Equivalence of Wave Linear Repetitive Processes and the Singular 2-D Roesser State-Space Model

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ABSTRACT
This paper develops a direct method for transforming a polynomial system matrix describing a discrete wave linear repetitive process to a 2-D singular state-space Roesser model description where all relevant properties, including the zero coprimeness properties of the system matrix, are retained. It is shown that the transformation is zero coprime system equivalence. The structure of the resulting system matrix in singular form and the transformation are also established.

KEYWORDS
Linear wave repetitive processes, 2-D discrete systems, System matrix, 2-D singular Roesser form, Zero-coprime system equivalence.

1. Introduction

Research on systems with $n > 1$ independent variables (vector-valued for multiple-input multiple-output examples), commonly known as n-D systems, using a state-space setting and, for linear dynamics, a transfer-function or transfer-function matrix whose entries are defined in terms of n independent is a well established area in control systems analysis. Extensively used state-space models for these systems are due Roesser (1975) and Fornasini and Marchesini (1976), where various forms of the latter can be considered. Also the singular general model was developed in Kaczorek (1988). In, e.g., Zak (1984), the use of such models to describe the dynamics of systems described by partial differential equations is considered.

Realization theory for linear systems with time as the single independent variable and hence also known as 1-D systems in some of the n-D systems literature is of fundamental importance. Basically, given the controllability and observability properties, a 1-D linear time-invariant system is equivalently described by the corresponding state-space model and transfer-function (or transfer-function matrix). This general problem has also been addressed for nD linear systems for both nonsingular and singular state-space model descriptions of the dynamics, see, e.g., Boudellioua (2012); Galkowski (2001); Xu, Yan, Lin, and ya Matsushita (2012) and references therein. The analysis in this case is more complicated due to the complexity of the underlying analysis tools, e.g., for polynomials in $n > 1$ indeterminates primeness is no longer a single concept.

Linear repetitive processes make a series of sweeps, termed passes (or trials), through dynamics defined over a finite duration known as the pass (or trial) length. Once each pass is complete, the process resets to the starting point and the next pass can begin, either immediately after resetting is complete or after a further period of time has elapsed. The output on each pass is termed that pass profile and also the profile produced on the previous pass (or, more generally, on a finite number of previous passes) acts as a forcing function on the next pass and thereby contributes to the associated pass profile. The result can be oscillations that increase in amplitude from pass-to-pass and analysis and control law design for these processes requires the development of a stability theory and associated control law design algorithms.

Background on linear repetitive processes, including their origins in the modeling and control of industrial processes such as long-wall coal cutting with references to the original work, is given in Rogers, Galkowski, and Owens (2007). These processes have structural links with 2-D linear systems recursive over the upper-right quadrant of the 2-D plane. A critical difference, however, is that in repetitive processes information propagation in one independent variable (time) only occurs over a finite duration and this is an inherent property of the dynamics as opposed to an assumption.

A considerable volume of research has investigated the use of 2-D systems theory to solve control problems for linear repetitive processes, where use has been made of both the nonsingular and singular versions of these systems. Early work established that stability tests can be exchanged Rogers et al. (2007) (but only provided the boundary conditions are of a particular form). Also in Galkowski, Rogers, and Owens (1998) it has been shown that conditions for local controllability of discrete linear repetitive processes can be obtained by writing the dynamics of these processes in the form of a singular 2-D Roesser state-space model.

This last result, in particular, stimulated research to establish the connection between the state-space models of liner repetitive processes and those of the Roesser/Fornasini 2-D models with early results in, e.g., Galkowski, Rogers, and Owens (1999); Rogers et al. (2007). More recently in Boudellioua, Galkowski, and Rogers (2017a) it was shown that the dynamics of a linear repetitive process can, under certain conditions, represented by a 2-D singular Roesser model. An elementary operations based method for transforming a polynomial matrix description of a wave linear repetitive processes, with dynamics motivated by industrial examples that are not captured by the basic model, to a 2-D nonsingular Roesser model was developed in Boudellioua, Galkowski, and Rogers (2017b). This method was extended in Galkowski, Boudellioua, and Rogers (2017) to write wave linear repetitive processes in the 2-D singular Roesser state-space form, where the transformation required is Input/Output equivalence.

In this paper, a method based on a stronger type of equivalence is developed to transform wave linear repetitive process dynamics, which are characterized by the noncausal dynamics along the pass (information from left-to-right and right-to-left), to those described by a 2-D Roesser singular model description such that both the input/output properties of the system and the zero structure are preserved. Furthermore the exact equivalence transformation linking the original system with its associated singular model is established. The type of equivalence used has been the subject of considerable attention in the literature see, e.g., El-Nabrawy (2006); Levy (1981); Pugh, McInerney, and El-Nabrawy (2005), Johnson (1993) and Pugh, McInerney, Hou, and Hayton (1996); Pugh, McInerney, Boudellioua, Johnson, and Hayton (1998).

In this paper, the notation I_h denotes the $h \times h$ identity matrix and $O_{p,q}$ the $p \times q$ null matrix. In cases where the dimensions of the latter matrix is obvious, the subscript will be deleted.

2. 2-D Discrete Linear Systems and Repetitive Processes

A singular version of the Roesser state-space model (SR) Roesser (1975) is

$$
\begin{aligned}
E\begin{bmatrix} x^h(i+1,j) \\ x^v(i,j+1) \end{bmatrix} &= \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}\begin{bmatrix} x^h(i,j) \\ x^v(i,j) \end{bmatrix} + \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} u(i,j), \\
y(i,j) &= \begin{bmatrix} C_1 & C_2 \end{bmatrix}\begin{bmatrix} x^h(i,j) \\ x^v(i,j) \end{bmatrix} + Du(i,j),
\end{aligned} \tag{1}
$$

where $x^h(i,j) \in \mathbb{R}^{n_1}$ is the horizontal state vector and $x^v(i,j) \in \mathbb{R}^{n_2}$, is the vertical state vector, $y(i,j) \in \mathbb{R}^m$ is the output vector, $u(i,j) \in \mathbb{R}^l$, is the input vector and the matrix E is square and singular. In this model, static in both directions i and j links between sub-vectors are allowed. If $E = I$ in this model the most commonly used form of the Roesser model is obtained, which is also termed nonsingular in some of the literature. Boundary conditions are defined as $x^h(0,j) = f(j)$, $j \geq 0$ and $x^v(i,0) = d(i)$, $i \geq 0$, where the vectors $f(j) \in \mathbb{R}^{n_1}$ and $d(i) \in \mathbb{R}^{n_2}$ have known constant entries.

Discrete linear repetitive processes evolve over the subset of the positive quadrant in the 2-D plane defined by $\{(p,k) : 0 \leq p \leq \alpha - 1, k \geq 0\}$, and the most basic state-space model for their dynamics has the following form Rogers et al. (2007)

$$
\begin{aligned}
x_{k+1}(p+1) &= Ax_{k+1}(p) + Bu_{k+1}(p) + B_0 y_k(p), \\
y_{k+1}(p) &= Cx_{k+1}(p) + Du_{k+1}(p) + D_0 y_k(p).
\end{aligned} \tag{2}
$$

where α denotes the number of samples along the pass (α times the sampling period gives the pass length). On pass k, $x_k(p) \in \mathbb{R}^n$ is the state vector, $y_k(p) \in \mathbb{R}^m$ is the pass profile vector, and $u_k(p) \in \mathbb{R}^l$ is the vector of control inputs. The simplest form of boundary conditions are $x_{k+1}(0) = d_{k+1}$, $k \geq 0$, where the vector $d_{k+1} \in \mathbb{R}^n$ has known constant entries, and $y_0(p) = f(p)$, where $f(p) \in \mathbb{R}^m$ has entries that are known functions of p.

The 2-D systems structure of a repetitive process arises from the influence of the previous pass profile on the current pass state and pass profile vectors, i.e., due, respectively, to the presence of the terms $B_0 y_k(p)$ and $D_0 y_k(p)$ in (2). The updating structures of a discrete linear system described by a 2-D Roesser model and a discrete linear repetitive process are illustrated schematically in Figure 1.

It was shown in, e.g., Galkowski et al. (1999) that particular properties of discrete linear repetitive processes described by (2) can be analyzed using the 2-D Roesser model setting (1).

In the repetitive process model (2), the only previous pass (k) contribution to the dynamics at p on the current pass ($k+1$) comes from the same instance. A more general discrete linear repetitive process that also evolves over $\{(p,k) : 0 \leq p \leq \alpha - 1, k \geq 0\}$ but the previous pass (k) contribution to the dynamics at the given sample p on the current pass ($k+1$) comes from a pre-specified window of points is described by the

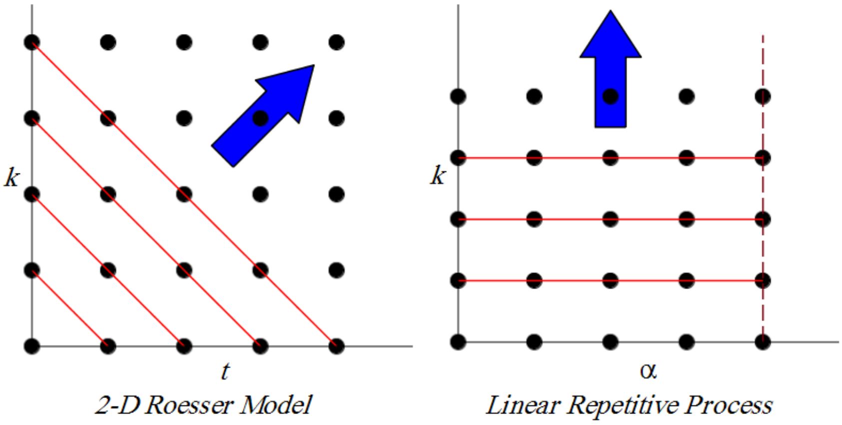

Figure 1. Evolution of the dynamics of the Roesser model and a discrete linear repetitive process.

state-space model

$$
\begin{aligned}
x_{k+1}(p+1) &= Ax_{k+1}(p) + Bu_{k+1}(p) + \sum_{i=-w_L}^{w_H} B_i y_k(p+i), \\
y_{k+1}(p) &= Cx_{k+1}(p) + Du_{k+1}(p) + \sum_{i=-w_L}^{w_H} D_i y_k(p+i),
\end{aligned} \tag{3}
$$

where on pass k, $x_k(p) \in \mathbb{R}^n$ is the state vector, $y_k(p) \in \mathbb{R}^m$ is the pass profile vector, $u_k(p) \in \mathbb{R}^l$ is the vector of control inputs and w_L, w_H are positive integers.

On each pass in (3), the previous pass (k) 'window' of previous pass profile sample values $p - w_L \leq p \leq p + w_H$ contribute to the current pass profile at sample p on pass $k+1$, where this window moves along the pass with p. This has led the term 'wave'linear repetitive process Galkowski, Cichy, Rogers, and Lam (2006) to describe examples represented by this model. Also setting $w_L = 0$ and $w_H = 0$ recovers the previous state-space model 2.

The state and pass profile updating structures, respectively, of a wave linear repetitive process are shown schematically in Figures 2 and 2.

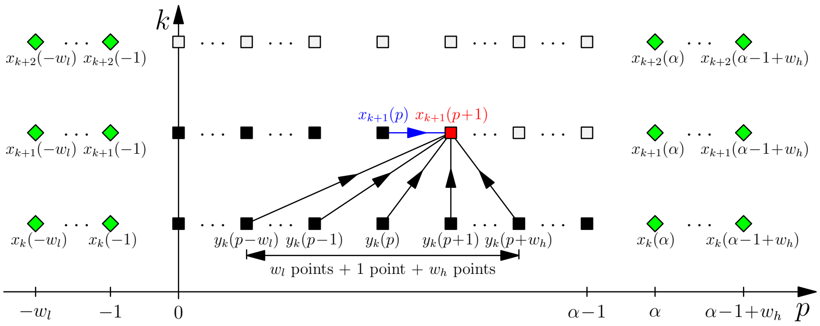

Figure 2. Illustrating the updating structure of the current state vector in (3).

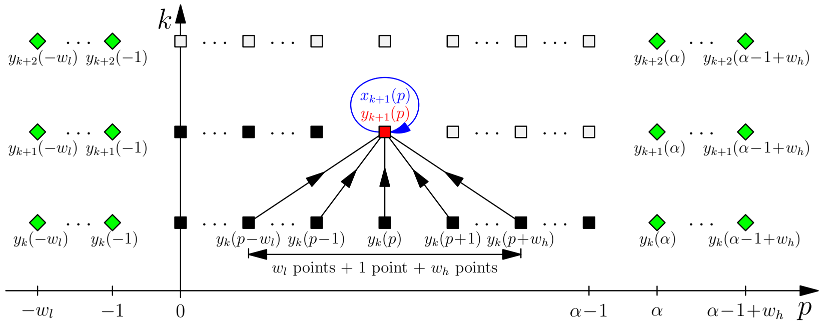

Figure 3. Illustrating the updating structure of the current pass profile vector in (3).

The boundary conditions for a wave linear repetitive process are of the form

$$\begin{aligned} x_{k+1}(0) &= d_{k+1}, \quad k \geq 0, \\ y_0(p) &= f(p), \quad 0 \leq p \leq \alpha - 1, \\ x_{k+1}(i) &= 0, \quad y_k(i) = 0, \\ i \in \{-w_L, \ldots, -1\} &\cup \{\alpha, \ldots, \alpha - 1 + w_H\}, \ k \geq 0, \end{aligned} \tag{4}$$

where the vector $d_{k+1} \in \mathbb{R}^n$ has known constant entries and $f(p) \in \mathbb{R}^m$ has entries that are known functions of p. In the remainder of this paper, this model is denoted by WLRP.

One area where a wave linear repetitive process state-space model description of the dynamics arises is iterative learning control (ILC). This design method has been especially developed for the many applications where the same finite duration task is performed over and over again. Each repetition is termed a pass (or trial in some of the literature) and the duration of each pass is termed the pass length. The survey papers Ahn, Chen, and Moore (2007); Bristow, Tharayil, and Alleyne (2006) can be used as a starting point for the literature.

In the simplest form of operation the system resets to the starting location at the end of each pass and the next pass can begin either immediately resetting is complete or after a further period of time has elapsed. Once a pass is complete all information generated over the pass length is available for use in constructing the input for the next pass. The core task in ILC design, therefore, is how to use this information to best effect in improving performance from pass-to-pass and the most common route is to construct the input for the next pass as the sum of the previous pass input and a correction term computed using previous pass data (or a finite number of previous passes).

Let $u_k(p)$ denote the input to system in the ILC setting on pass k and let $e_k(p)$ denote the error between a supplied reference signal and the output (pass profile) $y_k(p)$. Then the design problem is to force the sequence $\{e_k\}$ to converge in k to either zero or to within an acceptable bound. One basic ILC law has the form

$$u_{k+1}(p) = u_k(p) + Ke_k(p+1), \tag{5}$$

where K is a scalar gain (or matrix in the multi-input multi-output case). This law is known as phase lead ILC due to the shift advance in p in the last term (this unit advance can be replaced by $\lambda > 1$). This is also the novel feature of ILC, i.e., the inclusion of non-causal temporal information from the previous pass (or a finite number therefore). It is also known that if such information is not included then the ILC law can be replaced by an equivalent feedback control law (i.e., no learning), see the discussion in Ahn et al. (2007); Bristow et al. (2006) and the relevant cited articles in these references.

For a control law of the form (5) it is possible to write the resulting control dynamics in the form (2) where the pass profile on pass k is the corresponding error $e_k(p)$ and hence the stability theory for discrete linear repetitive processes can be used for ILC design. This is a single step design for error convergence from pass-to-pass and along the pass dynamics. This setting has led to experimental verification, see, e.g., Paszke, Rogers, and Gałkowski (2016) which reports the design and experimental verification of an iterative learning control law designed in the repetitive process setting.

The design referred to above requires a re-definition of the state vector in the repetitive process as a shifted version of the difference between these vectors on two successive passes. This removes the terms in $p+1$ from the repetitive process model arising from the second term in (5) and enables direct use of linear repetitive process stability theory for design. However, this operation is not possible if (5) contains further phase-lead terms, e.g., at $p+2$ or 'phase-lag', e.g., at $p-1$ terms. Such a control law fits naturally within a wave linear repetitive process state-space model.

The similarity between wave linear repetitive processes and the Roesser model is much less obvious than for processes described by (2). However, there still remains the question: is it possible to convert a wave linear repetitive process to Roesser form? This question is the subject of the rest of this paper with the emphasis on system matrix equivalence.

3. System Equivalence

Following the formulation of Rosenbrock Rosenbrock (1970) for 1D linear systems, a general 2-D linear system can be represented by the following system of equations assuming zero boundary conditions:

$$\begin{array}{rcl} T(z_1,z_2)x & = & U(z_1,z_2)u \\ y & = & V(z_1,z_2)x + W(z_1,z_2)u. \end{array} \tag{6}$$

where $x \in \mathbb{R}^n$ is the state vector, $u \in \mathbb{R}^l$ is the input vector and $y \in \mathbb{R}^m$ is the output vector, T, U, V and W are polynomial matrices with elements in $\mathbb{R}[z_1, z_2]$ of dimensions $n \times n, n \times l, m \times n$ and $m \times l$ respectively. In this system of equations z_1 and z_2 are shift operators, which for the Roesser model horizontal and vertical state vectors (and similarly for other cases) are defined as

$$z_1 x^h(i,j) = x^h(i+1,j), \quad z_2 x^v(i,j) = x^v(i,j+1). \tag{7}$$

The system of equations (6) gives rise to the system matrix

$$P(z_1,z_2) = \begin{bmatrix} T(z_1,z_2) & U(z_1,z_2) \\ -V(z_1,z_2) & W(z_1,z_2) \end{bmatrix}, \tag{8}$$

where

$$P(z_1,z_2)\begin{bmatrix} x \\ -u \end{bmatrix} = \begin{bmatrix} 0 \\ -y \end{bmatrix}. \tag{9}$$

In the case when $T(z_1,z_2)$ invertible, (8) is said to be regular and the corresponding 2-D transfer-function matrix is

$$G(z_1,z_2) = V(z_1,z_2)T^{-1}(z_1,z_2)U(z_1,z_2) + W(z_1,z_2). \tag{10}$$

There are a number of concepts of equivalence in n-D systems theory and this particular area has been the subject of much research, e.g., in more recent work a transformation based on a module theoretic approach was considered Bachelier and Cluzeau (2017). A basic transformation proposed for the study of 2-D system matrices is zero coprime system equivalence as given in Johnson (1993); Levy (1981). This transformation may be regarded as an extension of Fuhrmann's strict system equivalence Fuhrmann (1977) from the 1-D to the 2-D case (and n-D, $n \geq 3$) and is characterized by the following definition.

Definition 3.1. Let $\mathbb{P}(m,l)$ denote the class of $(n+m)\times(n+l)$ polynomial system matrices in z_1 and z_2 with real coefficients. Two polynomial system matrices $P_1(z_1,z_2)$ and $P_2(z_1,z_2) \in \mathbb{P}(m,l)$, are said to be zero coprime system equivalent if they satisfy

$$\underbrace{\begin{bmatrix} M & 0 \\ X & I_m \end{bmatrix}}_{S_1(z_1,z_2)} \underbrace{\begin{bmatrix} T_1 & U_1 \\ -V_1 & W_1 \end{bmatrix}}_{P_1(z_1,z_2)} = \underbrace{\begin{bmatrix} T_2 & U_2 \\ -V_2 & W_2 \end{bmatrix}}_{P_2(z_1,z_2)} \underbrace{\begin{bmatrix} N & Y \\ 0 & I_l \end{bmatrix}}_{S_2(z_1,z_2)}, \tag{11}$$

where $P_1(z_1,z_2), S_2(z_1,z_2)$ are zero right coprime and $P_2(z_1,z_2), S_1(z_1,z_2)$ are zero left coprime and $M(z_1,z_2)$, $N(z_1,z_2)$, $X(z_1,z_2)$ and $Y(z_1,z_2)$ are polynomial matrices of compatible dimensions.

The properties of controllability and observability lie at the heart of linear systems theory and they are characterized by the zero structure of their associated system matrices, see, e.g. Zerz (2000). The transformation of zero coprime system equivalence plays a key role in many aspects of 2-D systems theory, see, e.g., Johnson (1993), Levy (1981), Pugh et al. (1998) and Pugh et al. (1996)). One result is the following lemma.

Lemma 3.2 (Johnson, Johnson (1993))**.** *For the description of (11), the transformation of zero coprime system equivalence preserves the transfer-function matrix and the zero structure of the following matrices*

$$T_i(z_1,z_2), \quad P_i(z_1,z_2), \quad \begin{bmatrix} T_i(z_1,z_2) & U_i(z_1,z_2) \end{bmatrix}, \quad \begin{bmatrix} T_i(z_1,z_2) \\ -V_i(z_1,z_2) \end{bmatrix}, i = 1,2.$$

4. Polynomial System Matrix Descriptions

Consider first the singular Roesser (SR) state-space model of (1). Then under zero boundary conditions the polynomial equations describing the system dynamics are

$$P_{SR}(z_1, z_2) \begin{bmatrix} x^h \\ x^v \\ -u \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ -y \end{bmatrix} \tag{12}$$

where

$$P_{SR}(z_1, z_2) = \left[\begin{array}{c|c} z_1 E_1 + z_2 E_2 - A & B \\ \hline -C & D \end{array} \right], \tag{13}$$

with

$$\begin{aligned} E &= \begin{bmatrix} E_{11} & E_{12} \\ E_{21} & E_{22} \end{bmatrix}, \; E_1 = \begin{bmatrix} E_{11} & 0 \\ E_{21} & 0 \end{bmatrix}, \\ E_2 &= \begin{bmatrix} 0 & E_{12} \\ 0 & E_{22} \end{bmatrix}, \; A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, \\ B^T &= \left[\; [B_1^T \quad ; B_2^T \; \right]^T, \; C = \left[\; [C_1 \quad C_2 \; \right] \end{aligned}$$

is the corresponding system matrix. Also if the matrix $[z_1 E_1 + z_2 E_2 - A]$ is invertible, the transfer-function matrix corresponding to (1) is

$$G_{SR}(z_1, z_2) = C \left[z_1 E_1 + z_2 E_2 - A \right]^{-1} B + D. \tag{14}$$

Consider the wave linear repetitive process (3) and introduce the state vector

$$\nu(k, p) = [x_k^T(p) \;\; y_k^T(p)]^T. \tag{15}$$

Then, on applying the shift operators of (7), the system matrix associated with (3) is

$$P_{WR}(z_1, z_2) = \left[\begin{array}{cc|c} z_1 z_2 I_n - z_1 A & -\sum\limits_{i=-w_L}^{w_H} z_2^i B_i & z_1 B \\ -z_1 C & z_1 I_m - \sum\limits_{i=-w_L}^{w_H} z_2^i D_i & z_1 D \\ \hline 0_{m,n} & -I_m & 0_{m,l} \end{array} \right], \tag{16}$$

with transfer-function matrix

$$G_{WR}(z_1, z_2) = \left[\; 0 \quad I_m \; \right] \begin{bmatrix} z_1 z_2 I_n - z_1 A & -\sum\limits_{i=-w_L}^{w_H} z_2^i B_i \\ -z_1 C & z_1 I_m - \sum\limits_{i=-w_L}^{w_H} z_2^i D_i \end{bmatrix}^{-1} \begin{bmatrix} z_1 B \\ z_1 D \end{bmatrix}, \tag{17}$$

provided the matrix inverse in (17) exists.

An alternative description of (3) is to multiply the system equations by the shift operator $z_2^{w_L}$. This corresponds to multiplying the system matrix P_{WR} in (16) on the left by the following matrix that preserves the transfer-function matrix

$$\left[\begin{array}{cc|c} z_2^{w_L} I_n & 0 & 0 \\ 0 & z_2^{w_L} I_m & 0 \\ \hline 0 & 0 & I_l \end{array}\right].$$

Now let $j = i + w_L$ and $q = w_L + w_H$ and the polynomial system matrix $\tilde{P}_{WR}(z_1, z_2)$ is

$$\begin{aligned} \tilde{P}_{WR}(z_1, z_2) &\equiv \begin{bmatrix} \tilde{T}_{WR} & \tilde{U}_{WR} \\ -\tilde{V}_{WR} & 0_{m,l} \end{bmatrix} \\ &= \left[\begin{array}{cc|c} z_1 z_2^{w_L+1} I_n - z_1 z_2^{w_L} A & -\sum_{j=0}^{q} z_2^j B_{j-w_L} & z_1 z_2^{w_L} B \\ -z_1 z_2^{w_L} C & z_1 z_2^{w_L} I_m - \sum_{j=0}^{q} z_2^j D_{j-w_L} & z_1 z_2^{w_L} D \\ \hline 0_{m,n} & -I_m & 0_{m,l} \end{array}\right]. \end{aligned} \tag{18}$$

5. Transforming a WLRP to a SR

Let $t = n + m$ and let $\tilde{P}_{WR}(z_1, z_2)$ be a $(t + m) \times (t + l)$ polynomial system matrix of the form (18). Then this matrix can be written as:

$$\tilde{P}_{WR}(z_1, z_2) = \sum_{i=0}^{1} \sum_{j=0}^{q} z_1^i z_2^j \tilde{P}_{i,j}, \quad (\tilde{P}_{1,0} = 0), \tag{19}$$

where $\tilde{P}_{i,j}$ are $(t+m) \times (t+l)$ are matrices with constant entries. Next, construct the matrices

$$
\tilde{E} = \begin{bmatrix}
0_{t+l,t+l} & 0_{t+l,t+l} & \cdots & 0_{t+l,t+l} & 0_{t+l,m} \\
0_{t+l,t+l} & 0_{t+l,t+l} & \cdots & 0_{t+l,t+l} & 0_{t+l,m} \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0_{t+l,t+l} & 0_{t+l,t+l} & \cdots & 0_{t+l,t+l} & 0_{t+l,m} \\
\tilde{P}_{1,q} & \tilde{P}_{1,q-1} & \cdots & \tilde{P}_{1,1} & 0_{t+m,m} \\
0_{l,t+l} & 0_{l,t+l} & \cdots & 0_{l,t+l} & 0_{l,m}
\end{bmatrix},
$$

$$
\tilde{A}_0 = \begin{bmatrix}
I_{t+l} & 0_{t+l,t+l} & \cdots & 0_{t+l,t+l} & 0_{t+l,m} \\
0_{t+l,t+l} & I_{t+l} & \cdots & 0_{t+l,t+l} & 0_{t+l,m} \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0_{t+l,t+l} & 0_{t+l,t+l} & \cdots & 0_{t+l,t+l} & 0_{t+l,m} \\
0_{t+m,t+l} & 0_{t+m,t+l} & \cdots & -\tilde{P}_{0,0} & -\mathcal{F}^T_{m,t+m} \\
0_{l,t+l} & 0_{l,t+l} & \cdots & \mathcal{F}_{l,t+l} & 0_{l,m}
\end{bmatrix}, \tag{20}
$$

$$
\tilde{A}_2 = \begin{bmatrix}
0_{t+l,t+l} & I_{t+l} & \cdots & 0_{t+l,t+l} & 0_{t+l,m} \\
0_{t+l,t+l} & 0_{t+l,t+l} & \cdots & 0_{t+l,t+l} & 0_{t+l,m} \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0_{t+l,t+l} & 0_{t+l,t+l} & \cdots & I_{t+l} & 0_{t+l,m} \\
-\tilde{P}_{0,q} & -\tilde{P}_{0,q-1} & \cdots & -\tilde{P}_{0,1} & 0_{t+m,m} \\
0_{l,t+l} & 0_{l,t+l} & \cdots & 0_{l,t+l} & 0_{l,m}
\end{bmatrix},
$$

where $\mathcal{F}_{r,k} = [0_{r,k-r}\ I_r]$. The system matrix in singular Roesser form now is

$$
\begin{aligned}
P_{SR}(z_1,z_2) &\equiv \begin{bmatrix} T_{SR} & U_{SR} \\ -V_{SR} & 0_{m,l} \end{bmatrix} \\
&= \left[\begin{array}{cccc|c}
I_{q(t+l)+m} & -z_1 I_{q(t+l)+m} & 0_{q(t+l)+l,l} & 0_{q(t+l)+l,m} & 0_{q(t+l)+l,l} \\
z_2\tilde{E} & -z_2\tilde{A}_2 - \tilde{A}_0 & \mathcal{F}^T_{q(t+l)+m,l} & 0_{q(t+l)+m,m} & 0_{q(t+l)+m,l} \\
0_{m,q(t+l)+m} & -\mathcal{F}_{m,q(t+l)+m} & 0_{m,l} & I_m & 0_{m,l} \\
0_{l,q(t+l)+m} & 0_{l,q(t+l)+m} & -I_l & 0_{l,m} & I_l \\
\hline
0_{m,q(t+l)+m} & 0_{m,q(t+l)+m} & 0_{m,l} & -I_m & 0_{m,l}
\end{array}\right].
\end{aligned} \tag{21}
$$

Theorem 5.1. *Let $P_{SR}(z_1,z_2)$ be a $\{2[q(t+l)+l+m]+m\} \times \{2[q(t+m)+l+m]+l\}$ system matrix as constructed as in (21). Then $P_{SR}(z_1,z_2)$ is related to the system matrix $\tilde{P}_{WR}(z_1,z_2)$ in the form (18) by zero coprime system equivalence, i.e.,*

$$
S_1\tilde{P}_{WR} = P_{SR}S_2, \tag{22}
$$

where

$$S_1 = \left[\begin{array}{c|c} 0 & 0 \\ I_t & 0 \\ 0 & 0 \\ 0 & 0 \\ \hline 0 & I_m \end{array}\right], \quad S_2 = \left[\begin{array}{c|c} z_1 z_2^{q-1} I_t & 0 \\ 0 & z_1 z_2^{q-1} I_l \\ z_1 z_2^{q-2} I_t & 0 \\ 0 & z_1 z_2^{q-2} I_l \\ \vdots & \vdots \\ z_1 I_t & 0 \\ 0 & z_1 I_l \\ z_1 \mathcal{F}_{m,t} & 0 \\ z_2^{q-1} I_t & 0 \\ 0 & z_2^{q-1} I_l \\ z_2^{q-2} I_t & 0 \\ 0 & z_2^{q-2} I_l \\ \vdots & \vdots \\ I_t & 0 \\ 0 & I_l \\ \mathcal{F}_{m,t} & 0_{m,l} \\ 0 & I_l \\ \mathcal{F}_{l,t} & 0 \\ \hline 0 & I_l \end{array}\right]. \tag{23}$$

Proof. It can be verified that

$$S_1 \tilde{P}_{WR} = P_{SR} S_2 = \begin{bmatrix} 0 & 0 \\ \tilde{T}_{WR} & \tilde{U}_{WR} \\ 0 & 0 \\ 0 & 0 \\ \tilde{V}_{WR} & 0_{m,l} \end{bmatrix} \tag{24}$$

and hence it remains to establish zero coprimeness of the matrices. Zero right coprimeness of $\tilde{P}_{WR}$ and S_2 follows from the fact that the matrix

$$\begin{bmatrix} \tilde{P}_{WR} \\ S_2 \end{bmatrix},$$

contains a highest order minor of order $t + l$ which is equal to 1. Also zero left coprimeness of P_{SR} and S_1 since the matrix

$$\begin{bmatrix} P_{SR} & S_1 \end{bmatrix} = \left[\begin{array}{ccccc|cc} I_{q(t+l)+m} & -z_1 I_{q(t+l)+m} & 0 & 0 & 0 & 0 & 0 \\ z_2 \tilde{E} & -z_2 \tilde{A}_2 - \tilde{A}_0 & \mathcal{F}^T_{q(t+l)+m,l} & 0 & 0 & I_{q(t+l)+m} & 0 \\ 0 & -\mathcal{F}_{m,q(t+l)+m} & 0 & I_m & 0 & 0 & 0 \\ 0 & 0 & -I_l & 0 & I_l & 0 & 0 \\ 0 & 0 & 0 & -I_m & 0 & 0 & I_m \end{array}\right], \tag{25}$$

has a highest order minor of order $2[q(t+l)+l+m]+m$ which is equal to ± 1, obtained by deleting the second and third block columns of the matrix in (25). $\square$

Example 5.2. Consider the system matrix in (18) with $w_L = w_H = 1, (q = 2)$, i.e.,

$$\tilde{P}_{WR} = \left[\begin{array}{cc|c} z_1 z_2^2 I_n - z_1 z_2 A & -B_{-1} - B_0 z_2 - B_1 z_2^2 & z_1 z_2 B \\ -z_1 z_2 C & z_1 z_2 I_m - D_{-1} - D_0 z_2 - D_1 z_2^2 & z_1 z_2 D \\ \hline 0_{m,n} & -I_m & 0_{m,l} \end{array}\right]. \tag{26}$$

In this case

$$\tilde{P}_{WR} = \tilde{P}_{0,0} + \tilde{P}_{0,1} z_2 + \tilde{P}_{0,2} z_2^2 + \tilde{P}_{1,1} z_1 z_2 + \tilde{P}_{1,2} z_1 z_2^2, \quad (\tilde{P}_{1,0} = 0). \tag{27}$$

where

$$\begin{aligned} \tilde{P}_{0,0} &= \left[\begin{array}{cc|c} 0 & -B_{-1} & 0 \\ 0 & -D_{-1} & 0 \\ \hline 0 & -I_m & 0 \end{array}\right], \ \tilde{P}_{0,1} = \left[\begin{array}{cc|c} 0 & -B_0 & 0 \\ 0 & -D_0 & 0 \\ \hline 0 & 0 & 0 \end{array}\right], \ \tilde{P}_{0,2} = \left[\begin{array}{cc|c} 0 & -B_1 & 0 \\ 0 & -D_1 & 0 \\ \hline 0 & 0 & 0 \end{array}\right], \\ \tilde{P}_{1,1} &= \left[\begin{array}{cc|c} -A & 0 & B \\ -C & I_m & D \\ \hline 0 & 0 & 0 \end{array}\right], \ \tilde{P}_{1,2} = \left[\begin{array}{cc|c} I_n & 0 & 0 \\ 0 & 0 & 0 \\ \hline 0 & 0 & 0 \end{array}\right]. \end{aligned} \tag{28}$$

Then the matrices in (20) are given by

$$\tilde{E} = \begin{bmatrix} 0 & 0 & 0 \\ \tilde{P}_{1,2} & \tilde{P}_{1,1} & 0 \\ 0 & 0 & 0 \end{bmatrix}, \ \tilde{A}_0 = \begin{bmatrix} -I & 0 & 0 \\ 0 & -\tilde{P}_{0,0} & 0 \\ 0 & 0 & 0 \end{bmatrix}, \ \tilde{A}_2 = \begin{bmatrix} 0 & I & 0 \\ -\tilde{P}_{0,2} & -\tilde{P}_{0,1} & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

The resulting system matrix $P_{SR}(z_1, z_2)$ is

$$P_{SR} = \left[\begin{array}{ccccccccc} I_n & 0 & 0 & 0 & 0 & 0 & 0 & -z_1 I_n & 0 \\ 0 & I_m & 0 & 0 & 0 & 0 & 0 & 0 & -z_1 I_m \\ 0 & 0 & I_l & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & I_n & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & I_m & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & I_l & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & I_m & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & I_n & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & I_m \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ z_2 I_n & 0 & 0 & -A z_2 & 0 & B z_2 & 0 & 0 & -B_1 z_2 \\ 0 & 0 & 0 & -C z_2 & z_2 I & D z_2 & 0 & 0 & -D_1 z_2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right.$$

$$
\left.\begin{array}{ccccccc|c}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-z_1 I_l & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -z_1 I_n & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -z_1 I_m & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -z_1 I_l & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -z_1 I_m & 0 & 0 & 0 \\
0 & -z_2 I_n & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -z_2 I_m & 0 & 0 & 0 & 0 & 0 \\
I_l & 0 & 0 & -z_2 I_l & 0 & 0 & 0 & 0 \\
0 & 0 & -B_0 z_2 - B_{-1} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -D_0 z_2 - D_{-1} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -I_m & 0 & I_m & 0 & 0 & 0 \\
0 & 0 & 0 & -I_l & 0 & I_l & 0 & 0 \\
0 & 0 & 0 & 0 & -I_m & 0 & I_m & 0 \\
0 & 0 & 0 & 0 & 0 & -I_l & 0 & I_l \\
\hline
0 & 0 & 0 & 0 & 0 & 0 & -I_m & 0
\end{array}\right], \tag{29}
$$

where the transformation matrices S_1 and S_2 are

$$
S_1 = \left[\begin{array}{cc|c}
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
I_n & 0 & 0 \\
0 & I_m & 0 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
\hline
0 & 0 & I_m
\end{array}\right], \quad
S_2 = \left[\begin{array}{cc|c}
z_1 z_2 I_n & 0 & 0 \\
0 & z_1 z_2 I_m & 0 \\
0 & 0 & z_1 z_2 I_l \\
z_1 I_n & 0 & 0 \\
0 & z_1 I_m & 0 \\
0 & 0 & z_1 I_l \\
0 & z_1 I_m & 0 \\
z_2 I_n & 0 & 0 \\
0 & z_2 I_m & 0 \\
0 & 0 & z_2 I_l \\
I_n & 0 & 0 \\
0 & I_m & 0 \\
0 & 0 & I_l \\
0 & I_m & 0 \\
0 & 0 & I_l \\
0 & I_l & 0 \\
\hline
0 & 0 & I_l
\end{array}\right] \tag{30}
$$

and hence

$$S_1\tilde{P}_{WR} = P_{SR}S_2 = \left[\begin{array}{cc|c} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ -Az_1z_2 + z_1{z_2}^2 & -B_1{z_2}^2 - B_0z_2 - B_{-1} & z_2z_1B \\ -z_2z_1C & -D_1{z_2}^2 - D_0z_2 + z_1z_2 - D_{-1} & z_2z_1D \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & -I_m & 0 \end{array}\right]$$

$$\equiv \begin{bmatrix} 0 & 0 \\ \tilde{T}_{WR} & \tilde{U}_{WR} \\ 0 & 0 \\ 0 & 0 \\ \tilde{V}_{WR} & 0_{m,l} \end{bmatrix}. \tag{31}$$

Finally, routine manipulations give

$$G_{SR}(z_1, z_2) = \tilde{G}_{WR}(z_1, z_2).$$

Conclusions

In this paper, an equivalent representation is derived for wave linear repetitive processes as a 2-D singular Roesser state-space model starting from a given system matrix. The exact connection between the original system matrix and the equivalent singular 2-D systems representation has been established as zero coprime system equivalence. This permits, where relevant, the exchange of analysis tools between these two areas to solve currently open systems theoretic question and hence, where appropriate, transfer to applications areas for wave linear repetitive processes such as ILC design.

To preserve the zero structure, the resulting system matrix has somewhat large dimensions. One possible area for future research is to find ways of reducing the dimensions of this matrix without, of course, changing its zero structure.

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