Simplicial G-complexes and representation stability of polyhedral products

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Representation stability in the sense of Church-Farb is concerned with stable properties of representations of sequences of algebraic structures, in particular of groups. We study this notion on objects arising in toric topology. With a simplicial G-complex K and a topological pair (X,A), a G-polyhedral product $(X,A)^K$ is associated. We show that the homotopy decomposition [2] of $\Sigma(X,A)^K$ is then G-equivariant after suspension. In the case of Σ_m-polyhedral products, we give criteria on simplicial Σ_m-complexes which imply representation stability of Σ_m-representations $\{H_i((X,A)^{K_m})\}$.

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1 Introduction

Church and Farb [10] introduced the theory of representation stability. The goal of representation stability is to provide a framework for generalising the classical homology stability to situations when each vector space V_m has an action of the symmetric group Σ_m (or other natural families of groups). We initiate the study of representation stability to toric topology.

Let K be a simplicial complex on m vertices. With K, and a topological pair (X,A), a polyhedral product $(X,A)^K$ can be associated in the following way

$$(X,A)^K = \bigcup_{I\in K}(X,A)^I$$

where $(X,A)^I = \{(x_1,\ldots,x_m) \in \prod_{j=1}^{m} X \mid x_j \in A \text{ for } j \notin I\}$.

In particular, when $(X,A)=(D^2,S^1)$, the polyhedral product $(D^2,S^1)^K$ is known as the moment-angle complex $\mathcal{Z}_K$. These objects and their topological and lately homotopy theoretical properties have been of main interest in toric topology.

If a finite group G acts simplicially on a simplicial complex K, then that action induces a G-action on polyhedral products, in particular on the moment-angle complex $\mathcal{Z}_K$. Notice that by acting simplicially on a simplicial complex K on m vertices, G is a subgroup of the symmetric group Σ_m.

In this paper we study Σ_m-representation stability of polyhedral products. We start by analysing G-equivariant properties of the stable homotopy decomposition of moment-angle complexes $\mathcal{Z}_K$ [13, 6] and polyhedral products $(X,A)^K$ [2]. These homotopy decompositions induce $\mathbf{k}G$-module decompositions of the cohomology of moment-angle complexes and polyhedral products, respectively. Recall [5, Corollary 5.4] that for a G-module $N \cong \oplus_{i\in I} N_i$, with the G-action permuting the summands of N according to some G-action on I, there exists a G-isomorphism

$$N \cong \bigoplus_{i\in E} \mathrm{Ind}_{G_i}^{G} N_i \tag{1}$$

where E is a set of representatives of orbits of I and G_i is the stabiliser of i in G.

Specialising to $G=\Sigma_m$, we describe several non-trivial constructions of families of simplicial Σ_m-complexes $\mathcal{K}=\{K_m\}$ (see Constructions 4.3 and 4.4) and describe conditions on these families which together with decomposition (1) and Hemmer's result [12] imply uniform representation stability of Σ_m-representation of $\{\widetilde{H}_*((X,A)^{K_m};\mathbf{k})\}$ (see Theorem 4.8 and Corollary 4.10).

In the case of moment-angle complexes, we construct a sequence of Σ_m-manifolds which are uniformly representation stable although not homology stable (see Proposition 4.11).

The uniform representation stability influences the behaviour of the Betti numbers of the i-th homology groups $\{\widetilde{H}_i((X,A)^{K_m};\mathbb{Q})\}$ and we show that in this case their growth is eventually polynomial with respect to m (see Theorem 5.3).

2 Moment-angle complexes associated with simplicial G-complexes

Moment-angle complexes $\mathcal{Z}_K=(D^2,S^1)^K$ are considered as spaces on which a torus T^l, $l\le m$ acts. The action of the torus is induced by an S^1-action on (D^2,S^1). Extensive literature exists on the study of this action. The problem we are studying is how symmetries of a simplicial complex K influence the symmetries of the moment-angle complex $\mathcal{Z}_K$.

2.1 kG-module structures on $H^*(\mathcal{Z}_K;\mathbf{k})$

Let $\mathbf{k}$ be a field or $\mathbb{Z}$, let G be a finite group, and let K be a simplicial G-complex. We shall describe G-actions on the moment-angle complex $\mathcal{Z}_K$ induced by a simplicial G-action on K.

A *G-complex* is a CW-complex X together with a group action G on it which permutes the cells. A *simplicial G-complex* is a simplicial complex K on a vertex set $[m]$ with a G-action on $[m]$ such that the induced action on subsets of $[m]$ preserves K. Thus, the geometrical realisation of a simplicial G-complex K is a G-complex. For a simplicial G-complex K, each chain group $C_n(K;\mathbf{k})$ is a direct sum of copies of $\mathbf{k}$, each summand corresponding to an n-simplex of K on which G acts. Denote by G_σ the stabiliser of σ, and let E_n be a set of representatives of the G-orbits of n-simplices of K. Thus, by (1),

$$C_n(K;\mathbf{k}) \cong \bigoplus_{\sigma\in E_n} \mathrm{Ind}_{G_\sigma}^G \mathbf{k}.$$

A moment-angle complex $\mathcal{Z}_K$ can be given the following cellular decomposition. The disc D^2 has three cells e^0, e^1 and e^2 of dimensions 0, 1 and 2, respectively. The cells of $D^{2m} \cong (D^2)^m$ are parametrised by subsets $I, L \subseteq [m]$ with $I \cap L = \emptyset$, so that a cell denoted by $\kappa(L, I)$ is equal to $e_1 \times \ldots \times e_m$ in D^{2m}, where e_i is the 2-dimensional cell e^2 if $i \in I$, e_i is the 1-dimensional cell e^1 if $i \in L$, and e_i is the point e^0 if $i \in [m] \setminus (I \cup L)$. Since $\mathcal{Z}_K$ is a subcomplex of D^{2m} determined by the simplicial complex K, the cells of $\mathcal{Z}_K$ are those cells $\kappa(L, I)$, where $I \in K$.

We start by showing that if K is a simplicial G-complex, the corresponding moment-angle complex $\mathcal{Z}_K$ is a G-complex. Let $2^{[m]}$ be the power set of $[m]$. Then the G-action on K can be extended to an action Φ on $2^{[m]}$. Specifically, $\Phi\colon G \times 2^{[m]} \longrightarrow 2^{[m]}$ is given by $\Phi(g, \{i_1, \ldots, i_l\}) = \{g \cdot i_1, \ldots, g \cdot i_l\}$, where $g \in G$ and $\{i_1, \ldots, i_l\} \subset [m]$.

The simplicial G-action on K induces a G-action on $\mathcal{Z}_K$, $\rho : G \times \mathcal{Z}_K \longrightarrow \mathcal{Z}_K$, through homeomorphisms of $\mathcal{Z}_K$ given by

$$\rho_g \cdot (z_1, \ldots, z_m) = (z_{g\cdot 1}, \ldots, z_{g\cdot m}). \tag{2}$$

Lemma 2.1 *For a simplicial G-complex K, the moment-angle complex $\mathcal{Z}_K$ is a G-complex.*

Proof A cell $\kappa(L, I)$, $I \in K$ of $\mathcal{Z}_K$ is mapped by $g \in G$ to $g \cdot \kappa(L, I) = \kappa(g \cdot L, g \cdot I)$ which is again a cell of $\mathcal{Z}_K$ as a simplicial G-action maps simplices to simplices and non-simplices to non-simplices. Thus, $\mathcal{Z}_K$ is a G-complex. □

Geometralising the famous Hochster decomposition [13], Buchstaber and Panov [6, 15] together with Baskakov [3] showed that $H^*(\mathcal{Z}_K;\mathbf{k}) \cong \bigoplus_{J\subseteq[m]} \widetilde{H}^*(K_J;\mathbf{k})$ as $\mathbf{k}$-algebras, where K_J is the full subcomplex of K on J defined by $K_J = \{\sigma \cap J \mid \sigma \in K\}$. We aim to show that this is a $\mathbf{k}G$-algebra isomorphism.

Lemma 2.2 *Let K be a simplicial G-complex on $[m]$. Then for any subset $J \subseteq [m]$ and $g \in G$, the set $g \cdot K_J = \{g \cdot \sigma \mid \sigma \in K_J\}$ is the full subcomplex $K_{g\cdot J}$.*

Proof Since K_J is a subcomplex of K, every subset τ of σ is in K_J if $\sigma \in K_J$. Hence for $\sigma \in K_J$, every subset τ' of $g \cdot \sigma$ is $g \cdot \tau$ for some $\tau \leq \sigma$ and therefore is in $g \cdot K_J$. Thus $g \cdot K_J$ is a subcomplex of K.

To check that $g \cdot K_J$ is the full subcomplex $K_{g\cdot J}$, we observe that $g \cdot K_J = g \cdot (K \cap J) = g \cdot K \cap g \cdot J = K \cap g \cdot J = K_{g\cdot J}$. □

Denote by $\{i_0,\dots,i_p\}$ an unoriented simplex in K and by $[i_0,\dots,i_p]$ an oriented simplex in K. For an oriented p-simplex $\sigma=[i_0,\dots,i_p]$, let $\sigma^*=[i_0,\dots,i_p]^*$ denote the basis cochain in $C^p(K;\mathbf{k})$.

Next, we show that a simplicial G-action on K induces a G-action on $\bigoplus\limits_{J\subseteq[m]}\widetilde{H}^*(K_J;\mathbf{k})$.

Lemma 2.3 *Let K be a simplicial G-complex. For every $g\in G$ and $J\subseteq[m]$,*

$$g\cdot\widetilde{H}^*(K_J;\mathbf{k})=\widetilde{H}^*(K_{g\cdot J};\mathbf{k}).$$

Proof Let $\sigma=[i_0,\dots,i_p]$ be an oriented simplex in K_J and σ^* be the corresponding base cochain in $C^p(K_J;\mathbf{k})$. Since g gives a bijection between the basis of $C^*(K_J;\mathbf{k})$ and the basis of $C^*(K_{g\cdot J};\mathbf{k})$ by $\sigma^*\mapsto g\cdot\sigma^*=\epsilon(g,\sigma)(g\cdot\sigma)^*$, the cochain complex $C^*(K_J;\mathbf{k})$ is isomorphic to $C^*(K_{g\cdot J};\mathbf{k})$ as abelian groups. As the coboundary operator d is given by

$$d\sigma^*=\sum\varepsilon_j\tau_j^*$$

where the summation of the coboundary operator extends over all $(p+1)$-simplices τ_j having σ as a face, and $\varepsilon_j=\pm1$ is the sign with which σ appears in the expression for $\partial\tau$, we obtain the commutative diagram

$$\begin{array}{ccc} C^*(K_J;\mathbf{k}) & \xrightarrow{\cong} & C^*(K_{g\cdot J};\mathbf{k}) \\ \downarrow{\scriptstyle d} & & \downarrow{\scriptstyle d} \\ C^*(K_J;\mathbf{k}) & \xrightarrow{\cong} & C^*(K_{g\cdot J};\mathbf{k}). \end{array}$$

Therefore g induces an isomorphism between $\widetilde{H}^*(K_J;\mathbf{k})$ and $\widetilde{H}^*(K_{g\cdot J};\mathbf{k})$. □

We continue by showing that the G-actions on $H^*(\mathcal{Z}_K;\mathbf{k})$ and $\bigoplus\limits_{J\subseteq[m]}\widetilde{H}^*(K_J;\mathbf{k})$ are compatible.

On $\mathcal{C}^*(\mathcal{Z}_K;\mathbf{k})$ a multigrading can be defined. Consider a subset $J\subseteq[m]$ as a vector in $\mathbb{N}^m$ whose j-th coordinate is 1 if $j\in J$, or is 0 if $j\notin J$. Define a $\mathbb{Z}\oplus\mathbb{N}^m$-grading on $\mathcal{C}^*(\mathcal{Z}_K;\mathbf{k})$ as

$$\mathcal{C}^*(\mathcal{Z}_K;\mathbf{k})=\bigoplus_{J\subseteq[m]}\mathcal{C}^{*,2J}(\mathcal{Z}_K;\mathbf{k})$$

where $\mathcal{C}^{*,2J}(\mathcal{Z}_K;\mathbf{k})$ is the subcomplex spanned by cochains $\kappa(J\setminus I,I)^*$ with $I\subseteq J$ and $I\in K$ whose multidegree is $\mathrm{mg}\kappa(J\setminus I,I)^*=(-|J\setminus I|,J)$.

Buchstaber and Panov [7, Theorem 3.2.9] showed that there are isomorphisms between $\widetilde{H}^{p-1}(K_J;\mathbf{k})$ and $H^{p-|J|,2J}(\mathcal{Z}_K;\mathbf{k})$ which are functorial with respect to simplicial maps and are induced by the cochain isomorphisms $f_J\colon C^{p-1}(K_J;\mathbf{k})\to\mathcal{C}^{p-|J|,2J}(\mathcal{Z}_K;\mathbf{k})$ given by

$$f_J(\sigma^*)=\epsilon(\sigma,J)\kappa(J\setminus\sigma,\sigma)^* \tag{3}$$

where $\sigma\in K_J$ and $\epsilon(\sigma,J)=\prod\limits_{j\in\sigma}\epsilon(j,J)$ with $\epsilon(j,J)=(-1)^{r-1}$ if j is the r-th element of J.

The functorial property induces a commutative diagram

$$\begin{array}{ccc} C^{p-1}(K_J;\mathbf{k}) & \xrightarrow{f_J} & \mathcal{C}^{p-|J|,2J}(\mathcal{Z}_K;\mathbf{k}) \\ \downarrow g & & \downarrow g \\ C^{p-1}(K_{g\cdot J};\mathbf{k}) & \xrightarrow{f_{g\cdot J}} & \mathcal{C}^{p-|g\cdot J|,2g\cdot J}(\mathcal{Z}_K;\mathbf{k}) \end{array}$$

implying the following statement.

Lemma 2.4 *If K is a simplicial G-complex, then $\mathcal{C}^*(\mathcal{Z}_K;\mathbf{k})$ is multigraded isomorphic to $\bigoplus_{J\subseteq[m]} C^*(K_J;\mathbf{k})$ as $\mathbf{k}G$-modules.*

Passing to cohomology, we obtain the following corollary.

Corollary 2.5 *For a simplicial G-complex, $H^*(\mathcal{Z}_K;\mathbf{k})$ is isomorphic to $\bigoplus_{J\subseteq[m]} \widetilde{H}^*(K_J;\mathbf{k})$ as $\mathbf{k}G$-algebras.*

Proof By [7, Theorem 4.5.8], the multiplication on $\bigoplus_{J\subseteq[m]} \widetilde{H}^*(K_J;\mathbf{k})$ is given by

$$H^i(K_I;\mathbf{k})\otimes H^j(K_J;\mathbf{k})\to H^{i+j}(K_{I\cup J};\mathbf{k})$$

which is induced by the simplicial inclusions $K_{I\cup J}\to K_I * K_J$ for $I\cap J=\emptyset$ and zero otherwise. Under this multiplication, the maps f_J induce a $\mathbf{k}$-algebraic isomorphism $\bigoplus_{J\subseteq[m]} \widetilde{H}^*(K_J;\mathbf{k})\to H^*(\mathcal{Z}_K;\mathbf{k})$. Since $f_{g\cdot J}\circ g = g\circ f_J$, the maps f_J induce a $\mathbf{k}G$-algebraic isomorphism. □

Now we state the main result of this section.

Proposition 2.6 *Let K be a simplicial G-complex. Then there are $\mathbf{k}G$-algebra isomorphisms*

$$H^*(\mathcal{Z}_K;\mathbf{k})\cong \bigoplus_{J\in[m]/G}\ \bigoplus_{g\in G/G_J} g\cdot \widetilde{H}^{*-|J|-1}(K_J;\mathbf{k})$$

where $G_J=\{g\in G\mid g\cdot J=J\}$ is the stabiliser of J and $[m]/G$ is a set of representatives of G-orbits of $2^{[m]}$.

The multiplication on $\bigoplus_{J\in[m]/G}\bigoplus_{g\in G/G_J} g\cdot\widetilde{H}^*(K_J;\mathbf{k})$ is given so that for any $I,J\in[m]/G$ and $g\in G/G_J,\ h\in G/G_I$, there is a map

$$\mu\colon g\cdot H^{k-|J|-1}(K_J;\mathbf{k})\otimes h\cdot H^{l-|I|-1}(K_I;\mathbf{k}) = H^{k-|J|-1}(K_{g\cdot J};\mathbf{k})\otimes H^{l-|I|-1}(K_{h\cdot I};\mathbf{k})\to H^{k+l-|I|-|J|-1}(K_{g\cdot J\cup h\cdot I};\mathbf{k})$$

which is induced by the simplicial inclusion $K_{g\cdot J\cup h\cdot I} \longrightarrow K_{g\cdot J} * K_{h\cdot I}$ if $g\cdot J\cap h\cdot I=\emptyset$ and is a zero map otherwise.

Proof Since by Corollary 2.5 $H^*(\mathcal{Z}_K;\mathbf{k}) \cong \bigoplus_{J\subseteq[m]} \widetilde{H}^*(K_J;\mathbf{k})$ as $\mathbf{k}G$-algebras, it suffices to show that the G-isomorphism

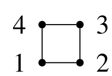
$$\bigoplus_{J\subseteq[m]} \widetilde{H}^*(K_J;\mathbf{k}) \cong \bigoplus_{J\in[m]/G}\ \bigoplus_{g\in G/G_J} g\cdot \widetilde{H}^*(K_J;\mathbf{k})$$

preserves the multiplications on both sides. The multiplication on $\bigoplus_{J\in[m]/G}\ \bigoplus_{g\in G/G_J} g\cdot \widetilde{H}^*(K_J;\mathbf{k})$ is induced by the multiplication on $\bigoplus_{J\subseteq[m]} \widetilde{H}^*(K_J;\mathbf{k})$ via the above G-isomorphism. Therefore,

$$H^*(\mathcal{Z}_K;\mathbf{k}) \cong \bigoplus_{J\in[m]/G}\ \bigoplus_{g\in G/G_J} g\cdot \widetilde{H}^{*-|J|-1}(K_J;\mathbf{k})$$

as $\mathbf{k}G$-algebras. □

We illustrate Proposition 2.6 on several examples.

Example 2.7 Let K be the boundary of a square, 

. It is a simplicial C_4-complex, where C_4 is the cyclic group of order 4. Write $C_4=\{(1),(1234),(13)(24),(1432)\}$ as a subgroup of the permutation group Σ_4. A set of representatives of $2^{[4]}$ under C_4 is given by

$$E=\{\emptyset,\{1\},\{1,2\},\{1,3\},\{1,2,3\},\{1,2,3,4\}\}.$$

Taking J to be an element in E, observe that

$$\widetilde{H}^p(K_J;\mathbf{k}) = \begin{cases} \mathbf{k} & \text{when } J=\emptyset \text{ and } p=-1\\ \mathbf{k} & \text{when } J=\{1,3\} \text{ and } p=0\\ \mathbf{k} & \text{when } J=\{1,2,3,4\} \text{ and } p=1\\ 0 & \text{otherwise.}\end{cases}$$

The stabilisers G_J corresponding to $J=\emptyset$, $J=\{1,3\}$ and $J=\{1,2,3,4\}$ are $G_\emptyset=C_4$, $G_{13}=\{(1),(13)(24)\}$ and $G_{1234}=C_4$, respectively. Therefore, the cohomology groups of $\mathcal{Z}_K$ are given by

$$H^i(\mathcal{Z}_K;\mathbf{k}) = \begin{cases} \mathbf{k}\oplus\mathbf{k} & \text{for } i=3\\ \mathbf{k} & \text{for } i=0,6.\end{cases}$$

Example 2.8 Let $K = \Delta_m^k$ be the full k-skeleton of Δ^{m-1} which consists all subsets of $[m]$ with cardinality at most $k+1$. The permutation group Σ_m acts on K simplicially. A set of representatives of $2^{[m]}$ under the action of Σ_m can be also chosen as

$$E = \{\emptyset, \{1\}, \{1,2\}, \dots, \{1,\dots,m\}\}.$$

For any $J = \{1,2,\dots,|J|\} \in E$, the stabiliser of J is the Young subgroup $\Sigma_{|J|} \times \Sigma_{m-|J|}$. If $J \in E$ with $|J| \leq k+1$, then $K_J = \Delta^{|J|-1}$. Thus $\widetilde{H}^*(K_J;\mathbf{k}) = 0$.

If $J \in E$ with $k+2 \leq |J| \leq m$, then K_J is the full k-skeleton of $\Delta^{|J|-1}$. Recall that $\widetilde{H}^*(K_J;\mathbf{k}) = \bigoplus_c \mathbf{k}$, where $c = \binom{|J|-1}{k+1}$ if $* = k$; otherwise $\widetilde{H}^*(K_J;\mathbf{k}) = 0$. Therefore,

$$H^i(\mathcal{Z}_K;\mathbf{k}) = \begin{cases} \mathbf{k} & \text{where } i = 0 \\ \bigoplus_c \mathbf{k} & \text{where } c = \binom{m}{|J|}\binom{|J|-1}{k+1} \text{ and } i = |J|+k+1 \\ 0 & \text{otherwise.} \end{cases}$$

Let us remark that for $k = 0$, the simplicial complex K consists of m disjoint points and denote by $\mathcal{Z}_m$ the moment-angle complex corresponding to it. By Proposition 2.6, $H^3(\mathcal{Z}_m;\mathbf{k})$ has a basis $\{a_{ij} \mid 1 \leq i < j \leq m\}$ and identifying $a_{ji} = -a_{ij}$, the symmetric group Σ_m acts on $H^3(\mathcal{Z}_m;\mathbf{k})$ by a permutation of the indices.

For $K_m = \Delta_m^k$ with k fixed and m increasing, we get a sequence of moment-angle complexes $\{\mathcal{Z}_{K_m}\}$. There exist retractions $p_m\colon \mathcal{Z}_{K_{m+1}} \longrightarrow \mathcal{Z}_{K_m}$ obtained by restricting the projection map $(D^2)^{m+1} \longrightarrow (D^2)^m$ to $\mathcal{Z}_K$. We shall consider the representation stability of the sequence $\{H^i(\mathcal{Z}_K;\mathbf{k}), p_m^i\}$ in Section 4.

3 Polyhedral products associated with simplicial G-complexes

Moment-angle complexes are specific examples of polyhedral products $(X,A)^K$ which are constructed from combinatorial information of a simplicial complex K and a topological pair (X,A). Our next aim is to study symmetries of polyhedral products induced by the symmetries of K. The geometric and homological properties of polyhedral products arising from simplicial Aut(K)-complexes have been studied by Ali Al-Raisi in his PhD thesis [1]. Al-Raisi proved that the map $(X,A)^K \longrightarrow \Omega\Sigma(\bigvee_{I\subseteq[m]} (X,A)^{\wedge K_I})$ is homotopy Aut(K)-equivariant.

In this section, we will give a different method for studying homotopy G-decompositions of polyhedral product $(X,A)^K$ associated with a simplicial G-complex K by studying the adjoint of the Al-Raisi map, known as the Bahri-Bendersky-Cohen-Gitler (BBCG) map, after several suspensions. We start with the BBCG homotopy decomposition for polyhedral products $(X,A)^K$

(see [2]). For any subset $I = \{i_1, \dots, i_l\} \subseteq [m]$, and a pair of connected based CW-complexes (X,A), recall the following notation

$$\begin{aligned}(X,A)^I &= \{(x_1,\dots,x_m) \in \prod_{j=1}^{m} X \mid x_j \in A \text{ for } j \notin I\},\\ (X,A)^{\wedge I} &= \{x_1 \wedge \dots \wedge x_m \in X \wedge \dots \wedge X \mid x_j \in A \text{ for } j \notin I\},\\ X^{\wedge I} &= X_{i_1} \wedge \dots \wedge X_{i_l}, \text{ where each } X_{i_j} = X.\end{aligned}$$

For a simplicial complex K on m vertices, the *polyhedral product* with respect to (X,A) is defined as

$$(X,A)^K = \bigcup_{I \in K} (X,A)^I.$$

Analogously, the *polyhedral smash product* of a topological pair (X,A) and a simplicial complex K is defined as

$$(X,A)^{\wedge K} = \bigcup_{I \in K} (X,A)^{\wedge I}.$$

In [2], it was shown that the classical homotopy equivalence $\Sigma(X_1 \times \dots \times X_m) \to \Sigma \bigvee_{I \subseteq [m]} X^{\wedge I}$ induces the following homotopy decomposition.

$$\Sigma(X,A)^K \simeq \Sigma \bigvee_{I \subseteq [m]} (X,A)^{\wedge K_I} \tag{4}$$

when (X,A) is a topological pair of connected and based CW-complexes.

If K is a simplicial G-complex, then the G-action on K induces a cellular G-action on the corresponding polyhedral product $(X,A)^K$ with respect to a pair of CW-complexes (X,A), $A \subseteq X$. Explicitly, for $\underline{x} = (x_1,\dots,x_m) \in (X,A)^K$, $g \cdot (x_1,\dots,x_m) = (x_{g\cdot 1},\dots,x_{g\cdot m})$. Thus $(X,A)^K$ is a G-complex.

If Y is a G-CW-complex, then each i-th homology group $H_i(Y;R)$ is an RG-module. Consider a natural G-action on ΣY by $g \cdot (\langle y,t\rangle) = \langle g \cdot y, t\rangle$ for $g \in G$. The naturality of long exact sequence for the topological pair (CY,Y) implies that the isomorphism $H_{i+1}(\Sigma Y;R) \cong \widetilde{H}_i(Y;R)$ is an RG-isomorphism.

Consider X^m as a Σ_m-space given by $g \cdot \underline{x} = g \cdot (x_1,\dots,x_m) = (x_{g\cdot 1},\dots,x_{g\cdot m})$ for $g \in \Sigma_m$ and $x_i \in X$. There exists a Σ_m-action on the based spaces ΣX^m and $\Sigma(\bigvee_{I \subseteq [m]} X^{\wedge I})$, where I runs over the non-empty subset of $[m]$. Explicitly, for every $g \in \Sigma_m$ and $\langle \underline{x}, t\rangle \in \Sigma X^m$, $g \cdot \langle \underline{x}, t\rangle = \langle g \cdot \underline{x}, t\rangle$. For any non-empty subset $I = \{i_1,\dots,i_l\} \subseteq [m]$, each map $g\colon \Sigma X^{\wedge I} \to \Sigma X^{\wedge g\cdot I}$ sending $\langle x_{i_1} \wedge \dots \wedge x_{i_l}, t\rangle$ to $\langle x_{g\cdot i_1} \wedge \dots \wedge x_{g\cdot i_l}, t\rangle$ induces a Σ_m-action on $\Sigma \bigvee_{I \subseteq [m]} X^{\wedge I}$.

Lemma 3.1 *There exists a homotopy equivalence*

$$\Sigma\theta_m \colon \Sigma^2 X^m \longrightarrow \Sigma^2 \bigvee_{I\subseteq[m]} X^{\wedge I}$$

that is Σ_m-equivariant.

Proof For a non-empty set $I = \{i_1, \dots, i_l\} \subseteq [m]$, define maps $\Sigma p^{\wedge I}$ by

$$\begin{aligned} \Sigma p^{\wedge I} \colon \Sigma X^m &\longrightarrow \Sigma X^{\wedge I} \\ \langle x_1, \dots, x_m, t\rangle &\longmapsto \langle x_{i_1} \wedge \dots \wedge x_{i_l}, t\rangle \end{aligned}$$

Let $L = 2^m - 1$. Define a comultiplication map $\delta_m \colon \Sigma X^m \longrightarrow \bigvee_{j=1}^{L} \Sigma X^m$ on ΣX^m such that if $t \in [\frac{i}{L}, \frac{i+1}{L}]$ $(0 \le i \le L-1)$,

$$\delta_m(\langle x_1, \dots, x_m, t\rangle) = (*, \dots, *, \langle x_1, \dots, x_m, Lt - i\rangle, *, \dots, *)$$

where $\langle x_1, \dots, x_m, Lt - i\rangle$ is in the $(i+1)$-st wedge summand of $\bigvee_{j=1}^{L} \Sigma X^m$.

Fix an order $I_1 > I_2 > \cdots > I_L$ on the finite set $\{I \subseteq [m] \mid I \neq \emptyset\}$. Let each I_j contain elements written in an increasing order. Rewrite $\Sigma(\bigvee_{I\subseteq[m]} X^{\wedge I})$ as $\Sigma X^{\wedge I_1} \vee \dots \vee \Sigma X^{\wedge I_L}$.

Consider a map $\bigvee_{I\in 2^{[m]}\setminus\emptyset} \Sigma p^{\wedge I} \colon \bigvee_{j=1}^{L} \Sigma X^m \longrightarrow \Sigma(\bigvee_{I\subseteq[m]} X^{\wedge I})$ given by

$$\bigvee_{I\in 2^{[m]}\setminus\emptyset} \Sigma p^{\wedge I} = \Sigma p^{\wedge I_1} \vee \dots \vee \Sigma p^{\wedge I_L} \colon \bigvee_{j=1}^{L} \Sigma X^m \longrightarrow \Sigma X^{\wedge I_1} \vee \dots \vee \Sigma X^{\wedge I_L}.$$

Thus the map

$$\theta_m = \bigvee_{I\in 2^{[m]}\setminus\emptyset} \Sigma p^{\wedge I} \circ \delta_m.$$

Let $g \in \Sigma_m$ and $\langle x_1, \dots, x_m, t\rangle \in \Sigma X^m$. For $t \in [\frac{i}{L}, \frac{i+1}{L}]$ $(0 \le i \le L-1)$, there is

$$\begin{aligned} \theta_m \circ g(\langle x_1, \dots, x_m, t\rangle) &= \bigvee_{I\in 2^{[m]}\setminus\emptyset} \Sigma p^{\wedge I} \circ \delta_m(\langle x_{g\cdot 1}, \dots, x_{g\cdot m}, t\rangle) \\ &= \bigvee_{I\in 2^{[m]}\setminus\emptyset} \Sigma p^{\wedge I}(*, \dots, *, \underbrace{\langle x_{g\cdot 1}, \dots, x_{g\cdot m}, Lt - i\rangle}_{i+1}, *, \dots, *) \\ &= (*, \dots, *, \underbrace{\langle x_{g\cdot m_1^{(i+1)}} \wedge \dots \wedge x_{g\cdot m_s^{(i+1)}}, Lt - i\rangle}_{i+1}, *, \dots, *) \end{aligned}$$

where $I_{i+1} = \{m_1^{(i+1)}, \dots, m_s^{(i+1)}\}$ with $m_1^{(i+1)} < \cdots < m_s^{(i+1)}$.

Recall that $\bigvee\limits_{I\in 2^{[m]}\setminus\emptyset}\Sigma p^{\wedge I}=\Sigma p^{\wedge I_1}\vee\ldots\vee\Sigma p^{\wedge I_L}$ and define by $\bigvee\limits_{I\in 2^{[m]}\setminus\emptyset}\Sigma p^{\wedge(g\cdot I)}=\Sigma p^{\wedge(g\cdot I_1)}\vee\ldots\vee\Sigma p^{\wedge(g\cdot I_L)}$. Hence, $\theta_m\circ g=\bigvee\limits_{I\in 2^{[m]}\setminus\emptyset}\Sigma p^{\wedge(g\cdot I)}\circ\delta_m$.

On the other hand, there exits a permutation T of summand $\bigvee\limits_{j=1}^{L}\Sigma X^m$ induced by g such that $g\circ\theta_m=\bigvee\limits_{I\in 2^{[m]}\setminus\emptyset}\Sigma p^{\wedge(g\cdot I)}\circ T\circ\delta_m$. Since g acts on a set $\{1,\ldots,L\}$ by $g\cdot i$ being the unique number satisfying $I_{g\cdot i}=g\cdot I_i$ as sets, this action on $\{1,\ldots,L\}$ induces a permutation T of $\bigvee\limits_{j=1}^{L}\Sigma X^m$. Note that for $t\in[\frac{i}{L},\frac{i+1}{L}]$ $(0\le i\le L-1)$,

$$\begin{aligned}
&\bigvee_{I\in 2^{[m]}\setminus\emptyset}\Sigma p^{\wedge(g\cdot I)}\circ T\circ\delta_m(\langle x_1,\ldots,x_m,t\rangle)\\
=&\bigvee_{I\in 2^{[m]}\setminus\emptyset}\Sigma p^{\wedge(g\cdot I)}\circ T(*,\ldots,*,\underbrace{\langle x_1,\ldots,x_m,Lt-i\rangle}_{i+1},*,\ldots,*)\\
=&\bigvee_{I\in 2^{[m]}\setminus\emptyset}\Sigma p^{\wedge(g\cdot I)}(*,\ldots,*,\underbrace{\langle x_1,\ldots,x_m,Lt-i\rangle}_{g\cdot(i+1)},*,\ldots,*)\\
=&(*,\ldots,*,\underbrace{\langle x_{g\cdot m_1^{(i+1)}}\wedge\ldots\wedge x_{g\cdot m_s^{(i+1)}},Lt-i\rangle}_{g\cdot(i+1)},*,\ldots,*)
\end{aligned}$$

where $I_{i+1}=\{m_1^{(i+1)},\ldots,m_s^{(i+1)}\}$ with $m_1^{(i+1)}<\cdots<m_s^{(i+1)}$.

Also, for $t\in[\frac{i}{L},\frac{i+1}{L}]$ $(0\le i\le L-1)$,

$$\begin{aligned}
g\circ\theta_m(\langle x_1,\ldots,x_m,t\rangle)&=g(*,\ldots,*,\underbrace{\langle x_{m_1^{(i+1)}}\wedge\ldots\wedge x_{m_s^{(i+1)}},Lt-i\rangle}_{i+1},*,\ldots,*)\\
&=(*,\ldots,*,\underbrace{\langle x_{g\cdot m_1^{(i+1)}}\wedge\ldots\wedge x_{g\cdot m_s^{(i+1)}},Lt-i\rangle}_{g\cdot(i+1)},*,\ldots,*).
\end{aligned}$$

Thus we have $g\circ\theta_m=\bigvee\limits_{I\in 2^{[m]}\setminus\emptyset}\Sigma p^{\wedge(g\cdot I)}\circ T\circ\delta_m$.

Since $\Sigma\delta_m$ is cocommutative, $\Sigma(g\circ\theta_m)\simeq\Sigma(\theta_m\circ g)$. □

The following statement is a consequence of Lemma 3.1.

Lemma 3.2 *a) For $g \in \Sigma_m$ and $I \subseteq [m]$, there is the homotopy commutative diagram*

(5)
$$\begin{array}{ccc}
\Sigma^2(X,A)^I & \xrightarrow{\simeq} & \bigvee_{J\subseteq[m]} \Sigma^2(X,A)^{\wedge(I\cap J)} \\
\downarrow{\scriptstyle g} & & \downarrow{\scriptstyle g} \\
\Sigma^2(X,A)^{g\cdot I} & \xrightarrow{\simeq} & \bigvee_{g\cdot J\subseteq[m]} \Sigma^2(X,A)^{\wedge g\cdot(I\cap J)}
\end{array}$$

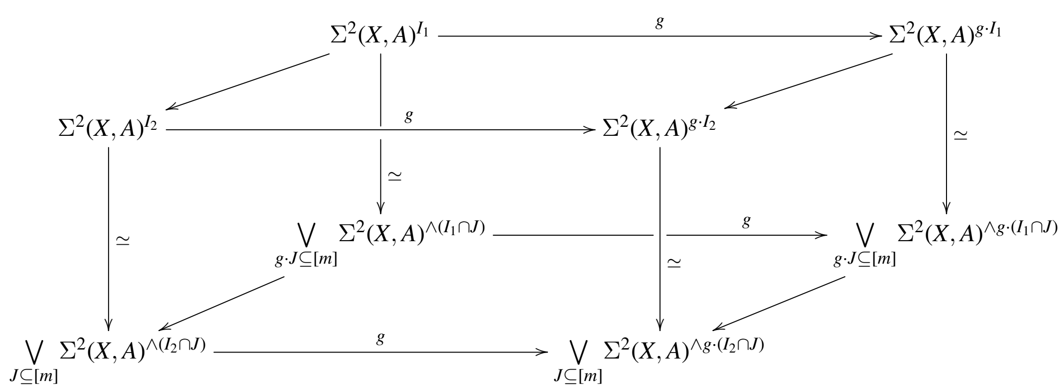
where the vertical map g on the left is given by

$$g\cdot\langle x_1,\dots,x_m,t,s\rangle = \langle x_{g\cdot 1},\dots,x_{g\cdot m},t,s\rangle$$

and the vertical map g on the right maps each element in $\Sigma^2(X,A)^{\wedge(I\cap J)}$ into the corresponding one in $\Sigma^2(X,A)^{\wedge g\cdot(I\cap J)}$ via a coordinate permutation by g.

b) For an inclusion $I_1 \subseteq I_2 \subseteq [m]$, there is the diagram

(6)



where the four side diagrams are homotopy commutative and the top and bottom diagrams are commutative. □

Since the homotopy decomposition $\Sigma^2(X,A)^K \simeq \Sigma^2 \bigvee_{J\subseteq[m]} (X,A)^{\wedge K_J}$ is natural with respect to inclusions in K ([2, Theorem 2.10]), the next result follows immediately from the lemma above.

Theorem 3.3 *Let K be a simplicial G-complex with m vertices. Then there is a homotopy G-decomposition*

(7)
$$\theta\colon \Sigma^2(X,A)^K \simeq \Sigma^2 \bigvee_{J\subseteq[m]} (X,A)^{\wedge K_J}$$

where the G-action on $\Sigma^2(X,A)^K$ is induced by the G-action on X^m, and the G-action on the right hand side is induced by (5).

Proof Let CAT(K) be the face category of K consisting of simplices of K and simplicial inclusions in K. Define two functors D and E from CAT(K) to CW_* by $D(\sigma) = (X,A)^\sigma$ and $E(\sigma) = \bigvee_{J\subseteq[m]} (X,A)^{\wedge(\sigma\cap J)}$ for $\sigma \in$ CAT(K). For every $\sigma \in$ CAT(K) and $g \in G$, diagram (5) implies that there exists a homotopy

$$H_g(\sigma)\colon \Sigma^2 D(\sigma)\times \mathbb{I} \longrightarrow \Sigma^2 E(g\cdot\sigma)$$

such that $H_g(\sigma)(x,0) = \theta(\sigma)$ and $H_g(\sigma)(x,1) = \theta(g\cdot\sigma)$, where $\theta(\sigma)$ is the natural homotopy equivalence between $\Sigma^2 D(\sigma)$ and $\Sigma^2 E(\sigma)$ and $\mathbb{I}$ is the interval $[0,1]$. Diagram (6) implies that if $\sigma,\tau \in$ CAT(K), then $H_g(\sigma\cap\tau) = H_g(\sigma)|_{\Sigma^2 D(\sigma\cap\tau)\times\mathbb{I}} = H_g(\tau)|_{\Sigma^2 D(\sigma\cap\tau)\times\mathbb{I}}$. $H_g(\cdot)$ induces a natural transformation from $\Sigma^2 D(\cdot)\times\mathbb{I}$ to $\Sigma^2 E(\cdot)$.

With g fixed, $H_g(\sigma)$ will induce a continuous map $H_g\colon \operatorname{colim}\Sigma^2 D\times\mathbb{I} \longrightarrow \operatorname{colim}\Sigma^2 E$ such that $H_g(x,0) = g\theta(x)$ and $H_g(x,1) = \theta(g\cdot x)$. Therefore, θ is a homotopy G-decomposition. □

Example 3.4 Let K be the k-skeleton of a simplex Δ^{m-1} on which Σ_m acts by permuting vertices. By Porter [16], Grbić-Theriault [11], the homotopy type of $(\mathrm{Cone}A,A)^K$ is the wedge

$$(\mathrm{Cone}A,A)^K \simeq \bigvee_{j=k+2}^{m}\Big(\bigvee_{1\le i_1<\ldots<i_j\le m}\binom{j-1}{k+1}\Sigma^{k+1}A_{i_1}\wedge\ldots\wedge A_{i_j}\Big).$$

Although Σ_m acts on both sides this homotopy equivalence might not be a homotopy Σ_m-equivalence. However after suspending it twice, by Theorem 3.3 it is a homotopy equivariant map.

Considering G-equivalence (7) and observing the induced G-actions on the reduced homology groups, we have the following result.

Theorem 3.5 *Let K be a simplicial G-complex on m vertices. Then there exists a $\mathbf{k}G$-module isomorphism*

$$\widetilde{H}_i((X,A)^K;\mathbf{k}) \cong \bigoplus_{J\subseteq[m]}\widetilde{H}_i((X,A)^{\wedge K_J};\mathbf{k}) \cong \bigoplus_{J\in[m]/G}\mathrm{Ind}_{G_J}^{G}\widetilde{H}_i((X,A)^{\wedge K_J};\mathbf{k})$$

where G acts on the middle term by permuting the summands such that $g\cdot\widetilde{H}_i((X,A)^{\wedge K_J};\mathbf{k}) = \widetilde{H}_i((X,A)^{K_{g\cdot J}};\mathbf{k})$, $[m]/G$ is a set of representatives of orbit of $2^{[m]}\setminus\emptyset$ under G and G_J is the stabiliser of J. □

4 Representation stability for polyhedral products

Let G be a finite group and $\mathbf{k}$ be a field of characteristic zero. Then a G-action on a simplicial complex K induces a G-complex structure on the corresponding polyhedral product $(X,A)^K$ and therefore its homology is a $\mathbf{k}G$-module. Since every $\mathbf{k}G$-module is a G-representation over $\mathbf{k}$, we are able to use representation theory to study the homology groups of polyhedral products associated with simplicial G-complexes. Representation stability studies a sequence of finite dimensional vector spaces such that each vector space V_m is equipped with a G_m-action and each $V_m \xrightarrow{\psi_m} V_{m+1}$ is G_m-equivariant. Here groups G_m are not arbitrary; they all belong to a fixed family of groups whose $\mathbf{k}$-linear irreducible representations are determined by some datum λ which is independent of G_m and therefore of m. One such family consists of symmetric groups Σ_m, which we will consider in this section. The idea of representation stability was firstly introduced by Church and Farb in [10, Section 2.3]. Stability in representation theory generalises a classical homological stability. A sequence $\{Y_m\}$ of groups, manifolds or topological spaces with maps $Y_m \xrightarrow{\psi_m} Y_{m+1}$ for each $i \geq 0$ is called homology stable if the map $H_i(Y_m) \xrightarrow{(\psi_m)_*} H_i(Y_{m+1})$ is an isomorphism for a sufficiently large m.

We recall the precise definition of uniformly representation stability of representations of symmetric groups according to Church and Farb [10, Definition 2.6]. Let $\{V_m, \psi_m\}$ be a sequence of Σ_m-representations so that the group Σ_m acts on V_{m+1} as a subgroup of Σ_{m+1}. Then it is *consistent* if each V_m decomposes as a direct sum of finite-dimensional irreducible representations. If given any partition $\lambda = (\lambda_1, \dots, \lambda_l) \vdash k$, then for $m \geq \lambda_1 + k$, the partition $\lambda[m] = (m-k, \lambda_1, \dots, \lambda_l)$ is called *padded partition*. Its corresponding irreducible representation is denoted by $V(\lambda)_m$.

Let now $\{V_m, \psi_m\}$ be a consistent sequence of Σ_m-representations over a field $\mathbf{k}$ of characteristic 0. The sequence $\{V_m, \psi_m\}$ is *uniformly representation stable* with stable range $m \geq N$ if each of the following conditions holds for all $m \geq N$.

1. Injectivity: The natural map $\psi_m \colon V_m \to V_{m+1}$ is injective.
2. Surjectivity: The Σ_{m+1}-orbit of $\psi_m(V_m)$ spans V_{m+1}.
3. Multiplicities (uniform): Decompose V_m into irreducible representations as

$$V_m = \bigoplus_{\lambda} c_{\lambda,m} V(\lambda)_m$$

with multiplicities $0 \leq c_{\lambda,m} \leq \infty$. There is some M, not depending on λ, so that for $m \geq M$ the multiplicities $c_{\lambda,m}$ are independent of m for all λ.

Hemmer [12] proved the uniform representation stability of Σ_m-representations $\{\mathrm{Ind}_{H\times\Sigma_{m-k}}^{\Sigma_m} V \boxtimes \mathbf{k}\}$ induced by an H-representation V, where $H \leq \Sigma_k$. Note that V can be seen as an $(H \times \Sigma_{m-k})$-representation, where Σ_{m-k} acts on V trivially, denoted by $V \boxtimes \mathbf{k}$ for any $m \geq k$.

In this section, we study the representation stability arising in polyhedral products over a sequence of finite simplicial Σ_m-complexes.

Definition 4.1 A sequence of finite simplicial complexes

$$K_0 \subseteq K_1 \subseteq K_2 \subseteq \ldots \subseteq K_m \subseteq K_{m+1} \subseteq \ldots$$

where $K_0 = \emptyset$ and each K_m is a simplicial Σ_m-complex and the simplicial inclusion $i_m\colon K_m \subseteq K_{m+1}$ is Σ_m-equivariant (Σ_m acts on K_{m+1} via $\Sigma_m \hookrightarrow \Sigma_{m+1}$) is called a *consistent sequence*.

We start by considering few families of consistent sequences of finite simplicial complexes. The main aim of the paper is to show that these consistent sequences induce the consistent sequence of Σ_m-representations of the homology of polyhedral products which are representation stable.

Example 4.2 (k-skeleton sequences) Fix an integer $k \geq 0$. To each m assign the k-skeleton Δ_m^k of a standard $(m-1)$-simplex,

$$\emptyset \subseteq \Delta_1^k \subseteq \ldots \subseteq \Delta_m^k \subseteq \Delta_{m+1}^k \subseteq \ldots. \tag{8}$$

The action of Σ_m on K_m is induced by permutations of all m vertices. Each simplicial inclusion $i_m\colon \Delta_m^k \longrightarrow \Delta_{m+1}^k$ is Σ_m-equivariant. Therefore (8) is consistent.

In general, if K and L are simplicial G-complexes on $V(K)$ and $V(L)$ respectively, then the G-action can be extended to the join $K * L$, as a complex on $V(K) \cup V(L)$ vertices, diagonally.

Construction 4.3 Fix integers $s \geq 1$ and $k_1, \ldots, k_s \geq 0$. For each $m \geq 0$, let K_m be a simplicial complex on sm vertices given by the join of $\Delta_m^{k_1}, \Delta_m^{k_2}, \ldots, \Delta_m^{k_s}$. Since each $\Delta_m^{k_i}$ is a simplicial Σ_m-complex, then K_m is also a simplicial Σ_m-complex with the Σ_m-action given by $g \cdot (\sigma_1 \sqcup \ldots \sqcup \sigma_s) = g \cdot \sigma_1 \sqcup \ldots \sqcup g \cdot \sigma_s$ for $g \in \Sigma_m$ and for each $\sigma_i \in \Delta_m^{k_i}$. Let us consider the sequence

$$\emptyset \subseteq \Delta_1^{k_1} * \ldots * \Delta_1^{k_s} \subseteq \ldots \subseteq \Delta_m^{k_1} * \ldots * \Delta_m^{k_s} \subseteq \Delta_{m+1}^{k_1} * \ldots * \Delta_{m+1}^{k_s} \subseteq \ldots.$$

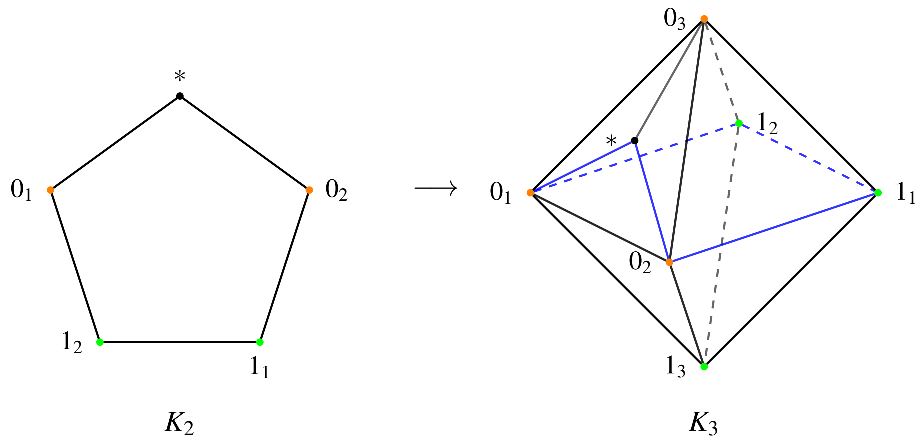
The inclusion $K_m \subseteq K_{m+1}$ is given as a join of coordinate Σ_m-equivariant inclusions $\Delta_m^{k_i} \subseteq \Delta_{m+1}^{k_i}$ and therefore it is Σ_m-equivariant.

Notice that for $s = 1$ we recover the family of k-skeleton sequences of Example 4.2.

Next we construct a non-tivial example of consistent sequence of finite simplicial Σ_m-complexes.

Construction 4.4 Let I^m be an m-cube. Consider the simplicial complex K_m obtained by taking the boundary of the dual of a simple polytope $vc(I^m)$, where $vc(I^m)$ is obtained by cutting a vertex from I^m. Note that K_1 consists of two disjoint points. K_m can also be constructed as follows. Let $S_{2m} = S_1^0 * \ldots * S_m^0$ be the join of m copies of two disjoint points, where $S_i^0 = \{0_i, 1_i\}$. Notice that S_{2m} is a triangulation of an $(m-1)$-sphere on $2m$ vertices. Then K_m is obtained from S_{2m} by deleting the interior of the $(m-1)$-face on vertices $0_1, \ldots, 0_m$ and taking the cone on it. The natural Σ_m-action on K_m is given by $g \cdot 0_i = 0_{g \cdot i}$, $g \cdot 1_i = 1_{g \cdot i}$, and g fixes the cone vertex. The inclusions $K_m \subseteq K_{m+1}$ are induced by the inclusions $S_1^0 * \ldots * S_m^0 \subseteq S_1^0 * \ldots * S_m^0 * S_{m+1}^0$ with the cone vertex mapping to itself.

For example, when $m = 2$, K_2 is simplicially isomorphic to a pentagon, and the Σ_2-action on K_2 is given by 0_1 mapping to 0_2 and 1_1 mapping to 1_2 keeping the cone vertex fixed. As shown in the picture below, the blue colour lines represent how K_2 is included into K_3.



Vertices with the same color belong to the same orbit of symmetric actions.

Definition 4.5 Given an integer $r \geq 0$, a consistent sequence $\mathcal{K} = \{K_m, i_m\}$ of finite simplicial Σ_m-complexes is called *r-face-stable* at degree d if for $m \geq d$ and every $\sigma \in K_m$ with $\dim \sigma = r$ there exist a $g \in \Sigma_m$ and $\tau \in K_d$ such that $g \cdot i_{d,m}(\tau) = \sigma$, where $i_{d,m} = i_m \circ \ldots \circ i_d$ is a composite of the inclusions $i_d, \ldots, i_m$.

Similarly, a consistent sequence $\mathcal{K} = \{K_m, i_m\}$ of finite simplicial Σ_m-complexes is called *r-vertex-stable* at degree d if for $m \geq d$ and any collection $\{v_0, \ldots, v_r\}$ of $r+1$ vertices of K_m there exist a $g \in \Sigma_m$ and a collection $\{u_0, \ldots, u_r\}$ of $r+1$ vertices in K_d such that $g \cdot i_{d,m}(u_i) = v_i$. In particular, if $r = 0$ then $\mathcal{K}$ is called *vertex-stable*.

If a consistent sequence $\{K_m, i_m\}$ of finite simplicial Σ_m-complexes is r-vertex-stable (resp. r-face-stable) for every $r \geq 0$, we call it *completely surjective* (resp. *simplicially surjective*).

Remark 4.6 Note that Construction 4.3 and Construction 4.4 are completely surjective.

(i) Let $K_m = \Delta^k_m$. For $r \geq 0$, let $E_{m,r+1}$ consist of all the subsets of $[m]$ with cardinality $r+1$. Then the transitivity of Σ_m-action on $E_{m,r+1}$ implies that $\{K_m\}$ is r-vertex-stable at degree $r+1$.

(ii) Let $K_m = \Delta^{k_1}_m * \ldots * \Delta^{k_s}_m$. If $s = 2$, then $\Delta^{k_1}_m * \Delta^{k_2}_m$ is r-vertex-stable at degree $r+1$. Let J_1, J_2 be two subsets of $[m]$ with $|J_1| + |J_2| = r+1$ and $m \geq r+1$. If $J_1 \cap J_2 \neq \emptyset$, $J_1 \cap J_2$ can be seen as a subset of vertices of $\Delta^{k_1}_m$ and $\Delta^{k_2}_m$, respectively. Let $J^c_1 = J_1 \setminus J_1 \cap J_2$ and $J^c_2 = J_2 \setminus J_1 \cap J_2$ with cardinalities r_1 and r_2 and let $r_0 = |J_1 \cap J_2|$.

Define $g \in \Sigma_m$ by sending $\{1, \ldots, r_0\}$ to $J_1 \cap J_2$, $\{r_0+1, \ldots, r_0+r_1\}$ to J^c_1 and $\{r_0+r_1+1, \ldots, r_0+r_1+r_2\}$ to J^c_2 and the complement of $\{1, \ldots, r_0+r_1+r_2\}$ in $[m]$ to the complement of $J_1 \cup J_2$ in $[m]$, respecting to the initial order of vertices.

Now take the subset of vertices $\{1, \ldots, r_0+r_1\}$ of $\Delta^{k_1}_{r+1}$ and the subset of vertices $\{1, \ldots, r_0, r_0+r_1+1, \ldots, r_0+r_1+r_2\}$ of $\Delta^{k_2}_{r+1}$ satisfying $g \cdot (\{1, \ldots, r_0+r_1\} \sqcup \{1, \ldots, r_0, r_0+r_1+1, \ldots, r_0+r_1+r_2\}) = J_1 \sqcup J_2$.

If $J_1 \cap J_2 = \emptyset$, then $r_0 = 0$ and $g \in \Sigma_m$ sending $\{1, \dots, r_1\}$ to J_1 and $\{r_1+1, \dots, r_1+r_2\}$ to J_2 and the complement of $\{1, \dots, r_1+r_2\}$ in $[m]$ to the complement of $J_1 \cup J_2$ in $[m]$. Inductively, K_m is completely surjective.

(iii) For any $r \geq 0$, $K_m = \partial vc(I^m)^*$ is r-vertex-stable at degree $d = r+1$. With $m \geq d$, let J be a subset of vertices of K_m and $|J| = r+1$. Write $J = J_* \sqcup J_1 \sqcup \dots \sqcup J_m$, where J_* is either empty or the cone vertex $\{*\}$ and each $J_i \subseteq \{0_i, 1_i\}$. Since $|J| = r+1$, there are at most $r+1$ nonempty components of J, say $J_{t_1}, \dots, J_{t_{r+1}}$. If $* \notin J$, define $g \in \Sigma_m$ by sending i to t_i if $i \leq r+1$ and to k_{i-r-1} otherwise where $\{k_1, \dots, k_{m-r-1}\}$ is the complement of $\{t_1, \dots, t_{r+1}\}$ in $[m]$. Now let $J' = J'_1 \sqcup \dots \sqcup J'_{r+1}$ from the vertex set of K_{r+1} where J'_i contains 0_i or 1_i if and only if J_{t_i} contains 0_{t_i} or 1_{t_i}. If $* \in J$, consider $\widetilde{J} = J \setminus \{*\}$ and repeat the above procedure to find $g \in \Sigma_m$ and $\widetilde{J}' \in \mathrm{Ver}(K_{r+1})$ for $\widetilde{J}$. Then let $J' = J_* \sqcup \widetilde{J}'$ and $g \cdot J' = J$.

By Theorem 3.5, for a simplicial G-complex K on m vertices

$$\widetilde{H}_i((X,A)^K; \mathbf{k}) \cong \bigoplus_{J \in [m]/G} \mathrm{Ind}_{G_J}^G \widetilde{H}_i((X,A)^{\wedge K_J}; \mathbf{k}). \tag{9}$$

If a consistent sequence $\{K_m, i_m\}$ of Σ_m-complexes K_m on the vertex set $V(K_m)$ is completely surjective then the summands in (9) do not depend on m for sufficiently large m. We shall use Hemmer's result to study the uniformly representation stability of polyhedral products. For that the stabiliser $(\Sigma_m)_J$ needs to be of the form $H \times \Sigma_{m-k}$ for some $H \leq \Sigma_k$. Therefore we proceed by studying the stabiliser of $J \in \mathcal{P}(V(K_m))$ in Σ_m which we denote by $\mathrm{stab}(J,m)$.

Observe that for a fixed integer d, for all $m \geq d$ and for some $J \in \mathcal{P}(V(K_d))$, as J also belongs to the Σ_m-set $\mathcal{P}(V(K_m))$, there is a sequence of stabilisers

$$\begin{array}{ccccccc} \dots & \longrightarrow & \Sigma_m & \longrightarrow & \Sigma_{m+1} & \longrightarrow & \dots \\ & & \uparrow & & \uparrow & & \\ \dots & \longrightarrow & \mathrm{stab}(J,m) & \longrightarrow & \mathrm{stab}(J,m+1) & \longrightarrow & \dots. \end{array}$$

For instance, in Example 4.2, if $m \geq |J|$, then $J \in \mathcal{P}(V(\Delta_m^k))$ and $\mathrm{stab}(J,m) = \Sigma_{|J|} \times \Sigma_{m-|J|}$.

In Construction 4.3, $K_m = \Delta_m^{k_1} * \Delta_m^{k_2} * \dots * \Delta_m^{k_s}$. Write J as a disjoint union of $J_1, \dots, J_s$, where each J_t $(1 \leq t \leq s)$ is from the t-th component $\mathcal{P}(V(\Delta_m^{k_t}))$. Let $b(J) = \max_{1 \leq t \leq s} |J_t|$. For $m \geq b(J)$, we observe the stabilisers of J in Σ_m,

$$\mathrm{stab}(J,m) = \{g \in \Sigma_m \mid g \cdot J_t = J_t, 1 \leq t \leq s\} = \bigcap_{1 \leq t \leq s} \mathrm{stab}(J_t, m)$$

where, as in Example 4.2, each $\mathrm{stab}(J_t,m)$ is isomorphic to $\Sigma_{|J_t|} \times \Sigma_{m-|J_t|}$. For integers $a \leq b \leq m$, we have

$$\Sigma_a \times \Sigma_{m-a} \cap \Sigma_b \times \Sigma_{m-b} = \Sigma_a \times \Sigma_{b-a} \times \Sigma_{m-b}.$$

Therefore $\mathrm{stab}(J,m) = \mathrm{stab}(J,b(J)) \times \Sigma_{m-b(J)}$ for $m \geq b(J)$, and $\Sigma_{m-b(J)}$ acts on J trivially.

We call such sequences *stabiliser consistent*.

Definition 4.7 A consistent sequence $\mathcal{K} = \{K_m, i_m\}$ of finite simplicial Σ_m-complexes is called *stabiliser consistent* if for every d and every finite set $J \in \mathcal{P}(V(K_d))$ there exists an integer $b(J)$, such that if $m \geq d$ and $m \geq b(J)$, then either Σ_m acts on J trivially or the stabiliser $\mathrm{stab}(J,m)$ is isomorphic to $\mathrm{stab}(J,b(J)) \times \Sigma_{m-b(J)}$, where $\Sigma_{m-b(J)}$ acts on J trivially.

Construction 4.4 also provides a stabiliser consistent sequence. If $J \in \mathcal{P}(V(K_d))$ for some d, write $J = J_* \sqcup J_1 \sqcup \ldots \sqcup J_d$, where J_* is either empty or $\{*\}$ and each $J_i \subseteq \{0_i, 1_i\}$. Since Σ_m acts on $*$ trivially, $\mathrm{stab}(J,m) = \mathrm{stab}(\tilde{J},m)$ where $\tilde{J} = J_1 \sqcup \ldots \sqcup J_d$. Let $b(J)$ be the number of non-empty components J_t. Then for $m \geq b(J)$, $\mathrm{stab}(J,m) \cong \mathrm{stab}(J,b(J)) \times \Sigma_{m-b(J)}$ where $\Sigma_{m-b(J)}$ acts on J trivially.

As a consequence, we have the following result that states conditions on a sequence of finite simplicial complexes that will induce in homology a uniformly representation stable sequence.

Theorem 4.8 *Let $\{K_m, i_m\}$ be a consistent sequence of finite simplicial complexes and X be a connected, based CW-complexes of finite type with a based subcomplex A. Suppose that $\{K_m, i_m\}$ is completely surjective and stabiliser consistent. Then the consistent sequence of Σ_m-representations $\{\widetilde{H}_i((X,A)^{K_m};\mathbf{k}), i_{m*}\}$ for* $\mathrm{char}\,\mathbf{k} = 0$ *is uniformly representation stable.*

Proof By Theorem 3.5, we have

$$\widetilde{H}_i((X,A)^{K_m};\mathbf{k}) \cong \bigoplus_{J\in E_m} \mathrm{Ind}^{\Sigma_m}_{\mathrm{stab}(J,m)} \widetilde{H}_i((X,A)^{\wedge K_{m,J}};\mathbf{k}) \tag{10}$$

where E_m is a set of representatives of $\mathcal{P}(V(K_m))$ under the action Σ_m, and $\mathrm{stab}(J,m)$ is the stabiliser of J under Σ_m.

We prove that if $|J| \geq i+1$ then $\widetilde{H}_i((X,A)^{\wedge K_{m,J}};\mathbf{k})$ is trivial. By the reduced Künneth formula for path-connected spaces, it is obvious that $\widetilde{H}_i(Y_1 \wedge \ldots \wedge Y_{|J|};\mathbf{k}) = 0$ if $|J| \geq i+1$, where each Y_i is either X or A. This implies that for any $\sigma \in K_{m,J}$, $\widetilde{H}_i((X,A)^{\wedge\sigma};\mathbf{k}) = 0$ if $|J| \geq i+1$. If X_1 and X_2 are connected CW-complexes with a non-empty intersection such that $\widetilde{H}_i(X_1;\mathbf{k}) = \widetilde{H}_i(X_2;\mathbf{k}) = \widetilde{H}_i(X_1 \cap X_2;\mathbf{k}) = 0$ for $i \leq l$, then $\widetilde{H}_i(X_1 \cup X_2;\mathbf{k}) = 0$ for $i \leq l$. As $(X,A)^{K_{m,J}}$ is a union of $(X,A)^{\sigma}$ over all $\sigma \in K_{m,J}$, inductively $\widetilde{H}_i((X,A)^{\wedge K_J};\mathbf{k})$ is trivial if $|J| \geq i+1$.

Since $\{K_m, i_m\}$ is completely surjective, if $|J| \leq i$ there exists an integer $N \geq 1$ such that if $m \geq N$, we have $E_{m+1,i} = E_{m,i} = \ldots = E_{N,i}$, where $E_{m,i} = \{J \in E_m \mid |J| \leq i\}$. Therefore the summands in (10) do not depend on m for $m \geq N$. On the other hand, for each $J \in E_*$ there exists an integer $b(J)$ such that for $m \geq b(J)$, either Σ_m acts on J trivially or the stabiliser $\mathrm{stab}(J,m) = \mathrm{stab}(J,b(J)) \times \Sigma_{m-b(J)}$, where $\Sigma_{m-|J|}$ acts on J trivially. In the first case, if Σ_m acts on J trivially for $m \geq b(J)$, then for any $k \leq b(J)$, Σ_k acts on J trivially because Σ_k acts on J as a subgroup of $\Sigma_{b(J)}$. As the vertex support set J of $K_{m,J}$ is fixed, the space $(X,A)^{\wedge K_{m,J}}$ will stay the same when m increases. Thus, $\widetilde{H}_i((X,A)^{\wedge K_{m,J}};\mathbf{k})$ is a fixed finite-dimensional trivial Σ_m-representation even though m varies. It follows that $\{\widetilde{H}_i((X,A)^{\wedge K_{m,J}};\mathbf{k})\}$ is uniformly representation stable.

If $\mathrm{stab}(J,m) = \mathrm{stab}(J,b(J)) \times \Sigma_{m-b(J)}$, then $\widetilde{H}_i((X,A)^{\wedge K_{m,J}};\mathbf{k})$ is a $\mathrm{stab}(J,m)$-representation with a trivial $\Sigma_{m-b(J)}$-action. By [12], we have that $\mathrm{Ind}^{\Sigma_m}_{\mathrm{stab}(J,m)}\widetilde{H}_i((X,A)^{\wedge K_{m,J}};\mathbf{k})$ is uniformly representation stable.

Therefore, the sequence of Σ_m-representations $\{\widetilde{H}_i((X,A)^{K_m};\mathbf{k}), i_{m*}\}$ is uniformly representation stable as the summands do not depend on m eventually. □

Remark 4.9 In general, we require the simplicial maps i_m in the consistent sequence of finite simplicial complexes to be inclusions so that they induce maps of polyhedral products. However, in the case when (X,A) is a pair of topological monoids, as it is for moment-angle complexes when $(X,A) = (D^2,S^1)$, any Σ_m-simplicial map, not necessary a simplicial inclusion, can be chosen for i_m. A simplicial map $f\colon K \longrightarrow L$ induces a continuous map $(X,A)^K \longrightarrow (X,A)^L$ defined by $(x_1,\ldots,x_p) = (y_1,\ldots,y_q)$, where $y_j = \prod\limits_{i\in f^{-1}(j)} x_i$. Here p and q are the number of vertices of K and L, respectively.

We have proved that the sequences in Constructions 4.3 and 4.4 are completely surjective and stabiliser consistent. Applying Theorem 4.8, we conclude the following statement.

Corollary 4.10 *Let $\mathcal{K}$ be one of the consistent sequences in Constructions 4.3 and 4.4 and X be a connected, based CW-complexes of finite type with a based subcomplex A. Then the consistent sequence of Σ_m-representations $\{\widetilde{H}_i((X,A)^{K_m};\mathbf{k}), i_{m*}\}$ for $\mathrm{char}\,\mathbf{k} = 0$ is uniformly representation stable.* □

Note that since the sequence in Construction 4.4 provides a consistent sequence of finite simplicial complexes, given by taking the boundary of dual of simple polytopes, the corresponding moment-angle complexes are a sequence of manifolds.

Proposition 4.11 *Let $\mathcal{K}$ be the consistent sequence in Construction 4.4. Then for the moment-angle manifolds $\mathcal{Z}_{\mathcal{K}}$, the consistent sequence of Σ_m-representations $\{H_*(\mathcal{Z}_{K_m};\mathbf{k}), i_{m*}\}$ for $\mathrm{char}\,\mathbf{k} = 0$ is uniformly representation stable.* □

Moreover, due to [4, 8], the manifold $\mathcal{Z}_{K_m}$ is diffeomorphic to $\partial((\prod\limits_m S^3 - D^{3m}) \times D^2) \# \mathop{\#}\limits_{j=1}^{m} \binom{m}{j}(S^{j+2} \times S^{3m-j-1})$. Therefore, $H_3(\mathcal{Z}_{K_m};\mathbf{k})$ has Betti number m which means that the sequence of moment-angle manifolds $\mathcal{Z}_{K_m}$ with the maps $\mathcal{Z}_{K_m} \longrightarrow \mathcal{Z}_{K_{m+1}}$ induced by simplicial maps $K_m \longrightarrow K_{m+1}$ is not homology stable.

Let $K_m = \Delta_m^k$. Since every K_m is a full subcomplex of K_{m+1}, the moment-angle complex $\mathcal{Z}_{K_m}$ retracts off $\mathcal{Z}_{K_{m+1}}$, and the retraction map $p_m \colon \mathcal{Z}_{K_{m+1}} \longrightarrow \mathcal{Z}_{K_m}$ is Σ_m-equivariant. The uniform stability of Σ_m-representations $\{H^i(\mathcal{Z}_{K_m};\mathbf{k}), p_m^i\}$ follows immediately.

Proposition 4.12 *For $i \geq 2k+3$, the sequence $\{H^i(\mathcal{Z}_{K_m};\mathbf{k}), p_m^i\}$ of Σ_m-representations is uniformly representation stable.*

Proof By Proposition 2.6, we have

$$H^i(\mathcal{Z}_{K_m};\mathbf{k}) \cong \bigoplus_{J\in E_m} \mathrm{Ind}_{\Sigma_{|J|}\times\Sigma_{m-|J|}}^{\Sigma_m} \widetilde{H}^{i-|J|-1}(K_{J,m};\mathbf{k})$$

where $E_m = \{\{1\},\{1,2\},\ldots,\{1,2,\ldots,m\}\}$ and $K_{J,m} = J\cap\Delta_m^k$. Thus K_J is a $(|J|-1)$-face of K, if $|J|\leq k+1$ and is the k-skeleton of $(|J|-1)$-simplex with J as its vertex set if $|J|\geq k+2$. The latter one allows an $\Sigma_{|J|}$-action. Therefore, if $|J|\leq k+1$, then $\widetilde{H}^*(K_{J,m};\mathbf{k}) = 0$. If $k+2\leq |J|\leq m$, $\widetilde{H}^p(K_{J,m};\mathbf{k}) = \mathbf{k}$ if $p=k$, and is 0, otherwise.

The nontrivial cohomology group of $K_{J,m}$ implies that $i-|J|-1 = k$ and $k+2\leq |J|\leq m$. Thus if $2k+3\leq i\leq m+k+1$, we have a Σ_m-representation isomorphism that

$$H^i(\mathcal{Z}_{K_m};\mathbf{k}) \cong \mathrm{Ind}_{\Sigma_{|J|}\times\Sigma_{m-|J|}}^{\Sigma_m} \widetilde{H}^k(K_{J,m};\mathbf{k}),\ \text{with } |J| = i-k-1.$$

Hemmer [12] implies the uniform representation stability of the sequence of Σ_m-representations $\{H^i(\mathcal{Z}_{K_m};\mathbf{k}), p_m^i\}$. □

Example 4.13 When K_m consists of m disjoint points and for $i\geq 3$, as a Σ_m-representation $H^i(\mathcal{Z}_m;\mathbf{k})$ can be written explicitly as

$$H^i(\mathcal{Z}_m;\mathbf{k}) = \mathrm{Ind}_{\Sigma_{i-1}\times\Sigma_{m-i+1}}^{\Sigma_m} V_{(i-2,1)} \boxtimes \mathbf{k}$$

where $V_{(i-2,1)}$ is the standard representation of Σ_{i-1}.

In particular,

$$\begin{aligned}
H^3(\mathcal{Z}_m;\mathbf{k}) =& V_{(m-1,1)} \oplus V_{(m-2,1,1)} \ \text{ for } m\geq 3;\\
H^4(\mathcal{Z}_m;\mathbf{k}) =& V_{(m-1,1)} \oplus V_{(m-2,2)} \oplus V_{(m-2,1,1)} \oplus V_{(m-3,2,1)} \ \text{ for } m\geq 5;\\
H^5(\mathcal{Z}_m;\mathbf{k}) =& V_{(m-1,1)} \oplus V_{(m-2,2)} \oplus V_{(m-2,1,1)} \oplus V_{(m-3,3)} \oplus V_{(m-3,2,1)} \oplus V_{(m-4,3,1)} \ \text{ for } m\geq 7;\\
H^6(\mathcal{Z}_m;\mathbf{k}) =& V_{(m-1,1)} \oplus V_{(m-2,2)} \oplus V_{(m-2,1,1)} \oplus V_{(m-3,3)} \oplus V_{(m-3,2,1)} \oplus V_{(m-4,4)}\\
&\oplus V_{(m-4,3,1)} \oplus V_{(m-5,4,1)} \ \text{ for } m\geq 9.
\end{aligned}$$

5 Applications of uniformly representation stability of polyhedral products

We finish the paper by investigating what kind of structural properties of $H_i((X,A)^K;\mathbb{Q})$ are implied by representation stability.

One of the key properties of a sequence of uniformly stable Σ_m-representations over $\mathbb{Q}$ is that their characters are eventually polynomials [9, Definition 1.4].

Denote by $\lambda \vdash m$ a partition $\lambda = (\lambda_1, \dots, \lambda_l)$ with $\lambda_1 \geq \dots \lambda_l > 0$ and $\lambda_1 + \dots + \lambda_l = m$. Let $|\lambda|$ be the sum $\lambda_1 + \dots + \lambda_l$. Given any partition λ, for any $m \geq |\lambda| + \lambda_1$, denote by $\lambda[m] = (m - |\lambda|, \lambda_1, \dots, \lambda_l)$ (see [9, Definition 2.2.5]). Denote by $V(\lambda)_m$ the irreducible representation corresponding to partition $\lambda[m]$. The *weight* of a consistent sequence of Σ_m-representations $\{V_m, \psi_m\}$ is the maximum of $|\lambda|$ over all irreducible constituents $V(\lambda)_m$ that appears in V_m.

Example 5.1 For any partition $\mu \vdash n$ and $m \geq n$, applying Proposition 3.2.4 in [9], the consistent sequence $\{\mathrm{Ind}_{\Sigma_n\times\Sigma_{m-n}}^{\Sigma_m} V_\mu \boxtimes \mathbf{k}\}$ has weight n, where V_μ is the irreducible representation corresponding to μ.

Hemmer [12] constructed a sequence of Σ_m-representations that is uniformly representation stable. Next we calculate the weight of this sequence applying the result from Example 5.1.

Lemma 5.2 *Fix an integer $n \geq 0$. Let H be a subgroup of Σ_n and V is a Σ_n-representation over a field $\mathbf{k}$ of characteristic 0. For $m \geq n$, the consistent sequence $\{\mathrm{Ind}_{H\times\Sigma_{m-n}}^{\Sigma_m} V \boxtimes \mathbf{k}\}$ has weight n.*

Proof Observe that

$$\mathrm{Ind}_{H\times\Sigma_{m-n}}^{\Sigma_m} V \boxtimes \mathbf{k} = \mathrm{Ind}_{\Sigma_n\times\Sigma_{m-n}}^{\Sigma_m}(\mathrm{Ind}_{H\times\Sigma_{m-n}}^{\Sigma_n\times\Sigma_{m-n}}(V\boxtimes\mathbf{k})) = \mathrm{Ind}_{\Sigma_n\times\Sigma_{m-n}}^{\Sigma_m}(\mathrm{Ind}_H^{\Sigma_n} V)\boxtimes\mathbf{k}.$$

As Σ_n-representations, $\mathrm{Ind}_H^{\Sigma_n} V$ is decomposed as $\bigoplus_{\mu\vdash n} V_\mu^{\oplus c_\mu}$, where c_μ are multiplicities. By Example 5.1, $\{\mathrm{Ind}_{\Sigma_n\times\Sigma_{m-n}}^{\Sigma_m} V_\mu \boxtimes \mathbf{k}\}$ has weight n. Then $\{\mathrm{Ind}_{H\times\Sigma_{m-n}}^{\Sigma_m} V \boxtimes \mathbf{k}\}$ has weight n, as each $\mathrm{Ind}_{H\times\Sigma_{m-n}}^{\Sigma_m} V \boxtimes \mathbf{k}$ is decomposed into a finite direct sum as Σ_m-representations

$$\mathrm{Ind}_{H\times\Sigma_{m-n}}^{\Sigma_m} V \boxtimes \mathbf{k} \cong \bigoplus_{\mu\vdash n} c_\mu \mathrm{Ind}_{\Sigma_n\times\Sigma_{m-n}}^{\Sigma_m} V_\mu \boxtimes \mathbf{k}.$$

□

Given a uniformly representation stable sequence $\{V_m, \psi_m\}$, the uniform multiplicity stability implies that there exists an integer $M \geq 0$ such that V_m is decomposed into $\bigoplus_\lambda c_\lambda V(\lambda)_m$ for $m \geq M$. A classical result ([14, Example I.7.14]) in representation theory states that the character of $V(\lambda)_m$ is polynomial if $m \geq |\lambda| + \lambda_1$. Explicitly, let $a_1, a_2, \ldots$ be class functions $a_j \colon \Sigma_i \longrightarrow \mathbb{N}$ for any $i \geq 0$ such that $a_j(g)$ is the number of j-cycles in the cycle decomposition of g. Then, for each partition λ there exists a polynomial $P_\lambda \in \mathbb{Q}[a_1, a_2, \ldots]$, called the character polynomial corresponding to the partition λ, such that P_λ has degree $|\lambda|$ and the character $\chi_{V(\lambda)_m}(g) = P_\lambda(g)$ for all $m \geq |\lambda| + \lambda_1$ and $g \in \Sigma_m$.

We finish our paper by looking at the growth of Betti numbers of polyhedral products.

Theorem 5.3 *Let $\{K, i_m\}$ and (X, A) be as in Theorem 4.8. Then for each $i \geq 0$, the consistent sequence $\{\widetilde{H}_i((X,A)^{K_m}; \mathbb{Q}), i_{m_*}\}$ has a finite weight. Moreover, the growth of Betti numbers of $\{\widetilde{H}_i((X,A)^{K_m}; \mathbb{Q}), i_{m_*}\}$ is eventually polynomial with respect to m.*

Proof By Theorem 4.8, $\{\widetilde{H}_i((X,A)^{K_m}; \mathbb{Q}), i_{m_*}\}$ is uniformly representation stable. Thus the uniformly multiplicity stability implies that there exists an integer $N > 0$, not depending on λ, such that for all $m \geq N$, there are constant integers c_λ such that

$$\widetilde{H}_i((X,A)^{K_m}; \mathbb{Q}) \cong \bigoplus_{\lambda} c_\lambda V(\lambda)_m$$

and are uniquely given by multiplicities defined in the irreducible components of $\widetilde{H}_i((X,A)^{K_N}; \mathbb{Q})$. Therefore, the weight ω_i of sequence $\{\widetilde{H}_i((X,A)^{K_m}; \mathbb{Q}), i_{m_*}\}$ is the maximum $|\lambda|$ that forms a irreducible component of $\widetilde{H}_i((X,A)^{K_N}; \mathbb{Q})$. Since $\widetilde{H}_i((X,A)^{K_N}; \mathbb{Q})$ has finite dimension over $\mathbb{Q}$, ω_i is finite.

In particular, if $m \geq 2\omega_i$, then for all λ appearing in the above equation, $m \geq |\lambda| + \lambda_1$. Then there exists a polynomial character of $\{\widetilde{H}_i((X,A)^{K_m}; \mathbb{Q})$ given by $\sum_\lambda P_\lambda$. Take g to be the identity of symmetric groups. This gives that the growth of Betti numbers of $\{\widetilde{H}_i((X,A)^{K_m}; \mathbb{Q}), i_{m_*}\}$ is eventually polynomial with respect to m. □

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