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INTRODUCTION

Slender body methods for predicting ship motions response to sea waves have proved to be very successful and are now used by naval architects in design work. There have been a number of papers seeking to evaluate forces and moments on slenderbodies in uniform motion that could form the basis of a treatment of forces and moments arising when a ship is manoeuvring in calm water. However, these techniques have not been developed to the point of forming a usable design tool for the prediction of course keeping stability and steady rates of turn.

The manoeuvring situation differs from the ship motion case in that the significant factor in the generation of lateral forces on the vessel is the formation of vortex sheets either downstream of a lifting surface (such as a yacht keel or a rudder) or due to separation of the flow round the hull itself. The latter phenomenon gives rise to non-linear forces and moments on the vessel that become important at larger angles of yaw.

It is usual in slender body theory, as applied to lifting bodies, to treat fully submerged bodies and to represent a ship operating in a free surface by a double model, symmetric about a horizontal plane representing the nominal free surface. This treatment of the free surface as a rigid surface can be justified by order of magnitude arguments, as being consistent with the assumption of a slenderbody, and this implies that wave making effects can be expected to be small compared to the basic double model forces and moments. This is probably correct in relation to moments acting on the vessel, but there are circumstances in which the double model forces approach zero and in such cases body wave making may need to be taken into account in estimating lateral forces acting on the vessel.

A sailing vessel normally adopts a heeled attitude, whilst most vessels turning at high speed will heel because of the vertical separation of the centre of gravity and the line of action of the lateral force acting on the below water hull form. When the vessel is heeled the equivalent double body has sections which are not symmetrical about the vertical centreline plane.

This paper presents a treatment of the flow of an ideal fluid past a slender body for which

(a) Transverse sections are symmetric about a horizontal plane

(b) The sections are not necessarily symmetric about a vertical plane

(c) The body may be fitted with lifting surface appendages

(d) Vortex sheets may exist due to the presence of these appendages or due to flow separation along the body length.

It will be assumed that the body is moving forward at uniform speed, but that it may be executing lateral motions. The primary aim is to evaluate total forces and moments acting on the body including any historical effects implied in the timewise development of the vortex sheets. The paper draws together a number of topics treated in other earlier papers, but includes a treatment of the effects of asymmetry as a major extension of previous theory.

1. BASIC FORMULATION OF THE POTENTIAL PROBLEM

The problem is treated as a potential flow problem in which the body produces a disturbance to a uniform stream of speed U defined by a velocity potential $\phi(x, y, \xi, t)$. The direction of the free stream flow and the coordinate system chosen is indicated in fig. 1. At this stage the surface of the body will be defined by the equation

$$y = Y(x - d(\xi,t), \xi) \qquad 1.1$$

where $d(\xi,t)$ represents a lateral displacement of the body relative to some nominal datum position. In this form the displacement is allowed to vary along the body length as well as with time. There is no formal need for this displacement function to represent a rigid body motion, although this will normally be the case.

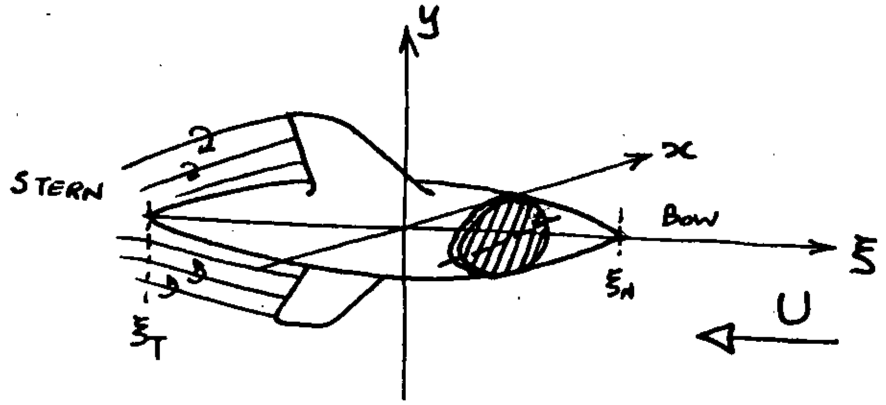


fig 1.

In fig. 1 a typical cross section is shaded. This section is supposed symmetric about the (x,ξ) plane, but may be asymmetric about the (y,ξ) plane.

Orders of magnitude of the various quantities occurring in the problem will be expressed in terms of a slenderness parameter ε which is to be thought of as the ratio of a typical transverse body dimension (e.g. body beam) to a typical longitudinal dimension (e.g. waterline length). The parameter ε is assumed small (i.e. $\varepsilon << 1.0$).

For the general three dimensional case the disturbance potential satisfies the Laplace equation

$$\nabla^2\phi = \phi_{\xi\xi} + \phi_{xx} + \phi_{yy} = 0 \qquad 1.2$$

subject to the kinematic boundary condition (no flow through the hull surface) which takes the form

$$\phi_y = \left[\phi_x + d_\xi .(U - \phi_\xi) - d_t\right].\ Y_x - (U - \phi_\xi).\ Y_\xi \qquad 1.3$$

In equations 1.2 and 1.3 and in all subsequent equations a subscript denotes partial differentiation.

That is

$$\phi_{\xi\xi} \equiv \frac{\partial^2\phi}{\partial\xi^2} \quad \text{etc.}$$

Equation 1.3 is to apply everywhere over the body surface. The disturbance potential will naturally decay to zero remote from the body and from any trailing vortex system.

Once a disturbance potential has been found, pressures within the fluid and hence forces acting on the body can be found from Bernoulli's equation in its unsteady form

$$\frac{p}{\rho} + \phi_t + \frac{1}{2}(U - \phi_\xi)^2 + \frac{1}{2}\phi_x^2 + \frac{1}{2}\phi_y^2 = \frac{1}{2}U^2 \qquad 1.4$$

Here p is the fluid pressure above its hydrostatic value, so that force estimates using 1.4 will not include buoyancy effects.

The essence of the slenderbody method is to examine the order of magnitude of each term in eqns 1.2, 1.3 and 1.4 either by a process of scaling the coordinates or by reference to approximate solutions for the streamflow past a body of revolution. A consistent scaling process yields the following orders of magnitude :

Terms of order $\varepsilon^0 = 1$: U, ξ, Y_x, ϕ_{xx}, ϕ_{yy}
Terms of order ε : x, y, d, Y, Y_ξ, ϕ_x, ϕ_y
Terms of order ε^2 : ϕ, ϕ_ξ, $\phi_{\xi\xi}$, ϕ_t

Some texts give the order of magnitude of ϕ as $\varepsilon^2 \ln \varepsilon$ (based on flow past a body of revolution), but arrive at the same reduced equations as those given below, so that the simpler form will be taken here.

A systematic rejection of the smallest terms in the basic equations leads to the following equations :

1.2 reduces to $$\phi_{xx} + \phi_{yy} = 0 \qquad 1.5$$

1.3 reduces to $$\phi_y = [\phi_x + U.d_\xi - d_t] \cdot Y_x - U.Y_\xi \qquad 1.6$$

1.4 reduces to $$\frac{p}{\rho} + \phi_t - U.\phi_\xi + \frac{1}{2}\phi_x^2 + \frac{1}{2}\phi_y^2 = 0 \qquad 1.7$$

In each equation the ratio $\dfrac{\text{neglected terms}}{\text{retained terms}} = O(\varepsilon^2)$

so that, if $\varepsilon = {}^1/_7$ is regarded as typical, the neglected terms should be of the order of 2% of those retained. The importance of this reduction is that at each crossflow plane along the body length, the disturbance potential is obtained as a solution to a two dimensional problem to which very powerful conformal mapping methods may be applied.

2. THE LOCAL LATERAL FORCE

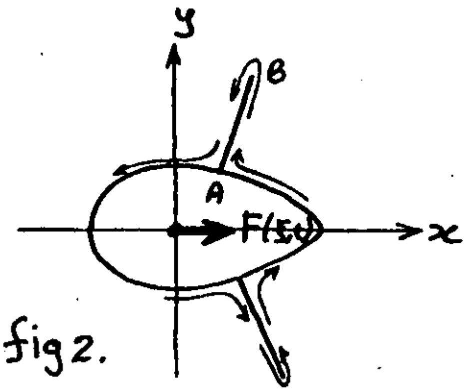


fig 2.

fig. 2 represents one particular cross section of the body. The line AB could represent either a flat plate fin attached to the body or it could represent a cut through a vortex sheet shed in the wake of some upstream lifting appendage.

The lateral force per unit length of body experienced by the body at this cross section is given by

$$F(\xi,t) = - \int_c p dy \qquad 2.1$$

The integration can be carried out round the contour indicated in fig. 2 since there is zero pressure difference across any portion of AB that represents a vortex sheet rather than part of an appendage. The inclusion of all vortex sheets within the contour c in this way makes it possible to evaluate $F(\xi,t)$ by integration round any contour reconcilable with c and in many circumstances considerable simplification can result by taking a very large contour enclosing the section.

Using equation 1.7 it is now possible to write

$$F(\xi,t) = \rho \int_c \left[\phi_t - U\phi_\xi + \frac{1}{2}\phi_x^2 + \frac{1}{2}\phi_y^2\right] dy \qquad 2.2$$

Now define $A(\xi,t)$ by the equation

$$A(\xi,t) = \int_c \phi \, dy \qquad 2.3$$

On $y = Y(x - d(\xi,t), \xi)$ at fixed y

$$0 = Y_x\left[\delta x - d_\xi \delta\xi - d_t\ \delta t\right] + Y_\xi . \delta\xi$$

Thus at fixed time $\dfrac{\delta x}{\delta \xi} = d_\xi - \dfrac{Y_\xi}{Y_x}$

and at fixed longitudinal position $\dfrac{\delta x}{\delta t} = d_t$

It follows that

$$\frac{\partial A}{\partial \xi} = \int_c \left[\phi_\xi + \left(d_\xi - \frac{Y_\xi}{Y_x}\right) \cdot \phi_x\right] dy \qquad 2.4$$

and $$\frac{\partial A}{\partial t} = \int_c \left[\phi_t + d_t . \phi_x\right] dy \qquad 2.5$$

Equation 2.4 can be rewritten by making use of the body boundary conditions 1.6 :

$$U\frac{\partial A}{\partial \xi} = \int_c \left\{U\phi_\xi + Ud_\xi . \phi_x - \frac{\phi_x}{Y_x}\left[(\phi_x + Ud_x - d_t)Y_x - \phi_y\right]\right\} dy$$

which simplifies to give

$$U\frac{\partial A}{\partial \xi} = \int_c \left[(U\phi_\xi + \phi_x . d_t - \phi_x^2)\ dy + \phi_x\phi_y\ dx\right] \qquad 2.6$$

Thus, on using 2.2, 2.5 and 2.6 :

$$F(\xi,t) + \rho U\frac{\partial A}{\partial \xi} - \rho\frac{\partial A}{\partial t} = \rho \int_c \left[\tfrac{1}{2}(\phi_y^2 - \phi_x^2)\ dy + \phi_x\phi_y\ dx\right] \qquad 2.7$$

Now, as is demonstrated in appendix 1, the contour integral on the right hand side of 2.7 vanishes identically. This means that the load per unit length is given simply by

$$F(\xi,t) = -\ \rho U\frac{\partial A}{\partial \xi} + \rho\frac{\partial A}{\partial t} \qquad 2.8$$

This remarkable result reduces the evaluation of hydrodynamic forces and moments simply to the evaluation of

$$A(\xi,t) = \int_C \phi \, dy$$

The total force and the moment acting on the body are then obtained by integration along the body length.

3. THE DETERMINATION OF THE DISTURBANCE POTENTIAL

The complete disturbance potential can be regarded as the linear sum of three components, each of which provides its own separate contribution to the total force on the body.

Thus $\phi = \phi_o + \phi_1 + \phi_2$

where ϕ_o = general streaming flow past a stationary body
ϕ_1 = flow generated by body lateral motions
ϕ_2 = flow due to external vortex sheets.

The position and strength of the external vortex sheets at any given station along the length and at any given time will, of course, depend on the convection processes occurring upstream of the given section taking into account all three components of ϕ.

The general boundary condition 1.6 can thus be split into the following three sub-problem conditions, each to be satisfied on the body surface :

$$\phi_{oy} = \phi_{ox}.Y_x - UY_\xi \qquad 3.1$$

$$\phi_{1y} = (\phi_{1x} + Ud_\xi - d_t)Y_x \qquad 3.2$$

and $$\phi_{2y} = \phi_{2x}.Y_x \qquad 3.3$$

4. SECTION MAPPING FUNCTION

In order to arrive at a complex potential function corresponding to each contribution to the disturbance potential ϕ it will be assumed that the body section can be mapped from the unit circle $\zeta = e^{i\theta}$ by a mapping function of the form

$$z = x + i\,y = x_o(\xi) + a_o(\xi).\zeta + \sum_{n=1}^{\infty} \frac{a_n(\xi)}{\zeta^n} \qquad 4.1$$

For convenience x_o will be written in place of $x_o(\xi)$ and x_o' will be written in place of

$$\frac{dx_o}{d\xi}$$

A similar convention will be adopted for the coefficients a_o, a_n. It should be noted that symmetry about the real (x) axis implies that x_o, a_o, a_n are all real and that the even order coefficients a_{2m} arise because of a lack of symmetry about the imaginary (y) axis. In an application of this method 4.1 would be truncated suitably and the coefficients would be determined numerically from the actual section shapes.

As a further convenient notation the body shape can be represented variously as

$$z = Z(\xi,\theta) \qquad 4.2$$

or $\quad x = X(\xi,\theta)$ and $y = Y(\xi,\theta) \qquad 4.3$

In terms of the mapping function 4.1 the partial derivatives of Z can be written as

$$Z_\theta = i\left[a_o\zeta - \sum_{n=1}^{\infty} \frac{na_n}{\zeta^n}\right] \qquad 4.4$$

and $$Z_\xi = x_o' + a_o'\zeta + \sum_{n=1}^{\infty} \frac{a'_n}{\zeta^n} \qquad 4.5$$

In deriving the complex potential for ϕ_o the following relation will prove useful

$$Z_\xi . \bar{Z}_\theta = (X_\xi + iY_\xi)(X_\theta - iY_\theta)$$

$$= (X_\xi X_\theta + Y_\xi Y_\theta) + i(X_\theta . Y_\xi - X_\xi . Y_\theta)$$

whence $Z_\xi . \bar{Z}_\theta - \bar{Z}_\xi . Z_\theta = 2i(X_\theta . Y_\xi - X_\xi . Y_\theta)$ 4.6

The product $Z_\xi . \bar{Z}_\theta$ can be written out in terms of the mapping function, using the result that for points on the body $\zeta\bar{\zeta} = 1.0$:

$$Z_\xi . \bar{Z}_\theta = - i \left[x'_o + \frac{a'_o}{\zeta} + \sum_{n=1}^{\infty} \frac{a'_n}{\zeta^n} \right] . \left[\frac{a_o}{\zeta} - \sum_{n=1}^{\infty} \frac{na_n}{\bar{\zeta}^n} \right] \qquad 4.7$$

Appendix 2 shows that this series may be rearranged in inverse powers of ζ and $\bar{\zeta}$ as

$$Z_\xi . \bar{Z}_\theta = - i \left[a_o a'_o + \frac{x'_o a_o}{\zeta} - \sum_{n=1}^{\infty} na_n a'_n + \sum_{n=1}^{\infty} \frac{a_o a'_n}{\zeta^{n+1}} - \sum_{n=1}^{\infty} \frac{na_n x'_o}{\bar{\zeta}^n} \right.$$

$$\left. - \sum_{n=1}^{\infty} \frac{na_n a_o}{\bar{\zeta}^{n+1}} - \sum_{n=1}^{\infty} \sum_{s=1}^{\infty} \frac{(n+s) a_{n+s} a'_n}{\bar{\zeta}^s} - \sum_{n=1}^{\infty} \sum_{s=1}^{\infty} \frac{na_n a'_{n+s}}{\zeta^s} \right] \qquad 4.8$$

This result is valid for points on the body for which $\zeta = e^{i\theta}$.

5. SOLUTION OF THE GENERAL STREAMFLOW

Since ϕ_o is a solution of a two dimensional problem, governed by equation 1.5 it is possible to define a corresponding stream function ψ_o and to write

$$\phi_o + i\psi_o = W_o \; (\zeta, \xi) \qquad 5.1$$

The appropriate boundary condition on the body ($\zeta = e^{i\theta}$) is from eq. 3.1

$$\phi_{oy} = \phi_{ox} . \left. \frac{\partial Y}{\partial X} \right|_{\xi \text{ Const}} - U \left. \frac{\partial Y}{\partial \xi} \right|_{X \text{ Const}}$$

or $$-\psi_{ox} = \psi_{oy} \frac{\delta y}{\delta x} - U\frac{\partial y}{\partial \xi}\bigg|_{x\ \mathrm{const}}$$ 5.2

Now, from equations 4.3

$$\delta x = X_\xi\ \delta\xi + X_\theta\ \delta\theta = 0$$

and $$\delta y = Y_\xi\ \delta\xi + Y_\theta\ \delta\theta = \frac{Y_\xi . X_\theta - X_\xi . Y_\theta}{X_\theta}\ \delta\xi$$

i.e. $$\frac{\partial y}{\partial \xi}\bigg|_{x\ \mathrm{const}} = \frac{Y_\xi . X_\theta - X_\xi Y_\theta}{X_\theta}$$ 5.3

Using this result the body boundary condition 5.2 can be rewritten as

$$\frac{\partial \psi_o}{\partial \theta} = U(Y_\xi . X_\theta - X_\xi . Y_\theta)$$ 5.4

It follows that the boundary condition finally takes the form

$$\frac{\partial W_o}{\partial \theta} - \frac{\partial \bar{W}_o}{\partial \theta} = \frac{2i\partial \psi_o}{\partial \theta} = U\left[Z_\xi . \bar{Z}_\theta - \bar{Z}_\xi . Z_\theta\right]$$ 5.5

To summarize the position at this stage, it is necessary to find a function $W_o(\zeta,\xi)$ which satisfies equation 5.5 on $\zeta = e^{i\theta}$ and which represents a flow which dies away as $|\zeta| \to \infty$. By considering the terms in equation 4.8 and in its complex conjugate, it can be seen that equation 5.5 is satisfied if, on the body

$$\frac{\partial W_o}{\partial \theta} = -iU\left[a_o a_o' - \sum_{n=1}^{\infty} na_n a_n' + \frac{a_o x_o'}{\zeta} + \sum_{n=1}^{\infty} \frac{a_o a_n'}{\zeta^{n+1}} - \sum_{n=1}^{\infty} \frac{na_n x_o'}{\zeta^n}\right.$$

$$\left. - \sum_{n=1}^{\infty} \frac{na_n a_o'}{\zeta^{n+1}} - \sum_{n=1}^{\infty}\sum_{s=1}^{\infty} \frac{(n+s)a_{n+s} a_s'}{\zeta^n} - \sum_{n=1}^{\infty}\sum_{s=1}^{\infty} \frac{sa_s a_{n+s}'}{\zeta^n}\right]$$

The last two terms are obtained by interchanging the roles of n and s in the double summations and reversing the order of summation.

This result can now be integrated round the body contour to obtain a complex potential given by

$$W_o(\zeta,\xi) = -U\left[(a_o a_o' - \sum_{n=1}^{\infty} na_n a_n')\ln\zeta - \frac{a_o x_o'}{\zeta} + \sum_{n=1}^{\infty}\frac{na_n a_o' - a_o a_n'}{(n+1)\zeta^{n+1}} + \sum_{n=1}^{\infty}\frac{na_n x_o' + \sum_{n=1}^{\infty}\{(n+s)a_{n+s}a_s' + s\,a_s a_{n+s}'\}}{n.\ \zeta^n}\right]$$

This equation can be conveniently written as

$$W_o(\zeta,\xi) = -U\left[A_o\ \ln\zeta + \sum_{n=1}^{\infty}\frac{A_n}{\zeta^n}\right] \qquad 5.6$$

where $A_o = a_o a_o' - \sum_{n=1}^{\infty} na_n a_n'$

$$A_1 = -a_o x_o' + a_1 x_o' + \sum_{n=1}^{\infty}\{(s+1)a_{s+1}a_s' + sa_s a_{s+1}'\}$$

and for $n \geq 2$

$$A_n = \frac{1}{n}\left[(n-1)a_{n-1}a_o' - a_o a_{n-1}' + na_n x_o' + \sum_{s=1}^{\infty}\{(n+s)a_{n+s}a_s' + sa_s a_{n+s}'\}\right]$$

This result gives an explicit solution for the complex potential representing the general streaming motion past a slender body with transversely asymmetric body sections represented by the mapping function 4.1. It should be noted that since the coefficients x_o, a_o and a_n occurring in the mapping function are all real the coefficients A_o and A_n occurring in 5.6 are also real. The complex potential can be used to compute transverse velocity components at any field point in the cross flow plane and such a computation is required in order to follow through the convection process for any free vortex sheet present near the body. The contribution of this general streaming motion to the hydrodynamic lateral force can be obtained as follows :-

On the body contour $\phi_o = -U.\sum_{n=1}^{\infty} A_n \operatorname{Cos}n\theta$ 5.7

whilst from the mapping function

$$y = a_o \sin\theta - \sum_{m=1}^{\infty} a_m \sin m\theta$$

$$\therefore \int_c \phi_o \, dy = - U \int_o^{2\pi} \sum_{n=1}^{\infty} A_n \cos n\theta . \left[a_o \cos\theta - \sum_{m=1}^{\infty} m a_m \cos m\theta\right] d\theta$$

Using the identity $\int_o^{2\pi} \cos n\theta \cos m\theta \, d\theta = \begin{cases} \pi & (n = m \neq 0) \\ o & (n \neq m) \end{cases}$

it follows that

$$A_o(\xi) = \int_c \phi_o \, dy = - \pi U \left[a_o A_1 - \sum_{n=1}^{\infty} n a_n A_n\right] \qquad 5.8$$

and from equation 2.8 the corresponding cross force per unit body length is

$$F_o(\xi) = \rho \pi U^2 A_o'(\xi) \qquad 5.9$$

The total transverse force and yawing moment components from this streaming motion are

$$L_o = \int_{\xi_T}^{\xi_N} F_o(\xi) d\xi = \rho \pi U^2 \left[A_o(\xi_N) - A_o(\xi_T)\right] \qquad 5.10$$

and $$M_o = \int_{\xi_T}^{\xi_N} \xi F_o(\xi) d\xi = \rho \pi U^2 \left[\xi . A_o(\xi)\Big|_{\xi_T}^{\xi_N} - \int_{\xi_T}^{\xi_N} A_o(\xi) d\xi \right] \qquad 5.11$$

In these equations the nose (or bow) of the body is at $\xi = \xi_N$ and the tail (or stern) is at $\xi = \xi_T$ and M_o is the moment about $\xi = 0$. If the body sections shrink smoothly to a point at the body ends $A_o(\xi_N) = A_o(\xi_T) = 0$ so that $L_o = 0$ and the streaming motion produces a simple couple

$$M_o = - \rho \pi U^2 \int_{\xi_T}^{\xi_N} A_o(\xi) d\xi \qquad 5.12$$

Where the body is symmetric longitudinally about $\xi = 0$ the couple M_o will vanish on the basis that x_o, a_o and a_n are even functions of ξ, consequently a_o', a_n', A_o and A_n are all odd functions of ξ, which in turn implies that $A_o(\xi)$ is also an odd function of ξ for which

$$\int_{\xi_T}^{\xi_N} A_o(\xi) d\xi \equiv 0$$

Thus it is necessary for the body to be asymmetric longitudinally as well as transversely in order that the general streaming motion should produce a couple M_o acting on the body.

6. TRANSVERSELY SYMMETRIC BODIES YAWED

Previous texts on slenderbody theory have dealt with transversely symmetric bodies yawed at a small angle α to the main streamflow. This case is embedded in the theory presented here and can be extracted by noting that in the mapping function $a_{2m} = 0$ for transversely symmetric sections.

With this restriction only those terms corresponding to odd values of n contribute to the sum occurring in equation 5.8. Now, for odd n, terms like $a_{n+s}\, a_s' = 0$ since $a_{n+s} = 0$ if s is odd and $a_s' = 0$ if s is even. Thus the coefficients occurring in the complex potential (equation 5.6) reduce in this case (for odd n) to

$$A_1 = (a_1 - a_o)\, x_o'$$

$$A_n = a_n\, x_o' \qquad (\text{odd } n > 2)$$

Thus in 5.8 $A_o(\xi)$ reduces to

$$A_o(\xi) = + \pi U \left[(a_1 - a_o)^2 + \sum_{n=3}^{\infty} n\, a_n^2 \right] x_o'$$

where the summation involves odd n only.

Now the cross-sectional area enclosed by the body contour can be obtained from the mapping function as

$$\text{Area} = \int_c x\,dy = \pi\left[a_o^2 - a_1^2 - \sum_{n=3}^{\infty} n a_n^2\right] \qquad \text{(odd n only)}$$

Thus if the substitution $x_o' = \alpha$ is made the case of transverse symmetry reduces to

$$A_o(\xi) = -U\alpha\,\left[2\pi\, a_o(a_1 - a_o) + \text{Area}\right] \qquad 6.1$$

In other words in order to evaluate forces on a transversely symmetric double body it is only necessary to determine the mapping coefficients a_o and a_1 for each body section.

7. CONTRIBUTION DUE TO LATERAL MOTIONS

The contribution to the velocity potential due to body lateral motions (ϕ_1) satisfies the boundary condition 3.2 :

$$\phi_{1y} = (\phi_{1x} + Ud_\xi - d_t)\, Y_x$$

This is the boundary condition for a two dimensional body moving laterally with a speed $v = d_t - Ud_\xi$. By superinposing a lateral velocity $-v$ in the z plane to convert the problem to that of a uniform stream past a stationary body, the corresponding complex potential can be written down immediately as

$$w_1(z,t) = -a_o v\,\left[\zeta + \frac{1}{\zeta}\right] + vz$$

$$\text{or} \qquad w_1(z,t) = (d_t - Ud_\xi)\,\left[-\frac{a_o}{\zeta} + \sum_{n=1}^{\infty} \frac{a_n}{\zeta^n}\right] \qquad 7.1$$

Equation 7.1 follows on substituting the section mapping function (equation 4.1).

From this result it follows, using the same methods as in sections 5 and 6 that

$$A_1(\xi,t) = \int_c \phi_1 \, dy = (d_t - Ud_\xi) \left[2\pi \, a_o(a_1 - a_o) + \text{Area}\right] \qquad 7.2$$

If d_t is set to zero and d_ξ to a yaw angle α it can be seen that equation 7.2 is in agreement with equation 6.1. This clearly should be the case. In the case of a purely lateral motion $d_\xi = 0$ and the corresponding lateral body force is

$$F_1(\xi,t) = \rho \frac{\partial A_1}{\partial t} = \rho \left[2\pi \, a_o(a_1 - a_o) + \text{Area}\right] . \, d_{tt} \qquad 7.3$$

This corresponds to a section added mass in lateral motion given by

$$m(\xi) = - \rho\left[2\pi \, a_o(a_1 - a_o) + \text{Area}\right] \qquad 7.4$$

For the alternative case of a body set at a fixed yaw angle α 7.2 reduces to

$$\rho U \, A_1(\xi) = - \rho U^2 \alpha\left[2\pi \, a_o(a_1 - a_o) + \text{Area}\right] = m(\xi) \, U^2\alpha$$

The lateral body force is now

$$F_1(\xi) = - \rho U \, A_1'(\xi) = U^2\alpha . \, m'(\xi) \qquad 7.3$$

This relationship between lateral forces on a yawed body and the section added mass in lateral motion is in fact a well known slenderbody result.

8. CONTRIBUTION DUE TO EXTERNAL VORTEX SHEETS

The last contribution to the total flow at any cross section is that due to any vortex sheets present, generated either at the trailing edge of a lifting appendage or in some separation process taking place along the body length. Deferring for the moment the problem of the determination of the location and strength of the vortex sheets, the velocity potential at any given section and at any particular time is governed by the boundary condition 3.3 :

$$\phi_{2y} = \phi_{2x} . Y_x$$

This boundary corresponds to the flow generated by the vortex sheet round a stationary two dimensional body and on the body surface the corresponding stream function ψ_2 will have a constant value.

Consider first the evaluation of the integral $\int_c \phi_2 \, dy$ taken round a contour c as shown in figure 2 comprising the body contour and both surfaces of each attached vortex sheet.

$$\int_c \phi_2 \, dy = \text{Re} - i \int_c \phi_2 \, dz \quad = \quad \text{Re} - i \int_c \{w_2 - i\psi_2\} \, dz$$

Thus $$\int_c \phi_2 \, dy = - \text{Re} \; i \int_c w_2(z) dz - \text{Re} \int_c \psi_2 \, dz \qquad 8.1$$

Now, since ψ_2 is continuous across each vortex sheet and constant round the closed body contour it follows that

$$\int_c \psi_2 \, dz \equiv 0$$

so that $$\int_c \phi_2 \, dy = - \text{Re} \; i \int_c w_2(z) dz \qquad 8.2$$

The integral can be taken round any reconcilable contour, and in particular if there is no vorticity outside c a very large contour $z = Re^{i\theta}$ $(R \to \infty)$ may be used.

If as $|z| \to \infty$ w_2 can be expanded in the form

$$w_2(z) \to \sum_{n=1}^{\infty} \frac{A_n}{z^n} \qquad 8.3$$

the contour integral will reduce to the residue of the term in z^{-1} only.

That is $$\int_c \phi_2 \, dy = 2\pi \; \text{Re} \; A_1 \qquad 8.4$$

This extremely useful result can be found in Lamb's "Hydrodynamics", but it is not as well known as it deserves to be. It provides a powerful method of evaluating added mass coefficients for awkward shaped sections. Here it will be used to determine the contribution of the vortex sheets to the lateral force on the body.

If an element of a vortex sheet is idealized as a point vortex of strength K_m at a point $\zeta = \zeta_m$ in the circle plane, its contribution to w_2 is obtained by the method of images as

$$\delta w_2 = iK_m \ln (\zeta - \zeta_m) - i K_m \ln (\zeta - \frac{1}{\bar{\zeta}_m}) \qquad 8.5$$

At large $|z|$ the mapping function 4.1 can be partially inverted as

$$\zeta \doteqdot \frac{z - x_o}{a_o} + O\left(\frac{1}{z}\right) \qquad 8.6$$

whence $\ln(\zeta - \zeta_m) \rightarrow \ln \left\{ \frac{z}{a_o} \left(1 - \frac{x_o + a_o\zeta_m}{z} + O\left(\frac{1}{z^2}\right)\right)\right\}$

$$= \ln \frac{z}{a_o} - \frac{x_o + a_o\zeta_m}{z} + O\left(\frac{1}{z^2}\right)$$

It follows that $\delta w_2 = - iK_m \left\{ a_o \left(\zeta_m - \frac{1}{\bar{\zeta}_m}\right) . \frac{1}{z} + O\left(\frac{1}{z^2}\right)\right\}$ 8.7

Equation 8.7 is the beginnings of an expansion of δw_2 in a series of the form of equation 8.3 and from 8.4 it follows that

$$\int_c \delta\phi_2 \, dy = 2\pi \, Re \left\{- iK_m a_o \left(\zeta_m - \frac{1}{\bar{\zeta}_m}\right) \right\}$$

On writing $\zeta_m = r_m e^{i\theta m}$ this yields the result

$$\int_c \delta\phi_2 \, dy = 2\pi \; K_m a_o \left(r_m + \frac{1}{r_m}\right) \sin\theta_m \qquad 8.8$$

Equations 8.5 and 8.8 can be extended to the whole assembly of vortex sheets either by a process of integration or, perhaps more practically, by representing the sheets by a discrete set of point vortices.

9. THE CONVECTION OF VORTEX SHEETS

The development of the form of the vortex sheets depends on the fact that pressure must be continuous through the sheets (to avoid infinite fluid acceleration at the sheet). Adopting the notation f_+ and f_- for the values of a quantity f on opposite sides of a sheet at any particular point on the sheet, and writing $\Delta f = f_+ - f_-$, the behaviour of the sheet can be derived as follows :

Fluid pressure is governed by equation 1.7

$$\frac{p}{\rho} + \phi_t - U\phi_\xi + \frac{1}{2}\phi_x^2 + \frac{1}{2}\phi_y^2 = 0$$

Now $$\frac{1}{2}\Delta\phi_x^2 = (\phi_{x+} - \phi_{x-})\left(\frac{\phi_{x+} + \phi_{x-}}{2}\right) = \Delta\phi_x . \phi_{xv}$$

where $$\phi_{xv} = \frac{\phi_{x+} + \phi_{x-}}{2}$$

Thus $$\frac{1}{\rho}\Delta p + \Delta\phi_t - U\Delta\phi_\xi + \Delta\phi_x . \phi_{xv} + \Delta\phi_y . \phi_{yv} = 0 \qquad 9.1$$

or, since $\Delta p = 0$, $$\frac{d\Delta\phi}{dt} = 0 \qquad 9.2$$

where $$\frac{d}{dt} = \frac{\partial}{\partial t} - U\frac{\partial}{\partial \xi} + \phi_{xv}\frac{\partial}{\partial x} + \phi_{yv}\frac{\partial}{\partial y}$$

Thus $\Delta\phi$ = constant along a trajectory defined by the equations

$$\frac{d\xi}{dt} = - U \quad \frac{dx}{dt} = \phi_{xv} \quad \text{and} \quad \frac{dy}{dt} = \phi_{yv} \qquad 9.3$$

These relations are equivalent to

$$\frac{dx}{d\xi} = - \frac{\phi_{xv}}{U} \quad \text{and} \quad \frac{dy}{d\xi} = - \frac{\phi_{yv}}{U} \qquad 9.4$$

Integrations of these relations along the body length from the point of attachment of the trajectory or vortex line will determine the location of the vortex sheet in any crossflow plane abaft the point of attachment. At this location the jump in velocity potential $\Delta\phi$ is known from the value at the point of attachment, and hence the local strength of the vortex sheet is known from the relation

$$\gamma(s) = \frac{\partial \Delta\phi}{\partial s} \qquad 9.5$$

where s is an arc length measurement along the vortex sheet in the transverse plane. This gradient can be determined from the values of $\Delta\phi$ for neighbouring trajectories on the sheet. If the sheet is to be represented by a set of point vortices the strength of the substitution vortex between two neightbouring trajectories a distance δs apart would be

$$K = \gamma(s)\delta s = \Delta\phi_2 - \Delta\phi_1 \qquad 9.6$$

The strength K will then be constant at all transverse planes for which both trajectories exist.

Two final comments may be made concerning the formation of the vortex sheets. The first concerns the evaluation of the crossflow velocity components ϕ_{xv} and ϕ_{yv}. These components will have contributions from each of the velocity potentials ϕ_o, ϕ_1 and ϕ_2. If a contribution from ϕ_2 is included the processes by which the vortex sheets roll up into concentrated longitudinal vortex cores will be reproduced, if the contribution from ϕ_2 is omitted the roll up will be suppressed. The simpler form of calculation may be adequate for the discussion of linear effects due to infinitessimally small departures from uniform straight line motion, but the full calculation would be required to cope with non-linear behaviour at large yaw angles.

The second comment relates to the evaluation of $\Delta\phi$ at the point of attachment of the trajectory. Where the trajectory originates at a sharp trailing edge of a lifting appendage $\Delta\phi$ will be continuous as the trajectory crosses the trailing edge and can thus be obtained from a solution to the flow problem in the plane immediately upstream of the trailing edge. This is the equivalent of the well known "Kutta condition" principle. Where the trajectory originates at a line of separation along the body length the appropriate value of $\Delta\phi$ is much less clear, although the application of a Kutta condition may still be attempted. Some practical procedure based on experimental evidence may be required in this case.

10. CONCLUDING REMARKS

The purpose of this paper has been to present the solution to the asymmetric streamflow problem set in the general context of a slenderbody theory capable in principle of providing estimates of forces and moments acting on manoeuvring double bodies. The suggestion is that such a theory could be the basis of a rational hydrodynamic theory of ship manoeuvring.

Since the paper is concerned with general principles certain numerical details have been omitted, particularly in the evaluation of the mapping coefficients from the geometry of each body cross section and in the processes of integration along vortex trajectories to determine the geometry of trailing vortex sheets.

The use of slenderbody methods offers a tremendous simplification over any fully three dimensional method, but never the less still represents a very formidable computational task for any arbitrary manoeuvres. Practical computational procedures may need to be restricted to two simplified cases :

(a) A steady turn for which the lateral motion is described simply by

$$d_t - Ud_\xi = U\{\frac{\xi}{R} - \alpha\}$$

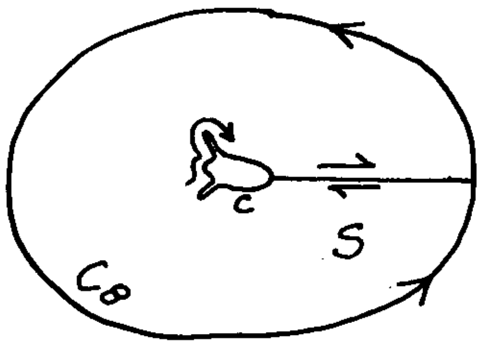
where R = radius of turn and α = drift angle

(b) Infinitessimal periodic motions about a straight path.

The steady turn case will correspond to a fully developed pattern of vortex sheets whose geometry is independent of time, whilst the infinitessimal periodic motion case is capable of further analytic development to avoid the necessity of following the history of the motion numerically, time step by time step. It is also likely that in the latter case body separation will not take place and vortex sheet roll up may be safely ignored.

A major feature of all slenderbody methods is the direct evaluation of total lateral force without the need for a numerical integration of pressure round the body contour. There are three independent contributions to the velocity potential and each provides a separate contribution to total force since the integration concerned is linear in ϕ. There is a practical need to evaluate hydrodynamic heeling moments on a real ship represented by one half of the double body and this can only be done by the direct evaluation of pressure loadings from equation 1.7. Since this equation is not linear in ϕ the heeling moment cannot be separated into independent contributions.

APPENDIX 1 RHS OF EQUATION 2.7



Consider a simple connected region S bounded by two closed contours C and C_∞ and rendered simply connected by a cut joining C to C_∞.

By applying Gauss theorem

$$\left\{ \int_{C_\infty} - \int_{C} \right\} \left[\tfrac{1}{2}(\phi_y^2 - \phi_x^2)\, dy + \phi_x\phi_y dx \right]$$

$$= \iint_S \{ \tfrac{1}{2}(\phi_y^2 - \phi_x^2)_x - (\phi_x\phi_y)_y \}\, dxdy$$

$$= \iint_S \{ \phi_y\phi_{xy} - \phi_x\phi_{xx} - \phi_{xy}\phi_y - \phi_x\phi_{yy} \}\, dxdy$$

$$= \iint_S \phi_x \{\phi_{xx} + \phi_{yy}\}\, dxdy$$

$$= 0$$

Since ϕ is a solution to the Laplace equation $\phi_{xx} + \phi_{yy} = 0$.

$$\therefore \int_C \left[\tfrac{1}{2}(\phi_y^2 - \phi_x^2)\, dy + \phi_x\phi_y\, dx\right] = \int_{C_\infty} \left[\tfrac{1}{2}(\phi_y^2 - \phi_x^2)\, dy + \phi_x\phi_y\, dx\right]$$

Now examination of ϕ_o, ϕ_1 and ϕ_2 shows that at large distances r from the body contour C $\phi = O(\ln r)$ whilst dx, dy = O(r)

Hence $$\int_{C_\infty} [\;] \to O(\tfrac{1}{r}) \to 0 \quad \text{as } r\to\infty$$

$$\therefore \quad \int_C \left[\tfrac{1}{2}(\phi_y^2 - \phi_x^2)\, dy + \phi_x\phi_y\, dx\right] \equiv 0.$$

This is the result required to eliminate the RHS of equation 2.7.

APPENDIX 2 REARRANGEMENT OF SERIES 4.7

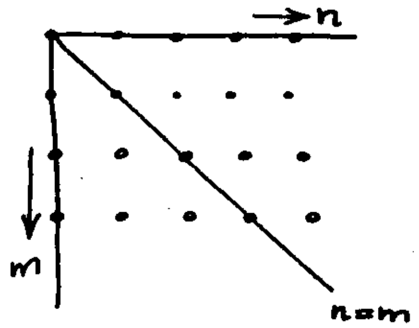
Equation 4.7 contains the product

$$P = \left[x'_o + \frac{a'_o}{\zeta} + \sum_{n=1}^{\infty} \frac{a'_n}{\zeta^n}\right] \cdot \left[\frac{a_o}{\zeta} - \sum_{m=1}^{\infty} \frac{m a_m}{\bar{\zeta}^m}\right]$$

This can be rearranged when $\zeta\bar{\zeta} = 1.0$ as follows :

$$P = a_o a'_o + \frac{x'_o a_o}{\zeta} + \sum_{n=1}^{\infty} \frac{a_o a'_n}{\zeta^{n+1}} - \sum_{m=1}^{\infty} \frac{m a_m x'_o}{\bar{\zeta}^m} - \sum_{m=1}^{\infty} \frac{m a_m a'_o}{\bar{\zeta}^{m+1}}$$

$$- \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} \frac{m a_m a'_n}{\zeta^n \bar{\zeta}^m}$$



The complete array of terms in the double sum, covering all values of n and m can be separated into

(i) The diagonal terms $n = m$

(ii) The terms below the diagonal $m > n$

(iii) The terms above the diagonal $n > m$

Taking each separately and substituting $s = m - n$ or $s = n - m$ as appropriate it is found that for $\zeta\bar{\zeta} = 1.0$

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty} \frac{m a_m a'_n}{\zeta^n \bar{\zeta}^m} = \sum_{n=1}^{\infty} n a_n a'_n + \sum_{n=1}^{\infty}\sum_{s=1}^{\infty} \frac{(n+s) a_{n+s} a'_n}{\bar{\zeta}^s}$$

$$+ \sum_{m=1}^{\infty}\sum_{s=1}^{\infty} \frac{m a_m a'_{m+s}}{\zeta^s}$$

Substituting this series for the double sum in P yields the RHS of equation 4.8.

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FACULTY OF ENGINEERING AND APPLIED SCIENCE

SLENDERBODY THEORY APPLIED TO ASYMMETRIC BODIES FITTED WITH LIFTING APPENDAGES

by J.F. Wellicome

Ship Science Report No. 8/81

INTRODUCTION

Slender body methods for predicting ship motions response to sea waves have proved to be very successful and are now used by naval architects in design work. There have been a number of papers seeking to evaluate forces and moments on slenderbodies in uniform motion that could form the basis of a treatment of forces and moments arising when a ship is manoeuvring in calm water. However, these techniques have not been developed to the point of forming a usable design tool for the prediction of course keeping stability and steady rates of turn.

The manoeuvring situation differs from the ship motion case in that the significant factor in the generation of lateral forces on the vessel is the formation of vortex sheets either downstream of a lifting surface (such as a yacht keel or a rudder) or due to separation of the flow round the hull itself. The latter phenomenon gives rise to non-linear forces and moments on the vessel that become important at larger angles of yaw.

It is usual in slender body theory, as applied to lifting bodies, to treat fully submerged bodies and to represent a ship operating in a free surface by a double model, symmetric about a horizontal plane representing the nominal free surface. This treatment of the free surface as a rigid surface can be justified by order of magnitude arguments, as being consistent with the assumption of a slenderbody, and this implies that wave making effects can be expected to be small compared to the basic double model forces and moments. This is probably correct in relation to moments acting on the vessel, but there are circumstances in which the double model forces approach zero and in such cases body wave making may need to be taken into account in estimating lateral forces acting on the vessel.

A sailing vessel normally adopts a heeled attitude, whilst most vessels turning at high speed will heel because of the vertical separation of the centre of gravity and the line of action of the lateral force acting on the below water hull form. When the vessel is heeled the equivalent double body has sections which are not symmetrical about the vertical centreline plane.

This paper presents a treatment of the flow of an ideal fluid past a slender body for which

(a) Transverse sections are symmetric about a horizontal plane

(b) The sections are not necessarily symmetric about a vertical plane

(c) The body may be fitted with lifting surface appendages

(d) Vortex sheets may exists due to the presence of these appendages or due to flow separation along the body length.

It will be assumed that the body is moving forward at uniform speed, but that it may be executing lateral motions. The primary aim is to evaluate total forces and moments acting on the body including any historical effects implied in the timewise development of the vortex sheets. The paper draws together a number of topics treated in other earlier papers, but includes a treatment of the effects of asymmetry as a major extension of previous theory.

1. BASIC FORMULATION OF THE POTENTIAL PROBLEM

The problem is treated as a potential flow problem in which the body produces a disturbance to a uniform stream of speed U defined by a velocity potential $\phi(x, y, \xi, t)$. The direction of the free stream flow and the coordinate system chosen is indicated in fig. 1. At this stage the surface of the body will be defined by the equation

$$y = Y(x - \mu d(\xi,t), \xi) \qquad 1.1$$

where $d(\xi,t)$ represents a lateral displacement of the body relative to some nominal datum position. In this form the displacement is allowed to vary along the body length as well as with time. There is no formal need for this displacement function to represent a rigid body motion, although this will normally be the case.

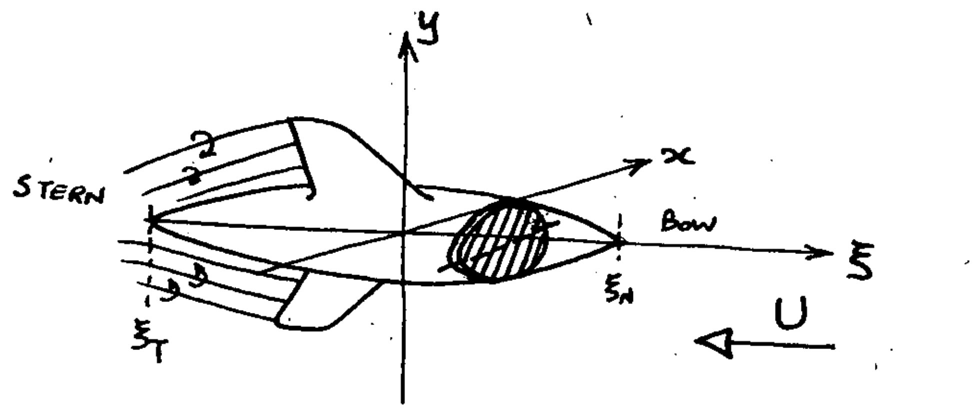


fig 1.

In fig. 1 a typical cross section is shaded. This section is supposed symmetric about the (x,ξ) plane, but may be asymmetric about the (y,ξ) plane.

Orders of magnitude of the various quantities occurring in the problem will be expressed in terms of a slenderness parameter ε which is to be thought of as the ratio of a typical transverse body dimension (e.g. body beam) to a typical longitudinal dimension (e.g. waterline length). The parameter ε is assumed small (i.e. $\varepsilon \ll 1.0$).

For the general three dimensional case the disturbance potential satisfies the Laplace equation

$$\nabla^2\phi = \phi_{\xi\xi} + \phi_{xx} + \phi_{yy} = 0 \qquad 1.2$$

subject to the kinematic boundary condition (no flow through the hull surface) which takes the form

$$\phi_y = [\phi_x + d_\xi .(U - \phi_\xi) - d_t] . Y_x - (U - \phi_\xi) . Y_\xi \qquad 1.3$$

In equations 1.2 and 1.3 and in all subsequent equations a subscript denotes partial differentiation.

That is

$$\phi_{\xi\xi} \equiv \frac{\partial^2\phi}{\partial\xi^2} \quad \text{etc.}$$

Equation 1.3 is to apply everywhere over the body surface. The disturbance potential will naturally decay to zero remote from the body and from any trailing vortex system.

Once a disturbance potential has been found, pressures within the fluid and hence forces acting on the body can be found from Bernoulli's equation in its unsteady form

$$\frac{p}{\rho} + \phi_t + \frac{1}{2}(U - \phi_\xi)^2 + \frac{1}{2}\phi_x^2 + \frac{1}{2}\phi_y^2 = \frac{1}{2}U^2 \qquad 1.4$$

Here p is the fluid pressure above its hydrostatic value, so that force estimates using 1.4 will not include buoyancy effects.

The essence of the slenderbody method is to examine the order of magnitude of each term in eqns 1.2, 1.3 and 1.4 either by a process of scaling the coordinates or by reference to approximate solutions for the streamflow past a body of revolution. A consistent scaling process yields the following orders of magnitude :

Terms of order $\varepsilon^0 = 1$: $U, \xi, Y_x, \phi_{xx}, \phi_{yy}$
Terms of order ε : $x, y, d, Y, Y_\xi, \phi_x, \phi_y$
Terms of order ε^2 : $\phi, \phi_\xi, \phi_{\xi\xi}, \phi_t$

Some texts give the order of magnitude of ϕ as $\varepsilon^2 \ln \varepsilon$ (based on flow past a body of revolution), but arrive at the same reduced equations as those given below, so that the simpler form will be taken here.

A systematic rejection of the smallest terms in the basic equations leads to the following equations :

1.2 reduces to $$\phi_{xx} + \phi_{yy} = 0 \qquad 1.5$$

1.3 reduces to $$\phi_y = [\phi_x + U.d_\xi - d_t] \, . \, Y_x - U.Y_\xi \qquad 1.6$$

1.4 reduces to $$\frac{p}{\rho} + \phi_t - U.\phi_\xi + \frac{1}{2}\phi_x^2 + \frac{1}{2}\phi_y^2 = 0 \qquad 1.7$$

In each equation the ratio $\dfrac{\text{neglected terms}}{\text{retained terms}} = O(\varepsilon^2)$

so that, if $\varepsilon = {}^{1}/_{7}$ is regarded as typical, the neglected terms should be of the order of 2% of those retained. The importance of this reduction is that at each crossflow plane along the body length, the disturbance potential is obtained as a solution to a two dimensional problem to which very powerful conformal mapping methods may be applied.

2. THE LOCAL LATERAL FORCE

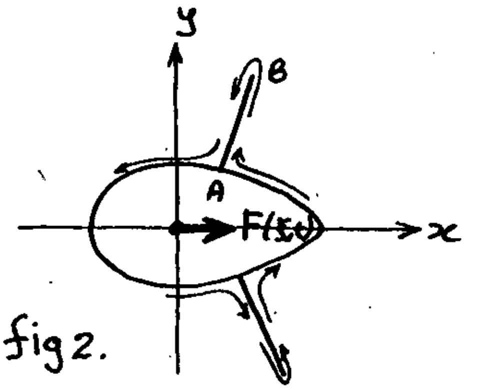


fig 2.

fig. 2 represents one particular cross section of the body. The line AB could represent either a flat plate fin attached to the body or it could represent a cut through a vortex sheet shed in the wake of some upstream lifting appendage.

The lateral force per unit length of body experienced by the body at this cross section is given by

$$F(\xi,t) = -\int_c p\,dy \qquad 2.1$$

The integration can be carried out round the contour indicated in fig. 2 since there is zero pressure difference across any portion of AB that represents a vortex sheet rather than part of an appendage. The inclusion of all vortex sheets within the contour c in this way makes it possible to evaluate $F(\xi,t)$ by integration round any contour reconcilable with c and in many circumstances considerable simplification can result by taking a very large contour enclosing the section.

Using equation 1.7 it is now possible to write

$$F(\xi,t) = \rho \int_c \left[\phi_t - U\phi_\xi + \frac{1}{2}\phi_x^2 + \frac{1}{2}\phi_y^2\right] dy \qquad 2.2$$

Now define $A(\xi,t)$ by the equation

$$A(\xi,t) = \int_c \phi\,dy \qquad 2.3$$

On $y = Y(x - d(\xi,t), \xi)$ at fixed y

$$0 = Y_x[\delta x - d_\xi \delta\xi - d_t \ \delta t] + Y_\xi . \delta\xi$$

Thus at fixed time $\frac{\delta x}{\delta \xi} = d_\xi - \frac{Y_\xi}{Y_x}$

and at fixed longitudinal position $\frac{\delta x}{\delta t} = d_t$

It follows that

$$\frac{\partial A}{\partial \xi} = \int_c \left[\phi_\xi + \left(d_\xi - \frac{Y_\xi}{Y_x}\right) \cdot \phi_x\right] dy \qquad 2.4$$

and $$\frac{\partial A}{\partial t} = \int_c [\phi_t + d_t . \phi_x] \ dy \qquad 2.5$$

Equation 2.4 can be rewritten by making use of the body boundary conditions 1.6 :

$$U\frac{\partial A}{\partial \xi} = \int_c \left\{U\phi_\xi + Ud_\xi . \phi_x - \frac{\phi_x}{Y_x} \left[(\phi_x + Ud_x - d_t)Y_x - \phi_y\right]\right\} dy$$

which simplifies to give

$$U\frac{\partial A}{\partial \xi} = \int_c \left[(U\phi_\xi + \phi_x . d_t - \phi_x^2) \ dy + \phi_x \phi_y \ dx\right] \qquad 2.6$$

Thus, on using 2.2, 2.5 and 2.6 :

$$F(\xi,t) + \rho U \frac{\partial A}{\partial \xi} - \rho\frac{\partial A}{\partial t} = \rho \int_c \left[\frac{1}{2}(\phi_y^2 - \phi_x^2) \ dy + \phi_x \phi_y \ dx\right] \qquad 2.7$$

Now, as is demonstrated in appendix 1, the contour integral on the right hand side of 2.7 vanishes identically. This means that the load per unit length is given simply by

$$F(\xi,t) = - \rho U\frac{\partial A}{\partial \xi} + \rho\frac{\partial A}{\partial t} \qquad 2.8$$

This remarkable result reduces the evaluation of hydrodynamic forces and moments simply to the evaluation of

$$A(\xi,t) = \int_c \phi \, dy$$

The total force and the moment acting on the body are then obtained by integration along the body length.

3. THE DETERMINATION OF THE DISTURBANCE POTENTIAL

The complete disturbance potential can be regarded as the linear sum of three components, each of which provides its own separate contribution to the total force on the body.

Thus $\phi = \phi_o + \phi_1 + \phi_2$

where ϕ_o = general streaming flow past a stationary body
ϕ_1 = flow generated by body lateral motions
ϕ_2 = flow due to external vortex sheets.

The position and strength of the external vortex sheets at any given station along the length and at any given time will, of course, depend on the convection processes occurring upstream of the given section taking into account all three components of ϕ.

The general boundary condition 1.6 can thus be split into the following three sub-problem conditions, each to be satisfied on the body surface :

$$\phi_{oy} = \phi_{ox}.Y_x - UY_\xi \quad 3.1$$

$$\phi_{1y} = (\phi_{1x} + Ud_\xi - d_t)Y_x \quad 3.2$$

and

$$\phi_{2y} = \phi_{2x}.Y_x \quad 3.3$$

4. SECTION MAPPING FUNCTION

In order to arrive at a complex potential function corresponding to each contribution to the disturbance potential ϕ it will be assumed that the body section can be mapped from the unit circle $\zeta = e^{i\theta}$ by a mapping function of the form

$$z = x + i\,y = x_o(\xi) + a_o(\xi).\zeta + \sum_{n=1}^{\infty} \frac{a_n(\xi)}{\zeta^n} \qquad 4.1$$

For convenience x_o will be written in place of $x_o(\xi)$ and x_o' will be written in place of

$$\frac{dx_o}{d\xi}$$

A similar convention will be adopted for the coefficients a_o, a_n. It should be noted that symmetry about the real (x) axis implies that x_o, a_o, a_n are all real and that the even order coefficients a_{2m} arise because of a lack of symmetry about the imaginary (y) axis. In an application of this method 4.1 would be truncated suitably and the coefficients would be determined numerically from the actual section shapes.

As a further convenient notation the body shape can be represented variously as

$$z = Z(\xi,\theta) \qquad 4.2$$

or $$x = X(\xi,\theta) \text{ and } y = Y(\xi,\theta) \qquad 4.3$$

In terms of the mapping function 4.1 the partial derivatives of Z can be written as

$$Z_\theta = i\left[a_o\zeta - \sum_{n=1}^{\infty} \frac{na_n}{\zeta^n}\right] \qquad 4.4$$

and $$Z_\xi = x_o' + a_o'\zeta + \sum_{n=1}^{\infty} \frac{a_n'}{\zeta^n} \qquad 4.5$$

In deriving the complex potential for ϕ_o the following relation will prove useful

$$Z_\xi . \bar{Z}_\theta = (X_\xi + iY_\xi)(X_\theta - iY_\theta)$$

$$= (X_\xi X_\theta + Y_\xi Y_\theta) + i(X_\theta . Y_\xi - X_\xi . Y_\theta)$$

whence $Z_\xi . \bar{Z}_\theta - \bar{Z}_\xi . Z_\theta = 2i(X_\theta . Y_\xi - X_\xi . Y_\theta)$ 4.6

The product $Z_\xi . \bar{Z}_\theta$ can be written out in terms of the mapping function, using the result that for points on the body $\zeta\bar{\zeta} = 1.0$:

$$Z_\xi . \bar{Z}_\theta = - i \left[x'_o + \frac{a'_o}{\zeta} + \sum_{n=1}^{\infty} \frac{a'_n}{\zeta^n} \right] . \left[\frac{a_o}{\zeta} - \sum_{n=1}^{\infty} \frac{na_n}{\bar{\zeta}^n} \right] \quad 4.7$$

Appendix 2 shows that this series may be rearranged in inverse powers of ζ and $\bar{\zeta}$ as

$$Z_\xi . \bar{Z}_\theta = - i \left[a_o a'_o + \frac{x'_o a_o}{\zeta} - \sum_{n=1}^{\infty} na_n a'_n + \sum_{n=1}^{\infty} \frac{a_o a'_n}{\zeta^{n+1}} - \sum_{n=1}^{\infty} \frac{na_n x'_o}{\bar{\zeta}^n} \right.$$

$$\left. - \sum_{n=1}^{\infty} \frac{na_n a_o}{\bar{\zeta}^{n+1}} - \sum_{n=1}^{\infty}\sum_{s=1}^{\infty} \frac{(n+s) a_{n+s} a'_n}{\bar{\zeta}^s} - \sum_{n=1}^{\infty}\sum_{s=1}^{\infty} \frac{na_n a'_{n+s}}{\zeta^s} \right] \quad 4.8$$

This result is valid for points on the body for which $\zeta = e^{i\theta}$.

5. SOLUTION OF THE GENERAL STREAMFLOW

Since ϕ_o is a solution of a two dimensional problem, governed by equation 1.5 it is possible to define a corresponding stream function ψ_o and to write

$$\phi_o + i\psi_o = W_o \ (\zeta, \xi) \quad 5.1$$

The appropriate boundary condition on the body ($\zeta = e^{i\theta}$) is from eq. 3.1

$$\phi_{oy} = \phi_{ox} . \left.\frac{\partial Y}{\partial X}\right|_{\xi \text{ Const}} - U \left.\frac{\partial Y}{\partial \xi}\right|_{X \text{ Const}}$$

or $$-\psi_{ox} = \psi_{oy} \frac{\delta y}{\delta x} - \frac{U\partial y}{\partial \xi}\Bigg|_{x\ \text{const}} \qquad 5.2$$

Now, from equations 4.3

$$\delta x = X_\xi\ \delta\xi + X_\theta\ \delta\theta = 0$$

and $$\delta y = Y_\xi\ \delta\xi + Y_\theta\ \delta\theta = \frac{Y_\xi . X_\theta - X_\xi . Y_\theta}{X_\theta}\ \delta\xi$$

i.e. $$\frac{\partial y}{\partial \xi}\Bigg|_{x\ \text{const}} = \frac{Y_\xi . X_\theta - X_\xi Y_\theta}{X_\theta} \qquad 5.3$$

Using this result the body boundary condition 5.2 can be rewritten as

$$\frac{\partial \psi_o}{\partial \theta} = U(Y_\xi . X_\theta - X_\xi . Y_\theta) \qquad 5.4$$

It follows that the boundary condition finally takes the form

$$\frac{\partial W_o}{\partial \theta} - \frac{\partial \bar{W}_o}{\partial \theta} = \frac{2i\partial\psi_o}{\partial \theta} = U\left[Z_\xi . \bar{Z}_\theta - \bar{Z}_\xi . Z_\theta\right] \qquad 5.5$$

To summarize the position at this stage, it is necessary to find a function $W_o(\zeta,\xi)$ which satisfies equation 5.5 on $\zeta = e^{i\theta}$ and which represents a flow which dies away as $|\zeta|\to\infty$. By considering the terms in equation 4.8 and in its complex conjugate, it can be seen that equation 5.5 is satisfied if, on the body

$$\frac{\partial W_o}{\partial \theta} = -\, iU\Bigg[a_o a_o' - \sum_{n=1}^{\infty} n a_n a_n' + \frac{a_o x_o'}{\zeta} + \sum_{n=1}^{\infty} \frac{a_o a_n'}{\zeta^{n+1}} - \sum_{n=1}^{\infty} \frac{n a_n x_o'}{\zeta^n}$$
$$- \sum_{n=1}^{\infty} \frac{n a_n a_o'}{\zeta^{n+1}} - \sum_{n=1}^{\infty}\sum_{s=1}^{\infty} \frac{(n+s) a_{n+s} a_s'}{\zeta^n} - \sum_{n=1}^{\infty}\sum_{s=1}^{\infty} \frac{s a_s a_{n+s}'}{\zeta^n}\Bigg]$$

The last two terms are obtained by interchanging the roles of n and s in the double summations and reversing the order of summation.

This result can now be integrated round the body contour to obtain a complex potential given by

$$W_o(\zeta,\xi) = -U\left[\left(a_o a'_o - \sum_{n=1}^{\infty} n a_n a'_n\right)\ln\zeta - \frac{a_o x'_o}{\zeta} + \sum_{n=1}^{\infty}\frac{n a_n a'_o - a_o a'_n}{(n+1)\zeta^{n+1}} + \sum_{n=1}^{\infty}\frac{n a_n x'_o + \sum_{n=1}^{\infty}\{(n+s)a_{n+s}a'_s + s\, a_s a'_{n+s}\}}{n.\ \zeta^n}\right]$$

This equation can be conveniently written as

$$W_o(\zeta,\xi) = -U\left[A_o \ln\zeta + \sum_{n=1}^{\infty}\frac{A_n}{\zeta^n}\right] \qquad 5.6$$

where $A_o = a_o a'_o - \sum_{n=1}^{\infty} n a_n a'_n$

$$A_1 = -a_o x'_o + a_1 x'_o + \sum_{n=1}^{\infty}\{(s+1)a_{s+1}a'_s + s a_s a'_{s+1}\}$$

and for $n \geq 2$

$$A_n = \frac{1}{n}\left[(n-1)a_{n-1}a'_o - a_o a'_{n-1} + n a_n x'_o + \sum_{s=1}^{\infty}\{(n+s)a_{n+s}a'_s + s a_s a'_{n+s}\}\right]$$

This result gives an explicit solution for the complex potential representing the general streaming motion past a slender body with transversely asymmetric body sections represented by the mapping function 4.1. It should be noted that since the coefficients x_o, a_o and a_n occurring in the mapping function are all real the coefficients A_o and A_n occurring in 5.6 are also real. The complex potential can be used to compute transverse velocity components at any field point in the cross flow plane and such a computation is required in order to follow through the convection process for any free vortex sheet present near the body. The contribution of this general streaming motion to the hydrodynamic lateral force can be obtained as follows :-

On the body contour $\phi_o = -U.\sum_{n=1}^{\infty} A_n \cos n\theta$ 5.7

whilst from the mapping function

$$y = a_o \sin\theta - \sum_{m=1}^{\infty} a_m \sin m\theta$$

$$\therefore \int_c \phi_o \, dy = - U \int_o^{2\pi} \sum_{n=1}^{\infty} A_n \operatorname{Cos} n\theta \left[a_o \operatorname{Cos}\theta - \sum_{m=1}^{\infty} m a_m \operatorname{Cos} m\theta\right] d\theta$$

Using the identity $\int_o^{2\pi} \operatorname{Cos} n\theta \operatorname{Cos} m\theta \, d\theta = \begin{cases} \pi & (n = m \neq 0) \\ o & (n \neq m) \end{cases}$

it follows that

$$A_o(\xi) = \int_c \phi_o \, dy = - \pi U\left[a_o A_1 - \sum_{n=1}^{\infty} n a_n A_n\right] \qquad 5.8$$

and from equation 2.8 the corresponding cross force per unit body length is

$$F_o(\xi) = \rho \, \pi U^2 A_o'(\xi) \qquad 5.9$$

The total transverse force and yawing moment components from this streaming motion are

$$L_o = \int_{\xi_T}^{\xi_N} F_o(\xi) d\xi = \rho \, \pi U^2 \left[A_o(\xi_N) - A_o(\xi_T)\right] \qquad 5.10$$

and

$$M_o = \int_{\xi_T}^{\xi_N} \xi \, F_o(\xi) d\xi = \rho \, \pi U^2 \left[\xi . A_o(\xi) \Big|_{\xi_T}^{\xi_N} - \int_{\xi_T}^{\xi_N} A_o(\xi) d\xi \right] \qquad 5.11$$

In these equations the nose (or bow) of the body is at $\xi = \xi_N$ and the tail (or stern) is at $\xi = \xi_T$ and M_o is the moment about $\xi = 0$. If the body sections shrink smoothly to a point at the body ends $A_o(\xi_N) = A_o(\xi_T) = 0$ so that $L_o = 0$ and the streaming motion produces a simple couple

$$M_o = - \rho \pi U^2 \int_{\xi_T}^{\xi_N} A_o(\xi) d\xi \qquad 5.12$$

Where the body is symmetric longitudinally about $\xi = 0$ the couple M_o will vanish on the basis that x_o, a_o and a_n are even functions of ξ, consequently a'_o, a'_n, A_o and A_n are all odd functions of ξ, which in turn implies that $A_o(\xi)$ is also an odd function of ξ for which

$$\int_{\xi_T}^{\xi_N} A_o(\xi) d\xi \equiv 0 .$$

Thus it is necessary for the body to be asymmetric longitudinally as well as transversely in order that the general streaming motion should produce a couple M_o acting on the body.

6. TRANSVERSELY SYMMETRIC BODIES YAWED

Previous texts on slenderbody theory have dealt with transversely symmetric bodies yawed at a small angle α to the main streamflow. This case is embedded in the theory presented here and can be extracted by noting that in the mapping function $a_{2m} = 0$ for transversely symmetric sections.

With this restriction only those terms corresponding to odd values of n contribute to the sum occurring in equation 5.8. Now, for odd n, terms like $a_{n+s}\, a'_s = 0$ since $a_{n+s} = 0$ if s is odd and $a'_s = 0$ if s is even. Thus the coefficients occurring in the complex potential (equation 5.6) reduce in this case (for odd n) to

$$A_1 = (a_1 - a_o)\, x'_o$$

$$A_n = a_n\, x'_o \qquad \text{(odd } n > 2\text{)}$$

Thus in 5.8 $A_o(\xi)$ reduces to

$$A_o(\xi) = + \pi\, U \left[(a_1 - a_o)^2 + \sum_{n=3}^{\infty} n\, a_n^2 \right] x'_o$$

where the summation involves odd n only.

Now the cross-sectional area enclosed by the body contour can be obtained from the mapping function as

$$\text{Area} = \int_c x\,dy = \pi\left[a_o^2 - a_1^2 - \sum_{n=3}^{\infty}{}' n a_n^2\right] \qquad \text{(odd n only)}$$

Thus if the substitution $\dot{x}_o' = \alpha$ is made the case of transverse symmetry reduces to

$$A_o(\xi) = -\,U\alpha\,\left[2\pi\, a_o(a_1 - a_o) + \text{Area}\right] \qquad 6.1$$

In other words in order to evaluate forces on a transversely symmetric double body it is only necessary to determine the mapping coefficients a_o and a_1 for each body section.

7. CONTRIBUTION DUE TO LATERAL MOTIONS

The contribution to the velocity potential due to body lateral motions (ϕ_1) satisfies the boundary condition 3.2 :

$$\phi_{1y} = (\phi_{1x} + Ud_\xi - d_t)\, Y_x$$

This is the boundary condition for a two dimensional body moving laterally with a speed $v = d_t - Ud_\xi$. By superinposing a lateral velocity $-v$ in the z plane to convert the problem to that of a uniform stream past a stationary body, the corresponding complex potential can be written down immediately as

$$w_1(z,t) = -\,a_o v\,\left[\zeta + \frac{1}{\zeta}\right] + vz$$

or

$$w_1(z,t) = (d_t - Ud_\xi)\,\left[-\frac{a_o}{\zeta} + \sum_{n=1}^{\infty}\frac{a_n}{\zeta^n}\right] \qquad 7.1$$

Equation 7.1 follows on substituting the section mapping function (equation 4.1).

From this result it follows, using the same methods as in sections 5 and 6 that

$$A_1(\xi,t) = \int_c \phi_1 \, dy = (d_t - Ud_\xi) \left[2\pi \, a_o(a_1 - a_o) + \text{Area}\right] \qquad 7.2$$

If d_t is set to zero and d_ξ to a yaw angle α it can be seen that equation 7.2 is in agreement with equation 6.1. This clearly should be the case. In the case of a purely lateral motion $d_\xi = 0$ and the corresponding lateral body force is

$$F_1(\xi,t) = \rho \frac{\partial A_1}{\partial t} = \rho \left[2\pi \, a_o(a_1 - a_o) + \text{Area}\right] . \, d_{tt} \qquad 7.3$$

This corresponds to a section added mass in lateral motion given by

$$m(\xi) = - \rho\left[2\pi \, a_o(a_1 - a_o) + \text{Area}\right] \qquad 7.4$$

For the alternative case of a body set at a fixed yaw angle α 7.2 reduces to

$$\rho U \, A_1(\xi) = - \rho U^2 \, \alpha\left[2\pi \, a_o(a_1 - a_o) + \text{Area}\right] = m(\xi) \, U^2\alpha$$

The lateral body force is now

$$F_1(\xi) = - \rho U \, A_1'(\xi) = U^2\alpha . \, m'(\xi) \qquad 7.3$$

This relationship between lateral forces on a yawed body and the section added mass in lateral motion is in fact a well known slenderbody result.

8. CONTRIBUTION DUE TO EXTERNAL VORTEX SHEETS

The last contribution to the total flow at any cross section is that due to any vortex sheets present, generated either at the trailing edge of a lifting appendage or in some separation process taking place along the body length. Deferring for the moment the problem of the determination of the location and strength of the vortex sheets, the velocity potential at any given section and at any particular time is governed by the boundary condition 3.3 :

$$\phi_{2y} = \phi_{2x} \cdot Y_x$$

This boundary corresponds to the flow generated by the vortex sheet round a stationary two dimensional body and on the body surface the corresponding stream function ψ_2 will have a constant value.

Consider first the evaluation of the integral $\int_c \phi_2 \, dy$ taken round a contour c as shown in figure 2 comprising the body contour and both surfaces of each attached vortex sheet.

$$\int_c \phi_2 \, dy = \text{Re} - i \int_c \phi_2 \, dz = \text{Re} - i \int_c \{w_2 - i\psi_2\} \, dz$$

Thus $$\int_c \phi_2 \, dy = - \text{Re}\, i \int_c w_2(z) dz - \text{Re} \int_c \psi_2 \, dz \qquad 8.1$$

Now, since ψ_2 is continuous across each vortex sheet and constant round the closed body contour it follows that

$$\int_c \psi_2 \, dz \equiv 0$$

so that $$\int_c \phi_2 \, dy = - \text{Re}\, i \int_c w_2(z) dz \qquad 8.2$$

The integral can be taken round any reconcilable contour, and in particular if there is no vorticity outside c a very large contour $z = Re^{i\theta}$ $(R \to \infty)$ may be used.

If as $|z| \to \infty$ w_2 can be expanded in the form

$$w_2(z) \to \sum_{n=1}^{\infty} \frac{A_n}{z^n} \qquad 8.3$$

the contour integral will reduce to the residue of the term in z^{-1} only.

That is $$\int_c \phi_2 \, dy = 2\pi \, \text{Re}\, A_1 \qquad 8.4$$

This extremely useful result can be found in Lamb's "Hydrodynamics", but it is not as well known as it deserves to be. It provides a powerful method of evaluating added mass coefficients for awkward shaped sections. Here it will be used to determine the contribution of the vortex sheets to the lateral force on the body.

If an element of a vortex sheet is idealized as a point vortex of strength K_m at a point $\zeta = \zeta_m$ in the circle plane, its contribution to w_2 is obtained by the method of imagesaas

$$\delta w_2 = iK_m \ln (\zeta - \zeta_m) - i K_m \ln (\zeta - \frac{1}{\bar{\zeta}_m}) \qquad 8.5$$

At large $|z|$ the mapping function 4.1 can be partially inverted as

$$\zeta \doteqdot \frac{z - x_o}{a_o} + O\left(\frac{1}{z}\right) \qquad 8.6$$

whence $\ln(\zeta - \zeta_m) \rightarrow \ln \left\{ \frac{z}{a_o} \left[1 - \frac{x_o + a_o\zeta_m}{z} + O\left(\frac{1}{z^2}\right)\right]\right\}$

$$= \ln \frac{z}{a_o} - \frac{x_o + a_o\zeta_m}{z} + O\left(\frac{1}{z^2}\right)$$

It follows that $\delta w_2 = - iK_m \left\{ a_o \left(\zeta_m - \frac{1}{\bar{\zeta}_m}\right) . \frac{1}{z} + O\left(\frac{1}{z^2}\right)\right\}$ 8.7

Equation 8.7 is the beginnings of an expansion of δw_2 in a series of the form of equation 8.3 and from 8.4 it follows that

$$\int_c \delta\phi_2 \, dy = 2\pi \, \mathrm{Re} \left\{- iK_m a_o \left(\zeta_m - \frac{1}{\bar{\zeta}_m}\right) \right\}$$

On writing $\zeta_m = r_m e^{i\theta m}$ this yields the result

$$\int_c \delta\phi_2 \, dy = 2\pi \; K_m a_o \left(r_m + \frac{1}{r_m}\right) \mathrm{Sin}\theta_m \qquad 8.8$$

Equations 8.5 and 8.8 can be extended to the whole assembly of vortex sheets either by a process of integration or, perhaps more practically, by representing the sheets by a discrete set of point vortices.

9. THE CONVECTION OF VORTEX SHEETS

The development of the form of the vortex sheets depends on the fact that pressure must be continuous through the sheets (to avoid infinite fluid acceleration at the sheet). Adopting the notation f_+ and f_- for the values of a quantity f on opposite sides of a sheet at any particular point on the sheet, and writing $\Delta f = f_+ - f_-$, the behaviour of the sheet can be derived as follows :

Fluid pressure is governed by equation 1.7

$$\frac{p}{\rho} + \phi_t - U\phi_\xi + \frac{1}{2}\phi_x^2 + \frac{1}{2}\phi_y^2 = 0$$

Now $$\frac{1}{2}\Delta\phi_x^2 = (\phi_{x+} - \phi_{x-})\left(\frac{\phi_{x+} + \phi_{x-}}{2}\right) = \Delta\phi_x . \phi_{xv}$$

where $$\phi_{xv} = \frac{\phi_{x+} + \phi_{x-}}{2}$$

Thus $$\frac{1}{\rho}\Delta p + \Delta\phi_t - U\Delta\phi_\xi + \Delta\phi_x . \phi_{xv} + \Delta\phi_y . \phi_{yv} = 0 \qquad 9.1$$

or, since $\Delta p = 0$, $$\frac{d\Delta\phi}{dt} = 0 \qquad 9.2$$

where $$\frac{d}{dt} = \frac{\partial}{\partial t} - U\frac{\partial}{\partial\xi} + \phi_{xv}\frac{\partial}{\partial x} + \phi_{yv}\frac{\partial}{\partial y}$$

Thus $\Delta\phi$ = constant along a trajectory defined by the equations

$$\frac{d\xi}{dt} = -U \quad \frac{dx}{dt} = \phi_{xv} \quad \text{and} \quad \frac{dy}{dt} = \phi_{yv} \qquad 9.3$$

These relations are equivalent to

$$\frac{dx}{d\xi} = -\frac{\phi_{xv}}{U} \quad \text{and} \quad \frac{dy}{d\xi} = -\frac{\phi_{yv}}{U} \qquad 9.4$$

Integrations of these relations along the body length from the point of attachment of the trajectory or vortex line will determine the location of the vortex sheet in any crossflow plane abaft the point of attachment. At this location the jump in velocity potential $\Delta\phi$ is known from the value at the point of attachment, and hence the local strength of the vortex sheet is known from the relation

$$\gamma(s) = \frac{\partial \Delta\phi}{\partial s} \qquad 9.5$$

where s is an arc length measurement along the vortex sheet in the transverse plane. This gradient can be determined from the values of $\Delta\phi$ for neighbouring trajectories on the sheet. If the sheet is to be represented by a set of point vortices the strength of the substitution vortex between two neighbouring trajectories a distance δs apart would be

$$K = \gamma(s)\delta s = \Delta\phi_2 - \Delta\phi_1 \qquad 9.6$$

The strength K will then be constant at all transverse planes for which both trajectories exist.

Two final comments may be made concerning the formation of the vortex sheets. The first concerns the evaluation of the crossflow velocity components ϕ_{xv} and ϕ_{yv}. These components will have contributions from each of the velocity potentials ϕ_o, ϕ_1 and ϕ_2. If a contribution from ϕ_2 is included the processes by which the vortex sheets roll up into concentrated longitudinal vortex cores will be reproduced, if the contribution from ϕ_2 is omitted the roll up will be suppressed. The simpler form of calculation may be adequate for the discussion of linear effects due to infinitessimally small departures from uniform straight line motion, but the full calculation would be required to cope with non-linear behaviour at large yaw angles.

The second comment relates to the evaluation of $\Delta\phi$ at the point of attachment of the trajectory. Where the trajectory originates at a sharp trailing edge of a lifting appendage $\Delta\phi$ will be continuous as the trajectory crosses the trailing edge and can thus be obtained from a solution to the flow problem in the plane immediately upstream of the trailing edge. This is the equivalent of the well known "Kutta condition" principle. Where the trajectory originates at a line of separation along the body length the appropriate value of $\Delta\phi$ is much less clear, although the application of a Kutta condition may still be attempted. Some practical procedure based on experimental evidence may be required in this case.

10. CONCLUDING REMARKS

The purpose of this paper has been to present the solution to the asymmetric streamflow problem set in the general context of a slenderbody theory capable in principle of providing estimates of forces and moments acting on manoeuvring double bodies. The suggestion is that such a theory could be the basis of a rational hydrodynamic theory of ship manoeuvring.

Since the paper is concerned with general principles certain numerical details have been omitted, particularly in the evaluation of the mapping coefficients from the geometry of each body cross section and in the processes of integration along vortex trajectories to determine the geometry of trailing vortex sheets.

The use of slenderbody methods offers a tremendous simplification over any fully three dimensional method, but never the less still represents a very formidable computational task for any arbitrary manoeuvres. Practical computational procedures may need to be restricted to two simplified cases :

(a) A steady turn for which the lateral motion is described simply by

$$d_t - Ud_\xi = U\{\frac{\xi}{R} - \alpha\}$$

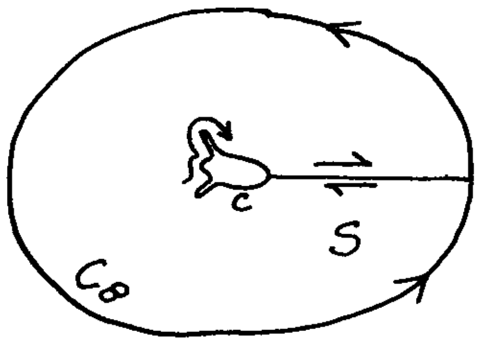
where R = radius of turn and α = drift angle

(b) Infinitessimal periodic motions about a straight path.

The steady turn case will correspond to a fully developed pattern of vortex sheets whose geometry is independent of time, whilst the infinitessimal periodic motion case is capable of further analytic development to avoid the necessity of following the history of the motion numerically, time step by time step. It is also likely that in the latter case body separation will not take place and vortex sheet roll up may be safely ignored.

A major feature of all slenderbody methods is the direct evaluation of total lateral force without the need for a numerical integration of pressure round the body contour. There are three independent contributions to the velocity potential and each provides a separate contribution to total force since the integration concerned is linear in ϕ. There is a practical need to evaluate hydrodynamic heeling moments on a real ship represented by one half of the double body and this can only be done by the direct evaluation of pressure loadings from equation 1.7. Since this equation is not linear in ϕ the heeling moment cannot be separated into independent contributions.

APPENDIX 1 RHS OF EQUATION 2.7



Consider a simple connected region S bounded by two closed contours C and C_∞ and rendered simply connected by a cut joining C to C_∞.

By applying Gauss theorem

$$\left\{ \int_{C_\infty} - \int_C \right\} \left[\tfrac{1}{2}(\phi_y^2 - \phi_x^2)\, dy + \phi_x\phi_y dx \right]$$

$$= \iint_S \left\{ \tfrac{1}{2}(\phi_y^2 - \phi_x^2)_x - (\phi_x\phi_y)_y \right\} dxdy$$

$$= \iint_S \left\{ \phi_y\phi_{xy} - \phi_x\phi_{xx} - \phi_{xy}\phi_y - \phi_x\phi_{yy} \right\} dxdy$$

$$= \iint_S \phi_x \left\{ \phi_{xx} + \phi_{yy} \right\} dxdy$$

$$= 0$$

Since ϕ is a solution to the Laplace equation $\phi_{xx} + \phi_{yy} = 0$.

$$\therefore \quad \int_C \left[\tfrac{1}{2}(\phi_y^2 - \phi_x^2)\, dy + \phi_x\phi_y\, dx \right] = \int_{C_\infty} \left[\tfrac{1}{2}(\phi_y^2 - \phi_x^2)\, dy + \phi_x\phi_y\, dx \right]$$

Now examination of ϕ_o, ϕ_1 and ϕ_2 shows that at large distances r from the body contour C $\phi = O(\ln r)$ whilst dx, dy = O(r)

$$\text{Hence} \qquad \int_{C_\infty} [\ ] \to O(\tfrac{1}{r}) \to 0 \qquad \text{as } r \to \infty$$

$$\therefore \qquad \int_C \left[\tfrac{1}{2}(\phi_y^2 - \phi_x^2)\, dy + \phi_x\phi_y\, dx \right] \equiv 0.$$

This is the result required to eliminate the RHS of equation 2.7.

APPENDIX 2 REARRANGEMENT OF SERIES 4.7

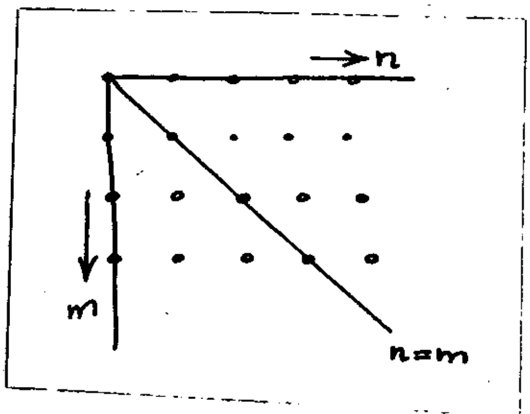
Equation 4.7 contains the product

$$P = \left[x'_o + \frac{a'_o}{\zeta} + \sum_{n=1}^{\infty} \frac{a'_n}{\zeta^n}\right] . \left[\frac{a_o}{\zeta} - \sum_{m=1}^{\infty} \frac{ma_m}{\bar{\zeta}^m}\right]$$

This can be rearranged when $\zeta\bar{\zeta} = 1.0$ as follows :

$$P = a_o a'_o + \frac{x'_o a_o}{\zeta} + \sum_{n=1}^{\infty} \frac{a_o a'_n}{\zeta^{n+1}} - \sum_{m=1}^{\infty} \frac{ma_m x'_o}{\bar{\zeta}^m} - \sum_{m=1}^{\infty} \frac{ma_m a'_o}{\bar{\zeta}^{m+1}}$$

$$- \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} \frac{ma_m a'_n}{\zeta^n \bar{\zeta}^m}$$



The complete array of terms in the double sum, covering all values of n and m can be separated into

(i) The diagonal terms n = m

(ii) The terms below the diagonal m > n

(iii) The terms above the diagonal n > m

Taking each separately and substituting s = m - n or s = n - m as appropriate it is found that for $\zeta\bar{\zeta} = 1.0$

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty} \frac{ma_m a'_n}{\zeta^n \bar{\zeta}^m} = \sum_{n=1}^{\infty} na_n a'_n + \sum_{n=1}^{\infty}\sum_{s=1}^{\infty} \frac{(n+s)a_{n+s} a'_n}{\bar{\zeta}^s}$$

$$+ \sum_{m=1}^{\infty}\sum_{s=1}^{\infty} \frac{ma_m a'_{m+s}}{\zeta^s}$$

Substituting this series for the double sum in P yields the RHS of equation 4.8.