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THE CHARACTER TABLE OF A SHARPLY 5-TRANSITIVE SUBGROUP OF THE ALTERNATING GROUP OF DEGREE 12

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Abstract. We calculate the character table of a sharply 5-transitive subgroup of Alt(12), and of a sharply 4-transitive subgroup of Alt(11). Our presentation of these calculations is new because we make no reference to the sporadic simple Mathieu groups, and instead deduce the desired character tables using only the existence of the stated multiply transitive permutation representations.

1. Introduction

Let $\mathrm{Alt}(n)$ denote the alternating group on n points. In this paper we will be interested in subgroups of $\mathrm{Alt}(n)$ that are sharply k-transitive, for some integer $k \geq 4$. Recall that a permutation group G acting on a set Ω is *sharply k-transitive* if, first, G acts transitively on the set of k-tuples of distinct elements of Ω and, second, the (point-wise) stabilizer of any k-tuple of distinct elements of Ω is trivial. Notice that "1-transitivity" is the same as "transitivity". In this paper we prove the following theorem.

Theorem 1.1.

(1) *If G is a sharply 5-transitive subgroup of* $\mathrm{Alt}(12)$*, then its character table is given by Table 1.*
(2) *If G is a sharply 4-transitive subgroup of* $\mathrm{Alt}(11)$*, then its character table is given by Table 2.*

A corollary of this theorem is that the subgroups in question are simple groups. This theorem is not new – item (1) is a consequence of the fact that the sporadic simple Mathieu group M_{12} is the unique sharply 5-transitive subgroup of Alt(12), and the fact that the character table of M_{12} is known; item (2) is a consequence of the two analogous statements for the sporadic simple Mathieu group M_{11}. Our proof of the theorem is new, however, because it makes no reference whatsoever to the groups M_{11}

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G_{12}	1^{12}	2^4	2^6	3^3	3^4	4^2	2^24^2	5^2	$2^13^16^1$	6^2	2^18^1	4^18^1	2^110^1	11^1_A	11^1_B
χ_0	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
χ_1	11	3	-1	2	-1	3	-1	1	0	-1	1	-1	-1	0	0
χ_2	11	3	-1	2	-1	-1	3	1	0	-1	-1	1	-1	0	0
χ_3	16	0	4	-2	1	0	0	1	0	1	0	0	-1	ω	$\overline{\omega}$
χ_4	16	0	4	-2	1	0	0	1	0	1	0	0	-1	$\overline{\omega}$	ω
χ_5	45	-3	5	0	3	1	1	0	0	-1	-1	-1	0	1	1
χ_6	54	6	6	0	0	2	2	-1	0	0	0	0	1	-1	-1
χ_7	55	7	-5	1	1	-1	-1	0	1	1	-1	-1	0	0	0
χ_8	55	-1	-5	1	1	-1	3	0	-1	1	-1	1	0	0	0
χ_9	55	-1	-5	1	1	3	-1	0	-1	1	1	-1	0	0	0
χ_{10}	66	2	6	3	0	-2	-2	1	-1	0	0	0	1	0	0
χ_{11}	99	3	-1	0	3	-1	-1	-1	0	-1	1	1	-1	0	0
χ_{12}	120	-8	0	3	0	0	0	0	1	0	0	0	0	-1	-1
χ_{13}	144	0	4	0	-3	0	0	-1	0	1	0	0	-1	1	1
χ_{14}	176	0	-4	-4	-1	0	0	1	0	-1	0	0	1	0	0

TABLE 1. The character table of G_{12}, where $\omega = \frac{1}{2}(-1+\sqrt{-11})$.

G_{11}	1^{11}	2^4	3^3	4^2	5^2	$2^13^16^1$	$2^18^1_A$	$2^18^1_B$	11^1_A	11^1_B
χ_0	1	1	1	1	1	1	1	1	1	1
χ_1	10	2	1	2	0	-1	0	0	-1	-1
χ_2	10	-2	1	0	0	1	α	$\overline{\alpha}$	-1	-1
χ_3	10	-2	1	0	0	1	$\overline{\alpha}$	α	-1	-1
χ_4	11	3	2	-1	1	0	-1	-1	0	0
χ_5	16	0	-2	0	1	0	0	0	β	$\overline{\beta}$
χ_6	16	0	-2	0	1	0	0	0	$\overline{\beta}$	β
χ_7	44	4	-1	0	-1	1	0	0	0	0
χ_8	45	-3	0	1	0	0	-1	-1	1	1
χ_9	55	-1	1	-1	0	-1	1	1	0	0

TABLE 2. The character table of G_{11}, where $\alpha = \sqrt{-2}$ and $\beta = \frac{1}{2}(-1+\sqrt{-11})$.

and M_{12} but, instead, deduces the character table using nothing more than the stated assumptions about sharp multiple transitivity.

The motivation in proving this theorem starts with Frobenius' famous 1904 paper in which (amongst many other things) he calculates the character table of M_{12} and M_{24} [Fro04]. Understanding Frobenius' methods is important because, in the late nineteenth century, there appeared to be some lingering doubt as to the "existence" of the Mathieu groups (that is to say, people questioned whether the permutations that Mathieu wrote down in his original paper [Mat73] generated alternating groups, rather than any genuinely "new" groups). As late as 1897, Miller published a paper claiming that M_{24} did not exist [Mil97], although he retracted this claim soon after [Mil00].

In fact, studying Frobenius' 1904 paper, it seems that Frobenius was in no doubt as to the existence of the Mathieu groups and, indeed, he uses specific properties of these groups when he calculates their character tables (see the `MathOverflow` discussion on this subject for more detail [Lad]). Nonetheless, it is natural to ask whether, in principle, Frobenius could have calculated the character table of M_{12} using nothing more than the property of sharp 5-transitivity – the main result of this paper confirms that the answer to this question is "yes"! In particular, note that the proof of Theorem 1.1 below uses little more than the basics of character theory, all of which would have been available to Frobenius in 1904 – the results that we make use of are summarized at the start of §3.

There is interest from others in results similar to our main theorem; note, for example [LP10, §2.10], where the authors calculate the character table of the Mathieu group M_{11} using only its simplicity, and its order. Note that our hypothesis is slightly different and that we also deal with the Mathieu groups M_9 and M_{10} as well as the sporadic simple group M_{12}. In addition, our methods are somewhat more elementary: in [LP10], the theory of Frobenius-Schur indicators is employed; this theory was not introduced until 1906 [FS06], some time after the publication of Frobenius' 1904 paper.

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2. Conjugacy classes

In what follows we will consider groups G_9, G_{10}, G_{11} and G_{12}. For $i = 9, \ldots, 12$, the group G_i denotes any sharply $(i-7)$-transitive subgroup of $\mathrm{Alt}(i)$ on i points. In this section we calculate the conjugacy class structure of G_9, G_{10}, G_{11} and G_{12}.

We label conjugacy classes via the cycle structure of their elements. Where there is more than one conjugacy class with the same cycle structure, they are distinguished with subscript Roman letters –

Type	1^9	2^4	3^3	4_A^2	4_B^2	4_C^2
Size	1	9	8	18	18	18

TABLE 3. Conjugacy class sizes in G_9

see, for instance, Table 3 which lists the conjugacy classes of G_9, three of which contain elements of type 4^2.

We remind the reader that a conjugacy class C of G is called **real** if any element of C is conjugate to its inverse; similarly, a character of G is called **real** if it takes real values for all elements of G. The connection between these two concepts is given in Theorem 3.4, and will be exploited in §3.

2.1. **The conjugacy classes of G_9.** Let N be a minimal normal subgroup of G_9. Since $|G_9| = 72$, and G_9 is sharply 2-transitive on 9 points, N must be elementary-abelian of size 9; moreover N must act regularly on the set $\{1, \ldots, 9\}$. Now 2-transitivity implies that the stabilizer of the point 1, call it H, acts transitively on the set $\{2, \ldots, 9\}$; since H is of order 8, we conclude that H acts regularly on the set $\{2, \ldots, 9\}$ and so H acts (by conjugation) regularly on the non-identity elements of N. This action induces a 1-1 homomorphism $H \to \mathrm{Aut}(N) = \mathrm{GL}_2(3)$.

Thus H is a group of order 8 that is isomorphic to a subgroup of $\mathrm{GL}_2(3)$ that acts regularly on the set of non-zero vectors in the associated vector space. An easy matrix calculation confirms that $H \cong Q_8$; thus $G \cong (C_3 \times C_3) \rtimes Q_8$.

The conjugacy classes of G_9 can now be written down in Table 3. Note that all classes are real.

2.2. **The conjugacy classes of G_{10}.** The group $G = G_{10}$ has order 720, and elements that fix at least one point must have cycle structure in the list given in Table 3, with the possibility of fusion for the elements of cycle type 4^2.

When one considers the cycle type of fixed-point-free elements of G, one must exclude all elements that have non-identity powers that fix elements and that are not of a type listed in Table 3. One obtains immediately that the only possible new cycle types are 5^2 and $2^1 8^1$.

There must be an element $g \in G$ of type 5^2, since 5 divides $|G|$. What is more, since $C_{\mathrm{Alt}(10)}(g) = \langle g \rangle$, we conclude that $C_G(g) = \langle g \rangle$ and so the conjugacy class containing g has size 144. Let $P = \langle g \rangle$, a Sylow 5-subgroup of G_{10}. Since $C_G(P) = P$, and $N_G(P)/C_G(P) \lesssim \mathrm{Aut}(P) \cong C_4$. We conclude that $N_G(P)$ has order dividing 20. Sylow's theorems tell us that $|G : N_G(P)| \equiv 1 \pmod 5$ and we conclude that $|N_G(P)| = 20$. Since all conjugates of P intersect trivially, we conclude that G contains 144 elements of order 5, hence there is precisely one conjugacy class of elements of type 5^2.

We now consider the elements of type 3^3 (these come from the G_9 subgroup, see Table 3). For elements of type 3^3, we observe first that an element of this type fixes a unique point and, since the stabilizer of a point contains a unique Sylow 3-subgroup of G, we conclude that each element of type 3^3 lies in a unique Sylow 3-subgroup of G. Now we use the fact that the stabilizer of a point is maximal in G to conclude that there are precisely 10 Sylow 3-subgroups and, therefore, 80 elements of type 3^3 in G.

Type	1^{10}	2^4	3^3	4_A^2	4_B^2	5^2	$2^1 8_A^{1\,*}$	$2^1 8_B^{1\,*}$
Size	1	45	80	90	180	144	90	90

TABLE 4. Conjugacy class sizes in G_{10}

All remaining elements, of which there are 450, must be of type 4^2 or $2^1 8^1$. Now let P be a Sylow 2-subgroup of G and observe that P has two orbits, of size 2 and 8. Let $\{9, 10\}$ be the smaller orbit; then the stabilizer of 9 is equal to the stabilizer of 10 and is equal to Q_8, a Sylow 2-subgroup of G_9. The points in the other orbit each have stabilizers in P of size 2, and there are four distinct stabilizers; this leaves four elements which must be fixed-point-free, and hence are of type $2^1 8^1$. What is more these elements cannot be central in P, otherwise P would be abelian. We conclude that there are at most two conjugacy classes in G of elements of type $2^1 8^1$, and they have size 90 (since they do not commute with any non-identity elements of odd order).

In fact, it is clear that any Sylow 2-subgroup of G_{10} is characterized by its orbit of size 2. This implies, first, that, since the same is true of elements of type $2^1 8^1$, each element of type $2^1 8^1$ is in a unique Sylow 2-subgroup; it implies, second, that there are precisely 45 Sylow 2-subgroups of G_{10}, and so there are 180 elements of type $2^1 8^1$, split into two conjugacy classes of size 90.

Now there are at most three conjugacy classes in G of elements of type 4^2 containing a total of 270 elements; since these elements are real in G_9, they are real in G_{10} and so the classes have even order. One of these classes must be squares of elements of order 8, and so at least one conjugacy class has size 90; the others have size 90 or 180. There are, therefore, two possibilities for elements of type 4^2: three conjugacy classes of size 90, or two of size 90 and 180. Note that, since there are 180 elements of type $2^1 8^1$, there are only 90 elements that are squares of these. Thus if there are three conjugacy classes of size 90, then two of these must have a centralizer isomorphic to $C_4 \times C_2$. It is easy to check, though, that this is not possible, given that all involutions are of type 2^4.

The conjugacy classes of G_{10} are summarized in Table 4. Note that we have asterisked the two conjugacy classes that are not real – it is clear that they are the only conjugacy classes that have a chance of being non-real; to see that they are not real, simply observe that there are no elements of type 4^2 in Alt(8) that send an 8-cycle to its inverse.[1]

2.3. **The conjugacy classes of G_{11}.** The group $G = G_{11}$ has order 7920, and elements that fix at least one point must have cycle structure given in Table 4, with the possibility of fusion for the elements of cycle type 4^2 or $2^1 8^1$.

Taking into account the same sort of considerations as before, we obtain that any fixed-point-free elements of G must have cycle type 11^1 or $2^1 3^1 6^1$.

[1]It is well-known that the group G_{10} is, in fact, M_{10}, the unique non-split extension of Alt(6). One can compare our enumeration of the conjugacy classes of G_{10} with the enumeration of conjugacy classes of M_{10} that appears in the ATLAS [CCN$^+$85]; note that our class 4_B^2 is labeled $4C$ in the ATLAS; similarly the two classes of elements of order 8 are labeled $8C$ and D^{**} in the ATLAS.

Type	1^{11}	2^4	3^3	4^2	5^2	$2^1 3^1 6^1$	$2^1 8_A^{1\,*}$	$2^1 8_B^{1\,*}$	$11_A^{1\,*}$	$11_B^{1\,*}$
Size	1	165	440	990	1584	1320	990	990	720	720

TABLE 5. Conjugacy class sizes in G_{11}

An element, g, of type 11^1 has centralizer in Alt(11) equal to $\langle g \rangle$, hence the same is true in G_{11}. We conclude that a conjugacy class of this type has size 720. Recall, next, that in Alt(11) there are exactly two conjugacy classes of elements of cycle type 11^1, so we deduce there are at least two in G_{11}. Moreover, a Sylow 11-subgroup of Alt(11) has a normalizer of size 55, hence $N_G(\langle g \rangle)$ has size 11 or 55. If the former, then one immediately concludes that there are 10 conjugacy classes of this type; this means that there are a total of 720 elements in G that are not of type 11^1. Since a point-stabilizer of G_{11} has size 720 and does not contain any elements of type 11^1, we conclude that a point-stabilizer of G_{11} is normal, a contradiction. Thus $|N_G(\langle g \rangle)| = 55$ and we conclude, furthermore, that there are exactly two conjugacy classes in G of type 11^1.

We know that there is a unique conjugacy class of elements of type 5^2 in G_{10}, hence the same is true in G_{11}. What is more these elements are self-centralizing, hence this class has size 1584.

Now consider an element, g, of type 3^3. It is clear that $C_G(g)$ contains an involution if and only if there is an element of type $2^1 3^1 6^1$. If this is **not** the case, then the conjugacy class of type 3^3 has size $7920/9 = 880$, and it is the only non-trivial conjugacy class that does not have size divisible by 3. But now $7920 - 880 - 1 = 7039$ is not divisible by 3, and we have a contradiction. Hence we conclude that $|C_G(g)|$ is even, and contains an element of type $2^1 3^1 6^1$. Since the Sylow 2-subgroup of $C_{\mathrm{Alt}(11)}(g)$ is of size 2, we conclude that $|C_G(g)| = 18$ and the conjugacy class of type 3^3 has size 440.

Let h be an element of type 2^4. Then g is central in a Sylow 2-subgroup of G, and is also centralized by an element of type 3^3; it is easy to check that it is not centralized by a Sylow 3-subgroup of G, hence the conjugacy class of involutions has size $7920/48 = 165$.

Elements of type $2^1 8^1$ are self-centralizing, so these conjugacy classes have size 990; any fusion of the two conjugacy classes would have to take place in a Sylow 2-subgroup and, since this Sylow 2-subgroup is the same as for G_{10}, we know that there are two such conjugacy classes.

The remaining elements are of type 4^2 and of type $2^1 3^1 6^1$ and they make up the remaining 2310 elements. The size of the respective conjugacy classes is $7920/8$ and $7920/6$ and since these two numbers sum to 2310 we know that there is a unique conjugacy class of each type.

The conjugacy classes of G_{11} are now written down in Table 5. Note that we have asterisked the four conjugacy classes that are not real. It is clear that all the other classes are real, and it clear that the classes of type 11^1 are non-real, since they are non-real in Alt(11); similarly, the classes of type $2^1 8^1$ are non-real since they are non-real in G_{10} and any putative "reversing element" in G_{11} would have to lie in G_{10}.

2.4. **The conjugacy classes of G_{12}.** The group $G = G_{12}$ has order 95040, and elements that fix at least one point must have cycle structure given in Table 5, with the possibility of fusion for the elements of order 8 and 11.

Taking into account the same considerations as before, we obtain that any fixed-point-free elements of G must have cycle type $2^1 10^1$, $3^1 9^1$, $4^1 8^1$, 6^2, $2^2 4^2$, 3^4, 2^6. In order to exclude elements of type $3^1 9^1$, we need a lemma:

Lemma 2.1. *Let P be a Sylow 3-subgroup of G. Then $P \cong He(3)$.*

Here $He(3)$ is the Heisenberg group over the field of order 3, a group of order 27. Note that all non-trivial elements of $He(3)$ have order 3, hence there are no elements of order 9 in G.

Proof. Let G act sharply 5-transitively on Ω, a set of 12 points. Observe that $|P| = 27$. There are five groups of order 27, three abelian and two non-abelian.

Let us suppose first that P is abelian. Suppose that Λ is an orbit of P in its action on $\{1, \ldots, 12\}$, and let $\lambda \in \Lambda$. Then, since P is abelian, any elements that fixes λ must fix every element in Λ. Since no element of G fixes more than 4 points, this means that $|\Lambda| = 1$ or 3. The group P must not fix more than 4 points, thus there are at least two orbits of P of size 3, call these Λ_1 and Λ_2. Now the orbit-stabilizer theorem asserts that P has a subgroup P_1, of order 9, that fixes every element of Λ_1; similarly P has a subgroup P_2, of order 9, that fixes every element of Λ_2. But now $P_1 \cap P_2$ is non-trivial (by order considerations) and an element in the intersection fixes at least the 6 points of $\Lambda_1 \cup \Lambda_2$. This is a contradiction.

Suppose, then, that P is the non-abelian group of order 27 that is not $He(3)$. This is the extraspecial group of exponent 9; it has center, Z, of order 3; it has a normal elementary-abelian subgroup, P_0, of order 9; and all the elements in $P \setminus P_0$ are of order 9. Since all of the elements of order 9 are fixed-point-free, and since a stabilizer of a point in G has order divisible by 9, we conclude that P_0 stabilizes a point, indeed it must fix 3 points. But, since the elements of order 3 in P have cycle type 3^3 this implies that all elements of P of order 3 fix the same 3 points. This is impossible. □

The same argument as before gives two conjugacy classes of self-centralizing elements of type 11^1. Similarly there is a single conjugacy class of self-centralizing elements of type $2^1 3^1 6^1$. In addition there is an easy counting argument that says that the number of elements that fix exactly four points is $\binom{12}{4} \times 7 = 3465$. There are only two conjugacy classes that do this – of type 2^4 and 4^2; what is more, using the fact that the stabilizer of 4 points is Q_8, we see that there are six times as many elements of type 4^2 as there are of type 2^4. This leaves us with 495 of type 2^4 and 2970 of type 4^2.

Let g be an element of type 5^2. Then $|C_G(g)|$ is even if and only if there is an element of type $2^1 10^1$. Suppose that this is not the case – then $C_G(g)$ is of order 5 and all other non-trivial conjugacy classes have order divisible by 5. But $95040 - 1 - 95040/5$ is not divisible by 5, a contradiction. We conclude that $|C_G(g)| = 10$, and there exist elements of type $2^1 10^1$. What is more $N_G(\langle g \rangle)$ must be a group of size 40 with $C_G(g)$ a normal cyclic subgroup of order 10. If h is a generator of $C_G(g)$, then $N_G(\langle g \rangle)$ acts by conjugation on $C_G(g) = \langle h \rangle$ and is transitive on the elements of order 10 in $\langle h \rangle$; we conclude that there is a unique conjugacy class of elements of type $2^1 10^1$, and this conjugacy class has size 9504.

Let us examine the Sylow 3-subgroup of G in more detail. The orbits of P must be of size 3 and 9, and a count of elements in the stabilizers immediately yields that P contains 14 elements of type 3^3 and 12 that are fixed-point-free, i.e. of type 3^4. Since $14 \not\equiv 0 \pmod 3$, we conclude that the non-trivial central elements of P are of type 3^3, and the centralizer of an element of type 3^3 is divisible by 27. In addition, for g of type 3^3, $|C_{\mathrm{Alt}(12)}(g)|$ is not divisible by 4, and so we conclude that $|C_{G_{12}}(g)| = 54$.

A straightforward counting argument tells us that there are 35310 fixed-point-free elements in G; we currently have 47190 elements unaccounted for, of which the only type that is not fixed-point-free are those of type $2^1 8^1$. We conclude that there are 11880 of these; on the other hand, consulting Table 4, there are 90 elements in conjugacy class $2^1 8^1_A$ for the stabilizer of any two letters. Thus each conjugacy class of this type has size $\binom{12}{2} \times 90 = 11880$, and we conclude that there is a unique conjugacy class of this type.

We are left with the fixed-point-free elements of G – there are 35310 of these, and they are of types 2^6, 3^4, $2^2 4^2$, 6^2, $4^1 8^1$ and $2^1 10^1$ (although we do not yet know if all of these occur). We already know that there are 9504 elements of type $2^1 10^1$ and this leave us 25806 elements for the rest.

Our earlier calculations imply that the centralizer of an element of type 3^4 is divisible by 9 but not 27; we conclude that the centralizer is of size 9, 18 or 36. Suppose that g is of type 3^4, and that $C_G(g)$ is not of size 9 – then there is an element of type 2^6 that centralizes g and we conclude that G contains an element h of type 6^2. Now $C_G(h)$ is of size at most 12. But now, since $\frac{1}{9} = \frac{1}{12} + \frac{1}{36}$, we conclude that in any case there are at least $|G|/9 = 10560$ elements of type 3^4 or 6^2.

The remaining fixed-point-free elements of G all have order a power of 2. Let us, therefore, examine P, a Sylow 2-subgroup of G. Since P is of size 64, we know that P does not fix any points; thus the orbit structure of P is either $8{-}4$ or $8{-}2{-}2$ (since a Sylow 2-subgroup of G_9 has orbit structure $8{-}1$).

Suppose that the orbit structure is $8{-}2{-}2$. If α is a point in an orbit of size 2, then P_α is a normal index 2 subgroup of P, and so P_α fixes all points in the orbit. If β is in the other orbit, then P_β does likewise, and so $P_\alpha \cap P_\beta$ has size at least 16 and fixes 4 points, a contradiction.

Thus the orbit structure is $8{-}4$. Let Λ be the orbit of size 4, let Γ be the orbit of size 8, and let K be the kernel of the action on Λ; then $K \cong Q_8$, the Sylow 2-subgroup of G_9. Since $N = N_{\mathrm{Sym}(8)}(K)$ has size $192 = 64 \times 3$, we conclude that P is isomorphic to a Sylow 2-subgroup of N. What is more the natural map $P \to \mathrm{Sym}(4)$, given by considering the action of P on Λ, yields that $P/K \cong D_4$, the dihedral group of order 8.

Next observe that $C = C_{\mathrm{Sym}(8)}(K)$ is isomorphic to Q_8, and $Z(C) = Z(K)$; indeed all non-central elements of C and K act on Γ as elements of type $4{-}4$ that all square to the same element. Then the non-central elements of C must be of type $4{-}4{-}2{-}2$ in the action on Ω; in particular C induces the normal Klein 4-subgroup of $\mathrm{Sym}(4)$ in the action on Λ. But now one can easily check that the four cosets of K in $N \setminus C$ all contain an 8-cycle; since two of these cosets must induce a 4-cycle on Λ, we conclude that G contains elements of type $8{-}4$. These elements are self-centralizing in $\mathrm{Alt}(12)$, and so likewise in G; what is more there are not enough elements left unaccounted for to allow for more than one such conjugacy class. Thus there is a unique conjugacy class of type $8{-}4$ and it has size $|G|/8 = 11880$.

Type	1^{12}	2^4	2^6	3^3	3^4	4^2	2^24^2	5^2	$2^13^16^1$	6^2	2^18^1	4^18^1	2^110^1	$11_A^{1\,*}$	$11_B^{1\,*}$
Size	1	495	396	1760	2640	2970	2970	9504	15840	7920	11880	11880	9504	8640	8640

TABLE 6. Conjugacy class sizes in G_{12}

There are, in addition, elements of type 2^24^2, and these have a centralizer of size at most 32; again, a count of remaining elements leads us to conclude that there is a unique conjugacy class of elements of this type and it has size $|G|/32 = 2970$.

At this stage, then, we have $\frac{83}{720}|G|$ elements unaccounted for; these are of type 3^4, 6^2 and/or 2^6, and we know that at least $\frac{80}{720}|G|$ of them are of type 3^4 or 6^2. Thus there are at most $|G|/240$ elements of type 2^6. If there are no elements of type 2^6 centralized by an element of order 3, then we conclude that there is a unique conjugacy class of elements of type 2^6, and it must have size $|G|/320$. But this also means that there are no elements of type 6^2, and that the elements of type 3^4 all have centralizers of size 9. This does not yield the correct number of elements.

We conclude that there are elements of type 2^6 centralized by elements of order 3. By counting remaining elements, we conclude that there is a unique conjugacy class of elements of type 6^2, and it has size $|G|/12$; similarly, there is a unique conjugacy class of elements of type 3^4, and it has size $|G|/36$.

There are, therefore, $|G|/240$ elements of type 2^6; let g be one such. Notice, first, that a Sylow 3-subgroup of G does not have a subgroup of order 9 for which all elements are of type 3^4; we conclude that $|C_G(g)|$ is not divisible by 9. Next, notice by our arguments above, that we can take $g \notin Z(KC)$; it is now an easy matter to check that $|C_P(g)| \leq 16$, where P is a Sylow 2-subgroup of G. Thus $|C_G(g)|$ has size at most 240 and we conclude that there is exactly one conjugacy class of elements of type 2^6.

We summarise what we have worked out in Table 6. Note that we have asterisked the two conjugacy classes that are not real; it is clear that these are the only possible conjugacy classes that have a chance of being non-real, and it is equally clear that they **are** non-real, since they are non-real in Alt(12).

3. Character tables

In this section we work out the character tables of G_9, G_{10}, G_{11} and G_{12}. To do this we will need nothing more than the basics of ordinary character theory, along with enough information about the ordinary characters of the symmetric group to calculate irreducible characters for G_{12}.

As a reminder we note down five results that will be particularly useful in what follows.

Theorem 3.1. [JL01, p. 342] *Let G act 2-transitively on a finite set Ω, let $g \in G$ and define* $\mathrm{fix}(g)$ *to be the set of points in Ω fixed by g. Then the function $\chi(g) = |\mathrm{fix}(g)| - 1$ is an irreducible character.*

Theorem 3.2. [JL01, p. 196] *Let G be a finite group and let $g \in G$. Let χ be a character of G, then $\chi^2(g)$ decomposes as the direct sum of two characters, a symmetric part $\frac{1}{2}(\chi^2(g) + \chi(g^2))$ and an antisymmetric part $\frac{1}{2}(\chi^2(g) - \chi(g^2))$.*

Theorem 3.3. [JL01, p. 236] *Let G be a finite group, let $H \leq G$ and suppose that χ is a character of H. For $x \in G$, we set ℓ to be the number of H-conjugacy classes making up $H \cap x^G$ and, provided $\ell > 0$, we define*

$$(\chi \uparrow G)(x) = |C_G(x)| \left(\frac{\chi(x_1)}{|C_H(x_1)|} + \cdots + \frac{\chi(x_\ell)}{|C_H(x_\ell)|} \right), \tag{3.1}$$

where $x_1, \ldots, x_\ell \in H$ are representative elements of the ℓ classes of H. If $\ell = 0$, then we define $(\chi \uparrow G)(x) = 0$. Then $\chi \uparrow G$ is a character of G.

Theorem 3.4. [JL01, p. 264] *The number of real characters of G is the same as the number of real conjugacy classes of G.*

Next we recall the definition of the inner product on characters: if χ and ϕ are characters of G, then we define

$$\langle \chi, \phi \rangle = \sum_{i=1}^{k} \frac{\chi(g_i)\overline{\phi}(g_i)}{|C_G(g_i)|}.$$

The next theorem is a reminder of Schur's Orthogonality Relations; in particular it tells us how to use this inner product to recognise irreducible characters.

Theorem 3.5. [JL01, p. 161] *Let $\chi_1, \ldots, \chi_k$ be the irreducible characters of G and let $g_1, \ldots, g_k$ be representative elements of the conjugacy classes of G, then for every $r, s \in \{1, \ldots, k\}$ we have:*

$$\langle \chi_r, \chi_s \rangle = \delta_{rs}; \tag{3.2}$$

$$\sum_{i=1}^{k} \chi_i(g_r)\overline{\chi_i}(g_s) = \delta_{rs}|C_G(g_r)|. \tag{3.3}$$

Note that (3.2) implies that if χ is a character of G such that $\langle \chi, \chi \rangle = 1$, then χ is irreducible.

3.1. **The character table of G_9.** We know that G_9 has 6 conjugacy classes and hence 6 irreducible characters. We also know that G_9 has a normal subgroup $N \cong C_3 \times C_3$, with $G/N \cong Q_8$. We can lift 3 linear characters and a 2 dimensional character from Q_8. The final character χ_5 is given by the 2-transitive action of G_9 on 9 points (Theorem 3.1). The character table is given in Table 7.

G_9	1^9	2^4	3^3	4_A^2	4_B^2	4_C^2
χ_0	1	1	1	1	1	1
χ_1	1	1	1	-1	1	-1
χ_2	1	1	1	1	-1	-1
χ_3	1	1	1	-1	-1	1
χ_4	2	-2	2	0	0	0
χ_5	8	0	-1	0	0	0

TABLE 7. The character table of G_9

3.2. **The character table of G_{10}.** Note that G_{10} has 8 conjugacy classes, hence 8 irreducible characters. Let χ_0 denote the trivial character of G_{10} and let χ_2 denote the irreducible character obtained by applying Theorem 3.1 to the 3-transitive action of G_{10}.

G_{10}	1^{10}	2^4	3^3	4_A^2	4_B^2	5^2	$2^1 8_A^1$	$2^1 8_B^1$
χ_0	1	1	1	1	1	1	1	1
χ_2	9	1	0	1	1	-1	-1	-1

We will now try inducing characters from the point stabilizer G_9. Let χ be a character of G_9, then using the centralizer orders and Theorem 3.3 we have:

$$(\chi \uparrow G_{10})(g) = \begin{cases} 10\chi(g) & \text{if } g \in 1^{10}; \\ 2\chi(g) & \text{if } g \in 2^4,\ 4_A^2; \\ \chi(g_B) + \chi(g_C) & \text{if } g \in 4_B^2; \\ \chi(g) & \text{if } g \in 3^3; \\ 0 & \text{otherwise.} \end{cases}$$

(Here g_B and g_C are elements from the G_9-classes 4_B^2 and 4_C^2 respectively.) Let χ_A, χ_B and χ_C denote the lifts of the characters χ_1, χ_2 and χ_4 for G_9 respectively (see Table 7).

χ_A	10	2	1	-2	0	0	0	0
χ_B	10	2	1	2	-2	0	0	0
χ_C	20	-4	2	0	0	0	0	0

Taking inner products of the characters we find that χ_A is irreducible, $\langle \chi_B, \chi_B \rangle = 2$ and $\langle \chi_C, \chi_C \rangle = 2$. To keep with our naming convention, relabel χ_A to χ_4. Our next strategy will be to construct the antisymmetric parts of both χ_2^2 and χ_4^2, we will skip over the symmetric decomposition of these characters because they do not yield information that is useful to our endeavours. Let χ_D and χ_E denote the antisymmetric components of χ_2^2 and χ_4^2 respectively.

χ_D	36	-4	0	0	0	1	0	0
χ_E	45	-3	0	1	-1	0	0	0

Taking inner products of χ_D with both itself and χ_C we find that $\langle \chi_D, \chi_D \rangle = 3$ and $\langle \chi_C, \chi_D \rangle = 2$. Let $\chi_7 = \chi_D - \chi_C$; then $\langle \chi_7, \chi_7 \rangle = 1$ and, since $\chi_7(1) > 0$, we conclude that χ_7 is an irreducible character. Repeating the process with χ_E we find that $\langle \chi_E, \chi_E \rangle = 4$, $\langle \chi_7, \chi_E \rangle = 1$, $\langle \chi_B, \chi_E \rangle = 1$ and $\langle \chi_C, \chi_E \rangle = 2$. Let $\chi_3 = \chi_E - \chi_7 - \chi_C = \chi_E - \chi_D$, then $\langle \chi_3, \chi_3 \rangle = 1$ and, since $\chi_3(1) > 0$, we conclude that χ_3 is an irreducible character

χ_3	9	1	0	1	-1	-1	1	1
χ_7	16	0	-2	0	0	1	0	0

Finally, observe that $\langle \chi_B, \chi_3 \rangle = 1$. Then $\chi_1 = \chi_B - \chi_3$ is irreducible. Using Theorem 3.4 and the fact that G_{10} has two non-real classes, we conclude that the remaining two characters occur as a complex conjugate pair. These can, then, be calculated using the Schur orthogonality relations.

G_{10}	1^{10}	2^4	3^3	4_A^2	4_B^2	5^2	$2^1 8_A^1$	$2^1 8_B^1$
χ_0	1	1	1	1	1	1	1	1
χ_1	1	1	1	1	-1	1	-1	-1
χ_2	9	1	0	1	1	-1	-1	-1
χ_3	9	1	0	1	-1	-1	1	1
χ_4	10	2	1	-2	0	0	0	0
χ_5	10	-2	1	0	0	0	ω	$\overline{\omega}$
χ_6	10	-2	1	0	0	0	$\overline{\omega}$	ω
χ_7	16	0	-2	0	0	1	0	0

TABLE 8. The character table of G_{10}, where $\omega = \sqrt{-2}$.

3.2.1. *The structure of G_{10}.* Using the character table, the following result can be easily derived about the structure of G_{10}.

Theorem 3.6. *The group G_{10} is a non-split extension of C_2 by a normal subgroup $K = \ker(\chi_1)$.*

Proof. As the kernels of characters are normal subgroups we see that $K \lhd G_{10}$ and moreover this is the only non-trivial proper normal subgroup of G_{10}. Observe that $G_{10}/K \cong C_2$. Since there are no involutions in $G_{10} \setminus K$, we conclude that we have a non-split extension. □

3.3. **The character table of G_{11}.** By assumption G_{11} acts 4-transitively on a set of size 11. By considering the number of fixed points of each conjugacy class we obtain a 10 dimensional irreducible character χ_1 (Theorem 3.1).

G_{11}	1^{11}	2^4	3^3	4^2	5^2	$2^1 3^1 6^1$	$2^1 8_A^1$	$2^1 8_B^1$	11_A^1	11_B^1
χ_0	1	1	1	1	1	1	1	1	1	1
χ_1	10	2	1	2	0	-1	0	0	-1	-1

Now, let χ_S and χ_A be the symmetric and antisymmetric decomposition of χ_1^2.

χ_S	55	7	1	3	0	1	1	1	0	0
χ_A	45	-3	0	1	0	0	-1	-1	1	1

A quick calculation gives $\langle \chi_S, \chi_S \rangle = 3$ and $\langle \chi_A, \chi_A \rangle = 1$. Moreover, $\langle \chi_S, \chi_0 \rangle = 1$ and $\langle \chi_S, \chi_1 \rangle = 1$. Define $\chi_8 = \chi_A$ and $\chi_7 = \chi_S - \chi_0 - \chi_1$ and note that $\langle \chi_7, \chi_7 \rangle = 1$. Hence we have found two new irreducible characters of G_{11}.

χ_7	44	4	-1	0	-1	1	0	0	0	0
χ_8	45	-3	0	1	0	0	-1	-1	1	1

3.3.1. *Induction from G_{10}.* We will now try inducing characters from the 1-point stabilizer subgroup G_{10}. Let χ be a character of G_{10}; then, using the centralizer orders and Theorem 3.3, we have:

$$(\chi \uparrow G_{11})(g) = \begin{cases} 11\chi(g) & \text{if } g \in 1^{10}; \\ 3\chi(g) & \text{if } g \in 2^4; \\ 2\chi(g) & \text{if } g \in 3^3; \\ \chi(g_A) + 2\chi(g_B) & \text{if } g \in 4^2; \\ \chi(g) & \text{if } g \in 5^2,\ 2^1 8^1_A,\ 2^1 8^1_B; \\ 0 & \text{otherwise.} \end{cases}$$

(Here g_A and g_B are elements from the G_{10}-classes 4^2_A and 4^2_B respectively.) Let χ_4 be the induced character of the non-trivial linear character of G_{10} and let χ_9 be the the anti-symmetric decomposition of χ_4^2.

χ_4	11	3	2	-1	1	0	-1	-1	0	0
χ_9	55	-1	1	-1	0	-1	1	1	0	0

We find that $\langle \chi_4, \chi_4 \rangle = 1$ and $\langle \chi_9, \chi_9 \rangle = 1$, hence, they are both irreducible characters of G_{11}.

3.3.2. *Schur orthogonality.* The remaining four characters come in complex conjugate pairs. We can deduce this by using the fact that elements of cycle type $2^1 8^1$ and 11^1 are not real and applying Theorem 3.4. Given that each pair will have the same dimension, we can attempt to calculate the dimension of these 4 remaining characters. Let the dimension of the first pair be d_1 and the dimension of the second pair be d_2.

We have $d_1^2 + d_1^2 + d_2^2 + d_2^2 = 712$, hence $d_1^2 + d_2^2 = 356$. By an exhaustive search we find that 356 can be expressed as the sum of two squares in exactly one way, that is $356 = 10^2 + 16^2$. It immediately follows that $d_1 = 10$ and $d_2 = 16$.

Let $\chi_2(1) = \chi_3(1) = 10$ and $\chi_5(1) = \chi_6(1) = 16$. We will now use the column relations to calculate the character values for the remaining conjugacy classes. We use the fact that $\chi(g) = \overline{\chi(g^{-1})}$, and let the characters take the following values:

χ_2	11	x_1	x_2	x_3	x_4	x_5	x_6	$\overline{x_6}$	x_7	$\overline{x_7}$
χ_3	11	x_1	x_2	x_3	x_4	x_5	$\overline{x_6}$	x_6	$\overline{x_7}$	x_7
χ_5	16	y_1	y_2	y_3	y_4	y_5	y_6	$\overline{y_6}$	y_7	$\overline{y_7}$
χ_6	16	y_1	y_2	y_3	y_4	y_5	$\overline{y_6}$	y_6	$\overline{y_7}$	y_7

Substituting the column containing x_i for $i = 1, \dots 5$ and the first column of the character table into (3.3) we can obtain values for x_i and y_i. We shall demonstrate this with x_1 as follows

$$1 + 20 + 2 \times 10x_1 + 33 + 2 \times 16y_1 + 176 - 135 - 55 = 0$$

which simplifies to

$$5x_1 + 8y_1 = -10. \tag{3.4}$$

Substituting the column containing x_1 into (3.3) twice gives

$$1 + 4 + 2x_1^2 + 9 + 2y_1^2 + 16 + 9 + 1 = 48$$

which simplifies to

$$x_1^2 + y_1^2 = 4. \tag{3.5}$$

Solving (3.4) and (3.5), we obtain two solutions $x_1 = -2,\ y_1 = 0$ and $x_1 = \frac{78}{89},\ y_1 = \frac{-160}{89}$. The second set of these cannot be expressed as a sum of 2nd roots of unity. Hence, $x_1 = -2$ and $y_1 = 0$. Continuing in this manner we obtain:

χ_2	10	-2	1	0	0	1	x_6	$\overline{x_6}$	x_7	$\overline{x_7}$
χ_5	16	0	-2	0	1	0	y_6	$\overline{y_6}$	y_7	$\overline{y_7}$

The remaining values $x_6,\ x_7,\ y_6$, and y_7 can be calculated by repeat applications of the row and column relations. We find that $x_6 = \sqrt{-2}$, $x_7 = -1$, $y_6 = 0$ and $y_7 = \frac{1}{2}(-1 + \sqrt{-11})$. Table 2 shows the complete character table.

3.4. **The character table of G_{12}.** We begin by noting that G_{12} has 15 conjugacy classes and 15 irreducible characters, one of which is the trivial character χ_0.

3.4.1. *The permutation character and tensor products.* By assumption, G_{12} acts 5-transitively on a set of size 12. Hence, by Theorem 3.1, we get the permutation character χ_1.

G_{12}	1^{12}	2^4	2^6	3^3	3^4	4^2	$2^2 4^2$	5^2	$2^1 3^1 6^1$	6^2	$2^1 8^1$	$4^1 8^1$	$2^1 10^1$	11_A^1	11_B^1
χ_0	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
χ_1	11	-1	3	2	-1	3	-1	1	0	-1	-1	1	-1	0	0

Now, let χ_S and χ_A be the symmetric and antisymmetric decomposition of χ_1^2.

χ_S	66	10	6	3	0	6	2	1	1	0	2	0	1	0	0
χ_A	55	-1	-5	1	1	3	-1	0	-1	1	-1	1	0	0	0

A quick calculation gives $\langle \chi_S, \chi_S \rangle = 3$ and $\langle \chi_A, \chi_A \rangle = 1$. Moreover, $\langle \chi_S, \chi_0 \rangle = 1$ and $\langle \chi_S, \chi_1 \rangle = 1$. Define $\chi_8 = \chi_A$ and $\chi_6 = \chi_S - \chi_0 - \chi_1$. We have found two new irreducible characters of G_{12}. Squaring these characters, however, is not a viable plan; the characters obtained have dimensions 2916 and 3025.

χ_6	54	6	6	0	0	2	2	-1	0	0	0	0	1	-1	-1
χ_8	55	-1	-5	1	1	3	-1	0	-1	1	-1	1	0	0	0

3.4.2. *Induction from* G_{11}. We will now try inducing characters from the subgroup G_{11}. Let χ be a character of G_{11}; then using the centralizer orders and Theorem 3.3 we have:

$$(\chi \uparrow G_{12})(g) = \begin{cases} 12\chi(g) & \text{if } g \in 1^{12}; \\ 4\chi(g) & \text{if } g \in 2^4, 4^2; \\ 3\chi(g) & \text{if } g \in 3^3; \\ 2\chi(g) & \text{if } g \in 5^2; \\ \chi(g) & \text{if } g \in 2^1 3^1 6^1, 11^1_A, 11^1_B; \\ \chi(g_A) + \chi(g_B) & \text{if } g \in 2^1 8^1; \\ 0 & \text{otherwise.} \end{cases}$$

(Here g_A and g_B are elements from the G_{11}-classes $2^1 8^1_A$ and $2^1 8^1_B$ respectively.) Inducing the trivial character of G_{11} gives a character equal to $\chi_0 + \chi_1$. Inducing the integer valued 10-dimensional character gives a character equal to $\chi_1 + \chi_6 + \chi_8$. Now, let χ_{12} be the induced character of a complex valued 10-dimensional character of G_{11}. We see that $\langle \chi_{12}, \chi_{12} \rangle = 1$, therefore, χ_{12} is irreducible.

Let χ_V be the character obtained by inducing the 55-dimensional character of G_{11}. Finally, let χ_B be the induced complex valued 16-dimensional character of G_{11} and let $\omega = \frac{1}{2}(-1 + \sqrt{-11})$, then we have:

χ_{12}	120	-8	0	3	0	0	0	0	0	1	0	0	0	-1	-1
χ_V	660	-4	0	3	0	0	-4	0	0	-1	2	0	0	0	0
χ_B	192	0	0	-6	0	0	0	2	0	0	0	0	0	ω	$\overline{\omega}$

Note that $\langle \chi_B, \chi_B \rangle = 2$ and $\langle \chi_V, \chi_V \rangle = 6$, but the inner product of χ_B with any known irreducible is 0. We have $\langle \chi_V, \chi_{12} \rangle = 1$, but the inner product is 0 for any other known irreducibles. These do not give us any new irreducible characters, but we will use these characters later.

3.4.3. *Restriction from* Sym(12). Using the Frobenius character formula, it is possible to construct low dimensional characters of Sym(12) evaluated over the conjugacy classes of G_{12}; recall that these characters are labeled with partitions of 12. In this section we consider the restriction of some of these characters to the group G_{12}; for instance we note that $(\chi_{(11,1)} \downarrow G_{12}) = \chi_1$, $(\chi_{(10,2)} \downarrow G_{12}) = \chi_2$ and $(\chi_{(10,1,1)} \downarrow G_{12}) = \chi_3$.

The table below gives six new characters that we have constructed in this way. Note that we abuse notation here: given a partition, λ, of 12, we would normally write χ_λ for the associated character of Sym(12) whereas here we write χ_λ for the restriction of the associated character to G_{12}. Note too that, for ease of notation, we let $\lambda_A = (9, 1, 1, 1)$ and $\lambda_B = (8, 1, 1, 1, 1)$.[2]

[2]The history of this sort of restriction is worth a note: a classical result of Frobenius asserts that if t is a natural number with $t < n/2$, then a subgroup $G < \mathrm{Sym}(n)$ is $2t$-transitive if and only if every character of $\mathrm{Sym}(n)$ labeled by a partition $(\lambda_1, \lambda_2, \ldots, \lambda_a)$ with $\lambda_2 + \cdots + \lambda_a \leq t$ remains irreducible when restricted to G. This result appears in [Fro04], the same paper in which Frobenius calculates the character tables of M_{12} and M_{24}; indeed Frobenius makes use

G_{12}	1^{12}	2^4	2^6	3^3	3^4	4^2	2^24^2	5^2	$2^13^16^1$	6^2	2^18^1	4^18^1	2^110^1	11^1_A	11^1_B
$\chi_{(9,3)}$	154	10	−6	1	4	−2	−2	−1	1	0	0	0	−1	0	0
χ_{λ_A}	165	−11	5	3	3	1	1	0	1	−1	−1	−1	0	0	0
$\chi_{(8,4)}$	275	11	15	5	−4	−1	3	0	−1	0	−1	1	0	0	0
$\chi_{(7,5)}$	297	9	−15	0	0	5	−3	2	0	0	−1	−1	0	0	0
$\chi_{(3,2,1)}$	320	0	0	−4	−4	0	0	0	0	0	0	0	0	1	1
χ_{λ_B}	330	−6	10	6	−3	−2	−2	0	1	0	0	0	0	0	0

We first check the inner product of each character with itself and then with each of the known irreducibles. We find that $\langle \chi_{\lambda_A}, \chi_{\lambda_A} \rangle = 2$ and $\langle \chi_{\lambda_A}, \chi_{12} \rangle = 1$, define $\chi_5 = \chi_{\lambda_A} - \chi_{12}$ and note that $\langle \chi_5, \chi_5 \rangle = 1$. Hence, we have found a new irreducible character of G_{12}.

χ_5	45	−3	5	0	3	1	1	0	0	−1	−1	−1	0	1	1

We find that $\langle \chi_{\lambda_B}, \chi_{\lambda_B} \rangle = 3$ and that $\langle \chi_{\lambda_B}, \chi_{12} \rangle = 1$, hence we define $\chi_X = \chi_{\lambda_B} - \chi_{12}$. Similarly we find that $\langle \chi_{(8,4)}, \chi_{(8,4)} \rangle = 4$ and that $\langle \chi_{(8,4)}, \chi_6 \rangle = 1$, hence we define $\chi_Y = \chi_{(8,4)} - \chi_6$. Checking the inner products of χ_X and χ_Y with themselves and each other we obtain $\langle \chi_X, \chi_X \rangle = 2$, $\langle \chi_Y, \chi_Y \rangle = 3$ and $\langle \chi_X, \chi_Y \rangle = 2$. Define $\chi_2 = \chi_Y - \chi_X$, we find that $\langle \chi_2, \chi_2 \rangle = 1$; we have found a new irreducible character of G_{12}.

χ_Y	221	5	9	5	−4	−3	1	1	−1	0	−1	1	−1	1	1
χ_X	210	2	10	3	−3	−2	−2	0	1	−1	0	0	0	1	1
χ_2	11	3	−1	2	−1	−1	3	1	0	−1	−1	1	−1	0	0

Let χ_S and χ_A be the symmetric and antisymmetric decomposition of χ_2^2. We find that $\chi_S = \chi_0 + \chi_2 + \chi_6$ and $\chi_9 = \chi_A$ is a new irreducible.

χ_9	55	−1	−5	1	1	3	−1	0	−1	1	1	−1	0	0	0

of this result in his calculation of these tables. Note that Theorem 3.1 is a special case of Frobenius' result. A modern version of Frobenius result, making use of the Classification of Finite Simple Groups, was given in a beautiful paper of Saxl [Sax87] – his theorem considers G, a subgroup of $\mathrm{Sym}(n)$ or of $\mathrm{Alt}(n)$ and χ, an ordinary character of $\mathrm{Sym}(n)$ or of $\mathrm{Alt}(n)$, and he describes all pairs (χ, G) where the restriction of χ to G is irreducible. In our proof we do not make use of Frobenius' (or Saxl's) result – it is enough for us to be able to calculate the restriction of various characters of Sym(12) directly and naively.

We will now check the inner product of every restricted character and χ_V with our known irreducible characters of G_{12}.

Character	χ_0	χ_1	χ_2	χ_5	χ_6	χ_8	χ_9	χ_{12}
$\chi_{(9,3)}$	0	0	0	0	0	0	0	0
χ_{λ_A}	0	0	0	1	0	0	0	1
$\chi_{(8,4)}$	0	0	1	0	1	0	0	0
$\chi_{(7,5)}$	0	1	0	0	1	0	0	0
$\chi_{(3,2,1)}$	0	0	0	0	0	0	0	0
χ_{λ_B}	0	0	0	0	0	0	0	1
χ_V	0	0	0	1	0	0	0	1

Define the following characters:

$$\chi_C = \chi_{(9,3)}$$
$$\chi_D = \chi_{(7,5)} - \chi_1 - \chi_6$$
$$\chi_E = \chi_{(3,2,1)}$$
$$\chi_F = \chi_{\lambda_B} - \chi_{12}$$
$$\chi_W = \chi_V - \chi_5 - \chi_{12}$$

The values of these are as follows.

G_{12}	1^{12}	2^4	2^6	3^3	3^4	4^2	$2^2 4^2$	5^2	$2^1 3^1 6^1$	6^2	$2^1 8^1$	$4^1 8^1$	$2^1 10^1$	11_A^1	11_B^1
χ_C	154	10	−6	1	4	−2	−2	−1	1	0	0	0	−1	0	0
χ_D	231	7	−9	−1	0	−1	−1	1	1	0	−1	−1	1	0	0
χ_E	320	0	0	−4	−4	0	0	0	0	0	0	0	0	1	1
χ_F	210	2	10	3	−3	−2	−2	0	1	−1	0	0	0	1	1
χ_W	485	5	5	−1	−1	−3	−3	0	−1	−1	1	1	0	1	1

Taking the inner products of each of these new characters with each other gives the following:

	χ_C	χ_D	χ_E	χ_F	χ_W
χ_C	2	1	0	0	1
χ_D	1	2	1	0	1
χ_E	0	1	2	1	2
χ_F	0	0	1	2	2
χ_W	1	1	2	2	4

Note, first, that this table of values implies that each of these characters is the sum of distinct irreducibles. Writing these irreducibles as α, β, γ, and so on, we see immediately that we can write

$$\chi_C = \alpha + \beta;$$
$$\chi_D = \beta + \gamma;$$
$$\chi_E = \gamma + \delta;$$
$$\chi_F = \gamma + \epsilon;$$
$$\chi_W = \alpha + \gamma + \delta + \epsilon.$$

Now one obtains that $\alpha = \frac{1}{2}(\chi_W - \chi_F + \chi_C - \chi_D)$. Once we have α it is an easy matter to obtain the other four irreducibles using the equalities just given. We therefore have five new irreducibles which we label as follows:

$$\alpha = \chi_{11},\ \beta = \chi_7,\ \gamma = \chi_{14},\ \delta = \chi_{13},\ \epsilon = \chi_{10}.$$

Finally, we return to the character χ_B from earlier. We find that $\langle \chi_B, \chi_{14} \rangle = 1$, and so we define $\chi_3 = \chi_B - \chi_{14}$. Letting χ_4 be the complex conjugate of χ_3, we obtain our final two irreducibles. The full character table is given in Table 1.

4. Final remarks

One can read off many properties of the groups G_{11} and G_{12} by looking at the character tables that we have constructed. Note, for instance, that all of the irreducibles of the two groups have trivial kernel; one concludes immediately that G_{11} and G_{12} are simple.

We saw above, in §2.1, that $G_9 \cong (C_3 \times C_3) \rtimes Q_8$. We should note that, although we have not deduced the isomorphism types of G_{10}, G_{11} and G_{12}, in each case it is well-known that they are unique up to group isomorphism. Indeed $G_{10} \cong M_{10}$, the unique non-split degree 2 extension of Alt(6), while $G_{11} \cong M_{11}$ and $G_{12} \cong M_{12}$, the two smallest sporadic simple groups of Mathieu.

One wonders about proving the analogous theorem for the simple Mathieu group M_{24}, the theorem would be similar to: "If G is a 5-transitive subgroup of Alt(24) of order 244823040, then the character table of G is as follows..." It would be desirable to prove such a theorem using methods similar to the arguments laid out here, although one might expect that the details would be rather onerous.

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