RATIONAL GROWTH AND DEGREE OF COMMUTATIVITY OF GRAPH PRODUCTS

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Abstract. Let G be an infinite group and let X be a finite generating set for G such that the growth series of G with respect to X is a rational function; in this case G is said to have rational growth with respect to X. In this paper a result on sizes of spheres (or balls) in the Cayley graph $\Gamma(G, X)$ is obtained: namely, the size of the sphere of radius n is bounded above and below by positive constant multiples of $n^\alpha \lambda^n$ for some integer $\alpha \geq 0$ and some $\lambda \geq 1$.

As an application of this result, a calculation of degree of commutativity (d. c.) is provided: for a finite group F, its d. c. is defined as the probability that two randomly chosen elements in F commute, and Antolín, Martino and Ventura have recently generalised this concept to all finitely generated groups. It has been conjectured that the d. c. of a group G of exponential growth is zero. This paper verifies the conjecture (for certain generating sets) when G is a right-angled Artin group or, more generally, a graph product of groups of rational growth in which centralisers of non-trivial elements are "uniformly small".

1. Introduction

Let G be a group which has a finite generating set X. For any element $g \in G$, let $|g| = |g|_X$ be the word length of g with respect to X. For any $n \in \mathbb{Z}_{\geq 0}$, let

$$B_{G,X}(n) := \{g \in G \mid |g|_X \leq n\}$$

be the *ball* in G with respect to X of radius n, and let

$$S_{G,X}(n) := \{g \in G \mid |g|_X = n\}$$

be the *sphere* in G with respect to X of radius n. One writes $B_G(n)$ or $B(n)$ for the ball (and $S_G(n)$ or $S(n)$ for the sphere) if the generating set or the group itself is clear. A group G is said to have *exponential growth* if

$$\liminf_{n\to\infty} \frac{\log |B_{G,X}(n)|}{n} > 0 \tag{1}$$

and *subexponential growth* otherwise; note that as there are at most $(2|X|)^n$ words over $X^{\pm 1}$ of length n, the limit in (1) is finite, so the group cannot have 'superexponential' growth. A group G is said to have *polynomial growth of degree d* if

$$d := \limsup_{n\to\infty} \frac{\log |B_{G,X}(n)|}{\log n} < \infty$$

and *superpolynomial growth* otherwise. It is well-known that having exponential growth or polynomial growth of degree d is independent of the generating set X.

The pairs (G, X) as above considered in this paper will have some special properties. In particular, consider the (*spherical*) *growth series* $s_{G,X}(t)$ of a finitely generated group G with a finite generating set X, defined by

$$s_{G,X}(t) = \sum_{g\in G} t^{|g|_X} = \sum_{n=0}^{\infty} |S_{G,X}(n)| t^n.$$

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Cases of particular interest includes pairs (G, X) for which $s_{G,X}(t)$ is a rational function, i.e. a ratio two polynomials; in this case G is said to have *rational growth* with respect to X. In general, this property depends on the chosen generating set: for instance, the higher Heisenberg group $G = H_2(\mathbb{Z})$ has two finite generating sets X_1, X_2 such that $s_{G,X_1}(t)$ is rational but $s_{G,X_2}(t)$ is not [19].

Rational growth series implies some nice properties on the growth of a group. In particular, one can obtain the first main result of this paper:

Theorem 1. *Let G be an infinite group with a finite generating set X such that $s_{G,X}(t)$ is a rational function. Then there exist constants $\alpha \in \mathbb{Z}_{\geq 0}$, $\lambda \in [1, \infty)$ and $D > C > 0$ such that*

$$Cn^\alpha \lambda^n \leq |S_{G,X}(n)| \leq Dn^\alpha \lambda^n$$

for all $n \geq 1$.

Some of the ideas that go into the proof of Theorem 1 appear in the work of Stoll [19], where asymptotics of ball sizes are used to show that the higher Heisenberg group $G = H_2(\mathbb{Z})$ has a finite generating set X such that the series $s_{G,X}(t)$ is transcendental.

Remark 2. It is clear that, with the assumptions and notation as above, Theorem 1 implies

$$\liminf_{n\to\infty} \frac{|S_{G,X}(n)|}{n^\alpha \lambda^n} \geq C > 0 \qquad \text{and} \qquad \limsup_{n\to\infty} \frac{|S_{G,X}(n)|}{n^\alpha \lambda^n} \leq D < \infty.$$

It is easy to check that the converse implication is also true. In particular, the conclusion of Theorem 1 is equivalent to the statement that there exist $\alpha \in \mathbb{Z}_{\geq 0}$ and $\lambda \in [1, \infty)$ such that

$$\liminf_{n\to\infty} \frac{|S_{G,X}(n)|}{n^\alpha \lambda^n} > 0 \qquad \text{and} \qquad \limsup_{n\to\infty} \frac{|S_{G,X}(n)|}{n^\alpha \lambda^n} < \infty.$$

Theorem 1 agrees with the result for hyperbolic groups. Indeed, it is known that if G is a hyperbolic group and X is a finite generating set, then $s_{G,X}(t)$ is rational [14, Theorem 8.5.N]. In this case the Theorem gives a weaker version of [8, Théorème 7.2], which states that the conclusion of Theorem 1 holds with $\alpha = 0$.

As an application of Theorem 1 a calculation of degree of commutativity is provided. For a finite group F, the *degree of commutativity* of F was defined by Erdős and Turán [10] and Gustafson [15] as

$$\mathrm{dc}(F) := \frac{|\{(x, y) \in F^2 \mid [x, y] = 1\}|}{|F|^2}, \tag{2}$$

i.e. the probability that two elements of F chosen uniformly at random commute. In [1], Antolín, Martino and Ventura generalise this definition to infinite finitely generated groups:

Definition 3. Let G be a finitely generated group and let X be a finite generating set for G. The *degree of commutativity* for G with respect to X is

$$\begin{aligned}\mathrm{dc}_X(G) &:= \limsup_{n\to\infty} \frac{|\{(x, y) \in B_{G,X}(n)^2 \mid [x, y] = 1\}|}{|B_{G,X}(n)|^2} \\ &= \limsup_{n\to\infty} \frac{\sum_{x\in B_{G,X}(n)} |C_G(x) \cap B_{G,X}(n)|}{|B_{G,X}(n)|^2},\end{aligned}$$

where $C_G(x)$ is the centraliser of x in G.

Note that if G is finite then for any generating set X one has $B_{G,X}(N) = G$ for all sufficiently large N, so this definition agrees with (2).

It is known that $\mathrm{dc}_X(G) = 0$ when G is either a non-virtually-abelian residually finite group of subexponential growth [1, Theorem 1.3] or a non-elementary hyperbolic group [1, Theorem 1.7], independently of the generating set X. It has been conjectured that indeed $\mathrm{dc}_X(G) = 0$ whenever G has superpolynomial growth [1, Conjecture 1.6].

The interest of this paper is the degree of commutativity of graph products of groups.

Definition 4. Let Γ be a finite simple (undirected) graph, and let $\mathbf{H} : V(\Gamma) \to \mathcal{G}$ be a map from the vertex set of Γ to the category $\mathcal{G}$ of groups; suppose that $\mathbf{H}(v) \not\cong \{1\}$ for each $v \in V(\Gamma)$. Let

$$\tilde{G}(\Gamma, \mathbf{H}) := *_{v \in V(\Gamma)} \mathbf{H}(v)$$

be a free product of groups, and let

$$R(\Gamma, \mathbf{H}) := \{[g,h] \mid g \in \mathbf{H}(v), h \in \mathbf{H}(w), \{v,w\} \in E(\Gamma)\}.$$

Then the *graph product* associated with Γ and $\mathbf{H}$ is defined to be the group

$$G(\Gamma, \mathbf{H}) := \tilde{G}(\Gamma, \mathbf{H}) / \langle\!\langle R(\Gamma, \mathbf{H}) \rangle\!\rangle^{\tilde{G}(\Gamma, \mathbf{H})}.$$

In particular, this is the construction of *right-angled Artin* (respectively *Coxeter*) *groups* if $\mathbf{H}(v) \cong \mathbb{Z}$ (respectively $\mathbf{H}(v) \cong C_2$) for all $v \in \Gamma$.

This paper considers groups G which, together with their finite generating sets X, belong to a certain class, defined as follows.

Definition 5. Say a pair (G, X) with a group G and a finite generating set X of G is a *rational pair with small centralisers* if the following two conditions hold:

(i) $s_{G,X}(t)$ is a rational;
(ii) there exist constants $P, \beta \in \mathbb{Z}_{\geq 1}$ such that $|C_G(g) \cap B_{G,X}(n)| \leq P n^\beta$ for all $n \geq 1$ and all non-trivial elements $g \in G$.

Note that condition (ii) is independent of the choice of a generating set X: indeed, as any word metrics on G associated with generating sets X and $\hat{X}$ are bi-Lipschitz equivalent, the inequality $|C_G(g) \cap B_{G,X}(n)| \leq P n^\beta$ implies the inequality $|C_G(g) \cap B_{G,\hat{X}}(n)| \leq \hat{P} n^\beta$ for some $\hat{P} \in \mathbb{Z}_{\geq 1}$ depending only on $\hat{X}$ and P.

It was shown in [7] that, given a finite simple graph Γ with a group $\mathbf{H}(v)$ and a finite generating set $X(v) \subseteq \mathbf{H}(v)$ associated to every vertex $v \in V(\Gamma)$, if $s_{\mathbf{H}(v),X(v)}(t)$ is rational for each $v \in V(\Gamma)$ then so is $s_{G(\Gamma,\mathbf{H}),X(\Gamma,\mathbf{H})}(t)$, where $X(\Gamma, \mathbf{H}) = \bigsqcup_{v \in V(\Gamma)} X(v)$.

If $G(\Gamma, \mathbf{H})$ has exponential growth, then, together with an explicit form of centralisers in $G(\Gamma, \mathbf{H})$, described in [2], Theorem 1 can be used to compute the degree of commutativity of $G(\Gamma, \mathbf{H})$:

Theorem 6. *Let Γ be a finite simple graph, and for each vertex $v \in V(\Gamma)$, let $(\mathbf{H}(v), X(v))$ be a rational pair with small centralisers. Suppose that $G(\Gamma, \mathbf{H})$ has exponential growth, and let $X = \bigsqcup_{v \in V(\Gamma)} X(v)$. Then*

$$\mathrm{dc}_X(G(\Gamma, \mathbf{H})) = 0.$$

Remark 7. Theorem 6 is enough to confirm [1, Conjecture 1.6] in this setting: that is, either $G = G(\Gamma, \mathbf{H})$ is virtually abelian, or $\mathrm{dc}_X(G) = 0$. Indeed, $G(\Gamma, \mathbf{H})$ has subexponential growth if and only if all the $\mathbf{H}(v)$ have subexponential growth, the complement Γ^C of Γ contains no length 2 paths, and $\mathbf{H}(v) \cong C_2$ for every non-isolated vertex v of Γ^C. In this case, rationality of $s_{\mathbf{H}(v),X(v)}(t)$ implies that the $\mathbf{H}(v)$ all have polynomial growth (by Theorem 1, for instance). Thus $G(\Gamma, \mathbf{H})$ is a direct product of groups of polynomial growth: namely, the group $\mathbf{H}(v)$ for each isolated vertex v of Γ^C, and an infinite dihedral group for each edge in Γ^C. Consequently, $G(\Gamma, \mathbf{H})$ itself has polynomial growth, and so [1, Corollary 1.5] implies that either $G(\Gamma, \mathbf{H})$ is virtually abelian, or $\mathrm{dc}_X(G(\Gamma, \mathbf{H})) = 0$.

Cases of particular interest of Theorem 6 include right-angled Artin groups and graph products of finite groups. More generally, let us note two special cases of pairs of (G, X) satisfying Definition 5:

(i) Let G be virtually nilpotent, and X be a finite generating set with $s_{G,X}(t)$ rational: in particular, this holds whenever G is virtually abelian [3] and for $G = H_1$, the integral Heisenberg group [9]. It was shown that by Wolf [20] that if G is virtually nilpotent then it has polynomial growth (by Gromov's Theorem [13], the converse is also true), and so part (ii) of Definition 5 holds trivially by bounding growth of centralisers by the growth of G itself.

(ii) Let G be a torsion-free hyperbolic group, and X be any finite generating set. Cannon [6] and Gromov [14, Theorem 8.5.N] have shown that hyperbolic groups have rational growth with respect to any generating set, and all infinite-order elements have virtually cyclic centralisers. Moreover, for any torsion-free hyperbolic group G with a finite generating set X, there is a constant $P > 0$ such that $|C_G(g) \cap B_{G,X}(n)| \leq Pn$ for all $n \geq 1$ and all non-trivial $g \in G$: see the proof of Theorem 1.7 in [1] for details and references.

The paper is structured as follows. Section 2 applies to all infinite groups with rational spherical growth series and is dedicated to a proof of Theorem 1. Section 3 is used to prove Theorem 6.

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2. Groups with rational growth series

This section provides a proof of Theorem 1. Let G be an infinite group, and suppose that the growth series of G with respect to a finite generating set X is a rational function. In particular, the spherical growth series is

$$s(t) = s_{G,X}(t) = \sum_{n=0}^{\infty} \mathfrak{S}(n) t^n = \frac{p(t)}{q(t)}$$

where $\mathfrak{S}(n) = \mathfrak{S}_G(n) = \mathfrak{S}_{G,X}(n) := |S_{G,X}(n)|$, and

$$q(t) = q_0 t^c \prod_{i=1}^{r} (1 - \lambda_i t)^{\alpha_i + 1} \quad \text{and} \quad p(t) = p_0 t^{\tilde{c}} \prod_{i=1}^{\tilde{r}} (1 - \tilde{\lambda}_i t)^{\tilde{\alpha}_i + 1}$$

are non-zero polynomials with no common roots (and so either $c = 0$ or $\tilde{c} = 0$), with $\alpha_i, \tilde{\alpha}_i \in \mathbb{Z}_{\geq 0}$ for all i. Since the series $(\mathfrak{S}(n))_{n=0}^{\infty}$ grows at most exponentially, $s(t)$ is analytic (and so continuous) at 0, hence one has

$$1 = \mathfrak{S}(0) = \lim_{t \to 0} s(t) = \frac{p_0}{q_0} \lim_{t \to 0} t^{\tilde{c} - c}$$

and so $c = \tilde{c}$ and $p_0 = q_0$. Thus $c = \tilde{c} = 0$ and, without loss of generality, $q_0 = p_0 = 1$.

Coefficients of such a series are described in [16, Lemma 1]; in particular, it follows that

$$\mathfrak{S}(n) = \sum_{i=1}^{r} \sum_{j=0}^{\alpha_i} b_{i,j} n^j \lambda_i^n \tag{3}$$

for n large enough, with $b_{i,\alpha_i} \neq 0$ for all i.

Now consider the terms of (3) that give a non-negligible contribution to $\mathfrak{S}(n)$ for large n. In particular, one may assume without loss of generality that

$$\lambda := |\lambda_1| = |\lambda_2| = \cdots = |\lambda_{\tilde{k}}| > |\lambda_{\tilde{k}+1}| \geq |\lambda_{\tilde{k}+2}| \geq \cdots \geq |\lambda_r|$$

for some $\tilde{k} \leq r$ and that

$$\alpha := \alpha_1 = \alpha_2 = \cdots = \alpha_k > \alpha_{k+1} \geq \alpha_{k+2} \geq \cdots \geq \alpha_{\tilde{k}}$$

for some $k \leq \tilde{k}$. Note that one must have $\lambda \geq 1$: otherwise the radius of convergence of $s(t)$ is $\lambda^{-1} > 1$ and so the series $\sum_n \mathfrak{S}(n)$ converges, contradicting the fact that G is infinite.

For $n \in \mathbb{Z}_{\geq 0}$, define

$$c_n = \sum_{j=1}^{k} b_{j,\alpha} \exp(i\varphi_j n)$$

where $\lambda_j = \lambda \exp(i\varphi_j)$ for some $\varphi_j \in (-\pi, \pi]$, for $1 \leq j \leq k$. It follows that

$$\mathfrak{S}(n) = n^\alpha \lambda^n (c_n + o(1)) \tag{4}$$

as $n \to \infty$. In particular, since $\mathfrak{S}(n) \in (0, \infty) \subseteq \mathbb{R}$ for all n, it follows that

$$\liminf_{n\to\infty} \operatorname{Re}(c_n) \geq 0 \quad \text{and} \quad \lim_{n\to\infty} \operatorname{Im}(c_n) = 0. \tag{5}$$

It is clear that

$$\limsup_{n\to\infty} \frac{\mathfrak{S}(n)}{n^\alpha \lambda^n} \leq \sum_{j=1}^{k} |b_{j,\alpha}|,$$

which shows existence of the constant D in Theorem 1; in order to prove the Proposition, it is enough to show that $\liminf_{n\to\infty} \mathfrak{S}(n)/(n^\alpha \lambda^n) > 0$. However, this bound does not follow solely from the fact that $s(t)$ is a rational function: see Example 12 (i) at the end of this section.

Remark 8. Clearly, for any $n_1, n_2 \geq 0$, if $g \in G$ has $|g|_X = n_1 + n_2$ (respectively $|g|_X \leq n_1 + n_2$), then one can write $g = g_1 g_2$ where $|g_j|_X = n_j$ (respectively $|g_j|_X \leq n_j$) for $j \in \{1, 2\}$. This gives injections $S(n_1 + n_2) \to S(n_1) \times S(n_2)$ and $B(n_1 + n_2) \to B(n_1) \times B(n_2)$ by mapping $g \mapsto (g_1, g_2)$. In particular, it follows that

$$\mathfrak{S}(n_1 + n_2) \leq \mathfrak{S}(n_1)\mathfrak{S}(n_2) \quad \text{and} \quad \mathfrak{B}(n_1 + n_2) \leq \mathfrak{B}(n_1)\mathfrak{B}(n_2)$$

for any $n_1, n_2 \in \mathbb{Z}_{\geq 0}$. This property is called *submultiplicativity* of sphere and ball sizes in G.

The aim is now to show that submultiplicativity of the sequence $(\mathfrak{S}(n))_{n=0}^\infty$, together with rationality of $s(t)$, implies the conclusion of Theorem 1. As the $b_{j,\alpha}$ are non-zero and the φ_j are distinct, given (5) the following result seems highly likely:

Lemma 9. *The numbers c_n are real, and for some constant $\delta > 0$, the set*

$$E_\delta := \{n \in \mathbb{Z}_{\geq 0} \mid c_n \geq \delta\}$$

is relatively dense in $[0, \infty)$, i.e. the inclusion $E_\delta \hookrightarrow [0, \infty)$ is a $(1, K)$-quasi-isometry for some $K \geq 0$.

However, the author has been unable to come up with a straightforward proof of Lemma 9 without using some additional theory on 'quasi-periodicity' of the sequence $(c_n)_{n=0}^\infty$. Before giving a proof, let us deduce Theorem 1 from Lemma 9.

Assuming Lemma 9, one can find $N \in \mathbb{Z}_{\geq 1}$ such that for all n, there exists a $\beta = \beta_n \in \{0, \ldots, N\}$ with $c_{n+\beta} \geq \delta$. Define

$$R := \max\{\lambda^{-\beta}\mathfrak{S}(\beta) \mid 0 \leq \beta \leq N\},$$

and let $M \in \mathbb{Z}_{\geq 1}$ be such that for all $n \geq M$, one has

$$\mathfrak{S}(n) \geq n^\alpha \lambda^n \left(c_n - \frac{\delta}{2}\right)$$

(such an M exists by (4)). Then submultiplicativity of sphere sizes implies that for all $n \geq M$,

$$\frac{\delta}{2}(n+\beta_n)^\alpha \lambda^{n+\beta_n} \leq \left(c_{n+\beta_n} - \frac{\delta}{2}\right)(n+\beta_n)^\alpha \lambda^{n+\beta_n}$$
$$\leq \mathfrak{S}(n+\beta_n) \leq \mathfrak{S}(n)\mathfrak{S}(\beta_n) \leq \mathfrak{S}(n) R \lambda^{\beta_n}.$$

It follows that

$$\mathfrak{S}(n) \geq \frac{\delta}{2R}(n+\beta_n)^\alpha \lambda^n \geq \frac{\delta}{2R} n^\alpha \lambda^n$$

for $n \geq M$, showing that

$$\liminf_{n\to\infty} \frac{\mathfrak{S}(n)}{n^\alpha \lambda^n} \geq \frac{\delta}{2R} > 0,$$

which shows existence of the constant $C > 0$ in Theorem 1. Thus in order to prove Theorem 1 it is now enough to prove Lemma 9.

Proof of Lemma 9. To prove the Lemma, one may employ a digression into a certain class of functions from $\mathbb{R}$ to $\mathbb{C}$, called 'uniformly almost periodic functions'. The theory for these functions is presented in a book by Besicovitch [5].

Let $f : \mathbb{R} \to \mathbb{C}$ be a function. Given $\varepsilon > 0$, define the set $E(f,\varepsilon) \subseteq \mathbb{R}$ to be the set of all numbers $\tau \in \mathbb{R}$ (called the *translation numbers* for f belonging to ε) such that

$$\sup_{x\in\mathbb{R}} |f(x+\tau) - f(x)| \leq \varepsilon.$$

The function f is said to be *uniformly almost periodic* (*u. a. p.*) if, for any $\varepsilon > 0$, the set $E(f,\varepsilon)$ is relatively dense in $\mathbb{R}$, i.e. the inclusion $E(f,\varepsilon) \hookrightarrow \mathbb{R}$ is a $(1,K)$-quasi-isometry for some $K \geq 0$. It is easy to see that any periodic function is u. a. p., and that every continuous u. a. p. function is bounded.

Now note that the function

$$c : \mathbb{R} \to \mathbb{C}$$
$$t \mapsto \sum_{j=1}^{k} b_{j,\alpha} \exp(i\varphi_j t)$$

is a sum of continuous periodic functions, and so is a continuous u. a. p. function by [5, Section 1.1, Theorem 12]. By definition, $c_n = c(n)$ for any $n \in \mathbb{Z}_{\geq 0}$.

The aim is to show that the function $\bar{c} : t \mapsto c(\lfloor t \rfloor)$ is also u. a. p. For this, note that c is everywhere differentiable and the derivative $c'(t)$ is a sum of continuous periodic functions, so is continuous and u. a. p. – in particular, it is bounded, by some $R > 0$, say. For a given $\varepsilon \in (0,R)$, set a constant $M := \varepsilon / \left(2\sin\left(\frac{\pi\varepsilon}{2R}\right)\right)$ and define $f : \mathbb{R} \to \mathbb{R}$ by $f(t) = M\sin(\pi t)$. It is easy to check that

$$E\left(f, \frac{\varepsilon}{2}\right) \subseteq \bigcup_{n\in\mathbb{Z}} \left[n - \frac{\varepsilon}{2R}, n + \frac{\varepsilon}{2R}\right]. \tag{6}$$

For any $\tau \in \mathbb{R}$, define $n_\tau = \left\lfloor \tau + \frac{1}{2} \right\rfloor \in \mathbb{Z}$ to be the nearest integer to τ. Pick $\tau \in E\left(f, \frac{\varepsilon}{2}\right) \cap E\left(c, \frac{\varepsilon}{2}\right)$ – then $|c(x+\tau) - c(x)| \leq \frac{\varepsilon}{2}$ for all $x \in \mathbb{R}$, and, by (6), $|\tau - n_\tau| \leq \frac{\varepsilon}{2R}$, so in particular $|c(x+\tau) - c(x+n_\tau)| \leq \frac{\varepsilon}{2}$ for all $x \in \mathbb{R}$ by the choice of R. Thus $|c(x+n_\tau) - c(x)| \leq \varepsilon$ for all $x \in \mathbb{R}$, i.e. $n_\tau \in E(c,\varepsilon)$.

But by [5, Section 1.1, Theorem 11], the set $E\left(f, \frac{\varepsilon}{2}\right) \cap E\left(c, \frac{\varepsilon}{2}\right)$ is relatively dense, hence (by the previous paragraph) so is the set $E(c,\varepsilon) \cap \mathbb{Z}$. However, for any $n \in E(c,\varepsilon) \cap \mathbb{Z}$ and any $x \in \mathbb{R}$ one has

$$|\bar{c}(x+n) - \bar{c}(x)| = |c(\lfloor x+n \rfloor) - c(\lfloor x \rfloor)| = |c(\lfloor x \rfloor + n) - c(\lfloor x \rfloor)| \leq \varepsilon$$

and so $E(c,\varepsilon) \cap \mathbb{Z} \subseteq E(\bar{c},\varepsilon) \cap \mathbb{Z}$. It follows that $E(\bar{c},\varepsilon) \cap \mathbb{Z}$ is relatively dense (and so the function $\bar{c} : t \mapsto c(\lfloor t \rfloor)$ is u. a. p.).

Now recall that (5) provides constraints for limits of sequences $(\mathrm{Re}(c_n))$ and $(\mathrm{Im}(c_n))$: namely,

$$\liminf_{n\to\infty} \mathrm{Re}(c_n) \geq 0 \quad \text{and} \quad \lim_{n\to\infty} \mathrm{Im}(c_n) = 0. \tag{7}$$

It is easy to see that $c_n \in \mathbb{R}_{\geq 0}$ for all n: indeed, if either $\mathrm{Re}(c_n) = -\delta < 0$ or $|\mathrm{Im}(c_n)| = \delta > 0$ for some n then the fact that the set $E(\bar{c}, \delta/2) \cap \mathbb{Z}$ is relatively dense contradicts (7). Similarly, if $c_N > 0$ for some N then the set $E(\bar{c}, \delta) \cap \mathbb{Z}$ is a relatively dense set contained in the set $\{n \in \mathbb{Z} \mid c(n) \geq \delta\}$, where $\delta = c_N/2$. To prove Lemma 9 it is therefore enough to show that the sequence $(c_n)_{n=0}^{\infty}$ is not identically zero.

Now recall that the sequence (c_n) is defined by

$$c_n = \sum_{j=1}^{k} b_{j,\alpha} \exp(i\varphi_j n),$$

and suppose for contradiction that $c_n = 0$ for all $n \in \mathbb{Z}_{\geq 0}$, and in particular for $0 \leq n \leq k-1$. This is the same as saying that $Mv = 0$, where

$$M = \begin{pmatrix} 1 & 1 & \cdots & 1 \\ \exp(i\varphi_1) & \exp(i\varphi_2) & \cdots & \exp(i\varphi_k) \\ \vdots & \vdots & \ddots & \vdots \\ \exp(i\varphi_1)^{k-1} & \exp(i\varphi_2)^{k-1} & \cdots & \exp(i\varphi_k)^{k-1} \end{pmatrix}$$

and

$$v = \begin{pmatrix} b_{1,\alpha-1} \\ b_{2,\alpha-1} \\ \vdots \\ b_{k,\alpha-1} \end{pmatrix}.$$

Thus M has a zero eigenvalue and so $\det M = 0$. But M^t is a Vandermonde matrix with pairwise distinct rows, so $\det M \neq 0$. This gives a contradiction which completes the proof. □

Remark 10. A stronger conclusion of Theorem 1 holds if in addition $s_{G,X}(t)$ is a *positive* rational function, i.e. it is contained in the smallest sub-semiring of $\mathbb{C}(t)$ containing the semiring $\mathbb{Z}_{\geq 0}[t]$ and closed under quasi-inversion, $f(t) \mapsto (1 - f(t))^{-1}$ (for $f(t) \in \mathbb{C}(t)$ with $f(0) = 0$). This is the case in particular if there exists a language $\mathcal{L}$ in $(X \cup X^{-1})^*$ that is regular (i.e. recognised by a finite state automaton), the monoid homomorphism $\Phi : \mathcal{L} \to G$ extending the inclusion $X \cup X^{-1} \hookrightarrow G$ is a bijection, and $\mathcal{L}$ consists only of geodesic words in the Cayley graph of G with respect to X, i.e. the length of any word $l \in \mathcal{L}$ is $|\Phi(l)|_X$. If $s_{G,X}(t)$ is a positive rational function, then the numbers φ_j above are in fact rational multiples of π [4], and as a consequence the sequence (c_n) is periodic.

However, the author has not been able to find a reason why the function $s_{G,X}(t)$, in case it is rational, must also be positive. In particular, one can find pairs (G, X) such that $s_{G,X}(t)$ is rational but there are no regular languages $\mathcal{L}$ as above, and one can even find groups G such that this holds for (G, X) for any generating set X. For instance, it can be shown that growth of the 2-step nilpotent Heisenberg group

$$G = H_3 = \langle a, b, c \mid [a, b] = c, [a, c] = [b, c] = 1 \rangle$$

is rational with respect to any generating set [9, Theorem 1], but there are no languages $\mathcal{L}$ as above when G is a 2-step nilpotent group that is not virtually abelian [18, Corollary 3].

It is easy to check that the conclusion of Theorem 1 implies that

$$\liminf_{n\to\infty} \frac{|B_{G,X}(n)|}{n^{\hat{\alpha}}\lambda^n} > 0 \qquad \text{and} \qquad \limsup_{n\to\infty} \frac{|B_{G,X}(n)|}{n^{\hat{\alpha}}\lambda^n} < \infty, \tag{8}$$

where $\hat{\alpha} = \alpha + 1$ if $\lambda = 1$ and $\hat{\alpha} = \alpha$ otherwise. Asymptotics similar to these have been obtained for nilpotent groups, even without the condition on rational growth. In particular, in [17] Pansu showed that given a nilpotent group G with a finite generating set X, there exists $\hat{\alpha} \in \mathbb{Z}_{\geq 0}$ such that $\frac{|B_{G,X}(n)|}{n^{\bar{\alpha}}} \to C$ as $n \to \infty$ for some $C > 0$. Moreover, in [19] Stoll calculates the constant C for certain 2-step nilpotent groups G explicitly to show that the corresponding growth series $s_{G,X}(t)$ *cannot* be rational. However, in general – for groups that are not virtually nilpotent – one cannot expect lim sup and lim inf in (8) to be equal, as the hyperbolic group $C_2 * C_3$ shows: see [12, §3].

Finally, note that the same proof indeed shows a more general result:

Theorem 11. *Let $(a_n)_{n=0}^{\infty}$ be a submultiplicative sequence of numbers in $\mathbb{Z}_{\geq 1}$ such that $s(t) = \sum a_n t^n$ is a rational function. Then there exist constants $\alpha \in \mathbb{Z}_{\geq 0}$, $\lambda \in [1, \infty)$ and $D > C > 0$ such that for all $n \geq 1$,*

$$Cn^{\alpha}\lambda^n \leq a_n \leq Dn^{\alpha}\lambda^n.$$

The example below shows that both submultiplicativity and rationality are necessary requirements.

Example 12. *(i) Let*

$$p(t) = 1 + 12t^2 - 16t^3$$

and

$$q(t) = (1 - t)(1 - 2t)(1 - 2\omega t)(1 - 2\bar{\omega}t),$$

where ω is a 6^{th} primitive root of unity. Let $s(t)$, (a_n), λ, α and (c_n) be as above. Then $\lambda = 2$ and $\alpha = 0$, and [16, Lemma 1] *can be used to calculate*

$$a_n = c_n 2^n + 1$$

where

$$c_n = 4 - 2\omega^n - 2\bar{\omega}^n = \begin{cases} 0, & n \equiv 0 \pmod 6, \\ 2, & n \equiv \pm 1 \pmod 6, \\ 6, & n \equiv \pm 2 \pmod 6, \\ 8, & n \equiv 3 \pmod 6. \end{cases}$$

But as $c_n = 0$ for infinitely many values of n, one has

$$\liminf_{n\to\infty} a_n/(n^{\alpha}\lambda^n) = 0.$$

Note that in this case $a_7 = 257 > 5 = a_1 a_6$, so the sequence (a_n) is not submultiplicative.

(ii) For $n \geq 0$, let $a_n = 2^{b(n)}$, where $b(n)$ is the sum of digits in the binary representation of n. Then (a_n) is a submultiplicative sequence, but $\sum a_n t^n$ is not a rational function. For each $n \geq 0$, one has $a_{2^n-1} = 2^n$ and $a_{2^n} = 2$. Thus

$$\liminf_{n\to\infty} \frac{a_n}{n} \leq \liminf_{n\to\infty} \frac{2}{2^n} = 0$$

and

$$\limsup_{n\to\infty} a_n \geq \limsup_{n\to\infty} 2^n = \infty,$$

so (a_n) does not satisfy the conclusion of Theorem 11 for any $\lambda \geq 1$ and $\alpha \in \mathbb{Z}_{\geq 0}$.

3. Degree of commutativity

The aim of this section is to prove Theorem 6. For this, let Γ be a finite simple graph and for each $v \in V(\Gamma)$, let $(\mathbf{H}(v), X(v))$ be a rational pair with small centralisers (see Definition 5). To simplify notation, suppose in addition that the sets $X(v)$ are symmetric and do *not* contain the identity $1 \in \mathbf{H}(v)$: clearly this does not affect the results. Suppose in addition that $G = G(\Gamma, \mathbf{H})$ is a group of exponential growth. One thus aims to show that $\mathrm{dc}_X(G) = 0$, where $X = \bigsqcup_{v \in V(\Gamma)} X(v)$.

3.1. **Preliminaries.** This subsection collects the terminology and preliminary results used in the proof of Theorem 6.

Let $\ell_n : X^* \to \mathbb{Z}_{\geq 0}$ be the *normal form length* function (n in ℓ_n stands for 'normal'): for $w \in X^*$, set $\ell_n(w) := m$ where m is the minimal integer for which $w \equiv w_1 w_2 \cdots w_m$ as words, where $w_i \in X(v_i)^*$ for some $v_i \in V(\Gamma)$. Moreover, let $\ell_w : X^* \to \mathbb{Z}_{\geq 0}$ be the *word length* function (w in ℓ_w stands for 'word'), i.e. let $\ell_w(w)$ be the number of letters in $w \in X^*$.

The following result says that given any word $w \in X^*$ representing $g \in G$, there is a simple algorithm to transform it into a word $\hat{w}$ representing g with $\ell_n(\hat{w})$ or $\ell_w(\hat{w})$ small. This follows quite easily from a result of Green [11].

Proposition 13. *Let $\ell : X^* \to \mathbb{Z}_{\geq 0}$ be either $\ell = \ell_n$ or $\ell = \ell_w$. Let $w \in X^*$ be a word representing an element $g \in G$, and let $\hat{w}$ be a word representing g with $(\ell(\hat{w}), \ell_w(\hat{w}))$ minimal (in the lexicographical ordering) among such words. Then $\hat{w}$ can be obtained from w by applying a sequence of moves of two types:*

(i) for some $w_u \in X(u)^*$ and $w_v \in X(v)^*$ with $\{u, v\} \in E(\Gamma)$, replacing a subword $w_u w_v$ with $w_v w_u$;
(ii) for some $v \in V(\Gamma)$ and some subword $w_1 \in X(v)^*$, replacing the subword w_1 with a word $w_0 \in X(v)^*$ representing the same element in $\mathbf{H}(v)$, such that $\ell_w(w_0) \leq \ell_w(w_1)$.

Proof. Suppose first that $\ell = \ell_n$, and let $\hat{w} \equiv w_1 \cdots w_m$, where $w_i \in X(v_i)^*$ for some $v_i \in V(\Gamma)$ and $m = \ell_n(w)$. In [11, Theorem 3.9], Green showed that by using moves (i) and (ii) we can transform w into a word $\hat{w}' \equiv w_1' \cdots w_m'$ where $w_i' \in X(v_i)^*$ and w_i, w_i' represent the same element of $\mathbf{H}(v)$. Notice that we have $\ell_w(w_i) \leq \ell_w(w_i')$ for each i: otherwise, existence of the word $w_1 \cdots w_{i-1} w_i' w_{i+1} \cdots w_m$ would contradict the minimality of $\hat{w}$. Thus a sequence of moves (ii) allows us to transform $\hat{w}'$ into $\hat{w}$, as required.

Suppose now that $\ell = \ell_w$. Let $\hat{w}_n \in X^*$ be a word representing g with $(\ell_n(\hat{w}_n), \ell_w(\hat{w}_n))$ minimal among all such words. Then the result for $\ell = \ell_n$ says that $\hat{w}$ can be transformed into $\hat{w}_n$ by using the moves (i)–(ii). Notice that if $w' \in X^*$ is obtained from $w \in X^*$ by applying move (i) or (ii), then $\ell_w(w') \leq \ell_w(w)$, and if the equality holds then there exists a move that transforms w' back into w. By definition of $\hat{w}$, no moves strictly decreasing the word length are used when transforming $\hat{w}$ to $\hat{w}_n$, and so there exists a sequence of moves transforming $\hat{w}_n$ into $\hat{w}$ as well. Thus we may apply moves (i)–(ii) to obtain $\hat{w}_n$ from w and subsequently $\hat{w}$ from $\hat{w}_n$, as required. ☐

Note that it follows from the proof of Proposition 13 that minimal values of $\ell_n(w)$ and $\ell_w(w)$ can be obtained simultaneously. This justifies the following:

Definition 14. For $g \in G$, define a *normal form* of g to be a word $w \in X^*$ with both $\ell_n(w)$ and $\ell_w(w)$ minimal (so that $\ell_w(w) = |g|_X$). Write $w = w_1 w_2 \cdots w_n$ for $w_i \in X$, and define the *support* of g as

$$\operatorname{supp}(g) := \{v \in V(\Gamma) \mid w_i \in X(v) \text{ for some } i\};$$

by Proposition 13 this does not depend on the choice of w.

Now suppose for contradiction that $\mathrm{dc}_X(G) > 0$. That means that for some constant $\varepsilon > 0$, one has

$$\sum_{g \in B(n)} \frac{|C_G(g) \cap B(n)|}{\mathfrak{B}(n)^2} \geq \varepsilon \tag{9}$$

for infinitely many values of n, where $C_G(g)$ denotes the centraliser of an element $g \in G$, and $\mathfrak{B}(n) = \mathfrak{B}_G(n) = \mathfrak{B}_{G,X}(n) := |B_{G,X}(n)|$.

In the proof certain conjugates of elements in G will be considered. In particular, let $g \in G$, and pick a conjugate $\tilde{g} \in G$ of g such that $g = p_g^{-1} \tilde{g} p_g$ with $|g| = 2|p_g| + |\tilde{g}|$ and such that $|\tilde{g}|$

is minimal subject to this. If $p_g = 1$, then g is called *cyclically reduced*; hence $\tilde{g}$ is cyclically reduced. Note that being cyclically reduced is a weaker condition than being cyclically normal in the sense of [2].

For any subset $A \subseteq V(\Gamma)$, let G_A denote $G(\Gamma(A), \mathbf{H}|_A)$, where $\Gamma(A)$ is the full subgraph of Γ spanned by A. These will be viewed as subgroups (called the *special subgroups*) of G. One may also define the *link* of A to be

$$\operatorname{link} A = \{u \in V(\Gamma) \mid (u, v) \in E(\Gamma) \text{ for all } v \in A\}.$$

Before carrying on with the proof, consider the sequence $(d_n)_{n=0}^{\infty}$ where

$$d_n := \frac{|\{(x, y) \in B_{G,X}(n)^2 \mid [x, y] = 1\}|}{\mathfrak{B}_{G,X}(n)^2}.$$

One aims to show that $d_n \to 0$ as $n \to \infty$. Note that for many groups of exponential growth, including all the non-elementary hyperbolic groups [1], the sequence $(d_n)_{n=0}^{\infty}$ converges to zero exponentially fast. However, the following example shows that this is not always the case for graph products. The result of Theorem 6 may be therefore more delicate than one might think.

Example 15. *Suppose Γ is a complete bipartite graph $K_{k,k}$, i.e. Γ has vertex set*

$$V(\Gamma) = \{u_1, \ldots, u_k, v_1, \ldots, v_k\}$$

and edge set

$$E(\Gamma) = \{\{u_i, v_j\} \mid 1 \le i, j \le k\},$$

and let $\mathbf{H}(u) \cong \mathbb{Z}$ with generators $X(u) = \{x_u, x_u^{-1}\}$ for each $u \in V(\Gamma)$. In this case one has $G(\Gamma, \mathbf{H}) \cong F_k \times F_k$ (direct product of two free groups of rank k) and so one can calculate sphere sizes in $G(\Gamma, \mathbf{H})$ and its special subgroups easily. Note that clearly (by the definition of link) every element of $G_A \le G$ commutes with every element of $G_{\operatorname{link} A} \le G$. Now consider the case where $A = \{u_1, \ldots, u_k\}$ and so $\operatorname{link} A = \{v_1, \ldots, v_k\}$. It follows that

$$\{(x, y) \in B(n)^2 \mid [x, y] = 1\} \supseteq B_{G_A}(n) \times B_{G_{\operatorname{link} A}}(n).$$

An explicit computation shows that

$$\mathfrak{B}_{G_A}(n) = \mathfrak{B}_{G_{\operatorname{link} A}}(n) = \frac{k(2k-1)^n - 1}{k - 1}$$

and

$$\mathfrak{B}_G(n) = \frac{2k^2 n (2k-1)^n}{(k-1)(2k-1)} + e_1 (2k-1)^n + e_2$$

where $e_1 = e_1(k)$ and $e_2 = e_2(k)$ are some constants. It follows that

$$d_n \ge \frac{\mathfrak{B}_{G_A}(n) \mathfrak{B}_{G_{\operatorname{link} A}}(n)}{\mathfrak{B}_G(n)^2} \sim \left(\frac{2k-1}{2kn} \right)^2$$

as $n \to \infty$. In particular, the sequence $(d_n)_{n=0}^{\infty}$ converges to zero only at a polynomial rate for $G = G(\Gamma, \mathbf{H})$.

The proof of Theorem 6 is based on the fact that if (9) held for infinitely many n then there would exist a subset $A \subseteq V(\Gamma)$ such that the growth of both G_A and $G_{\operatorname{link} A}$ would be comparable to that of G. More precisely, the outline of the proof is as follows:

(i) finding such a subset $A \subseteq V(\Gamma)$ and showing that G_A is not negligible in G, i.e. $\frac{\mathfrak{B}_{G_A}(n)}{\mathfrak{B}_G(n)} \not\to 0$ as $n \to \infty$ (subsection 3.2);
(ii) finding a collection $\mathcal{H}$ of subgroups of G having (uniformly) polynomial growth such that, for all $H \in \mathcal{H}$, $G_{\operatorname{link} A} \times H$ is a subgroup of G and $\frac{|(G_{\operatorname{link} A} \times H) \cap B_G(n)|}{\mathfrak{B}_G(n)}$ is uniformly bounded below as $n \to \infty$ (subsection 3.3);
(iii) using the embedding $G_A \times G_{\operatorname{link} A} \subseteq G$ and Theorem 1 to obtain a contradiction (subsection 3.4).

3.2. **A non-negligible special subgroup.** Note that (9) can be rewritten as

$$\sum_{A\subseteq V(\Gamma)} \sum_{\substack{g\in B(n)\\ \mathrm{supp}(\tilde{g})=A}} \frac{|C_G(g)\cap B(n)|}{\mathfrak{B}(n)^2} \geq \varepsilon \tag{10}$$

and so (10) holds for infinitely many n. But as Γ is finite, there are only $2^{|V(\Gamma)|} < \infty$ subsets of $V(\Gamma)$, thus in particular there exists a subset $A \subseteq V(\Gamma)$ such that

$$\sum_{\substack{g\in B(n)\\ \mathrm{supp}(\tilde{g})=A}} \frac{|C_G(g)\cap B(n)|}{\mathfrak{B}(n)^2} \geq 2^{-|V(\Gamma)|}\varepsilon \tag{11}$$

holds for infinitely many n. One may restrict the subset of elements $g \in G$ considered even further:

Lemma 16. *There exist constants $\tilde{\varepsilon} > 0$ and $s \in \mathbb{Z}_{\geq 0}$ such that*

$$\sum_{\substack{g\in B(n)\\ \mathrm{supp}(\tilde{g})=A\\ |p_g|\leq s}} \frac{|C_G(g)\cap B(n)|}{\mathfrak{B}(n)^2} \geq \tilde{\varepsilon}$$

for infinitely many n.

Proof. As G has rational spherical growth series by [7], Theorem 1 says that there exist constants $\alpha \in \mathbb{Z}_{\geq 0}$, $\lambda \geq 1$, $C = C_G > 0$ and $D = D_G > C$ such that

$$Cn^\alpha\lambda^n \leq \mathfrak{S}(n) \leq Dn^\alpha\lambda^n \tag{12}$$

for all $n \geq 1$. As it is also assumed that G has exponential growth, one has $\lambda > 1$. It is easy to show that in this case

$$Cn^\alpha\lambda^n < \mathfrak{B}(n) < \frac{D\lambda}{\lambda-1}n^\alpha\lambda^n \tag{13}$$

for all $n \geq 1$.

Now one can bound the number of terms in (11) corresponding to elements $g \in G$ with $|p_g|$ large (even without requiring $\mathrm{supp}(\tilde{g}) = A$). Indeed, as any $g \in G$ can be written as $g = p_g^{-1}\tilde{g}p_g$ with $|g| = 2|p_g| + |\tilde{g}|$, (12) and (13) imply

$$\begin{aligned}\frac{1}{\mathfrak{B}(n)} \sum_{\substack{g\in B(n)\\ |p_g|>s}} \frac{|C_G(g)\cap B(n)|}{\mathfrak{B}(n)} &\leq \frac{|\{g\in B(n) \mid |p_g|>s\}|}{\mathfrak{B}(n)} \leq \sum_{i=s+1}^{\lfloor \frac{n}{2}\rfloor} \frac{\mathfrak{S}(i)\mathfrak{B}(n-2i)}{\mathfrak{B}(n)}\\ &\leq \frac{D}{C}\left(\frac{1}{2}\right)^\alpha \lambda^{-\frac{n}{2}} + \frac{D^2\lambda}{C(\lambda-1)} \sum_{i=s+1}^{\lfloor \frac{n-1}{2}\rfloor} \left(\frac{i(n-2i)}{n}\right)^\alpha \lambda^{-i}.\end{aligned} \tag{14}$$

The first term of the sum above clearly tends to zero as $n \to \infty$, and the second term is bounded above by the infinite sum $\sum_{i=s+1}^{\infty} i^\alpha\lambda^{-i}$, which tends to zero as $s \to \infty$ since the series $\sum_i i^\alpha\lambda^{-i}$ converges. Hence there exists a value of $s \in \mathbb{Z}_{\geq 0}$ which ensures that the right hand side in (14) is less than $2^{-|V(\Gamma)|-1}\varepsilon$ for n large enough. This means that

$$\sum_{\substack{g\in B(n)\\ \mathrm{supp}(\tilde{g})=A\\ |p_g|\leq s}} \frac{|C_G(g)\cap B(n)|}{\mathfrak{B}(n)^2} \geq 2^{-|V(\Gamma)|-1}\varepsilon$$

for infinitely many n, so setting $\tilde{\varepsilon} := 2^{-|V(\Gamma)|-1}\varepsilon$ completes the proof. □

Now note that one may write

$$\sum_{\substack{g\in B(n)\\ \mathrm{supp}(\tilde{g})=A\\ |p_g|\leq s}} \frac{|C_G(g)\cap B(n)|}{\mathfrak{B}(n)^2} \leq \frac{|\{g\in B(n)\mid \mathrm{supp}(\tilde{g})=A, |p_g|\leq s\}|}{\mathfrak{B}(n)} \times \max\left\{ \frac{|C_G(g)\cap B(n)|}{\mathfrak{B}(n)} \;\middle|\; g\in B(n), \mathrm{supp}(\tilde{g})=A, |p_g|\leq s \right\}$$

where both terms in the product are bounded above by 1. It follows by Lemma 16 that both

$$(*)\qquad \frac{|\{g\in B(n)\mid \mathrm{supp}(\tilde{g})=A, |p_g|\leq s\}|}{\mathfrak{B}(n)} \geq \tilde{\varepsilon}$$

and

$$(\dagger)\qquad \max\left\{ \frac{|C_G(g)\cap B(n)|}{\mathfrak{B}(n)} \;\middle|\; g\in B(n), \mathrm{supp}(\tilde{g})=A, |p_g|\leq s \right\} \geq \tilde{\varepsilon}$$

hold for infinitely many n.

The aim is now to show that $(*)$ and $(\dagger)$ imply that the special subgroups G_A and $G_{\mathrm{link}\,A}$ (respectively) are non-negligible in G. For the latter, one may consider explicit forms of centralisers of G: see the next subsection. For the former, note that the set in the numerator consists of elements $g\in B_G(n)$ which have an expression $g=p_g^{-1}\tilde{g}p_g$ with $p_g\in B_G(s)$ and $\tilde{g}\in B_{G_A}(n)$. It follows that

$$|\{g\in B(n)\mid \mathrm{supp}(\tilde{g})=A, |p_g|\leq s\}| \leq \mathfrak{B}_G(s)\mathfrak{B}_{G_A}(n)$$

and so $(*)$ implies that

$$(**)\qquad \frac{\tilde{\varepsilon}}{\mathfrak{B}_G(s)} \leq \frac{\mathfrak{B}_{G_A}(n)}{\mathfrak{B}_G(n)} \leq 1$$

for infinitely many n, where the second inequality comes from the fact that $B_{G_A}(n)\subseteq B_G(n)$.

3.3. **Centralisers in G.** In order to use $(\dagger)$, one needs to consider forms of centralisers of elements $g\in G$ with $\mathrm{supp}(\tilde{g})=A$. Fix an element $g\in G$ with $\mathrm{supp}(\tilde{g})=A$ and note that one clearly has $C_G(g)=p_g^{-1}C_G(\tilde{g})p_g$, so if $|p_g|\leq s$ then one has

$$(15)\qquad |C_G(g)\cap B(n)| \leq |C_G(\tilde{g})\cap B(n+2s)|.$$

In particular, it follows from $(\dagger)$ that for infinitely many n, there exists an element $g\in B(n)$ with $\mathrm{supp}(\tilde{g})=A$ and $|p_g|\leq s$ such that

$$(\dagger\dagger)\qquad \tilde{\varepsilon} \leq \frac{|C_G(\tilde{g})\cap B_G(n+2s)|}{\mathfrak{B}_G(n)} \leq \mathfrak{B}_G(2s);$$

here the second inequality comes from the fact that $|C_G(\tilde{g})\cap B(n+2s)| \leq \mathfrak{B}(n+2s) \leq \mathfrak{B}(n)\mathfrak{B}(2s)$.

Now define an element $g\in G$ to be *cyclically normal* (in the sense of [2]) if either $\ell_n(g)\leq 1$, or $n:=\ell_n(g)\geq 2$ and for any normal form $w=w_1\cdots w_n\in X^*$ of g, where $w_i\in X(v_i)^*$ for some $v_i\in V(\Gamma)$, one has $v_1\neq v_n$. Then one has

Lemma 17. *For any $g\in G$ with $\mathrm{supp}(\tilde{g})=A$, there exists an element $\tilde{p}_g\in G_A$ such that $\hat{g}:=\tilde{p}_g\tilde{g}\tilde{p}_g^{-1}$ is cyclically normal and $\mathrm{supp}(\hat{g})=A$.*

Proof. If $\ell_n(\tilde{g})\leq 1$ then $\tilde{p}_g=1$ does the job. Thus suppose that $n:=\ell_n(\tilde{g})\geq 2$. Let

$$E(\tilde{g}) := \{g_0 \mid w=w_1\cdots w_n\in X^* \text{ is a normal form for } \tilde{g} \text{ where } w_i\in X(v_i)^* \text{ for some } v_i\in V(\Gamma), \text{ and } w_n \text{ represents } g_0\}$$

be a finite subset of G_A. By Proposition 13, any two elements in $E(\tilde{g})$ commute, and so, for any two distinct elements $g_1\in\mathbf{H}(v_1)$ and $g_2\in\mathbf{H}(v_2)$ of $E(\tilde{g})$, one has $v_1\neq v_2$. Now define $\tilde{p}_g := \prod_{g_n\in E(\tilde{g})} g_n$. Then $\tilde{p}_g\in G_A$, and following the proof of [2, Lemma 23] one can see that $\hat{g} := \tilde{p}_g \tilde{g} \tilde{p}_g^{-1}$ is cyclically normal. Since $\mathrm{supp}(\tilde{p}_g) \subseteq A$ and $\mathrm{supp}(\tilde{g}) = A$, it is clear that $\mathrm{supp}(\hat{g}) \subseteq A$. It also follows by [2, Lemma 18] that $\mathrm{supp}(\hat{g}) \cup \mathrm{supp}(\tilde{p}_g) \supseteq A$. Thus one only needs to check that $\mathrm{supp}(\tilde{p}_g) \subseteq \mathrm{supp}(\hat{g})$.

Suppose for contradiction that there exists some $v \in \mathrm{supp}(\tilde{p}_g) \backslash \mathrm{supp}(\hat{g})$, and let $g_v \in E(\tilde{g}) \cap \mathbf{H}(v)$ be the (unique) element. It is easy to see that $v \notin \mathrm{link}(A \setminus \{v\})$: otherwise any normal form of $\hat{g}$ would contain a subword in $X(v)^*$ representing g_v and so $v \in \mathrm{supp}(\hat{g})$. Then, following again the proof of [2, Lemma 23], one has $\tilde{n} := \ell_n(g_v \tilde{g} g_v^{-1}) \leq n-1$, with $\tilde{n} = n-1$ if and only if $\tilde{g}$ has no normal form $w_1 \cdots w_n$, where $w_i \in X(v_i)^*$ for some $v_i \in V(\Gamma)$, with w_1 and w_n representing g_v^{-1} and g_v, respectively. Thus, by minimality of $|\tilde{g}|$, clearly $\tilde{n} = n-1$; but this cannot happen by [2, Lemma 18], since by assumption $v \notin \mathrm{supp}(\hat{g})$. Hence $\mathrm{supp}(\tilde{p}_g) \subseteq \mathrm{supp}(\hat{g})$, as required. □

The following Proposition describes growth of centralisers in G.

Proposition 18. *Let $g, \tilde{g} \in G$ and $A \subseteq V(\Gamma)$ be as above. Then*

$$C_G(\tilde{g}) = H_1 \times \cdots \times H_k \times G_{\mathrm{link}\,A}$$

for some subgroups $H_1, \ldots, H_k \leq G$, and the following hold:

(i) for any $h_1 \in H_1, \ldots, h_k \in H_k$ and $c \in G_{\mathrm{link}\,A}$,

$$|h_1 \cdots h_k c|_X = |h_1|_X + \cdots + |h_k|_X + |c|_X;$$

(ii) there exist constants $D_1, \ldots, D_k, \alpha_1, \ldots, \alpha_k \in \mathbb{Z}_{\geq 1}$ such that

$$|H_i \cap B_{G,X}(n)| \leq D_i n^{\alpha_i}$$

for all $n \geq 1$.

Furthermore, the number $k \in \mathbb{Z}_{\geq 1}$, the D_i and the α_i only depend on A and not on g.

Proof. Let $A_1, \ldots, A_k \subseteq A$ form a partition of A such that the graphs $\Gamma(A_i)^C$ are precisely the connected components of the graph $\Gamma(A)^C$, where Δ^C denotes the complement of a graph Δ. Let $\tilde{p}_g, \hat{g} \in G_A$ be as in Lemma 17. Then $\mathrm{supp}(\hat{g}) = A$ and so $\hat{g}$ can be expressed as

$$\hat{g} = \hat{g}_1 \cdots \hat{g}_k$$

where $\mathrm{supp}(\hat{g}_i) = A_i$.

Now suppose without loss of generality that for some m, the sets $A_i = \{v_i\}$ are singletons for $1 \leq i \leq m$, and $|A_i| \geq 2$ for $m+1 \leq i \leq k$. Then Proposition 25, Theorem 32 and Theorem 52 in [2] state that the centraliser of $\hat{g}$ in G is

$$C_G(\hat{g}) = C_{\mathbf{H}(v_1)}(\hat{g}_1) \times \cdots \times C_{\mathbf{H}(v_m)}(\hat{g}_m) \times \langle h_{m+1} \rangle \times \cdots \times \langle h_k \rangle \times G_{\mathrm{link}\,A}$$

where $h_{m+1}, \ldots, h_k \in G$ are some infinite order elements with $\mathrm{supp}(h_i) = A_i$ (in fact, one has $\hat{g}_i = h_i^{\beta_i}$ for some $\beta_i \in \mathbb{Z} \setminus \{0\}$).

In particular, since $\tilde{p}_g \in G_A$, one has $\tilde{p}_g = p_1 \cdots p_k$ for some $p_i \in G_{A_i}$. Thus $\tilde{p}_g^{-1} q_i \tilde{p}_g = p_i^{-1} q_i p_i$ for any $q_i \in G_{A_i}$, and $\tilde{p}_g^{-1}(G_{\mathrm{link}\,A})\tilde{p}_g = G_{\mathrm{link}\,A}$, hence

$$\begin{aligned} C_G(\tilde{g}) = \tilde{p}_g^{-1} C_G(\hat{g}) \tilde{p}_g = C_{\mathbf{H}(v_1)}(\tilde{g}_1) \times \cdots \times C_{\mathbf{H}(v_m)}(\tilde{g}_m) \\ \times \langle \tilde{g}_{m+1} \rangle \times \cdots \times \langle \tilde{g}_k \rangle \times G_{\mathrm{link}\,A} \end{aligned}$$

where $\tilde{g}_i := p_i^{-1} \hat{g}_i p_i$ for $1 \leq i \leq m$, and $\tilde{g}_i := p_i^{-1} h_i p_i$ for $m+1 \leq i \leq k$. Hence, by setting $H_i := C_{\mathbf{H}(v_i)}(\tilde{g}_i)$ for $1 \leq i \leq m$ and $H_i := \langle \tilde{g}_i \rangle \cong \mathbb{Z}$ for $m+1 \leq i \leq k$ one obtains the required expression. By construction, k depends only on A (and not on g).

To show (i), it is enough to note that $H_i \leq G_{A_i}$ for each i, and that by construction the subsets A_i are pairwise disjoint and disjoint from $\mathrm{link}\,A$. Indeed, then it follows from Proposition 13 that if w_i (respectively u) is a normal form for an element $h_i \in G_{A_i}$ (respectively $c \in G_{\mathrm{link}\,A}$), then $w_1 \cdots w_k u$ is a normal form for the element $h_1 \cdots h_k c$. This implies (i).

To show (ii) and the last part of the Proposition, one may consider cases $1 \leq i \leq m$ and $m+1 \leq i \leq k$ separately. For $1 \leq i \leq m$, note that, as a consequence of Proposition 13, $|h|_X = |h|_{X(v_i)}$ for all $h \in H_i$, and therefore $|H_i \cap B_{G,X}(n)| = |H_i \cap B_{\mathbf{H}(v_i),X(v_i)}(n)|$ for all $n \geq 1$. Thus, (ii) follows from the facts that $\tilde{g}_i \neq 1$ and that $(\mathbf{H}(v_i), X(v_i))$ is a rational pair with small centralisers; it also follows that D_i, α_i do not depend on g. For $m+1 \leq i \leq k$, it follows from the proof of [2, Lemma 37] that since $\hat{g}_i$ is cyclically normal and since $\Gamma(\operatorname{supp}(\hat{g}_i))^C = \Gamma(A_i)^C$ is connected, one has $\ell_n(\tilde{g}_i^\gamma) \geq \ell_n(\hat{g}_i^\gamma) = |\gamma|\ell_n(\hat{g}_i)$ for all $\gamma \in \mathbb{Z}$. In particular, $|\tilde{g}_i^\gamma|_X \geq \ell_n(\tilde{g}_i^\gamma) \geq |\gamma|$ for any $\gamma \in \mathbb{Z}$ and so $|H_i \cap B_{G,X}(n)| \leq 2n+1 \leq 3n$ for all $n \geq 1$. Thus taking $D_i = 3$ and $\alpha_i = 1$ shows (ii); independence from g is clear. □

3.4. **Products of special subgroups.** To finalise the proof, one employs the following general result:

Lemma 19. *Let G be a group with a finite generating set X. Let $H, K \leq G$ be subgroups such that $H \times K$ is also a subgroup of G, i.e. the map $H \times K \to G, (h,k) \mapsto hk$ is an injective group homomorphism. Suppose that there exist constants $\alpha_H, \alpha_K \in \mathbb{Z}_{\geq 0}$, $\lambda_H, \lambda_K \in [1,\infty)$ and $D > C \geq 0$ such that*

$$Cn^{\alpha_H}\lambda_H^n \leq |H \cap S_{G,X}(n)| \leq Dn^{\alpha_H}\lambda_H^n$$
$$\text{and} \qquad Cn^{\alpha_K}\lambda_K^n \leq |K \cap S_{G,X}(n)| \leq Dn^{\alpha_K}\lambda_K^n$$

for all $n \geq 1$. Furthermore, suppose that $|hk|_X = |h|_X + |k|_X$ for all $h \in H^{(n)}$, $k \in K^{(n)}$, and that $\lambda_H \geq \lambda_K$. If $\lambda_H > \lambda_K$, then there exists constant $\widetilde{D} = \widetilde{D}(D, \alpha_H, \alpha_K, \lambda_H, \lambda_K) > 0$, which does not depend on H or K, such that

$$|(H \times K) \cap S_{G,X}(n)| \leq \widetilde{D}n^{\alpha_H}\lambda_H^n$$

for all $n \geq 1$. Furthermore, if $\lambda_H = \lambda_K$ and $C > 0$, then no such constant $\widetilde{D}$ exists.

Proof. Suppose first that $\lambda_H > \lambda_K$. Clearly it is enough to show that

$$\limsup_{n\to\infty} \frac{|(H \times K) \cap S_{G,X}(n)|}{n^{\alpha_H}\lambda_H^n} < \infty.$$

Fix $n \geq 1$. As $|hk|_X = |h|_X + |k|_X$ for any $h \in H$, $k \in K$, one has

$$\frac{|(H \times K) \cap S_{G,X}(n)|}{n^{\alpha_H}\lambda_H^n} = \frac{1}{n^{\alpha_H}\lambda_H^n}\sum_{i=0}^{n} |H \cap S_{G,X}(n-i)| \times |K \cap S_{G,X}(i)|$$
$$\leq D^2\left(1 + \sum_{i=1}^{n-1}\left(\frac{\lambda_K}{\lambda_H}\right)^i\left(\frac{n-i}{n}\right)^{\alpha_H} i^{\alpha_K} + \left(\frac{\lambda_K}{\lambda_H}\right)^n n^{\alpha_K-\alpha_H}\right).$$

As $\lambda_K/\lambda_H < 1$, limits of the first and third term above as $n \to \infty$ are D^2 and 0, respectively. The second term can be bounded above by an upper bound for the series $D^2\sum_i(\lambda_K/\lambda_H)^i i^{\alpha_K}$, which converges by the ratio test. Hence indeed $\limsup_{n\to\infty} |(H\times K)\cap S_{G,X}(n)|/(n^{\alpha_H}\lambda_H^n) < \infty$, which implies the result. It is also clear from the inequality above that $\widetilde{D}$ depends only on D, α_H, α_K, λ_H and λ_K.

Conversely, suppose that $C > 0$ and $\lambda_H = \lambda_K =: \lambda$. Let $n \geq 20$, so that $\lceil\sqrt{n}\rceil \leq n/4$. Then

$$\frac{|(H \times K) \cap S_{G,X}(n)|}{n^{\alpha_H}\lambda^n} = \frac{1}{n^{\alpha_H}\lambda^n}\sum_{i=0}^{n} |H \cap S_{G,X}(n-i)| \times |K \cap S_{G,X}(i)|$$
$$\geq C^2\sum_{i=1}^{n-1}\left(\frac{n-i}{n}\right)^{\alpha_H} i^{\alpha_K} \geq C^2\sum_{i=\lceil\sqrt{n}\rceil}^{\lfloor n/2\rfloor}\left(\frac{1}{2}\right)^{\alpha_H}(\sqrt{n})^{\alpha_K}$$
$$\geq C^2 2^{-(\alpha_H+2)} n^{\frac{\alpha_K}{2}+1}.$$

In particular, one has $|(H \times K) \cap S_{G,X}(n)|/(n^{\alpha_H}\lambda_H^n) \to \infty$ as $n \to \infty$, implying the result. □

Given this Lemma, the proof can be finalised as follows. Recall (see (12) and (13)) that one has constants $\alpha \in \mathbb{Z}_{\geq 0}$, $\lambda > 1$ and $D_{V(\Gamma)} > C_{V(\Gamma)} > 0$ such that

$$\begin{aligned} &C_{V(\Gamma)} n^\alpha \lambda^n \leq \mathfrak{S}_G(n) \leq D_{V(\Gamma)} n^\alpha \lambda^n \\ \text{and} \quad &C_{V(\Gamma)} n^\alpha \lambda^n < \mathfrak{B}_G(n) < \frac{D_{V(\Gamma)}\lambda}{\lambda - 1} n^\alpha \lambda^n \end{aligned} \tag{16}$$

for all $n \geq 1$. Now $(**)$ implies that, for infinitely many n,

$$\widetilde{C}_A n^\alpha \lambda^n \leq \mathfrak{B}_{G_A}(n) \leq \widetilde{D}_A n^\alpha \lambda^n \tag{17}$$

for some $\widetilde{D}_A > \widetilde{C}_A > 0$. But as G_A has rational growth with respect to $\bigsqcup_{v \in A} X(v)$, it follows from Theorem 1 that in fact, after modifying the constants $\widetilde{D}_A$ and $\widetilde{C}_A$ if necessary, (17) holds for all $n \geq 1$, and since $\lambda > 1$, after further modifying $\widetilde{C}_A$, one has

$$\widetilde{C}_A n^\alpha \lambda^n \leq \mathfrak{S}_{G_A}(n) \leq \widetilde{D}_A n^\alpha \lambda^n \tag{$***$}$$

for all $n \geq 1$.

Moreover, $(\dagger\dagger)$ implies that for infinitely many $n \geq 2s+1$ there exists $g \in B(n)$ such that

$$\widetilde{C}'(n-2s)^\alpha \lambda^{n-2s} \leq |C_G(\tilde{g}) \cap B_{G,X}(n)| \leq \widetilde{D}'(n-2s)^\alpha \lambda^{n-2s}$$

for some $\widetilde{D}' > \widetilde{C}' > 0$. After decreasing the constant $\widetilde{C}' > 0$ if necessary, one may therefore assume that, for infinitely many n,

$$\widetilde{C}' n^\alpha \lambda^n \leq |C_G(\tilde{g}) \cap B_{G,X}(n)| \leq \widetilde{D}' n^\alpha \lambda^n \tag{18}$$

for some $g \in B(n)$ with $\mathrm{supp}(\widetilde{g}) = A$ and $|p_g| \leq s$.

Note that $G_{\mathrm{link}\, A}$ has rational growth with respect to $\bigsqcup_{v \in \mathrm{link}\, A} X(v)$ as it is a special subgroup of G, and so by Theorem 1 it follows that, for all $n \geq 1$,

$$\widetilde{C}_{\mathrm{link}\, A} n^{\alpha_0} \lambda_0^n \leq \mathfrak{S}_{G_{\mathrm{link}\, A}}(n) \leq \widetilde{D}_{\mathrm{link}\, A} n^{\alpha_0} \lambda_0^n \tag{19}$$

for some $\widetilde{D}_{\mathrm{link}\, A} > \widetilde{C}_{\mathrm{link}\, A} > 0$ and some $\alpha_0 \in \mathbb{Z}_{\geq 0}$, $\lambda_0 \geq 1$.

One may now show that $(\lambda_0, \alpha_0) = (\lambda, \alpha)$. Indeed, as $\mathfrak{S}_{G_{\mathrm{link}\, A}}(n) \subseteq \mathfrak{S}_G(n)$, it follows from (16) that either $\lambda_0 < \lambda$ or $\lambda_0 = \lambda$ and $\alpha_0 \leq \alpha$. Let $g \in G$ be such that $\mathrm{supp}(\widetilde{g}) = A$ for all n. By Proposition 18, one has an expression

$$C_G(\widetilde{g}) = H_1 \times \cdots \times H_k \times G_{\mathrm{link}\, A}.$$

One now applies Lemma 19 k times. In particular, for each $i = k, k-1, \ldots, 1$ in order, it follows from Proposition 18 that Lemma 19 can be applied for

$$\begin{aligned} H &:= H_{i+1} \times \cdots \times H_k \times G_{\mathrm{link}\, A}, \\ K &:= H_i \\ (\alpha_H, \lambda_H) &:= \begin{cases} (\alpha_0, \lambda_0) & \text{if } \lambda_0 > 1, \\ (0, \frac{\lambda+1}{2}) & \text{if } \lambda_0 = 1, \end{cases} \\ (\alpha_K, \lambda_K) &:= \left(0, \frac{\lambda_H + 1}{2}\right), \\ C &:= 0, \\ \text{and} \qquad D = \overline{D}_i &:= \max\{\widetilde{D}_i, \widetilde{D}'_i\}. \end{aligned}$$

Here $\widetilde{D}'_i > 0$ is such that $D_i n^{\alpha_i} \leq \widetilde{D}'_i \lambda_K^n$ for each $n \geq 1$, where D_i and α_i are as in Proposition 18, $\widetilde{D}_k$ is such that $\mathfrak{S}_{G_{\mathrm{link}\, A}}(n) \leq \widetilde{D}_k n^{\alpha_H} \lambda_H^n$ for all $n \geq 1$, and, for each $i = k-1, k-2, \ldots, 1$, $\widetilde{D}_i = \widetilde{D}(\overline{D}_{i+1}, \alpha_H, \alpha_K, \lambda_H, \lambda_K)$ is the constant given by Lemma 19.

It then follows that, for all $g \in G$ with $\mathrm{supp}(\widetilde{g}) = A$ and $|p_g| \leq s$,

$$|C_G(\widetilde{g}) \cap S_{G,X}(n)| \leq \widetilde{D} n^{\alpha_H} \lambda_H^n \tag{20}$$

for all $n \geq 1$, where $\widetilde{D} = \widetilde{D}(\overline{D}_1, \alpha_H, \alpha_K, \lambda_H, \lambda_K)$ is the constant, independent from g, given by Lemma 19. Since $\lambda_H > 1$, by further increasing $\widetilde{D}$ we may replace $S_{G,X}(n)$ with $B_{G,X}(n)$ in (20). But by construction, one has either $\lambda_H < \lambda$ or $\lambda_H = \lambda$ and $\alpha_H \leq \alpha$, and so together with (18) this implies that $(\lambda_H, \alpha_H) = (\lambda, \alpha)$. Thus, by the choice of (λ_H, α_H), one has $(\lambda_0, \alpha_0) = (\lambda, \alpha)$, as claimed. In particular, (19) can be rewritten as

$$(\dagger\dagger\dagger) \qquad \widetilde{C}_{\operatorname{link} A} n^\alpha \lambda^n \leq \mathfrak{S}_{G_{\operatorname{link} A}}(n) \leq \widetilde{D}_{\operatorname{link} A} n^\alpha \lambda^n.$$

Finally, note that the group $G_{A \cup \operatorname{link} A} = G_A \times G_{\operatorname{link} A}$ is a special subgroup of G and so one has $S_{G_{A \cup \operatorname{link} A}}(n) \subseteq S_G(n)$. It then follows from $(***)$, $(\dagger\dagger\dagger)$ and Lemma 19 that for any $\widetilde{D} > 0$ one has

$$\mathfrak{S}_G(n) \geq \mathfrak{S}_{G_{A \cup \operatorname{link} A}}(n) > \widetilde{D} n^\alpha \lambda^n$$

for some n, which contradicts (16). This completes the proof of Theorem 6.

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