequence similar to the first row to give a necessary condition for $\mathcal{G}_k(S^4) \simeq \mathcal{G}_l(S^4)$. In Sect. 5 we follow the same approach and use the first row to give a necessary condition for $\mathcal{G}_k(\mathbb{CP}^2) \simeq \mathcal{G}_l(\mathbb{CP}^2)$.

We remark that it is difficult to use only one of the two rows to prove both Theorems 1.5 and 1.6. On the one hand, $\partial'_1 \circ \epsilon$ factors through a map $\bar{\partial} : \Sigma C \to \mathrm{Map}^*(\mathbb{CP}^2, BSU(n))$. There is no obvious method to show that $\bar{\partial}$ and $\partial'_1 \circ \epsilon$ have the same orders except direct calculation. Therefore we cannot compare the orders of $\bar{\partial}$ and ∂'_1 to prove Theorem 1.5 without calculating the order of $\partial'_1 \circ \epsilon$. On the other hand, applying the method used in Sect. 5 to the second row gives a much weaker conclusion than Theorem 1.6. This is because $[\Sigma C, B\mathcal{G}_k(\mathbb{CP}^2)]$ is a much smaller group than $[\Sigma\mathbb{CP}^{n-1}, B\mathcal{G}_k(\mathbb{CP}^2)]$ and much information is lost by the map $\bar{q}^*$.

4 A lower bound on the order of ∂_1'

The restriction of ∂_1 to $\Sigma\mathbb{CP}^{n-1}$ is $\partial_1 \circ \epsilon$, which is the triple adjoint of the composition

$$\langle \imath, \epsilon\rangle : S^3 \wedge \Sigma\mathbb{CP}^{n-1} \xrightarrow{\imath\wedge\epsilon} SU(n) \wedge SU(n) \xrightarrow{\langle \mathbb{1},\mathbb{1}\rangle} SU(n).$$

Since $SU(n) \simeq \Omega BSU(n)$, we can further take its adjoint and get

$$\rho : \Sigma S^3 \wedge \Sigma\mathbb{CP}^{n-1} \xrightarrow{\Sigma\imath\wedge\epsilon} \Sigma SU(n) \wedge SU(n) \xrightarrow{[ev,ev]} BSU(n),$$

where $[ev, ev]$ is the Whitehead product of the evaluation map

$$ev : \Sigma SU(n) \simeq \Sigma\Omega BSU(n) \to BSU(n)$$

with itself. Similarly, the restriction $\partial_1' \circ \epsilon$ is adjoint to the composition

$$\rho' : \mathbb{CP}^2 \wedge \Sigma\mathbb{CP}^{n-1} \xrightarrow{q\wedge\mathbb{1}} S^4 \wedge \Sigma\mathbb{CP}^{n-1} \xrightarrow{\Sigma\imath\wedge\epsilon} \Sigma SU(n) \wedge SU(n) \xrightarrow{[ev,ev]} BSU(n).$$

Since we will frequently refer to the facts established in [4,5], it is easier to follow their setting and consider its adjoint

$$\gamma = \tau(\rho' \circ T) : \mathbb{CP}^2 \wedge \mathbb{CP}^{n-1} \to SU(n),$$

where $T : \Sigma\mathbb{CP}^2 \wedge \mathbb{CP}^{n-1} \to \mathbb{CP}^2 \wedge \Sigma\mathbb{CP}^{n-1}$ is the swapping map and $\tau : [\Sigma\mathbb{CP}^2 \wedge \mathbb{CP}^{n-1}, BSU(n)] \to [\mathbb{CP}^2 \wedge \mathbb{CP}^{n-1}, SU(n)]$ is the adjunction. By adjunction, the orders of $\partial_1' \circ \epsilon$, ρ' and γ are the same. We will calculate the order of γ using unstable K-theory to prove Theorem 1.5.

Apply $[\mathbb{CP}^2 \wedge \mathbb{CP}^{n-1}, -]$ to fibration (8) to obtain the exact sequence

$$\tilde{K}^0(\mathbb{CP}^2 \wedge \mathbb{CP}^{n-1}) \xrightarrow{p_*} [\mathbb{CP}^2 \wedge \mathbb{CP}^{n-1}, \Omega W_n] \longrightarrow [\mathbb{CP}^2 \wedge \mathbb{CP}^{n-1}, SU(n)] \longrightarrow 0.$$

Since $\mathbb{CP}^2 \wedge \mathbb{CP}^{n-1}$ is a CW-complex with even dimensional cells, the last item $[\mathbb{CP}^2 \wedge \mathbb{CP}^{n-1}, SU(\infty)] \cong \tilde{K}^1(\mathbb{CP}^2 \wedge \mathbb{CP}^{n-1})$ is zero. First we identify the term $[\mathbb{CP}^2 \wedge \mathbb{CP}^{n-1}, \Omega W_n]$.

Lemma 4.1 *We have the following:*

- $[\Sigma^{2n-4}\mathbb{CP}^2, \Omega W_n] \cong \mathbb{Z}$*;*
- $[\Sigma^{2n-3}\mathbb{CP}^2, \Omega W_n] = 0$ *for n odd;*
- $[\Sigma^{2n-2}\mathbb{CP}^2, \Omega W_n] \cong \mathbb{Z} \oplus \mathbb{Z}$*.*

Proof First, apply $[\Sigma^{2n-4}-, \Omega W_n]$ to cofibration (4) to obtain the exact sequence

$$\pi_{2n}(W_n) \longrightarrow \pi_{2n+1}(W_n) \longrightarrow [\Sigma^{2n-4}\mathbb{CP}^2, \Omega W_n] \longrightarrow \pi_{2n-1}(W_n).$$

We refer to Table (9) freely for the homotopy groups of W_n. Since $\pi_{2n-1}(W_n)$ and $\pi_{2n}(W_n)$ are zero, $[\Sigma^{2n-4}\mathbb{CP}^{n-1}, \Omega W_n]$ is isomorphic to $\pi_{2n+1}(W_n) \cong \mathbb{Z}$.

Second, apply $[\Sigma^{2n-3}-, \Omega W_n]$ to (4) to obtain

$$\pi_{2n+2}(W_n) \longrightarrow [\Sigma^{2n-3}\mathbb{CP}^2, \Omega W_n] \longrightarrow \pi_{2n}(W_n).$$

Since $\pi_{2n}(W_n)$ and $\pi_{2n+2}(W_n)$ are zero for n odd, so is $[\Sigma^{2n-3}\mathbb{CP}^2, \Omega W_n]$.

Third, apply $[\Sigma^{2n-2}-, \Omega W_n]$ to (4) to obtain

$$\begin{aligned}\pi_{2n+2}(W_n) \xrightarrow{\eta_1} \pi_{2n+3}(W_n) \longrightarrow [\Sigma^{2n-2}\mathbb{CP}^2, \Omega W_n]\\ \xrightarrow{j} \pi_{2n+1}(W_n) \xrightarrow{\eta_2} \pi_{2n+2}(W_n),\end{aligned}$$

where η_1 and η_2 are induced by Hopf maps $\Sigma^{2n}\eta : S^{2n+3} \to S^{2n+2}$ and $\Sigma^{2n-1}\eta : S^{2n+2} \to S^{2n+1}$, and j is induced by the inclusion $S^{2n+1} \hookrightarrow \Sigma^{2n-2}\mathbb{CP}^2$ of the bottom cell. When n is odd, $\pi_{2n+2}(W_n)$ is zero and $\pi_{2n+1}(W_n)$ and $\pi_{2n+3}(W_n)$ are $\mathbb{Z}$, so $[\Sigma^{2n-2}\mathbb{CP}^{n-1}, \Omega W_n]$ is $\mathbb{Z} \oplus \mathbb{Z}$. When n is even, the $(2n+4)$-skeleton of W_n is $S^{2n+1} \vee S^{2n+3}$. The inclusions

$$i_1 : S^{2n+1} \to S^{2n+1} \vee S^{2n+3} \quad \text{and} \quad i_2 : S^{2n+3} \to S^{2n+1} \vee S^{2n+3}$$

generate $\pi_{2n+1}(W_n)$ and the $\mathbb{Z}$-summand of $\pi_{2n+3}(W_n)$, and the compositions

$$j_1 : S^{2n+2} \xrightarrow{\Sigma^{2n-1}\eta} S^{2n+1} \xrightarrow{i_1} W_n \text{ and } j_2 : S^{2n+3} \xrightarrow{\Sigma^{2n}\eta} S^{2n+2} \xrightarrow{\Sigma^{2n-1}\eta} S^{2n+1} \xrightarrow{i_1} W_n$$

generate $\pi_{2n+2}(W_n)$ and the $\mathbb{Z}/2\mathbb{Z}$-summand of $\pi_{2n+3}(W_n)$ respectively. Since η_1 sends j_1 to j_2, the cokernel of η_1 is $\mathbb{Z}$. Similarly, η_2 sends i_1 to j_1, so $\eta_2 : \mathbb{Z} \to \mathbb{Z}/2\mathbb{Z}$ is surjective. This implies the preimage of j is a $\mathbb{Z}$-summand. Therefore $[\Sigma^{2n-2}\mathbb{CP}^2, \Omega W_n] \cong \mathbb{Z} \oplus \mathbb{Z}$. □

Let C be the quotient $\mathbb{CP}^{n-1}/\mathbb{CP}^{n-3}$. Since ΩW_n is $(2n-1)$-connected, $[\mathbb{CP}^2 \wedge \mathbb{CP}^{n-1}, \Omega W_n]$ is isomorphic to $[\mathbb{CP}^2 \wedge C, \Omega W_n]$ which is easier to determine.

Lemma 4.2 *The group* $[\mathbb{CP}^2 \wedge \mathbb{CP}^{n-1}, \Omega W_n] \cong [\mathbb{CP}^2 \wedge C, \Omega W_n]$ *is isomorphic to* $\mathbb{Z}^{\oplus 3}$.

Proof When n is even, C is $S^{2n-2} \vee S^{2n-4}$. By Lemma 4.1, $[\mathbb{CP}^2 \wedge C, \Omega W_n]$ is $[\Sigma^{2n-2}\mathbb{CP}^2, \Omega W_n] \oplus [\Sigma^{2n-4}\mathbb{CP}^2, \Omega W_n] \cong \mathbb{Z}^{\oplus 3}$.

When n is odd, C is $\Sigma^{2n-6}\mathbb{CP}^2$. Apply $[\Sigma^{2n-6}\mathbb{CP}^2 \wedge -, \Omega W_n]$ to cofibration (4) to obtain the exact sequence

$$\begin{aligned}[\Sigma^{2n-3}\mathbb{CP}^2, \Omega W_n] \longrightarrow [\Sigma^{2n-2}\mathbb{CP}^2, \Omega W_n] \longrightarrow [\Sigma^{2n-6}\mathbb{CP}^2 \wedge \mathbb{CP}^2, \Omega W_n]\\ \longrightarrow [\Sigma^{2n-4}\mathbb{CP}^2, \Omega W_n] \longrightarrow [\Sigma^{2n-3}\mathbb{CP}^2, \Omega W_n]\end{aligned}$$

By Lemma 4.1, the first and the last terms $[\Sigma^{2n-3}\mathbb{CP}^2, \Omega W_n]$ are zero, while the second term $[\Sigma^{2n-2}\mathbb{CP}^2, \Omega W_n]$ is $\mathbb{Z} \oplus \mathbb{Z}$ and the fourth $[\Sigma^{2n-4}\mathbb{CP}^2, \Omega W_n]$ is $\mathbb{Z}$. Therefore $[\mathbb{CP}^2 \wedge C, \Omega W_n]$ is $\mathbb{Z}^{\oplus 3}$. $\square$

Define $a : [\mathbb{CP}^2 \wedge \mathbb{CP}^{n-1}, \Omega W_n] \to H^{2n}(\mathbb{CP}^2 \wedge \mathbb{CP}^{n-1}) \oplus H^{2n+2}(\mathbb{CP}^2 \wedge \mathbb{CP}^{n-1})$ to be a map sending $f \in [\mathbb{CP}^2 \wedge \mathbb{CP}^{n-1}, \Omega W_n]$ to $a(f) = f^*(a_{2n}) \oplus f^*(a_{2n+2})$. The cohomology class $\bar{x}_{2n+1}$ represents a map $\bar{x}_{2n+1} : W_n \to K(\mathbb{Z}, 2n+1)$ and $a_{2n} = \sigma(\bar{x}_{2n+1})$ represents its loop $\Omega\bar{x}_{2n+1} : \Omega W_n \to \Omega K(\mathbb{Z}, 2n+1)$. Similarly $a_{2n+2} = \sigma(\bar{x}_{2n+3})$ represents a loop map. This implies a is a group homomorphism. Furthermore, a_{2n} and a_{2n+2} induce isomorphisms between $H^i(\Omega W_n)$ and $H^i(K(2n, \mathbb{Z}) \times K(2n+2, \mathbb{Z}))$ for $i = 2n$ and $2n+2$. Since $[\mathbb{CP}^2 \wedge \mathbb{CP}^{n-1}, \Omega W_n]$ is a free $\mathbb{Z}$-module by Lemma 4.2, a is a monomorphism. Consider the diagram

$$\begin{tikzcd}
\tilde{K}^0(\mathbb{CP}^2 \wedge \mathbb{CP}^{n-1}) \arrow[r, "p_*"] \arrow[d, equal] & {[\mathbb{CP}^2 \wedge \mathbb{CP}^{n-1}, \Omega W_n]} \arrow[r] \arrow[d, "a"] & {[\mathbb{CP}^2 \wedge \mathbb{CP}^{n-1}, SU(n)]} \arrow[r] \arrow[d, "b"] & 0 \\
\tilde{K}^0(\mathbb{CP}^2 \wedge \mathbb{CP}^{n-1}) \arrow[r, "\Phi"] & \bigoplus_{i=0,2} H^{2n+i}(\mathbb{CP}^2 \wedge \mathbb{CP}^{n-1}) \arrow[r, "\psi"] & Coker(\Phi) \arrow[r] & 0
\end{tikzcd} \tag{10}$$

In the left square, Φ is defined to be $a \circ p^*$. In the right square, ψ is the quotient map and b is defined as follows. Any $f \in [\mathbb{CP}^2 \wedge \mathbb{CP}^{n-1}, SU(n)]$ has a preimage $\tilde{f}$ and $b(f)$ is defined to be $\psi(a(\tilde{f}))$. An easy diagram chase shows that b is well-defined and injective. Since b is injective, the order of $\gamma \in [\mathbb{CP}^2 \wedge \mathbb{CP}^{n-1}, SU(n)]$ equals the order of $b(\gamma) \in Coker(\Phi)$. In [4], Hamanaka and Kono gave an explicit formula for Φ.

Theorem 4.3 (Hamanaka and Kono [4]) *Let Y be a CW-complex. For any $f \in \tilde{K}^0(Y)$ we have*

$$\Phi(f) = n!ch_{2n}(f) \oplus (n+1)!ch_{2n+2}(f),$$

where $ch_{2i}(f)$ is the $2i$th part of $ch(f)$.

Let u and v be the generators of $H^2(\mathbb{CP}^2)$ and $H^2(\mathbb{CP}^{n-1})$. For $1 \le i \le n-1$, denote L_i and L_i' as the generators of $\tilde{K}^0(\mathbb{CP}^2 \wedge \mathbb{CP}^{n-1})$ with Chern characters $ch(L_i) = u^2(e^v - 1)^i$ and $ch(L_i') = (u + \frac{1}{2}u^2) \cdot (e^v - 1)^i$. By Theorem 4.3 we have

$$\begin{aligned}
\Phi(L_i) &= n(n-1)A_i u^2 v^{n-2} + n(n+1)B_i u^2 v^{n-1}, \\
\Phi(L_i') &= \frac{n(n-1)}{2} A_i u^2 v^{n-2} + nB_i u v^{n-1} + \frac{n(n+1)}{2} B_i u^2 v^{n-1},
\end{aligned}$$

where

$$A_i = \textstyle\sum_{j=1}^{i} (-1)^{i+j} \binom{i}{j} j^{n-2} \quad \text{and} \quad B_i = \sum_{j=1}^{i} (-1)^{i+j} \binom{i}{j} j^{n-1}.$$

Write an element $xu^2v^{n-2} + yuv^{n-1} + zu^2v^{n-1} \in H^{2n}(\mathbb{CP}^2 \wedge \mathbb{CP}^{n-1}) \oplus H^{2n+2}(\mathbb{CP}^2 \wedge \mathbb{CP}^{n-1})$ as (x, y, z). Then the coordinates of $\Phi(L_i)$ and $\Phi(L_i')$ are $(n(n-1)A_i, 0, n(n+1)B_i)$ and $(\frac{n(n-1)}{2}A_i, nB_i, \frac{n(n+1)}{2}B_i)$ respectively.

Lemma 4.4 *For* $n \geq 3$, $Im(\Phi)$ *is spanned by* $(\frac{n(n-1)}{2}, n, \frac{n(n+1)}{2})$, $(n(n-1), 0, 0)$ *and* $(0, 2n, 0)$.

Proof By definition, $Im(\Phi) = span\{\Phi(L_i), \Phi(L_i')\}_{i=1}^{n-1}$. For $i = 1$, $A_1 = B_1 = 1$. Then

$$\begin{aligned}\Phi(L_1) &= (n(n-1), 0, n(n+1))\\ &= 2\left(\frac{1}{2}n(n-1), n, \frac{1}{2}n(n+1)\right) - (0, 2n, 0)\\ &= 2\Phi(L_1') - (0, 2n, 0)\end{aligned}$$

Equivalently $(0, 2n, 0) = 2\Phi(L_1') - \Phi(L_1)$, so $span\{\Phi(L_1), \Phi(L_1')\} = span\{\Phi(L_1'), (0, 2n, 0)\}$. For other i's,

$$\begin{aligned}\Phi(L_i) &= (n(n-1)A_i, 0, n(n+1)B_i)\\ &= 2\left(\frac{1}{2}n(n-1)A_i, nB_i, \frac{1}{2}n(n+1)B_i\right) - (0, 2nB_i, 0)\\ &= 2\Phi(L_i') - B_i(0, 2n, 0)\end{aligned}$$

is a linear combination of $\Phi(L_i')$ and $(0, 2n, 0)$, so

$$Im(\Phi) = span\{\Phi(L_1'), \ldots, \Phi(L_{n-1}'), (0, 2n, 0)\}.$$

We claim that $span\{\Phi(L_i')\}_{i=1}^{n-1} = span\{\Phi(L_1'), (n(n-1), 0, 0)\}$. Observe that

$$\begin{aligned}\Phi(L_i') &= \left(\frac{n(n-1)}{2}A_i, nB_i, \frac{n(n+1)}{2}B_i\right)\\ &= \left(\frac{n(n-1)}{2}B_i, nB_i, \frac{n(n+1)}{2}B_i\right) + \left(\frac{n(n-1)}{2}(A_i - B_i), 0, 0\right)\\ &= B_i\Phi(L_1') + \frac{A_i - B_i}{2} \cdot (n(n-1), 0, 0).\end{aligned}$$

The difference

$$\begin{aligned}A_i - B_i &= \sum_{j=1}^{i}(-1)^{i+j}\binom{i}{j}j^{n-2} - \sum_{j=1}^{i}(-1)^{i+j}\binom{i}{j}j^{n-1}\\ &= \sum_{j=1}^{i}(-1)^{i+j+1}\binom{i}{j}(j^{n-1} - j^{n-2})\\ &= \sum_{j=1}^{i}(-1)^{i+j+1}\binom{i}{j}(j-1)j^{n-2}\end{aligned}$$

is even since each term $(j-1)j^{n-2}$ is even and $n \geq 3$. Therefore $\frac{A_i - B_i}{2}$ is an integer and $\Phi(L_i')$ is a linear combination of $\Phi(L_1')$ and $(n(n-1), 0, 0)$.

Furthermore,

$$\begin{aligned}\Phi(L_2') &= B_2\Phi(L_1') + (A_2 - B_2)\left(\frac{n(n-1)}{2}, 0, 0\right)\\ &= B_2\Phi(L_1') - 2^{n-3}(n(n-1), 0, 0)\end{aligned}$$

and

$$\begin{aligned}\Phi(L_3') &= B_3\Phi(L_1') + (A_3 - B_3)\left(\frac{n(n-1)}{2}, 0, 0\right)\\ &= B_3\Phi(L_1') - (3^{n-2} - 3\cdot 2^{n-3})(n(n-1), 0, 0).\end{aligned}$$

For $n = 3$, $B_2 = 2$ and $\Phi(L_2') = 2\Phi(L_1') - (n(n-1), 0, 0)$, so we have

$$span\{\Phi(L_i')\}_{i=1}^{n-1} = span\{\Phi(L_1'), \Phi(L_2')\} = span\{\Phi(L_1'), (n(n-1), 0, 0)\}.$$

For $n \geq 4$, since 2^{n-3} and $3^{n-2} - 3\cdot 2^{n-3}$ are coprime to each other, there exist integers s and t such that $2^{n-3}s + (3^{n-2} - 3\cdot 2^{n-3})t = 1$ and

$$(n(n-1), 0, 0) = (sB_2 + tB_3)\Phi(L_1') - s\Phi(L_2') - t\Phi(L_3').$$

Therefore $(n(n-1), 0, 0)$ is a linear combination of $\Phi(L_1')$, $\Phi(L_2')$ and $\Phi(L_3')$. This implies $span\{\Phi(L_1'), (n(n-1), 0, 0)\} = span\{\Phi(L_i')\}_{i=1}^{n-1}$.

Combine all these together to obtain

$$\begin{aligned}Im(\Phi) &= span\{\Phi(L_i), \Phi(L_i')\}_{i=1}^{n-1}\\ &= span\{\Phi(L_1'), (n(n-1), 0, 0), (0, 2n, 0)\}\\ &= span\left\{\left(\frac{n(n-1)}{2}, n, \frac{n(n+1)}{2}\right), (n(n-1), 0, 0), (0, 2n, 0)\right\}.\end{aligned}$$

□

Back to diagram (10). The map γ has a lift $\tilde{\gamma} : \mathbb{CP}^2 \wedge \mathbb{CP}^{n-1} \to \Omega W_n$. By exactness, the order of γ equals the minimum number m such that $m\tilde{\gamma}$ is contained in $Im(p_*)$. Since a and b are injective, the order of γ equals the minimum number m' such that $m'a(\tilde{\gamma})$ is contained in $Im(\Phi)$.

Lemma 4.5 *Let $\alpha : \Sigma X \to SU(n)$ be a map for some space X. If $\alpha' : \mathbb{CP}^2 \wedge X \to SU(n)$ is the adjoint of the composition*

$$\mathbb{CP}^2 \wedge \Sigma X \xrightarrow{q\wedge \mathbb{1}} \Sigma S^3 \wedge \Sigma X \xrightarrow{\Sigma\iota\wedge\alpha} \Sigma SU(n) \wedge SU(n) \xrightarrow{[ev,ev]} BSU(n),$$

then there is a lift $\tilde{\alpha}$ of α' such that $\tilde{\alpha}^*(a_{2i}) = u^2 \otimes \Sigma^{-1}\alpha^*(x_{2i-3})$, where Σ is the cohomology suspension isomorphism.

$$\begin{tikzcd} & \Omega W_n \arrow[d] \\ \mathbb{CP}^2 \wedge X \arrow[r, "\alpha'"] \arrow[ur, dashed, "\tilde{\alpha}"] & SU(n) \end{tikzcd}$$

Proof In [4,5], Hamanaka and Kono constructed a lift $\Gamma : \Sigma SU(n) \wedge SU(n) \to W_n$ of $[ev, ev]$ such that $\Gamma^*(\bar{x}_{2i+1}) = \sum_{j+k=i-1} \Sigma x_{2j+1} \otimes x_{2k+1}$. Let $\tilde{\Gamma}$ be the composition

$$\tilde{\Gamma} : \mathbb{CP}^2 \wedge \Sigma X \xrightarrow{q\wedge \mathbb{1}} \Sigma S^3 \wedge \Sigma X \xrightarrow{\Sigma\iota\wedge\alpha} \Sigma SU(n) \wedge SU(n) \xrightarrow{\Gamma} W_n.$$

Then we have

$$\begin{aligned} \tilde{\Gamma}^*(\bar{x}_{2i+1}) &= (q \wedge \mathbb{1})^*(\Sigma\iota \wedge \alpha)^*\Gamma^*(\bar{x}_{2i+1}) \\ &= (q \wedge \mathbb{1})^*(\Sigma\iota \wedge \alpha)^* \left(\sum_{j+k=i-1} \Sigma x_{2j+1} \otimes x_{2k+1} \right) \\ &= (q \wedge \mathbb{1})^*(\Sigma u_3 \otimes \alpha^*(x_{2i-3})) \\ &= u^2 \otimes \alpha^*(x_{2i-3}), \end{aligned}$$

where u_3 is the generator of $H^3(S^3)$.

Let $T : \Sigma\mathbb{CP}^2 \wedge X \to \mathbb{CP}^2 \wedge \Sigma X$ be the swapping map and let $\tau : [\Sigma\mathbb{CP}^2 \wedge X, W_n] \to [\mathbb{CP}^2 \wedge X, \Omega W_n]$ be the adjunction. Take $\tilde{\alpha} : \mathbb{CP}^2 \wedge X \to \Omega W_n$ to be the adjoint of $\tilde{\Gamma}$, that is $\tilde{\alpha} = \tau(\tilde{\Gamma} \circ T)$. Then $\tilde{\alpha}$ is a lift of α'. Since

$$(\tilde{\Gamma} \circ T)^*(\bar{x}_{2i+1}) = T^* \circ \tilde{\Gamma}^*(\bar{x}_{2i+1}) = T^*(u^2 \otimes \alpha^*(x_{2i-3})) = \Sigma u^2 \otimes \Sigma^{-1}\alpha^*(x_{2i-3}),$$

we have $\tilde{\alpha}^*(a_{2i}) = u^2 \otimes \Sigma^{-1}\alpha^*(x_{2i-3})$. □

Lemma 4.6 *In diagram (10), γ has a lift $\tilde{\gamma}$ such that $a(\tilde{\gamma}) = u^2v^{n-2} \oplus u^2v^{n-1}$.*

Proof Recall that γ is the adjoint of the composition

$$\rho' : \mathbb{CP}^2 \wedge \Sigma\mathbb{CP}^{n-1} \xrightarrow{q\wedge\mathbb{1}} \Sigma S^3 \wedge \Sigma\mathbb{CP}^{n-1} \xrightarrow{\Sigma\iota\wedge\epsilon} \Sigma SU(n) \wedge SU(n) \xrightarrow{[ev,ev]} BSU(n).$$

Now we use Lemma 4.5 and take α to be $\epsilon : \Sigma\mathbb{CP}^{n-1} \to SU(n)$. Then γ has a lift $\tilde{\gamma}$ such that $\tilde{\gamma}^*(a_{2i}) = u^2 \otimes \Sigma^{-1}\epsilon^*(x_{2i-3}) = u^2 \otimes v^{i-2}$. This implies

$$a(\tilde{\gamma}) = \tilde{\gamma}^*(a_{2n}) \oplus \tilde{\gamma}^*(a_{2n+2}) = u^2v^{n-2} \oplus u^2v^{n-1}.$$

□

Now we can calculate the order of $\partial_1' \circ \epsilon$, which gives a lower bound on the order of ∂_1'.

Theorem 4.7 *When $n \geq 3$, the order of $\partial_1' \circ \epsilon$ is $\frac{1}{2}n(n^2-1)$ for n odd and $n(n^2-1)$ for n even.*

Proof Since $\partial_1' \circ \epsilon$ is adjoint to γ , it suffices to calculate the order of γ. By Lemma 4.4, $Im(\Phi)$ is spanned by $(\frac{1}{2}n(n-1), n, \frac{1}{2}n(n+1))$, $(n(n-1), 0, 0)$ and $(0, 2n, 0)$. By Lemma 4.6, $a(\tilde{\gamma})$ has coordinates $(1, 0, 1)$. Let m be a number such that $ma(\tilde{\gamma})$ is contained in $Im(\Phi)$. Then

$$m(1, 0, 1) = s\left(\frac{1}{2}n(n-1), n, \frac{1}{2}n(n+1)\right) + t(n(n-1), 0, 0) + r(0, 2n, 0)$$

for some integers s, t and r. Solve this to get

$$m = \tfrac{1}{2}tn(n^2-1),\ s = -2r,\ s = t(n-1).$$

Since $s = -2r$ is even, the smallest positive value of t satisfying $s = t(n-1)$ is 1 for n odd and 2 for n even. Therefore m is $\frac{1}{2}n(n^2-1)$ for n odd and $n(n^2-1)$ for n even. □

For $SU(n)$-gauge groups over S^4, the order m of ∂_1 has the form $m = n(n^2-1)$ for $n = 3$ and 5 [5,19]. If p is an odd prime and $n < (p-1)^2 + 1$, then m and $n(n^2-1)$ have the same p-components [9,20]. These facts suggest it may be the case that $m = n(n^2-1)$ for any $n > 2$. In fact, one can follow the method Hamanaka and Kono used in [5] and calculate the order of $\partial \circ \epsilon$ to obtain a lower bound $n(n^2-1)$ for n odd. However, it does not work for the n even case since $[S^4 \wedge \mathbb{CP}^{n-1}, \Omega W_n]$ is not a free $\mathbb{Z}$-module. An interesting corollary of Theorem 4.7 is to give a lower bound on the order of ∂_1 for n even.

Corollary 4.8 *When n is even and greater than 2, the order of ∂_1 is at least $n(n^2-1)$.*

Proof The order of $\partial_1' \circ \epsilon$ is a lower bound on the order of ∂_1', which is either the same as or half of the order of ∂_1 by Lemma 2.2. The corollary follows from Theorem 4.7. □

5 A necessary condition for $\mathcal{G}_k(\mathbb{CP}^2) \simeq \mathcal{G}_l(\mathbb{CP}^2)$

In this section we follow the approach in [5] to prove Theorem 1.6. The techniques used are similar to that in Sect. 4, except we are working with the quotient $\Sigma C = \Sigma\mathbb{CP}^{n-1}/\Sigma\mathbb{CP}^{n-1}$ instead of $\Sigma\mathbb{CP}^{n-1}$. When n is odd, C is $\Sigma^{2n-6}\mathbb{CP}^2$, and when n is even, C is $S^{2n-2} \vee S^{2n-4}$. Apply $[\Sigma C, -]$ to fibration (3) to obtain the exact sequence

$$\begin{aligned}&[\Sigma C, SU(n)] \xrightarrow{(\partial_k')_*} [\Sigma C, \mathrm{Map}_0^*(\mathbb{CP}^2, BSU(n))]\\ &\longrightarrow [\Sigma C, B\mathcal{G}_k(\mathbb{CP}^2)] \longrightarrow [\Sigma C, BSU(n)],\end{aligned}$$

where $(\partial'_k)_*$ sends $f \in [\Sigma C, SU(n)]$ to $\partial'_k \circ f \in [\Sigma C, \mathrm{Map}_0^*(\mathbb{CP}^2, BSU(n))]$. Since $BSU(n) \to BSU(\infty)$ is a $2n$-equivalence and ΣC has dimension $2n-1$, $[\Sigma C, BSU(n)]$ is $\tilde{K}^0(\Sigma C)$ which is zero. Similarly, $[\Sigma C, SU(n)] \cong [\Sigma^2 C, BSU(n)]$ is $\tilde{K}^0(\Sigma^2 C) \cong \mathbb{Z} \oplus \mathbb{Z}$. Furthermore, by adjunction we have $[\Sigma C, \mathrm{Map}_0^*(\mathbb{CP}^2, BSU(n))] \cong [\Sigma C \wedge \mathbb{CP}^2, BSU(n)]$. The exact sequence becomes

$$\tilde{K}^0(\Sigma^2 C) \xrightarrow{(\partial'_k)_*} [\Sigma C \wedge \mathbb{CP}^2, BSU(n)] \longrightarrow [\Sigma C, B\mathcal{G}_k(\mathbb{CP}^2)] \longrightarrow 0. \tag{11}$$

This implies $[\Sigma C, B\mathcal{G}_k(\mathbb{CP}^2)] \cong [C, \mathcal{G}_k(\mathbb{CP}^2)]$ is $Coker(\partial'_k)_*$. Also, apply $[\mathbb{CP}^2 \wedge C, -]$ to fibration (8) to obtain the exact sequence

$$\begin{aligned}
&[\mathbb{CP}^2 \wedge C, \Omega SU(\infty)] \xrightarrow{p_*} [\mathbb{CP}^2 \wedge C, \Omega W_n] \\
&\quad \longrightarrow [\mathbb{CP}^2 \wedge C, SU(n)] \longrightarrow [\mathbb{CP}^2 \wedge C, SU(\infty)].
\end{aligned} \tag{12}$$

Observe that $[\mathbb{CP}^2 \wedge C, \Omega SU(\infty)] \cong \tilde{K}^0(\mathbb{CP}^2 \wedge C)$ is $\mathbb{Z}^{\oplus 4}$ and $[\mathbb{CP}^2 \wedge C, SU(\infty)] \cong \tilde{K}^1(\mathbb{CP}^2 \wedge C)$ is zero. Combine exact sequences (11) and (12) to obtain the diagram

$$\begin{tikzcd}
 & \tilde{K}^0(\mathbb{CP}^2 \wedge C) \arrow[d, "p_*"] \arrow[dr, "\Phi"] & & \\
 & {[\mathbb{CP}^2 \wedge C, \Omega W_n]} \arrow[r, "a"] \arrow[d] & H^{2n}(\mathbb{CP}^2 \wedge C) \oplus H^{2n+2}(\mathbb{CP}^2 \wedge C) & \\
\tilde{K}^0(\Sigma^2 C) \arrow[r, "(\partial'_k)_*"] & {[\mathbb{CP}^2 \wedge C, SU(n)]} \arrow[r] \arrow[d] & {[C, B\mathcal{G}_k(\mathbb{CP}^2)]} \arrow[r] & 0 \\
 & 0 & &
\end{tikzcd}$$

where $a(f) = f^*(a_{2n}) \oplus f^*(a_{2n+2})$ for any $f \in [\mathbb{CP}^2 \wedge C, \Omega W_n]$, and Φ is defined to be $a \circ p_*$. By Lemma 4.2 $[\mathbb{CP}^2 \wedge C, \Omega W_n]$ is free. Following the same argument in Sect. 4 implies the injectivity of a.

Our strategy to prove Theorem 1.6 is as follows. If $\mathcal{G}_k(\mathbb{CP}^2)$ is homotopy equivalent to $\mathcal{G}_l(\mathbb{CP}^2)$, then $[C, \mathcal{G}_k(\mathbb{CP}^2)] \cong [C, \mathcal{G}_l(\mathbb{CP}^2)]$ and exactness in (12) implies that $Im(\partial'_k)_*$ and $Im(\partial'_l)_*$ have the same order in $[\mathbb{CP}^2 \wedge C, SU(n)]$, resulting in a necessary condition for a homotopy equivalence $\mathcal{G}_k(\mathbb{CP}^2) \simeq \mathcal{G}_l(\mathbb{CP}^2)$. To calculate the order of $Im(\partial'_k)_*$, we will find a preimage $\tilde{\partial}_k$ of $Im(\partial'_k)_*$ in $[\mathbb{CP}^2 \wedge C, \Omega W_n]$. Since a is injective, we can embed $\tilde{\partial}_k$ into $H^{2n}(\mathbb{CP}^2 \wedge C) \oplus H^{2n+2}(\mathbb{CP}^2 \wedge C)$ and work out the order of $Im(\partial'_k)_*$ there.

Let u, v_{2n-4} and v_{2n-2} be generators of $H^2(\mathbb{CP}^2)$, $H^{2n-4}(C)$ and $H^{2n-2}(C)$. Then we write an element $xu^2v_{2n-4} + yuv_{2n-2} + zu^2v_{2n-2} \in H^{2n}(\mathbb{CP}^2 \wedge C) \oplus H^{2n+2}(\mathbb{CP}^2 \wedge C)$ as (x, y, z). First we need to find the submodule $Im(a)$.

Lemma 5.1 *For n odd, $Im(a)$ is $\{(x, y, z) | x + y \equiv z \pmod 2\}$, and for n even, $Im(a)$ is $\{(x, y, z) | y \equiv 0 \pmod 2\}$.*

Proof When n is odd, C is $\Sigma^{2n-6}\mathbb{CP}^2$ and the $(2n+3)$-skeleton of ΩW_n is $\Sigma^{2n-2}\mathbb{CP}^2$. To say $(x, y, z) \in Im(a)$ means there exists $f \in [\mathbb{CP}^2 \wedge C, \Omega W_n]$ such that

$$f^*(a_{2n}) = xu^2 v_{2n-4} + yuv_{2n-2} \text{ and } f^*(a_{2n+2}) = zu^2 v_{2n-2}. \tag{13}$$

Reducing to homology with $\mathbb{Z}/2\mathbb{Z}$-coefficients, we have

$$Sq^2(u) = u^2,\ Sq^2(v_{2n-4}) = v_{2n-2},\ Sq^2(a_{2n}) = a_{2n+2}.$$

Apply Sq^2 to (13) to get $x + y \equiv z \pmod 2$. Therefore $Im(a)$ is contained in $\{(x, y, z)|x + y \equiv z \pmod 2\}$. To show that they are equal, we need to show that $(1, 0, 1)$, $(0, 1, 1)$ and $(0, 0, 2)$ are in $Im(a)$. Consider maps

$$\begin{aligned}
f_1 &: \mathbb{CP}^2 \wedge C \xrightarrow{q_1} S^4 \wedge C \simeq \Sigma^{2n-2}\mathbb{CP}^2 \hookrightarrow \Omega W_n \\
f_2 &: \mathbb{CP}^2 \wedge C \xrightarrow{q_2} \mathbb{CP}^2 \wedge S^{2n-2} \hookrightarrow \Omega W_n \\
f_3 &: \mathbb{CP}^2 \wedge C \xrightarrow{q_3} S^{2n+2} \xrightarrow{\theta} \Omega W_n
\end{aligned}$$

where q_1, q_2 and q_3 are quotient maps and θ is the generator of $\pi_{2n+3}(W_n)$. Their images are

$$a(f_1) = (1, 0, 1)\ a(f_2) = (0, 1, 1)\ a(f_3) = (0, 0, 2)$$

respectively, so $Im(a) = \{(x, y, z)|x + y \equiv z \pmod 2\}$.

When n is even, C is $S^{2n-2} \vee S^{2n-4}$ and the $(2n + 3)$-skeleton of ΩW_n is $S^{2n+2} \vee S^{2n}$. Reducing to homology with $\mathbb{Z}/2\mathbb{Z}$-coefficients, $Sq^2(v_{2n-4}) = 0$ and $Sq^2(a_{2n}) = 0$. Apply Sq^2 to (13) to get $y \equiv 0 \pmod 2$. Therefore $Im(a)$ is contained in $\{(x, y, z)|y \equiv 0 \pmod 2\}$. To show that they are equal, we need to show that $(1, 0, 0)$, $(0, 2, 0)$ and $(0, 0, 1)$ are in $Im(a)$. The maps

$$\begin{aligned}
f_1' &: \mathbb{CP}^2 \wedge C \xrightarrow{q_1'} S^4 \wedge (S^{2n-2} \vee S^{2n-4}) \xrightarrow{p_1} S^4 \wedge S^{2n-4} \hookrightarrow \Omega W_n \\
f_2' &: \mathbb{CP}^2 \wedge C \xrightarrow{q_2'} S^4 \wedge (S^{2n-2} \vee S^{2n-4}) \xrightarrow{p_2} S^4 \wedge S^{2n-2} \hookrightarrow \Omega W_n
\end{aligned}$$

where q_1' and q_2' are quotient maps and p_1 and p_2 are pinch maps, have images $a(f_1') = (1, 0, 0)$ and $a(f_2') = (0, 0, 1)$. To find $(0, 2, 0)$, apply $[- \wedge S^{2n-2}, \Omega W_n]$ to cofibration (4) to obtain the exact sequence

$$\pi_{2n+3}(W_n) \longrightarrow [\mathbb{CP}^2 \wedge S^{2n-2}, \Omega W_n] \xrightarrow{i^*} \pi_{2n+1}(W_n) \xrightarrow{\eta^*} \pi_{2n+2}(W_n)$$

where i^* is induced by the inclusion $i : S^2 \hookrightarrow \mathbb{CP}^2$ and η^* is induced by Hopf map η. The third term $\pi_{2n+1}(W_n) \cong \mathbb{Z}$ is generated by $i' : S^{2n+1} \to W_n$, the inclusion of the bottom cell, and the fourth term $\pi_{2n+2}(W_n) \cong \mathbb{Z}/2\mathbb{Z}$ is generated by $i' \circ \eta$, so $\eta^* : \mathbb{Z} \to \mathbb{Z}/2\mathbb{Z}$ is a surjection. By exactness $[\mathbb{CP}^2 \wedge S^{2n-2}, \Omega W_n]$ has a $\mathbb{Z}$-summand with the property that i^* sends its generator g to $2i'$. Therefore the composition

$$f_3' : \mathbb{CP}^2 \wedge (S^{2n-2} \vee S^{2n-4}) \xrightarrow{pinch} \mathbb{CP}^2 \wedge S^{2n-2} \xrightarrow{g} \Omega W_n$$

has image $(0, 2, 0)$. It follows that $Im(a) = \{(x, y, z) | y \equiv 0 \pmod 2\}$. $\square$

Now we split into the n odd and n even cases to calculate the order of $Im(\partial_k')_*$.

5.1 The order of $Im(\partial_k')_*$ for n odd

When n is odd, C is $\Sigma^{2n-6}\mathbb{CP}^2$. First we find $Im(\Phi)$ in $Im(a)$. For $1 \leq i \leq 4$, let L_i be the generators of $\tilde{K}^0(\mathbb{CP}^2 \wedge C) \cong \mathbb{Z}^{\oplus 4}$ with Chern characters

$$\begin{array}{ll} ch(L_1) = \left(u + \frac{1}{2}u^2\right) \cdot \left(v_{2n-4} + \frac{1}{2}v_{2n-2}\right) & ch(L_2) = \left(u + \frac{1}{2}u^2\right) v_{2n-2} \\ ch(L_3) = u^2 \left(v_{2n-4} + \frac{1}{2}v_{2n-2}\right) & ch(L_4) = u^2 v_{2n-2}. \end{array}$$

By Theorem 4.3, we have

$$\begin{aligned} \Phi(L_1) &= \frac{n!}{2}u^2 v_{2n-4} + \frac{n!}{2}u v_{2n-2} + \frac{(n+1)!}{4}u^2 v_{2n-2} \\ \Phi(L_2) &= n! u v_{2n-2} + \frac{(n+1)!}{2}u^2 v_{2n-2} \\ \Phi(L_3) &= n! u^2 v_{2n-4} + \frac{(n+1)!}{2}u^2 v_{2n-2} \\ \Phi(L_4) &= (n+1)! u^2 v_{2n-2}. \end{aligned}$$

By Lemma 5.1, $Im(a)$ is spanned by $(1, 0, 1)$, $(0, 1, 1)$ and $(0, 0, 2)$. Under this basis, the coordinates of the $\Phi(L_i)$'s are

$$\begin{array}{ll} \Phi(L_1) = \left(\frac{n!}{2}, \frac{n!}{2}, \frac{(n-3)\cdot n!}{8}\right), & \Phi(L_2) = \left(0, n!, \frac{(n-1)\cdot n!}{4}\right), \\ \Phi(L_3) = \left(n!, 0, \frac{(n-1)\cdot n!}{4}\right), & \Phi(L_4) = \left(0, 0, \frac{(n+1)!}{2}\right). \end{array}$$

We represent their coordinates by the matrix

$$M_\Phi = L \begin{pmatrix} \frac{n(n-1)}{2} & \frac{n(n-1)}{2} & \frac{n(n-1)(n-3)}{8} \\ 0 & n(n-1) & \frac{n(n-1)^2}{4} \\ n(n-1) & 0 & \frac{n(n-1)^2}{4} \\ 0 & 0 & \frac{n(n^2-1)}{2} \end{pmatrix},$$

where $L = (n-2)!$. Then $Im(\Phi)$ is spanned by the row vectors of M_Φ.

Next, we find a preimage of $Im(\partial'_k)_*$ in $[\mathbb{CP}^2 \wedge C, \Omega W_n]$. In exact sequence (11) $\tilde{K}^0(\Sigma^2 C)$ is $\mathbb{Z} \oplus \mathbb{Z}$. Let α_1 and α_2 be its generators with Chern classes

$$\begin{array}{ll} c_{n-1}(\alpha_1) = (n-2)!\Sigma^2 v_{2n-4} & c_n(\alpha_1) = \frac{(n-1)!}{2}\Sigma^2 v_{2n-2} \\ c_{n-1}(\alpha_2) = 0 & c_n(\alpha_2) = (n-1)!\Sigma^2 v_{2n-2}. \end{array}$$

Lemma 5.2 *For $i = 1, 2$, $(\partial'_k)_*(\alpha_i)$ has a lift $\tilde{\alpha}_{i,k} : \mathbb{CP}^2 \wedge C \to \Omega W_n$ such that*

$$a(\tilde{\alpha}_{i,k}) = ku^2 \otimes \Sigma^{-2} c_{n-1}(\alpha_i) \oplus ku^2 \otimes \Sigma^{-2} c_n(\alpha_i).$$

Proof For dimension and connectivity reasons, $\alpha_i : \Sigma^2 C \to BSU(\infty)$ lifts through $BSU(n) \to BSU(\infty)$. Label the lift $\Sigma^2 C \to BSU(n)$ by α_i as well. Let $\alpha'_i : \Sigma C \to SU(n)$ be the adjoint of α_i. Then $(\partial'_k)_*(\alpha_i)$ is the adjoint of the composition

$$\mathbb{CP}^2 \wedge \Sigma C \xrightarrow{q \wedge \mathbb{1}} \Sigma S^3 \wedge \Sigma C \xrightarrow{\Sigma k\imath \wedge \alpha'_i} \Sigma SU(n) \wedge SU(n) \xrightarrow{[ev,ev]} BSU(n).$$

By Lemma 4.5, $(\partial'_k)_*(\alpha_i)$ has a lift $\tilde{\alpha}_{i,k}$ such that $\tilde{\alpha}^*_{i,k}(a_{2j}) = ku^2 \otimes \Sigma^{-1}(\alpha')^*(x_{2j-3})$. Since $\sigma(c_{j-1}) = x_{2j-3}$, we have $\tilde{\alpha}^*_{i,k}(a_{2j}) = ku^2 \otimes \Sigma^{-2} c_{j-1}(\alpha_i)$ and

$$a(\tilde{\alpha}_{i,k}) = ku^2 \otimes \Sigma^{-2} c_{n-1}(\alpha_i) \oplus ku^2 \otimes \Sigma^{-2} c_n(\alpha_i).$$

□

By Lemma 5.2, $(\partial'_k)_*(\alpha_1)$ and $(\partial'_k)_*(\alpha_2)$ have lifts

$$\tilde{\alpha}_{1,k} = (n-2)!ku^2 v_{2n-4} + \frac{(n-1)!}{2} ku^2 v_{2n-2} \quad \text{and} \quad \tilde{\alpha}_{2,k} = (n-1)!ku^2 v_{2n-2}.$$

We represent their coordinates by the matrix

$$M_\partial = kL \begin{pmatrix} 1 & 0 & \frac{n-3}{4} \\ 0 & 0 & \frac{n-1}{2} \end{pmatrix}.$$

Let $\tilde{\partial}_k = span\{\tilde{\alpha}_{1,k}, \tilde{\alpha}_{2,k}\}$ be the preimage of $Im(\partial'_k)_*$ in $[\mathbb{CP}^2 \wedge C, \Omega W_n]$. Then $\tilde{\partial}_k$ is spanned by the row vectors of M_∂.

Lemma 5.3 *When n is odd, the order of $Im(\partial'_k)_*$ is*

$$|Im(\partial'_k)_*| = \frac{\frac{1}{2}n(n^2-1)}{(\frac{1}{2}n(n^2-1), k)} \cdot \frac{n}{(n,k)}.$$

Proof Suppose $n = 4m + 3$ for some integer m. Then

$$M_\Phi = (4m+3)L \begin{pmatrix} 2m+1 & 2m+1 & 2m^2+m \\ 0 & 4m+2 & 4m^2+4m+1 \\ 4m+2 & 0 & 4m^2+4m+1 \\ 0 & 0 & 8m^2+12m+4 \end{pmatrix}$$

and

$$M_{\partial} = kL \begin{pmatrix} 1 & 0 & m \\ 0 & 0 & 2m+1 \end{pmatrix}.$$

Transform M_{Φ} into Smith normal form

$$A \cdot M_{\Phi} \cdot B = (4m+3)L \begin{pmatrix} (2m+1) & & \\ & (2m+1) & \\ & & (2m+1)(4m+4) \\ & & 0 \end{pmatrix},$$

where

$$A = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -2 & 0 & 1 & 0 \\ 4m+2 & 1 & -(2m+1) & 0 \\ 4 & -2 & -2 & 1 \end{pmatrix} \text{ and } B = \begin{pmatrix} 1 & -m & -(2m+1) \\ 0 & 0 & 1 \\ 0 & 1 & 2 \end{pmatrix}.$$

The matrix B represents a basis change in $Im(a)$ and A represents a basis change in $Im(\Phi)$. Therefore $[\mathbb{CP}^2 \wedge C, SU(n)]$ is isomorphic to

$$\frac{\mathbb{Z}}{\frac{1}{2}(4m+3)!\mathbb{Z}} \oplus \frac{\mathbb{Z}}{\frac{1}{2}(4m+3)!\mathbb{Z}} \oplus \frac{\mathbb{Z}}{\frac{1}{2}(4m+4)!\mathbb{Z}}.$$

We need to find the representation of $\tilde{\partial}_k$ under the new basis represented by B. The new coordinates of $\tilde{\alpha}_{1,k}$ and $\tilde{\alpha}_{2,k}$ are the row vectors of the matrix

$$M_{\partial} \cdot \begin{pmatrix} 1 & -m & -(2m+1) \\ 0 & 0 & 1 \\ 0 & 1 & 2 \end{pmatrix} = \begin{pmatrix} kL & 0 & -kL \\ 0 & (2m+1)kL & (4m+2)kL \end{pmatrix}.$$

Apply row operations to get

$$\begin{pmatrix} 1 & 0 \\ 4m+2 & 1 \end{pmatrix} \cdot \begin{pmatrix} kL & 0 & -kL \\ 0 & (2m+1)kL & (4m+2)kL \end{pmatrix} = \begin{pmatrix} kL & 0 & -kL \\ (4m+2)kL & (2m+1)kL & 0 \end{pmatrix}.$$

Let $\mu = (kL, 0, -kL)$ and $\nu = ((4m+2)kL, (2m+1)kL, 0)$. Then

$$\tilde{\partial}_k = \{x\mu + y\nu \in [\mathbb{CP}^2 \wedge C, \Omega W_n] | x, y \in \mathbb{Z}\}.$$

If $x\mu + y\nu$ and $x'\mu + y'\nu$ are the same modulo $Im(\Phi)$, then we have

$$\begin{cases} xkL + (4m+2)ykL \equiv x'kL + (4m+2)y'kL & (\text{mod } (2m+1)(4m+3)L) \\ (2m+1)ykL \equiv (2m+1)y'kL & (\text{mod } (2m+1)(4m+3)L) \\ xkL \equiv x'kL & (\text{mod } (2m+1)(4m+3)(4m+4)L) \end{cases}$$

These conditions are equivalent to

$$\begin{cases} xk \equiv x'k & (\text{mod } (2m+2)(4m+3)(4m+2)) \\ yk \equiv y'k & (\text{mod } (4m+3)) \end{cases}$$

This implies that there are $\dfrac{(2m+2)(4m+3)(4m+2)}{((2m+2)(4m+3)(4m+2), k)}$ distinct values of x and $\dfrac{4m+3}{(4m+3, k)}$ distinct values of y, so we have

$$|Im(\partial'_k)_*| = \frac{(2m+2)(4m+3)(4m+2)}{((2m+2)(4m+3)(4m+2), k)} \cdot \frac{4m+3}{(4m+3, k)}.$$

When $n = 4m+1$, we can repeat the calculation above to obtain

$$|Im(\partial'_k)_*| = \frac{2m(4m+2)(4m+1)}{(2m(4m+2)(4m+1), k)} \cdot \frac{4m+1}{(4m+1, k)}.$$

□

5.2 The order of $Im(\partial'_k)_*$ for n even

When n is even, C is $S^{2n-2} \vee S^{2n-4}$. For $1 \le i \le 4$, let L_i be the generators of $\tilde{K}^0(\mathbb{CP}^2 \wedge C) \cong \mathbb{Z}^{\oplus 4}$ with Chern characters

$$\begin{aligned} ch(L_1) &= \left(u + \tfrac{1}{2}u^2\right) v_{2n-4} \; ch(L_2) = u^2 v_{2n-4} \\ ch(L_3) &= \left(u + \tfrac{1}{2}u^2\right) v_{2n-2} \; ch(L_4) = u^2 v_{2n-2}. \end{aligned}$$

By Theorem 4.3, we have

$$\begin{aligned} \Phi(L_1) &= \frac{n!}{2}u^2 v_{2n-4} \\ \Phi(L_2) &= n! u^2 v_{2n-4} \\ \Phi(L_3) &= n! u v_{2n-2} + \frac{(n+1)!}{2}u^2 v_{2n-2} \\ \Phi(L_4) &= (n+1)! u^2 v_{2n-2}. \end{aligned}$$

By Lemma 5.1, $Im(a)$ is spanned by $(1, 0, 0)$, $(0, 2, 0)$ and $(0, 0, 1)$. Under this basis, the coordinates of the $\Phi(L_i)$'s are

$$\begin{aligned} \Phi(L_1) &= (\tfrac{n!}{2}, 0, 0), & \Phi(L_2) &= (n!, 0, 0), \\ \Phi(L_3) &= \left(0, \tfrac{n!}{2}, \tfrac{(n+1)!}{2}\right), & \Phi(L_4) &= (0, 0, (n+1)!). \end{aligned}$$

We represent the coordinates of $\Phi(L_i)$'s by the matrix

$$M_\Phi = \frac{n(n-1)}{2} L \begin{pmatrix} 1 & 0 & 0 \\ 2 & 0 & 0 \\ 0 & 1 & n+1 \\ 0 & 0 & 2n+2 \end{pmatrix}$$

Then $Im(\Phi)$ is spanned by the row vectors of M_Φ.

In exact sequence (11) $\tilde{K}^0(\Sigma^2 C)$ is $\mathbb{Z} \oplus \mathbb{Z}$. Let α_1 and α_2 be its generators with Chern classes

$$\begin{array}{ll} c_{n-1}(\alpha_1) = (n-2)!\Sigma^2 v_{2n-4} & c_n(\alpha_1) = 0 \\ c_{n-1}(\alpha_2) = 0 & c_n(\alpha_2) = (n-1)!\Sigma^2 v_{2n-2}. \end{array}$$

By Lemma 5.2, $(\partial_k')_*(\alpha_1)$ and $(\partial_k')_*(\alpha_2)$ have lifts

$$\tilde{\alpha}_{1,k} = (n-2)!ku^2 v_{2n-4} \text{ and } \tilde{\alpha}_{2,k} = (n-1)!ku^2 v_{2n-2}.$$

We represent their coordinates by a matrix

$$M_\partial = kL \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & n-1 \end{pmatrix}.$$

Then the preimage $\tilde{\partial}_k = span\{\tilde{\alpha}_{1,k}, \tilde{\alpha}_{2,k}\}$ of $Im(\partial_k')_*$ is spanned by the row vectors of M_∂. We calculate as in the proof of Lemma 5.3 to obtain the following lemma.

Lemma 5.4 *When n is even, the order of $Im(\partial_k')_*$ is*

$$|Im(\partial_k')_*| = \frac{\frac{1}{2}n(n-1)}{\left(\frac{1}{2}n(n-1), k\right)} \cdot \frac{n(n+1)}{(n(n+1), k)}.$$

5.3 Proof of Theorem 1.6

Before comparing the orders of $Im(\partial_k')_*$ and $Im(\partial_k')_*$, we prove a preliminary lemma.

Lemma 5.5 *Let n be an even number and let p be a prime. Denote the p-component of t by $\nu_p(t)$. If there are integers k and l such that*

$$\nu_p\left(\frac{1}{2}n, k\right) \cdot \nu_p(n,k) = \nu_p\left(\frac{1}{2}n, l\right) \cdot \nu_p(n,l),$$

then $\nu_p(n,k) = \nu_p(n,l)$.

Proof Suppose p is odd. If p does not divide n, then $\nu_p(n,k) = \nu_p(n,l) = 1$, so the lemma holds. If p divides n, then $\nu_p(\frac{1}{2}n, k) = \nu_p(n,k)$. The hypothesis becomes $\nu_p(n,k)^2 = \nu_p(n,l)^2$, implying that $\nu_p(n,k) = \nu_p(n,l)$.

Suppose $p = 2$. Let $\nu_2(n) = 2^r$, $\nu_2(k) = 2^t$ and $\nu_2(l) = 2^s$. Then the hypothesis implies

$$min(r-1, t) + min(r, t) = min(r-1, s) + min(r, s). \tag{14}$$

To show $\nu_2(n, k) = \nu_2(n, l)$, we need to show $min(r, t) = min(r, s)$. Consider the following cases: (1) $t, s \geq r$, (2) $t, s \leq r-1$, (3) $t \leq r-1, s \geq r$ and (4) $s \leq r-1, t \geq r$.

Case (1) obviously gives $min(r, t) = min(r, s)$. In case (2), when $t, s \leq r-1$, equation (14) implies $2t = 2s$. Therefore $t = s$ and $min(r, t) = min(r, s)$.

It remains to show cases (3) and (4). For case (3) with $t \leq r-1$, $s \geq r$, equation (14) implies

$$2t = min(r-1, s) + r.$$

Since $s \geq r$, $min(r-1, s) = r-1$ and the right hand side is $2r-1$ which is odd. However, the left hand side is even, leading to a contradiction. This implies that this case does not satisfy the hypothesis. Case (4) is similar. Therefore $\nu_2(n, k) = \nu_2(n, l)$ and the asserted statement follows. □

Proof of Theorem 1.6 In exact sequence (11), $[C, \mathcal{G}_k(\mathbb{CP}^2)]$ is $Coker(\partial'_k)_*$. By hypothesis, $\mathcal{G}_k(\mathbb{CP}^2)$ is homotopy equivalent to $\mathcal{G}_l(\mathbb{CP}^2)$, so $|Im(\partial'_k)_*| = |Im(\partial'_k)_*|$. The n odd and n even cases are proved similarly, but the even case is harder.

When n is even, by Lemma 5.4 the order of $Im(\partial'_k)_*$ is

$$|Im(\partial'_k)_*| = \frac{\frac{1}{2}n(n-1)}{\left(\frac{1}{2}n(n-1), k\right)} \cdot \frac{n(n+1)}{(n(n+1), k)},$$

so we have

$$\left(\frac{1}{2}n(n-1), k\right) \cdot (n(n+1), k) = \left(\frac{1}{2}n(n-1), l\right) \cdot (n(n+1), l). \tag{15}$$

We need to show that

$$\nu_p(n(n^2-1), k) = \nu_p(n(n^2-1), l) \tag{16}$$

for all primes p. Suppose p does not divide $\frac{1}{2}n(n^2-1)$. Equation (16) holds since both sides are 1. Suppose p divides $\frac{1}{2}n(n^2-1)$. Since $n-1$, n and $n+1$ are coprime, p divides only one of them. If p divides $n-1$, then $\nu_p(\frac{1}{2}n, k) = \nu_p(n, k) = \nu_p(n+1, k) = 1$. Equation (15) implies $\nu_p(n-1, k) = \nu_p(n-1, l)$. Since

$$\nu_p(n(n^2-1), k) = \nu_p(n-1, k) \cdot \nu_p(n, k) \cdot \nu_p(n+1, k),$$

this implies equation (16) holds. If p divides $n+1$, then equation (16) follows from a similar argument. If p divides n, then equation (15) implies $\nu_p(\frac{1}{2}n, k) \cdot \nu_p(n, k) = \nu_p(\frac{1}{2}n, l) \cdot \nu_p(n, l)$. By Lemma 5.5 $\nu_p(n, k) = \nu_p(n, l)$, so equation (16) holds.

When n is odd, by Lemma 5.3 the order of $Im(\partial_k')_*$ is

$$|Im(\partial_k')_*| = \frac{\frac{1}{2}n(n^2-1)}{\left(\frac{1}{2}n(n^2-1), k\right)} \cdot \frac{n}{(n, k)},$$

so we have

$$\left(\frac{1}{2}n(n^2-1), k\right) \cdot (n, k) = \left(\frac{1}{2}n(n^2-1), l\right) \cdot (n, l).$$

We can argue as above to show that for all primes p,

$$\nu_p\left(\frac{1}{2}n(n^2-1), k\right) = \nu_p\left(\frac{1}{2}n(n^2-1), l\right).$$

$\square$

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