Cohomology of Fuchsian Groups and Non-Euclidean Crystallographic Groups

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Abstract

For each geometrically finite non-Euclidean crystallographic group (NEC group), we compute the cohomology groups. In the case where the group is a Fuchsian group, we also determine the ring structure of the cohomology. Finally, we compute the L^2-Betti numbers of the NEC groups.

1 Introduction

Let Γ be a geometrically finite non-Euclidean crystallographic group (NEC group), i.e. a discrete subgroup of $\mathrm{PGL}_2(\mathbb{R})$ with a finite sided fundamental domain for the action of Γ on the hyperbolic plane $\mathbb{H}^2$. Throughout we let $\Lambda(\Gamma)$ denote the *limit set* of Γ. In this paper, we will calculate the cohomology of Γ. In the case where Γ is a Fuchsian group, i.e. Γ is contained in $\mathrm{PSL}_2(\mathbb{R})$, we will also calculate the cohomology ring. Our proof will involve finding a suitable fundamental domain for the action of the group in $\mathbb{H}^2 \cup \Lambda(\Gamma)$ and then applying the following Γ equivariant spectral sequence.

Theorem 1.1. [Bro94, Chapter VII (7.10)] *Let X be a Γ-complex and let $\Omega(p)$ be a set of representatives of Γ-orbits of p-cells in X. Let Γ_σ be the stabiliser of σ. For each p-cell σ we have a Γ_σ module $\mathbb{Z}_\sigma$ which Γ_σ acts on via $\chi_\sigma : \Gamma_\sigma \to \{\pm 1\}$. Then,*

$$E^1_{pq} = \bigoplus_{\sigma \in \Omega(p)} H_q(\Gamma_\sigma; \mathbb{Z}_\sigma) \Rightarrow H^\Gamma_{p+q}(X; \mathbb{Z}).$$

Moreover, a description of $d^1 : E^1_{p,*} \to E^1_{p-1,*}$ is given in [Bro94, Chapter VII.8].

Since $X = \mathbb{H}^2 \cup \Lambda(\Gamma)$ is contractible, this sequence converges to the cohomology of Γ. Using knowledge of the abelianization of Γ, it is easy to compute with the spectral sequence. We will now set the convention that an omission of coefficients in the (co)homology functors should be assumed to be $\mathbb{Z}$ coefficients.

Theorem 1.2. *Let Γ be an NEC group of signature*

$$(g, s, \epsilon, [m_1, \dots, m_r], \{(n_{1,1}, \dots, n_{1,s_1}), \dots, (n_{k,1}, \dots, n_{k,s_k}), (), \dots, ()\}).$$

Where the number of empty cycles equals d. Let C_E denote the number of even $n_{i,l}$ and let C_O denote the number of period cycles for which every $n_{i,l}$ is odd.

(a) If $\epsilon = +$ and $d = k = s = 0$ (i.e. Γ is a cocompact Fuchsian group) then

$$H^q(\Gamma) = \begin{cases} \mathbb{Z} & \textit{for } q = 0, \\ \mathbb{Z}^{2g} & \textit{for } q = 1, \\ \mathbb{Z} \oplus \left(\bigoplus_{j=1}^{r-1} \mathbb{Z}_{t_j}\right) & \textit{for } q = 2, \\ \bigoplus_{j=1}^{r} \mathbb{Z}_{m_j} & \textit{for } q = 2l, \textit{ where } l \geq 2, \\ 0 & \textit{otherwise.} \end{cases}$$

(b) If $\epsilon = +$ and $d + k + s > 0$ then

$$H^q(\Gamma) = \begin{cases} \mathbb{Z} & q = 0, \\ \mathbb{Z}^{2g+s+k+d-1} & q = 1, \\ \mathbb{Z}_2^{\frac{1}{2}qC_E+C_O+d} \oplus \left(\bigoplus_{j=1}^{r} \mathbb{Z}_{m_j}\right) & q \equiv 2 \pmod 4, \\ \mathbb{Z}_2^{\frac{1}{2}(q-1)C_E+C_O+d} & q = 2p+1 \textit{ where } p \geq 1, \\ \mathbb{Z}_2^{\frac{1}{2}qC_E+C_O+d} \oplus \left(\bigoplus_{i=1}^{k} \bigoplus_{l=1}^{s_i} \mathbb{Z}_{n_{i,l}}\right) \oplus \left(\bigoplus_{j=1}^{r} \mathbb{Z}_{m_j}\right) & q > 0 \textit{ and } q \equiv 0 \pmod 4. \end{cases}$$

(c) If $\epsilon = -$ then

$$H^q(\Gamma) = \begin{cases} \mathbb{Z} & q = 0, \\ \mathbb{Z}^{g+s+k+d-1} & q = 1, \\ \bigoplus_{p=1}^{r+d+k+\sum_{i=1}^{k} s_i} \mathbb{Z}_{w_p} & q = 2, \\ \mathbb{Z}_2^{\frac{1}{2}(q-1)C_E+C_O+d} & q = 2p+1 \textit{ where } p \geq 1, \\ \mathbb{Z}_2^{\frac{1}{2}qC_E+C_O+d} \oplus \left(\bigoplus_{i=1}^{k} \bigoplus_{l=1}^{s_i} \mathbb{Z}_{n_{i,l}}\right) \oplus \left(\bigoplus_{j=1}^{r} \mathbb{Z}_{m_j}\right) & q > 0 \textit{ and } q \equiv 0 \pmod 4, \\ \mathbb{Z}_2^{\frac{1}{2}qC_E+C_O+d} \oplus \left(\bigoplus_{j=1}^{r} \mathbb{Z}_{m_j}\right) & q > 2 \textit{ and } q \equiv 2 \pmod 4. \end{cases}$$

Where the constants t_j and w_p can be deduced from the abelianization of Γ.

In fact, the following definition will give a precise description of the constants t_j. In the case where Γ is a Fuchsian group we also compute the ring structure (Theorem 1.4).

Definition 1.3. For $j = 1, \ldots, r-1$, let $\hat{t}_j$ be the greatest common divisor of the set of products of $m_1, \ldots, m_r$ taken j at a time. Then, let $t_1 = \hat{t_1}$ and for $j = 2, \ldots, r-1$ let $t_j = \hat{t}_j / \hat{t}_{j-1}$. We will write $\bigoplus_{j=1}^{r} \mathbb{Z}_{m_j} = (\bigoplus_{j=1}^{r-1} \mathbb{Z}_{t_j}) \oplus (\bigoplus_{k=1}^{l} \mathbb{Z}_{q_k})$, where the $\bigoplus_{k=1}^{l} \mathbb{Z}_{q_k}$ term is decomposed via the classification of finite abelian groups. Finally, define R_q to be the ring $\mathbb{Z}[x,y]/(x^3 = y^2, qx, qy)$ where $|x| = 4$ and $|y| = 6$. Note that R_q is isomorphic to the subring of $\mathbb{Z}[z]/(qz)$ generated by z^2 and z^3.

Theorem 1.4. *Let Γ be a Fuchsian group of signature $[g, s; m_1, \ldots, m_r]$.*

(a) If $s = 0$ then $H^*(\Gamma) \cong H^*(\Sigma_g) \oplus \left(\bigoplus_{j=1}^{r-1} H^*(\mathbb{Z}_{t_j})\right) \oplus (\bigoplus_{k=1}^{l} R_{q_k})$.

(b) If $s > 0$ then $H^*(\Gamma) \cong \mathbb{Z}[x_1, \ldots, x_{2g+s-1}]/(x_i x_j \ \forall i, j) \oplus \left(\bigoplus_{j=1}^{r} H^*(\mathbb{Z}_{m_j})\right)$ where $|x_i| = 1$.

We remark that some of the results have appeared in the literature before. The case where Γ is a cocompact Fuchsian group, so $\epsilon = +$ and $d = k = s = 0$, was considered by Majumdar [Maj70], however, our computation of the ring structure is new. The case $\epsilon = +$ and $d = k = 0$ is a corollary of a result of Huebschmann [Hue79] and the case $\epsilon = -$ and $d = k = s = 0$ was considered by Akhter and Majumdar [AM16]. Each of these previous results used different methods to the ideas here.

Other interpretations of the cohomology of Fuchsian groups have appeared in the literature. These have primarily dealt with lifting phenomena [Pat75], with Eichler cohomology [Eic57, Cur70] or with K-theory in relation to the Baum-Connes conjecture [LS00, BJPP02].

The paper is structured as follows. In Section 2 we define the signature of an NEC group. In Section 3 we prove Theorem 1.2 and Theorem 1.4. Finally, in Section 4 we compute the L^2-Betti numbers of each NEC group (Corollary 4.3) using a relation to the rational Euler characteristic $\chi_{\mathbb{Q}}$ of a group.

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2 Non-Euclidean crystallographic groups

We will first describe Wilkie and Macbeath's NEC signatures [Wil66, Mac67], then the associated fundamental domain in $\mathbb{H}^2 \cup \Lambda(\Gamma)$, and finally we will give a presentation for an NEC group in terms of its signature.

An *NEC signature* consists of a sign $\epsilon = \pm$, and several sequences of integers grouped in the following manner:

1. Two integers $g, s \geq 0$.
2. An ordered set of integer *periods* $[m_1, \dots, m_r]$.
3. An ordered set of k *period cycles* $\{C_i := (n_{i,1}, \dots, n_{i,s_i}) : 1 \leq i \leq k\}$.
4. A further d empty period cycles $(), \dots, ()$.

The sequences and sign are then combined into the NEC signature, which is written as

$$(g, s, \epsilon, [m_1, \dots, m_r], \{(n_{1,1}, \dots, n_{1,s_1}), \dots, (n_{k,1}, \dots, n_{k,s_k}), (), \dots, ()\}).$$

We let C_E denote the number of even $n_{i,l}$ and we let C_O denote the number of C_i for which every $n_{i,l}$ is odd.

Associated to each NEC signature is a *surface symbol* describing a fundamental domain for the associated NEC group. The surface symbol is a list of edges travelling around the polygon anticlockwise. Two edges paired orientably will be indicated by the same letter and a prime. Two edges paired non-orientably will be indicated by the same letter and an asterisk. When $\epsilon = +$, we have the surface symbol

$$\xi_1\xi_1' \dots \xi_r\xi_r'\epsilon_1\gamma_{1,0} \dots, \gamma_{1,s_1}\epsilon_1'\epsilon_2 \dots \epsilon_k\gamma_{k,0} \dots, \gamma_{k,s_k}\epsilon_k'\alpha_1\beta_1'\alpha_1'\beta_1' \dots \alpha_g\beta_g'\alpha_g'\beta_g'.$$

When $\epsilon = -$, we have the surface symbol

$$\xi_1\xi_1' \dots \xi_{r+s}\xi_{r+s}'\epsilon_1\gamma_{1,0} \dots, \gamma_{1,s_1}\epsilon_1'\epsilon_2 \dots \epsilon_k\gamma_{k,0} \dots, \gamma_{k,s_k}\epsilon_k'\alpha_1\alpha_1^* \dots \alpha_g\alpha_g^*.$$

For $j = 1, \dots, r$, the period m_j is attached to the vertex v_j common to the edges ξ_j and ξ'_j. The cycle period $n_{i,l}$ is associated with the vertex $w_{i,l}$ common to the edges $\gamma_{i,l-1}$ and $\gamma_{i,l}$. The vertices v_j for $j = r+1, \dots, r+s$ lie on the boundary $\partial\mathbb{H}^2$.

Under the action of the associated NEC group, the stabiliser of the vertex v_j is a cyclic group of order m_j for $1 \leq j \leq r$, or $\mathbb{Z}$ if v_j lies on $\partial\mathbb{H}^2$. The stabiliser of the vertex $w_{i,l}$ is a dihedral group $D_{2n_{i,l}}$ of order $2n_{i,l}$. The stabiliser of the edge $\gamma_{i,l}$ is a reflection group $\mathbb{Z}_2$. No other points of the polygon are fixed points of the NEC group.

If $\epsilon = +$, and $2g - 2 + s + r + d + k + \frac{1}{2}\sum_{i=1}^{k} s_i - \sum_{j=1}^{r}\frac{1}{m_i} - \frac{1}{2}\sum_{i=1}^{k}\sum_{j=1}^{s_i}\frac{1}{n_{i,j}} > 0$ or if $\epsilon = -$ and $g - 2 + s + r + d + k + \frac{1}{2}\sum_{i=1}^{k} s_i - \sum_{j=1}^{r}\frac{1}{m_i} - \frac{1}{2}\sum_{i=1}^{k}\sum_{j=1}^{s_i}\frac{1}{n_{i,j}} > 0$, then there exists an NEC group with the corresponding signature. We can now give a presentation for an NEC group. Due to the large number of generators and relations, we detail this in Table 1.

Signature element	Generator(s)	Relation(s)
Period m_j	x_j	$x_j^{m_j} = 1$
Cycle $(n_{i,1}, \dots, n_{i,s_i})$	e_i $c_{i,0} \dots c_{i,s_i}$	$c_{i,s_i} = e_i^{-1} c_{i,0} e_i$ $c_{i,l-l}^2 = c_{i,l}^2 = (c_{i,l-1}c_{i,l})^2 = 1$
s	$x_r, \dots, x_{r+s}$	See $g \pm$
$g\ +$	$a_1, b_1, \dots, a_g, b_g$	$\prod_{j=1}^{r+s} x_j \prod_{i=1}^{k} e_i \prod_{t=1}^{g} [a_t, b_t] = 1$
$g\ -$	$a_1, \dots, a_g$	$\prod_{j=1}^{r+s} x_j \prod_{i=1}^{k} e_i \prod_{t=1}^{g} a_t^2 = 1$

Table 1: Generators and relations for an NEC group.

If $d = k = 0$ and $\epsilon = +$, then we write the signature of Γ as $[g, s; m_1, \dots, m_r]$ and we refer to Γ a *Fuchsian group* (i.e. a discrete subgroup of $\mathrm{PSL}_2(\mathbb{R})$. If $s = 0$, we say that Γ is *cocompact*.

3 Cohomology

3.1 The cocompact Fuchsian case

First, we will deal with the computation of the ring structure. Recall that R_q is the ring $\mathbb{Z}[x, y]/(x^3 = y^2, qx, qy)$ where $|x| = 4$ and $|y| = 6$.

Proof of Theorem 1.4. The general strategy here is to look at induced maps from quotients of Γ. In the case $s = 0$, the map from $\Gamma \to \pi_1(\Sigma_g)$, given by setting each $d_j = 1$, induces an inclusion of cohomology rings $H^*(\Sigma_g) \hookrightarrow H^*(\Gamma)$. The map $\pi : \Gamma \to T(\Gamma^{ab})$ induces an inclusion of cohomology rings $\bigoplus_{j=1}^{r-1} H^*(\mathbb{Z}_{t_j}) \hookrightarrow H^*(\Gamma)$.

Now, let $\bigoplus_{j=1}^{r} \mathbb{Z}_{m_j} = (\bigoplus_{j=1}^{r-1} \mathbb{Z}_{t_j}) \oplus (\bigoplus_{k=1}^{l} \mathbb{Z}_{q_k})$ be the decomposition given in Definition 1.3. Each inclusion of $\mathbb{Z}_{q_k}$ into Γ induces a map $H^*(\Gamma) \to H^*(\mathbb{Z}_{q_k})$ which is surjective in every dimension except 2 where it is 0. In particular, it is surjective onto a ring isomorphic to R_{q_k}. By considering preimages of elements, we see the map restricts to an isomorphism and we conclude $H^*(\Gamma)$ has a summand isomorphic to R_{q_k}. This completes the proof of the first case.

In the case where $s > 0$, the map $\Gamma \to T(\Gamma^{ab})$ induces an inclusion of rings $\bigoplus_{j=1}^{r} H^*(\mathbb{Z}_{m_j}) \hookrightarrow H^*(\Gamma)$. The proof is then finished by a simple calculation to determine that any two (not necessarily distinct) infinite additive order classes in $H^1(\Gamma)$ cup to give 0. $\square$

Next, we will calculate the cohomology of cocompact Fuchsian groups. We note that the proof here is new, except for we calculate the abelianization using Smith normal form in the same way as Majumdar [Maj70].

Proof of Theorem 1.2(a). We will use Theorem 1.1 and then apply the universal coefficient theorem. In this case $X = \mathbb{H}^2$ endowed with the induced cell structure from the Wilkie-Macbeath polygon. To set up the spectral sequence we observe for each m_j there is a Γ-orbit of 0-cells, where each cell has stabiliser $\mathbb{Z}_{m_j}$. Now, by Theorem 1.1 the E^1-page of the spectral sequence has the form given by Figure 1.

$$\begin{tikzcd}
3 & \bigoplus_{j=1}^{r} \mathbb{Z}_{m_j} & 0 & 0 & 0 \\
2 & 0 & 0 & 0 & 0 \\
1 & \bigoplus_{j=1}^{r} \mathbb{Z}_{m_j} & 0 & 0 & 0 \\
0 & \mathbb{Z}^{r+1} & \mathbb{Z}^{2g+r} \arrow[l] & \mathbb{Z} \arrow[l] & 0 \\
 & 0 & 1 & 2 & 3
\end{tikzcd}$$

Figure 1: The E^1-page of the spectral sequence.

The only non-trivial differentials are along the bottom row. Fixing a basis for the chain groups we have a sequence

$$0 \longleftarrow \langle \alpha_0, \ldots, \alpha_r \rangle \xleftarrow{d^1_{1,0}} \langle \beta_1, \ldots, \beta_{2g+r} \rangle \xleftarrow{d^1_{2,0}} \langle \gamma \rangle \longleftarrow 0.$$

We have $d^1_{1,0}(\beta_i) = \alpha_i - \alpha_0$ for $1 \le i \le r$ and $d^1_{1,0}(\beta_i) = \alpha_0$ for $r+1 \le i \le 2g+r$ and, $d^1_{2,0} = 0$. In particular, $\mathrm{Im}(d^1_{1,0}) \cong \mathbb{Z}^r$, $\mathrm{Ker}(d^1_{1,0}) \cong \mathbb{Z}^{2g}$, $\mathrm{Im}(d^1_{2,0}) = 0$ and $\mathrm{Ker}(d^1_{2,0}) \cong \mathbb{Z}$. From this calculation we deduce the E^2 page is as in Figure 2.

$$\begin{tikzcd}
3 & \bigoplus_{j=1}^{r} \mathbb{Z}_{m_j} & 0 & 0 & 0 \\
2 & 0 & 0 & 0 & 0 \\
1 & \bigoplus_{j=1}^{r} \mathbb{Z}_{m_j} & 0 & 0 & 0 \\
0 & \mathbb{Z} & \mathbb{Z}^{2g} & \mathbb{Z} \arrow[ull] & 0 \\
 & 0 & 1 & 2 & 3
\end{tikzcd}$$

Figure 2: The E^2-page of the spectral sequence.

The only non-trivial differential is the map drawn in Figure 2. Moreover, the spectral sequence clearly collapses after the computation of this differential. We can easily deduce what this differential is using the knowledge of $H_1(\Gamma)$. We will compute the abelianization using the same method as Majumdar [Maj70].

To compute the abelianization we write out the presentation matrix M of Γ and then compute the Smith normal form. We find $H_1(\Gamma) = \mathbb{Z}^{2g} \oplus \left(\bigoplus_{j=1}^{r-1} \mathbb{Z}_{t_j} \right)$. The constants t_j (defined above) come from

Theorem 6 in Ferrar's book 'Finite Matrices' [Fer51]. In particular, $\prod_{j=1}^{p} t_j$ is equal to the greatest common divisor of the p-rowed minors of M.

The map is a surjection onto the factor $\bigoplus_{k=1}^{l} \mathbb{Z}_{q_k}$ from the decomposition $\bigoplus_{j=1}^{r} \mathbb{Z}_{m_j} = (\bigoplus_{j=1}^{r-1} \mathbb{Z}_{t_j}) \oplus (\bigoplus_{k=1}^{l} \mathbb{Z}_{q_k})$. To complete the proof, we apply the universal coefficient theorem to $H_0(\Gamma) = \mathbb{Z}$, $H_1(\Gamma) = \mathbb{Z}^{2g} \oplus \left(\bigoplus_{j=1}^{r-1} \mathbb{Z}_{t_j}\right)$, $H_2(\Gamma) = \mathbb{Z}$, $H_{2l-1}(\Gamma) = \bigoplus_{j=1}^{m} \mathbb{Z}_{m_j}$ and $H_{2l}(\Gamma) = 0$ for $l \geq 2$. $\square$

3.2 Orientable NEC groups with at least one cusp or boundary component

The remaining proofs will use the homology of finite dihedral groups. We record them here for the convenience of the reader.

Theorem 3.1. [Han93] *Let D_{2n} denote a dihedral group of order $2n$. In the case n is odd we have*

$$H_q(D_{2n};\mathbb{Z}) \;=\; \begin{cases} \mathbb{Z} & \text{if } q = 0, \\ \mathbb{Z}_2 & \text{if } q \equiv 1 \pmod 4, \\ \mathbb{Z}_{2n} & \text{if } q \equiv 3 \pmod 4, \\ 0 & \text{otherwise.} \end{cases} \qquad\qquad H_q(D_{2n};\mathbb{Z}_2) = \mathbb{Z}_2 \text{ for } q \geq 0.$$

In the case n is even we have

$$H_q(D_{2n};\mathbb{Z}) \;=\; \begin{cases} \mathbb{Z} & \text{if } q = 0, \\ \mathbb{Z}_2^{\frac{1}{2}(q+3)} & \text{if } q \equiv 1 \pmod 4, \\ \mathbb{Z}_2^{\frac{1}{2}q} & \text{if } q > 0 \text{ is even}, \\ \mathbb{Z}_2^{\frac{1}{2}(q+1)} \oplus \mathbb{Z}_n & \text{if } q \equiv 3 \pmod 4. \end{cases} \qquad\qquad H_q(D_{2n};\mathbb{Z}_2) = \mathbb{Z}_2^{q+1} \text{ for } q \geq 0.$$

We will now compute the cohomology of an NEC group with orientable quotient space with at least one boundary component or cusp.

Proof of Theorem 1.2(b). Let $k, d, s \geq 0$ such that $k + d + s > 0$ and let $\epsilon = +$. We will use Theorem 1.1; here our space X is $\mathbb{H}^2 \cup \Lambda(\Gamma)$ endowed with the induced cell structure from the Wilkie-Macbeath polygon. To set up the sequence, observe that the stabiliser of a marked point v_j in the interior of the quotient space is a cyclic group $\mathbb{Z}_{m_j}$. If the vertex v_j lies on $\partial\mathbb{H}^2$ then the stabiliser is $\mathbb{Z}$. The stabiliser of a marked point $w_{i,l}$ on the boundary of the quotient space is a dihedral group $D_{2n_{i,l}}$, and edges along the boundary are stabilised by reflection groups isomorphic to $\mathbb{Z}_2$. It follows that the E^1-page has the form given in Figure 3.

We will first deal with the differentials $d^1_{*,0}$. We have a sequence

$$\begin{tikzcd}
0 & \left\langle v_j, w_{i,l} \;\middle|\; \begin{matrix} 0 \leq j \leq r+s \\ 1 \leq i \leq k+d \\ 0 \leq l \leq s_i \end{matrix} \right\rangle \arrow[l] & \left\langle \alpha_t, \beta_t, \xi_j, \gamma_{i,l}, \epsilon_i \;\middle|\; \begin{matrix} 1 \leq t \leq 2g,\ 1 \leq j \leq r+s \\ 1 \leq i \leq k+d,\ 0 \leq l \leq s_i \end{matrix} \right\rangle \arrow[l, "d^1_{1,0}"'] \\
\langle f \rangle \arrow[urr, "d^1_{2,0}"] & 0. \arrow[l] &
\end{tikzcd}$$

$$
\begin{array}{c|cccccc}
5 & \mathbb{Z}_2^{k+d} \oplus \left(\bigoplus_{j=1}^{r} \mathbb{Z}_{m_j}\right) \oplus \left(\bigoplus_{i=1}^{k} \bigoplus_{l=1}^{s_i} H_5(D_{2n_{i,l}})\right) & \longleftarrow & \mathbb{Z}_2^{k+d+\sum_{i=1}^{k} s_i} & & 0 & 0 \\
4 & \bigoplus_{i=1}^{k} \bigoplus_{l=1}^{s_i} H_4(D_{2n_{i,l}}) & & 0 & & 0 & 0 \\
3 & \mathbb{Z}_2^{k+d} \oplus \left(\bigoplus_{j=1}^{r} \mathbb{Z}_{m_j}\right) \oplus \left(\bigoplus_{i=1}^{k} \bigoplus_{l=1}^{s_i} H_3(D_{2n_{i,l}})\right) & \longleftarrow & \mathbb{Z}_2^{k+d+\sum_{i=1}^{k} s_i} & & 0 & 0 \\
2 & \bigoplus_{i=1}^{k} \bigoplus_{l=1}^{s_i} H_2(D_{2n_{i,l}}) & & 0 & & 0 & 0 \\
1 & \mathbb{Z}^s \oplus \mathbb{Z}_2^{k+d} \oplus \left(\bigoplus_{j=1}^{r} \mathbb{Z}_{m_j}\right) \oplus \left(\bigoplus_{i=1}^{k} \bigoplus_{l=1}^{s_i} H_1(D_{2n_{i,l}})\right) & \longleftarrow & \mathbb{Z}_2^{k+d+\sum_{i=1}^{k} s_i} & & 0 & 0 \\
0 & \mathbb{Z}^{1+r+s+k+d+\sum_{i=1}^{k} s_i} & \longleftarrow & \mathbb{Z}^{2g+r+s+2k+2d+\sum_{i=1}^{k} s_i} & \leftarrow & \mathbb{Z} & 0 \\
\hline
 & 0 & & 1 & & 2 & 3
\end{array}
$$

Figure 3: The E^1-page of the spectral sequence.

Computing the image of the differential $d^1_{2,0}$ on the $\mathbb{Z}$-basis element f, we obtain that up to sign

$$f \mapsto \sum_{i=1}^{k} \sum_{l=0}^{s_i} \gamma_{i,l}.$$

So, we find $\mathrm{Im}(d^1_{2,0}) = \mathbb{Z}$ and $E^2_{2,0} = 0$. For $d^1_{1,0}$ we have the following

$$
\begin{array}{ll}
\alpha_t \mapsto v_0 - v_0 = 0 & \text{for } 1 \leq t \leq 2g; \\
\beta_t \mapsto v_0 - v_0 = 0 & \text{for } 1 \leq t \leq 2g; \\
\xi_j \mapsto v_j - v_0 & \text{for } 1 \leq j \leq r+s; \\
\gamma_{i,l} \mapsto w_{i,(l+1 \mod s_i)} - w_{i,l} & \text{for } 1 \leq i \leq k, \text{ and } 0 \leq l \leq s_i; \\
\gamma_{i,0} \mapsto w_{i,0} - w_{i,0} = 0 & \text{for } k+1 \leq i \leq k+d; \\
\epsilon_i \mapsto w_{i,0} - v_0 & \text{for } 1 \leq i \leq k+d.
\end{array}
$$

In particular, we have $\mathrm{Im}(d^1_{1,0}) = \mathbb{Z}^{r+s+k+\sum_{i=1}^{k} s_i}$ and $\mathrm{Ker}(d^1_{1,0}) = \mathbb{Z}^{2g+k+d}$. It then follows that $E^2_{1,0} = \mathbb{Z}^{2g+k+d-1}$ and $E^2_{0,0} = \mathbb{Z}$. At this point, it is easy to see that the spectral sequence will collapse trivially once we have computed the differentials $d^1_{1,*}$.

We will begin with the differential $d^1_{1,q}$ where $q \equiv 1 \pmod 4$. Since the edges connected to the vertices corresponding to the $\mathbb{Z}_{m_j}$ summands have trivial stabilisers, the $\mathbb{Z}_{m_j}$ summands will survive to the E^2-page. In the case $q = 1$, the $\mathbb{Z}$ summands also survive by the same reasoning.

We now draw our focus to the other summands. Let each $D_{2n_{i,l}}$ be generated by a reflection $r_{i,l}$ and a rotation $t_{i,l}$ of order $n_{i,l}$. We have that $H_1(D_{2n_{i,l}})$ is generated by $r^1_{i,l}, t^1_{i,l}$, the images of $t_{i,l}$ and $r_{i,l}$ under the abelianization map. For $q > 1$ there will be extra generators whenever an $n_{i,l}$ is even; we will suppress this from the notation. Note that $t^1_{i,l} = 0$ if n is odd. For each $q \equiv 1 \pmod 4$ we now have a sequence (modulo the extra classes arising from dihedral groups where $n_{i,l}$ is even and when $q > 1$)

$$0 \longleftarrow \left\langle w^q_{i,0}, w^q_{p,0}, r^q_{i,l}, t^q_{i,l} \;\middle|\; \begin{matrix} 1 \le i \le k \\ 1 \le l \le s_i \\ 1 \le p \le d \end{matrix} \right\rangle \xleftarrow{d^1_{1,q}} \left\langle \gamma^q_{i,l}, \gamma^q_{p,0} \;\middle|\; \begin{matrix} 1 \le i \le k \\ 0 \le l \le s_i \\ 1 \le p \le d \end{matrix} \right\rangle \longleftarrow 0.$$

We will break the map $d^1_{1,q}$ into several cases depending on the adjacent edges in the fundamental domain and the cycle type of the boundary component. First, we will consider each 'end' of a boundary component with non-empty period of cycles. We have that $\gamma^q_{i,0} \mapsto w^q_{i,0} + t^q_{i,1}$ and $\gamma^q_{i,s_i} \mapsto w^q_{i,0} + t^q_{i,s_i} + r^q_{i,s_i}$ (plus potentially some extra classes from the even dihedral groups). For the intermediary edges we have $\gamma^q_{i,l} \mapsto t^q_{i,l} + t^q_{i,l+1} + r^q_{i,l}$ (plus potentially some extra classes from the even dihedral groups). To justify this, note that we are looking at the maps induced by the inclusions $\langle r_{i,l} t_{i,l} \rangle \hookrightarrow D_{2n_{i,l}}$ and $\langle r_{i,l+1} \rangle \hookrightarrow D_{2n_{i,l+1}}$.

The reader may be wondering why we are suppressing the extra order 2 classes arising from the even dihedral groups from the notation. The key point is that the generator of a $\mathbb{Z}_2$ reflection group either maps to exactly one generator, or it maps to a sum of generators (and we only need to know it maps to more than 1). Thus, the two maps are linearly independent and we can safely omit the classes.

If the boundary component only contains odd cycles, then $\gamma^q_{i,s_i} = \sum_{l=0}^{s_i-1} \gamma^q_{i,l}$, so we have an order 2 element in the kernel of $d^1_{1,q}$. If the boundary component has an empty period of cycles, then we have exactly one edge $\gamma_{i,0}$ with vertex $w_{i,0}$ at each end. In particular $\gamma^q_{i,0} \mapsto w^q_{i,0} - w^q_{i,0} = 0$. From this analysis we deduce that $\mathrm{Ker}(d^1_{1,q}) = \mathbb{Z}_2^{C_O+d}$ and $\mathrm{Im}(d^1_{1,q}) \cong \mathbb{Z}_2^{k+\sum_{i=1}^k s_i - C_O}$. It then follows from a simple calculation that $E^2_{1,q} = \mathbb{Z}_2^{C_O+d}$ and $E^1_{0,q} \cong \mathbb{Z}_2^{\frac{1}{2}(q+1)C_E+C_O+d} \oplus \left(\bigoplus_{j=1}^r \mathbb{Z}_{m_j}\right)$ for $q \equiv 1 \pmod 4$, $q > 1$. When $s > 0$ we have an additional $\mathbb{Z}^s$ summand in $E^2_{0,1}$.

An alternative way of considering these maps is as follows. Let C_{E_i} denote the number of even periods in the ith period cycle. Observe that each period cycle contributes $\frac{1}{2}(q+1)C_{E_i} - 1$ summands of $\mathbb{Z}_2$ to $E^2_{0,q}$. The C_O summands of $\mathbb{Z}_2$ contained in $\mathrm{Ker}(d^1_{1,q})$ cause an additional C_O summands of $\mathbb{Z}_2$ to survive to $E^2_{0,q}$. From, above we then have that

$$k + \sum_{i=1}^k \left(\frac{1}{2}(q+1)C_{E_i} - 1\right) + C_O = k + \frac{1}{2}(q+1)C_E - k + C_O = \frac{1}{2}(q+1)C_E + C_O.$$

We now need to compute the maps $d^1_{1,q}$ for $q \equiv 3 \pmod 4$. We have essentially the same cases and proof as when $q \equiv 1 \pmod 4$ except that $\mathrm{Coker}(d^1_{1,q})$ contains a summand $\mathbb{Z}_{n_{i,l}}$ for each $n_{i,l}$. When $n_{i,l}$ is odd this is immediate, we will now prove this when $n_{i,l}$ is even.

Let $n := n_{i,l}$ be even and consider $H^{q+1}(D_{2n};\mathbb{Z})$ where $q \equiv 3 \pmod 4$. There is an element of order n in $H^{q+1}(D_{2n};\mathbb{Z})$ that corresponds to a power of the second Chern class of a 2-dimensional linear representation ρ of $D_{2n} = \langle r, t \rangle$. Restricting ρ to the subgroup $\langle rt \rangle$ gives the regular representation of $\mathbb{Z}_2 \cong \langle rt \rangle$. Now, the total Chern class of $\mathbb{Z}_2$ is equal to 0 in degree 4. It follows that the map $H^{q+1}(D_{2n};\mathbb{Z}) \to H^{q+1}(\langle rt \rangle)$ has kernel containing a $\mathbb{Z}_n$ summand. In particular, the map $H^q(\langle rt \rangle) \to H^q(D_{2n})$ has cokernel containing a $\mathbb{Z}_n$ summand.

We conclude the description of E^2 as follows. First, when $q \equiv 3 \pmod 4$ we have $\mathrm{Ker}(d^1_{1,q}) \cong \mathbb{Z}_2^{C_O+d}$ and $\mathrm{Im}(d^1_{1,q}) \cong \mathbb{Z}_2^{k+\sum_{i=1}^k s_i - C_O}$. It follows $E^2_{1,q} \cong \mathbb{Z}^{C_O+d}$ and $E^2_{0,q} \cong \mathbb{Z}_2^{\frac{1}{2}(q-1)C_E+C_O+d} \oplus \left(\bigoplus_{i=1}^k \bigoplus_{l=1}^{s_i} \mathbb{Z}_{n_{i,l}}\right) \oplus \left(\bigoplus_{j=1}^k \mathbb{Z}_{m_j}\right)$. Every other entry on the E^2-page is 0 trivially.

The theorem follows from resolving the extension problems $0 \to E^2_{1,q-1} \to H_q(\Gamma) \to E^2_{0,q} \to 0$, where $q > 0$ is even. Then, applying the universal coefficient theorem. To resolve the extension problems, we will compute the homology of Γ with $\mathbb{Z}_2$ coefficients and then compare the $\mathbb{Z}_2$-rank of $H_q(\Gamma;\mathbb{Z}_2)$ with the $\mathbb{Z}_2$-rank of $(E^2_{1,q-1} \oplus E^2_{0,q}) \otimes \mathbb{Z}_2 \oplus \mathrm{Tor}(E^2_{0,q-1}, \mathbb{Z}_2)$. Note that the latter is equal to $(q+1)C_E + 2C_O + 2d$. If the ranks are equal, then the extension will split.

Recall that $H_n(\mathbb{Z}_2;\mathbb{Z}_2) = \mathbb{Z}_2$ for $n \geq 0$. Combining this with the $\mathbb{Z}_2$-homology groups of the Dihedral groups (Theorem 3.1) and the Γ-equivariant spectral sequence (Theorem 1.1), we can set up a spectral sequence calculation. To simplify things, note we are only interested in the maps $d^1_{1,q}$ for $q > 0$.

Let $q > 0$ and let C_T denote the number of odd cycles, so $C_T + C_E = \sum_{i=1}^k s_i$. We then have a sequence

$$0 \longleftarrow \mathbb{Z}_2^{(q+1)C_E+C_T+d+k} \xleftarrow{d^1_{1,q}} \mathbb{Z}_2^{C_E+C_T+d+k} \longleftarrow 0.$$

By essentially using the same calculations as above we have that $\mathrm{Im}(d^1_{1,q}) \cong \mathbb{Z}_2^{C_E+C_T+k-C_O}$. From this we conclude that $E^2_{0,q} = \mathbb{Z}_2^{(q+1)C_E+C_O+d}$ and that $E^2_{1,q} = \mathbb{Z}_2^{C_O+d}$. This gives a $\mathbb{Z}_2$-rank of $(q+1)C_E+2C_O+2d$. Thus, the extension splits.

The theorem now follows from applying the universal coefficient theorem to

$$H_q(\Gamma) = \begin{cases} \mathbb{Z} & q = 0, \\ \mathbb{Z}^{2g+s+k+d-1} \oplus \mathbb{Z}_2^{C_E+C_O+d} \oplus \left(\bigoplus_{j=1}^r \mathbb{Z}_{m_j}\right) & q = 1, \\ \mathbb{Z}_2^{\frac{1}{2}qC_E+C_O+d} & q = 2p > 0, \\ \mathbb{Z}_2^{\frac{1}{2}(q-1)C_E+C_O+d} \oplus \left(\bigoplus_{i=1}^k \bigoplus_{l=1}^{s_i} \mathbb{Z}_{n_{i,l}}\right) \oplus \left(\bigoplus_{j=1}^r \mathbb{Z}_{m_j}\right) & q \equiv 3 \pmod 4, \\ \mathbb{Z}_2^{\frac{1}{2}(q+1)C_E+C_O+d} \oplus \left(\bigoplus_{j=1}^r \mathbb{Z}_{m_j}\right) & q > 1 \text{ and } q \equiv 1 \pmod 4. \end{cases}$$

□

3.3 Non-orientable NEC groups

We will now compute the cohomology of an NEC group with non-orientable quotient space. The proof is almost exactly the same as the proof of Theorem 1.2(b) so we will only provide a brief sketch and highlight the differences. A description of the numbers w_p is detailed in the proof below.

Proof of Theorem 1.2(c) (sketch). The key differences between the orientable and non-orientable cases is the $E^1_{1,0}$ term and the map $d^1_{2,0}$. The $E^1_{1,0}$ now contains a $\mathbb{Z}^g$ summand instead of a $\mathbb{Z}^{2g}$ summand. The map $d^1_{2,0}$ now sends the generator to the sum of boundary components plus 2 times each generator of the aforementioned $\mathbb{Z}^g$ summand. In particular, $E^2_{1,0} = \mathbb{Z}^{g+k+d-1} \oplus \mathbb{Z}_2$.

The proof goes through identically from here, except that now we have an additional splitting problem in $H_1(\Gamma)$ (note the other splitting problems are easily resolved by computing homology with $\mathbb{Z}_2$-coefficients as before). To resolve this splitting problem, we instead turn to the Smith normal form of the presentation matrix M for Γ.

Indeed, we easily deduce there is a $\mathbb{Z}^{g+s+k+d-1}$ summand in $H_1(\Gamma)$. The remaining finite cyclic summands can be determined by Theorem 6 of [Fer51]. We denote these by $\mathbb{Z}_{w_p}$ where $\prod_{a=1}^p w_a$ is equal to the greatest common divisor of the p-rowed minors of M.

The theorem now follows from applying the universal coefficient theorem to

$$H_q(\Gamma) = \begin{cases} \mathbb{Z} & q = 0, \\ \mathbb{Z}^{g+s+k+d-1} \oplus \left(\bigoplus_{p=1}^{r+d+k+\sum_{i=1}^{k} s_i} \mathbb{Z}_{w_p} \right) & q = 1, \\ \mathbb{Z}_2^{\frac{1}{2}qC_E + C_O + d} & q = 2p > 0, \\ \mathbb{Z}_2^{\frac{1}{2}(q-1)C_E + C_O + d} \oplus \left(\bigoplus_{i=1}^{k} \bigoplus_{l=1}^{s_i} \mathbb{Z}_{n_{i,l}} \right) \oplus \left(\bigoplus_{j=1}^{r} \mathbb{Z}_{m_j} \right) & q \equiv 3 \pmod 4, \\ \mathbb{Z}_2^{\frac{1}{2}(q+1)C_E + C_O + d} \oplus \left(\bigoplus_{j=1}^{r} \mathbb{Z}_{m_j} \right) & q > 1 \text{ and } q \equiv 1 \pmod 4. \end{cases}$$

□

4 Rational Euler characteristics and L^2-Betti numbers

Let Γ be a group of finite homological type (or less generally of type VF). Let $\Gamma' \trianglelefteq \Gamma$ be a finite-index torsion-free normal subgroup and let $d = |\Gamma : \Gamma'|$. The *rational Euler characteristic* of Γ is defined to be

$$\chi_{\mathbb{Q}}(\Gamma) := \frac{1}{d}\chi(\Gamma') = \frac{1}{d}\sum_{p \geq 0}(-1)^p \beta_p(\Gamma),$$

where $\beta_p(\Gamma) = \dim_{\mathbb{Q}}(H_p(\Gamma; \mathbb{Q}))$ is the pth Betti-number of G. We note that if Λ is a finite index subgroup of Γ then $\chi_{\mathbb{Q}}(\Lambda) = |\Gamma : \Lambda|\chi_{\mathbb{Q}}(\Gamma)$.

A complete introduction to and survey of L^2-Betti numbers can be found in [Lü02]. We will denote the *pth L^2-Betti number* of a topological space X by $\beta_p^{(2)}(X)$ and for a group Γ we take $\beta_p^{(2)}(\Gamma) = \beta_p^{(2)}(K(\Gamma, 1))$. We note two important facts about L^2-Betti numbers. Firstly, $\chi(X) = \sum_{p \geq 0}(-1)^p \beta_p^{(2)}(X)$ and secondly, if $\tilde{X} \to X$ is a finite cover of degree d, then $\beta_p^{(2)}(\tilde{X}) = d\beta_p^{(2)}(X)$.

Rather than give the technical definition in terms of chain complexes, we will consider L^2-Betti numbers as an asymptotic invariant of towers of finite-index normal subgroups. To view L^2-Betti numbers this way, we use Lück's Approximation Theorem.

Theorem 4.1. [Lü94] *Let Γ be a finitely presented group and let $\Gamma = \Gamma_1 > \Gamma_2 > \dots$ be a sequence of finite-index normal subgroups of G that intersect in the identity. The pth L^2-Betti number of Γ is then given by*

$$\beta_p^{(2)}(G) = \lim_{k \to \infty} \frac{\beta_p(\Gamma_k)}{|\Gamma : \Gamma_k|}.$$

The following elementary proposition extends the remark about Euler characteristics of spaces to rational Euler characteristics of groups.

Proposition 4.2. *Let Γ be a finitely presented residually finite group of finite homological type. Then*

$$\chi_{\mathbb{Q}}(\Gamma) = \sum_{p \geq 0}(-1)^p \beta_p^{(2)}(\Gamma).$$

Applying this proposition to the rational Euler characteristic of a NEC group Γ (see for instance [Con03]), we obtain that $\beta_1^{(2)} = -\chi_{\mathbb{Q}}(\Gamma)$. More precisely we have:

Corollary 4.3. *Let Γ be a NEC group of signature*

$$(g, s, \epsilon, [m_1, \ldots, m_r], \{(n_{1,1}, \ldots, n_{1,s_1}), \ldots, (n_{k,1}, \ldots, n_{k,s_k}), (), \ldots, ()\}),$$

where the number of empty period cycles is equal to d.

(a) If $\epsilon = +$ then

$$\beta_p^{(2)}(\Gamma) = \begin{cases} 2g - 2 + s + r + d + k + \frac{1}{2}\sum_{i=1}^{k} s_i - \sum_{j=1}^{r} \frac{1}{m_i} - \frac{1}{2}\sum_{i=1}^{k}\sum_{j=1}^{s_i} \frac{1}{n_{i,j}} & \text{if } p = 1, \\ 0 & \text{otherwise.} \end{cases}$$

(b) If $\epsilon = -$ then

$$\beta_p^{(2)}(\Gamma) = \begin{cases} g - 2 + s + r + d + k + \frac{1}{2}\sum_{i=1}^{k} s_i - \sum_{j=1}^{r} \frac{1}{m_i} - \frac{1}{2}\sum_{i=1}^{k}\sum_{j=1}^{s_i} \frac{1}{n_{i,j}} & \text{if } p = 1, \\ 0 & \text{otherwise.} \end{cases}$$

In the case where Γ is a Fuchsian group, the first L^2-Betti number was calculated in [BCR16] by directly using Lück's Approximation Theorem.

Corollary 4.4. *Let Γ be a Fuchsian group of signature $[g, s; m_1, \ldots, m_r]$. Then,*

$$\beta_p^{(2)}(\Gamma) = \begin{cases} 2g - 2 + s + r - \sum_{i=1}^{r} \frac{1}{m_i} & \text{if } p = 1, \\ 0 & \text{otherwise.} \end{cases}$$

We finish with a couple of remarks. Firstly, by a result of D. Osin [Osi15], this provides another (admittedly indirect) proof that NEC groups are acylindrically hyperbolic (of course this also follows from the fact a NEC group is word hyperbolic). As a consequence, for $V = \ell^p(\Gamma)$, where $p \in [1, \infty)$ or $V = \mathbb{R}$, the 2nd bounded cohomology group $H_b^2(\Gamma; V)$ is infinite-dimensional. We also note that the methods here should be adaptable to computing the cohomology of various families of generalised triangle groups [EW05] and more generally, geometrically finite Kleinian groups.

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