Context variation and syntax nuances of the equal sign in elementary school mathematics

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Existing research suggests that young children can develop a partial understanding of the equal sign as an operator rather than as a relational symbol of equivalence. This partial understanding can be the result of overemphasis on canonical equation syntaxes of the type a + b = c in elementary school mathematics. This paper presents an examination of context and syntax nuances of relevant sections from the Grade-1 Greek series of textbooks and workbooks. Using a conceptual framework of context variation, the analysis shows qualitative differences between equations of similar syntax and provides a nuanced determination of contextual and structural aspects of 'variation' in how the equal sign is presented in elementary mathematics. The paper proposes that since equations have context-specific meanings, context variations should constitute a separate element of analysis when investigating how the equal sign is presented. The implication for practice and future research is that nuanced considerations of equation syntax within varied contexts are needed for elaborating analyses of the equal sign presentation that move beyond dichotomized categorizations of canonical/non-canonical syntaxes.

Keywords: equal sign; equations; arithmetic; symbols; textbook analysis; elementary education.

Introduction

The equal sign, as a symbol that is associated with mathematical equivalence, has a pervasive role across all levels of mathematics (McNeil, Hornburg, Devlin, Carrazza, & McKeever, 2019) and is used in equations as a relation between mathematical objects that are the same and interchangeable (Jones, Inglis, Gilmore, & Dowens, 2012; McNeil, 2008). Development of deep understanding of this fundamental notion is closely associated with children's performance in arithmetic as well as learning of algebra (Alibali, Knuth, Hattikudur, McNeil, & Stephens, 2007; Byrd, McNeil, Chesney, & Matthews, 2015).

Research has indicated that children often develop a narrow understanding of the equal sign in that they often interpret it as an operational symbol that denotes "do the operation" or "find the answer" rather than a relational symbol that has the meaning of "the same as" (Byrd et al., 2015; Matthews & Rittle-Johnson, 2009; Stephens et al., 2013). Children's often partial or incorrect understanding of the equal sign has been attributed to early experiences with arithmetic at school that emphasize procedural rather than relational knowledge in mathematics and not to age-related conceptual constraints (McNeil, 2008; McNeil & Alibali, 2005). Seo and Ginsburg (2003), for instance, have argued that formal instruction introduces the equal sign predominantly within number sentences and equations of a 'canonical' form such as: a + b = __, where the operation appears only on the left side of the equal sign. They point out that children have very few opportunities to see and use the equal sign in other, 'non-canonical' formats (e.g., 5 + 5 = 7 + __).

Mathematics textbooks can have a significant impact on students' mathematics experiences (Bryant et al., 2008; Schubring & Fan, 2018). On this basis, important research has been conducted to explore the presentation of the equal sign in textbooks and its influence on children's learning. This research has predominantly analyzed US and Chinese textbooks. For example, overemphasis on presenting the equal sign in association with the four operations and within particular equation formats has been evidenced by analysis of US middle-school curriculum materials and has been highlighted as one of the factors that lead children to generalize 'operational patterns' which narrow their knowledge of the meanings of the equal sign and may in turn lead to the development of persistent misconceptions (McNeil, Fyfe, Petersen, Dunwiddie, & Brletic-Shipley, 2011; McNeil et al., 2006; Powell, 2012). Furthermore, in comparing US with Chinese student textbooks, Li, Ding, Capraro and Capraro (2008) found that, in contrast to US textbooks, Chinese textbooks presented the equal sign as a relational symbol within varied contexts.

Building upon the existing research on the presentation of the equal sign, its meanings and use in arithmetic, the study that is presented in this paper employed a directed text analysis approach

(Potter & Levine-Donnerstein, 1999) for the analysis of equation syntaxes and contexts identified in the Greek Grade-1 series of core textbooks and accompanying workbooks for 6-7 year old elementary school children. The analysis goes beyond a solely quantitative investigation of equation syntax that has been the primary focus of previous textbook research (McNeil et al., 2006; Powell, 2012). Influenced by Li et al's (2008) emphasis of the importance of context variability in presenting the equal sign in Chinese textbooks as a determining factor for Chinese students’ appropriate understanding of the equal sign, the current study presents a qualitative view of equation presentation with a particular focus on context as an element that is distinct from equation syntax. While the term ‘syntax’ is used in this study to refer to the structure of the equation and the position of the equal sign within it, ‘context’ is used in the extended use of the term (Seo & Ginsburg, 2003) to denote different kinds of problems, topics, statements and activities within which the equal sign may be used to indicate equivalence.

The study seeks an answer for the following research question: What interrelations of equation context and syntax can be revealed through a qualitative analysis of the presentation of the equal sign in an elementary school textbook and accompanied workbooks?

The paper presents a close examination of qualitative syntax nuances and context variation in textbook examples which have not been captured sufficiently by existing, dichotomized categorizations of equation structure that focus solely on canonical/non-canonical syntaxes. Influenced by the change-resistance theoretical account, which suggests that children’s difficulties in developing a deep understanding of the equal sign are the result of “inappropriate generalization of knowledge constructed from overly narrow experience in arithmetic” (McNeil, 2014, p. 42), and calls for increased variation of problem types and contexts within which the equal sign is presented (Seo & Ginsburg, 2003), the paper proposes a nuanced consideration of qualitative, contextual and structural aspects of ‘variation’, when presenting the equal sign. Such qualitative aspects may easily be overlooked and yet are highly important for inviting a relational view of the symbol.

The Greek Grade-1 set of textbooks is used as a resource that provides genuine textbook examples for exploring context variation and syntax nuances of the equal sign. The analysis presented here is significant in highlighting the importance of attending to interrelations between context and syntax when introducing the equal sign in early school mathematics and thus aims to inform future textbook analyses as well as the development of mathematics curriculum resources and early learning environments. Exploring and analyzing early learning environment influences on children's learning of the equal sign is highly significant as it is children's experiences in the early years of education that shape the often deeply ingrained understandings and misunderstandings that children develop (McNeil & Alibali, 2005).

Background: Theoretical and empirical perspectives

Mathematical equivalence in its symbolic form involves mathematical sentences or equations within which the position of the equal sign may vary. The terms 'canonical' and 'non-canonical' have been employed to denote the most typical and less typical forms for presenting equations respectively. For this study, the adopted definition is that canonical equations are equations of the syntactical form a + b = c where the arithmetical expression is on the left side of the equal sign and is followed by its outcome (McNeil et al., 2019), and therefore the equal sign is in the second-to-last position (Jones & Pratt, 2006; Powell, 2015). Non-canonical equations include different syntaxes such as the following: c = a + b, a + b = c + d where the expression is on the right side or both sides of the equal sign or in identity statements of the type a = a (Jones & Pratt, 2006; Powell, 2015). The terms 'equivalence problems' or 'equivalence context' have been used by McNeil and Alibali (2005) when referring to equations where the equal sign is presented in the middle of the number sentence with numbers or operations or missing numbers on either side (e.g., 4 + 3 + 2 = 4 + __). When asked to make judgements about the correctness of equations, children who have a predominantly operator view of the equal sign as a symbol that denotes 'find the answer' or 'do the operation' only recognize canonical equations as correct and find it difficult to assign a meaning to non-canonical equations (e.g., Kieran, 1981; McNeil et al., 2011; Stephens et al., 2013).

Influenced by their early experiences in mathematics, young children move from an initially 'operational' understanding of the equal sign associated with completing an action or arithmetical operation to a 'relational' understanding which allows a view of the equal sign as a symbol that denotes equivalence and has the meaning of ‘the same as’ (Baroody & Ginsburg, 1983; Behr, Erlwanger, & Nichols, 1980). Jones, Inglis, Gilmore, and Evans (2013) have further refined this definition by arguing that a relational understanding includes the notion of ‘sameness’ as well as the notion of ‘substitution’ which entails understanding that the equal sign also means ‘can be substituted for’.

However, research has indicated that some elementary school children find it difficult to move beyond an operator view of the equal sign (e.g., Byrd et al., 2015; Kieran, 1981; Knuth, Stephens, McNeil, & Alibali, 2006; Sherman & Bisanz, 2009) and that often, young children associate symbols such as +, - and = exclusively with sums and find it difficult to connect them with anything outside the arithmetic part of mathematics (Hughes, 1998).

McNeil (2014) explains that while early theories have attributed children’s difficulties with the concept of equivalence solely to under-developed domain-general cognitive structures (e.g., Piaget and Szeminska 1941/1995 as cited in McNeil, 2014), evidence has shown that young children can grasp the concept of equivalence following appropriate instruction that focuses on developing children’s conceptual understanding of this notion (e.g., Carpenter, Franke, & Levi, 2003; Warren, 2007). This indicates that factors associated with children’s early learning experiences also play a significant role in their understanding of equivalence.

On this basis, a change-resistance theoretical account has been proposed, suggesting that children’s difficulties with the notion of equivalence are primarily due “to constraints and misconceptions that emerge as a consequence of prior learning, rather than to general conceptual, procedural and working memory limitations in childhood” (McNeil, 2014, p. 43). The change-resistance theory proposes that prior knowledge structures that young children have developed as a result of early exposure to limited problem types that emphasize an operational, “outcome-based way of thinking” for arithmetic questions, can become entrenched, hindering the development of a deep, meaningful understanding of the equal sign (McNeil et al., 2019, p. 941).

Within the framework of the change-resistant theoretical account, Byrd et al. (2015) have argued that repeated practice with traditional arithmetic problems of the type: 4 + 3 = __, leads young children to develop rigid operational patterns that resist change and are in competition with other ideas and interpretations that subsequent teaching may introduce. Consequently, this may impede children's future development of algebraic thinking. Knuth et al. (2006, p. 309) provided evidence for a strong positive relation between middle school children's understanding of the equal sign and successful performance in solving algebraic equations "whether using an algebraic strategy or not" and argued for the critical importance of teaching that explicitly supports the development of children's relational understanding of the equal sign throughout elementary and middle school mathematics. These findings have been further supported by McNeil et al. (2019) who found that children's understanding of equivalence in second grade in the US, predicted their future achievement and that in line with the change-resistant account, children's poor outcomes were related to entrenched misconceptions linked to reliance on traditional arithmetic that reinforces the operator view of the equal sign.

McNeil et al. (2011, p. 1628) observed the following three entrenched patterns when analyzing the representations of the equal sign that children with an operational understanding hold: "The 'operations on left side' problem format; the 'perform all given operations on all given numbers' strategy; and the 'calculate the total' concept of the equal sign". It was shown that modifying the format in which equations and problems are presented, and offering children the opportunity to practice problems presented in non-canonical formats, can have a positive impact on supporting the development of a relational understanding of the equal sign and therefore an improved understanding of mathematical equivalence at the early stages of children's education. Intervention studies have demonstrated that young children are capable of grasping the relational meaning of the equal sign at an early age (Molina, Castro, & Castro, 2009; Warren, 2007) and have emphasized that developing a more sophisticated understanding of equal does not have to do with replacing an operational view with a relational view, but with complementing operator notions with relational conceptions (Jones et al., 2013).

To address the resistance of knowledge structures resulting from children's over-exposure to limited problem types, researchers have highlighted the need to provide young children with early learning experiences, including instruction practices, task design and textbook content (Capraro, Ding, Matteson, Capraro, & Li, 2007; McNeil et al., 2019; Stephens et al., 2013), that expose them to an increased variation of problem types, encouraging them to develop in-depth understanding of equivalence and algebraic thinking. On the basis that the notion of 'variation' in how the equal sign is presented has been emphasized as being significant for addressing change resistance, the present paper aims at examining and proposing a nuanced determination of qualitative variation when considering the presentation of the equal sign in elementary mathematics textbooks.

The equal sign presentation in textbooks

Analyses of mathematics textbooks in the US have revealed that the equal sign is mostly presented within arithmetic statements of 'canonical' or 'typical' syntax (3 + 4 = 7) and that such over-emphasis does not support the development of an equivalence relation view (Li et al., 2008; McNeil et al., 2006; Seo & Ginsburg, 2003). In examining Grade-1 to Grade-6 US textbooks between 1970 and 2000, Capraro et al. (2007) reported that they identified only limited or no guidance provided to teachers about the teaching of the equal sign. Furthermore, problem type variation was greater in earlier textbooks and completely disappeared in the more recent textbooks. In contrast, analysis of three Chinese Grade-1 textbooks by Capraro and colleagues revealed that the equal sign was introduced in all textbooks alongside the symbols > and < and before introducing the notions of addition and subtraction. The researchers argued that the introduction of the equal sign in Grade-1 Chinese textbooks within comparison contexts and alongside the use of multiple representations of equivalence offers Chinese students the opportunity to encounter the relational meaning of the symbol very early in their mathematics learning while U.S. textbooks do very little to address the misconceptions that students might develop in relation to the meanings of the equal sign (Capraro et al., 2007). Li et al. (2008) further proposed that the reason why Chinese children seem to develop different, and in many occasions better, understanding of the equal sign than children in the US may lie in the high diversity of contexts within which the equal sign is presented and the use of open-ended problems of the type: __ + 5 = __

Textbook analyses with a focus on the presentation of the equal sign have mainly adopted a quantitative approach that investigates the number of instances where the equal sign is presented in a particular position within different equation formats and problems. Researchers who have studied the impact that the presentation of the equal sign in textbooks and instruction may have on children's learning and understanding (e.g., McNeil et al., 2006; Powell, 2012) have used the terms 'context', 'format', 'syntax' or 'equation types' almost interchangeably to refer to equation structure mainly, as well as to the position of the equal sign within different equations such as: a + b = __ or a = b + c.

In particular, in a study that investigated how students' understanding of the equal sign changes in relation to their mathematics experience, McNeil and Alibali (2005, p. 302) found that undergraduate and graduate students were "relatively immune to context effects." Elementary school students who encountered the equal sign in the context of addition problems mainly, appeared to hold an operational understanding of the equal sign and lack knowledge of its relational meaning. Most interestingly, seventh-grade students for whom a relational understanding was still under development and who perhaps were in a transition phase of moving from an initially operational to a relational understanding of the equal sign (Baroody & Ginsburg, 1983; Behr, et al., 1980), appeared to be highly influenced by the problem format when interpreting and defining the meaning of the equal sign. Seventh-grade students provided operational interpretations when the equal sign was presented in addition problems (such as: 4 + 3 + 6 = __) and relational interpretations when the equal sign was presented within equivalence problems (such as 4 + 3 + 6 = 4 + __). The researchers concluded that context plays a central role "in eliciting newly emerging knowledge" and so it is important that early mathematics instruction presents the equal sign within different contexts, "especially ones that dissuade the operational interpretation" (McNeil & Alibali, 2005, p. 304). While the importance of context is underlined and supported by strong evidence in the aforementioned study, the term 'context' is used in a way that denotes equation structure, format or syntax in relation to the position of the equal sign in the equation.

The term 'context' has been differentiated from the term 'format' in subsequent work by McNeil and colleagues. In a study that aimed at improving children's understanding of mathematical equivalence through practice with different problem formats, McNeil et al. (2011, p. 1624) used the term 'format' (and not 'context' as they had done in previous work) to denote differences in equation structure. For example, problems of the type 9 + 3 = __ were considered as having a 'traditional' format (the researchers' term) while problems of the type __ = 7 + 5 were considered as having a 'nontraditional' format (the researchers' term). The researchers found that children in the 'nontraditional' condition developed better understanding of mathematical equivalence than children in the 'traditional' condition. However, children in the 'nontraditional' condition did not demonstrate the gains that the researchers expected overall. They concluded that operational interpretations of the equal sign are resistant to change and presenting the equal sign mostly within arithmetic contexts activates children's operational knowledge regardless of format. Therefore, they suggested that exposing children first to the use of the equal sign in contexts other than arithmetic (e.g., 36 = 36) may help them establish a relational understanding before engaging with different arithmetic problem formats.

Seo and Ginsburg (2003) have extended the use of the term 'context' beyond reference to equation syntax and structure. They have identified different examples of context in mathematics within which the equal sign might be used to denote equivalence. For example, the equal sign can be used within the context of operations in canonical (e.g., 4 + 3 + 7 = 14, 3 × __ = 12) or non-canonical syntaxes (e.g., 15 = 10 + 5) but it can also be used within non-arithmetic contexts. For example, a non-arithmetic context would involve using the equal sign to denote that 1 meter = 100 centimeters or that 1 hour = 60 minutes.

As mentioned earlier, syntax is differentiated from context in the current study. The term 'syntax' is used to refer to the structure of the equation and the position of the equal sign within it and the term 'context' is used in accordance to Seo and Ginsburg's (2003) definition to denote different kinds of problems, topics, statements and activities within which the equal sign may be used to indicate equivalence.

Method of analysis

The Greek Grade-1 set of textbooks is used here as a resource that provides genuine textbook examples for exploring context and syntax variations when introducing the equal sign in the first year of formal schooling. The textbooks are published by the Pedagogical Institute of the governmental

Department of Education. The Grade-1 textbook series consists of two core volumes and four workbooks that were analyzed for this study. The table below presents an overview of the content and structure of the two core volumes (translation into English by the author).

Table 1. The structure of the Greek Grade-1 mathematics textbooks

Volume 1		
Part A: Numbers: Numbers up to 20; Operations: Addition with numbers to 10; Geometry: Space orientation, Shapes; Measurement: Magnitude comparison, Coins up to 10.		
Unit 1 Numbers up to 5; Space and Shapes.	Unit 2 Addition, Analysis of numbers up to 5.	Unit 3 Numbers up to 20, Sums up to 10, Coins.
Part B: Numbers: Numbers up to 50, Place value, Subtraction with numbers up to 10; Operations: Sums with multiple terms, Addition bridging over ten; Geometry: Drawing of lines, Position and movement on squared paper, Shapes; Measurement: Patterns, Time.		
Unit 4 Subtraction, Drawing of lines, Patterns.		
Volume 2		
Part B continues		
Unit 5 Numbers up to 50, Units and Tens, Squared paper.	Unit 6 Units and Tens, Shapes, Time.	
Part C: Numbers: Numbers up to 100; Operations: Addition and Subtraction of two-digit and on-digit numbers, Addition bridging over ten, Multiplication; Geometry: Drawing lines, Puzzles, Tessellations, Shapes, Symmetry; Measurement: Measurement of continuous properties, Weight, Coins.		
Unit 7 Drawing of lines, Puzzles, Addition and Subtraction, Bridging over 10.	Unit 8 Numbers up to 70, Operations, Measurement, Symmetry.	Unit 9 Numbers up to 100, Operations, Weight, Shapes.

The structure and focus of the activities included in the four accompanying workbooks follow the sequence of topics that are covered in the core volumes.

Guided by the research question, the study adopted a pragmatic approach to methodology whereby “different steps of analysis within their different logics” were put together to lead “to the solution of the research question” (Mayring, 2014, p. 8). Adopting the position that text analysis involves both quantitative and qualitative steps (Mayring, 2014) was appropriate for this study because it allowed the process of analysis to zoom into an initially broad picture of quantitative examination of syntaxes that occur in the textbook series, and examine closely, in subsequent analytical steps, qualitative variations and interrelations of equation structure and context. A directed text analysis approach (Potter & Levine-Donnerstein, 1999) was employed, whereby key concepts and variables indicated by previous research provided a guide for the categorization of equation syntaxes and contexts across the Grade-1 mathematics textbook series. The process of analysis has been informed and guided by prior literature, as explained in the following paragraphs.

The analysis was conducted in two phases. In the initial phase, instances where the equal sign is presented were separated from instances where children are encouraged to insert the equal sign to complete an equation or use it in writing a complete number sentence. Subsequently, instances of canonical or non-canonical syntax were categorized in accordance with the types of syntaxes identified by Jones and Pratt (2006), Powell (2015) and McNeil et al. (2019). Hence, equations where the expression is on the left side of the equal sign followed by the outcome and therefore have the equal sign in the second-to-last position were categorized as ‘canonical’ (e.g., 4 + 1 = __, 1 + __ = 2, 1 + 2 + 1 = __). Equations where the expression is on the right side or both sides of the equal sign were categorized as ‘non-canonical’ (e.g., 4 = 1 + 3, __ = 2 + 1, 9 + 4 = 9 + 1 + 3).

The number of instances where the equal sign is presented and instances where children are encouraged to use the equal sign (i.e. insert it between numbers or sets to denote equivalence or use it when writing a number sentence) in canonical or non-canonical syntaxes were aggregated and allowed a broad, quantitative view of the occurrence of canonical and non-canonical equations within the particular textbook series. It should be noted that the books include a number of activities, such as word problems, that require children to provide the answer but are not always accompanied by a particular example of the way in which the answer can be worked out or the type of notation that is to be followed. Even though equations may be written down in the process of solving these problems, the canonical or non-canonical syntax of such notation cannot be assumed so such instances were not included in the count of ‘equal sign use’. Furthermore, the books include a significant number of arithmetic sums that according to the textbook suggestion should be solved mentally. These were not included in the count either.

A second phase of analysis involved the qualitative exploration of the different kinds of mathematics contexts within which equations are presented, the context within which the equal sign is first introduced in the textbook and nuances of canonical or non-canonical syntax. Utilizing Seo and Ginsburg's (2003) outline of different contexts as a guide, the analysis explored the occurrence of arithmetic contexts where the equal sign was presented within operations (thereafter referred to as operation-related contexts), non-operation-related contexts where the equal sign was presented to compare sets of objects or numbers (e.g., 6 = 6) as well as contexts such as money and measurement (Seo & Ginsburg, 2003).

A subsequent review of the qualitative categorization that resulted from this phase of analysis revealed examples of equal sign presentation that would require a more nuanced approach than a dichotomized, canonical / non-canonical, categorization of syntax. Therefore, in addition to presenting the outcomes of the quantitative and qualitative phases of analysis, the following section presents a discussion of specific examples that require some problematization.

Findings

The results of the quantitative analysis of instances where the equal sign is *presented*, and instances where children are encouraged to *use* (i.e., insert by writing), the equal sign in canonical or non-canonical syntaxes across the two core volumes and four workbooks is presented in Table 2.

Table 2. Occurrence of canonical and non-canonical syntaxes for equation presentation and use.

Syntax	**Presentation**	**Use**
Canonical	88.1% (617)	89.2% (58)
Non-canonical	11.9% (83)	10.8% (7)
Total	100% (700)	100% (65)

The percentages in Table 2 show that the equal sign is predominantly presented within canonical equations. This is in line with findings reported by previous international textbook analyses that were discussed earlier in the paper. The instances where children have the opportunity to insert by writing the equal sign in equivalence expressions are limited.

The equal sign is first introduced in Core Volume 1-Unit 2, in a chapter titled: Number Comparison, The Symbols =, > and <. Children are presented with pictures of two different-in-magnitude groups of fish. Following a completed example, children are asked to determine the quantity and write the relevant number under each group. Children need to compare the two numbers and place the appropriate symbol (=, > or <) between them to denote whether they are equal or whether one set is larger or smaller than the other. Children are asked to place the appropriate symbol (=, > or <) between pairs of numbers such as: 4…3, 2…6, 8…8, 6….9, 5…5.

Similarly, the equal sign is first introduced in workbook 1 in the context of paper and pencil activities that are related to number and magnitude comparison and is presented alongside the symbols > and <. Children are asked to compare the size of two sets of objects, or the magnitude of two stacks of cubes, or two numbers, and use the appropriate symbol between them to denote their relation.

It is noteworthy that contrary to indications from previous research (e.g., Seo & Ginsburg, 2003), the introduction of the notion of equivalence and the equal sign occurs here within the context of magnitude and number comparison and not in the context of performing operations. Furthermore, the equal sign is not presented with the plus or minus sign; rather, it is presented alongside symbols that denote strict inequality. On this basis, it could be argued that the equal sign is introduced in the particular textbook within a context that aims to convey the equivalence relation and not an operator view of the symbol (Hattikudur & Alibali, 2010).

Context and syntax variations

Figures 1 and 2 present an outline of indicative examples that have been categorized as cases where the equal sign is presented within operation-related and non-operation-related contexts and in equations of canonical or non-canonical syntaxes. To facilitate the presentation of more than one different type of examples under each category, the figures include reproductions of textbook examples.

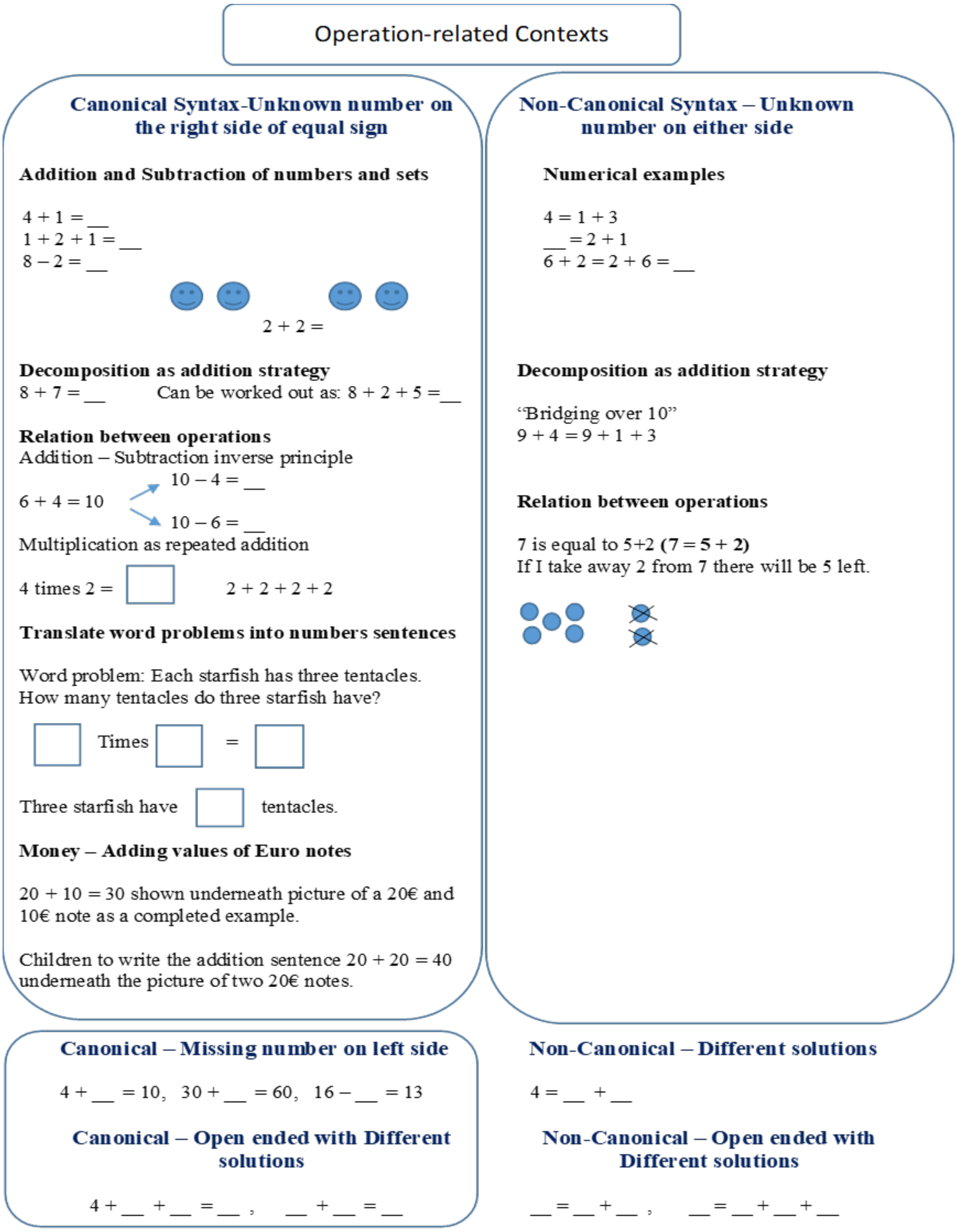

Figure 1. Syntax nuances within operation-related contexts

The predominance of canonical syntaxes and operation-related contexts is noteworthy (Figure 1). However, the analysis reveals great within-context variation. Equations appear in a number of different operation-related contexts such as activities that focus on explaining and encouraging the application of particular strategies, numerical expressions that correspond to word statements and problems (including addition related to money), as well as conceptual relations, such as the inverse relationship between addition and subtraction, the relation between addition and multiplication and also word problems that introduce the relation between multiplication and division.

Equations presented in the context of decomposition (e.g., 8 + 7 = can be worked out as 8 + 2 + 5 =, and 9 + 4 = 9 + 1 + 3) are presented in canonical as well as non-canonical syntaxes, and, as such, evoke the idea of 'substitution' that Jones et al. (2012) highlighted as a notion that is often overlooked and yet is required for a sophisticated conception of the equal sign.

Careful observation and consideration of nuances of syntax variation that exist within the above-presented operation-related contexts could support an argument that an operation-related context does not necessarily invite a 'do the operation' solving approach. This is illustrated by examples where the equal sign is presented in canonical equations with an unknown number on the left side of the equal sign (e.g., 4 + __ = 10 and 5 + 2 + __ = 10).

Strong evidence shows that such types of equation where the operation is on the left side of the equal sign (canonical) are less challenging for children than equations where the operation is on right side of the equal sign (non-canonical) even when they include an unknown number on the left (Matthews, Rittle-Johnson, McEldoon, & Taylor, 2012). However, when equations of the type 5 + 2 + __ = 10 are compared with other equations of canonical syntax that do not include an unknown number on the left (e.g., 1 + 2 + 1 = __), it could be argued that canonical equations with the unknown number on the left invite a different approach than a 'do the operation' approach, as it would be in cases of number sentences such as: 4 + 1 = __ or 1 + 2 + 1 = __. The basis for this argument is that determining the missing number in numbers sentences such as 1 + __ = 2 or 5 + 2 + __ = 10, invites according to Schoenfeld (2017, p. 488), some "understanding of the semantics underlying the equation." Schoenfeld (2017) has linked such understanding to flexible thinking when discussing how a number sentence such as 8 + 4 = __ + 5 can be solved by rewriting it as 12 = __ + 5 or as __ + 5 = 12 which would then necessitate thinking “for the number that has the property that when I add five to it, I get 12” and so solving the equation is a sense making activity (p. 488).

Furthermore, canonical equations where the operation appears on the left side of the equal sign but unknown numbers appear on both sides (e.g., 4 + __ + __ = __ or __ + __ = __), are also qualitatively different than canonical equations of the type 4 + 1 = __, or the above discussed example 1 + __ = 2 because of their ‘open-ended’ nature that invites more than one solution. Open-ended equations have also been identified in Chinese textbooks by Li et al. (2008) who argued that because such equations can have more than one solution, they “encourage more fluid and dynamic understanding of both the operators and equal sign” and “although this more open-ended format resembled the operation equals answers as defined by McNeil et al. (2006), students must demonstrate some relational thinking in order to obtain admissible solutions for this context.” (p. 206). (Li et al., 2008, p. 208) further elaborate that open-ended equations encourage “inductive bridging” that supports the transition between standard and non-standard presentations of the equal sign.

On this basis, the argument could also be that canonical equations such as 4 + __ + __ = __, or __ + __ = __, are qualitatively similar to non-canonical equations of the type __ = __ + __. This is because both types, irrespectively of whether the operation is on the left or right side of the equal sign, are difficult to solve when applying a ‘do the operation’ approach. In summary, when categorizing particular examples of statements of equivalence in textbook analyses, focusing on the unknown number(s) (i.e., how many unknown numbers are included and what their position is in the equation), is important to consider alongside the position of the operation in relation to the equal sign, because it allows for more nuanced syntactical and contextual analysis of the presentation of the equal sign. For these reasons, such equations appear as separate subcategories of syntax within operation-related contexts in Figure 1.

Figure 2 presents examples where the equal sign is presented within non-operation-related contexts. These included magnitude, number or money-related comparisons, composition of two-digit numbers with a focus on place value, partitioning and grouping in tens and units.

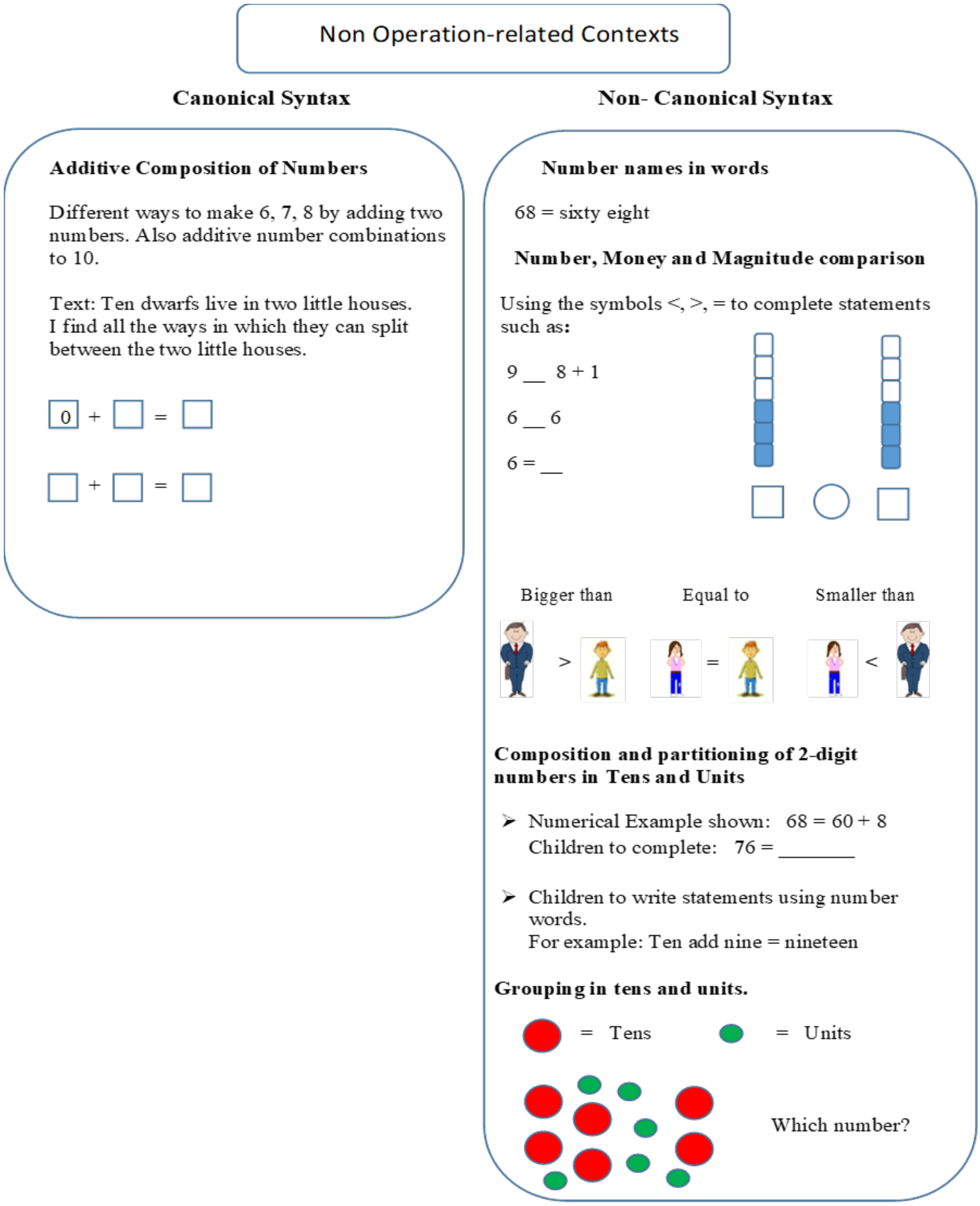

Figure 2. Syntax nuances within non-operation-related contexts

Non-operation-related contexts include activities that may or may not be numerical (e.g., 68 = 60 + 8; equal sign presented between the pictures of two children who have the same height or age, Ten add nine = nineteen) and may or may not be complete (e.g., 68 = sixty eight, 6 …. 6). It is interesting to observe that non-operation-related contexts are predominantly associated with non-canonical syntaxes. An exception worth noting is the activities related to the additive composition of single-digit numbers where children are asked to think of all possible two-number additive combinations that result to numbers such as 7, 8 or 10 and complete open-ended statements of canonical syntax. Open-ended statements such as: 0 + __ = __ and __ + __ = __, that are part of this activity resemble open-ended equations that are presented as part of operation-related contexts, in the sense that children have to identify more than one possible solution. The difference in the case of additive composition activities is that the outcome of the equation is a specified rather than an unspecified number. Nevertheless, the positioning of missing numbers on the left side of the equal sign would require children to determine which two numbers can be added together so that a total of 10 (for example) appears on both sides of the equal sign to achieve numerical equivalence (Molina et al., 2009). This would require retrieving from memory or constructing through calculation strategies an additive combination that results to the same number as the one appearing on the other side of the equal sign. Therefore, such statements, positioned either in operation or non-operation-related contexts invite a view of the equal sign different than 'work out the sum', for identifying different combinations that satisfy the equivalence relation.

Not only do canonical syntaxes appear to traverse operation and non-operation-related activities but also contexts such as money appear both in operation-related and non-operation-related activities. The equal sign appears as one of the symbols that can be chosen amongst the symbols: <, > or =, to compare amounts of money presented in € coins, but is also used elsewhere in the textbook as part of operation-related activities for which children are asked to write the corresponding additions underneath the pictures of different € notes combinations. The following section presents a discussion of the significance of the identified syntax nuances and context variations and implications for further research and future analyses.

Discussion and concluding remarks

While the quantitative exploration of the occurrence of different equation syntaxes across the textbook series confirms the predominance of numerical canonical equations and operation-related contexts that has also been found in previous textbook research (e.g., McNeil et al., 2006; Powell, 2012), qualitative analysis reveals syntax nuances and great within-context variation that warrant attention.

In relation to syntax nuances, the examples analyzed in this paper suggest that the positioning of the unknown number(s) in equations underlies qualitative nuances between equations of similar syntax that need to be considered in the analysis of curriculum materials. Research that has explored children’s understanding of the equal sign, and the level of difficulty that children experience in solving equations of canonical syntax, has provided strong evidence that canonical equations of the type a + b = c, with the unknown number either on the left or right side of the equal side, are of similar difficulty for children and can be solved by most children who demonstrate operational-only understanding of the equal sign (Matthews et al., 2012). Whilst this is the case, and this paper does not dispute this evidence because analysis of children’s understanding is beyond the scope of this study, this paper supports the argument that this nuance of syntax within canonical equations (i.e., the unknown number on the left side) is significant for consideration in analyses of tasks and equations presented in textbooks and in classroom instruction and also in research focusing on the equal sign. The reason is that, in line with Schoenfeld's (2017) position, canonical equations with the unknown number on the left, even if they are of the same difficulty as canonical equations with the unknown number on the right, invite a qualitatively different approach than a ‘do the operation’ approach and at least some appreciation of other concepts alongside the concept of equivalence (e.g., understanding of part-whole relations). Consideration of such qualitative differences between canonical syntaxes and the subtle differences in the approach that they may invoke is needed in analyses of tasks and the presentation of the equal sign and may also be a useful avenue for further extension of the comprehensive assessment measures of children’s understanding of the equal sign developed by

(Matthews et al., 2012), which, as the researchers state, do not currently address other concepts that may be involved in solving successfully specific types of equations.

In relation to context variation, non-operation-related contexts included activities that present the equal sign as a symbol that is used for making comparisons between sets, magnitudes and numbers and, therefore, as an indicator of correspondence between mathematical and non-mathematical objects (Molina et al., 2009, p. 348). Operation-related contexts included examples that focused on relations between operations and also examples of equations that depict the notion of substitution which has been associated with a relational conception of the equal sign (Jones et al., 2013). Despite their positioning within operation-related contexts, examples that focus on exemplifying relations, indicate that 'operation-related' contexts do not necessarily and exclusively involve 'operational'-only depictions of the equal sign but can also include the use of equal sign in equations that delineate conceptual relations. Therefore, in textbook analyses, a significant distinction needs to be made between arithmetic 'operation-related' contexts which, despite focusing on arithmetic operations, may not exclusively include depictions of the equal sign as an operator but may also use it to exemplify relations, and arithmetic contexts where the equal sign is presented solely as an operator.

Qualitative analysis of real textbook examples revealed a variety of operation-related and non-operation-related contexts as well as great variety of task-types within operation-related and non-operation-related contexts. The paper presents the argument that nuances of syntax that go beyond a dichotomy of canonical and non-canonical equation formats can be revealed when equations of similar syntax are considered alongside the context within which they are presented and across contexts. Context variation and syntax nuances within varied contexts can be fruitfully taken into account by future textbook analyses and analyses of instructional practices with a focus on the equal sign. Furthermore, consideration of the qualitatively different thinking approaches that subtle differences in equation structure may invite - including the positioning of the unknown numbers in either canonical or non-canonical equations - can offer a useful platform for teachers to operationalize the notion of 'variation' of problem types in their teaching. It can also inform teacher educators who aim to increase pre-service teachers' awareness of, and sensitivity to, the meanings of the equal sign that different equation structures and contexts may evoke.

In using a textbook as the instrument for exploring interrelations of context and syntax in how the equal sign is presented, the present study did not consider how the textbook is used in classrooms. Research has indicated that the way teachers use textbooks is situated in particular local and global contexts (Remillard, 2005). Therefore, teachers' use of textbooks and affordances for adaptation are particularly important to explore in future research when analyzing features of curriculum materials that are not the object of choice by schools but are tightly associated with statutory national curricula, such as the case of the textbook analyzed here. Examining the use of textbooks in the classroom could help researchers and educators understand relations between presentation, use and interpretation of equation context variations and syntax nuances that can best support the development of children's understanding of the equal sign, in a holistic way, within the learning environment.

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