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Robust Newsvendor Games with Ambiguity in Demand Distributions

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In classical newsvendor games, vendors collaborate to serve their aggregate demand whose joint distribution is assumed known with certainty. We investigate a new class of newsvendor games with ambiguity in the joint demand distributions, which is represented by a Fréchet class of distributions with some, possibly overlapping, marginal information. To model this new class of games, we use ideas from distributionally robust optimization to handle distributional ambiguity and study the robust newsvendor games. We provide conditions for the existence of core solutions of these games using the structural analysis of the worst-case joint demand distributions of the corresponding distributionally robust newsvendor optimization problem.

Key words: Cooperative games; uncertain payoffs; newsvendor games; robust optimization; stability

1. Introduction

A joint venture is usually an effective approach for individual players in the market to share costs, reduce risk, and increase the total joint revenue or profit. For example, individual retailers can decide whether to order inventories together and share the (expected) profit from selling ordered products later. Cooperative game theory provides a mathematical framework for addressing this problem, which is modeled as a *newsvendor centralization game* (or newsvendor game for short) (see, e.g., Hartman [15]).

Traditionally, the underlying assumption in newsvendor games is that the joint demand distribution is known with certainty. In reality, it can be sometimes difficult to justify this assumption, especially when retailers need to decide whether to cooperate with each other before observing and sharing records of joint demands. In this paper, we focus on newsvendor games with *uncertain payoffs*, i.e., expected profits, with uncertainty captured and represented in the payoff functions of the games. Our research departs from existing models of uncertain cooperative games with a different approach of how to represent and handle uncertainty. We are going to model the uncertainty of the payoffs through an *uncertainty set* derived directly from the ambiguity of the joint demand distribution as in the framework of distributionally robust optimization.

Given that individual retailers usually collect historical demands independently before they join any coalition, it is reasonable to assume the knowledge of joint demand distributions of subsets of retailers, e.g., those that are located close to each other. Therefore, we focus on uncertainty sets constructed with known (multivariate) marginal distributions in this paper. The sets of joint distributions with fixed marginal distributions are referred to as Fréchet classes of distributions (see, e.g., Rüschendorf [25]) with applications in risk management [11], project management [7] and portfolio optimization [8]. We provide further details about how our research fits in and departs from the broader literature of uncertain cooperative games and newsvendor games in Section 1.2.

1.1. Contributions and paper outline

In this paper, we focus on the newsvendor games with ambiguity in the joint demand distribution represented by Fréchet classes of distributions. Specifically, our contributions and the structure of the paper are as follows:

(1) We propose a new framework based on the principles of robust optimization to handle newsvendor games with ambiguity in demand distributions, which we call robust newsvendor games, in Section 2.

(2) We provide conditions of the existence of individually rational and stable payoff distributions of these robust newsvendor games under distributional ambiguity represented by Fréchet classes of distributions in Section 3. These conditions are shown using the structural analysis of the worst-case joint demand distributions of the corresponding distributionally robust newsvendor optimization problem.

1.2. Related Literature and Modeling Choices

The newsvendor games were first introduced by Hartman [15], which uses expected profit as the payoff. Dror et al. [9] refer to these newsvendor games as *newsvendor expected games* to distinguish them from *newsvendor realization games*, which focus on the realized profits given arbitrary demand realization. For newsvendor expected games, the uncertainty of random demand is absorbed in the calculation of the expected profit as payoff, which makes these games deterministic if the joint demand distribution is known with certainty. If one allows distributional ambiguity, the payoff becomes uncertain. In the literature, there are several different approaches of how to represent

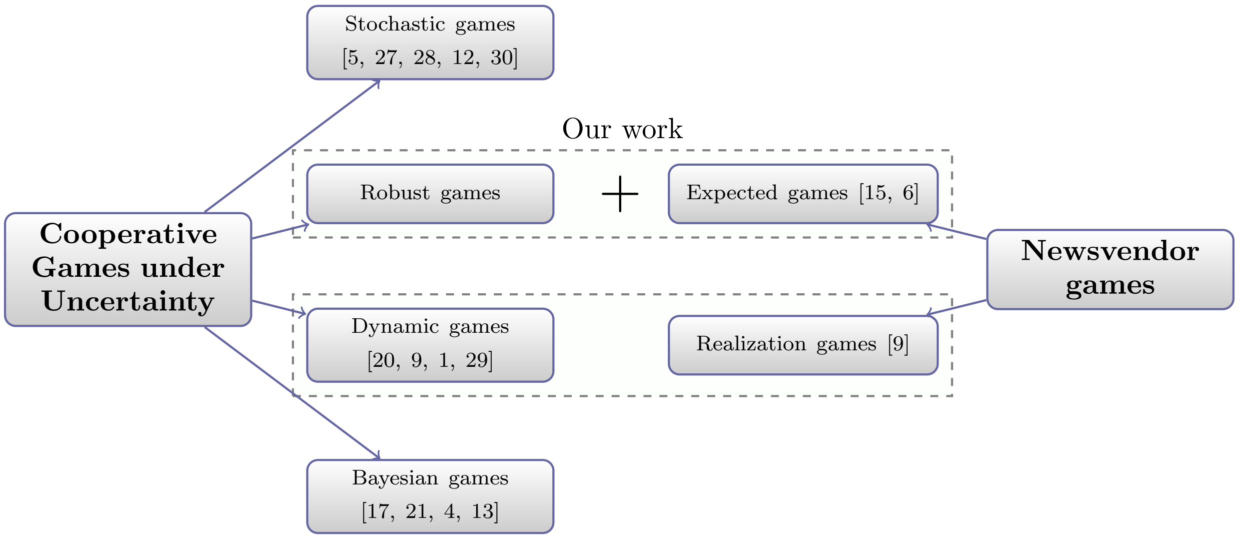

Figure 1 Modeling choices for cooperative games under uncertainty and newsvendor games

and handle uncertainty in cooperative games. Figure 1 provides a picture of what these modeling choices are and how they might be applied in the context of newsvendor games.

The first approach is stochastic cooperative games where the payoff of each individual coalition is assumed to be a *random variable* with known distribution. Charnes and Granot [5] propose a two-stage payoff distribution scheme. This approach assumes risk-neutral behaviors among players. Suijs et al. [27] study a different payoff distribution scheme for stochastic cooperative games using preference orders for random payoffs, which can handle different types of risk behavior. Timmer et al. [28] propose a similar payoff distribution scheme for stochastic cooperative games without monetary exchange in which a multiple of the random coalition payoff is allocated to each individual. Fernández et al. [12] also consider this payoff distribution scheme as a special case while investigating general stochastic payoff distributions using stochastic orders for stochastic cooperative games. Uhan [30] generalizes the payoff distribution scheme for stochastic linear programming games with applications in inventory centralization and network fortification.

Recently, motivated by the centralized inventory allocation problem, new cooperative game models have been proposed to handle the dynamic nature of the games. Dror et al. [9] study repeated stochastic cooperative games and apply them to dynamic newsvendor realization games. Bauso and Timmer [1], on the other hand, study dynamic cooperative games under the setting of a family of games whose coalition values are uncertain. Toriello and Uhan [29] investigate the dynamic linear programming games with risk-averse players. Lehrer [20] investigates payoff distribution processes in repeated deterministic cooperative games and shows that these processes converge to some well-known solutions of cooperative games under appropriate allocation rules.

The third approach is Bayesian cooperative games that study cooperative games with incomplete information [17, 21, 4, 13]. This body of literature models situations where there is asymmetry in what the players know about the other players, their contributions, and the states of the world.

In this paper, we focus on payoff sharing schemes of newsvendor expected games under distributional ambiguity. As discussed in Chen and Zhang [6], these sharing schemes of newsvendor expected games can be used to allocate actual profits given an arbitrary demand realization. We use principles of robust optimization to find payoff allocation schemes which makes these games stable under all possible realizations of the joint demand distribution, i.e., to immunize against the uncertainty. We call the resulting games from this approach robust games and in this paper, we shall analyze the existence of core solutions of robust newsvendor games.

2. Framework for Robust Newsvendor Games

2.1. Newsvendor Games

Consider the set $\mathcal{N}$ of N retailers and let $\tilde{d}_i \in \mathbb{R}_+$ be the random demand for retailer i, $i \in \mathcal{N}$. In the setting of newsvendor games, we assume that the unit ordering cost c and the unit selling price p are the same for all retailers, $0 < c < p$. Given an ordering quantity y, the expected profit (or payoff) of retailer i is

$$v_i(y) = \mathbb{E}_{P_i}\left[p\min\{\tilde{d}_i, y\} - cy\right], \quad i \in \mathcal{N}, \tag{1}$$

where P_i is the distribution function of $\tilde{d}_i$ for all $i \in \mathcal{N}$. Individual retailer i needs to decide the optimal ordering quantity y_i^* to maximize the expected profit (or payoff), $y_i^* \in \arg\max_{y\geq 0} v_i(y)$, which is the $(p-c)/p$-quantile of P_i. The optimal expected profit is $\bar{v}_i = v_i(y_i^*) = \mathbb{E}_{P_i}\left[p\min\{\tilde{d}_i, y_i^*\} - cy_i^*\right]$.

In the setting of newsvendor games, individual retailers consider whether they should form a coalition to make orders together, share inventories with each other, and serve their aggregate demand. For a coalition $\mathcal{S} \subseteq \mathcal{N}$, the aggregate demand is $\tilde{d}(\mathcal{S}) = \sum_{i\in\mathcal{S}} \tilde{d}_i$, which follows the joint distribution $P(\mathcal{S})$ of $\tilde{d}_i$, $i \in \mathcal{S}$. Given an ordering quantity y, the total expected profit is

$$v(y, \mathcal{S}) = \mathbb{E}_{P(\mathcal{S})}\left[p\min\{\tilde{d}(\mathcal{S}), y\} - cy\right]. \tag{2}$$

Similarly, coalition $\mathcal{S}$ of retailers needs to decide the optimal ordering quantity $y^*(\mathcal{S})$ to maximize the total expected profit,

$$y^*(\mathcal{S}) \in \arg\max_{y\geq 0} v(y, \mathcal{S}), \tag{3}$$

which again is the $(p-c)/p$-quantile of the distribution of $\tilde{d}(\mathcal{S})$. The optimal expected profit is

$$\bar{v}(\mathcal{S}) = v(y^*(\mathcal{S}), \mathcal{S}) = \mathbb{E}_{P(\mathcal{S})}\left[p\min\{\tilde{d}(\mathcal{S}), y^*(\mathcal{S})\} - cy^*(\mathcal{S})\right]. \tag{4}$$

This expected profit function $\bar{v}(\cdot)$ is the *characteristic function* of newsvendor games under the assumption that the joint demand distribution is known with certainty. We are interested in finding an allocation $\boldsymbol{x} \in \mathbb{R}^N$ to distribute the total expected profit $\bar{v}(\mathcal{N})$ among individual retailers. As discussed in Chen and Zhang [6], if one focuses on the allocation of the actual profit given each demand realization, any allocation with respect to the expected profit of newsvendor games can be considered as a profit *allocation rule* that retailers can agree on before the demand is realized.

Among allocations of newsvendor games, we are interested in *imputations*, allocations which are *efficient*, i.e., $\sum_{i\in\mathcal{N}} x_i = \bar{v}(\mathcal{N})$, and *individually rational*, i.e., $x_i \geq \bar{v}(\{i\})$ for all $i \in \mathcal{N}$ or equivalently, all individual retailers are better off in joining the grand coalition $\mathcal{N}$. Individual rationality is not sufficient to guarantee that some retailers would prefer the grand coalition $\mathcal{N}$ to a smaller coalition $\mathcal{S} \subsetneq \mathcal{N}$. We are therefore, also interested in *core allocations*, allocations which are efficient and *stable*, i.e., $\sum_{i\in\mathcal{S}} x_i \geq \bar{v}(\mathcal{S})$ for all coalitions $\mathcal{S} \subsetneq \mathcal{N}$.

For newsvendor games, Hartman [15] shows that the characteristic function $\bar{v}$ is super-additive. This shows that newsvendor games always have imputations. For normally distributed demands, Hartman et al. [16] show that the cores of these games are non-empty. Müller et al. [23] prove that every newsvendor game has a non-empty core no matter what distributions of random demands are. Chen and Zhang [6] use stochastic linear programming duality to show the non-emptiness of the core in inventory centralization games and to provide a constructive approach for finding one. For newsvendor games, this dual approach provides a simple closed-form solution. Similarly, Montrucchio and Scarsini [22] also show how to construct an allocation in the core of newsvendor games. In the next section, we study solution concepts of newsvendor games under the ambiguity of the joint demand distribution.

2.2. Robust Newsvendor Games

We consider the situation when the joint demand distribution P is uncertain, i.e., P belongs to an ambiguity set $\mathcal{P}$ such as a Fréchet class of distributions. This makes the payoffs of coalitions uncertain and the above framework of newsvendor games cannot be applied any more. Sujis et al. [27] propose a general framework for cooperative games with stochastic payoffs which requires additional elements such as actions of coalitions, payoff allocation rules, and individual preferences of stochastic payoffs which are used to define the stability of the grand coalition. We use this extended structure to develop a new framework for newsvendor games under distributional ambiguity with a relevant payoff allocation rule and a new approach to define stability based on the principles of robust optimization.

2.2.1. Retailer Actions For newsvendor games, the action which retailers can take is to determine the ordering quantity. Here, we note that, if retailers in a coalition $\mathcal{S} \subseteq \mathcal{N}$ do not fully know their joint demand distribution, there is no ordering quantity to achieve the "optimal" expected profit, i.e., the retailers can take different actions when facing uncertainty. Given an ordering quantity y, the total expected profit $v_P(y,\mathcal{S}) = \mathbb{E}_{P(\mathcal{S})}\left[p\min\{\tilde{d}(\mathcal{S}), y\} - cy\right]$ is uncertain since in general, the marginal demand distribution $P(\mathcal{S})$ of $P \in \mathcal{P}$ also belongs to an ambiguity set. This leads us to the discussion of payoff allocation rules and the break-away incentive next.

2.2.2. Profit Allocation Rule Facing the uncertain expected profit $v_P(y,\mathcal{S})$, $P \in \mathcal{P}$, we are interested in finding a profit allocation rule that retailers can agree on before joining the coalition. In this paper, we use a proportional payoff distribution scheme which follows the natural process of joint-venture negotiation where the stakeholders agree on some proportions of their shares of the future unknown profits. It is also motivated by the discussion on how the actual profit might be allocated (proportionally) given each demand realization mentioned in Chen and Zhang [6, Remark 3] using the allocation of expected profit in newsvendor games. In the context of cooperative games, the proportional rule is first used by Timmer et al. [28]. Formally, for a coalition $\mathcal{S}$, an allocation rule is represented by $\boldsymbol{z} \in \mathbb{R}^{|\mathcal{S}|}$ such that for a given ordering quantity y, the uncertain allocation for each retailer i, $i \in \mathcal{S}$, is $x_i^P = v_P(y,\mathcal{S}) \cdot z_i$ for $P \in \mathcal{P}$. An allocation rule is *efficient* if $\sum_{i\in\mathcal{S}} z_i = 1$, which implies the uncertain allocation $\boldsymbol{x}^P$ is efficient given any realization of $P \in \mathcal{P}$.

REMARK 1. When $\mathcal{P}$ is a singleton, i.e., $\mathcal{P} = \{\bar{P}\}$, the set of all possible allocation rules $\boldsymbol{z} \in \mathbb{R}^{|\mathcal{S}|}$ covers every possible allocation $\boldsymbol{x}_{\mathcal{S}}$ with the corresponding expected profit of $\bar{v}(\mathcal{S}) = v_{\bar{P}}(y^*(\mathcal{S}),\mathcal{S})$ for a coalition $\mathcal{S}$ if $\bar{v}(\mathcal{S}) \neq 0$. Restrictions happen only when $\bar{v}(\mathcal{S}) = 0$. In this case, any allocation rule $\boldsymbol{z} \in \mathbb{R}^{|\mathcal{S}|}$ will result in $\boldsymbol{x}_{\mathcal{S}} = \mathbf{0}$. However, for newsvendor games, $\bar{v}(\mathcal{S})$ is generally postive under some mild assumptions on $\bar{P}$ and we can claim that in general the proposed allocation rule for uncertain newsvendor games with the class of distribution $\mathcal{P}$ reduces to the actual allocation for deterministic newsvendor games when $\mathcal{P}$ is a singleton.

REMARK 2. With the proportional allocation rule, we make an implicit assumption that there is no monetary exchange in the contract agreement between retailers in our proposed newsvendor game model. Monetary exchange is considered in some other allocation rules such as the one proposed by Suijs et al. [27]. The allocation rule proposed by Suijs et al. [27] is represented by $(\boldsymbol{d},\boldsymbol{r}) \in \mathbb{R}^{|\mathcal{S}|} \times \mathbb{R}_+^{|\mathcal{S}|}$ with d_i as the monetary exchange component in the uncertain allocation $x_i^P = d_i + v_P(y,\mathcal{S}) \cdot r_i$ for retailer i, $i \in \mathcal{S}$, given an ordering quantity y. The proposed framework can be developed with Suijs' allocation rule (or any other allocation rules) together with the definition of break-away incentive in Section 2.2.3; however, the analysis of existence of imputations and core decisions established in the paper using $v_{\max}(y, \mathcal{S})$ is only applied to the proportional allocation rule (see Lemmas 1 and 3 for more details). It cannot be applied directly to other allocation rules and different analysis techniques will be required.

2.2.3. Break-Away Incentive Another important consideration of newsvendor games is how to characterize the incentive to break away from the grand coalition of individuals as well as coalitions of retailers. When the expected profit function $\bar{v}(\cdot)$ is known, an individual retailer i has the incentive to break away if $\bar{v}(\{i\}) > x_i$, where $\boldsymbol{x}$ is the allocation considered by the grand coalition. A coalition $\mathcal{S} \subsetneq \mathcal{N}$ has the incentive to break away if there exists an efficient allocation $\hat{\boldsymbol{x}}$ of $\bar{v}(\mathcal{S})$, i.e., $\sum_{i \in \mathcal{S}} \hat{x}_i = \bar{v}(\mathcal{S})$, which *strongly dominates* the allocation $\boldsymbol{x}$ of the grand coalition, i.e., $\hat{x}_i > x_i$ for all $i \in \mathcal{S}$ (see, e.g., Gillies [14]). The concept of dominance (and Pareto optimality) has been widely used in economics since it was first introduced by Pareto [24]. In mathematical optimization, it was used to describe *efficient* solutions of (vector) optimization problems in the seminal paper by Kuhn and Tucker [19] and eventually became a cornerstone of multi-objective optimization (see, e.g. Ehrgott [10]).

When facing uncertainty in multi-objective optimization problems, Bitran [3] defined the concepts of *necessary* and *sufficient efficiency* by considering efficiency in each realization of uncertainty. These realization-based concepts of efficiency for multi-objective optimization under uncertainty appear again under different names, *flimsily* and *highly robust efficiency*, in Ide and Schöbel [18]. In this paper, we use a similar realization-based approach to define the break-away incentive of coalitions of retailers. This approach is different from the individual-based approaches where preferences of retailers are redefined based on their uncertain payoffs (see, e.g., Sujis and Borm [26]). Our proposed realization-based approach uses the standard allocation domination for fixed distributions $P \in \mathcal{P}$ to define the break-away incentive as follows.

Let us consider an *efficient decision* $(y, \boldsymbol{z})$ with ordering quantity y and efficient allocation rule $\boldsymbol{z}$ for the grand coalition $\mathcal{N}$. Given a fixed distribution $\bar{P} \in \mathcal{P}$, the resulting allocations, $x_i^{\bar{P}} = v_{\bar{P}}(y, \mathcal{N}) \cdot z_i$, $i \in \mathcal{N}$, are known with certainty. Similarly, a coalition $\mathcal{S} \subsetneq \mathcal{N}$ with an efficient decision $(\hat{y}, \hat{\boldsymbol{z}})$ will also obtain well-defined allocation with $\hat{x}_i^{\bar{P}} = v_{\bar{P}(\mathcal{S})}(\hat{y}, \mathcal{S}) \cdot \hat{z}_i$ for all $i \in \mathcal{S}$, given the fixed distribution $\bar{P} \in \mathcal{P}$. Assuming $\bar{P}$ is the realized distribution, as in the deterministic newsvendor games with a known joint demand distribution, it is clear that $\mathcal{S}$ has the incentive to break away if there exists a decision $(\hat{y}, \hat{\boldsymbol{z}})$ whose resulting allocation $\hat{\boldsymbol{x}}$ strongly dominates the allocation $\boldsymbol{x}$ obtained from the grand coalition, i.e., $\hat{x}_i^{\bar{P}} > x_i^{\bar{P}}$ for all $i \in \mathcal{S}$. We can use this standard break-away incentive given realized distributions to define the break-away incentive when facing the uncertain distribution $P \in \mathcal{P}$ as follows.

Definition 1. Given the ambiguity set $\mathcal{P}$ and an efficient decision $(y, \boldsymbol{z})$ of the grand coalition $\mathcal{N}$, a coalition $\mathcal{S} \subsetneq \mathcal{N}$ has the incentive to break away from the grand coalition $\mathcal{N}$ if there exists an efficient decision $(\hat{y}, \hat{\boldsymbol{z}})$ for the coalition $\mathcal{S}$ such that the resulting allocation $\hat{\boldsymbol{x}}^{\bar{P}}(\hat{y}, \hat{\boldsymbol{z}})$ strongly dominates the allocation $\boldsymbol{x}^{\bar{P}}(y, \boldsymbol{z})$ of the grand coalition *under at least one realization of* $\bar{P} \in \mathcal{P}$.

Remark 3. When $\mathcal{P}$ is a singleton, i.e., $\mathcal{P} = \{\bar{P}\}$, given the relationship between allocation rules and their resulting allocations, it is clear that the proposed definition of the break-away incentive reduces to that of the standard break-away incentive since there is only a single distribution.

Given this definition of break-away incentive, we can define imputations and core decisions. An imputation is an *individually rational* decision $(y, \boldsymbol{z})$ if no individual retailer has the incentive to break away from the grand coalition and $\boldsymbol{z}$ is an efficient allocation rule. Similarly, a core decision is defined as a *stable* decision $(y, \boldsymbol{z})$, i.e., no coalition $\mathcal{S} \subsetneq \mathcal{N}$ has the incentive to break away from the grand coalition, where $\boldsymbol{z}$ is again an efficient allocation rule. A coalition $\mathcal{S} \subsetneq \mathcal{N}$ has no incentive to break away if no matter which efficient decision $(\hat{y}, \hat{\boldsymbol{z}})$ it takes and no matter which distribution $P \in \mathcal{P}$ is realized, not all of its members are better off. With this interpretation of the break-away incentive, the stability of the grand coalition is clearly immunized against uncertainty, which follows the principle of robust optimization (see Ben-Tal et al. [2] and references therein). We therefore call newsvendor games with this definition of the break-away incentive robust newsvendor games. In this paper, we are interested in the existence of imputations and core decisions of the proposed robust newsvendor games with Fréchet classes of demand distributions as ambiguity sets.

3. Robust Newsvendor Games with Fréchet Classes of Demand Distributions

Retailers usually need to make a decision whether to join a coalition without knowing the complete joint demand distribution. They normally collect their demand data independently and it is reasonable to assume that we know some marginal distributions, e.g., those of retailers located close to each other and serving customers from the same area. We use the information of these marginal distributions to represent the ambiguity in the joint demand distributions as follows.

Let us consider a cover of $\mathcal{N}$ with R subsets $\mathcal{N}_1, \ldots, \mathcal{N}_R$, $\mathcal{N}_r \subsetneq \mathcal{N}$ for all $r = 1, \ldots, R$, and $\mathcal{N} = \bigcup_{r=1}^{R} \mathcal{N}_r$. Given a vector $\boldsymbol{d} \in \mathbb{R}^N$, let $\boldsymbol{d}_r \in \mathbb{R}^{N_r}$ denote the sub-vector formed with the elements in the rth subset $\mathcal{N}_r$ where $N_r = |\mathcal{N}_r|$ is the size of the subset. We assume that probability measures P_r of random vectors $\tilde{\boldsymbol{d}}_r$ are known for all $r = 1, \ldots, R$. The Fréchet class of joint probability measures of the random vector $\tilde{\boldsymbol{d}}$ consistent with the prescribed probability measures of the random vectors $\tilde{\boldsymbol{d}}_r$ for all $r = 1, \ldots, R$ can be written as $\mathcal{P}(P_1, \ldots, P_R) = \left\{P \,|\, \mathrm{proj}_{\mathcal{N}_r}(P) = P_r,\, r = 1, \ldots, R\right\}$, where $\mathrm{proj}_{\mathcal{N}_r}(P)$ is the corresponding marginal joint distribution of $\tilde{d}_i,\ i \in \mathcal{N}_r$, derived from P.

Remark 4. We assume that the ambiguity set $\mathcal{P}(P_1,\ldots,P_R)$ is non-empty, which implies the necessary pairwise consistency requirements for the marginal distributions P_r, i.e., for all $r \neq s$, if $\mathcal{N}_r \cap \mathcal{N}_s \neq \emptyset$, $\text{proj}_{\mathcal{N}_r\cap\mathcal{N}_s}(P_r) = \text{proj}_{\mathcal{N}_r\cap\mathcal{N}_s}(P_s)$. Note that if the cover is a partition, i.e., $\mathcal{N}_r \cap \mathcal{N}_s$ for all $r \neq s$, we do not need the pairwise consistency requirements and the ambiguity set $\mathcal{P}(P_1,\ldots,P_R)$ is always non-empty with the independent measure among the sub-vectors as a feasible distribution.

Given a coalition $\mathcal{S} \subseteq \mathcal{N}$, the distribution $P(\mathcal{S}) := \text{proj}_{\mathcal{S}}(P)$, which is the corresponding marginal joint distribution of $\tilde{d}_i$, $i \in \mathcal{S}$, derived from P, is unknown in general and coalition $\mathcal{S}$ can choose the ordering quantity from $\mathcal{Y}(\mathcal{S})$, the set of feasible ordering quantities. To keep it simple, we shall let $\mathcal{Y}(\mathcal{S}) = \mathbb{R}_+$ given the fact that the ordering quantities are non-negative for all $\mathcal{S} \subsetneq \mathcal{N}$. If $\mathcal{S} \subseteq \mathcal{N}_r$ for some r, we shall restrict $\mathcal{Y}(\mathcal{S}) = \{y^*(\mathcal{S})\}$ since the joint distribution of $\tilde{d}_i$, $i \in \mathcal{S}$, is completely known. Given an ordering quantity $y \in \mathcal{Y}(\mathcal{S})$, as in (2), the expected profit of coalition $\mathcal{S}$ for $P \in \mathcal{P}(P_1,\ldots,P_R)$ is

$$v_P(y,\mathcal{S}) = \mathbb{E}_{P(\mathcal{S})}\left[p\min\{\tilde{d}(\mathcal{S}),y\} - cy\right] = (p-c)y - p\,\mathbb{E}_{P(\mathcal{S})}\left[\left(y - \tilde{d}(\mathcal{S})\right)^+\right]. \tag{5}$$

For the grand coalition $\mathcal{N}$, we shall consider only ordering quantities which guarantee positive profits under any circumstances, i.e., $\mathcal{Y}(\mathcal{N}) = \{y \in \mathbb{R}_+ \,|\, v_P(y,\mathcal{N}) > 0, \forall\, P \in \mathcal{P}(P_1,\ldots,P_R)\}$. This is a reasonable consideration, which implies that in general, the grand coalition would be beneficial for retailers to join in. The condition will be technically necessary for the analysis of the proposed allocation rule. In order to make sure that $\mathcal{Y}(\mathcal{N}) \neq \emptyset$, we impose the following simple assumption.

Assumption 1. *There is at least one subset $\mathcal{N}_r$ of retailers, $r = 1,\ldots,R$, whose joint demand is positive almost surely, i.e., $\mathbb{P}(\tilde{d}(\mathcal{N}_r) > 0) = 1$.*

This assumption of positive demand is quite reasonable in reality for a variety of essential products which everyone needs to buy regularly. We provide the detailed proof of the non-emptiness of $\mathcal{Y}(\mathcal{N})$ under Assumption 1 in Lemma 5 in Appendix A. Throughout the paper, for clarity of exposition, we only provide the statements of the theoretical results, their implications, and brief proof ideas where necessary, while leaving the detailed proofs in the appendices. We are now ready to characterize the properties of imputations and core solutions of the robust newsvendor games with the ambiguity set $\mathcal{P}(P_1,\ldots,P_R)$.

3.1. Existence of Imputations

The newsvendor games with known joint demand distributions always have imputations since the corresponding characteristic function is super-additive. We will show the same imputation existence result for the robust newsvendor game. For each $\mathcal{S} \subsetneq \mathcal{N}$, let us define

$$v_{\max}(y,\mathcal{S}) = \max_{P\in\mathcal{P}(P_1,\ldots,P_R)} \left\{\frac{\max_{y_s\in\mathcal{Y}(\mathcal{S})} v_P(y_s,\mathcal{S})}{v_P(y,\mathcal{N})}\right\}.$$

We first need the following lemma.

LEMMA 1. *Given a decision $(y, \boldsymbol{z})$ of the grand coalition, a retailer $i \in \mathcal{N}$ has no incentive to break away if and only if $z_i \geq v_{\max}(y, \{i\})$.*

The proof of the lemma is provided in Appendix B.1. Here, we literally transform Definition 1 into the condition for a retailer to break away from the grand coalition. Next, we study the worst-case optimal ordering quantity $y^*_{wc}(\mathcal{S})$, which will allow us to construct individually rational decisions used later to prove the existence of imputations of the proposed robust newsvendor games:

$$y^*_{wc}(\mathcal{S}) \in \arg\max_{y \geq 0} \left\{ (p-c)y - p \max_{P \in \mathcal{P}(P_1,\ldots,P_R)} \mathbb{E}_P \left[\left(y - \tilde{d}(\mathcal{S}) \right)^+ \right] \right\}. \tag{6}$$

Let $P_{wc}(\mathcal{S})$ denote the worst-case distribution and let us also define $v_{wc}(\mathcal{S})$ as the maximum worst-case expected profit for coalition $\mathcal{S}$, i.e.,

$$v_{wc}(\mathcal{S}) = (p-c)y^*_{wc}(\mathcal{S}) - p \cdot \mathbb{E}_{P_{wc}(\mathcal{S})} \left[\left(y^*_{wc}(\mathcal{S}) - \tilde{d}(\mathcal{S}) \right)^+ \right]. \tag{7}$$

When $\mathcal{S} \subseteq \mathcal{N}_r$ for some r, clearly, $y^*_{wc}(\mathcal{S}) = y^*(\mathcal{S})$, the $(p-c)/p$-quantile of the known distribution of $\tilde{d}(\mathcal{S})$, and $v_{wc}(\mathcal{S}) = \bar{v}(\mathcal{S})$ as defined in (4). In general, it will be difficult to analytically compute $y^*_{wc}(\mathcal{S})$ and $v_{wc}(\mathcal{S})$ given a general Fréchet class of distributions $\mathcal{P}(P_1, \ldots, P_R)$. However, when $\mathcal{N}_1, \ldots, \mathcal{N}_R$ is a partition or equivalently, a non-overlapping cover, i.e., $\mathcal{N}_r \cap \mathcal{N}_s = \emptyset$ for all $r \neq s$, we can calculate the worst-case optimal ordering quantity $y^*_{wc}(\mathcal{S})$ and the worst-case expected profit $v_{wc}(\mathcal{S})$ and characterize the structure of the worst-case distribution $P_{wc}(\mathcal{S})$ for an arbitrary $\mathcal{S} \subseteq \mathcal{N}$ using the following lemma.

LEMMA 2. *Assuming that $\mathcal{N}_1, \ldots, \mathcal{N}_R$ is a partition, i.e., $\mathcal{N}_r \cap \mathcal{N}_s = \emptyset$ for all $r \neq s$, then the following statements hold:*

(a) The worst-case optimal ordering quantity $y^*_{wc}(\mathcal{S})$ for a coalition $\mathcal{S} \subseteq \mathcal{N}$ as defined in (6) can be calculated as $y^*_{wc}(\mathcal{S}) = \sum_{r=1}^{R} y^*(\mathcal{S}_r)$, where $\mathcal{S}_r = \mathcal{S} \cap \mathcal{N}_r$ for all $r = 1, \ldots, R$, $y^*(\mathcal{S}_r)$ is the $(p-c)/p$-quantile of the known distribution of $\tilde{d}(\mathcal{S}_r)$, and $y^*(\emptyset) = 0$.

(b) The maximum worst-case expected profit is $v_{wc}(\mathcal{S}) = \sum_{r=1}^{R} v_{wc}(\mathcal{S}_r) = \sum_{r=1}^{R} \bar{v}(\mathcal{S}_r)$.

(c) For all $\boldsymbol{d} \in supp(P_{wc}(\mathcal{S}))$, i.e., $\boldsymbol{d}$ belongs to the support of $P_{wc}(\mathcal{S})$, either $d(\mathcal{S}_r) \leq y^*(\mathcal{S}_r)$ for all $r = 1, \ldots, R$, or $d(\mathcal{S}_r) \geq y^*(\mathcal{S}_r)$ for all $r = 1, \ldots, R$.

The proof of the lemma is provided in Appendix B.2. The proof exploits structural properties of the Fréchet class of distribution with non-overlapping covers. Results from Lemmas 1 and 2 allow us to show the existence of imputations of robust newsvendor games with the ambiguity set $\mathcal{P}(P_1, \ldots, P_R)$ as stated in the following theorem.

THEOREM 1. *Robust newsvendor games with the Fréchet ambiguity set $\mathcal{P}(P_1, \ldots, P_R)$ always have imputations.*

The proof of Theorem 1 is provided in Appendix B.3. The key idea is to use the fact that the demand distribution for each individual retailer is known for any joint demand distribution $P \in \mathcal{P}(P_1, \ldots, P_R)$. Theorem 1 shows that individual retailers would prefer to join the grand coalition even when the joint demand distribution is not known with certainty. It is clear that any two retailers ($N = 2$) would join up for better expected profits no matter what the joint demand distribution is. However, we cannot claim the same for more retailers, i.e., when $N > 2$. We will discuss the existence of core solutions of these robust newsvendor games in the next section.

3.2. Existence of Core Solutions

In order to study the existence of the core, we first analyse the condition for a coalition to break away from the grand coalition as stated in the following lemma.

LEMMA 3. *Given an individually rational decision $(y, \boldsymbol{z})$ of the grand coalition, a coalition $\mathcal{S} \subsetneq \mathcal{N}$ has the incentive to break away if and only if $\sum_{i \in \mathcal{S}} z_i < v_{\max}(y, \mathcal{S})$.*

The proof of the lemma is provided in Appendix C.1, which is similar to that of Lemma 1. Note that here, we focus only on individually rational decisions given that core decisions always need to be individually rational. Now, given the ambiguity of joint demand distribution, the existence of core solutions depends on how much distributional information is provided. As previously mentioned, newsvendor games with known joint demand distributions always have core solutions. We will show that core solutions also exist if sufficient marginal information is given. Let $\mathcal{C}_i = \mathcal{N} \setminus \{i\}$ for all $i \in \mathcal{N}$ and assume that N marginal distributions Q_i^c, $i \in \mathcal{N}$, of the cover $(\mathcal{C}_1, \ldots, \mathcal{C}_N)$ are known. Note that the ambiguity set $\mathcal{P}(Q_1^c, \ldots, Q_N^c)$, which is constructed using the marginal information of these large overlapping subsets, is not a singleton in general. The following theorem shows the existence of core solutions of the robust newsvendor games given the ambiguity set $\mathcal{P}(Q_1^c, \ldots, Q_N^c)$.

THEOREM 2. *Robust newsvendor games with the Fréchet ambiguity set $\mathcal{P}(Q_1^c, \ldots, Q_N^c)$ always have core solutions.*

The detailed proof is provided in Appendix C.2. Similar to Theorem 1, the key idea in the proof of Theorem 2 is to use the fact that the demand distribution of any coalition except the grand coalition is known to show the existence of core solutions. Next, on the other extreme, we will show that when $\mathcal{N}_1, \ldots, \mathcal{N}_R$ is a partition, i.e., a non-overlapping cover, core solutions in general do not exist. To that end, we first show a property of the core decision in the following lemma (with detailed proof provided in Appendix C.3.)

LEMMA 4. *Assuming that $\mathcal{N}_1, \ldots, \mathcal{N}_R$ is a partition, i.e., $\mathcal{N}_r \cap \mathcal{N}_s = \emptyset$ for all $r \neq s$, if $(y, \boldsymbol{z})$ is a core solution of a robust newsvendor game defined with the Fréchet ambiguity set $\mathcal{P}(P_1, \ldots, P_R)$, then the ordering quantity y is the worst-case optimal ordering quantity, i.e., $y = y^*_{wc}(\mathcal{N})$.*

The result of Lemma 4 can be regarded as an extension of that from Lemma 2. We now utilise both lemmas to show that the core solutions in general do not exist for non-overlapping covers. For the sake of simplicity, we shall consider discrete demand distributions in the following theorem even though its proof can be modified to accommodate continuous demand distributions. In addition, when demand distributions are discrete, further analyses can be done as shown in Propositions 1 and 2.

THEOREM 3. *Assuming that $\mathcal{N}_1,\ldots,\mathcal{N}_R$ is a partition, i.e., $\mathcal{N}_r \cap \mathcal{N}_s = \emptyset$ for all $r \neq s$, and $R \geq 3$, then the robust newsvendor game defined with the ambiguity set $\mathcal{P}(P_1,\ldots,P_R)$, where P_r are disrete distributions for all $r = 1,\ldots,R$, has no core solution if $y^*(\mathcal{N}_r) > 0$ for all $r = 1,\ldots,R$, and there exist $r_1, r_2 \in \{1,\ldots,R\}$, $r_1 \neq r_2$, such that $y^*(\mathcal{N}_{r_1}) > d_{\min}(\mathcal{N}_{r_1})$ and $y^*(\mathcal{N}_{r_2}) < d_{\max}(\mathcal{N}_{r_2})$, where $d_{\min}(\mathcal{S}) = \min\{d(\mathcal{S}) \mid \boldsymbol{d} \in P(\mathcal{S})\}$ and $d_{\max}(\mathcal{S}) = \max\{d(\mathcal{S}) \mid \boldsymbol{d} \in P(\mathcal{S})\}$ for $\mathcal{S} \subseteq \mathcal{N}$.*

The detailed proof is provided in Appendix C.4. The key idea is to perturb a worst-case distribution and construct a new distribution in the ambiguity set such that conditions for the existence of core solutions are violated. To this end, we exploit results from Lemma 2(c) on the structure of worst-case distributions, which indicates that a demand vector $\boldsymbol{d}$ belonging to the support of a worst-case distribution if any only if $y^*(\mathcal{S}_r) - d(\mathcal{S}_r)$ has the same sign for all $r = 1,\ldots,R$. The perturbed distribution violates this condition with some demand vectors $\boldsymbol{d}$ in its support such that $(y^*(\mathcal{S}_{r_1}) - d(\mathcal{S}_{r_1}))(y^*(\mathcal{S}_{r_2}) - d(\mathcal{S}_{r_2})) < 0$ for some r_1, r_2. With this perturbed distribution, we are able to establish a lower bound on $v_{\max}(y_{wc}(\mathcal{N}), \mathcal{N}_{r_1} \cup \mathcal{N}_{r_2})$, which is then used to show the instability of the grand coalition.

REMARK 5. i) The condition $y^*(\mathcal{N}_r) > 0$ is equivalent to $\mathbb{P}\left(\tilde{d}(\mathcal{N}_r) = 0\right) < \dfrac{p-c}{p}$ given that $y^*(\mathcal{N}_r)$ is the $(p-c)/p$-quantile of the known distribution of $\tilde{d}(\mathcal{N}_r)$. One simple sufficient condition could be that the demand $\tilde{d}(\mathcal{N}_r)$ is positive almost surely, i.e., $\mathbb{P}\left(\tilde{d}(\mathcal{N}_r) > 0\right) = 1$, for all $r = 1,\ldots,R$.

ii) The condition $y^*(\mathcal{N}_r) > d_{\min}(\mathcal{N}_r)$ holds if the probability $\mathbb{P}\left(\tilde{d}(\mathcal{N}_r) = d_{\min}(\mathcal{N}_r)\right)$ is small enough, $\mathbb{P}\left(\tilde{d}(\mathcal{N}_r) = d_{\min}(\mathcal{N}_r)\right) < \dfrac{p-c}{p}$. Similarly, the condition $y^*(\mathcal{N}_r) < d_{\max}(\mathcal{N}_r)$ holds if the probability $\mathbb{P}\left(\tilde{d}(\mathcal{N}_r) = d_{\max}(\mathcal{N}_r)\right)$ is small enough, $\mathbb{P}\left(\tilde{d}(\mathcal{N}_r) = d_{\max}(\mathcal{N}_r)\right) < \dfrac{c}{p}$. These conditions are usually satisfied, i.e., the optimal ordering quantities are not extremal values.

iii) The proof of this theorem can be modified to accommodate continuous demand distributions using the same idea of worst-case distribution perturbation. Given that $y^*(\mathcal{N}_r) \in (d_{\min}(\mathcal{N}_r), d_{\max}(\mathcal{N}_r))$ if P_r is continuous for all $r = 1,\ldots,R$, one can show that robust newsvendor games defined as in Theorem 3 with continuous demand distributions always has no core when $R \geq 3$.

In the next proposition, we utilize Remark 5ii) to show that if random demands have heavy skew discrete distributions in some extreme cases, core solutions of the robust newsvendor game when $\mathcal{N}_1,\ldots,\mathcal{N}_R$ is a partition could exist.

PROPOSITION 1. *Assuming that $\mathcal{N}_1,\ldots,\mathcal{N}_R$ is a partition, i.e., $\mathcal{N}_r \cap \mathcal{N}_s = \emptyset$ for all $r \neq s$, then core solutions of the robust newsvendor game defined with the ambiguity set $\mathcal{P}(P_1,\ldots,P_R)$, where P_r are disrete distributions for all $r = 1,\ldots,R$, exist if $\mathbb{P}\left(\tilde{d}(\mathcal{S}_r) = d_{\min}(\mathcal{S}_r)\right) \geq 1 - \dfrac{c}{Rp}$ for all $\mathcal{S}_r \subseteq \mathcal{N}_r$, $r = 1,\ldots,R$, or $\mathbb{P}\left(\tilde{d}(\mathcal{S}_r) = d_{\max}(\mathcal{S}_r)\right) \geq 1 - \dfrac{p-c}{Rp}$ for all $\mathcal{S}_r \subseteq \mathcal{N}_r$, $r = 1,\ldots,R$.*

There is still a gap between the conditions derived in Theorem 3 and Proposition 1 for the existence and non-existence of core solutions under the general settings. The following proposition shows that the gap can be closed for some special cases.

PROPOSITION 2. *Assuming that $\mathcal{N}_i = \{i\}$ and $P_i \equiv P_0$ with supp$(P_0) = [d_{\min}, d_{\max}]$, where $d_{\min} < d_{\max}$, for all $i = 1,\ldots,N$, then the robust newsvendor game defined with the ambiguity set $\mathcal{P}(P_1,\ldots,P_N)$, where P_i are disrete distributions for all $i = 1,\ldots,N$, has no core solution if and only if $N \geq 3$, $\mathbb{P}_{P_0}\left(\tilde{d} = d_{\min}\right) < 1 - \dfrac{c}{(N-1)p}$, and $\mathbb{P}_{P_0}\left(\tilde{d} = d_{\max}\right) < 1 - \dfrac{p-c}{(N-1)p}$.*

The proofs of Propositions 1 and 2 are provided in Appendices C.5 and C.6, respectively. Theorems 2, 3 and Propositions 1, 2 show different conditions for the existence (and non-existence) of robust solutions. In general, robust solutions exist when there is enough distributional information or under some special cases.

4. Conclusion

In this paper, we develop a framework for newsvendor expected games with ambiguity in demand distributions, which we call robust newsvendor games. We use Fréchet classes of distributions to handle practical situations in the newsvendor games where only partial information of general (overlapping) marginal distributions is available. We are able to derive conditions for the existence (and non-existence) of core solutions of these robust newsvendor games using structural analysis of the worst-case distributions of the corresponding distributionally robust newsvendor optimization problem. Future research directions include how to generalize the concept of break-away incentive and use it to handle the conservativeness of the proposed framework of robust cooperative games. In addition, other allocation rules should also be considered. Development of a computational framework to find core (and least core) decisions of these robust cooperative games is another challenging research topic.

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References

[1] D. Bauso and J. Timmer. Robust dynamic cooperative games. *International Journal of Game Theory*, 38(1):23–36, 2009.

[2] A. Ben-Tal, L. El Ghaoui, and A. Nemirovski. *Robust optimization.* Princeton University Press, 2009.

[3] G. Bitran. Linear multiple objective problems with interval coefficients. *Management Science*, 26(7): 694–706, 1980.

[4] G. Chalkiadakis and G. Boutilier. Bayesian reinforcement learning for coalition formation under uncertainty. In *Proceedings of the 3rd International Joint Conference on Autonomous Agents and Multiagent Systems - Volume 3*, pages 1090–1097. IEEE Computer Society, 2004.

[5] A. Charnes and D. Granot. Prior solutions: Extensions of convex nucleus solutions to chance-constrained games. Technical report, DTIC Document, 1973.

[6] X. Chen and J. Zhang. A stochastic programming duality approach to inventory centralization games. *Operations Research*, 57(4):840–851, 2009.

[7] X. V. Doan and K. Natarajan. On the complexity of nonoverlapping multivariate marginal bounds for probabilistic combinatorial optimization problems. *Operations Research*, 60(1):138–149, 2012.

[8] X. V. Doan, X. Li, and K. Natarajan. Robustness to dependency in portfolio optimization using overlapping marginals. *Operations Research*, 63(6):1468–1488, 2015.

[9] M. Dror, L. Guardiola, A. Meca, and J. Puerto. Dynamic realization games in newsvendor inventory centralization. *International Journal of Game Theory*, 37(1):139–153, 2008.

[10] M. Ehrgott. *Multicriteria optimization.* Springer, Berlin, Germany, 2nd edition, 2005.

[11] P. Embrechts and G. Puccetti. Bounds for functions of dependent risks. *Finance and Stochastics*, 10 (3):341–352, 2006.

[12] F. R. Fernández, J. Puerto, and M. J. Zafra. Cores of stochastic cooperative games with stochastic orders. *International Game Theory Review*, 4(03):265–280, 2002.

[13] F. Forges and R. Serrano. Cooperative games with incomplete information: Some open problems. *International Game Theory Review*, 15(2), 2013.

[14] D. Gillies. Solutions to general non-zero-sum games. In A. Tucker and R. Luce, editors, *Contributions to the Theory of Games*, volume 4, pages 47–86. Princeton University Press, 1959.

[15] B. C. Hartman. *Cooperative games and inventory cost allocation.* PhD thesis, College of Business and Public Administration, The University of Arizona, 1994.

[16] B. C. Hartman, M. Dror, and M. Shaked. Cores of inventory centralization games. *Games and Economic Behavior*, 31(1):26–49, 2000.

[17] S. Ieong and Y. Shoham. Bayesian coalitional games. In *Proceedings of the 23rd AAAI Conference on Artificial Intelligence*, pages 95–100, 2008.

[18] Jonas J. Ide and A. Schöbel. Robustness for uncertain multi-objective optimization: a survey and analysis of different concepts. *OR spectrum*, 38(1):235–271, 2016.

[19] H. Kuhn and A. Tucker. Nonlinear programming. In J. Neyman, editor, *Second Berkeley Symposium on Mathematical Statistics and Probability*, pages 481–492, 1951.

[20] E. Lehrer. Allocation processes in cooperative games. *International Journal of Game Theory*, 31(3): 341–351, 2003.

[21] Y. Li and V. Conitzer. Cooperative game solution concepts that maximize stability under noise. In G. Chalkiadakis, E. Elkind, and R. Meir, editors, *Proceedings of the 5th Workshop on Cooperative Games in Multiagent Systems*, 2014.

[22] L. Montrucchio and M. Scarsini. Large newsvendor games. *Games and Economic Behavior*, 58(2): 316–337, 2007.

[23] A. Müller, M. Scarsini, and M. Shaked. The newsvendor game has a nonempty core. *Games and Economic Behavior*, 38(1):118–126, 2002.

[24] V. Pareto. *Manuale di economia politica*, volume 13. Societa Editrice Libraria, 1906.

[25] L. Rüschendorf. Bounds for distributions with multivariate marginals. In K. Mosler and M. Scarsini, editors, *Stochastic Orders and Decision under Risk*, pages 285–310. IMS Lecture Notes - Monograph Series 19, 1991.

[26] J. Suijs and P. Borm. Stochastic cooperative games: superadditivity, convexity, and certainty equivalents. *Games and Economic Behavior*, 27(2):331–345, 1999.

[27] J. Suijs, P. Borm, A. De Waegenaere, and S. Tijs. Cooperative games with stochastic payoffs. *European Journal of Operational Research*, 113(1):193–205, 1999.

[28] J. Timmer, P. Borm, and S. Tijs. Convexity in stochastic cooperative situations. *International Game Theory Review*, 7(1):25–42, 2005.

[29] A. Toriello and N. Uhan. Dynamic linear programming games with risk-averse players. *Mathematical Programming*, 163(1-2):25–56, 2017.

[30] N. Uhan. Two-stage allocations for stochastic linear programming games with concave preferences. *European Journal of Operational Research*, 243(2):637–646, 2015.

Appendix A: Non-emptiness of $\mathcal{Y}(\mathcal{N})$

LEMMA 5. *Under Assumption 1,* $\mathcal{Y}(\mathcal{N}) \neq \emptyset$*.*

Proof. We have $\tilde{d}(\mathcal{N}) = \sum_{r=1}^{R} \tilde{d}(\mathcal{N}_r)$. Under Assumption 1, $\max_{r=1,\dots,R} d_{\min}(\mathcal{N}_r) > 0$, where $d_{\min}(\mathcal{S}) = \min\{d(\mathcal{S}) \mid \boldsymbol{d} \in \operatorname{supp}(P(\mathcal{S}))\}$ for $\mathcal{S} \subseteq \mathcal{N}$. Thus, we have

$$d_{\min}(\mathcal{N}) \geq \sum_{r=1}^{R} d_{\min}(\mathcal{N}_r) > 0.$$

For $y \in [0, d_{\min}(N)]$, $v_P(y,\mathcal{N}) = (p-c)y$, which is strictly increasing for any $P \in \mathcal{P}(P_1,\dots,P_R)$ given the fact that $p > c$. In addition, for $y \geq d_{\max}(\mathcal{N})$, where $d_{\max}(\mathcal{S}) = \max\{d(\mathcal{S}) \mid \boldsymbol{d} \in \operatorname{supp}(P(\mathcal{S}))\}$ for $\mathcal{S} \subseteq \mathcal{N}$, $v_P(y,\mathcal{N}) = -cy + p\sum_{r=1}^{R} \mathbb{E}_{P_r}\left[\tilde{d}(\mathcal{N}_r)\right]$, which is strictly decreasing for any $P \in \mathcal{P}(P_1,\dots,P_R)$ given the fact that $c > 0$.

Now consider the function $\bar{v}(y,\mathcal{N}) = \min_{P \in \mathcal{P}(P_1,\dots,P_R)} v_P(y,\mathcal{N})$. It is clear that $\bar{v}(\cdot,\mathcal{N})$ is again strictly increasing in $[0, d_{\min}(\mathcal{N})]$ and strictly decreasing in $[d_{\max}(\mathcal{N}), +\infty)$. Thus, we have

$$\arg\max_{y \geq 0} \bar{v}(y,\mathcal{N}) \in [d_{\min}(\mathcal{N}), d_{\max}(\mathcal{N})],$$

and

$$\max_{y \geq 0} \bar{v}(y,\mathcal{N}) \geq \bar{v}(d_{\min}(\mathcal{N}),\mathcal{N}) = (p-c)d_{\min}(\mathcal{N}) > 0.$$

Thus, there exists $y \geq 0$ such that $\bar{v}(y,\mathcal{N}) > 0$, or equivalently, $v_P(y,\mathcal{N}) > 0$ for all $P \in \mathcal{P}(P_1,\dots,P_R)$. This shows that $\mathcal{Y}(\mathcal{N}) \neq \emptyset$. □

Appendix B: Proofs for Results Related to Imputation Existence

B.1. Proof of Lemma 1

Proof. The profit allocation of retailer i, $i \in \mathcal{N}$, is $x_i^P = v_P(y,\mathcal{N}) \cdot z_i$ for any $P \in \mathcal{P}(P_1,\dots,P_R)$. According to Definition 1, retailer i does not have the incentive to break away if and only if

$$v_P(y_i,\{i\}) \leq v_P(y,\mathcal{N}) \cdot z_i, \quad \forall\, y_i \in \mathcal{Y}(\{i\}),\, P \in \mathcal{P}(P_1,\dots,P_R).$$

Since $v_P(y,\mathcal{N}) > 0$ for $P \in \mathcal{P}(P_1,\dots,P_R)$, the above condition is equivalent to $z_i \geq \max_{y_i \in \mathcal{Y}(\{i\})} \dfrac{v_P(y_i,\{i\})}{v_P(y,\mathcal{N})}$ for all $P \in \mathcal{P}(P_1,\dots,P_R)$, i.e.,

$$z_i \geq \max_{P \in \mathcal{P}(P_1,\dots,P_R)} \left\{ \frac{\max_{y_i \in \mathcal{Y}(\{i\})} v_P(y_i,\{i\})}{v_P(y,\mathcal{N})} \right\} = v_{\max}(y,\{i\}).$$

□

B.2. Proof of Lemma 2

Proof. Consider the optimization problem in (6). For $y \leq d_{\min}(\mathcal{S}) := \min\{\tilde{d}(\mathcal{S})\}$, we can write $(p-c)y - p\,\mathbb{E}_P\left[\left(y - \tilde{d}(\mathcal{S})\right)^+\right] = (p-c)y$ for any distribution P. Since $(p-c) > 0$, this is an increasing function in y in $(-\infty; d_{\min}(\mathcal{S})]$. Since $d_{\min}(\mathcal{S}) \geq 0$, we can then remove the non-negative constraint $y \geq 0$ from (6) when calculating $y^*_{wc}(\mathcal{S})$. Now, consider the inner optimization problem of (6). This is an instance of the distributionally robust optimization problem studied in Doan and Natarajan [7] with Fréchet classes of distributions under the setting that $\mathcal{N}_1, \ldots, \mathcal{N}_R$ is a partition. Without loss of generality, we can assume that $\mathcal{S}_r \neq \emptyset$ for all $r = 1, \ldots, R$ knowing that $y^*(\emptyset) = 0$. Applying Proposition 1(ii) from [7], we obtain the following reformulation:

$$\max_{P \in \mathcal{P}(P_1, \ldots, P_R)} \mathbb{E}_P\left[\left(y - \tilde{d}(\mathcal{S})\right)^+\right] = \begin{aligned}[t] \min_{\boldsymbol{x}} \; & \sum_{r=1}^R \mathbb{E}_{P_r}\left[\left(x_r - \tilde{d}(\mathcal{S}_r)\right)^+\right] \\ \text{s.t.} \; & \sum_{r=1}^R x_r = y. \end{aligned}$$

Thus, in order to find $y^*_{wc}(\mathcal{S})$, we can solve the following optimization problem

$$\begin{aligned} \max_{y, \boldsymbol{x}} \; & (p-c)y - p\sum_{r=1}^R \mathbb{E}_{P_r}\left[\left(x_r - \tilde{d}(\mathcal{S}_r)\right)^+\right] \\ \text{s.t.} \; & \sum_{r=1}^R x_r = y. \end{aligned}$$

The optimal ordering quantity $y^*_{wc}(\mathcal{S})$ can then be calculated as $y^*_{wc}(\mathcal{S}) = \sum_{r=1}^R x^*_r$, where $\boldsymbol{x}^*$ is the optimal solution of the following separable optimization problem:

$$\max_{\boldsymbol{x}} \sum_{r=1}^R \left((p-c)x_r - p\sum_{r=1}^R \mathbb{E}_{P_r}\left[\left(x_r - \tilde{d}(\mathcal{S}_r)\right)^+\right]\right),$$

or equivalently,

$$\sum_{r=1}^R \max_{x_r} \left((p-c)x_r - p\,\mathbb{E}_{P_r}\left[\left(x_r - \tilde{d}(\mathcal{S}_r)\right)^+\right]\right).$$

For each sub-problem, x^*_r is the $(p-c)/p$-quantile of the distribution of $\tilde{d}(\mathcal{S}_r)$, which means $x^*_r = y^*(\mathcal{S}_r)$ for all $r = 1, \ldots, R$, according to (3). Thus, we have $y^*_{wc}(\mathcal{S}) = \sum_{r=1}^R y^*(\mathcal{S}_r)$. This also leads to $v_{wc}(\mathcal{S}) = \sum_{r=1}^R v_{wc}(\mathcal{S}_r) = \sum_{r=1}^R \bar{v}(\mathcal{S}_r)$. Here, the joint demand distribution for players in $\mathcal{S}_r$ is known and hence the worst-case expected profit $v_{wc}(S_r)$ is exactly the same with the expected profit $\bar{v}(\mathcal{S}_r)$ shown in (4).

Now, given the fact that $x^+ + y^+ \geq (x+y)^+$ for all $x, y \in \mathbb{R}$, we have

$$\mathbb{E}_{P_{wc}(\mathcal{S})}\left[\left(y^*_{wc}(\mathcal{S}) - \tilde{d}(\mathcal{S})\right)^+\right] \leq \sum_{r=1}^R \mathbb{E}_{P(\mathcal{S}_r)}\left[\left(y^*(\mathcal{S}_r) - \tilde{d}(\mathcal{S}_r)\right)^+\right].$$

The optimality of $(y^*_{wc}(\mathcal{S}), P_{wc}(\mathcal{S}))$ implies the equality of the above inequality, which holds if and only

$$(y^*_{wc}(\mathcal{S}) - d(\mathcal{S}))^+ = \sum_{r=1}^{R} (y^*(\mathcal{S}_r) - d(\mathcal{S}_r))^+, \quad \forall \boldsymbol{d} \in \operatorname{supp}(P_{wc}(\mathcal{S})).$$

This condition is equivalent to the fact that for all $\boldsymbol{d} \in \operatorname{supp}(P_{wc}(\mathcal{S}))$, $y^*(\mathcal{S}_r) - d(\mathcal{S}_r)$ has to be either non-negative or non-positive for all $r = 1, \ldots, R$. This is precisely what is stated in the lemma regarding the support of the worst-case distribution $P_{wc}(\mathcal{S})$. $\square$

B.3. Proof of Theorem 1

Proof. We will show the existence of imputations by constructing an individually rational decision. Let Q_i be the marginal distribution of retailer i, $i = 1, \ldots, N$. Since $\mathcal{N}_1, \ldots, \mathcal{N}_R$ is a cover of $\mathcal{N}$, i.e., $\bigcup_{r=1}^{R} \mathcal{N}_r = \mathcal{N}$, all the marginal distributions Q_i, $i = 1, \ldots, N$, are known. In addition, Q_i is the marginal distribution of some distribution P_r, $r = 1, \ldots, R$, for all $i = 1, \ldots, N$. Thus, $\mathcal{P}(P_1, \ldots, P_R) \subseteq \mathcal{P}(Q_1, \ldots, Q_N)$.

For each individual retailer i, the demand distribution Q_i is known; therefore $\mathcal{Y}(\{i\}) = \{y^*_i\}$, where y^*_i be the $(p-c)/p$-quantile of the distribution function of $\tilde{d}_i$. Thus,

$$\max_{y_i \in \mathcal{Y}(\{i\})} v_P(y_i, \{i\}) = v_i(y^*_i) = \bar{v}_i \geq 0, \quad \forall P \in \mathcal{P}(Q_1, \ldots, Q_N).$$

Consider $v_{\max}(y, \{i\})$, $i \in \mathcal{N}$, for $y \in \mathcal{Y}(\mathcal{N})$, we have

$$v_{\max}(y, \{i\}) = \max_{P \in \mathcal{P}(P_1, \ldots, P_R)} \left\{ \frac{\max_{y_i \in \mathcal{Y}(\{i\})} v_P(y_i, \{i\})}{v_P(y, \mathcal{N})} \right\} = \frac{\bar{v}_i}{\min_{P \in \mathcal{P}(P_1, \ldots, P_R)} v_P(y, \mathcal{N})}.$$

In order to construct an individually rational decision, we consider the worst-case optimal ordering quantity $y^*_{wc}(\mathcal{N})$ as defined in (6). Clearly, we have $y^*_{wc}(\mathcal{N}) \in \mathcal{Y}(\mathcal{N})$ and $v_{wc}(\mathcal{N}) = \min_{P \in \mathcal{P}(P_1, \ldots, P_R)} v_P(y^*_{wc}(\mathcal{N}), \mathcal{N}) > 0$. We then have

$$v_{\max}(y^*_{wc}(\mathcal{N}), \{i\}) = \frac{\bar{v}_i}{v_{wc}(\mathcal{N})}.$$

Let $\boldsymbol{x}$ be an imputation of the corresponding deterministic newsvendor game with demand distribution $P^*_{wc}(\mathcal{N})$; that is, $\boldsymbol{x}$ solves $\sum_{i \in \mathcal{N}} x_i = v_{wc}(\mathcal{N})$ and $x_i \geq \bar{v}_i$. Let us define $\boldsymbol{z} = \boldsymbol{x}/\bar{v}_{wc}(\mathcal{N})$. We then have $\boldsymbol{z}(\mathcal{N}) = \boldsymbol{x}(\mathcal{N})/v_{wc}(\mathcal{N}) = 1$ and hence $\boldsymbol{z}$ is an efficient allocation rule. We also have

$$z_i = \frac{x_i}{v_{wc}(\mathcal{N})} \geq \frac{\bar{v}_i}{v_{wc}(\mathcal{N})} = v_{\max}(y^*_{wc}(\mathcal{N}), \{i\}).$$

According to Lemma 1, the decision $(y^*_{wc}(\mathcal{N}), \boldsymbol{z})$ is individually rational. Thus, it is an imputation given that allocation rule $\boldsymbol{z}$ is also efficient. This shows that robust newsvendor games with an arbitrary ambiguity set $\mathcal{P}(P_1, \ldots, P_R)$ always have imputations.

$\square$

Appendix C: Proofs for Results Related to Robust Core Existence

C.1. Proof of Lemma 3

Proof. Given an individually rational decision $(y,\boldsymbol{z})$, a coalition $\mathcal{S}$ has the incentive to break away if there exists an efficient decision $(\hat{y},\hat{\boldsymbol{z}})$ for $\mathcal{S}$ and a demand distribution $P\in\mathcal{P}(P_1,\ldots,P_R)$ such that for all retailers i, $i\in\mathcal{S}$,

$$v_P(\hat{y},\mathcal{S})\cdot\hat{z}_i > v_P(y,\mathcal{N})\cdot z_i.$$

Since $\hat{\boldsymbol{z}}$ is an efficient allocation for $\mathcal{S}$, we have: $\sum_{i\in\mathcal{S}}\hat{z}_i=1$. Summing the inequality above over all $i\in\mathcal{S}$, we then obtain the following statement:

$$\exists\, P\in\mathcal{P}(P_1,\ldots,P_R) : v_P(y,\mathcal{N})\cdot\sum_{i\in\mathcal{S}}z_i < v_P(\hat{y},\mathcal{S})\cdot\sum_{i\in\mathcal{S}}\hat{z}_i = v_P(\hat{y},\mathcal{S}).$$

Equivalently, since $v_P(y,\mathcal{N})>0$ for $P\in\mathcal{P}(P_1,\ldots,P_R)$, we have

$$\exists\, P\in\mathcal{P}(P_1,\ldots,P_R) : \sum_{i\in\mathcal{S}}z_i < \frac{v_P(\hat{y},\mathcal{S})}{v_P(y,\mathcal{N})} \Leftrightarrow \sum_{i\in\mathcal{S}}z_i < \max_{P\in\mathcal{P}(P_1,\ldots,P_R)}\left\{\frac{v_P(\hat{y},\mathcal{S})}{v_P(y,\mathcal{N})}\right\}.$$

We have

$$\begin{aligned}
\max_{P\in\mathcal{P}(P_1,\ldots,P_R)}\left\{\frac{v_P(\hat{y},\mathcal{S})}{v_P(y,\mathcal{N})}\right\} &\le \max_{y_{\mathcal{S}}\in\mathcal{Y}(\mathcal{S})}\max_{P\in\mathcal{P}(P_1,\ldots,P_R)}\left\{\frac{v_P(y_{\mathcal{S}},\mathcal{S})}{v_P(y,\mathcal{N})}\right\}\\
&= \max_{P\in\mathcal{P}(P_1,\ldots,P_R)}\max_{y_{\mathcal{S}}\in\mathcal{Y}(\mathcal{S})}\left\{\frac{v_P(y_{\mathcal{S}},\mathcal{S})}{v_P(y,\mathcal{N})}\right\}\\
&= \max_{P\in\mathcal{P}(P_1,\ldots,P_R)}\left\{\frac{\max_{y_{\mathcal{S}}\in\mathcal{Y}(\mathcal{S})} v_P(y_{\mathcal{S}},\mathcal{S})}{v_P(y,\mathcal{N})}\right\}.
\end{aligned}$$

The second equality is due to the fact that $v_P(y,\mathcal{N})>0$ for all $P\in\mathcal{P}(P_1,\ldots,P_R)$ given that $y\in\mathcal{Y}(\mathcal{N})$. Thus, if a coalition $\mathcal{S}$ has the incentive to break away, then

$$\sum_{i\in\mathcal{S}}z_i < \max_{P\in\mathcal{P}(P_1,\ldots,P_R)}\left\{\frac{\max_{y_{\mathcal{S}}\in\mathcal{Y}(\mathcal{S})} v_P(y_{\mathcal{S}},\mathcal{S})}{v_P(y,\mathcal{N})}\right\} = v_{\max}(y,\mathcal{S}).$$

Now, suppose $\sum_{i\in\mathcal{S}}z_i < v_{\max}(y,\mathcal{S})$. We will show that coalition $\mathcal{S}$ has the incentive to break away. We need to show the existence of an efficient decision $(\hat{y},\hat{\boldsymbol{z}})$ and a demand distribution $P\in\mathcal{P}(P_1,\ldots,P_R)$ such that for all retailers i, $i\in\mathcal{S}$,

$$v_P(\hat{y},\mathcal{S})\cdot\hat{z}_i > v_P(y,\mathcal{N})\cdot z_i.$$

Let $(\hat{y},\hat{P})\in\arg\max_{y_{\mathcal{S}}\in\mathcal{Y}(\mathcal{S})}\max_{P\in\mathcal{P}(P_1,\ldots,P_R)}\left\{\frac{v_P(y_{\mathcal{S}},\mathcal{S})}{v_P(y,\mathcal{N})}\right\}$. Clearly, we have

$$\frac{v_{\hat{P}}(\hat{y},\mathcal{S})}{v_{\hat{P}}(y,\mathcal{N})} = \max_{y_{\mathcal{S}}\in\mathcal{Y}(\mathcal{S})}\max_{P\in\mathcal{P}(P_1,\ldots,P_R)}\left\{\frac{v_P(y_{\mathcal{S}},\mathcal{S})}{v_P(y,\mathcal{N})}\right\} = \max_{P\in\mathcal{P}(P_1,\ldots,P_R)}\left\{\frac{\max_{y_{\mathcal{S}}\in\mathcal{Y}(\mathcal{S})} v_P(y_{\mathcal{S}},\mathcal{S})}{v_P(y,\mathcal{N})}\right\} = v_{\max}(y,\mathcal{S}).$$

From Lemma 1, we have $z_i \geq v_{\max}(y, \{i\}) \geq 0$ since $(y, \boldsymbol{z})$ is an individually rational decision (i.e., an imputation). Thus, $v_{\max}(y, \mathcal{S}) > \sum_{i \in \mathcal{S}} z_i \geq 0$ and hence $v_{\hat{P}}(\hat{y}, \mathcal{S}) \neq 0$.

Let $\epsilon = \frac{1}{|\mathcal{S}|}\left(v_{\max}(y, \mathcal{S}) - \sum_{i \in \mathcal{S}} z_i\right) > 0$ and define $\hat{z}_i = \frac{z_i + \epsilon}{v_{\max}(y, \mathcal{S})}$ for all $i \in \mathcal{S}$. Clearly, $\sum_{i \in \mathcal{S}} \hat{z}_i = 1$ given the definition of ϵ. In addition, for all $i \in \mathcal{S}$, we have:

$$v_{\hat{P}}(\hat{y}, \mathcal{S}) \cdot \hat{z}_i = v_{\hat{P}}(y, \mathcal{N}) \cdot z_i + v_{\hat{P}}(y, \mathcal{N}) \cdot \epsilon > v_{\hat{P}}(y, \mathcal{N}) \cdot z_i.$$

Thus, $(\hat{y}, \hat{\boldsymbol{z}})$ is an efficient decision and the inequality above shows that coalition $\mathcal{S}$ indeed has the incentive to break away. $\square$

C.2. Proof of Theorem 2

Proof. For an arbitrary coalition $\mathcal{S} \subsetneq \mathcal{N}$, there always exists $i \in \mathcal{N}$ such that $\mathcal{S} \subseteq \mathcal{C}_i$. Thus, the joint demand distribution $P(\mathcal{S})$ is known since it is a marginal distribution of Q_i^c for some $i \in \mathcal{N}$. Consider the worst-case optimal ordering quantity $y_{wc}^*(\mathcal{N})$ as defined in (6). We have

$$\begin{aligned}
v_{\max}(y_{wc}^*(\mathcal{N}), \mathcal{S}) &= \max_{P \in \mathcal{P}(Q_1^c, \ldots, Q_N^c)} \left\{ \frac{\max\limits_{y_{\mathcal{S}} \in \mathcal{Y}(\mathcal{S})} v_P(y_{\mathcal{S}}, \mathcal{S})}{v_P(y_{wc}^*(\mathcal{N}), \mathcal{N})} \right\} \\
&= \frac{\max\limits_{y_{\mathcal{S}} \in \mathcal{Y}(\mathcal{S})} v_{P(\mathcal{S})}(y_{\mathcal{S}}, \mathcal{S})}{\min\limits_{P \in \mathcal{P}(Q_1^c, \ldots, Q_N^c)} v_P(y_{wc}^*(\mathcal{N}), \mathcal{N})} \\
&= \frac{v_{wc}(\mathcal{S})}{v_{wc}(\mathcal{N})} = \frac{\bar{v}(\mathcal{S})}{v_{wc}(\mathcal{N})}.
\end{aligned}$$

Now consider a worst-case distribution $P_{wc}^*(\mathcal{N}) \in \arg\min_{P \in \mathcal{P}(Q_1^c, \ldots, Q_N^c)} v_P(y_{wc}^*(\mathcal{N}), \mathcal{N})$. We have $v_{P_{wc}^*(\mathcal{N})}(y_{wc}^*(\mathcal{N}), \mathcal{N}) = v_{wc}(\mathcal{N})$.

Let $\boldsymbol{x}$ be a core solution of the corresponding deterministic newsvendor game with demand distribution $P_{wc}^*(\mathcal{N})$. We then have $\sum_{i \in \mathcal{N}} x_i = v_{wc}(\mathcal{N})$ and $\sum_{i \in \mathcal{S}} x_i \geq \max_{y_{\mathcal{S}} \in \mathcal{Y}(\mathcal{S})} v_{P(\mathcal{S})}(y_{\mathcal{S}}, \mathcal{S}) = \bar{v}(\mathcal{S})$ for all $\mathcal{S} \subsetneq \mathcal{N}$. Let us define $\boldsymbol{z} = \boldsymbol{x}/\bar{v}_{wc}(\mathcal{N})$. We then have $\boldsymbol{z}(\mathcal{N}) = \boldsymbol{x}(\mathcal{N})/v_{wc}(\mathcal{N}) = 1$ and hence $\boldsymbol{z}$ is an efficient allocation rule. For each $\mathcal{S} \subsetneq \mathcal{N}$, we have

$$\sum_{i \in \mathcal{S}} z_i = \frac{\sum\limits_{i \in \mathcal{S}} x_i}{v_{wc}(\mathcal{N})} \geq \frac{\bar{v}(\mathcal{S})}{v_{wc}(\mathcal{N})} = v_{\max}(y_{wc}^*(\mathcal{N}), \mathcal{S}). \tag{8}$$

Thus, the decision $(y_{wc}^*(\mathcal{N}), \boldsymbol{z})$ is a core decision of the robust newsvendor game with the ambiguity set $\mathcal{P}(Q_1^c, \ldots, Q_N^c)$.

$\square$

C.3. Proof of Lemma 4

Proof. Let us consider a core solution $(y,\boldsymbol{z})$ of the robust newsvendor game with the ambiguity set $\mathcal{P}(P_1,\ldots,P_r)$. For each coalition $\mathcal{N}_r$, $r=1,\ldots,R$, we have

$$\begin{aligned}
v_{\max}(y,\mathcal{N}_r) &= \max_{P\in\mathcal{P}(P_1,\ldots,P_R)}\left\{\frac{\max\limits_{y_r\in\mathcal{Y}(\mathcal{N}_r)} v_P(y_r,\mathcal{N}_r)}{v_P(y,\mathcal{N})}\right\}\\
&= \frac{\max\limits_{y_r\in\mathcal{Y}(\mathcal{N}_r)} v_{P_r}(y_r,\mathcal{N}_r)}{\min\limits_{P\in\mathcal{P}(P_1,\ldots,P_R)} v_P(y,\mathcal{N})}\\
&= \frac{v_{wc}(\mathcal{N}_r)}{\min\limits_{P\in\mathcal{P}(P_1,\ldots,P_R)} v_P(y,\mathcal{N})} = \frac{\bar{v}(\mathcal{N}_r)}{\min\limits_{P\in\mathcal{P}(P_1,\ldots,P_R)} v_P(y,\mathcal{N})}.
\end{aligned}$$

This is due to the fact that P_r is known with certainty and $v_{wc}(\mathcal{N}_r)\geq 0$. According to Lemma 3, for $(y,\boldsymbol{z})$ to be a core solution, we need to have

$$\sum_{i\in\mathcal{N}_r} z_i \geq v_{\max}(y,\mathcal{N}_r) = \frac{\bar{v}(\mathcal{N}_r)}{\min\limits_{P\in\mathcal{P}(P_1,\ldots,P_R)} v_P(y,\mathcal{N})}.$$

Summing this over $r=1,\ldots,R$, we obtain

$$1=\sum_{i\in\mathcal{N}} z_i = \sum_{r=1}^{R}\sum_{i\in\mathcal{N}_r} z_i \geq \frac{\sum_{r=1}^{R}\bar{v}(\mathcal{N}_r)}{\min\limits_{P\in\mathcal{P}(P_1,\ldots,P_R)} v_P(y,\mathcal{N})}.$$

Given that $y\in\mathcal{Y}(\mathcal{N})$, we have $\min\limits_{P\in\mathcal{P}(P_1,\ldots,P_R)} v_P(y,\mathcal{N})>0$. Hence,

$$\sum_{r=1}^{R}\bar{v}(\mathcal{N}_r) \leq \min_{P\in\mathcal{P}(P_1,\ldots,P_R)} v_P(y,\mathcal{N}).$$

This leads to

$$\begin{aligned}
\sum_{r=1}^{R}\bar{v}(\mathcal{N}_r) &\leq \max_{y\geq 0}\min_{P\in\mathcal{P}(P_1,\ldots,P_R)} v_P(y,\mathcal{N})\\
&\equiv v_{wc}(\mathcal{N})\\
&= \sum_{r=1}^{R} v_{wc}(\mathcal{N}_r) = \sum_{r=1}^{R}\bar{v}(\mathcal{N}_r),
\end{aligned}$$

where the last equality comes from Lemma 2. It must then be the case that the ordering quantity y is the worst-case optimal ordering quantity, $y=y^*_{wc}(\mathcal{N})$, for the above inequalities to be tight. □

C.4. Proof of Theorem 3

Proof. First, without loss of generality, let $r_1 = 1$ and $r_2 = 2$. Let $\mathcal{H}_r(\mathcal{S}) = \underline{\mathcal{H}}_r(\mathcal{S}) \cap \overline{\mathcal{H}}_r(\mathcal{S}) = \{\boldsymbol{d} \in \text{supp}(P(\mathcal{S}_r)) \,|\, d(\mathcal{S}_r) = y^*(\mathcal{S}_r)\}$, where $\underline{\mathcal{H}}_r(\mathcal{S}) = \{\boldsymbol{d} \in \text{supp}(P(\mathcal{S}_r)) \,|\, d(\mathcal{S}_r) \leq y^*(\mathcal{S}_r)\}$, and similarly, $\overline{\mathcal{H}}_r(\mathcal{S}) = \{\boldsymbol{d} \in \text{supp}(P(\mathcal{S}_r)) \,|\, d(\mathcal{S}_r) \geq y^*(\mathcal{S}_r)\}$, for all $r = 1, \ldots, R$ and $\mathcal{S} \subseteq \mathcal{N}$. First, $\mathcal{H}_r(\mathcal{N}) \neq \emptyset$ given that $y^*(\mathcal{N}_r)$ is the $(p-c)/p$-quantile of the known distribution of $\tilde{d}(\mathcal{N}_r)$ for $r = 1, \ldots, R$. Next, if $y^*(\mathcal{N}_1) > d_{\min}(\mathcal{N}_1)$ and $y^*(\mathcal{N}_2) < d_{\max}(\mathcal{N}_2)$, we have $\underline{\mathcal{H}}_1(\mathcal{N}) \setminus \mathcal{H}_1(\mathcal{N}) \neq \emptyset$ and $\overline{\mathcal{H}}_2(\mathcal{N}) \setminus \mathcal{H}_2(\mathcal{N}) \neq \emptyset$. Thus, there exist $d_1 < y^*(\mathcal{N}_1)$ and $d_2 > y^*(\mathcal{N}_2)$ such that $\mathbb{P}_{P_r}\left(\tilde{\boldsymbol{d}}_r \in \mathcal{G}_r(d_r)\right) > 0$, where $\mathcal{G}_r(d_r) = \{\boldsymbol{d} \in \text{supp}(P_r) : d(\mathcal{N}_r) = d_r\}$ for $r = 1, 2$.

The main steps of the proof are as follows.

Step 1: From any worst-case joint demand distribution, we construct a new distribution $P(d_{r_1}, d_{r_2})$ in the uncertainty set whose support includes demand vector(s) containing both $\mathcal{G}_1(d_1)$ and $\mathcal{G}_2(d_2)$. This will be formally shown in details in Claim 1 (in Appendix C.4). The main idea used to prove this claim is to exploit the definition of $y^*(\mathcal{N}_r)$ being the $(p-c)/p$-quantile of $d(\mathcal{S}_r)$ in order to construct three columns of $(d_i)_{i \in \mathcal{N}_r}$, as shown in Figure 2, each of which has the same total probabilities for all $r = 1, \ldots, R$. Here, the middle column $\mathcal{H}_r^2$ are copies of $\mathcal{H}_r(\mathcal{N})$ and are formally defined in the proof of Claim 1.

Step 2: The expected profits $v_{P(d_1,d_2)}(y^*_{wc}(\mathcal{N}), \mathcal{N})$ and $v_{P(d_1,d_2)}(y^*_{wc}(\mathcal{N}_1 \cup \mathcal{N}_2), \mathcal{N}_1 \cup \mathcal{N}_2)$ for the grand coalition and for coalition $(\mathcal{N}_1 \cup \mathcal{N}_2)$ can be computed analytically given that $P(d_1, d_2)$ is only slightly different from a worst-case distribution.

Step 3: The newly constructed distribution is used to find a lower bound for $v_{\max}(y^*_{wc}(\mathcal{N}), \mathcal{N}_1 \cup \mathcal{N}_2)$. Finally, this lower bound is used in conjunction with results from Lemma 3 to show that conditions for the existence of core solutions are violated.

We now show the details of each step in the proof sketch of the theorem.

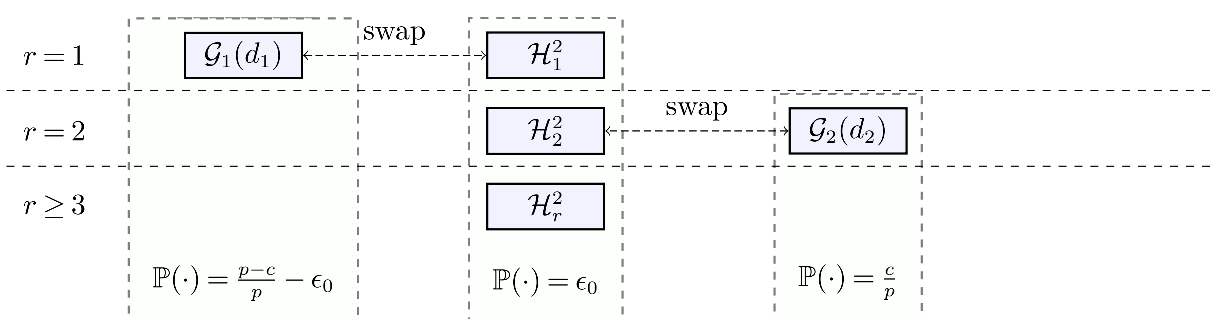

Figure 2 Construction of distributions $P(d_1, d_2)$.

Step 1: Let $\mathcal{G}(d_1, d_2) = \prod_{r=1}^{2} \mathcal{G}_r(d_r) \times \prod_{r=3}^{R} \mathcal{H}_r(\mathcal{N})$. Clearly, $\mathcal{G}(d_1, d_2) \cap \text{supp}(P_{wc}(\mathcal{N})) = \emptyset$ by Lemma 2(c). We make the following claim.

Claim 1: *There exists a distribution $P(d_1,d_2) \in \mathcal{P}(P_1,\ldots,P_R)$ whose support belongs to*

$$\left(\prod_{r=1}^{R} \underline{\mathcal{H}}_r(\mathcal{N})\right) \cup \left(\prod_{r=1}^{R} \overline{\mathcal{H}}_r(\mathcal{N})\right) \cup \mathcal{G}(d_1,d_2),$$

with $\mathbb{P}_{P(d_1,d_2)}\left(\tilde{\boldsymbol{d}} \in \mathcal{G}(d_1,d_2)\right) > 0$.

The proof of this claim is accomplished through two main steps:

Step 1a: Construction of worst-case distribution whose support can be divided into three columns, each with a fixed total probability.

We first construct a (worst-case) distribution $P \in \mathcal{P}(P_1,\ldots,P_r)$ whose support belongs to $\left(\prod_{r=1}^{R} \underline{\mathcal{H}}_r(\mathcal{N})\right) \cup \left(\prod_{r=1}^{R} \overline{\mathcal{H}}_r(\mathcal{N})\right)$. For each $r = 1,\ldots,R$, given the distribution P_r, we construct an equivalent distribution P_r' as follows. Since $y^*(\mathcal{N}_r)$ is the $(p-c)/p$-quantile of the known distribution of $\tilde{d}(\mathcal{N}_r)$, we have

$$\mathbb{P}_{P_r}\left(\tilde{\boldsymbol{d}}_r \in \underline{\mathcal{H}}_r(\mathcal{N}) \setminus \mathcal{H}_r(\mathcal{N})\right) < \frac{p-c}{p} \le \mathbb{P}_{P_r}\left(\tilde{\boldsymbol{d}}_r \in \underline{\mathcal{H}}_r(\mathcal{N}) \setminus \mathcal{H}_r(\mathcal{N})\right) + \mathbb{P}_{P_r}\left(\tilde{\boldsymbol{d}}_r \in \mathcal{H}_r(\mathcal{N})\right),$$

which also implies that $\mathbb{P}_{P_r}\left(\tilde{\boldsymbol{d}}_r \in \overline{\mathcal{H}}_r(\mathcal{N}) \setminus \mathcal{H}_r(\mathcal{N})\right) \le \frac{c}{p}$, for all $r = 1,\ldots,R$. This means we can 'redistribute' the probabilities of $\mathbb{P}_{P_r}\left(\tilde{\boldsymbol{d}}_r \in \mathcal{H}_r(\mathcal{N})\right)$ to both sides of $\mathbb{P}_{P_r}\left(\tilde{\boldsymbol{d}}_r \in \underline{\mathcal{H}}_r(\mathcal{N}) \setminus \mathcal{H}_r(\mathcal{N})\right)$ and $\mathbb{P}_{P_r}\left(\tilde{\boldsymbol{d}}_r \in \overline{\mathcal{H}}_r(\mathcal{N}) \setminus \mathcal{H}_r(\mathcal{N})\right)$ such that the three new parts now have the same total probabilities for all $r = 1,\ldots,R$, as shown in Figure 2. The key observation here is that any demand vector $\tilde{\boldsymbol{d}}_r \in \text{supp}(P_r)$ with a probability $\mathbb{P}_{P_r}\left(\tilde{\boldsymbol{d}}_r)\right)$ can be viewed as several demand vectors (copies) with the same value $\tilde{\boldsymbol{d}}$, but with different probabilities as long as their sum is equal to $\mathbb{P}_{P_r}\left(\tilde{\boldsymbol{d}}_r)\right)$, without changing the newsvendor games.

Formally, we generate three identical copies $\boldsymbol{d}_r^1 = \boldsymbol{d}_r^2 = \boldsymbol{d}_r^3 = \boldsymbol{d}_r$ for all $\boldsymbol{d}_r \in \mathcal{H}_r(\mathcal{N})$ and form three sets $\mathcal{H}_r^1$, $\mathcal{H}_r^2$, and $\mathcal{H}_r^3$, respectively. We construct $\underline{\mathcal{H}}_r'$ by replacing all $\boldsymbol{d}_r \in \mathcal{H}_r(\mathcal{N})$ with their identical copies $\boldsymbol{d}_r^1$ in $\underline{\mathcal{H}}_r(\mathcal{N})$, i.e., $\underline{\mathcal{H}}_r' = \underline{\mathcal{H}}_r(\mathcal{N}) \setminus \mathcal{H}_r(\mathcal{N}) \cup \mathcal{H}_r^1$. $\mathcal{H}_r'$ consists of all second identical copies $\boldsymbol{d}_r^2$ of $\boldsymbol{d}_r \in \mathcal{H}_r$ or equivalently, $\mathcal{H}_r' = \mathcal{H}_r^2$. Finally, we replace all $\boldsymbol{d}_r \in \mathcal{H}_r(\mathcal{N})$ by their third identical copies $\boldsymbol{d}_r^3$ to construct $\overline{\mathcal{H}}_r'$ from $\overline{\mathcal{H}}_r(\mathcal{N})$, i.e., $\overline{\mathcal{H}}_r' = \overline{\mathcal{H}}_r(\mathcal{N}) \setminus \mathcal{H}_r(\mathcal{N}) \cup \mathcal{H}_r^3$. The support of P_r' is defined as $\text{supp}(P_r') = \underline{\mathcal{H}}_r' \cup \mathcal{H}_r' \cup \overline{\mathcal{H}}_r'$. For all $\boldsymbol{d}_r \in \underline{\mathcal{H}}_r(\mathcal{N}) \setminus \mathcal{H}_r(\mathcal{N})$ or $\boldsymbol{d}_r \in \overline{\mathcal{H}}_r(\mathcal{N}) \setminus \mathcal{H}_r(\mathcal{N})$, let $\mathbb{P}_{P_r'}\left(\tilde{\boldsymbol{d}}_r = \boldsymbol{d}_r\right) = \mathbb{P}_{P_r}\left(\tilde{\boldsymbol{d}}_r = \boldsymbol{d}_r\right)$.

Let

$$\epsilon_0 = \frac{1}{2} \min_{r=1,\ldots,R} \left\{ \frac{p-c}{p} - \mathbb{P}_{P_r}\left(\tilde{\boldsymbol{d}}_r \in \underline{\mathcal{H}}_r(\mathcal{N}) \setminus \mathcal{H}_r(\mathcal{N})\right) \right\} > 0,$$

and for identical copies of $\boldsymbol{d}_r \in \mathcal{H}_r(\mathcal{N})$, we are able to redistribute their probabilities as follows:

$$\mathbb{P}_{P_r'}\left(\tilde{\boldsymbol{d}}_r = \boldsymbol{d}_r^1\right) = \mathbb{P}_{P_r}\left(\tilde{\boldsymbol{d}}_r = \boldsymbol{d}_r\right) \cdot \frac{\frac{p-c}{p} - \mathbb{P}_{P_r}\left(\tilde{\boldsymbol{d}}_r \in \underline{\mathcal{H}}_r(\mathcal{N}) \setminus \mathcal{H}_r(\mathcal{N})\right) - \epsilon_0}{\mathbb{P}_{P_r}\left(\tilde{\boldsymbol{d}}_r \in \mathcal{H}_r(\mathcal{N})\right)},$$

$$\mathbb{P}_{P'_r}\left(\tilde{\boldsymbol{d}}_r = \boldsymbol{d}_r^2\right) = \mathbb{P}_{P_r}\left(\tilde{\boldsymbol{d}}_r = \boldsymbol{d}_r\right) \cdot \frac{\epsilon_0}{\mathbb{P}_{P_r}\left(\tilde{\boldsymbol{d}}_r \in \mathcal{H}_r(\mathcal{N})\right)},$$

$$\mathbb{P}_{P'_r}\left(\tilde{\boldsymbol{d}}_r = \boldsymbol{d}_r^3\right) = \mathbb{P}_{P_r}\left(\tilde{\boldsymbol{d}}_r = \boldsymbol{d}_r\right) \cdot \frac{\frac{c}{p} - \mathbb{P}_{P_r}\left(\tilde{\boldsymbol{d}}_r \in \overline{\mathcal{H}}_r(\mathcal{N}) \setminus \mathcal{H}_r(\mathcal{N})\right)}{\mathbb{P}_{P_r}\left(\tilde{\boldsymbol{d}}_r \in \mathcal{H}_r(\mathcal{N})\right)}.$$

Clearly, $\sum_{k=1}^{3} \mathbb{P}_{P'_r}\left(\tilde{\boldsymbol{d}}_r = \boldsymbol{d}_r^k\right) = \mathbb{P}_{P_r}\left(\tilde{\boldsymbol{d}}_r = \boldsymbol{d}_r\right)$ for all $\boldsymbol{d}_r \in \mathcal{H}_r(\mathcal{N})$, which shows P'_r is equivalent to P_r (and belongs to $\mathcal{P}(P_1, \ldots, P_R)$). We also have

$$\begin{aligned}
\mathbb{P}_{P'_r}\left(\tilde{\boldsymbol{d}}_r \in \underline{\mathcal{H}}'_r\right) &= \frac{p-c}{p} - \epsilon_0,\\
\mathbb{P}_{P'_r}\left(\tilde{\boldsymbol{d}}_r \in \mathcal{H}'_r\right) &= \epsilon_0,\\
\mathbb{P}_{P'_r}\left(\tilde{\boldsymbol{d}}_r \in \overline{\mathcal{H}}'_r\right) &= \frac{c}{p},
\end{aligned}$$

for all $r = 1, \ldots, R$. This allows us to construct a probability $P \in \mathcal{P}(P_1, \ldots, P_R)$ or equivalently, $P \in \mathcal{P}(P'_1, \ldots, P'_R)$ with

$$\operatorname{supp}(P) \subseteq \left(\prod_{r=1}^{R} \underline{\mathcal{H}}'_r\right) \cup \left(\prod_{r=1}^{R} \mathcal{H}'_r\right) \cup \left(\prod_{r=1}^{R} \overline{\mathcal{H}}'_r\right).$$

For $\boldsymbol{d} \in \prod_{r=1}^{R} \underline{\mathcal{H}}'_r$, we let

$$\mathbb{P}_P\left(\tilde{\boldsymbol{d}} = \boldsymbol{d}\right) = \frac{1}{\left(\frac{p-c}{p} - \epsilon_0\right)^{R-1}} \cdot \prod_{r=1}^{R} \mathbb{P}_{P'_r}\left(\tilde{\boldsymbol{d}}_r = \boldsymbol{d}_r\right).$$

Similarly, for $\boldsymbol{d} \in \prod_{r=1}^{R} \mathcal{H}'_r$, the probability is set to

$$\mathbb{P}_P\left(\tilde{\boldsymbol{d}} = \boldsymbol{d}\right) = \frac{1}{\epsilon_0^{R-1}} \cdot \prod_{r=1}^{R} \mathbb{P}_{P'_r}\left(\tilde{\boldsymbol{d}}_r = \boldsymbol{d}_r\right),$$

and for $\boldsymbol{d} \in \prod_{r=1}^{R} \overline{\mathcal{H}}'_r$,

$$\mathbb{P}_P\left(\tilde{\boldsymbol{d}} = \boldsymbol{d}\right) = \left(\frac{p}{c}\right)^{R-1} \cdot \prod_{r=1}^{R} \mathbb{P}_{P'_r}\left(\tilde{\boldsymbol{d}}_r = \boldsymbol{d}_r\right).$$

It is straightforward to check that $P \in \mathcal{P}(P_1, \ldots, P_R)$.

Step 1b: We now construct $P(d_1, d_2)$ by modifying $\underline{\mathcal{H}}'_1$ and $\mathcal{H}'_1$ as well as $\overline{\mathcal{H}}'_2$ and $\mathcal{H}'_2$ through 'swapping' $\mathcal{G}_1(d_1)$ with $\mathcal{H}_1^2$ and $\mathcal{G}_2(d_2)$ with $\mathcal{H}_2^2$.

Formally, we generate two identical copies $\boldsymbol{d}_r^1 = \boldsymbol{d}_r^2 = \boldsymbol{d}_r$ for all $\boldsymbol{d}_r \in \mathcal{G}_r(d_r)$ and form two sets, $\mathcal{G}_r^1(d_r)$ and $\mathcal{G}_r^2(d_r)$, respectively, for $r = 1, 2$. We can now define $\underline{\mathcal{H}}''_1 = \underline{\mathcal{H}}'_1 \setminus \mathcal{G}_1(d_1) \cup \mathcal{G}_1^1(d_1)$. Similarly, let $\overline{\mathcal{H}}_2'' = \overline{\mathcal{H}}_2' \setminus \mathcal{G}_2(d_2) \cup \mathcal{G}_2^1(d_2)$. Finally, we set $\mathcal{H}_r'' = \mathcal{H}_r' \cup \mathcal{G}_r^2(d_r)$ for $r = 1, 2$. We now construct another equivalent distribution P_r'' for P_r (and P_r') whose support is $\mathrm{supp}(P_r'') = \underline{\mathcal{H}}_r'' \cup \mathcal{H}_r'' \cup \overline{\mathcal{H}}_r''$ for $r = 1, 2$. We have $\mathbb{P}_{P_r}\left(\tilde{\boldsymbol{d}}_r \in \mathcal{G}_r(d_r)\right) > 0$ for $r = 1, 2$. In addition $\mathbb{P}_{P_r'}\left(\tilde{\boldsymbol{d}}_r = \boldsymbol{d}_r\right) > 0$ for all $\boldsymbol{d}_r \in \mathcal{H}_r'$, $r = 1, 2$. Let

$$\epsilon_r = \frac{1}{2}\min\left\{\mathbb{P}_{P_r}\left(\tilde{\boldsymbol{d}}_r \in \mathcal{G}_r(d_r)\right), \min_{\boldsymbol{d}_r \in \mathcal{H}_r'} \mathbb{P}_{P_r'}\left(\tilde{\boldsymbol{d}}_r = \boldsymbol{d}_r\right)\right\} > 0, \quad r = 1, 2.$$

The probabilities of P_r'', $r = 1, 2$, can be defined as follows. For identical copies of $\boldsymbol{d}_r \in \mathcal{G}_r(d_r)$, $r = 1, 2$, we set

$$\begin{aligned}
\mathbb{P}_{P_r''}\left(\tilde{d}_r = \boldsymbol{d}_r^1\right) &= \mathbb{P}_{P_r}\left(\tilde{\boldsymbol{d}}_r = \boldsymbol{d}_r\right) \cdot \frac{\mathbb{P}_{P_r}\left(\tilde{\boldsymbol{d}}_r \in \mathcal{G}_r(d_r)\right) - \epsilon_r}{\mathbb{P}_{P_r}\left(\tilde{\boldsymbol{d}}_r \in \mathcal{G}_r(d_r)\right)}, \\
\mathbb{P}_{P_r''}\left(\tilde{d}_r = \boldsymbol{d}_r^2\right) &= \mathbb{P}_{P_r}\left(\tilde{\boldsymbol{d}}_r = \boldsymbol{d}_r\right) \cdot \frac{\epsilon_r}{\mathbb{P}_{P_r}\left(\tilde{\boldsymbol{d}}_r \in \mathcal{G}_r(d_r)\right)}.
\end{aligned}$$

For $\boldsymbol{d}_1^1 \in \mathcal{H}_1^1$ and $\boldsymbol{d}_1^2 \in \mathcal{H}_1^2$, the probabilities are

$$\begin{aligned}
\mathbb{P}_{P_1''}\left(\tilde{d}_1 = \boldsymbol{d}_1^1\right) &= \mathbb{P}_{P_1'}\left(\tilde{\boldsymbol{d}}_1 = \boldsymbol{d}_1^1\right) + \frac{\epsilon_1}{|\mathcal{H}_1|}, \\
\mathbb{P}_{P_1''}\left(\tilde{d}_1 = \boldsymbol{d}_1^2\right) &= \mathbb{P}_{P_1'}\left(\tilde{\boldsymbol{d}}_1 = \boldsymbol{d}_1^2\right) - \frac{\epsilon_1}{|\mathcal{H}_1|}.
\end{aligned}$$

Similarly, for $\boldsymbol{d}_2^2 \in \mathcal{H}_2^2$ and $\boldsymbol{d}_2^3 \in \mathcal{H}_2^3$, we define the probabilities as

$$\begin{aligned}
\mathbb{P}_{P_2''}\left(\tilde{d}_2 = \boldsymbol{d}_2^2\right) &= \mathbb{P}_{P_2'}\left(\tilde{\boldsymbol{d}}_2 = \boldsymbol{d}_2^2\right) - \frac{\epsilon_2}{|\mathcal{H}_2|}, \\
\mathbb{P}_{P_2''}\left(\tilde{d}_2 = \boldsymbol{d}_2^3\right) &= \mathbb{P}_{P_2'}\left(\tilde{\boldsymbol{d}}_2 = \boldsymbol{d}_2^3\right) + \frac{\epsilon_2}{|\mathcal{H}_2|}.
\end{aligned}$$

For the remaining $\boldsymbol{d}_r \in \mathrm{supp}(P_r'')$, we let $\mathbb{P}_{P_r''}\left(\tilde{\boldsymbol{d}}_r = \boldsymbol{d}_r\right) = \mathbb{P}_{P_r'}\left(\tilde{\boldsymbol{d}}_r = \boldsymbol{d}_r\right)$ for $r = 1, 2$. Clearly, $\mathbb{P}_{P_r''}\left(\tilde{\boldsymbol{d}}_r = \boldsymbol{d}_r^1\right) + \mathbb{P}_{P_r''}\left(\tilde{\boldsymbol{d}}_r = \boldsymbol{d}_r^2\right) = \mathbb{P}_{P_r}\left(\tilde{\boldsymbol{d}}_r = \boldsymbol{d}_r\right)$ for all $\boldsymbol{d}_r \in \mathcal{G}_r(d_r)$, $r = 1, 2$. In addition,

$$\mathbb{P}_{P_1''}\left(\tilde{\boldsymbol{d}}_1 = \boldsymbol{d}_1^1\right) + \mathbb{P}_{P_1''}\left(\tilde{\boldsymbol{d}}_1 = \boldsymbol{d}_1^2\right) = \mathbb{P}_{P_1'}\left(\tilde{\boldsymbol{d}}_1 = \boldsymbol{d}_1^1\right) + \mathbb{P}_{P_1'}\left(\tilde{\boldsymbol{d}}_1 = \boldsymbol{d}_1^2\right),$$

for $\boldsymbol{d}_1 \in \mathcal{H}_1$. Similarly,

$$\mathbb{P}_{P_2''}\left(\tilde{\boldsymbol{d}}_2 = \boldsymbol{d}_2^2\right) + \mathbb{P}_{P_2''}\left(\tilde{\boldsymbol{d}}_2 = \boldsymbol{d}_2^3\right) = \mathbb{P}_{P_2'}\left(\tilde{\boldsymbol{d}}_2 = \boldsymbol{d}_2^2\right) + \mathbb{P}_{P_2'}\left(\tilde{\boldsymbol{d}}_2 = \boldsymbol{d}_2^3\right),$$

for $\boldsymbol{d}_2 \in \mathcal{H}_2$. It shows P_r'' is equivalent to P_r' (and P_r) for $r = 1, 2$. We also have

$$\begin{aligned}
\mathbb{P}_{P_r''}\left(\tilde{\boldsymbol{d}}_r \in \underline{\mathcal{H}}_r''\right) &= \mathbb{P}_{P_r'}\left(\tilde{\boldsymbol{d}}_r \in \underline{\mathcal{H}}_r'\right) = \frac{p - c}{p} - \epsilon_0, \\
\mathbb{P}_{P_r''}\left(\tilde{\boldsymbol{d}}_r \in \mathcal{H}_r''\right) &= \mathbb{P}_{P_r'}\left(\tilde{\boldsymbol{d}}_r \in \mathcal{H}_r'\right) = \epsilon_0, \\
\mathbb{P}_{P_r''}\left(\tilde{\boldsymbol{d}}_r \in \overline{\mathcal{H}}_r''\right) &= \mathbb{P}_{P_r'}\left(\tilde{\boldsymbol{d}}_r \in \overline{\mathcal{H}}_r'\right) = \frac{c}{p},
\end{aligned}$$

for all $r = 1, 2$. For $r \geq 3$, let $P''_r \equiv P_r$ with $\underline{\mathcal{H}}''_r = \underline{\mathcal{H}}'_r$, $\mathcal{H}''_r = \mathcal{H}'_r$, and $\overline{\mathcal{H}}''_r = \overline{\mathcal{H}}'_r$, we can then again generate a distribution $P(d_1, d_2) \in \mathcal{P}(P''_1, \ldots, P''_R)$ using the same approach as above. We have

$$\text{supp}(P(d_1, d_2)) \subseteq \left(\prod_{r=1}^{R} \underline{\mathcal{H}}''_r\right) \cup \left(\prod_{r=1}^{R} \mathcal{H}''_r\right) \cup \left(\prod_{r=1}^{R} \overline{\mathcal{H}}''_r\right).$$

Note that $\mathcal{G}(d_1, d_2) \subset \prod_{r=1}^{R} \mathcal{H}''_r$. Since $\mathbb{P}_{P''_r}\left(\tilde{\boldsymbol{d}}_r = \boldsymbol{d}_r\right) > 0$ for all $\boldsymbol{d}_r \in \mathcal{H}''_r$, $r = 1, \ldots, R$, clearly $\mathbb{P}_{P(d_1,d_2)}\left(\tilde{\boldsymbol{d}} \in \mathcal{G}(d_1, d_2)\right) > 0$ given how $P(d_1, d_2)$ is constructed.

This completes the proof of Claim 1.

Step 2: Let $\epsilon = \mathbb{P}_{P(d_1,d_2)}\left(\tilde{\boldsymbol{d}} \in \mathcal{G}(d_1, d_2)\right) > 0$ and $\Delta_r = y^*(\mathcal{N}_r) - d_r$ for $r = 1, 2$. Clearly, $\Delta_1 > 0$ and $\Delta_2 < 0$ by the definition of d_1, d_2. We will show that

$$v_{P(d_1,d_2)}(y^*_{wc}(\mathcal{N}), \mathcal{N}) = v_{wc}(\mathcal{N}) + p\epsilon\left[\Delta_1 - (\Delta_1 + \Delta_2)^+\right].$$

The intuition for this result is that, the only two places where the calculation of $v_{P(d_1,d_2)}(y^*_{wc}(\mathcal{N}), \mathcal{N})$ differs from that of $v_{P_{wc}}(y^*_{wc}(\mathcal{N}), \mathcal{N})$ are (a) the addition of the vectors in $\mathcal{G}(d_1, d_2)$ with a probability of ϵ which introduces the new term $-p\epsilon(\Delta_1 + \Delta_2)^+$ and (b) the replacement of the component $\mathcal{G}(d_1)$ by $\mathcal{H}_1^2$ with a probability of ϵ which introduce the new term $p\epsilon\Delta_1$.

More formally, we have:

$$\begin{aligned}
v_{P(d_1,d_2)}(y^*_{wc}(\mathcal{N}), \mathcal{N}) &= (p - c)y^*_{wc}(\mathcal{N}) - p\epsilon(d_1 + d_2 - y^*(\mathcal{N}_1) - y^*(\mathcal{N}_2))^+ \\
&\quad - p \cdot \sum_{\boldsymbol{d} \in \text{supp}(P_{wc}(\mathcal{N}))} \mathbb{P}_{P(d_1,d_2))}\left(\tilde{\boldsymbol{d}} = \boldsymbol{d}\right)(y^*_{wc}(\mathcal{N}) - d(\mathcal{N}))^+ \\
&= (p - c)\sum_{r=1}^{R} y^*(\mathcal{N}_r) - p\epsilon(\Delta_1 + \Delta_2)^+ \\
&\quad - p \cdot \sum_{\boldsymbol{d} \in \prod_{r=1}^{R} \underline{\mathcal{H}}_r(\mathcal{N})} \mathbb{P}_{P(d_1,d_2)}\left(\tilde{\boldsymbol{d}} = \boldsymbol{d}\right) \sum_{r=1}^{R} (y^*(\mathcal{N}_r) - d(\mathcal{N}_r)) \\
&= (p - c)\sum_{r=1}^{R} y^*(\mathcal{N}_r) - p\epsilon(\Delta_1 + \Delta_2)^+ \\
&\quad - p \cdot \sum_{r=1}^{R} \sum_{\boldsymbol{d}_r \in \underline{\mathcal{H}}_r(\mathcal{N}) \setminus \mathcal{H}_r(\mathcal{N})} \mathbb{P}_{P(d_1,d_2)}\left(\tilde{\boldsymbol{d}}_r = \boldsymbol{d}_r, \tilde{\boldsymbol{d}} \in \prod_{r=1}^{R} \underline{\mathcal{H}}_r(\mathcal{N})\right) \delta_r(\boldsymbol{d}_r),
\end{aligned}$$

where $\delta_r(\boldsymbol{d}_r) = y^*(\mathcal{N}_r) - d_r(\mathcal{N}_r)$ and $\tilde{\boldsymbol{d}}_r := (\tilde{d}_i)_{i \in \mathcal{N}_r}$ is the projection of $\tilde{\boldsymbol{d}}$ on $\mathcal{N}_r$. Given that $P(d_1, d_2) \in \mathcal{P}(P_1, \ldots, P_R)$,

$$\mathbb{P}_{P(d_1,d_2)}\left(\tilde{\boldsymbol{d}}_r = \boldsymbol{d}_r, \tilde{\boldsymbol{d}} \in \prod_{r=1}^{R} \underline{\mathcal{H}}_r(\mathcal{N})\right) = \mathbb{P}_{P_r}\left(\tilde{\boldsymbol{d}}_r = \boldsymbol{d}_r\right),$$

for all $\boldsymbol{d}_r \in \underline{\mathcal{H}}_r(\mathcal{N}) \setminus \mathcal{H}_r(\mathcal{N})$, $r = 1, \ldots, R$, except for the case of $r = 1$ and when $\boldsymbol{d}_1 \in \mathcal{G}_1(d_1)$. For this special case, we have:

$$\mathbb{P}_{P(d_1,d_2)}\left(\tilde{\boldsymbol{d}}_1 \in \mathcal{G}_1(d_1),\, \tilde{\boldsymbol{d}} \in \prod_{r=1}^{R} \underline{\mathcal{H}}_r(\mathcal{N})\right) = \mathbb{P}_{P_1}\left(\tilde{\boldsymbol{d}}_1 \in \mathcal{G}_1(d_1)\right) - \epsilon.$$

Thus, we have:

$$\begin{aligned}
v_{P(d_1,d_2)}(y^*_{wc}(\mathcal{N}),\mathcal{N}) &= (p-c)\sum_{r=1}^{R} y^*(\mathcal{N}_r) - p\cdot\sum_{r=1}^{R}\sum_{\boldsymbol{d}_r \in \underline{\mathcal{H}}_r(\mathcal{N})\setminus\mathcal{H}_r(\mathcal{N})} \mathbb{P}_{P_r}\left(\tilde{\boldsymbol{d}}_r = \boldsymbol{d}_r\right)\delta_r(\boldsymbol{d}_r) \\
&\quad + p\epsilon\Delta_1 - p\epsilon(\Delta_1+\Delta_2)^+ \\
&= \sum_{r=1}^{R}\left((p-c)y^*(\mathcal{N}_r) - p\cdot\sum_{\boldsymbol{d}_r\in\operatorname{supp}(P_r)} \mathbb{P}_{P_r}\left(\tilde{\boldsymbol{d}}_r = \boldsymbol{d}_r\right)(\delta_r(\boldsymbol{d}_r))^+\right) \\
&\quad + p\epsilon\left[\Delta_1 - (\Delta_1+\Delta_2)^+\right] \\
&= \sum_{r=1}^{R}\left((p-c)y^*(\mathcal{N}_r) - p\,\mathbb{E}_{P_r}\left[(y^*(\mathcal{N}_r) - d_r(\mathcal{N}_r))^+\right]\right) + p\epsilon\left[\Delta_1 - (\Delta_1+\Delta_2)^+\right] \\
&= \sum_{r=1}^{R}\bar{v}(\mathcal{N}_r) + p\epsilon\left[\Delta_1 - (\Delta_1+\Delta_2)^+\right] \\
&= v_{wc}(\mathcal{N}) + p\epsilon\left[\Delta_1 - (\Delta_1+\Delta_2)^+\right]
\end{aligned}$$

Similarly, with $y^*_{wc}(\mathcal{N}_1\cup\mathcal{N}_2) = y^*(\mathcal{N}_1) + y^*(\mathcal{N}_2)$, we can derive $v_{P(d_1,d_2)}(y^*_{wc}(\mathcal{N}_1\cup\mathcal{N}_2),\mathcal{N}_1\cup\mathcal{N}_2)$ as

$$v_{P(d_1,d_2)}(y^*_{wc}(\mathcal{N}_1\cup\mathcal{N}_2),\mathcal{N}_1\cup\mathcal{N}_2) = \bar{v}(\mathcal{N}_1) + \bar{v}(\mathcal{N}_2) + p\epsilon\left[\Delta_1 - (\Delta_1+\Delta_2)^+\right].$$

Step 3: Now suppose that there exists a core solution $(y,\boldsymbol{z})$. Clearly, according to Lemma 4, $y = y^*_{wc}(\mathcal{N})$. In addition, given that no sub-coalition $\mathcal{S} \subsetneq \mathcal{N}$ has the incentive to break away from the core solution, using Lemma 3, we obtain the following condition:

$$1 = \sum_{i\in\mathcal{N}_1\cup\mathcal{N}_2} z_i + \sum_{r=3}^{R}\sum_{i\in\mathcal{N}_r} z_i \geq v_{\max}(y^*_{wc}(\mathcal{N}),\mathcal{N}_1\cup\mathcal{N}_2) + \sum_{r=3}^{R} v_{\max}(y^*_{wc}(\mathcal{N}),\mathcal{N}_r). \tag{9}$$

Using the definition of $v_{\max}$ and the fact that P_r is known, we have $v_{\max}(y^*_{wc}(\mathcal{N}),\mathcal{N}_r) = \dfrac{\bar{v}(\mathcal{N}_r)}{v_{wc}(\mathcal{N})}$ for all $r = 1,\ldots,R$. Now, we have

$$\begin{aligned}
v_{\max}(y^*_{wc}(\mathcal{N}),\mathcal{N}_1\cup\mathcal{N}_2) &= \max_{P\in\mathcal{P}(P_1,\ldots,P_R)}\left\{\frac{\max_{y\in\mathcal{Y}(\mathcal{N}_1\cup\mathcal{N}_2)} v_P(y,\mathcal{N}_1\cup\mathcal{N}_2)}{v_P(y^*_{wc}(\mathcal{N}),\mathcal{N})}\right\} \\
&\geq \frac{\max_{y\in\mathcal{Y}(\mathcal{N}_1\cup\mathcal{N}_2)} v_{P(d_1,d_2)}(y,\mathcal{N}_1\cup\mathcal{N}_2)}{v_{P(d_1,d_2)}(y^*_{wc}(\mathcal{N}),\mathcal{N})} \\
&\geq \frac{v_{P(d_1,d_2)}(y^*_{wc}(\mathcal{N}_1\cup\mathcal{N}_2),\mathcal{N}_1\cup\mathcal{N}_2)}{v_{P(d_1,d_2)}(y^*_{wc}(\mathcal{N}),\mathcal{N})}
\end{aligned}$$

$$
\begin{aligned}
&= \frac{\bar{v}(\mathcal{N}_1)+\bar{v}(\mathcal{N}_2)+p\epsilon\left[\Delta_1-(\Delta_1+\Delta_2)^+\right]}{v_{wc}(\mathcal{N})+p\epsilon\left[\Delta_1-(\Delta_1+\Delta_2)^+\right]}\\
&> \frac{\bar{v}(\mathcal{N}_1)+\bar{v}(\mathcal{N}_2)}{v_{wc}(\mathcal{N})}.
\end{aligned}
$$

The last inequality is due to the fact that $v_{wc}(\mathcal{N})=\sum_{r=1}^{R}\bar{v}(\mathcal{N}_r)>\bar{v}(\mathcal{N}_1)+\bar{v}(\mathcal{N}_2)>0$ for $R\geq 3$ given that $\bar{v}(\mathcal{N}_r)=\mathbb{E}_{P_r}\left[p\min\{\tilde{d}(\mathcal{N}_r),y^*(\mathcal{N}_r)\}-cy^*(\mathcal{N}_r)\right]>0$ with $y^*(\mathcal{N}_r)>0$ for all $r=1,\ldots,R$, and that $p\epsilon\left[\Delta_1-(\Delta_1+\Delta_2)^+\right]>0$ since $\Delta_1>0>\Delta_2$. Combining this with (9) leads to

$$
1\geq v_{\max}(y^*_{wc}(\mathcal{N}),\mathcal{N}_1\cup\mathcal{N}_2)+\sum_{r=3}^{R}v_{\max}(y^*_{wc}(\mathcal{N}),\mathcal{N}_r)>\frac{\bar{v}(\mathcal{N}_1)+\bar{v}(\mathcal{N}_2)}{v_{wc}(\mathcal{N})}+\sum_{r=3}^{R}\frac{\bar{v}(\mathcal{N}_r)}{v_{wc}(\mathcal{N})}=1.
$$

This is a contradiction. Thus, there is no core solution for the given robust newsvendor game.

□

C.5. Proof of Proposition 1

Proof. For an arbitrary $\mathcal{S}\subsetneq\mathcal{N}$, let $\mathcal{S}_r=\mathcal{S}\cap\mathcal{N}_r$, $r=1,\ldots,R$ and $\mathcal{R}(\mathcal{S})=\{r\,|\,\mathcal{S}_r\neq\emptyset\}$. We start with the first case when $\mathbb{P}\left(\tilde{d}(\mathcal{S}_r)=d_{\min}(\mathcal{S}_r)\right)\geq 1-\frac{c}{Rp}$ for all $\mathcal{S}_r\subseteq\mathcal{N}_r$, $r=1,\ldots,R$.

Using the fact that $\mathbb{P}(A\cap B)=\mathbb{P}(A)+\mathbb{P}(B)-\mathbb{P}(A\cup B)\geq\mathbb{P}(A)+\mathbb{P}(B)-1$, where A and B are arbitrary random events, we have

$$
\begin{aligned}
\mathbb{P}\left(\tilde{d}(\mathcal{S}_r)=d_{\min}(\mathcal{S}_r)\,|\,r\in\mathcal{R}(\mathcal{S})\right) &\geq \sum_{r\in\mathcal{R}(\mathcal{S})\neq\emptyset}\mathbb{P}\left(\tilde{d}(\mathcal{S}_r)=d_{\min}(\mathcal{S}_r)\right)-(|\mathcal{R}(\mathcal{S})|-1)\\
&\geq |\mathcal{R}(\mathcal{S})|\left(1-\frac{c}{Rp}\right)-(|\mathcal{R}(\mathcal{S})|-1)\\
&\geq \frac{c-p}{p}>0,
\end{aligned}
$$

where the third inequality is due to the fact that $|\mathcal{R}(\mathcal{S})|\leq R$. In addition, since $\tilde{\boldsymbol{d}}(\mathcal{S})\geq\sum_{r\in\mathcal{R}(\mathcal{S})}d_{\min}(\mathcal{S}_r)$ for all $\tilde{\boldsymbol{d}}(\mathcal{S})\in\mathrm{supp}(P)$, $P\in\mathcal{P}(P_1,\ldots,P_R)$, we have $\sum_{r=1}^{R}d_{\min}(\mathcal{S}_r)$ is the optimal ordering quantity for coalition $\mathcal{S}$.

Thus, we have

$$
\max_{y\in\mathcal{Y}(\mathcal{S})}v_P(y,\mathcal{S})=v_P(d_{\min}(\mathcal{S}),\mathcal{S})=(p-c)\sum_{r=1}^{R}d_{\min}(\mathcal{S}_r)
$$

for all $P\in\mathcal{P}(P_1,\ldots,P_R)$.

According to Lemma 4, in order to construct a core solution $(y,\boldsymbol{z})$, we need to consider $y=y^*_{wc}(\mathcal{N})=\sum_{r=1}^{R}y^*(\mathcal{N}_r)$. Since $\mathbb{P}\left(d(\mathcal{N}_r)=d_{\min}(\mathcal{N}_r)\right)\geq\frac{p-c}{p}$, we have $y^*(\mathcal{N}_r)=d_{\min}(\mathcal{N}_r)$. Thus,

$$
v_P(y^*_{wc}(\mathcal{N}),\mathcal{N})=\mathbb{E}_P\left[p\min\left\{\tilde{d}(\mathcal{N}),\sum_{r=1}^{R}d_{\min}(\mathcal{N}_r)\right\}-c\sum_{r=1}^{R}d_{\min}(\mathcal{N}_r)\right]=(p-c)\sum_{r=1}^{R}d_{\min}(\mathcal{N}_r)
$$

for all $P \in \mathcal{P}(P_1,\dots,P_R)$ since $\tilde{d}(\mathcal{N}) \geq d_{\min}(\mathcal{N}) \geq \sum_{r=1}^{R} d_{\min}(\mathcal{N}_r)$.

Thus, we have

$$v_{\max}(y^*_{wc}(\mathcal{N}),\mathcal{S}) = \max_{P\in\mathcal{P}(P_1,\dots,P_R)} \max_{y\in\mathcal{Y}(\mathcal{S})} \frac{v_P(y,\mathcal{S})}{v_P(y^*_{wc}(\mathcal{N}),\mathcal{N})} = \left(\sum_{r=1}^{R} d_{\min}(\mathcal{N}_r)\right)^{-1} \cdot \sum_{r=1}^{R} d_{\min}(\mathcal{S}_r).$$

Now, let $\boldsymbol{x}^r$ be a core solution of the deterministic newsvendor games defined on $\mathcal{N}_r$. Such a core solution exists since the joint demand distribution P_r is known for $r=1,\dots,R$. We can define $z_i = \left((p-c)\sum_{r=1}^{R} d_{\min}(\mathcal{N}_r)\right)^{-1} \cdot x_i^{r(i)}$ for all $i\in\mathcal{N}$, where $\mathcal{N}_{r(i)} \ni i$. It is straightforward to show that $\sum_{i\in\mathcal{N}} z_i = 1$ and since $\sum_{i\in\mathcal{S}_r} x_i^r \geq \bar{v}(\mathcal{S}_r) = (p-c)d_{\min}(\mathcal{S}_r)$ for all $r=1,\dots,R$, we have

$$\sum_{i\in\mathcal{S}} z_i = \sum_{r=1}^{R}\sum_{i\in\mathcal{S}_r} z_i \geq \sum_{r=1}^{R} d_{\min}(\mathcal{S}_r)\cdot\left(\sum_{r=1}^{R} d_{\min}(\mathcal{N}_r)\right)^{-1} = v_{\max}(y^*_w c(\mathcal{N}),\mathcal{S})$$

for all $\mathcal{S} \subsetneq \mathcal{N}$. This shows that $(y^*_{wc}(\mathcal{N}),\boldsymbol{z})$ is a core solution of the robust newsvendor game. For the second case where $\mathbb{P}\left(\tilde{d}(\mathcal{S}_r) = d_{\max}(\mathcal{S}_r)\right) \geq 1 - \frac{p-c}{Rp}$ for all $\mathcal{S}_r \subseteq \mathcal{N}_r$, $r=1,\dots,R$, the same arguments can be applied.

□

C.6. Proof of Proposition 2

Proof. For $N=2$, we have shown that the robust newsvendor game always has core solutions (i.e., imputations). We consider the following cases for $N \geq 3$ as follows.

Case 1: $\mathbb{P}_{P_0}\left(\tilde{d} = d_{\min}\right) \geq 1 - \frac{c}{(N-1)p}$ or $\mathbb{P}_{P_0}\left(\tilde{d} = d_{\max}\right) \geq 1 - \frac{p-c}{(N-1)p}$. Using the same approach as in the proof of Proposition 1 with the fact that for all $\mathcal{S} \subsetneq \mathcal{N}$, $|\mathcal{R}(\mathcal{S})| \leq N-1$, we can show that core solutions exist.

Case 2: $\mathbb{P}_{P_0}\left(\tilde{d} = d_{\min}\right) < 1 - \frac{c}{(N-1)p}$ and $\mathbb{P}_{P_0}\left(\tilde{d} = d_{\max}\right) < 1 - \frac{p-c}{(N-1)p}$. We will show that core solutions do not exist in this case by considering two smaller cases.

Case 2.1: $\mathbb{P}_{P_0}\left(\tilde{d} = d_{\min}\right) < 1 - \frac{c}{p}$ and $\mathbb{P}_{P_0}\left(\tilde{d} = d_{\max}\right) < 1 - \frac{p-c}{p}$. In this case, we have $\mathbb{P}\left(\tilde{d}_i = 0\right) < 1 - \frac{c}{p}$ since $d_{\min}(\{i\}) \geq 0$ for all $i=1,\dots,N$. Thus, according to Theorem 3, the robust newsvendor game has no core solution.

Case 2.2: $1 - \frac{c}{p} \leq \mathbb{P}_{P_0}\left(\tilde{d} = d_{\min}\right) < 1 - \frac{c}{(N-1)p}$ or $1 - \frac{p-c}{p} \leq \mathbb{P}_{P_0}\left(\tilde{d} = d_{\max}\right) < 1 - \frac{p-c}{(N-1)p}$. We are going to show that in this case, the robust newsvendor game also has no core solution. Let consider the case $1 - \frac{c}{p} \leq \mathbb{P}_{P_0}\left(\tilde{d} = d_{\min}\right) < 1 - \frac{c}{(N-1)p}$. Given the results of Lemma 4, we only need to consider $y = y^*_{wc}(\mathcal{N})$. Since $\mathbb{P}_{P_0}\left(\tilde{d} = d_{\min}\right) \geq 1 - \frac{c}{p}$, we have: $y^*(\mathcal{N}_i) = d_{\min}$ for all $i\in\mathcal{N}$. Thus,

$$v_P(y^*_{wc}(\mathcal{N},\mathcal{N})) = (p-c)N d_{\min}$$

for all $P \in \mathcal{P}(P_1,\ldots,P_N)$. Let $\alpha = \min\left\{\frac{q}{N-2}, 1-q\right\} > 0$ where $q = \mathbb{P}_{P_0}\left(\tilde{d} = d_{\min}\right)$ and let $\mathcal{S} \subsetneq \mathcal{N}$ be a coalition with $|\mathcal{S}| = N-1$. We make the following claim:

Claim 2: *There exists a probability distribution $P(\alpha,\mathcal{S}) \in \mathcal{P}(P_1,\ldots,P_N)$ such that*

$$\mathbb{P}_{P(\alpha,\mathcal{S})}\left(\tilde{d}(\mathcal{S}) = (N-1)d_{\min}\right) < 1 - \frac{c}{p}.$$

The proof of this claim is as follows. Without loss of generality, let $\mathcal{S} = \{1,\ldots,N-1\}$. We will first construct a feasible probability distribution P for the first $N-1$ demands $(\tilde{d}_1,\ldots,\tilde{d}_{N-1})$, which can then be used to construct $P(\alpha,\mathcal{S})$ such that $\tilde{d}_N$ is independent of $(\tilde{d}_1,\ldots,\tilde{d}_{N-1})$. Given the distribution P_0, we construct an equivalent distribution P_0' by making two identical copies of the set $\overline{\mathcal{H}} = \{d \,|\, d > d_{\min}\}$ as $\overline{\mathcal{H}}^1$ and $\overline{\mathcal{H}}^2$. We also denote $\mathcal{H} = \{d_{\min}\}$. We set the probabilities for $d^1 \in \overline{\mathcal{H}}^1$ and $d^2 \in \overline{\mathcal{H}}^2$ with $d^1 = d^2 = d$ as follows:

$$\mathbb{P}_{P_0'}\left(\tilde{d}' = d^1\right) = \frac{\alpha}{1-q} \cdot \mathbb{P}_{P_0}\left(\tilde{d} = d\right),$$

and

$$\mathbb{P}_{P_0'}\left(\tilde{d}' = d^2\right) = \frac{1-q-\alpha}{1-q} \cdot \mathbb{P}_{P_0}\left(\tilde{d} = d\right) \geq 0$$

since $\alpha \leq 1-q$. The distributions P_i are replaced by $P_i' \equiv P_0'$ for all $i = 1,\ldots,N$ and the support of P consists of three sets, $\prod_{i=1}^{N-1} \mathcal{H}_i$, $\bigcup_{i=1}^{N-1}\left(\overline{\mathcal{H}}_i^1 \times \prod_{j\neq i, j\leq N-1} \mathcal{H}_j\right)$, and finally, $\prod_{i=1}^{N-1} \overline{\mathcal{H}}_i^2$. For each $i = 1,\ldots,N-1$, we set the probabilities for elements in $\overline{\mathcal{H}}_i^1 \times \prod_{j\neq i, j\leq N-1} \mathcal{H}_j$ using the probabilities for the corresponding elements of $\overline{\mathcal{H}}_i^1$. According to our setting, the total probability for each element in $\overline{\mathcal{H}}_i^1, i = 1,\ldots,N-1$, are set to equal to α. Since each element in $\overline{\mathcal{H}}_i^1$ take $(N-2)$ elements from $\mathcal{H}_j, j \neq i, j \leq N-1$, the remaining probability for the first element $(d_{\min},\ldots,d_{\min})$ is $q-(N-2)\alpha \geq 0$ given that $\alpha \leq \frac{q}{N-2}$. Finally, if $\alpha < 1-q$, for $\boldsymbol{d}^2 \in \prod_{i=1}^{N-1} \overline{\mathcal{H}}_i^2$, we set its probability as follows:

$$\mathbb{P}\left(\tilde{\boldsymbol{d}}' = \boldsymbol{d}^2\right) = \frac{1}{(1-q-\alpha)^{N-2}} \prod_{i=1}^{N-1} \mathbb{P}_{P_i'}\left(\tilde{d}_i' = d_i^2\right).$$

If $\alpha = 1-q$, we can simply remove $\prod_{i=1}^{N-1} \overline{\mathcal{H}}_i^2$ from the support of P. It is straightforward to check that $\mathbb{P}_P\left(\tilde{d}_i = d_{\min}\right) = q$ and $\mathbb{P}_P\left(\tilde{d}_i = d\right) = \mathbb{P}_{P_0}\left(\tilde{d} = d\right)$ for all $d \in \overline{\mathcal{H}}$, $i = 1,\ldots,N-1$. Using this distribution P, we can construct $P(\alpha,\mathcal{S})$ as discussed above. Finally, we have

$$\begin{aligned}
\mathbb{P}_{P(\alpha,\mathcal{S})}\left(\tilde{d}(\mathcal{S}) = (N-1)d_{\min}\right) &= \mathbb{P}_{P(\alpha,\mathcal{S})}\left(\tilde{d}_i = d_{\min} \,|\, i = 1,\ldots,N-1\right) \\
&= q - (N-2)\alpha = q - (N-2)\min\left(\frac{q}{N-2}, 1-q\right)
\end{aligned}$$

$$
\begin{aligned}
&= \max(0, (N-1)q - (N-2)) \\
&< \max(0, (N-1)\left(1 - \frac{c}{(N-1)p}\right) - (N-2)) \\
&= \max(0, \frac{p-c}{p}) = \frac{p-c}{p}.
\end{aligned}
$$

This completes the proof of Claim 2.

Now, with the existence of $P(\alpha, \mathcal{S})$, this shows that $(N-1)d_{\min}$ is not the optimal ordering quantity for the coalition $\mathcal{S}$ with respect to the demand distribution $P(\alpha, \mathcal{S})$, i.e.,

$$\max_{y \in \mathcal{Y}(\mathcal{S})} v_{P(\alpha,\mathcal{S})}(y, \mathcal{S}) > v_{P(\alpha,\mathcal{S})}(d_{\min}(\mathcal{S}), \mathcal{S}) = (p-c)\,|\mathcal{S}|\,d_{\min} = (p-c)(N-1)d_{\min}.$$

Thus, we have

$$v_{\max}(y^*_{wc}(\mathcal{N}), \mathcal{S}) = \max_{P \in \mathcal{P}(P_1,\dots,P_N)} \max_{y \in \mathcal{Y}(\mathcal{S})} \frac{v_P(y, \mathcal{S})}{v_P(y^*_{wc}(\mathcal{N}, \mathcal{N})} > \frac{N-1}{N}.$$

Assuming there is core solution $(y^*_{wc}(\mathcal{N}), \boldsymbol{z})$, we then have $\sum_{i \in \mathcal{N}} z_i = 1$ and

$$\sum_{i \in \mathcal{S}} z_i \geq v_{\max}(y^*_{wc}(\mathcal{N}), \mathcal{S}) > \frac{N-1}{N}$$

for all $\mathcal{S} \subsetneq \mathcal{N}$ with $|\mathcal{S}| = N-1$. Thus, we have

$$N - 1 = (N-1)\sum_{i \in \mathcal{N}} z_i = \sum_{\mathcal{S} \subsetneq \mathcal{N} : |\mathcal{S}| = N-1} \sum_{i \in \mathcal{S}} z_i > N \cdot \frac{N-1}{N} = N - 1,$$

which is a contradiction. Thus, there is no core solution. The same arguments can be applied for the case when $1 - \frac{p-c}{p} \leq \mathbb{P}_{P_0}\left(\tilde{d} = d_{\max}\right) < 1 - \frac{p-c}{(N-1)p}$.

$\square$