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Generalised Energy Conservation Law of Chaotic Phenomena

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Chaotic phenomena are more and more observed in any fields of nature, when investigations on natural phenomena enter into nonlinearities to reveal their deep mechanisms. Chaos is normally understood as “a state of disorder”, for which until now there is no universally accepted a mathematical definition. Commonly used concept says that, for a dynamical system to be classified as chaotic, it must have the following properties: sensitive to initial conditions; topological transitivity; densely periodical orbits etc. To reveal the rules governing chaotic motions is an important unsolved task exploring nature. Here we show a generalised energy conservation law governing chaotic phenomena. Based on the two scalar variables: generalised potential and kinetic energies defined in the phase space describing nonlinear dynamical systems, we found that a chaotic motion is a periodical motion with infinite time period and its time-averaged generalised potential and kinetic energies are in conservation over its time period. This implies that with the average time increasing, the time averaged generalised potential and kinetic energies tend constants while their time-averaged energy flows, their time change rates, tend zero. The numerical simulations on the reported chaotic motions: forced Van der Pol’s system, forced duffing’s system, forced SD oscillator, Lorenz’s system and Rössler’s system, have demonstrated the above conclusion is correct, of which the obtained curves are given in the paper. The discover may advice that any chaotic phenomena of nature could be controlled, although their instant states are in disorder but the long-time averages are predicated.

I. NONLINEAR DYNAMICAL SYSTEMS

The *nonlinear dynamic systems* concerning chaos investigated in this letter are generally sufficient to regard a second order differential equation with its initial conditions, which can be transformed into the first order differential equation of the phase space in the following non-dimensional form [1],

$$d\boldsymbol{y}/dt \equiv \dot{\boldsymbol{y}} = \boldsymbol{f}(t,\boldsymbol{y}), \qquad \boldsymbol{y}(0) = \boldsymbol{y}_0. \tag{1}$$

Generally, we consider that $\boldsymbol{y} = \boldsymbol{y}(t) \in R^n$ is a vector valued function of an independent variable $t \in I = (t_1, t_2) \subseteq R$ and $\boldsymbol{f} : U \to R^n$ is a smooth function of the variable t and the vector $\boldsymbol{y}$ defined on some subset $U \subseteq R^n$, an n-dimensional phase space. Often we seek a solution $\boldsymbol{\varphi}(\boldsymbol{y}_0, t)$ such that

$$\boldsymbol{\varphi}(\boldsymbol{y}_0, 0) = \boldsymbol{y}_0. \tag{2}$$

The solution $\boldsymbol{\varphi}(\boldsymbol{y}_0, \cdot) : I \to R^n$ defines a solution curve, trajectory or orbit, of Eq. (1) based at $\boldsymbol{y}_0$ as shown in Fig. 1 (a). According to the local existence and uniqueness theorem of solutions [2], there exist no intersections of the trajectories of Eq. (1) in the solution space except at its fixed points. The all possible solution curves $\boldsymbol{\varphi}_t(U)$ generate the flow shown in Fig.1 (b).

Figure 1

FIG. 1 A solution curve and the flows: (a) the solution curve $\boldsymbol{\varphi}_t(\boldsymbol{y}_0)$ with its energy flow curve $E_t(\boldsymbol{y}_0)$, of which of their tangent vectors at a point $\boldsymbol{y}$ are $\dot{\boldsymbol{y}} = \boldsymbol{f}$ and $\dot{E}$, respectively; (b) the flow $\boldsymbol{\varphi}_t(U)$ and the energy flow $E_t(U)$ in R^n.

II. ENERGY FLOW VARIABLES AND EQUATIONS

To investigate the behaviours of nonlinear dynamical systems, based on Eq. (1) in the phase space Xing [3] introduced the three scalar energy flow variables

$$\begin{aligned} &\textbf{generalised potential energy:} \quad E = \boldsymbol{y}^T \boldsymbol{y}/2, \\ &\textbf{generalised kinetic energy:} \quad K = \dot{\boldsymbol{y}}^T \dot{\boldsymbol{y}}/2, \\ &\textbf{generalised force power:} \quad P = \boldsymbol{y}^T \boldsymbol{f}, \end{aligned} \tag{3}$$

of which the time averaged values during a time period $(0, T)$ are respectively defined by

$$\begin{aligned} \langle E \rangle &= \textstyle\int_0^T E dt / T, \\ \langle K \rangle &= \textstyle\int_0^T K dt / T, \\ \langle P \rangle &= \textstyle\int_0^T P dt / T. \end{aligned} \tag{4}$$

The corresponding energy flow equilibrium equations take the forms

$$\dot{E} = P, \qquad E_0 = \boldsymbol{y}_0^T \boldsymbol{y}_0 / 2, \tag{5}$$

$$\dot{K} = \dot{\boldsymbol{y}}^T(\partial \boldsymbol{f} / \partial t + \boldsymbol{J}\dot{\boldsymbol{y}}) = \dot{\boldsymbol{y}}^T \partial \boldsymbol{f} / \partial t + \dot{\boldsymbol{y}}^T \boldsymbol{E}\dot{\boldsymbol{y}},$$

$$K_0 = \dot{\boldsymbol{y}}_0^T \dot{\boldsymbol{y}}_0 / 2, \;\; \boldsymbol{J} = \partial \boldsymbol{f} / \partial \boldsymbol{y}^T, \tag{6}$$

$$\boldsymbol{E} = (\boldsymbol{J} + \boldsymbol{J}^T)/2, \;\; \boldsymbol{U} = (\boldsymbol{J} - \boldsymbol{J}^T)/2,$$

where $\boldsymbol{J}$ denotes the Jacobian matrix, $\boldsymbol{E}$ is a real symmetrical matrix called as the energy flow matrix and $\boldsymbol{U}$ is a skew-symmetrical matrix, so that $\dot{\boldsymbol{y}}^T\boldsymbol{U}\dot{\boldsymbol{y}} \equiv 0.$ The time averaged energy flows are defined as

$$\begin{aligned} \left\langle \dot{E} \right\rangle &= \int_0^T \dot{E}dt / T = [E(T) - E(0)]/T, \\ \left\langle E_0 \right\rangle &= \left\langle \boldsymbol{y}_0^T \boldsymbol{y}_0 \right\rangle / 2, \\ \left\langle \dot{K} \right\rangle &= \int_0^T \dot{K}dt / T = [K(T) - K(0)]/T, \\ \left\langle K_0 \right\rangle &= \left\langle \dot{\boldsymbol{y}}_0^T \dot{\boldsymbol{y}}_0 \right\rangle / 2. \end{aligned} \tag{7}$$

Geometrically, *the generalised potential energy E relates the position $\boldsymbol{y}$ of a point in the phase space, while the generalised kinetic energy K involves the velocity or tangent vector $\dot{\boldsymbol{y}}$ of the solution curve, and the generalised force power P gives the energy flow, i.e. the time change rate of the generalised potential energy*. As shown in Fig.1, an axis for generalised potential energy E is given, from which the potential energy flow curves are drawn.

III. ENERGY FLOW CHARACTERISTIC FACTORS AND VECTORS

The real symmetrical energy flow matrix $\boldsymbol{E}$ has its real eigenvalues λ_I and eigenvectors $\boldsymbol{\psi}_I$ satisfying the orthogonal relationships

$$\begin{aligned} &\boldsymbol{\Psi}^T \boldsymbol{E} \boldsymbol{\Psi} = \boldsymbol{\Lambda} = \mathbf{diag}(\lambda_I), \;\; \boldsymbol{\Psi}^T \boldsymbol{\Psi} = \boldsymbol{I}, \\ &\boldsymbol{\Psi} = [\boldsymbol{\psi}_1 \quad \boldsymbol{\psi}_2 \quad \cdots \quad \boldsymbol{\psi}_n]. \end{aligned} \tag{8}$$

These eigenvectors span a complete subspace in the neighbour of the point where the matrix $\boldsymbol{E}$ defined, so that the variation vector $\boldsymbol{\varepsilon}$ and a real quantity $\boldsymbol{\varepsilon}^T\boldsymbol{E}\boldsymbol{\varepsilon}$ around this point can be represented as

$$\boldsymbol{\varepsilon} = \boldsymbol{\Psi}\boldsymbol{\zeta}, \quad \boldsymbol{\varepsilon}^T \boldsymbol{E} \boldsymbol{\varepsilon} = \boldsymbol{\zeta}^T \boldsymbol{\Lambda} \boldsymbol{\zeta} = \sum_{I=1}^{n} \lambda_I \zeta_I^2. \tag{9}$$

This implies that the $\boldsymbol{\varepsilon}^T\boldsymbol{E}\boldsymbol{\varepsilon}$ is totally determined by the eigenvalues of the energy flow matrix, therefore we respectively name λ_I and $\boldsymbol{\psi}_I$ as the *energy flow characteristic factors* and the *energy flow characteristic vectors* of nonlinear dynamical systems.

IV. PHASE VOLUME STRAIN AND ITS TIME CHANGE RATE

Figure 2

FIG. 2 The integration of the divergence of vector field over a phase space volume**.**

As shown in Fig. 2, considering a phase space volume V closed by a surface S of unit outside normal ν_i, which moves to its new position represented by dashed line due to a displacement $\dot{\boldsymbol{y}}\Delta t$ caused by the flows of points on the surface S in a time interval Δt. Integrating the divergence of vector $\boldsymbol{f}$ over volume V and using the summation convention and Green theorem [4], we obtain

$$\int_V f_{i,i} dV = \int_V \dot{y}_{i,i} dV = \int_S \dot{y}_i \nu_i dS = \frac{\Delta V}{\Delta t}, \tag{10}$$

where ΔV represents a volume change of volume V in time interval Δt. Eq. (10) is valid for any size of volume V, so that we can consider a differential volume element V_0 and Eq. (10) is approximated to

$$\begin{aligned} &V_0 f_{i,i} = DV / Dt, \quad f_{i,i} = DV / (V_0 Dt) = \dot{\upsilon}, \\ &\upsilon = (V - V_0)/V_0. \end{aligned} \tag{11}$$

Here, υ and $\dot{\upsilon}$ are defined as the *phase volume stain* and its time change rate. Generally, they are functions of phase point and time. From Eqs. (6) and (11), it follows

$$f_{i,i} = \mathbf{tr}(\boldsymbol{J}) = \mathbf{tr}(\boldsymbol{E}) = \sum_{I=1}^{n} \lambda_I, \;\; \dot{\upsilon} = \sum_{I=1}^{n} \lambda_I. \tag{12}$$

Equation (12) represents that the time change rate of phase volume strain equals the summation of the energy flow characteristic factors of nonlinear dynamical system, from which we have

$$\dot{\upsilon} = \begin{cases} < 0, & \sum_{I=1}^{n} \lambda_I < 0, \\ = 0, & \sum_{I=1}^{n} \lambda_I = 0, \\ > 0, & \sum_{I=1}^{n} \lambda_I > 0, \end{cases} \tag{13}$$

which respectively correspond a contracting, isovolumetric and expanding phase space for the nonlinear dynamical systems around the point where $\boldsymbol{E}$ defined.

V. ZERO ENERGY FLOW SURFACE, FIXED POINTS AND STABILITIES

Generally, the energy flow of system is a function of time and position of a point in the phase space, which generates a scalar field called as the energy flow field of nonlinear dynamical system. The equation

$$\dot{E} = \boldsymbol{y}^T \boldsymbol{f}(t,\boldsymbol{y}) = P(t,\boldsymbol{y}) = 0, \tag{14}$$

defines a genralised surface or subspace in the phase space, called as a *zero-energy flow surface*, on which the energy flow vanishes. If an orbit of the nonlinear dynamical system is on a zero-energy flow surface, the distance of a point on the oribit is not changed with time. There are three cases satisfying $\dot{E} = 0$ in Eq.(14):

- Case 1: $\boldsymbol{y} = 0$, representing the origin of phase space, at which the genralised potential energy is defined as zero;
- Case 2: $\boldsymbol{f} = 0$, implying an equilibrium point of the system;
- Case 3: $P = 0, \ \boldsymbol{y} \neq 0 \neq \boldsymbol{f}$, correseponding a $\dot{E} = 0$ surface.

Assume that $\boldsymbol{y}$ denotes a orbit point on a zero-energy flow surface, $P(t,\boldsymbol{y}) = 0$, and $\boldsymbol{\varepsilon}$ is an small orbit variation around $\boldsymbol{y}$, genrally, the variation of energy flow caused by the oribit variation is given by

$$\Delta\dot{E} = \boldsymbol{\varepsilon}^T \boldsymbol{p} + \boldsymbol{\varepsilon}^T \boldsymbol{E}\boldsymbol{\varepsilon}, \qquad \boldsymbol{p} = \boldsymbol{f} + \boldsymbol{J}^T \boldsymbol{y}, \tag{15}$$

which reduces to

$$\Delta\dot{E} = \boldsymbol{\varepsilon}^T \boldsymbol{E}\boldsymbol{\varepsilon}, \tag{16}$$

for the equilibrium point $\boldsymbol{y} = 0$ or the case $\boldsymbol{\varepsilon}^T \boldsymbol{p} = 0$. The vector $\boldsymbol{p}$ is an energy flow gradient along the normal vector of zero energy flow surface, so that $\boldsymbol{\varepsilon}^T \boldsymbol{p} = 0$ implies the vector $\boldsymbol{\varepsilon}$ is on the surface and perpendicular to vector $\boldsymbol{p}$.

Based on Eq.(15) and the geometrical meaning of potential energy, we conclude:

- If $|\boldsymbol{y}+\boldsymbol{\varepsilon}| < |\boldsymbol{y}|$, then $E(t,\boldsymbol{y}+\boldsymbol{\varepsilon}) < E(t,\boldsymbol{y})$, so that $\Delta\dot{E} > 0$ implies the flow towards to the zero-energy flow surface, while $\Delta\dot{E} < 0$ indicates the flow backwards from the zero-energy flow surface;
- If $|\boldsymbol{y}+\boldsymbol{\varepsilon}| > |\boldsymbol{y}|$, then $E(t,\boldsymbol{y}+\boldsymbol{\varepsilon}) > E(t,\boldsymbol{y})$, so that $\Delta\dot{E} < 0$ implies the flow towards to the zero-energy flow surface, while $\Delta\dot{E} > 0$ indicates the flow backwards from the zero-energy flow surface;
- If the flows from both sides of the zero-energy flow surface toward it, this surface is an attracting surface.

Figure 3 shows a case where the orbit intersects at a point $\boldsymbol{y}$ on the energy flow surface, since $\Delta\dot{E} > 0$ above the surface and $\Delta\dot{E} < 0$ under the surface, so that the flow along the orbit backwards to this point and this is an unstable point.

At the fixed point $\boldsymbol{y} = 0$, from Eqs.(15) and (16), we have

$$\Delta\dot{E} = \dot{E}(\boldsymbol{\varepsilon}) = \boldsymbol{\varepsilon}^T \boldsymbol{E}\boldsymbol{\varepsilon} = \sum_{I=1}^{n} \lambda_I \zeta_I^2, \tag{17}$$

which implies the stability of a fixed point $\boldsymbol{y} = 0$ is determined by the energy flow characteristic factors of the energy flow matrix $\boldsymbol{E}$ as in the statement.

A fixed point $\boldsymbol{y} = 0$ of nonlinear dynamical systems governed by Eq.(1) is asymptotically stable, stable or unstable,

a) *if its quadratic form of energy flow in Eq.(17) is definitely negative $(\dot{E}(\boldsymbol{\varepsilon}) < 0)$, semi-negative $(\dot{E}(\boldsymbol{\varepsilon}) \le 0)$ or definitely positive $(\dot{E}(\boldsymbol{\varepsilon}) > 0)$ on the neighbourhood about the fixed point, respectively, or*

b) *if the energy flow characteristic factors of the system are all negative $(\lambda_I < 0)$, semi-negative $(\lambda_I \le 0)$ or there exists at least one positive factor $(\lambda_J > 0)$ on the neighbourhood about the fixed point, respectively.*

Figure 3

FIG. 3 The zero energy flow surface and an unstable point $\boldsymbol{y}$, the intersecting point of the orbit and this surface, determined by $\Delta\dot{E}$.

Liapunov function method [2] is often used to investigate the stability of a fixed points, which relies on finding a positive definite Liapunov function. However, to find a Liapunov function of generalised nonlinear dynamical system is very difficult. The defined generalised energy E of nonlinear dynamical systems plays a role of generalised Liapunov function to tackle the stability of fixed point of nonlinear systems in the phase space, which overcomes this difficulty.

Furthermore, the eigenvalues of the Jacobian matrix in Eq. (6) are widely used to give the stability solutions. As we know, the Jacobian matrix is a non-symmetrical matrix, of which the eigenvalues are complex. Instead of non-symmetrical Jacobian matrix, we adopt the real-symmetrical energy flow matrix with real eigenvalues to give the stability solution, which is more effective.

It may be necessary to mention that above statement about the stability of equilibrium point is based on linearized approximation at the zero point, for real linear systems, the conclusion is correct. However, for nonlinear systems, the zero energy flow characteristic factor cannot guarantee its stability characteristics and a higher order approximation analysis is needed. The central energy flow theory discussed in [1] provides the way and examples to further reveal the stability performance in this case.

VI. PERIODICAL ORBIT AND ITS ENERGY FLOW BEHAVIOUR

For a nonlinear dynamical system, a periodical orbit is a closed curve in the phase space, along which the phase point $\boldsymbol{y}(t)$ with its vector $\dot{\boldsymbol{y}}(t)$, starting from a position $\boldsymbol{y}(\hat{t})$ and $\dot{\boldsymbol{y}}(\hat{t})$ at time $t=\hat{t}$, moves to the same position $\boldsymbol{y}(\hat{t}+T)=\boldsymbol{y}(\hat{t})$ and $\dot{\boldsymbol{y}}(\hat{t}+T)=\dot{\boldsymbol{y}}(\hat{t})$ after a time period T and the motion repeats again, such as the case $\hat{t}=0$ shown in Fig.4.

Figure 4

FIG. 4 The periodical orbit and the unit normal / tangent vectors of line element ds on orbit**.**

We assume that ds denotes a differential line element with its unit outside normal vector ν_i and unit tangent vector τ_i at a point on the closed curve in Fig.4, so that

$$\tau_i ds = \dot{\boldsymbol{y}}\,|\,d\boldsymbol{y}\,|\,/\,|\,\dot{\boldsymbol{y}}\,| = \dot{\boldsymbol{y}}\,|\,\dot{\boldsymbol{y}}dt\,|\,/\,|\,\dot{\boldsymbol{y}}\,| = \dot{\boldsymbol{y}}dt, \quad (18)$$

based on which the following integrals along the curve hold.

Time averaged generalised potential energy

$$\langle E \rangle = \frac{1}{T}\int_0^T \boldsymbol{y}^T\boldsymbol{y}/2dt = \hat{E}, \ \ \textbf{constant,} \quad (19)$$

where $\hat{E}$ is a positive constant relating the averaged distance of the phase points on the orbit to the origin, since the motion repeats along the closed orbit.

Time averaged potential energy flow

$$\langle \dot{E} \rangle = \frac{1}{T}\int_{\hat{t}}^{T+\hat{t}} \dot{E}dt = \frac{E(T+\hat{t})-E(\hat{t})}{T} = 0, \quad (20)$$

since for the periodical orbit *S*, $\boldsymbol{y}(T+\hat{t})=\boldsymbol{y}(\hat{t})$.

Circulation integral: time averaged kinetic energy

$$\begin{aligned}\langle K \rangle &= \frac{1}{2T}\oint_S \dot{\boldsymbol{y}}\cdot\boldsymbol{\tau}dS = \frac{1}{2T}\oint_S \dot{\boldsymbol{y}}\cdot\dot{\boldsymbol{y}}dt \\ &= \frac{1}{T}\oint_S K dt = \hat{K},\end{aligned} \quad (21)$$

where $\hat{K}$ is a positive constant, since the phase point *periodically* moves along the curve and its generalised kinetic energy must be positive at all non-fixed points on the curve. Furthermore, the integral of Eq. (21) implies that $\textbf{curl}\dot{\boldsymbol{y}}=\textbf{curl}\boldsymbol{f}$ must not vanish, so that we have

$$\langle K \rangle = \frac{1}{T}\oint_S K dt = \hat{K}, \qquad \boldsymbol{U} \neq 0 \quad (22)$$

Time averaged kinetic energy flow

$$\langle \dot{K} \rangle = \frac{1}{T}\int_{\hat{t}}^{T+\hat{t}} \dot{K}dt = \frac{K(T+\hat{t})-K(\hat{t})}{T} = 0, \quad (23)$$

since for the periodical orbit S , $\dot{\boldsymbol{y}}(T+\hat{t})=\dot{\boldsymbol{y}}(\hat{t})$.

The averaged time change rates of both potential and kinetic energies vanish in a time period implies the time averaged mechanical energy $\langle E+K \rangle$ is conservative in a time period. Therefore, we can say: *for a periodical orbit of time period* T *in the phase space of a nonlinear dynamical system, its time averaged mechanical energy is conservative.*

VI. ENERGY FLOW CHARACTERISTICS OF CHAOTIC MOTIONS

Chaos is normally understood as “a state of disorder”. Although there is no universally accepted mathematical definition of chaos, a commonly used definition says that, for a dynamical system to be classified as chaotic, it must have the properties [5]: sensitive to initial conditions; topological transitivity; densely periodical orbits etc. From the energy flow point of view, chaotic motions have the characteristics.

Tractor around zero energy flow surface

A tractor of a nonlinear dynamical system should be a zero energy flow surface, towards which the phase points are attracted according to the following energy flow conditions

$$\dot{E}_d = \begin{cases} > 0, & d < d_{\dot{E}=0}, \\ = 0, & d = d_{\dot{E}=0}, \\ < 0, & d > d_{\dot{E}=0}, \end{cases} \tag{24}$$

where $d_{\dot{E}=0}$ and d respectively denote the distances of a point $\boldsymbol{y}$ on the zero energy flow surface and a neighbour point around this point $\boldsymbol{y}$ to the origin of phase space.

Negative time change rate of phase volume strain

Flows are restricted in a finite volume, so that the averaged time change rate of phase volume strain of the phase space must not be positive, i.e. from Eq.(12), we have

$$\dot{\upsilon}_V = \frac{1}{V}\int_V \dot{\upsilon} dV = \frac{1}{V}\int_V \sum_{I=1}^{n} \lambda_I dV \le 0. \tag{25}$$

Periodical motion with infinite time period

Based on above characteristics, a chaotic motion can be considered as a periodical motion with ***infinite time period***, so that it should have the characteristics of a periodical motion with infinite time period. From Eqs. (19-23) valid for periodical orbits, actually using the integral mean value theorem [6], we obtain:

$$\begin{aligned} \langle E \rangle &= \lim_{T\to\infty}\left\{\frac{1}{T}\int_{\hat{t}}^{T+\hat{t}} E dt\right\} \\ &= \lim_{T\to\infty}\left\{\frac{T\hat{E}}{T}\right\} \to \hat{E}, \quad \text{constant}, \end{aligned} \tag{26}$$

$$\langle \dot{E} \rangle = \lim_{T\to\infty}\left\{\frac{1}{T}\int_{\hat{t}}^{T+\hat{t}} \dot{E} dt\right\} \to 0, \tag{27}$$

$$\begin{aligned} \langle K \rangle &= \lim_{T\to\infty}\left\{\frac{1}{T}\int_{\hat{t}}^{T+\hat{t}} K dt\right\} \\ &= \lim_{T\to\infty}\left\{\frac{T\hat{K}}{T}\right\} \to \hat{K}, \quad \text{constant} \end{aligned}, \tag{28}$$

$$\langle \dot{K} \rangle = \lim_{T\to\infty}\left\{\frac{1}{T}\int_{\hat{t}}^{T+\hat{t}} \dot{K} dt\right\} \to 0. \tag{29}$$

This concludes that: *a chaotic motion of nonlinear dynamical systems can be considered as a periodical motion with infinite time period* $T \to \infty$, *of which the time averaged generalised energies are conservative with increasing the average time.*

VII. NUMERICAL EXAMINATIONS

Using the Matlab program [3], author has examined the chaotic motions of nonlinear dynamical systems reported in world publications, such as: forced Van der Pol's system [7]; Duffing's system [8]; Lorenz system [9, 10]; Rössler system [11] and SD oscillator [12-14]. The numerical results are shown in Figs. 6-10, of which the notations are defined as:

(I) the phase diagram (a), and the time change rate of phase volume strain (b);
(II) time histories of potential energy (GEP) (a) and its flow (PEF) (b), the curves of time averaged potential energy <GEP> (c) / potential flow (<*d*(GEP)/*dt*> (d) vs the averaged time *T*;
(III) time histories of kinetic energy (GKE) (a) and its flow (KEF) (b), the curves of time averaged kinetic energy <GKE> and kinetic energy flow (<*d*(GKE)/*dt*> vs the averaged time *T*.

The all numerical results have demonstrated that the revealed energy flow characteristics of chaotic motions of nonlinear dynamical systems are correct. Here, as example, we only give the detailed discussion on Lorenz system, but neglecting the related discussions for other systems except listing the governing equations with the used parameters.

Lorenz system: $\alpha = 10,\ \gamma = 8/3, \beta = 28$ [10]

$$\dot{\boldsymbol{y}} = \boldsymbol{f}, \quad \begin{cases} \dot{x} = \alpha(y - x) \\ \dot{y} = \beta x - y - xz, \\ \dot{z} = -\gamma z + xy \end{cases} \qquad \alpha, \beta, \gamma > 0, \quad \boldsymbol{y} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, \tag{30}$$

of which the Jacobian matrix, energy flow and spin matrix respectively are:

$$\boldsymbol{J} = \begin{bmatrix} -\alpha & \alpha & 0 \\ \beta - z & -1 & -x \\ y & x & -\gamma \end{bmatrix}, \tag{31}$$

$$\boldsymbol{E} = \begin{bmatrix} -\alpha & (\alpha+\beta-z)/2 & y/2 \\ (\alpha+\beta-z)/2 & -1 & 0 \\ y/2 & 0 & -\gamma \end{bmatrix}, \tag{32}$$

$$\boldsymbol{U} = \begin{bmatrix} 0 & (\alpha-\beta+z)/2 & -y/2 \\ -(\alpha-\beta+z)/2 & 0 & -x \\ y/2 & x & 0 \end{bmatrix}. \tag{33}$$

Energy flow equation and characteristic factors

The energy flow equation of Lorenz equation is

$$\dot{E} = [x \quad y \quad z]\boldsymbol{f} = [x \quad y \quad z]\boldsymbol{E}_0[x \quad y \quad z]^T,$$

$$\boldsymbol{E}_0 = \boldsymbol{E}(x = y = z = 0) \tag{34}$$

$$= \begin{bmatrix} -\alpha & (\alpha+\beta)/2 & 0 \\ (\alpha+\beta)/2 & -1 & 0 \\ 0 & 0 & -\gamma \end{bmatrix}.$$

For different bifurcation parameters, the energy flow matrix has the following three energy flow characteristic factors

$$\lambda_1 = \frac{-(\alpha+1)+B}{2} = \begin{cases} <0, & \beta < 2\sqrt{\alpha} - \alpha, \\ =0, & \beta = 2\sqrt{\alpha} - \alpha, \\ >0, & \beta > 2\sqrt{\alpha} - \alpha, \end{cases}$$

$$\lambda_2 = \frac{-(\alpha+1)-B}{2} < 0, \qquad \lambda_3 = -\gamma < 0. \tag{35}$$

$$B = \sqrt{(\alpha+1)^2 + (\alpha+\beta)^2 - 4\alpha}$$

Zero energy flow surface and its bifurcation

Figure 5

FIG. 5 The zero energy flow surface governed by $\dot{E} = 0$ from Eq. (36) in which $\lambda_1 > 0$, $\lambda_2 < 0$ and $\lambda_3 < 0$. Here $a = 1/\sqrt{\lambda_1}$, $b = 1/\sqrt{-\lambda_2}$ and $c = 1/\sqrt{-\lambda_3}$. The long dash dot-dot lines define an elliptical cylinder for the case of $\beta = 1$, $\alpha = 1$ and $\lambda_1 = 0$.

The energy flow Eq. (34) is represented in the energy flow space $o - \zeta_1\zeta_2\zeta_3$ span by the energy flow characteristic vectors in the form

$$\dot{E} = \lambda_1\zeta_1^2 + \lambda_2\zeta_2^2 + \lambda_3\zeta_3^2, \tag{36}$$

from which the zero energy flow surface $\dot{E} = 0$ and its bifurcation affected by the different parameter values can be obtained. Lanford [10] revealed the strange attractor of a Lorenz system with parameters $\alpha = 10,\ \gamma = 8/3$ and $\beta = 28$, which, by Eq. (35), has one positive ($\lambda_1 > 0$) and two negative ($\lambda_2 < 0$, $\lambda_3 < 0$) energy flow factors, so that its zero energy flow surface shown in Fig. 5. Due to the two negative energy flow factors, in the insider of the surface the energy flow $\dot{E} > 0$ and in the outside of surface the energy flow $\dot{E} < 0$. Therefore, *the flows at points in the outside of surface move towards the origin to reduce the potential and the flows in the inside of surface move backwards the origin to increase the potential. As result of this, this surface would be an attracting surface.*

Equilibrium points and stabilities

Origin (0, 0, 0).

The origin (0. 0, 0) is a fixed point, which is global stability if $\beta \le 2\sqrt{\alpha} - \alpha$ and unstable if $\beta > 2\sqrt{\alpha} - \alpha$, since three negative energy flow factors for the stable case while one positive factor $\lambda_1 > 0$ for the unstable case.

Nontrivial points

In the condition of $x = y$ and $\beta > 1$, there are two fixed points:

$$x = \pm\sqrt{\gamma(\beta-1)} = y, \qquad z = \beta - 1, \tag{37}$$

at which $\dot{E} = 0$. Since in the case $\beta > 1$, $\beta > 2\sqrt{\alpha} - \alpha$ is valid, these two fixed points are unstable. When $\beta \to 1$, these two points tend to the origin

Energy flow characteristics of chaotic motion

Time change rate of phase volume strain

The time change rate of phase volume strain of this Lorenz system can be obtained by using Eqs. (12) and (31), i.e.

$$\dot{\upsilon} = \mathbf{tr}(\boldsymbol{E}) = \sum_{I=1}^{3}\lambda_I = -1 - \alpha - \gamma < 0, \tag{38}$$

which holds at all points on the orbit, therefore, the phase volume of this Lorenz system is in contraction at every point in the phase space. Fig.6 (I) gives the phase diagram and the time change rate of phase volume train obtained by the numerical simulation for this system with an initial condition ($0.1,\ \ 0.1,\ \ 0$), which shows the style of attractor in a similar type drawn in Fig.5 and a constant negative time change rate of phase volume strain.

Figure 6

FIG. 6 Lorenz system ($\alpha = 10,\ \gamma = 8/3$ and $\beta = 28$).

Time averaged potential energy and its flow

From Eq.(36) and Eq.(27), the time averaged potential energy flow of this Lorenz system is given by

$$\left\langle \dot{E} \right\rangle = \lim_{T\to\infty}\left\{\frac{1}{T}\int_{\hat{t}}^{T+\hat{t}} \dot{E}dt\right\}$$
$$= \lim_{T\to\infty}\left\{\frac{1}{T}\int_{\hat{t}}^{T+\hat{t}} \sum_{I=1}^{3}\lambda_I \zeta_I^2 dt\right\} \to 0. \quad (39)$$

Due to one positive factor λ_1 and two negative factors λ_2 and λ_3 with the orbit restricted in a finite phase volume shown by Fig.6 (I), the value of Eq.(39) tends zero when $T \to \infty$. As shown in Fig. 6(II), the time history of generalised potential energy (GEP) oscillates around a positive mean value, so that the time averaged potential energy <GEP> tends the constant mean value when averaged time $T \to \infty$, while the potential energy flow (PEF) oscillates around the zero mean value and its time averaged one <*d*(GEP)/*dt*> tends to zero when averaged time $T \to \infty$.

Time averaged kinetic energy and its flow

The time change rate of kinetic energy defined by Eq.(6) is used to Eq.(30), which gives

$$\dot{K} = \dot{\boldsymbol{y}}^T \ddot{\boldsymbol{y}} = \dot{\boldsymbol{y}}^T \boldsymbol{J}\dot{\boldsymbol{y}} = \dot{\boldsymbol{y}}^T \boldsymbol{E}\dot{\boldsymbol{y}} = \dot{\boldsymbol{y}}^T \boldsymbol{E}_0 \dot{\boldsymbol{y}} + A,$$

$$A = \dot{\boldsymbol{y}}^T \begin{bmatrix} 0 & -z/2 & y/2 \\ -z/2 & 0 & 0 \\ y/2 & 0 & 0 \end{bmatrix} \dot{\boldsymbol{y}} \quad (40)$$

$$= \dot{x}\begin{bmatrix} \dot{y} & \dot{z} \end{bmatrix} \overline{\boldsymbol{U}} \begin{bmatrix} y \\ z \end{bmatrix}, \ \overline{\boldsymbol{U}} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}.$$

The eigenvectors of the spin matrix $\overline{\mathbf{U}}$ are

$$\boldsymbol{\Phi} = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ -\mathbf{i} & \mathbf{i} \end{bmatrix}, \quad (41)$$

based on which, without losing generalities, the real vector $\begin{bmatrix} y & z \end{bmatrix}^T$ and its time derivative can be decomposed into the complex forms

$$\begin{bmatrix} y \\ z \end{bmatrix} = \boldsymbol{\Phi}\begin{bmatrix} \tilde{\xi}_2 \\ \tilde{\xi}_3 \end{bmatrix} e^{\mathrm{i}\theta(t)}, \qquad \begin{bmatrix} \dot{y} \\ \dot{z} \end{bmatrix} = \mathbf{i}\dot{\theta}\boldsymbol{\Phi}\begin{bmatrix} \tilde{\xi}_2 \\ \tilde{\xi}_3 \end{bmatrix} e^{\mathrm{i}\theta(t)}, \quad (42)$$

since the spin matrix $\overline{\boldsymbol{U}}$ corresponds a rotation $\theta(t)$ [3]. As a result of this, we obtain that

$$\begin{bmatrix} \dot{y} & \dot{z} \end{bmatrix} \overline{\boldsymbol{U}} \begin{bmatrix} y \\ z \end{bmatrix} = -\mathbf{i}\dot{\theta}e^{-\mathrm{i}\theta(t)}\begin{bmatrix} \tilde{\xi}_2^* & \tilde{\xi}_3^* \end{bmatrix} \boldsymbol{\Phi}^{*T}\overline{\boldsymbol{U}}\boldsymbol{\Phi}\begin{bmatrix} \tilde{\xi}_2 \\ \tilde{\xi}_3 \end{bmatrix} e^{\mathrm{i}\theta(t)}$$

$$= \dot{\theta}(\tilde{\xi}_2^*\tilde{\xi}_2 - \tilde{\xi}_3^*\tilde{\xi}_3) = \dot{\theta}\left(\left|\tilde{\xi}_2\right|^2 - \left|\tilde{\xi}_3\right|^2\right), \quad (43)$$

$$= \textbf{a real number,}$$

Therefore, Eq.(40) becomes

$$\dot{K} = \dot{\boldsymbol{y}}^T \boldsymbol{E}_0 \dot{\boldsymbol{y}} + \dot{x}\dot{\theta}\left(\left|\tilde{\xi}_2\right|^2 - \left|\tilde{\xi}_3\right|^2\right)$$
$$= \sum_{I=1}^{3}\lambda_I \dot{\zeta}_I^2 + \dot{x}\dot{\theta}\left(\left|\tilde{\xi}_2\right|^2 - \left|\tilde{\xi}_3\right|^2\right), \quad (44)$$

of which, in a similar way as Eq.(39), the time averaged value

$$\left\langle \dot{K} \right\rangle = \lim_{T\to\infty}\left\{\frac{1}{T}\int_{\hat{t}}^{T+\hat{t}} \dot{K}dt\right\} \to 0. \quad (45)$$

The curves in Fig. 6(III) show that the time histories of kinetic energy (GKE) and its flow (KEF) oscillate respectively around their mean values, while the time averaged <GKE> and <*d*(GKE)/*d*t> respectively tend a constant and zero with averaging time $T \to \infty$.

Forced Van der Pol's system:

$\alpha = 5,\ F = 5,\ \omega = 2.446,\ \phi = 0$ [7]

$$\begin{bmatrix} \dot{y}_1 \\ \dot{y}_2 \end{bmatrix} = \begin{bmatrix} y_2 - \alpha(y_1^3/3 - y_1) \\ -y_1 \end{bmatrix} + \begin{bmatrix} 0 \\ F\cos(\omega t + \phi) \end{bmatrix}. \quad (46)$$

Figure 7 gives its numerical simulation results.

Figure 7

FIG. 7 Forced Van der Pol's system ($\alpha = 5,\ F = 5,\ \omega = 2.446,\ \phi = 0$).

Forced Duffing's system:

$F=0.3,\ \omega=1.0,\ \delta=0.15,\ \ \phi=0$ [8]

$$\begin{bmatrix}\dot{y}_1\\ \dot{y}_2\end{bmatrix}=\begin{bmatrix}y_2\\ y_1-y_1^3-\delta y_2\end{bmatrix}+\begin{bmatrix}0\\ F\cos(\omega t+\phi)\end{bmatrix}. \quad (47)$$

The simulation results are shown in Fig.8.

Figure 8

FIG. 8 Forced Duffing's system $F=0.3,\ \omega=1.0,\ \delta=0.15,\ \ \phi=0$.

Rössler system: $\alpha=0.1=\beta,\ \gamma=14$ [11]

$$\begin{bmatrix}\dot{y}_1\\ \dot{y}_2\\ \dot{y}_3\end{bmatrix}=\begin{bmatrix}-(y_2+y_3)\\ y_1+\alpha y_2\\ \beta+y_3(y_1-\gamma)\end{bmatrix},\ \begin{bmatrix}y_1\\ y_2\\ y_3\end{bmatrix}_0=\begin{bmatrix}1.0\\ 1.0\\ 0\end{bmatrix}, \quad (48)$$

of which the simulation results are shown in Fig.9.

Figure 9

FIG. 9 Rössler system $\alpha=0.1=\beta,\ \gamma=14$ with initial condition (1.0, 1.0, 0).

Forced SD oscillator: $\alpha=0.01,\ F=0.8,\ \delta=1,\ \omega=1.0605,\ \gamma=0.01415$, $g=0.5$ [12-13]

$$\begin{bmatrix}\dot{y}_1\\ \dot{y}_2\end{bmatrix}=\begin{bmatrix}y_2\\ -2\gamma y_2-y_1(1-\dfrac{1}{\sqrt{y_1^2+\alpha^2}}-\delta g\end{bmatrix}+\begin{bmatrix}0\\ F\cos(\omega t+\phi)\end{bmatrix}, \quad (49)$$

and the simulation results are shown in Fig.10.

Figure 10

FIG. 10 Forced SD oscillator $\alpha=0.01,\ F=0.8,\ \omega=1.0605,\ \gamma=0.01415$, $\delta=1,\ g=0.5$

VIII. Conclusion and discussion

The two real scalars, generalised potential and kinetic energies, are defined in the phase space to investigate the dynamical behaviour of nonlinear dynamical systems. The generalised potential energy plays a role of Liapunov function suitable to all nonlinear dynamical systems to examine stabilities about fixed points, and the real symmetrical energy flow matrix can replace the non-symmetrical Jacobian matrix to tackle the behaviour of linearised approximation at a point of nonlinear systems. Chaotic motions of nonlinear dynamical systems with negative time change rate of phase volume train are restricted in finite regions of phase space, so that they can be considered as periodical motions with infinite time period. With the average time tending to infinite, the time averaged potential and kinetic energies of chaotic motions tend to constants, while the time averaged potential and kinetic energy flows vanish, which provides an energy conservation law of chaotic motions in nature. The numerical simulations with given curves for the chaotic motions of 5 world recognised nonlinear dynamical systems have demonstrated that the above conclusions are correct.

It has been noticed: 1) that quasi-periodical motions are sometimes considered as the motion with infinite time period. For example, a motion consisting of two harmonic motions with an irrational ratio of their frequencies has no a common finite time period, so that its time period is considered as infinite. Concerning the relationship between chaotic motions and quasi-periodical motions, Wang and Hao [15] revealed the transition from quasi-periodical motions to chaotic motions; 2) the Casimir power of the Qi chaotic system [16] shows the same form of the generalised potential energy developed in [3] and used in this paper, which has been employed to uncover the mechanism of various dynamical systems, see for example [16-19]. These evidences further confirm applications of the developed generalised energy flow theory. In principle, any nonlinear dynamical systems defined by Eq.(1) can be investigated using the developed energy flow theory to reveal their some unknown phenomena.

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