Testing Continuous Spontaneous Localization with Fermi liquids

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Collapse models describe phenomenologically the quantum-to-classical transition by adding suitable nonlinear and stochastic terms to the Schrödinger equation, which implement the collapse of the wave function [1]. Such effects are negligible for microscopic systems, and become stronger when their mass increases. This is how the quantum-to-classical transition is described and the measurement problem solved, which is the main motivation why they were formulated in the first place.

The most studied model is the Continuous Spontaneous Localization (CSL) model, the most studied among collapse models, for Fermi liquids. We consider both the white and non-white noise case. Application to various astrophysical sources is presented.

I. INTRODUCTION

Collapse models provide a phenomenological description of quantum measurements, by adding stochastic and non-linear terms to the Schrödinger equation, which implement the collapse of the wave function [1]. Such effects are negligible for microscopic systems, and become stronger when their mass increases. This is how the quantum-to-classical transition is described and the measurement problem solved, which is the main motivation why they were formulated in the first place.

In this context, one important question is the origin of the collapse noise. While collapse models do not give an answer, as the collapse is inserted ‘by hand’ into the Schrödinger dynamics (but its mathematical structure is fully constrained by the request of no-superrunival-singling and norm-conservation [1]), several times it has been suggested that is related to gravity [32–42]. If there is truth in this conjecture, then the gravitational fluctuations responsible for the collapse add to the usual gravitational effects present in matter, in particular in strongly gravitationally bound systems as those we will consider in this paper.

A consequence of collapse models is a spontaneous heating, induced by the random collapse. This effect has been calculated for many types of systems [13, 14, 16–19, 23, 24], but not for Fermi liquids, an issue raised in a recent paper of Tilloy and Stace [43]. Here, we give a comprehensive treatment of CSL induced heating in Fermi liquids, including the experimentally relevant case of non-white noise, and apply our results to various astrophysical systems, including neutron stars.

II. CSL MODEL - PERTURBATIVE CALCULATION

Following [23], we consider the transition amplitude $c_{fi}(t)$ caused by a perturbation, from an initial state $|i\rangle$ of a quantum system to a final state $|f\rangle$, with associated energies $E_{i} = \hbar \omega_{i}$ and $E_{f} = \hbar \omega_{f}$ respectively. For the sake of simplicity we restrict the problem to the case of one fermion of mass $m_{A}$. The result for the $N$ particle case, either fermions or bosons, is given in Appendix A. We have:

$$c_{fi}(t) = -\frac{i}{\hbar} \int_{0}^{t} ds \langle f|e^{\frac{i}{\hbar} \hat{H}_{0}\tau} \hat{V}(\tau) e^{-\frac{i}{\hbar} \hat{H}_{0}\tau}|i\rangle,$$

(1)

where $\hat{H}_{0}$ is the free Hamiltonian and the perturbation, for the CSL process applied to a particle of mass $m_{A}$, is [23]:

$$\hat{V}(\tau) = \int dz w_{i}(z) \hat{V}(z),$$

$$\hat{V}(z) = -\frac{\hbar}{m_{0}} m_{A} g(z - \hat{x}_{A}),$$

(2)

where $m_{0}$ is the nucleon mass, $w_{i}(z)$ is a noise with zero mean ($\mathbb{E}[w_{i}(z)] = 0$) and correlator:

$$\mathbb{E}[w_{i}(z)w_{i}(x)] = \frac{1}{2\pi} \int_{-\infty}^{+\infty} d\omega \gamma(\omega)e^{-i\omega(t-s)} \delta(x - z),$$

(3)

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where \(\gamma(\omega) = \gamma(-\omega)\) is the frequency-dependent collapse strength. We denoted with \(\hat{x}_A\) the position operator of the particle, and:

\[
g(x) = \frac{e^{-x^2/2\sigma_c^2}}{(\sqrt{2\pi}\sigma_c)^3} = \frac{1}{(2\pi)^{3/2}} \int dq e^{-q^2\sigma_c^2/2-iq\cdot x}. \tag{4}
\]

We assume that the particle is free and confined in a box of side \(L\); the initial and final states read:

\[
\langle x| i \rangle = \frac{e^{i\mathbf{k}_i \cdot x}}{L^{3/2}}, \quad \text{and} \quad \langle x| f \rangle = \frac{e^{i\mathbf{k}_f \cdot x}}{L^{3/2}}. \tag{5}
\]

\[
E[|c_f|^2] = \frac{m_A^2}{m_0^2 L^3} \int dq e^{-q^2\sigma_c^2} \int d\omega \gamma(\omega) \delta(\mathbf{k}_f - \mathbf{k}_i - \mathbf{q}) t \delta(\omega_f - \omega), \tag{7}
\]

where we used the relations:

\[
[\delta(\mathbf{k}_f - \mathbf{k}_i - \mathbf{q})] \sim (L/(2\pi))^3 \delta(\mathbf{k}_f - \mathbf{k}_i - \mathbf{q}),
\]

\[
\int_0^t ds e^{i\omega_f(\omega_f - \omega)/2} = 2\pi e^{i\omega_f(\omega_f - \omega)/2} g(t)(\omega_f - \omega), \tag{8}
\]

\[
[\bar{g}(t)(\omega_f - \omega)] \sim (t/(2\pi))^3 \delta(t)(\omega_f - \omega).
\]

We now apply Eq. (7) to the system under study, i.e. a particle in a Fermi gas. The heating power \(P_{\text{CSL}}(t) = \frac{dE_{\text{ROT}}(t)}{dt}\) reads:

\[
P_{\text{CSL}}(t) = \frac{d}{dt} \sum_i \sum_f N(\mathbf{k}_i)(1 - N(\mathbf{k}_f)) h\omega_f \mathbb{E}[|c_f(t)|^2], \tag{9}
\]

where \(N(\mathbf{k}_i)\) is the probability of the initial state having momentum \(\mathbf{k}_i\), and \((1 - N(\mathbf{k}_f))\) is the probability for the final state with momentum \(\mathbf{k}_f\) not to be occupied, otherwise the particle could not jump there because of the Pauli exclusion principle. Since \(\mathcal{N}(\mathbf{k}_i)\mathcal{N}(\mathbf{k}_f)\) and \(\mathbb{E}[|c_f|^2]\) are even, whereas \(\omega_f\) is odd, under the interchange \(i \leftrightarrow f\), the term containing \(\mathcal{N}(\mathbf{k}_i)\mathcal{N}(\mathbf{k}_f)\) makes a vanishing contribution to Eq. (9). The above expression then simplifies to:

\[
P_{\text{CSL}}(t) = \frac{d}{dt} \sum_i \sum_f \mathcal{N}(\mathbf{k}_i) h\omega_f \mathbb{E}[|c_f(t)|^2]. \tag{10}
\]

Using the standard box-normalization prescription, according to which in the limit \(L \to +\infty\):

\[
\frac{1}{L^3} \sum_{\mathbf{p}} g(\mathbf{p}) \to \frac{1}{(2\pi)^3} \int d\mathbf{p} g(\mathbf{p}), \tag{11}
\]

one obtains

\[
P_{\text{CSL}}(t) = \frac{L^3}{(2\pi)^3} \frac{d}{dt} \sum_i \mathcal{N}(\mathbf{k}_i) \int |d\mathbf{k}_f h\omega_f \mathbb{E}[|c_f(t)|^2], \tag{12}
\]

where we used \(\gamma = \lambda(\sqrt{\pi}r_c)^3\) and \(\sum_i \mathcal{N}(\mathbf{k}_i) = 1\). For the \(N\) atom case, the calculation of Appendix A shows that \(m_A\) in Eq. (15) is replaced by the total mass \(M = Nm_A\). This is the same result obtained from the study of phononic modes in matter [23, 44, 45].

### III. Neutron Stars

Neutron stars are small (radius \(\sim 10\,\text{km}\)) and dense (mass \(M \sim 1.4 - 4.2 \times 10^{30}\,\text{kg}\) and density \(\mu \sim 10^{17}\,\text{kg/m}^3\)), resulting from the collapsed cores of stars with mass above the Chandrasekhar limit [46]. After a first stage next to their formation, where they cool through emission of baryonic matter, the main cooling process is dominated by thermal emission of radiation [47, 48], which is described by the Stefan-Boltzmann law:

\[
P_{\text{rad}} = \sigma S T^4, \tag{16}
\]

where \(S\) is the surface of the neutron star, \(\sigma = 5.6 \times 10^{-8}\,\text{W m}^{-2}\text{K}^{-4}\) is the Stefan’s constant and \(T\) is the effective black-body temperature of the star. As a
reference value for the temperature we can consider $T = 0.28^{+0.19}_{-0.12} \times 10^6$ K, which refers to the neutron star PSR J 1840–1419 [49]. The radius is $R = 10$ km and the mass $M = 2 \times 10^{30}$ kg, equal to the solar mass, giving a density $\mu = 4.8 \times 10^{17}$ kg/m$^3$. Variation of $R$ and $M$, for typical dimensions of a neutron star, do not imply significant changes in the bounds on the CSL parameters.

IV. RESULTS AND DISCUSSION

Assuming that the neutron star’s thermal radiation emission is balanced by the heating effect due to the CSL noise, we impose $P_{\text{rad}} = P_{\text{CSL}}$. This gives an estimate of collapse rate:

$$\lambda = \frac{16 R^2 m_0^2 \pi r_c^2 T^4 \sigma}{3 M r^2},$$

(17)

where we assumed that the neutron star can be approximated by a sphere of radius $R$. The corresponding upper bound is shown in red in Fig. 1.

It is interesting to apply Eq. (17) to other objects in the Universe. Table I shows the values of the ratio $P_{\text{rad}}/M$ and the corresponding value of $\lambda/r_c^2$ for the planets in the Solar system, the Moon, the Sun and, as a comparison, that of the neutron star PSR J 1840–1419 analyzed above. Numbers show that Neptune gives the best ratio $\lambda/r_c^2$, which is more than 4 orders smaller than the neutron star’s one. The corresponding upper bound is identifed by continuous blue line in Fig. 1. These bounds are weaker than the already existing bounds, and are further weakened if one assumes a high-frequency cut off in the noise spectrum following the methods of [23, 55–58], or dissipative modification of the CSL model as shown in [14, 20, 59, 60].

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Appendix A: Field-theoretical calculation

We perform the same analysis presented in the main text, within the framework of quantum field theory. Let us consider the CSL Hamiltonian:

\[ \hat{H} = \hat{H}_0 + \hat{V}_{\text{CSL}}, \]  

where

\[ \hat{H}_0 = \sum_i \sum_{\tau} \sum_p E_{p\tau\tau} \hat{b}_{p\tau\tau}^\dagger(t) \hat{b}_{p\tau\tau}(t), \]  

is the free Hamiltonian; the first sum is over the i-type of particle, the second sum over the spin \( \tau \) (i-th type of particle) and the third over momentum. Here \( \hat{b}_{p\tau\tau}^\dagger \) and \( \hat{b}_{p\tau\tau} \) are creation and annihilation operators respectively: since the final result is independent from the particle nature, they can be fermionic or bosonic. In fact, the derivation presented below depends only on the following commutation relations:

\[ \left[ \hat{b}_{p\tau\tau}^\dagger, \hat{b}_{p'\tau'\tau'} \right] = \delta^{(3)}(p' - k) \delta_{\tau'\tau} \delta_{p\tau\tau}, \]  

\[ \left[ \hat{b}_{p\tau\tau}, \hat{b}_{p'\tau'\tau'}^\dagger \right] = -\delta^{(3)}(p' - k) \delta_{\tau'\tau} \delta_{p\tau\tau}, \]
is [7]:

$$\hat{V}_{\text{ext}} = -\hbar \sum_j \sum_r \frac{m_j}{m_0} \int dx \hat{\Psi}_{\tau j}^\dagger (x, t) \hat{\Psi}_{\tau j} (x, t) \xi (x, t),$$  \hspace{1cm} (A3)$$

Here we introduced:

$$\xi (x, t) = \int dy e^{-\frac{(x-y)^2}{2\sigma_0^2}} \frac{u_t (y)}{(\sqrt{2\pi} \sigma_c)^3},$$  \hspace{1cm} (A4)$$

whose mean and correlator are:

$$\mathbb{E}[\xi (x, t)] = 0, \quad \text{and} \quad \mathbb{E}[\xi (x, t) \xi (y, s)] = \tilde{\gamma} (t-s) F (x-y),$$  \hspace{1cm} (A5)$$

where $\mathbb{E}$ denotes the stochastic average over the noise,

$$F (x) = \frac{e^{-x^2/4\sigma_0^2}}{(\sqrt{\pi} \sigma_c)^3}, \quad \text{and} \quad \tilde{\gamma} (t) = \frac{1}{2\pi} \int \omega \gamma (\omega) e^{-i\omega t}.$$  \hspace{1cm} (A6)$$

The relation between the operator $\hat{\Psi}_{\tau j} (x, t)$ and $\hat{b}_{p\tau j} (t)$ is given by

$$\hat{\Psi}_{\tau j} (x, t) = \sum_p \psi_{p\tau j} (x) \hat{b}_{p\tau j} (t),$$  \hspace{1cm} (A7)$$

$$\hat{b}_{p\tau j} (t) = \int dx \psi_{p\tau j}^* (x) \hat{\Psi}_{\tau j} (x, t),$$

with $\psi_{p\tau j} (x)$ denoting the Fourier coefficients of the transformation, spin $\tau$ and of momentum $p$. Below we specify the exact form of $\psi_{p\tau j} (x)$. The evolution of $\hat{b}_{p\tau i} (t)$ is determined by the Heisenberg equation $\frac{d\hat{b}_{p\tau j} (t)}{dt} = \frac{i}{\hbar} [\hat{H}, \hat{b}_{p\tau j} (t)],$ which gives

$$\frac{d\hat{b}_{p\tau j} (t)}{dt} = -\frac{i}{\hbar} E_{p\tau j} \hat{b}_{p\tau j} (t) + \frac{m_j}{m_0} \sum_k \int dx \psi_{p\tau j}^* (x) \psi_{k\tau j} (x) \xi (x, t) \hat{b}_{k\tau j} (t).$$  \hspace{1cm} (A8)$$

The solution is:

$$\hat{b}_{p\tau j} (t) = e^{-\frac{i}{\hbar} E_{p\tau j} t} \hat{b}_{p\tau j} (0) + \frac{m_j}{m_0} \sum_k \int dx \psi_{p\tau j}^* (x) \psi_{k\tau j} (x) \int_0^t ds e^{-\frac{i}{\hbar} E_{p\tau j} (t-s)} \xi (x, s) \hat{b}_{k\tau j} (s).$$  \hspace{1cm} (A9)$$

Since $\hat{b}_{k\tau j} (s)$ appears also in the last term, we need to solve perturbatively. We replace $\hat{b}_{k\tau j} (s)$ with the corresponding form given again by Eq. (A9), and truncate the expression to order $\gamma$:

$$\hat{b}_{p\tau j} (t) = \hat{A}_{p\tau j} (t) + \hat{B}_{p\tau j} (t) + \hat{C}_{p\tau j} (t) + \mathcal{O}(\gamma^{3/2}),$$  \hspace{1cm} (A10)$$

where

$$\hat{A}_{p\tau j} (t) = e^{-\frac{i}{\hbar} E_{p\tau j} t} \hat{b}_{p\tau j} (0),$$

$$\hat{B}_{p\tau j} (t) = \frac{m_j}{m_0} \sum_k \int dx \psi_{p\tau j}^* (x) \psi_{k\tau j} (x) \int_0^t ds e^{-\frac{i}{\hbar} E_{p\tau j} (t-s)} \xi (x, s) e^{-\frac{i}{\hbar} E_{k\tau j} s} \hat{b}_{k\tau j} (0),$$

$$\hat{C}_{p\tau j} (t) = -\left( \frac{m_j}{m_0} \right)^2 \sum_{kk'} \int dx \psi_{p\tau j}^* (x) \psi_{k\tau j} (x) \int_0^t ds e^{-\frac{i}{\hbar} E_{p\tau j} (t-s)} \xi (x, s) \int dy \psi_{k'\tau j}^* (y) \psi_{k'\tau j} (y) \times \int_0^t \int_0^{t'} ds' e^{-\frac{i}{\hbar} E_{k\tau j} (t-s') \xi (y, s') e^{-\frac{i}{\hbar} E_{k\tau j} s'} \hat{b}_{k\tau j} (0).$$

Given these expressions, we can compute the evolution of the energy expectation value, which is given by

$$E_{\text{tot}} (t) = \mathbb{E} [\hat{H}].$$  \hspace{1cm} (A12)$$
Due to the stochastic properties in Eq. (A5), we have $\mathbb{E}[\tilde{\mathcal{V}}_{csl}] = 0$, therefore only $\tilde{H}_0$ contributes to $E_{\text{tot}}(t)$. In particular

$$E_{\text{tot}}(t) = E_{\text{tot}}(0) + E_{\text{tot}}^{\text{CSL,1}}(t) + E_{\text{tot}}^{\text{CSL,2}}(t) + O(\gamma^{3/2}),$$  

(A13)

where

$$E_{\text{tot}}(0) = \sum_i \sum_\tau \sum_p E_{pr\tau} \langle \hat{B}_{pr\tau}(0) \hat{B}_{pr\tau} \rangle,$$

$$E_{\text{tot}}^{\text{CSL,1}}(t) = \sum_i \sum_\tau \sum_p E_{pr\tau} \langle \hat{B}_{pr\tau}^{\dagger}(t) \hat{B}_{pr\tau} \rangle,$$

$$E_{\text{tot}}^{\text{CSL,2}}(t) = \sum_i \sum_\tau \sum_p E_{pr\tau} \langle \hat{A}_{pr\tau}^{\dagger}(t) \hat{C}_{pr\tau} + \text{H.C.} \rangle,$$

(A14)

where H.C. stands for hermitian conjugate. We notice that there is no contribution from terms like $\hat{A}_{pr\tau}^{\dagger}(t) \hat{B}_{pr\tau}(t)$ or $\hat{B}_{pr\tau}^{\dagger}(t) \hat{C}_{pr\tau}(t)$: the first is zero under stochastic average and the second scales with $\gamma^{3/2}$ and can be then neglected. The above expressions, together with Eq. (A11), give:

$$E_{\text{tot}}(0) = \sum_i \sum_\tau \sum_p E_{pr\tau} \langle \hat{b}_{pr\tau}^{\dagger}(0) \hat{b}_{pr\tau}(0) \rangle,$$

$$E_{\text{tot}}^{\text{CSL,1}}(t) = \sum_i \sum_\tau \sum_p E_{pr\tau} \left( \frac{m_i}{m_0} \right)^2 \int \; d\mathbf{x} \int d\mathbf{y} \psi_{pr\tau}(\mathbf{x}) \psi_{pr\tau}^*(\mathbf{y}) f(\mathbf{x} - \mathbf{y}) \times$$

$$\int_0^t ds \int_0^t ds' \tilde{\gamma}(s - s') e^{-\frac{1}{\hbar} E_{pr\tau}(s-s')} e^{-\frac{1}{\hbar} E_{pr\tau}(s-s')} \langle \hat{b}_{pr\tau}^{\dagger}(0) \hat{b}_{pr\tau}^{\dagger}(0) \rangle,$$

$$E_{\text{tot}}^{\text{CSL,2}}(t) = -\sum_i \sum_\tau \sum_p E_{pr\tau} \left( \frac{m_i}{m_0} \right)^2 \int \; d\mathbf{x} \int \; d\mathbf{y} \psi_{pr\tau}(\mathbf{x}) \psi_{pr\tau}^*(\mathbf{y}) \int_0^t \; d\mathbf{s} \int_0^t \; d\mathbf{s}' \tilde{\gamma}(s - s') \times$$

$$\left[ \psi_{pr\tau}^*(\mathbf{x}) \psi_{pr\tau}(\mathbf{y}) e^{\frac{i}{\hbar} E_{pr\tau}(s-s')} e^{\frac{i}{\hbar} E_{pr\tau}(s-s')} \langle \hat{b}_{pr\tau}^{\dagger}(0) \hat{b}_{pr\tau}^{\dagger}(0) \rangle + \psi_{pr\tau}(\mathbf{y}) \psi_{pr\tau}^*(\mathbf{x}) e^{\frac{i}{\hbar} E_{pr\tau}(s-s')} e^{\frac{i}{\hbar} E_{pr\tau}(s-s')} \langle \hat{b}_{pr\tau}^{\dagger}(0) \hat{b}_{pr\tau}(0) \rangle \right].$$

(A15)

The above terms contain $\langle \hat{b}_{pr\tau}^{\dagger}(0) \hat{b}_{pr\tau}^{\dagger}(0) \rangle$. To compute it we consider a state of $N$ particles with density matrix diagonal in momentum and weight given by the occupation number $N(p)$. Then we have

$$\langle \hat{b}_{pr\tau}^{\dagger}(0) \hat{b}_{pr\tau}(0) \rangle = \delta_{pk} N(p).$$

(A16)

Although $N(p)$ is different in the fermionic and the bosonic case, as it should be clear from the calculations, the final result is independent from the type of statistics. Applying this result we obtain

$$E_{\text{tot}}(0) = \sum_i \sum_\tau \sum_p E_{pr\tau} N(p),$$

$$E_{\text{tot}}^{\text{CSL,1}}(t) = t \sum_i \sum_\tau \sum_{pk} \left( \frac{m_i}{m_0} \right)^2 N(k) \int \; d\mathbf{x} \int \; d\mathbf{y} \psi_{pr\tau}(\mathbf{x}) \psi_{pr\tau}^*(\mathbf{y}) \psi_{pr\tau}(\mathbf{y}) f(\mathbf{x} - \mathbf{y}) \times$$

$$\int \; d\omega \; \gamma(\omega) \delta(t) \left( \frac{E_{pr\tau} - E_{pr\tau}}{\hbar} - \omega \right),$$

(A17)

$$E_{\text{tot}}^{\text{CSL,2}}(t) = -t \sum_i \sum_\tau \sum_{pk} N(p) \left( \frac{m_i}{m_0} \right)^2 \int \; d\mathbf{x} \int \; d\mathbf{y} \psi_{pr\tau}(\mathbf{x}) \psi_{pr\tau}^*(\mathbf{y}) \psi_{pr\tau}(\mathbf{y}) f(\mathbf{x} - \mathbf{y}) \times$$

$$\int \; d\omega \; \gamma(\omega) \delta(t) \left( \frac{E_{pr\tau} - E_{pr\tau}}{\hbar} - \omega \right),$$

where we exploited the relations in Eq. (8) and Eq. (A6).
So far the result is general. We now apply it to the case of interest, i.e. $N$ particles in a cube box of length $L$. We apply the periodic boundary conditions and the box-normalization prescription

$$
\psi_{p\tau_i}(x) \rightarrow \phi_{q\tau}(x) = e^{i \frac{q \cdot x}{L^{3/2}}}, \quad \text{with} \quad q_{\tau} = \frac{2\pi}{L} n_{\tau_i},
$$

where $n_{\tau_i} \in \mathbb{Z}^3$. The wavefunctions $\phi_{q\tau}(x)$ are orthonormal

$$
\int_{-L/2}^{L/2} \int_{-L/2}^{L/2} \int_{-L/2}^{L/2} d^3x \phi_q(x) \phi_q^*(x) = \delta_{nn'}.
$$

In the $L \rightarrow +\infty$ limit (so that space-integrals extend over the whole space and can be performed exactly) we have

$$
E_{CSL,1}^{TOT}(t) = t \sum_i \sum_{\tau} \sum_p E_{p\tau} \left( \frac{m_i}{m_0} \right)^2 \mathcal{N}(p) \frac{1}{L^3} \sum_k e^{-\frac{(p-k)^2 r_c^2}{C}} \left( E_{k\tau_i} - E_{p\tau_i} \right) \gamma(t) \delta \left( \frac{E_{k\tau_i} - E_{p\tau_i}}{\hbar} - \omega \right),
$$

$$
E_{CSL,2}^{TOT}(t) = -t \sum_i \sum_{\tau} \sum_p E_{p\tau} \mathcal{N}(p) \left( \frac{m_i}{m_0} \right)^2 e^{-\frac{(p-k)^2 r_c^2}{C}} \left( E_{k\tau_i} - E_{p\tau_i} \right) \gamma(t) \delta \left( \frac{E_{k\tau_i} - E_{p\tau_i}}{\hbar} - \omega \right),
$$

The CSL heating power $P_{CSL} = \frac{d}{dt} E_{CSL,1}^{TOT}(t)$ in the long time limit is then given by

$$
P_{CSL} = \sum_i \sum_{\tau} \sum_p \mathcal{N}(p) \left( \frac{m_i}{m_0} \right)^2 \frac{1}{L^3} \sum_k e^{-\frac{(p-k)^2 r_c^2}{C}} \left( E_{k\tau_i} - E_{p\tau_i} \right) \frac{\gamma}{L^3} \sum_k e^{-\frac{(p-k)^2 r_c^2}{C}} (E_{k\tau_i} - E_{p\tau_i}) \gamma \left( \frac{E_{k\tau_i} - E_{p\tau_i}}{\hbar} \right).
$$

In the white noise case, where $\gamma(\omega) = \gamma = \lambda (2\sqrt{\pi r_c})^3$, by taking $E_{k\tau_i} = \frac{\hbar^2 k^2}{2m_i}$ we find

$$
\frac{\gamma}{L^3} \sum_k e^{-\frac{(p-k)^2 r_c^2}{C}} (E_{k\tau_i} - E_{p\tau_i}) \xrightarrow{L \rightarrow +\infty} \frac{3\hbar^2 \lambda}{4m_0 r_c^2}.
$$

By merging with Eq. (A21) we have:

$$
P_{CSL} = 3\frac{h^2 \lambda}{4m_0^2 r_c^2} \sum_i m_i \sum_{\tau} \sum_p \mathcal{N}(p) = 3\frac{h^2 \lambda M}{4m_0^2 r_c^2},
$$

since that $\sum_{\tau} \sum_p \mathcal{N}(p)$ gives the number of particle of type $i$. 