

Trace anomaly and infrared cutoffs

Tim R. Morris^{1a} and Roberto Percacci^{2,b,c}

^a*STAG Research Centre & Department of Physics and Astronomy,*

University of Southampton, Highfield, Southampton, SO17 1BJ, U.K.

^b*International School for Advanced Studies, via Bonomea 265, I-34136 Trieste, Italy*

^c*INFN, Sezione di Trieste, Italy*

Abstract. The Effective Average Action is a form of effective action which depends on a cutoff scale suppressing the contribution of low momentum modes in the functional integral. It reduces to the ordinary effective action when the cutoff scale goes to zero. We derive the modifications of the scale Ward identity due to this cutoff and show how the resulting identity then intimately relates the trace anomaly to the Wilsonian realisation of the renormalization group.

1 Introduction

A theory that is scale invariant at the classical level, is in general no longer scale invariant at the quantum level. The breaking of scale invariance is known as the trace anomaly. It has two causes. On the one hand a non-flat external metric leads to contributions proportional to integrals of curvatures of the metric [1, 2]. On the other hand, there is a part proportional to the beta functions of the theory [3–6]. The first type of contribution occurs even for a free field theory, while the second type of contribution appears even in flat space. In this paper we will be concerned almost exclusively with this last situation.

A scale transformation is a change of all lengths by a constant factor. One can interpret this either as a rescaling of the coordinates, or as a rescaling of the metric (see Appendix A). Even though we will not deal with gravity in this paper, we choose the latter interpretation. For simplicity we will deal mostly with a single scalar field ϕ . The infinitesimal transformation of the fields is then

$$\begin{aligned}\delta_\epsilon g_{\mu\nu} &= 2\epsilon g_{\mu\nu} \\ \delta_\epsilon \phi &= \epsilon d_\phi \phi\end{aligned}\tag{1.1}$$

where $d_\phi = -\frac{d-2}{2}$ is the canonical length dimension of ϕ in d spacetime dimensions. We work

¹e-mail address: T.R.Morris@soton.ac.uk

²e-mail address: percacci@sissa.it

in Euclidean signature where the energy-momentum tensor is defined by

$$T^{\mu\nu} = -\frac{2}{\sqrt{g}} \frac{\delta S}{\delta g_{\mu\nu}}. \quad (1.2)$$

In concordance with the quantum action principle [7], under a scale transformation the operator $-\delta_\epsilon S$ is inserted into correlation functions, where S is the bare action. If couplings in the theory are dimensionful, already at the classical level there will be a breaking of scale invariance through this contribution. We are interested in the case that the theory is scale invariant at the classical level, so we will have dimensionless couplings only. For the theory of a single scalar field in $d = 4$ spacetime dimensions, that means the interaction potential will just be

$$V(\phi) = \frac{\lambda}{4!} \phi^4, \quad (1.3)$$

where λ is the dimensionless coupling. When everything is written in renormalized terms, the result is then the insertion of the renormalised operator

$$\epsilon \int_x T^\mu{}_\mu = \epsilon \beta \int_x \frac{1}{4!} \phi^4, \quad (1.4)$$

where $\beta = \mu \partial_\mu \lambda(\mu)$ is the Renormalization Group (RG) beta function and we denote $\int_x = \int d^d x \sqrt{g}$ the integration over spacetime.³

Evidently this formula follows only if the breaking of scale invariance is solely due to the running at the quantum level of the dimensionless coupling λ , rather than any other mass scale introduced into the theory. On the other hand this breaking of scale invariance is a physical effect, and cannot be removed by improvements to the bare action. The bare action is scale invariant at the classical level, but at the quantum level scale invariance is broken by the regularisation. For example in dimensional regularisation, scale invariance is broken by the fact that the bare coupling λ_0 is dimensionful outside four dimensions, where it is rewritten as μ^{d-4} times a series in the renormalized coupling λ and $1/(d-4)$. With a physical (dimensionful) regulator, the breaking is more immediate in the generation of mass terms of order the regulator scale, for example. However the breaking in the bare action is only by these λ -dependent contributions whose effect, when added together with the quantum corrections, is such that the almost scale-invariant renormalized theory is recovered, *i.e.* the one in which only (1.4) is inserted.

To see what this means for the quantum theory, we note that inserting $\frac{1}{4!} \int_x \phi^4$ is achieved by differentiating with respect to λ . Recalling the extra sign in $e^{-\Gamma}$, we should therefore expect that for the Legendre Effective Action (EA) we have, in general,

$$\delta_\epsilon \Gamma = -\mathcal{A}(\epsilon) = -\epsilon \beta(\lambda) \partial_\lambda \Gamma, \quad (1.5)$$

³This anomaly is present also in curved space, as demonstrated for a spherical background in [8] and (using the background field method) on an arbitrary background in [9, 10].

where $\mathcal{A}(\epsilon) = \int_x \langle T^\mu{}_\mu \rangle$ stands for the insertion of the operator (1.4) into the functional integral defining the partition function. The sign on the far right of (1.5) can be understood when we recall that scale transformations (1.1) increase length scales for positive ϵ , and thus decrease mass scales, *i.e.* associate $-\epsilon$ with unit positive mass dimension, as we see from (1.1). In other words, we can think of δ_ϵ as generating a flow towards the infrared.

It is an old idea that mass scales in nature may be of quantum mechanical origin, as is indeed true to a large extent in QCD. For a scalar theory this is related to the Coleman-Weinberg potential [11]. This idea has seen a revival in recent years [12–14], see also [15–18] for similar ideas in a gravitational context. In this paper we will explore the implications of classical scale invariance in the context of the Effective Average Action (EAA) Γ_k , which is a version of the EA supplied with an infrared cutoff k , and reducing to the EA when $k \rightarrow 0$ [19, 20]. Our main result is the following: when the classical action is scale invariant, in addition to the RG flow for the EAA, there is a Ward Identity (WI) for scale transformations which takes the form

$$\delta_\epsilon \Gamma_k = -\mathcal{A}(\epsilon) + \epsilon \partial_t \Gamma_k. \quad (1.6)$$

The second term on the right hand side represents the RG flow due to k ($t = \log k$). In the rest of the paper we demonstrate in detail, using momentum cutoffs, how this anomaly arises and how it reduces to (1.5) in the limit $k \rightarrow 0$.

The paper is organized as follows. In section 2 we discuss the trace anomaly in the context of theories with momentum cutoffs: either UV, or IR or both. In section 3 we derive the WI (1.6) and recall how in some circumstances it can be applied to gain information on the EA. Section 4 deals with the form of the anomaly and the EAA in approximate treatment. We consider the one loop approximation and other popular approximations such as the derivative expansion or the vertex expansion. Section 5 is devoted specifically to the derivation of (1.5) from (1.6) in the limit $k \rightarrow 0$. In section 6 we briefly discuss fixed points and in section 7 we make some connections to other ideas in the literature and point to some possible developments.

2 UV cutoff and the trace anomaly

In order to develop some intuition for the workings of the anomaly when using momentum cutoff as regularization, it will be helpful to start from a perturbative treatment based on standard diagrammatic methods. We will consider the effect of both UV and IR cutoffs. If we work to one loop we can write the bare action as

$$S[\phi] = \int_x \left[\frac{1}{2} (\partial_\mu \phi)^2 + V(\phi) \right], \quad (2.1)$$

for some potential $V(\phi)$. Expanding the one loop EA, we can write

$$\Gamma[\phi] = S[\phi] + \sum_{n=1}^{\infty} \mathcal{V}_n[\phi] \quad (2.2)$$

where

$$\mathcal{V}_n[\phi] = -\frac{1}{2} \frac{(-1)^n}{n} \text{Tr} \left(\frac{1}{-\partial^2} V'' \right)^n, \quad (2.3)$$

and we have thrown away the field independent part. This is a sum over the Feynman diagrams as indicated in fig. 1. One finds

$$\mathcal{V}_n[\phi] = -\frac{(-1)^n}{2n} \int_{p_1, \dots, p_n} V''(p_1) \cdots V''(p_n) (2\pi)^d \delta(p_1 + \cdots + p_n) A(p_1, \dots, p_n). \quad (2.4)$$

The unconstrained momentum integral for each diagram takes the form

$$A(p_1, \dots, p_n) = \int_q \frac{1}{q^2 (q + P_1)^2 (q + P_2)^2 \cdots (q + P_{n-1})^2}, \quad (2.5)$$

where $P_j = \sum_{i=1}^j p_i$ ($P_n = 0$ being enforced by the momentum conserving δ -function) are partial sums of the external momenta injected into the diagram by

$$V''(p_i) = \int_x V''(\phi(x)) e^{ip_i \cdot x}, \quad (2.6)$$

and in the integrals over momenta we include the usual factor of $(2\pi)^{-d}$. The integrals (2.5) are infrared finite provided we choose non-exceptional external momenta, *i.e.* provided that $\sum_{i=j}^k p_i \neq 0$ for all $1 \leq j < k \leq n$.⁴ Furthermore for $n > 2$, these Feynman diagrams are ultraviolet finite. For a field $\phi(p)$ with some suitable smooth behaviour in momentum space, we can therefore define these $n > 2$ contributions rigorously. Thus provided that the limiting behaviour of $A(p_1, \dots, p_n)$ as momenta become exceptional, is still integrable when the complete vertex is considered, the $\mathcal{V}_{n>2}[\phi]$ are well defined.

Working in four dimensions, the insertions (2.6) have mass dimension -2. Taking into account all the other parts in (2.4), it is straightforward to verify that the $\mathcal{V}_n[\phi]$ are dimensionless, as they must be to be part of Γ . If further we use the potential (1.3), then no dimensionful coupling is included. Since the $n > 2$ contributions do not need regularisation, it follows that they are scale invariant, *i.e.* vanish under the operation δ_ϵ .

It will be helpful to show this in detail however, since we will then need to break the invariance with a cutoff. Recall that the scale variation is actually being carried by the fields and the

⁴In the case that this is violated we have in the denominator (at least) one $(q + P_j)^4$ term which is (at least) logarithmically IR divergent.

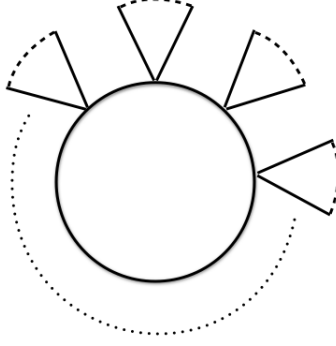


Figure 1: Feynman diagrams contributing to the effective action.

metric, as in eqn. (1.1). Our metric is currently flat: $g_{\mu\nu} = \delta_{\mu\nu}$, but its inverse is present in $\partial^2 = g^{\mu\nu}\partial_\mu\partial_\nu$, which means that its eigenvalues transform as

$$\delta_\epsilon q^2 = -2\epsilon q^2. \quad (2.7)$$

Since we should thus regard momentum as having a lower index, $p \cdot x$ does not contain the metric, and therefore the Fourier transform (2.6) transforms as $\delta_\epsilon V''(p_i) = 2\epsilon V''(p_i)$, thanks to the implicit \sqrt{g} included in the integral over x and the two fields included when we use the ϕ^4 vertex (1.3). Note that the same \sqrt{g} implies that $\delta(p_1 + \dots + p_n)$ transforms with a factor of 4ϵ . Therefore we see that integrals over momentum must transform as

$$\delta_\epsilon \int_q = -4\epsilon \int_q, \quad (2.8)$$

to be consistent with $\delta(q)$. Putting all this together we see that $A(p_1, \dots, p_n)$ transforms with a $(2n-4)\epsilon$ factor, and thus the well-defined vertices $\mathcal{V}_n[\phi]$ transform with a $(-4n+2n+4+2n-4) = 0$ factor (where the contributions from (2.4) are listed in order), *i.e.* are indeed invariant.

This is not true however of the $n \leq 2$ contributions. We do not consider the case of \mathcal{V}_0 , which only yields a field-independent contribution. For the vertex $\mathcal{V}_1[\phi]$, the Feynman integral is quadratically divergent:

$$A = \int_q \frac{1}{q^2}. \quad (2.9)$$

If we use a scale-free regularisation such as dimensional regularisation, then by dimensions the only possible answer is $A = 0$. For a physical regulator such as a UV momentum cutoff $q \leq \Lambda$, the result is a Λ^2 mass term that we have to remove by a local counterterm *i.e.* we have to start with a modified bare action $S[\phi]$ that includes precisely the opposite contribution. This then restores scale invariance for $\mathcal{V}_1[\phi]$, in this case by setting it to zero.

Let us now come to the case $n = 2$. The integral over q :

$$A(p_1, -p_1) = \int_q \frac{1}{q^2(q+p_1)^2}, \quad (2.10)$$

is ultraviolet divergent and thus not well defined.

As above, we will simply cut off the integral at $|q| = \Lambda$ for large Λ . Now the action of δ_ϵ on (2.10) picks up the boundary contribution

$$(+\epsilon\Lambda) \frac{2\Lambda^3}{(4\pi)^2} \frac{1}{\Lambda^4}. \quad (2.11)$$

To see this, note first that formally the ϵ contributions cancel, in the same way that they did rigorously for the $n > 2$ cases. The sole contribution thus comes from the boundary. Write the q integral as an integral over angles and over the radial direction $|q|$. By (2.7) we are instructed to replace $|q|$ with $(1 - \epsilon)|q|$ wherever we see it. But that implies that the ultraviolet boundary to the integral is now at $(1 - \epsilon)|q| = \Lambda$, or what is the same: $|q| = \Lambda(1 + \epsilon)$. The first factor in (2.11) is this extra contribution from the boundary, and the other factors are from the integrand as a function of $|q|$, in particular the second factor is the volume of the 3-sphere at $|q| = \Lambda$ divided by $(2\pi)^4$, and the final factor is the contribution from the integrand at the boundary (where since $|p_1| \ll \Lambda$ we can neglect p_1). Together with the $\lambda^2/4$ from the two insertions (2.6), and the $-1/4$ from (2.4), this gives

$$\delta_\epsilon \Gamma = -\epsilon \int_x \frac{\lambda^2}{128\pi^2} \phi^4, \quad (2.12)$$

which agrees with (1.5), once we recall that to one loop, the β function is:

$$\beta = \frac{3\lambda^2}{16\pi^2}. \quad (2.13)$$

Just as with \mathcal{V}_1 , since (2.10) is UV divergent, we need to modify the bare action. Adding to it the counterterm

$$+ \int_x \frac{\lambda^2}{128\pi^2} \log\left(\frac{\Lambda}{\mu}\right) \phi^4, \quad (2.14)$$

where μ is the usual arbitrary finite reference scale, ensures that overall the result is finite. Note however that under the global scale transformation (1.1), the counterterm is invariant. Thus the renormalized contribution still breaks scale invariance with the same result, namely (2.12).

On the other hand, now that the total contribution to the EA is finite, the breaking can be understood in a different way. The scale Λ has disappeared, but scale invariance is still broken, because of the appearance of the scale μ . To see that (1.4) emerges again, we note that by dimensions the amplitude (2.10) is proportional to $\log(p_1^2/\mu)$. In fact, the finite part of \mathcal{V}_2 is

$$+ \int_x \frac{\lambda^2}{256\pi^2} \phi^2 \log\left(\frac{-\partial^2}{\mu^2}\right) \phi^2, \quad (2.15)$$

up to a local, scale invariant ϕ^4 term which can be absorbed by the renormalization scheme. By (2.7), δ_ϵ clearly gives again minus the β -function times the ϕ^4 operator.

Note that the β -function is arising in a different way from the RG treatment. In the RG treatment, we associate the β -function as arising not directly from the integral (2.10) but from the counterterm (2.14) required to make it finite. Indeed the β -function for the renormalised coupling $\lambda(\mu)$ arises from the requirement that the bare coupling $\lambda(\Lambda)$ is independent of μ , where the bare coupling is the coupling in $S(\phi)$, and from (2.14) is now given by: ⁵

$$\lambda(\Lambda) = \lambda(\mu) + \frac{3\lambda^2(\mu)}{(4\pi)^2} \log\left(\frac{\Lambda}{\mu}\right). \quad (2.16)$$

We see in (2.15) that $\mathcal{V}_2[\phi]$ is non-local. Note that the $\mathcal{V}_{n>2}[\phi]$ are also non-local, as they must be by dimensions. For $\lambda\phi^4$ theory, the $n>2$ terms contain $2n$ fields and thus the vertex is a negative dimension function of the momenta p_i and clearly therefore must be non-local. ⁶ We will see in sec. 4.2 why these observations are important for the trace anomaly.

Finally note that according to (1.5), the $\mathcal{V}_{n>2}[\phi]$ vertices should also contribute to the anomaly, since they are proportional to λ^n . This is true, however since β already contains \hbar (starts at one loop), this is a higher loop effect that is thus neglected in this one loop computation (whereas (2.12) is a one loop effect on top of the classical contribution (1.3)). Indeed these contributions begin to show up once we include the diagrams shown in fig. 2.

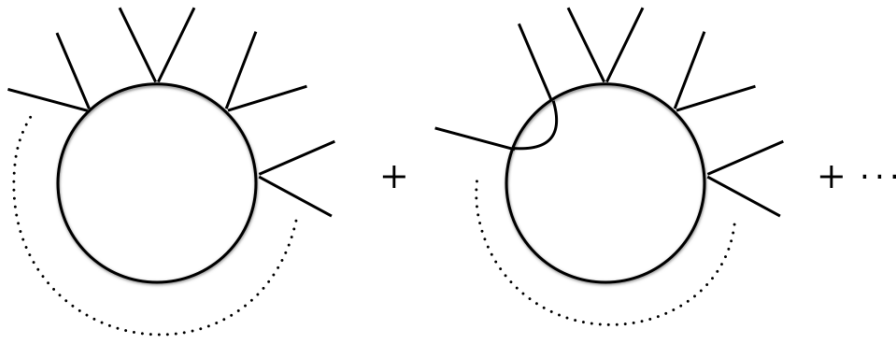


Figure 2: How higher-point vertices contribute to the trace anomaly beyond one loop.

⁵ Strictly speaking the notation $\lambda(\mu)$ cannot be correct: a dimensionless variable cannot depend on a dimensionful variable only. It must also depend on a second dimensionful variable and then through a dimensionless combination of the two. Thus $\lambda(\mu/\Lambda_{\text{dyn}})$ (where Λ_{dyn} is some dynamical scale) would be a better notation. We stick here to the notation that is common in the QFT literature.

⁶ Alternatively we can see this by noting that for all the $n \geq 2$ vertices, there is no Taylor expansion in the external momenta. The integrals from (2.5) that would give the coefficients of such a Taylor expansion are all infrared divergent and thus do not exist.

3 The WI of global scale transformations

3.1 Derivation

We now come to the Effective Average Action (EAA), which is defined as follows. Let

$$W_k(j; g_{\mu\nu}) = \log \int (d\phi) \exp \left[-S - S_k + \int d^d x \, j \cdot \phi \right] \quad (3.1)$$

be the generating functional obtained from the action $S + S_k$, where

$$S_k(\phi; g_{\mu\nu}) = \frac{1}{2} \int_x \phi R_k(\Delta) \phi \quad (3.2)$$

and $R_k(\Delta) \equiv k^2 r(y)$, with $y = \Delta/k^2$ and $\Delta = -\partial^2$, is a kernel suppressing the contribution of modes with momenta lower than k . It is quadratic in the fields and only affects the propagator.

The EAA is defined as a modified Legendre transform

$$\Gamma_k(\phi; g_{\mu\nu}) = -W_k(j; g_{\mu\nu}) + \int_x j \phi - S_k(\phi; g_{\mu\nu}) \quad (3.3)$$

where ϕ denotes here, by a slight abuse of language, the classical VEV of the corresponding quantum fields; the sources have to be interpreted as functionals of these classical fields and the last term subtracts the cutoff that had been inserted in the beginning in the bare action.

The main virtue of this functional is that it satisfies a simple equation [20, 21]

$$k \frac{\partial \Gamma_k}{\partial k} = \frac{1}{2} \text{Tr} \left(\Gamma_k^{(2)} + R_k \right)^{-1} k \frac{\partial R_k}{\partial k} \quad (3.4)$$

where $\Gamma_k^{(2)}$ is the second variation of the EAA with respect to the field. We note that this equation knows nothing about the action that entered in the functional integral.⁷ In particular, if we assume that S is scale invariant, as we shall henceforth do, (3.4) remains exactly the same.

We can now calculate the transformation of the cutoff term under rescaling. The Laplacian contains an inverse metric and therefore transforms under (1.1) by

$$\delta_\epsilon \Delta = -2\epsilon \Delta. \quad (3.5)$$

Since k does not change, we find $\delta_\epsilon R_k = -2\epsilon k^2 y r'$. On the other hand $\partial_t R_k = 2k^2 r - 2k^2 y r'$, so

$$\delta_\epsilon R_k = \epsilon (-2R_k + \partial_t R_k). \quad (3.6)$$

When we apply the variation to the cutoff action, all terms cancel except for the last term in (3.6), giving

$$\delta_\epsilon S_k = \frac{\epsilon}{2} \int_x \phi \partial_t R_k \phi, \quad (3.7)$$

⁷In principle the bare action could be reconstructed from the limit of a *given solution* Γ_k for $k \rightarrow \infty$ [20, 22, 23].

We now have all the ingredients that are needed to derive the WI. We subject W_k to a background scale transformation, with fixed sources and fixed k . Since the action S is assumed invariant, the only variations come from the measure, the cutoff and source terms:

$$\begin{aligned}\delta_\epsilon W_k &= \mathcal{A}(\epsilon) + \langle \delta_\epsilon S_k \rangle + \int_x j \langle \delta_\epsilon \phi \rangle \\ &= \mathcal{A}(\epsilon) + \epsilon \left[-\frac{1}{2} \int_x \frac{\delta W_k}{\delta j} \partial_t R_k \frac{\delta W_k}{\delta j} + \frac{1}{2} \text{Tr} \partial_t R_k \frac{\delta^2 W_k}{\delta j \delta j} + \int_x j d_\phi \frac{\delta W_k}{\delta j} \right].\end{aligned}\quad (3.8)$$

Here the first term comes from the variation of the measure and coincides with the trace anomaly that one always finds in the EA. It can be calculated for example by Fujikawa's method.⁸ The second term comes from the variations of the cutoff and the last comes from the variation of the source terms. The variation of the EAA can be computed from (3.3):

$$\delta_\epsilon \Gamma_k = -\delta_\epsilon W_k + \int_x j \delta_\epsilon \langle \phi \rangle - \delta_\epsilon S_k(\langle \phi \rangle). \quad (3.9)$$

Using (3.8), the source terms cancel out (since the variation is linear in the field we have $\langle \delta_\epsilon \phi \rangle = \delta_\epsilon \langle \phi \rangle$) and the first term in bracket in the r.h.s. of (3.8) cancels out with the last term in (3.9). The middle term in the same bracket can be rewritten in terms of Γ_k yielding

$$\delta_\epsilon \Gamma_k = -\mathcal{A}(\epsilon) + \epsilon \frac{1}{2} \text{Tr} \left(\frac{\delta^2 \Gamma_k}{\delta \phi \delta \phi} + R_k \right)^{-1} \partial_t R_k \quad (3.10)$$

Apart from the factor ϵ , the second term is exactly the FRGE. We can thus write as in (1.6):

$$\delta_\epsilon \Gamma_k = -\mathcal{A}(\epsilon) + \epsilon \partial_t \Gamma_k.$$

This is our main result. We see that in addition to the standard trace anomaly, which is the same as in (1.5), there is another source of variation due to the IR cutoff k , that is exactly proportional to the FRGE.

3.2 Applications

Let us recall how the WI (1.5) is often used in practice. If $\beta(\lambda)$ is known and if there is a sole reason for the breaking of scale invariance *e.g.* through a vacuum expectation value $\langle \phi \rangle$, or through constant background scalar curvature \mathcal{R} , or through summarising external momentum dependence by $-\partial^2$, then (1.5) can be integrated to give the exact answer for the physical EA Γ in terms of these quantities, provided its dependence on λ is also already known.⁹

⁸This is a somewhat abstract interpretation that stands for whatever UV regularization one is using. For example, it can be calculated by using an UV momentum cutoff, as we saw in section 2.

⁹To be clear, for the \mathcal{R} -dependence we are again considering only the interacting part of the anomaly [8].

To see this, let χ be the sole reason for breaking of scale invariance. Without loss of generality, we can set $\delta_\epsilon \chi = -\epsilon \chi$. Thus in the above examples we have chosen χ to be $\langle \phi \rangle$ (in $d = 4$ dimensions), or $\sqrt{\mathcal{R}}$, or $\sqrt{-\partial^2}$ respectively. Then

$$\delta_\epsilon \Gamma = -\epsilon \chi \partial_\chi \Gamma. \quad (3.11)$$

Combining this equation with (1.5) tells us that

$$\Gamma = \Gamma(\lambda(\chi)), \quad (3.12)$$

i.e. depends on χ only through its dependence on λ , where we suppress the dependence of Γ on the other quantities. In general $\lambda(\chi)$ is given by the implicit solution of the RG equation:

$$\int_{\lambda(\mu)}^{\lambda(\chi)} \frac{d\lambda}{\beta(\lambda)} = \log(\chi/\mu). \quad (3.13)$$

In a one-loop approximation this can be solved explicitly:

$$\lambda(\chi) = \lambda(\mu) + \frac{3\lambda^2(\mu)}{(4\pi)^2} \log\left(\frac{\chi}{\mu}\right). \quad (3.14)$$

Evidently (3.12) guarantees the standard form for the trace anomaly, since operating with (3.11) takes us back to (1.5).

As a concrete example, equations (3.12) and (3.14) imply that by setting $\chi = \langle \phi \rangle$ in the tree-level term (1.3), one obtains the Coleman-Weinberg potential:

$$V(\langle \phi \rangle) = \frac{\langle \phi \rangle^4}{4!} \left(\lambda(\mu) + \frac{3\lambda^2(\mu)}{(4\pi)^2} \log(\langle \phi \rangle/\mu) \right). \quad (3.15)$$

Similarly by setting $\chi^2 = \mathcal{R}$ one gets the interacting part of the conformal anomaly on a spherical background [8].

Let us now come to the EAA. If we again assume just one source of breaking of scale invariance, we have also in this case

$$\delta_\epsilon \Gamma_k = -\epsilon \chi \partial_\chi \Gamma_k, \quad (3.16)$$

Then combining this with the WI (1.6) and with the right half of (1.5)¹⁰ we have

$$[\beta(\lambda)\partial_\lambda - \chi\partial_\chi - \partial_t]\Gamma_k = 0. \quad (3.17)$$

This equation can be solved, *e.g.* by the method of characteristics. The solution implies that Γ_k also has a restricted functional form, which can for example be written as:

$$\Gamma_k = \hat{\Gamma}\left(\chi/k, \lambda(\chi^a k^{1-a})\right), \quad (3.18)$$

¹⁰We will show at the end of sec. 4.2 that it holds also when $k > 0$.

for any number a . We shall see in Section 5 how this reduces, in the limit $k \rightarrow 0$, to (3.12).

Notice that the anomalous WI (1.6) does not give us any information on the dependence of $\hat{\Gamma}$ on its arguments, nor on the dependence of λ on k . This information has to be obtained by other means, *e.g.* by solving the FRG in a some approximation.

4 Approximations

4.1 One-loop EAA

The one loop EAA is

$$\Gamma_k[\phi] = S[\phi] + \frac{1}{2} \text{Tr} \log \Delta_k \quad (4.1)$$

where

$$\Delta_k = P_k(-\partial^2) + V'' . \quad (4.2)$$

As usual we have defined $P_k(z) = z + R_k(z)$. Since $\delta_\epsilon P_k = -2\epsilon P_k + \epsilon \partial_t R_k$, we have

$$\delta_\epsilon \Gamma_k = \frac{1}{2} \text{Tr} \Delta_k^{-1} \delta_\epsilon \Delta_k = -\epsilon \text{Tr} \mathbf{1} + \frac{\epsilon}{2} \text{Tr} \partial_t R_k \Delta_k^{-1} . \quad (4.3)$$

We recognise that the second term is the flow equation, as expected from (3.10). It is describing the breaking of scale invariance by the infrared cutoff k . The first term contains the more fundamental breaking of scale invariance, as we have already seen in the previous sections. Despite appearances, it is actually field dependent, arising as it does from the ultraviolet modifications to the functional integral and bare action. Indeed, while the flow equation for the EAA is well defined of itself (since the ultraviolet regularisation of its right hand side is supplied by $\partial_t R_k$), solutions are not well defined without some overall ultraviolet regularisation. This is clearly the case for (4.1) for example, which is explicitly regularised only in the infrared.¹¹ Let us assume again that the ultraviolet regularisation, and counterterms if necessary, is put in place to ensure as much as possible that the $k = 0$ result is scale invariant. Then altogether at one loop we have

$$\delta_\epsilon \Gamma_k = -\epsilon \int d^4x \frac{\beta}{4!} \phi^4 + \frac{\epsilon}{2} \text{Tr} \partial_t R_k \cdot \Delta_k^{-1} . \quad (4.4)$$

Since the second term is the FRGE, it must also contain the β function. In fact, the t -differentiated ϕ^4 one-loop correction is regularised in the ultraviolet by $\partial_t R_k$, but it comes with the plus sign here, giving $\frac{\beta}{4!} \phi^4$. Therefore we see that δ_ϵ actually vanishes on the ϕ^4 piece of the one-loop EAA.

¹¹In practice this ultraviolet regularisation is typically supplied by assuming some initial ultraviolet EAA $\Gamma_{k=\Lambda}$ for the flow towards the infrared, starting from $k = \Lambda$. These issues have been discussed in detail in refs. [20, 23].

This result is somewhat unsettling if one wants to interpret $\delta_\epsilon \Gamma_k$ as a “total anomaly” but it is the right answer, as can be seen from both ends of the computation. On the one hand if we recall the correct metric factors, $\int_x \phi^4$ is invariant under (1.1). On the other hand the quantum correction that this part is computing is in essence (see (2.10) and below)

$$A_k(0, 0) = \frac{2}{(4\pi)^2} \int_k^\Lambda \frac{d|q|}{|q|}, \quad (4.5)$$

where now the momentum integral is cut off both in the UV and the IR.¹² Following the procedure as before to calculate δ_ϵ we now have a contribution from each boundary, namely $|q| = \Lambda(1 + \epsilon)$ as before, and now $|q| = k(1 + \epsilon)$. We see that the scale transformation induces the same expansion of each limit, as we should expect, however since one is an upper limit and the other a lower limit, of a formally scale invariant integral, the net result is exact cancellation.

Since the second term on the right hand side of (4.4) is the flow equation, it of course computes much more than just the ϕ^4 piece. With a smooth properly regularising IR cutoff R_k in place, the EAA is guaranteed an expansion in powers of the field and derivatives [25]; the flow equation computes all the coefficients.

We can get more insight if we focus on just the $\phi^2(-\partial^2)^n \phi^2$ operators. At one loop these all arise from inserting the IR regularisation into (2.10) to give:

$$A_k(p, -p) = \int_q \frac{1}{[q^2 + R_k(q)][(q+p)^2 + R_k(q+p)]}, \quad (4.6)$$

In general this integral is quite complicated, but since it is only logarithmically divergent we can get away with choosing the simple momentum independent $R_k = k^2$, *i.e.* a k -dependent mass term. This IR cutoff is not strong enough to work with more severe cases, but by inspecting this example we will be able easily to see what the general R_k will give. Using the Feynman trick the integral is

$$A_k(p, -p) = \int_0^1 d\alpha \int_q \frac{1}{[(1-\alpha)q^2 + \alpha(q+p)^2 + k^2]^2}. \quad (4.7)$$

Completing the square and shifting internal momentum we get

$$A_k(p, -p) = \int_0^1 d\alpha \int_q \frac{1}{[q^2 + k^2 + (1-\alpha)\alpha p^2]^2}. \quad (4.8)$$

This integral is now subject to the UV boundary condition that $|q - \alpha p| \leq \Lambda$, but replacing it with $|q| \leq \Lambda$ only introduces an error of order p^2/Λ^2 which vanishes as we take the UV limit.

Performing the q integral we thus find that the EAA contains

$$\int_x \frac{1}{8\pi^2} \left\{ \log \left(\frac{\Lambda^2}{k^2} \right) - 1 - \int_0^1 d\alpha \log [1 + (1-\alpha)\alpha p^2/k^2] \right\}, \quad (4.9)$$

¹²The infrared regularisation is more complicated than this but the result is the same for the action of δ_ϵ .

where again we discard terms that vanish as $\Lambda \rightarrow \infty$. Recalling from above (2.12), the factor of $-\lambda^2/16$, we see that the same counterterm (2.14) will render this finite. Indeed including the ϕ^4 contribution, (2.16), from the bare action $S[\phi]$ in (4.1), we have altogether:

$$\int_x \left\{ \left[\lambda(\mu) + \frac{3\lambda^2(\mu)}{32\pi^2} \left(1 + \log \left(\frac{k^2}{\mu^2} \right) \right) \right] \frac{\phi^4}{4!} + \frac{\lambda^2(\mu)}{256\pi^2} \phi^2 \int_0^1 d\alpha \log [1 - (1 - \alpha)\alpha \partial^2/k^2] \phi^2 \right\} \quad (4.10)$$

however for the FRG, it is more natural to choose as renormalization condition that the coefficient of $\phi^4/4!$ is the renormalized coupling $\lambda(k)$. In this way it is clear what it means to pick a solution that breaks scale invariance the least: we should pick the solution that breaks the invariance only through the running of this coupling [24]. It implies that the bare coupling $\lambda(\Lambda)$ is now set equal to:

$$\lambda(\Lambda) = \lambda(k) + \frac{3\lambda^2(k)}{32\pi^2} \left\{ \log \left(\frac{\Lambda^2}{k^2} \right) - 1 \right\}. \quad (4.11)$$

In this way we avoid introducing an explicit extra scale μ , whilst (4.11) and the β -function (2.13), now tells us that $\lambda(\Lambda)$ is independent of k up to terms of higher order, as it should be. Then (4.10) just reads:

$$\int_x \left\{ \frac{\lambda(k)}{4!} \phi^4 + \frac{\lambda^2(k)}{256\pi^2} \phi^2 \int_0^1 d\alpha \log [1 - (1 - \alpha)\alpha \partial^2/k^2] \phi^2 \right\}, \quad (4.12)$$

after using $\lambda^2(\mu) = \lambda^2(k) + O(\lambda^3)$. Finally, Taylor expanding the last term gives us the derivative expansion we were aiming for:

$$\int_x \left\{ \frac{\lambda(k)}{4!} \phi^4 + \frac{\lambda^2(k)}{256\pi^2} \sum_{n=1}^{\infty} a_n \phi^2 \left(\frac{-\partial^2}{k^2} \right)^n \phi^2 \right\}, \quad (4.13)$$

where we learn that with $R_k = k^2$, the a_n are given by:

$$a_n = (-1)^{n+1} \frac{n!(n-1)!}{(2n+1)!}. \quad (4.14)$$

Recalling the implicit metric factors, the action of $-\delta_\epsilon$ kills the first term and turns the sum into

$$\int_x \frac{\lambda^2(k)}{128\pi^2} \sum_{n=1}^{\infty} n a_n \phi^2 \left(\frac{-\partial^2}{k^2} \right)^n \phi^2. \quad (4.15)$$

This is unilluminating until we recognise that from (1.6), we have of course

$$\mathcal{A}(\epsilon) = \epsilon \partial_t \Gamma_k - \delta_\epsilon \Gamma_k. \quad (4.16)$$

We see that $\epsilon \partial_t \Gamma_k$ cancels (4.15) but turns the first term in (4.13) into $\epsilon \beta \phi^4/4!$, thus confirming the anomaly (1.4).

Now it is easy to see how the anomaly is reproduced at one loop in general in the EAA. Firstly, with a general IR cutoff R_k , the a_n are pure numbers that no longer satisfy (4.14) but

depend (non-universally) on the functional form of R_k . However the derivative expansion still has the form in (4.13). Therefore it is still the case that on applying (4.16), the two parts on the right hand side cancel each other for all the derivative operators, leaving the anomaly to be reproduced by the RG flow of $\lambda(k)\phi^4$ itself. Secondly, the one loop EAA also has $2n$ -point vertices $\mathcal{V}_{k,n}[\phi]$ where $n \neq 2$. Their expansion in local operators gives powers of derivatives and the field balanced by powers of k according to dimensions. (For $n = 1$ the tadpole integral yields exclusively a mass term proportional to k^2 .) As at the end of sec. 2, at one loop the λ^n factor does not run, being already proportional to \hbar . Therefore, as we will see confirmed also in the next section, the application of the right hand side of (4.16) just gives zero. From the (derivative expansion of the) whole of the one loop EAA, we are therefore left just with the one contribution coming from (1.4), which here is reproduced entirely from the RG running of the $\lambda(k)\phi^4$ term.

4.2 Local expansions

In practical applications of the EAA, one often assumes that it has the form

$$\Gamma_k = \sum_i \lambda_i(k) \mathcal{O}_i, \quad (4.17)$$

where the \mathcal{O}_i are integrals of local operators constructed with the fields, the metric and derivatives. For the purpose of counting, notice that the integral contains \sqrt{g} and therefore carries $d/2$ powers of the metric. Generically such approximations are called “truncations”. Systematic expansions are the derivative expansion and the vertex expansion, in which cases the sum in (4.17) is infinite and contains arbitrary powers of the field or of the derivative, respectively [25].

Differently from the previous section, we are here treating each operator as having its own separate coupling λ_i , and absorbing all powers of k into these couplings. Later on, we will specialise to the case where the continuum limit is controlled by just one marginal coupling.

For the WI (1.6), it is enough to consider one monomial at the time. Let \mathcal{O}_i involve n_ϕ powers of ϕ and, in total, n_g powers of the metric. The scaling dimension of \mathcal{O}_i under (1.1), is

$$\Delta = -2n_g + \frac{d-2}{2}n_\phi \quad (4.18)$$

and the scaling dimension of λ_i under (1.1) (which is minus its mass dimension) is $-\Delta$. We can also write

$$\lambda_i \mathcal{O}_i = \tilde{\lambda}_i \tilde{\mathcal{O}}_i$$

where

$$\tilde{\lambda}_i = k^\Delta \lambda_i; \quad \tilde{\mathcal{O}}_i = k^{-\Delta} \mathcal{O}_i, \quad (4.19)$$

which thus implies that $\tilde{\lambda}_i$ is dimensionless. The l.h.s. of the WI is

$$\delta_\epsilon(\lambda_i \mathcal{O}_i) = \epsilon \lambda_i \left(2n_g - \frac{d-2}{2} n_\phi \right) \mathcal{O}_i = -\epsilon \Delta \lambda_i \mathcal{O}_i \quad (4.20)$$

On the other hand, one has

$$\partial_t(\lambda_i \mathcal{O}_i) = \partial_t \lambda_i \mathcal{O}_i, \quad (4.21)$$

since the dimensionful operator in itself has no dependence on k . Thus the WI gives

$$-\epsilon \lambda_i \Delta \mathcal{O}_i = -\mathcal{A}(\epsilon) + \epsilon \partial_t \lambda_i \mathcal{O}_i \quad (4.22)$$

Bringing the l.h.s. to the r.h.s. it reconstructs the derivative of $\tilde{\lambda}_i$, times $k^{-\Delta}$, which can be rewritten as

$$\mathcal{A}(\epsilon) = \epsilon \partial_t \tilde{\lambda}_i \tilde{\mathcal{O}}_i. \quad (4.23)$$

Thus for an action of the form (4.17) the WI implies

$$\mathcal{A}(\epsilon) = \epsilon \sum_i \tilde{\beta}_i \tilde{\mathcal{O}}_i. \quad (4.24)$$

We see from this expression that the anomaly receives contributions from all the operators.

Now let us consider what form this must take when the continuum limit is controlled by a single marginal coupling λ , as in the one loop calculations in the previous sections. In the expansion over the local operators \mathcal{O}_i we have identified couplings $\lambda_i(k)$ as the parameters conjugate to these local operators. One of these λ_i is the coupling λ itself. In the continuum limit the other couplings are not independent but are functions of λ and k . The scaled couplings $\tilde{\lambda}_i = \tilde{\lambda}_i(\lambda)$ are dimensionless and thus have no explicit k dependence. They gain their k dependence only through their dependence on λ . (Note that, being marginal, $\tilde{\lambda} = \lambda$.) Therefore

$$\tilde{\beta}_i = \partial_t \tilde{\lambda}_i = \partial_t \lambda \partial_\lambda \tilde{\lambda}_i = \beta(\lambda) \partial_\lambda \tilde{\lambda}_i. \quad (4.25)$$

Notice that (4.25) also holds for the coupling λ_i that is λ itself since in this case it simply says $\tilde{\beta}_i = \beta_i = \beta$. Then (4.24) can be rewritten as

$$\mathcal{A}(\epsilon) = \epsilon \beta(\lambda) \partial_\lambda \sum_i \tilde{\lambda}_i \tilde{\mathcal{O}}_i, \quad (4.26)$$

or simply

$$\mathcal{A}(\epsilon) = \epsilon \beta(\lambda) \partial_\lambda \Gamma_k. \quad (4.27)$$

We see that this is the right half of the equation (1.5) that applies to the EA, and moreover it holds also at finite k *i.e.* for the EAA. What does not hold at finite k , is the left half, namely the statement that δ_ϵ induces the anomaly $\mathcal{A}(\epsilon)$ and only this anomaly.

5 Recovering the standard form of the trace anomaly

To recover the first part of the WI (1.5), we need to study the limit $k \rightarrow 0$, keeping all other quantities fixed. In this limit $\Gamma_k \rightarrow \Gamma$, and the breaking due the IR cutoff R_k should disappear. Comparing (1.6) and (1.5), we see that this is true provided that $\partial_t \Gamma_k \rightarrow 0$, which indeed must also hold in this limit as we show below. Note that the derivative expansion, or any approximation of Γ_k in terms of local operators, implies a Taylor expansion of the vertices in p_i/k , the dimensionless momenta. Since we hold p_i fixed and let $k \rightarrow 0$, such approximations are not valid in the regime we now need to study.

We will get insight by first inspecting the one loop case. At one loop, the term that contains the trace anomaly in the small k limit is in fact the non-local term (2.15), which indeed is missing from any local approximation to the EAA, in particular from the derivative expansion considered in the last section. Together we therefore have:

$$\Gamma_k \ni \frac{1}{4!} \int_x \left\{ \lambda(k) \phi^4 + \frac{3\lambda^2(k)}{32\pi^2} \phi^2 \log\left(\frac{-\partial^2}{k^2}\right) \phi^2 \right\}, \quad (5.1)$$

where the explicit k^2 is supplied by the counterterm in (4.11), in preference to the μ^2 supplied by (2.16). Now note that with the non-local term included, the ϕ^4 coefficient is actually independent of k (to the one-loop order in which we are working), the β -function contribution cancelling against the explicit k dependence in (5.1). We see that the non-local term is just what is needed in order to ensure that Γ_k has a sensible limit. Indeed we could have found the non-local term by insisting that Γ_k becomes independent of k as $k \rightarrow 0$, holding everything else finite. This implies a practical method for recovering such non-local terms from the flow of the couplings, as we will see shortly. By adding the missing non-local term as in (5.1), we now have a four-point vertex that satisfies

$$\partial_t \Gamma_k^{(4)} = 0, \quad (5.2)$$

but also gives the standard form of the trace anomaly. In this way we have reproduced the first part of the WI (1.5), *viz.* $\delta_\epsilon \Gamma_{k=0} = -\mathcal{A}(\epsilon)$.

Let us now set ourselves in the situation when there is a single source of scale symmetry breaking χ , as in section 3.2. To get an explicit answer for the four-point vertex in the limit as $k \rightarrow 0$, we can for example solve for $\lambda(k)$ in terms of some $\lambda(\mu)$. The solution is just (3.14) with χ replaced by k . Plugging this back in (5.1) we get the same expression as (5.1), but with $-\partial^2$ replaced by χ^2 and k replaced by μ , a consequence of the fact that the physical EA is actually an RG invariant, and thus independent of k or μ .

Now, if we want to go beyond the one loop approximation, in general we will have to solve equation (3.13). However in perturbation theory, by iteration, we can explicitly find this form

of the solution. For example to two loops, writing

$$\beta = \beta_1 \lambda^2 + \beta_2 \lambda^3, \quad (5.3)$$

where β_1 is the coefficient in (2.13), we must have

$$\lambda(\chi) = \lambda(k) + \beta_1 \lambda^2(k) \log(\chi/k) + \lambda^3(k) \alpha(\chi/k), \quad (5.4)$$

for some function α to be determined, where we recognise that (5.1) and (2.13) already fix the β_1 term. Differentiating the above with respect to t , using (5.3), and requiring that overall the result vanishes, we confirm again the λ^2 piece, and determine that

$$\partial_t \alpha(\chi/k) = -\beta_2 - 2\beta_1^2 \log(\chi/k). \quad (5.5)$$

Integrating we thus have

$$\alpha(\chi/k) = \beta_2 \log(\chi/k) + \beta_1^2 \log^2(\chi/k). \quad (5.6)$$

Note that the integration constant vanishes since by (5.4), we must have $\alpha(1) = 0$. Clearly our EA, (3.12), then does satisfy the anomalous WI, provided we recall that in (1.5) we have $\lambda = \lambda(\chi)$. In particular this means that the trace anomaly appears at this order as (1.5) where however

$$\beta = \beta_1 \lambda^2(\chi) + \beta_2 \lambda^3(\chi) = \beta_1 \lambda^2(k) + \lambda^3(k) \{ \beta_2 + 2\beta_1^2 \log(\chi/k) \}. \quad (5.7)$$

By design, and despite appearances, $\lambda(\chi)$ and Γ , are independent of k . Indeed, from (5.4) and (5.6), we know that substituting

$$\lambda(k) = \lambda(\mu) + \beta_1 \lambda^2(\mu) \log(k/\mu) + \beta_2 \lambda^3(\mu) \log(k/\mu) + \beta_1^2 \lambda^3(\mu) \log^2(k/\mu), \quad (5.8)$$

into (5.4) and (5.7), will eliminate k and $\lambda(k)$, in favour of μ and $\lambda(\mu)$, making explicit the fact that these formulae are actually independent of k .

By choosing λ to depend solely on k , setting $a = 0$ in (3.18), we return to the derivative expansion studied in sec. 4.2. For the small k limit, the appropriate choice is to take λ to depend solely on χ , setting $a = 1$ in (3.18), as we have above. The $k \rightarrow 0$ limit exists, because Γ in (3.18) can be expanded in its first argument in this case as a series in small k/χ (although not in general as a Taylor series).

To illustrate this last case, let us again use the example of one loop. The $(n \neq 4)$ -point vertices do not contribute to the anomaly in this case, but evidently they do have k dependence. The two-point vertex is proportional to $\lambda(k) k^2 \phi^2$, while $(n > 4)$ -point vertices in the limit of small k , are non-local, reflecting the discussion at the end of sec. 2. They come with a factor of $\lambda^{n/2}$

as we saw in fig. 1, which we can if we wish regard as having implicit k dependence through $\lambda = \lambda(k)$. However they also come with explicit k dependence. Since R_k acts like a mass term, in the regime of small k , holding all momenta fixed, we would naïvely conclude that the correction to the Feynman integral goes like k^2 . Actually since the coefficient of this k^2 is an infrared divergent integral, a more careful treatment establishes that it goes like $k^2 \log k$. By dimensions the coefficient integrals are even more non-local (*i.e.* have a higher overall inverse power of external momenta p_i). The coefficient integrals correspond to the same Feynman diagrams but with a mass-operator ϕ^2 insertion on the internal legs. The main point is that we see that in the limit that $k \rightarrow 0$, all the other explicit k dependence (which actually the four-point vertex also has) vanishes. This is the signature that this k dependence has no relation to the trace anomaly. Indeed for all these pieces, the t -derivative just gives a piece that still vanishes as $k \rightarrow 0$, since also $\partial_t(k^2 \log k) \rightarrow 0$. As we see in (5.1), the trace anomaly is accompanied by explicit k dependence that does not vanish, but rather diverges as $k \rightarrow 0$. In this limit, all other explicit k dependence disappears and only the explicit k dependence in the trace anomaly remains. In this example it comes exclusively from (5.1). As we have seen, it can better be written in k -independent form as (1.3) plus (2.15), utilising the fact that the explicit and implicit k dependencies actually cancel.

In the general case we arrive at this conclusion by writing Γ_k as in (3.18) and choosing $a = 1$, then as we illustrated, the remaining k/χ dependence vanishes in the limit as $k \rightarrow 0$.

Finally, we add a note to clarify the rôle of μ . Recall that the running of couplings with respect to the scale μ , is fundamentally different from the running of k in the Wilsonian RG. Whereas $k > 0$ parametrises an infrared cutoff, meaning that there are still low energy modes to be integrated out, μ is a dimensional parameter that remains even when the functional integral is completed. Then the RG is realised through μ , however only in the Callan-Symanzik sense: physical quantities must actually be independent of μ . To the extent that the EA is a physical quantity, the EA must therefore also be independent of μ . In this sense, dependence on μ is fake. It, and λ , can be eliminated in favour of a fixed dynamical scale (completing the so-called dimensional transmutation, *cf.* footnote 5). In the limit that $k \rightarrow 0$, we can only be left with this fake dependence on k , thus (only) in this limit k and μ appear on the same footing, as is exemplified explicitly in (5.8).

6 Fixed points

At a fixed point, (1.5) vanishes. The scale-invariance of the EA is then explicitly realised as invariance under the transformation δ_ϵ . Since (1.5) also implies $\beta(\lambda) = 0$, λ can no longer depend on μ and becomes a fixed number, as indeed is verified by (5.8) since now all the β_n vanish. For

the same reason, all the explicit μ dependence also disappears from the EA, as obviously it must in order for the EA to be overall independent of μ . (Again this is verified by (5.4) and (5.6).)

The connection to scale invariance is a little less direct for the EAA. In the presence of (generally) dimensionful couplings λ_i , a fixed point is defined by the vanishing of the beta functions of their dimensionless cousins $\tilde{\lambda}_i$, as in (4.19), *i.e.*

$$\tilde{\beta}_i(\tilde{\lambda}_j) = 0 .$$

One immediate consequence of (4.24) is then that the anomaly vanishes at a fixed point. This however does not lead to the statement that the EAA is scale-invariant at a fixed point. Indeed, if we look at equation (4.20) we see that the variation of the EAA under an infinitesimal scale transformation δ_ϵ is not zero in general. It is only zero in the case when $\Delta = 0$, *i.e.* when all the couplings λ_i are themselves dimensionless.

Consider, however, a different realization of scale invariance, namely one where we also transform the cutoff scale by [39–42]

$$\hat{\delta}_\epsilon k = -\epsilon k , \quad (6.1)$$

the action of $\hat{\delta}_\epsilon$ being the same as the action of δ_ϵ on all other quantities. Then, instead of (4.20) we have

$$\begin{aligned} \hat{\delta}_\epsilon \Gamma_k &= \delta_\epsilon \Gamma_k - \epsilon \sum_i k \partial_k \lambda_i \mathcal{O}_i \\ &= -\epsilon \sum_i \left(\Delta_i \tilde{\lambda}_i + \beta_i k^{\Delta_i} \right) \tilde{\mathcal{O}}_i = -\epsilon \sum_i \tilde{\beta}_i \tilde{\mathcal{O}}_i . \end{aligned} \quad (6.2)$$

This implies that at a fixed point one has invariance under the scale transformations generated by $\hat{\delta}_\epsilon$.

From this Wilsonian point of view, the relevant notion of scale transformation is one where the cutoff is also acted upon, and a fixed point is not a point where only dimensionless couplings are present, but rather one where all dimensionful couplings in the fixed point action appear as (non-universal) numbers times the appropriate power of k . It is this fact that ensures that the fixed point action does not vary with k , when all variables are written in dimensionless terms (using the appropriate scaling dimensions). Indeed we also recall that when written in these terms, the eigenoperators, which are integrated operators of definite scaling dimension $d_{\mathcal{O}}$, correspond to linearised perturbations about the fixed point action whose associated couplings carry power law k -dependence, namely $k^{d-d_{\mathcal{O}}}$. Thus the behaviour of these linearised couplings under change of scale is entirely given by (6.1).

We can further clarify the relation to the Wilsonian RG by the following argument. The partial derivative ∂_t gives zero when acting on \mathcal{O} , because all the k -dependence is assumed to be

in the coupling, and therefore $\partial_t \tilde{\mathcal{O}} = -\Delta \tilde{\mathcal{O}}$. Let us make this explicit by writing, for a monomial $\lambda \mathcal{O}$ in the EAA:

$$\partial_t(\lambda \mathcal{O})|_{\mathcal{O}} = (\tilde{\beta} - \Delta \tilde{\lambda}) \tilde{\mathcal{O}} .$$

If instead we take the derivative keeping $\tilde{\mathcal{O}}$ fixed, we get

$$\partial_t(\lambda \mathcal{O})|_{\tilde{\mathcal{O}}} = \tilde{\beta} \tilde{\mathcal{O}} .$$

This implies that for the EAA, which is a sum of terms of this type, the flow for scaled fields is

$$\epsilon \partial_t \Gamma_k|_{\tilde{\mathcal{O}}} = \epsilon \partial_t \Gamma_k|_{\mathcal{O}} - \delta_\epsilon \Gamma_k , \quad (6.3)$$

where (4.20) has been used. This equation is just the definition of an infinitesimal Wilsonian RG transformation in the way it was originally formulated [26]. Thus on the right hand side, the first term is an infinitesimal Kadanoff blocking transformation to the course grained scale, while the second term is the infinitesimal rescaling back to the original scale (hence the minus sign).

Now recall that in the WI (1.6), the t -derivative is taken at fixed \mathcal{O} :

$$\mathcal{A} = \epsilon \partial_t \Gamma_k|_{\mathcal{O}} - \delta_\epsilon \Gamma_k .$$

Comparing to (6.3), it is immediate to see that the anomaly, \mathcal{A} , and the Wilsonian RG transformation, $\epsilon \partial_t \Gamma_k|_{\tilde{\mathcal{O}}}$, are effectively the same thing. Indeed this is just eqn. (4.24) derived in a different way.

7 Concluding remarks

We have discussed the WI of global scale transformations for the EAA and its relation to the FRG. There are relations to several other strands of research and various natural extensions. One extension is to consider the WI of special conformal transformations. This has been discussed in [27, 28] and, more specifically in relation to the trace anomaly, in [29]. Another generalization is to make the scale transformations position-dependent. This can be used as a technical device in flat space physics [30] but is most natural in a gravitational context [31, 32].

Another point to be kept in mind is that interpreting the renormalization scale as a VEV of a dynamical field leads to a (typically non-renormalizable) theory where scale symmetry is not broken. This has been discussed recently in [33]. Related observations have been made for local Weyl transformations in the presence of a dilaton in [34–36] and for the EAA in [37, 38].

In a gravitational context, the result (1.6) bears some resemblance to our earlier results for the WI of split Weyl transformations [39–42]. The physical meaning is very different, though:

the split transformation is the freedom of shifting the background and the quantum field by equal and opposite amounts and is always an invariance of the classical action. The cutoff, however, introduces separate dependences on these two variables and breaks the split transformations. For transformations of the background metric of the form (1.1), with constant ϵ , the anomalous WI contains the term $\epsilon \partial_t \Gamma_k$ in the r.h.s. The difference with the physical scale transformations considered in this paper is highlighted by the invariance of the measure under split scale transformations, which results in the absence of the term $-\mathcal{A}$.

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A Rescaling the coordinates vs. rescaling the metric

A scale transformation is a change of all lengths by a common factor. Since physical lengths are defined by integrating the line element $ds^2 = g_{\mu\nu} dx^\mu dx^\nu$, a scale transformation can be interpreted either as a scaling of the metric or as a scaling of the coordinates. In the main text we have followed the former convention, which is more natural from the point of view of General Relativity. In flat space QFT it is customary to define the scale transformations as rescalings of the coordinates: $\delta_\epsilon x^\mu = \epsilon x^\mu$. Then the infinitesimal transformation of the fields is $\delta_\epsilon \phi = \epsilon(-x^\mu \partial_\mu + d_\phi)\phi$. The canonical dimension of a field, d_ϕ , which is determined by requiring scale invariance of the kinetic term, is the same in both cases. The canonical energy-momentum tensor comes from the Noether current associated to translation invariance. In general it does not coincide with the one defined in (1.2), but there are well-known “improvement” procedures that make them equal.

Equations (2.7, 2.8, 3.5) hold also when one rescales the coordinates, and so the derivation of the scale WI in section 3.1 goes through in the same way. The mass dimension of an operator \mathcal{O} , containing n_ϕ fields and n_∂ derivatives is $\Delta = -d + n_\partial + \frac{d-2}{2}n_\phi$, where $-d$ comes from $d^d x$. This is equal to the expression given in (4.18), because n_g is $d/2$, coming from \sqrt{g} , minus the number of inverse metrics, which must be equal to half the number of derivatives.

References

- [1] D. M. Capper and M. J. Duff, *Nuovo Cim. A* **23** (1974) 173.
- [2] M. J. Duff, *Class. Quant. Grav.* **11** (1994) 1387 [[hep-th/9308075](#)].
- [3] S. Coleman, “Dilatations”, Erice Lectures, publ. in “Aspects of Symmetry : Selected Erice Lectures,” Cambridge University Press (1985).
- [4] M. S. Chanowitz and J. R. Ellis, *Phys. Rev. D* **7** (1973) 2490.
- [5] S. L. Adler, J. C. Collins and A. Duncan, *Phys. Rev. D* **15** (1977) 1712.
- [6] J. C. Collins, A. Duncan and S. D. Joglekar, *Phys. Rev. D* **16** (1977) 438.
- [7] J. S. Schwinger, *Phys. Rev.* **82** (1951) 914.
- [8] I. T. Drummond and G. M. Shore, *Phys. Rev. D* **19** (1979) 1134.
- [9] M. S. Alves and J. Barcelos-Neto, *Mod. Phys. Lett. A* **4** (1989) 155.
- [10] H. Boschi-Filho and C. P. Natividade, *Phys. Rev. D* **46** (1992) 5458.
- [11] S. R. Coleman and E. J. Weinberg, *Phys. Rev. D* **7** (1973) 1888.
- [12] K. A. Meissner and H. Nicolai, *Phys. Lett. B* **660** (2008) 260 [[arXiv:0710.2840](#) [hep-th]].
- [13] C. T. Hill, *Phys. Rev. D* **89** (2014) no.7, 073003 [[arXiv:1401.4185](#) [hep-ph]].
- [14] H. Gies and L. Zambelli, *Phys. Rev. D* **92** (2015) no.2, 025016 [[arXiv:1502.05907](#) [hep-ph]].
- [15] A. Salvio and A. Strumia, *JHEP* **1406** (2014) 080 [[arXiv:1403.4226](#) [hep-ph]].
- [16] M. B. Einhorn and D. R. T. Jones, *JHEP* **1503** (2015) 047 [[arXiv:1410.8513](#) [hep-th]].
- [17] P. G. Ferreira, C. T. Hill and G. G. Ross, *Phys. Rev. D* **95** (2017) no.4, 043507 [[arXiv:1610.09243](#) [hep-th]].
- [18] I. Oda, *Eur. Phys. J. C* **78** (2018) no.10, 798 [[arXiv:1806.03420](#) [hep-th]].
- [19] C. Wetterich, *Phys. Lett. B* **301** (1993) 90 [[arXiv:1710.05815](#) [hep-th]].
- [20] T. R. Morris, *Int. J. Mod. Phys. A* **9** (1994) 2411 [[hep-ph/9308265](#)].
- [21] T. R. Morris, *Phys. Lett. B* **334** (1994) 355 [[arXiv:hep-th/9405190](#)].

- [22] E. Manrique and M. Reuter, Phys. Rev. D **79** (2009) 025008 [[arXiv:0811.3888](#) [hep-th]].
- [23] T. R. Morris and Z. H. Slade, JHEP **1511** (2015) 094 [[arXiv:1507.08657](#) [hep-th]].
- [24] T. R. Morris, Prog. Theor. Phys. Suppl. **131** (1998) 395f [[hep-th/9802039](#)].
- [25] T. R. Morris, Phys. Lett. B **329** (1994) 241 [[hep-ph/9403340](#)].
- [26] K. G. Wilson and J. B. Kogut, Phys. Rept. **12** (1974) 75.
- [27] O. J. Rosten, Eur. Phys. J. C **77** (2017) no.7, 477 [[arXiv:1411.2603](#) [hep-th]].
- [28] O. J. Rosten, [arXiv:1605.01729](#) [hep-th].
- [29] O. J. Rosten, [arXiv:1807.02181](#) [hep-th].
- [30] H. Osborn, Nucl. Phys. B **363** (1991) 486.
- [31] A. Codello, G. D’Odorico, C. Pagani, JHEP **1407** (2014) 040 [[arXiv:1312.7097](#) [hep-th]].
- [32] A. Codello, G. D’Odorico, Phys. Rev. D **92** (2015) no.2, 024026 [[arXiv:1412.6837](#) [gr-qc]].
- [33] Z. Lalak and P. Olszewski, [arXiv:1807.09296](#) [hep-th].
- [34] F. Englert, C. Truffin and R. Gastmans, Nucl. Phys. **B117** 407 (1976).
- [35] E.S. Fradkin, G.A. Vilkovitsky, Phys. Lett. **B73**, (1978), 209.
- [36] M. Shaposhnikov, D. Zenhausern, Phys. Lett. **B671**, 162 (2009) [arXiv:1104.1392](#) [hep-th];
M. Shaposhnikov, I. Tkachev, Phys. Lett. **B675**, 403 (2009) [arXiv:0811.1967](#) [hep-th].
- [37] R. Percacci, New J. Phys. **13** (2011) 125013 [[arXiv:1110.6758](#) [hep-th]].
- [38] A. Codello, G. D’Odorico, C. Pagani and R. Percacci, Class. Quant. Grav. **30** (2013) 115015 [[arXiv:1210.3284](#) [hep-th]].
- [39] T. R. Morris, JHEP **1611** (2016) 160 [[arXiv:1610.03081](#) [hep-th]].
- [40] R. Percacci and G. P. Vacca, Eur. Phys. J. C **77** (2017) no.1, 52 [[arXiv:1611.07005](#) [hep-th]].
- [41] N. Ohta, PTEP **2017** (2017) no.3, 033E02 [[arXiv:1701.01506](#) [hep-th]].
- [42] C. M. Nieto, R. Percacci and V. Skrinjar, Phys. Rev. D **96** (2017) no.10, 106019 [[arXiv:1708.09760](#) [gr-qc]].