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Some Superstring Amplitude Computations with the Non-Minimal Pure Spinor Formalism

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We use the non-minimal pure spinor formalism to compute in a super-Poincaré covariant manner the four-point massless one and two-loop open superstring amplitudes, and the gauge anomaly of the six-point one-loop amplitude. All of these amplitudes are expressed as integrals of ten-dimensional superfields in a "pure spinor superspace" which involves five θ coordinates covariantly contracted with three pure spinors. The bosonic contribution to these amplitudes agrees with the standard results, and we demonstrate identities which show how the t_8 and ϵ_{10} tensors naturally emerge from integrals over pure spinor superspace.

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1. Introduction

Although much has been learned about superstring amplitudes using the Ramond-Neveu-Schwarz (RNS) formalism, the need to sum over spin structures obscures the role of spacetime supersymmetry. Using the light-cone Green-Schwarz (GS) formalism, one can easily compute four-point tree and one-loop amplitudes with half of the supersymmetry manifest. But higher-point and higher-loop amplitudes are more difficult to compute in this light-cone formalism, especially amplitudes that involve the ten-dimensional ϵ tensor. Although a covariant version of the GS formalism has recently been developed by Lee and Siegel [1] [2], this covariant GS formalism has not been used to compute higher-loop amplitudes or amplitudes involving the ϵ tensor.

Over the last six years, a manifestly super-Poincaré covariant superstring formalism has been developed which involves bosonic ghost variables λ^α satisfying the pure spinor constraint $\lambda\gamma^m\lambda = 0$ [3]. Tree amplitudes and one-loop four-point amplitudes were computed in [4] using a "minimal" version of the formalism, and these computations were later extended to two-loop four-point amplitudes in [5] and to $d = 11$ one-loop computations in [6]. When all external states are bosons, these amplitudes were shown in [7] [8] [9] to coincide with the standard RNS result.

All of these amplitudes are expressed as integrals of superfields in "pure spinor superspace" which, in $d = 10$, involves five fermionic θ coordinates covariantly contracted with three bosonic pure spinors. When all superfields are on-shell, the superspace integrands are annihilated by the pure spinor BRST operator $Q = \lambda^\alpha D_\alpha$. As shown in [3], this implies that the amplitude expressions are invariant under all sixteen $d = 10$ supersymmetries even if the pure spinor superspace only involves five θ's.

More recently, a non-minimal version of the pure spinor formalism has been developed which involves both a pure spinor λ^α and its complex conjugate $\overline{\lambda}_\alpha$ [10]. The amplitude prescription using the non-minimal version is considerably simpler than in the minimal version since there are no picture-changing operators and Lorentz invariance is manifest at all stages in the computation. Furthermore, the amplitude prescription in the non-minimal formalism can be related to the prescription in topological string theory where the b ghost is replaced by a composite operator.

For tree amplitudes, it is trivial to show that the minimal and non-minimal pure spinor formalisms give the same answers. But for loop amplitudes, there are some differences between the minimal and non-minimal computations which makes it non-trivial to prove their equivalence. In the first part of this paper, the non-minimal pure spinor formalism will be used to re-compute the massless four-point one-loop and two-loop amplitudes and equivalence with the minimal computations will be proven. In terms of integrals over pure spinor superspace, the kinematic factors in these one-loop and two-loop amplitudes will be shown to be proportional to

$$K_{1-loop} = \langle(\lambda A)(\lambda\gamma^m W)(\lambda\gamma^n W)\mathcal{F}_{mn}\rangle, \tag{1.1}$$

$$K_{2-loop} = \langle(\lambda\gamma^{mnpqr}\lambda)\mathcal{F}_{mn}\mathcal{F}_{pq}\mathcal{F}_{rs}(\lambda\gamma^s W)\rangle,$$

where A_α, W^α, and $\mathcal{F}_{mn}$ are the spinor gauge superfield, spinor superfield-strength, and vector superfield-strength of super-Yang-Mills, and the pure spinor measure factor $\langle\ \rangle$ is defined such that $\langle(\lambda\gamma^m\theta)(\lambda\gamma^n\theta)(\lambda\gamma^p\theta)(\theta\gamma_{mnp}\theta)\rangle = 1$. Using the super-Yang-Mills equations of motion, it is easy to check that the integrands in (1.1) are annihilated by $\lambda^\alpha D_\alpha$, so these kinematic factors are supersymmetric.

The non-minimal formalism will then be used to compute in a supersymmetric manner the gauge variation of the massless six-point one-loop amplitude in Type-I superstring theory. Since this computation involves the ten-dimensional ϵ tensor, it has never been performed using the light-cone GS formalism. After expressing the gauge variation of the six-point amplitude as a term at the boundary of moduli space, it will be shown that the anomaly is proportional to the pure spinor superspace integral

$$K_{anomaly} = \langle(\lambda\gamma^m W)(\lambda\gamma^n W)(\lambda\gamma^p W)(W\gamma_{mnp}W)\rangle, \tag{1.2}$$

whose purely bosonic contribution is the standard $\epsilon_{10}F^5$ term.

Further investigation upon the appearance of ϵ_{10} in (1.2) led us to the discovery of a pure spinor superspace integral, namely,

$$\langle(\lambda\gamma^r W^1)(\lambda\gamma^s W^2)(\lambda\gamma^t W^3)(\theta\gamma^m\gamma^n\gamma_{rst}W^4)\rangle,$$

from which the t_8 and ϵ_{10} tensors naturally emerge in a unified manner, in the form $\eta^{mn}t_8^{m_1n_1\ldots m_4n_4} - \frac{1}{2}\epsilon_{10}^{mnm_1n_1\ldots m_4n_4}$. This differs from the RNS formalism where the t_8 and ϵ_{10} tensors come from different spin structures. It may be possible that this pure spinor superspace integral is related to the five-point one-loop amplitudes involving the heterotic $\epsilon_{10}B\mathrm{tr}F^4$ and Type IIA $\epsilon_{10}t_8BR^4$ terms, which would be useful for finding the supersymmetric completions of these terms.

It is interesting to compare these computations using the pure spinor formalism with the recently developed method of Lee and Siegel for computing one-loop amplitudes. The method of Lee and Siegel is based on the "ghost pyramid" covariant quantization of the Green-Schwarz superstring, in which the BRST operator has a complicated structure involving an infinite set of ghosts [1]. However, the vertex operators in the Lee-Siegel formalism are relatively simple and have a very similar structure to the integrated vertex operator in the pure spinor formalism.

In the one-loop computations performed in [2] using the Lee-Siegel method, all vertex operators are integrated and there are no picture-changing operators. Furthermore, there is no superspace integration using this method so the amplitudes are expressed in terms of the component fields. This is the analog of the $\mathcal{F}_1$ picture in the RNS formalism where all vertex operators are in the zero picture.

On the other hand, in the one-loop computations using the pure spinor formalism, one of the vertex operators is unintegrated, the b ghost is a composite operator playing the role of a picture-raising operator, and the amplitudes are expressed as integrals over pure spinor superspace. This is the analog of the $\mathcal{F}_2$ picture in the RNS formalism where the unintegrated vertex operator is in the -1 picture and the picture-raising operator is inserted on top of the b ghost.

For certain one-loop computations such as the four-point and five-point massless amplitudes computed in [2], there is no disadvantage in treating all vertex operators in integrated form. However, for the anomaly computation presented here, it is definitely more convenient to leave one unintegrated vertex operator in a "different picture" from the integrated vertex operators. It would be interesting to see how to compute this anomaly using the Lee-Siegel method, and if one needs to introduce some analog of picture-changing operators.

In section 2, we review the non-minimal amplitude prescription for one-loop and two-loop amplitudes. In section 3, we compute the massless four-point one-loop and two-loop amplitudes and show agreement with the computations using the minimal formalism. In section 4, we compute the gauge variation of the massless six-point one-loop amplitude. In section 5, we explain how t_8 and ϵ_{10} tensors naturally emerge from the integration over pure spinor superspace. And in appendix A, we list all the pure spinor superspace identities used in this paper and present two other representations for t_8 and ϵ_{10} tensors using pure spinors.

2. Non-Minimal Amplitude Prescription

The prescription in the non-minimal pure spinor formalism for computing N-point one-loop and two-loop scattering amplitudes is given by [10]

$$\mathcal{A}_{1-loop} = \int d\tau \langle \mathcal{N} \, (\int dw \mu(w) b(w)) V_1(z_1) \prod_{r=2}^{N} \int dz_r U_r(z_r) \rangle, \tag{2.1}$$

and

$$\mathcal{A}_{2-loop} = \int d\tau_1 d\tau_2 d\tau_3 \langle \mathcal{N} \, \prod_{s=1}^{3} (\int dw_s \mu_s(w_s) b(w_s)) \prod_{r=1}^{N} \int dz_r U_r(z_r) \rangle. \tag{2.2}$$

where τ_i are the Teichmuller parameters, μ_i are the Beltrami differentials, V_r and U_r are the unintegrated and integrated vertex operators, and $\langle \ \rangle$ denotes the functional integral over the Green-Schwarz-Siegel fields $[x^m, \theta^\alpha, d_\alpha]$, over the pure spinor ghosts λ^α and their conjugate momenta w_α, and over the non-minimal fields $[\overline{\lambda}_\alpha, r_\alpha]$ and their conjugate momenta $[\overline{w}^\alpha, s^\alpha]$.

As in topological string theory, the b-ghost is a composite operator satisfying $\{Q, b\} = T$ where T is the stress-tensor, and has the explicit form

$$b = s^\alpha \partial \overline{\lambda}_\alpha + \frac{2\Pi^m (\overline{\lambda}\gamma_m d) - N_{mn}(\overline{\lambda}\gamma^{mn}\partial\theta) - J(\overline{\lambda}\partial\theta) - (\overline{\lambda}\partial^2\theta)}{4(\overline{\lambda}\lambda)} \tag{2.3}$$

$$+\frac{(\overline{\lambda}\gamma^{mnp}r)(d\gamma_{mnp}d + 24 N_{mn}\Pi_p)}{192(\overline{\lambda}\lambda)^2} - \frac{(r\gamma^{mnp}r)(\overline{\lambda}\gamma_m d)N_{np}}{16(\overline{\lambda}\lambda)^3} + \frac{(r\gamma^{mnp}r)(\overline{\lambda}\gamma_{pqr}r)N_{mn}N^{qr}}{128(\overline{\lambda}\lambda)^4}$$

where $\Pi^m = \partial x^m + \frac{1}{2}(\theta\gamma^m\partial\theta)$ is the supersymmetric momentum and $N_{mn} = \frac{1}{2}(w\gamma_{mn}\lambda)$ and $J = \lambda w$ are the pure spinor Lorentz and ghost currents.

Integration over the zero modes of the bosonic and fermionic worldsheet fields naively gives 0/0, so it is necessary to insert a BRST-invariant operator $\mathcal{N} = e^{\{Q,\chi\}}$ which regularizes this zero mode integration. Since $\mathcal{N} = 1 + \{Q, \Omega\}$, the choice of χ does not affect the scattering amplitude. A convenient choice is $\chi = -\overline{\lambda}_\alpha\theta^\alpha - \sum_{I=1}^{g}(\frac{1}{2}N^I_{mn}(s^I\gamma^{mn}\overline{\lambda}) + J^I(s^I\overline{\lambda}))$, which implies that

$$\mathcal{N} = \exp(-\overline{\lambda}_\alpha\lambda^\alpha - r_\alpha\theta^\alpha) \tag{2.4}$$

$$\exp\Big(\sum_{I=1}^{g} \Big[-\frac{1}{2}N^I_{mn}\overline{N}^{mnI} - J^I\overline{J}^I \quad - \frac{1}{4}(s^I\gamma_{mn}\overline{\lambda})(\lambda\gamma^{mn}d^I) + (s^I\overline{\lambda})(\lambda d^I) \Big] \Big),$$

where $[N^I_{mn}, J^I, \overline{N}_{mn}, \overline{J}^I, d^I_\alpha, s^{I\alpha}]$ denote the g zero modes of these spin one fields on a genus g surface.

Finally, for massless external states, the unintegrated vertex operator is $V = \lambda^\alpha A_\alpha$ and the integrated vertex operator is

$$U = \partial\theta^\alpha A_\alpha + \Pi^m A_m + d_\alpha W^\alpha + \frac{1}{2} N^{mn} \mathcal{F}_{mn}.$$

The $[A_\alpha, A_n, W^\alpha, \mathcal{F}_{mn}]$ superfields describe super-Yang-Mills theory [11] and have the θ-expansions [12]

$$A_\alpha(x,\theta) = \frac{1}{2} a_m (\gamma^m\theta)_\alpha - \frac{1}{3}(\xi\gamma_m\theta)(\gamma^m\theta)_\alpha - \frac{1}{32} F_{mn}(\gamma_p\theta)_\alpha(\theta\gamma^{mnp}\theta) + \ldots$$

$$A_m(x,\theta) = a_m - (\xi\gamma_m\theta) - \frac{1}{8}(\theta\gamma_m\gamma^{pq}\theta)F_{pq} + \frac{1}{12}(\theta\gamma_m\gamma^{pq}\theta)(\partial_p\xi\gamma_q\theta) + \ldots$$

$$W^\alpha(x,\theta) = \xi^\alpha - \frac{1}{4}(\gamma^{mn}\theta)^\alpha F_{mn} + \frac{1}{4}(\gamma^{mn}\theta)^\alpha(\partial_m\xi\gamma_n\theta) + \frac{1}{48}(\gamma^{mn})^\alpha(\theta\gamma_n\gamma^{pq}\theta)\partial_m F_{pq} + \ldots$$

$$\mathcal{F}_{mn}(x,\theta) = F_{mn} - 2(\partial_{[m}\xi\gamma_{n]}\theta) + \frac{1}{4}(\theta\gamma_{[m}\gamma^{pq}\theta)\partial_{n]}F_{pq} + \ldots,$$

where $a_m(x)$ and $\xi^\alpha(x)$ describe the gluon and gluino fields, $F_{mn} = 2\partial_{[m}a_{n]}$, and $\ldots$ involve derivatives of a_m and ξ^α.

To compute the functional integral over the worldsheet fields, one first uses the free field OPE's to integrate out the non-zero modes. Note that as in topological string theory, computation of the partition function for the non-zero modes is trivial because of cancellations between bosonic and fermionic fields of equal spin. The worldsheet zero modes are then integrated out using the measure factors described in [10] and the regulator $\mathcal{N}$ of (2.4).

3. Four-Point One-Loop and Two-Loop Computations

As was shown in [4] and [9] using the minimal pure spinor formalism, the kinematic factors for the massless four-point one-loop and two-loop amplitudes are proportional to the pure spinor superspace integrals

$$K_{1-loop} = \langle(\lambda A)(\lambda\gamma^m W)(\lambda\gamma^n W)\mathcal{F}_{mn}\rangle, \quad (3.1)$$

$$K_{2-loop} = \langle(\lambda\gamma^{mnpqr}\lambda)\mathcal{F}_{mn}\mathcal{F}_{pq}\mathcal{F}_{rs}(\lambda\gamma^s W)\rangle, \quad (3.2)$$

where A_α, W^α, and $\mathcal{F}_{mn}$ are the spinor gauge superfield, spinor superfield-strength, and vector superfield-strength of the four external super-Yang-Mills multiplets, the expressions of (3.1) and (3.2) are summed over permutations of the four external superfields, and the pure spinor measure factor $\langle\ \rangle$ is defined such that $\langle(\lambda\gamma^m\theta)(\lambda\gamma^n\theta)(\lambda\gamma^p\theta)(\theta\gamma_{mnp}\theta)\rangle = 1$. In [8] and [9], the purely bosonic contributions to these pure spinor superspace integrals where shown to correctly reproduce the t_8 index contractions of the four Yang-Mills field-strengths.

It will now be shown that the non-minimal computation of the four-point massless one-loop and two-loop amplitudes contains the same kinematic factors as in [8][9]. Since the moduli space part of the amplitude computations in the minimal and non-minimal formalisms is the same, this proves the equivalence of the two prescriptions for these amplitudes.

3.1. One-loop computation

Using the one-loop prescription of (2.1), the regulator $\mathcal{N}$ of (2.4) can provide a maximum of eleven d_α zero modes, which are multiplied by the eleven s^α zero modes. So the remaining five d_α zero modes must come either from the vertex operators or from the single b ghost. Since the three integrated vertex operators can provide at most three d_α zero modes through the terms $(W^\alpha d_\alpha)$, the single b ghost of (2.3) must provide two d_α zero modes through the term

$$\frac{(\overline{\lambda}\gamma^{mnp}r)(d\gamma_{mnp}d)}{192(\lambda\overline{\lambda})^2}. \tag{3.3}$$

After integrating over the zero modes of the dimension one fields $(w_\alpha, \overline{w}^\alpha, d_\alpha, s^\alpha)$ using the measure factors described in [10], one is left with an expression proportional to

$$\int d^{16}\theta \int [d\lambda][d\overline{\lambda}][dr](\lambda\overline{\lambda})^{-2}(\lambda)^4(\overline{\lambda}\gamma^{mnp}r)AWWW\exp(-\lambda\overline{\lambda} - r\theta) \tag{3.4}$$

$$= \int d^{16}\theta \int [d\lambda][d\overline{\lambda}][dr]\exp(-\lambda\overline{\lambda} - r\theta)(\lambda\overline{\lambda})^{-2}(\lambda)^4(\overline{\lambda}\gamma^{mnp}D)AWWW \tag{3.5}$$

where $D_\alpha = \frac{\partial}{\partial\theta^\alpha} + (\gamma^m\theta)_\alpha\partial_m$ is the usual superspace derivative and the index contractions on

$$(\lambda)^4(\overline{\lambda}\gamma^{mnp}D)AWWW \tag{3.6}$$

have not been worked out. Note that (3.5) is obtained from (3.4) by writing $r_\alpha\exp(-r\theta) = \frac{\partial}{\partial\theta^\alpha}\exp(-r\theta)$, integrating by parts with respect to θ, and using conservation of momentum to ignore total derivatives with respect to x. Furthermore, the factor of $(\lambda)^4$ in (3.4) comes from the λ in the unintegrated vertex operator, the 11 factors of λ and $\overline{\lambda}$ which multiply the zero modes of d_α and s_α in $\mathcal{N}$, the factor of $(\lambda)^{-8}(\overline{\lambda})^{-8}$ in the measure factor of w_α and $\overline{w}^\alpha$, and the factor of $(\overline{\lambda})^{-3}$ in the measure factor of s^α.

Fortunately, it is easy to show there is a unique Lorentz-invariant way to contract the indices in (3.6). To show this, first choose a Lorentz frame in which the only non-zero component of λ^α is in the λ^+ direction. This choice preserves a $U(1)\times SU(5)$ subgroup of $SO(10)$, under which a Weyl spinor U^α and an anti-Weyl spinor V_α decompose as

$$U^\alpha \longrightarrow \left(U^+_{\frac{5}{2}}, U_{\frac{1}{2}[ab]}, U^a_{-\frac{3}{2}}\right), \quad V_\alpha \longrightarrow \left(V_{-\frac{5}{2}+}, V^{[ab]}_{-\frac{1}{2}}, V_{+\frac{3}{2}a}\right), \tag{3.7}$$

where the subscript denotes the $U(1)$ charge.

Since $(\lambda^+)^4$ carries $+10$ $U(1)$ charge, $(\overline{\lambda}\gamma^{mnp}D)AWWW$ must carry -10 $U(1)$ charge which is only possible if $(\overline{\lambda}\gamma^{mnp}D)$ carries -3 charge, A_α carries $-\frac{5}{2}$ charge, and each W^α carries $-\frac{3}{2}$ charge. Contracting the $SU(5)$ indices, one finds that the unique $U(1)\times SU(5)$ invariant contraction of the indices is

$$(\lambda^+)^4(\overline{\lambda}\gamma_{abc}D)A_+W^aW^bW^c. \tag{3.8}$$

Returing to covariant notation, one can easily see that (3.6) must be proportional to the Lorentz-invariant expression

$$(\overline{\lambda}\gamma_{mnp}D)(\lambda A)(\lambda\gamma^m W)(\lambda\gamma^n W)(\lambda\gamma^p W), \tag{3.9}$$

which reduces to (3.8) in the frame where λ^+ is the only non-zero component of λ^α.

However, to express the kinematic factor as an integral over pure spinor superspace as in (3.1), it is convenient to have an expression in which all $\overline{\lambda}_\alpha$'s appear in the combination $(\lambda^\alpha\overline{\lambda}_\alpha)$. If all $\overline{\lambda}$'s appear in this combination, one can use that

$$\int d^{16}\theta \int [d\lambda][d\overline{\lambda}][dr]\exp(-\lambda\overline{\lambda}-r\theta)(\lambda\overline{\lambda})^{-n}\lambda^\alpha\lambda^\beta\lambda^\gamma f_{\alpha\beta\gamma} \tag{3.10}$$

is proportional to

$$\langle\lambda^\alpha\lambda^\beta\lambda^\gamma f_{\alpha\beta\gamma}\rangle. \tag{3.11}$$

To convert (3.9) to this form, it is convenient to return to the frame in which λ^+ is the only non-zero component of λ^α and write (3.8) as

$$(\lambda^+)^4\epsilon_{abcde}(\overline{\lambda}^{[de]}D_+ - \overline{\lambda}_+ D^{[de]})A_+W^aW^bW^c. \tag{3.12}$$

Using the superspace equations of motion for A_α and W^α, it is easy to show that

$$D_+A_+ = D_+W^a = 0, \quad D^{[de]}A_+ + D_+A^{[de]} = 0, \quad \epsilon_{abcde}D^{[ab]}W^c = \mathcal{F}_{de}. \tag{3.13}$$

So (3.12) is proportional to two terms which are

$$(\lambda^+)^4\overline{\lambda}_+\epsilon_{abcde}(D_+A^{[de]})W^aW^bW^c \quad \text{and} \quad (\lambda^+)^4\overline{\lambda}_+A_+W^aW^b\mathcal{F}_{ab}. \tag{3.14}$$

The second term in (3.14) can be easily written in covariant language as

$$(\lambda\overline{\lambda})(\lambda A)(\lambda\gamma^m W)(\lambda\gamma^n W)\mathcal{F}_{mn}, \tag{3.15}$$

which produces the desired pure spinor superspace integral of (3.1). And the first term in (3.14) can be written in covariant language as

$$(\lambda\overline{\lambda})\big[(\lambda D)(\lambda\gamma^{mn}A)\big](\lambda\gamma^p W)(W\gamma_{mnp}W), \tag{3.16}$$

which produces the pure spinor superspace integral

$$\langle\big[(\lambda D)(\lambda\gamma^{mn}A)\big](\lambda\gamma^p W)(W\gamma_{mnp}W)\rangle. \tag{3.17}$$

But since BRST-trivial operators decouple,

$$\langle(\lambda D)\big[(\lambda\gamma^{mn}A)(\lambda\gamma^p W)(W\gamma_{mnp}W)\big]\rangle = 0,$$

which implies that (3.17) is equal to

$$\langle(\lambda\gamma^{mn}A)(\lambda D)\big[(\lambda\gamma^p W)(W\gamma_{mnp}W)\big]\rangle. \tag{3.18}$$

Finally, using the superspace equation that $D_\alpha W^\beta$ is proportional to $(\gamma_{mn})_\alpha^{\ \beta}\mathcal{F}^{mn}$, one finds that (3.18) is proportional to (3.1). So the non-minimal computation of the kinematic factor is proportional to the minimal computation of (3.1).

3.2. Two loops

To compute the kinematic factor at two loops using the non-minimal prescription of (2.2), first note that the regulator $\mathcal{N}$ can provide 22 d_α zero modes which are multiplied by the 22 zero modes of s^α. So the remaining 10 d_α zero modes must come from the four integrated vertex operators and the three b_α ghosts. This is only possible if each integrated vertex operators provides a d_α zero mode through the term $(W^\alpha d_\alpha)$ and each b ghost provides two d_α zero modes through the term of (3.3).

After integrating over the zero modes of the dimension one fields $(w^I_\alpha, \overline{w}^{I\alpha}, d^I_\alpha, s^{I\alpha})$ using the measure factors described in [10], one is left with an expression proportional to

$$\int d^{16}\theta \int [d\lambda][d\overline{\lambda}][dr](\lambda\overline{\lambda})^{-6}(\lambda)^6(\overline{\lambda}\gamma^{mnp}r)^3 WWWW \exp(-\lambda\overline{\lambda} - r\theta) \tag{3.19}$$

$$= \int d^{16}\theta \int [d\lambda][d\overline{\lambda}][dr] \exp(-\lambda\overline{\lambda} - r\theta)(\lambda\overline{\lambda})^{-6}(\lambda)^6(\overline{\lambda}\gamma^{mnp}D)^3 WWWW \tag{3.20}$$

where the index contractions on

$$(\lambda)^6(\overline{\lambda}\gamma^{mnp}D)^3 WWWW \tag{3.21}$$

have not been worked out. Note that the factor of $(\lambda)^6$ in (3.19) comes from the $11g$ factors of λ and $\overline{\lambda}$ which multiply the zero modes of d^I_α and s^I_α in $\mathcal{N}$, the factor of $(\lambda)^{-8g}(\overline{\lambda})^{-8g}$ in the measure factor of w^I_α and $\overline{w}^{I\alpha}$, and the factor of $(\overline{\lambda})^{-3g}$ in the measure factor of $s^{I\alpha}$.

As in the one-loop four-point amplitude, there is fortunately a unique way of contracting the indices of (3.21) in a Lorentz-invariant manner. Choosing the Lorentz frame where λ^+ is the only non-zero component of λ^α, one finds that $(\lambda^+)^6$ contributes $+15$ $U(1)$ charge so that each $(\overline{\lambda}\gamma^{mnp}D)$ must contribute -3 charge and each W must contribute $-\frac{3}{2}$ charge. Since the -3 component of $(\overline{\lambda}\gamma^{mnp}D)$ is $(\overline{\lambda}^{[ab]}D_+ - \overline{\lambda}_+ D^{[ab]})$, and since D_+ annihilates the $-\frac{3}{2}$ component of W^α, the only contribution to (3.21) comes from a term of the form

$$(\lambda^+)^6(\overline{\lambda}_+)^3(D^{[ab]}D^{[cd]}D^{[ef]})(W^g W^h W^j W^k) \tag{3.22}$$

where the ten $SU(5)$ indices are contracted with two ϵ_{abcde}'s.

The term of (3.22) produces three types of terms depending on how the three D's act on the four W's. If all three D's act on the same W, one gets a term proportional to $(\lambda^+)^6(\overline{\lambda}_+)^3 WWW\partial\mathcal{F}$, which by $U(1)\times SU(5)$ invariance must have the form

$$(\lambda^+)^6(\overline{\lambda}_+)^3 W^a W^b W^c \partial_a \mathcal{F}_{bc}. \tag{3.23}$$

And if two D's act on the same W, one gets a term proportional to $(\lambda^+)^6(\overline{\lambda}_+)^3\mathcal{F}WW\partial W$, which by $U(1)\times SU(5)$ invariance must have the form

$$(\lambda^+)^6(\overline{\lambda}_+)^3\mathcal{F}_{bc}W^aW^b\partial_aW^c. \tag{3.24}$$

Finally, if each D acts on a different W, one obtains a term that is proportional to $(\lambda^+)^6(\overline{\lambda}_+)^3W\mathcal{F}\mathcal{F}\mathcal{F}$, which by $U(1)\times SU(5)$ invariance must have the form

$$(\lambda^+)^6(\overline{\lambda}_+)^3\mathcal{F}_{ab}\mathcal{F}_{cd}\mathcal{F}_{ef}W^f\epsilon^{abcde}. \tag{3.25}$$

The first term in (3.23) vanishes by Bianchi identities. And the second term in (3.24) is proportional to the first term after integrating by parts with respect to ∂_a and using the equation of motion $\partial_aW^a=0$. So the only contribution to the kinematic factor comes from the third term of (3.25), which can be written in Lorentz-covariant notation as

$$(\lambda\overline{\lambda})^3(\lambda\gamma^{mnpqr}\lambda)\mathcal{F}_{mn}\mathcal{F}_{pq}\mathcal{F}_{rs}(\lambda\gamma^sW). \tag{3.26}$$

So the non-minimal computation of the two-loop kinematic factor agrees with the minimal computation of (3.2).

4. Type-I Anomaly with Pure Spinors

It will now be shown that the non-minimal pure spinor formalism computation of the hexagon gauge anomaly in the Type-I superstring is equivalent to the RNS result of [13]. As will be shown below, the kinematic factor of the hexagon gauge variation can be written as the pure spinor superspace integral

$$K=\langle(\lambda\gamma^mW^2)(\lambda\gamma^nW^3)(\lambda\gamma^pW^4)(W^5\gamma_{mnp}W^6)\rangle,$$

whose bosonic part is the well-known $\epsilon_{10}F^5$ RNS result of [13].

As discussed in [14] [15], the anomaly can be easily computed as a surface term which contributes at the boundary of moduli space. The result can be separated in two parts: the kinematic factor depending only on momenta and polarizations, and the moduli space part which depends on the worldsheet surface. We will be interested only in the kinematic factor, as the moduli space part uses identical computations as in the anomaly analysis using the RNS formalism[3].

[3] A pedagogical presentation of these computations can be found in [16].

4.1. Kinematic factor computation

In the type-I superstring theory with gauge group SO(N), the massless open string six-point one-loop amplitude is given by

$$\mathcal{A} = \sum_{top=P,NP,N} G_{top} \int_0^\infty dt \langle \mathcal{N} \int dw b(w)(\lambda A_1) \prod_{r=2}^{6} \int dz_r U_r(z_r) \rangle \tag{4.1}$$

where P, NP, N denotes the three possible different world-sheet topologies, each of which has a different group-factor G_{top} [17]. When all particles are attached to one boundary, we have a cylinder with $G_P = N\mathrm{tr}(t^{a_1}t^{a_2}t^{a_3}t^{a_4}t^{a_5}t^{a_6})$. When particles are attached to both boundaries, the diagram is a non-planar cylinder, where $G_{NP} = \mathrm{tr}(t^{a_1}t^{a_2})\mathrm{tr}(t^{a_3}t^{a_4}t^{a_5}t^{a_6})$. And finally, there is the non-orientable Möbius strip where $G_N = -\mathrm{tr}(t^{a_1}t^{a_2}t^{a_3}t^{a_4}t^{a_5}t^{a_6})$.

We will be interested in the amplitude when all external states are massless gluons with polarization e^r_m *i.e.*, $a^r_m(x) = e^r_m e^{ik\cdot x}$, where $m = 0, \ldots 9$ is the space-time vector index and r is the particle label [4]. To probe the anomaly, one can compute (4.1) and substitute one of the external polarizations for its respective momentum. However, instead of first computing the six-point amplitude and substituting $e_m \to k_m$ in the answer, we will first make the gauge transformation in (4.1) and then compute the resulting correlation function. This will give us the anomaly kinematic factor directly.

Under the super-Yang-Mills gauge transformation

$$\delta A_\alpha = D_\alpha \Omega, \quad \delta A_m = \partial_m \Omega, \tag{4.2}$$

the integrated vertex operator $\int dzU$ changes by the surface term $\int dz\delta U = \int dz\partial\Omega$, and the unintegrated vertex operator changes by the BRST-trivial quantity $\delta(\lambda A) = \lambda^\alpha D_\alpha \Omega = Q\Omega$. Choosing $\Omega(x,\theta) = e^{ik\cdot x}$ has the same effect as changing $e^m \to k^m$, which is the desired gauge transformation to probe the anomaly.

To compute the gauge anomaly, it will be convenient to choose the gauge transformation to act on the polarization e^1_m in the unintegrated vertex operator, so that the gauge variation of (4.1) is

$$\delta\mathcal{A} = \sum_{top=P,N,NP} G_{top} \int_0^\infty dt \langle \mathcal{N} \int dw b(w)(Q\Omega(z_1)) \prod_{r=2}^{6} \int dz_r U_r(z_r) \rangle. \tag{4.3}$$

[4] We will omit the adjoint gauge group index from the polarizations and field-strengths for the rest of this section.

"Integrating" Q by parts inside the correlation function will only get a contribution from the BRST variation of the b-ghost, which is a derivative with respect to the modulus [14,18]. So

$$\begin{aligned}\delta\mathcal{A} &= -\sum_{top} G_{top}\int_0^\infty dt\frac{d}{dt}\langle\Omega(z_1)\mathcal{N}\prod_{r=2}^{6}\int dz_r U_r(z_r)\rangle \\ &\equiv -K\sum_{top} G_{top}\Big[B_{top}(\infty)-B_{top}(0)\Big],\end{aligned} \tag{4.4}$$

where the moduli space part of the anomaly is encoded in the function

$$B_{top}(t)\equiv\int_0^t dz_6\int_0^{z_6} dz_5\int_0^{z_5} dz_4\int_0^{z_4} dz_3\int_0^{z_3} dz_2\,\langle\prod_{r=1}^{6} :e^{ik_r\cdot x_r}:\rangle_{top},$$

and $K = \langle\mathcal{N}U_2U_3U_4U_5U_6\rangle$. From (4.4), it is clear that the anomaly comes from the boundary of moduli space.

To compute the kinematic factor K, observe that there is an unique way to absorb the 16 zero modes of d_α, 11 of s^α and 11 of r_α. The regularization factor $\mathcal{N}$ must provide 11 d_α, 11 s^α and 11 r_α zero modes. The five remaining d_α zero modes must come from the external vertices[5] through $(d\mathcal{W})^5$. As in the computations of the previous section, the kinematic factor is thus given by a pure spinor superspace integral involving 3 λ's and 5 W's, as can be easily verified by integrating all the zero mode measures except $[d\lambda]$, $[d\overline{\lambda}]$ and $[dr]$. To find out how the indices are contracted in K, choose the reference frame where only $\lambda^+\neq 0$. Then one can easily check that the unique $U(1)\times SU(5)$-invariant contraction is

$$K = \langle(\lambda^+)^3\epsilon_{abcde}W_2^aW_3^bW_4^cW_5^dW_6^e\rangle,$$

which in SO(10)-covariant notation translates into

$$K = \langle(\lambda\gamma^m W_2)(\lambda\gamma^n W_3)(\lambda\gamma^p W_4)(W_5\gamma_{mnp}W_6)\rangle. \tag{4.5}$$

4.2. Bosonic contribution to kinematic factor

When all external states are gluons, there is only one possibility to saturate the pure spinor superspace correlation $\langle\lambda^3\theta^5\rangle$. Each superfield $W^\alpha(\theta)$ must contribute one θ through the term $-\frac{1}{4}(\gamma^{mn}\theta)^\alpha F_{mn}$. Thus, the kinematic factor (4.5) is proportional to

$$\langle(\lambda\gamma^p\gamma^{m_2n_2}\theta)(\lambda\gamma^q\gamma^{m_3n_3}\theta)(\lambda\gamma^r\gamma^{m_4n_4}\theta)(\theta\gamma^{m_5n_5}\gamma_{pqr}\gamma^{m_6n_6}\theta)\rangle F^2_{m_2n_2}\ldots F^6_{m_6n_6}. \tag{4.6}$$

---

[5] It follows from this zero mode counting that the anomaly trivially vanishes for amplitudes with less than six external massless particles.

We will now demonstrate the equivalence with the RNS anomaly result of [13] by proving that

$$\langle(\lambda\gamma^p\gamma^{m_1n_1}\theta)(\lambda\gamma^q\gamma^{m_2n_2}\theta)(\lambda\gamma^r\gamma^{m_3n_3}\theta)(\theta\gamma^{m_4n_4}\gamma_{pqr}\gamma^{m_5n_5}\theta)\rangle = \frac{1}{45}\epsilon^{m_1n_1\ldots m_5n_5}. \tag{4.7}$$

We will first show that the correlation in (4.7) is proportional to ϵ_{10} by checking its behavior under a parity transformation. Using the language of [4], we can rewrite (4.7) as

$$(T^{-1})^{(\alpha\beta\gamma)[\rho_1\rho_2\rho_3\rho_4\rho_5]}T_{(\alpha\beta\gamma)[\delta_1\delta_2\delta_3\delta_4\delta_5]}(\gamma^{m_1n_1})^{\delta_1}{}_{\rho_1}(\gamma^{m_2n_2})^{\delta_2}{}_{\rho_2}(\gamma^{m_3n_3})^{\delta_3}{}_{\rho_3}(\gamma^{m_4n_4})^{\delta_4}{}_{\rho_4}(\gamma^{m_5n_5})^{\delta_5}{}_{\rho_5}, \tag{4.8}$$

where T and T^{-1} are defined by

$$(T^{-1})^{(\alpha_1\alpha_2\alpha_3)[\delta_1\delta_2\delta_3\delta_4\delta_5]} = (\gamma^m)^{\alpha_1\delta_1}(\gamma^n)^{\alpha_2\delta_2}(\gamma^p)^{\alpha_3\delta_3}(\gamma_{mnp})^{\delta_4\delta_5} \tag{4.9}$$

$$T_{(\alpha_1\alpha_2\alpha_3)[\delta_1\delta_2\delta_3\delta_4\delta_5]} = \gamma^m_{\alpha_1\delta_1}\gamma^n_{\alpha_2\delta_2}\gamma^p_{\alpha_3\delta_3}(\gamma_{mnp})_{\delta_4\delta_5},$$

and the α-indices are symmetric and gamma matrix traceless, and the δ-indices are antisymmetric. Since a parity transformation has the effect of changing a Weyl spinor ψ^α to an anti-Weyl spinor ψ_α, it follows from the definitions of (4.9) that a parity transformation exchanges $T \leftrightarrow T^{-1}$. Furthermore, since a parity transformation also changes

$$(\gamma^{mn})^\delta{}_\rho \to (\gamma^{mn})_\delta{}^\rho = -(\gamma^{mn})^\rho{}_\delta,$$

it readily follows that the kinematic factor (4.8) is odd under parity, so it is proportional to ϵ_{10}. Finally, the proportionality constant of $\frac{1}{45}$ in (4.7) can be explicitly computed using the identities listed in Appendix A.

5. t_8 and ϵ_{10} from pure spinor superspace

In this section, we describe some interesting identities involving the t_8 and ϵ_{10} tensors and show how they are closely related when obtained from pure spinor superspace integrals. This is different from computations in the RNS formalism where t_8 and ϵ_{10} come from correlation functions with different spin structures.

Since the one-loop t_8F^4 and $\epsilon_{10}BF_4$ terms are expected to be related by non-linear supersymmetry, there might be a common superspace origin for the t_8 and ϵ_{10} tensors. This suggests looking for a BRST-closed pure spinor superspace integral involving four super-Yang-Mills superfields whose bosonic part involves both the t_8 and ϵ_{10} tensors. One such BRST-closed expression we found is

$$\langle(\lambda\gamma^r W^1)(\lambda\gamma^s W^2)(\lambda\gamma^t W^3)(\theta\gamma^m\gamma^n\gamma_{rst}W^4)\rangle. \tag{5.1}$$

Although (5.1) is not spacetime supersymmetric because of the explicit θ, it might be related to a supersymmetric expression in a constant background where the $N=1$ supergravity superfield $G_{m\alpha}$ satisfies $G_{m\alpha}=\gamma_{m\alpha\beta}\theta^\beta+b_{mn}(\gamma^n\theta)_\alpha$ for constant b_{mn}.

When restricted to its purely bosonic part, (5.1) defines the following 10-dimensional tensor:

$$t_{10}^{mnm_1n_1m_2n_2m_3n_3m_4n_4}=\langle(\lambda\gamma^a\gamma^{m_1n_1}\theta)(\lambda\gamma^b\gamma^{m_2n_2}\theta)(\lambda\gamma^c\gamma^{m_3n_3}\theta)(\theta\gamma^m\gamma^n\gamma_{abc}\gamma^{m_4n_4}\theta)\rangle. \tag{5.2}$$

Using $\gamma^m\gamma^n=\gamma^{mn}+\eta^{mn}$ we obtain

$$\begin{aligned} t_{10}^{mnm_1n_1m_2n_2m_3n_3m_4n_4} &= +\,\langle(\lambda\gamma^a\gamma^{m_1n_1}\theta)(\lambda\gamma^b\gamma^{m_2n_2}\theta)(\lambda\gamma^c\gamma^{m_3n_3}\theta)(\theta\gamma^{mn}\gamma_{abc}\gamma^{m_4n_4}\theta)\rangle \\ &\quad +\eta^{mn}\langle(\lambda\gamma^a\gamma^{m_1n_1}\theta)(\lambda\gamma^b\gamma^{m_2n_2}\theta)(\lambda\gamma^c\gamma^{m_3n_3}\theta)(\theta\gamma_{abc}\gamma^{m_4n_4}\theta)\rangle. \end{aligned} \tag{5.3}$$

And using the identities listed in appendix A, one can check that[6]

$$t_{10}^{mnm_1n_1m_2n_2m_3n_3m_4n_4}=-\frac{2}{45}\Big[\eta^{mn}t_8^{m_1n_1m_2n_2m_3n_3m_4n_4}-\frac{1}{2}\epsilon^{mnm_1n_1m_2n_2m_3n_3m_4n_4}\Big] \tag{5.4}$$

where the t_8 tensor is defined as usual by its contraction with four field-strengths to give

$$\begin{aligned} t_8^{m_1n_1\ldots m_4n_4}F^1_{m_1n_1}\ldots F^4_{m_4n_4} &= +\,8(F^1F^2F^3F^4)+8(F^1F^3F^2F^4)+8(F^1F^3F^4F^2) \\ &\quad -2(F^1F^2)(F^3F^4)-2(F^2F^3)(F^4F^1)-2(F^1F^3)(F^2F^4). \end{aligned}$$

It is also interesting to contrast the similarity between ϵ_{10} and t_8 when written in terms of the T and T^{-1} tensors:

$$\begin{aligned} \epsilon^{mnm_1n_1\ldots m_4n_4} &\propto (T^{-1})^{(\alpha\beta\gamma)[\rho_0\rho_1\rho_2\rho_3\rho_4]}T_{(\alpha\beta\gamma)[\delta_0\delta_1\delta_2\delta_3\delta_4]}(\gamma^{mn})^{\delta_0}{}_{\rho_0}\ldots(\gamma^{m_4n_4})^{\delta_4}{}_{\rho_4} \\ t_8^{m_1n_1\ldots m_4n_4} &\propto (T^{-1})^{(\alpha\beta\gamma)[\kappa\rho_1\rho_2\rho_3\rho_4]}T_{(\alpha\beta\gamma)[\kappa\delta_1\delta_2\delta_3\delta_4]}(\gamma^{m_1n_1})^{\delta_1}{}_{\rho_1}\ldots(\gamma^{m_4n_4})^{\delta_4}{}_{\rho_4}, \end{aligned} \tag{5.5}$$

which shows, in a pure spinor superspace language, how one can "obtain" the t_8 tensor from ϵ_{10}: it is a matter of removing $(\gamma^{mn})^{\delta_0}{}_{\rho_0}$ and contracting the associated spinorial indices in T and T^{-1}. So when using pure spinors, there is a close relation between these two different-looking tensors.

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[6] The sign in front of ϵ_{10} depends on the chirality of θ. For an anti-Weyl θ_α, the sign is "+".

Appendix A. Pure Spinor Superspace Identities

In this appendix we list all the identities used throughout this paper. They were obtained with the inestimable help of Ulf Gran's GAMMA package [19] along with some custom-made functions to handle ϵ_{10} tensors. The convention used for antisymmetrization of n indices is that one must divide by $n!$. Furthermore, it is sometimes more convenient to use the notation $\delta^{a_1\ldots a_n}_{m_1\ldots m_n} = \delta^{[a_1}_{m_1}\ldots\delta^{a_n]}_{m_1}$, e.g.,

$$\delta^{a_1a_2}_{m_1m_2} = \frac{1}{2!}\left(\delta^{a_1}_{m_1}\delta^{a_2}_{m_2} - \delta^{a_2}_{m_1}\delta^{a_1}_{m_2}\right),$$

and – for notational simplicity – not care about the difference between downstairs and upstairs indices in the formulæ.

A.1. Identities ad nauseam

The computation of a correlation like

$$\langle(\lambda\gamma^m\gamma^{m_1n_1}\theta)(\lambda\gamma^n\gamma^{m_2n_2}\theta)(\lambda\gamma^p\gamma^{m_3n_3}\theta)(\theta\gamma^{m_4n_4}\gamma_{mnp}\gamma^{m_5n_5}\theta)\rangle$$

or

$$\langle(\lambda\gamma^m\theta)(\lambda\gamma^a\gamma^{m_1n_1}\theta)(\lambda\gamma^{bcn}\gamma^{m_2n_2}\theta)(\theta\gamma^{m_3n_3}\gamma_{abc}\gamma^{m_4n_4}\theta)\rangle$$

requires a lot of identities, which will be listed below.

We first define $(\theta\gamma^{m_4n_4}\gamma_{mnp}\gamma^{m_5n_5}\theta) = G^{m_4n_4m_5n_5}_{mnpr_1r_2r_3}(\theta\gamma^{r_1r_2r_3}\theta)$. One can check that

$$G^{m_4n_4m_5n_5}_{mnpr_1r_2r_3} = +\frac{1}{6}\epsilon^{mm_4m_5nn_4n_5pr_1r_2r_3} - 24\delta^{np}_{n_4n_5}\delta^{mm_4m_5}_{r_1r_2r_3} + 12\delta^{m_5n_5}_{n_4p}\delta^{mm_4n}_{r_1r_2r_3} \tag{A.1}$$

$$-6\delta^{m_5n_5}_{np}\delta^{mm_4n_4}_{r_1r_2r_3} + 12\delta^{m_4n_4}_{n_5p}\delta^{mm_5n}_{r_1r_2r_3} - 6\delta^{m_4n_4}_{np}\delta^{mm_5n_5}_{r_1r_2r_3} - 2\delta^{m_4n_4}_{m_5n_5}\delta^{mnp}_{r_1r_2r_3} + [mnp] + [m_4n_4] + [m_5n_5],$$

and $+[mnp]+[m_4n_4]+[m_5n_5]$ means that one must antisymmetrize in those indices.

The computation t_8 also requires the identity $(\theta\gamma^{abc}\gamma^{mn}\theta) = (\theta\gamma^{r_1r_2r_3}\theta)K^{abcmn}_{r_1r_2r_3}$, where

$$K^{abcmn}_{r_1r_2r_3} = -\eta^{cn}\delta^{abm}_{r_1r_2r_3} + \eta^{cm}\delta^{abn}_{r_1r_2r_3} + \eta^{bn}\delta^{acm}_{r_1r_2r_3} - \eta^{bm}\delta^{acn}_{r_1r_2r_3} - \eta^{an}\delta^{bcm}_{r_1r_2r_3} + \eta^{am}\delta^{bcn}_{r_1r_2r_3}$$

The following identity is also useful[7]

$$\begin{aligned}(\lambda\gamma^{mnp}\theta)(\lambda\gamma^{qrs}\theta) &= -\frac{1}{96}(\theta\gamma^{tuv}\theta)(\lambda\gamma^{mnp}\gamma_{tuv}\gamma^{qrs}\lambda)\\ &\equiv -\frac{1}{96}(\lambda\gamma^{abcde}\lambda)(\theta\gamma^{tuv}\theta)f^{mnpqrs}_{abcdetuv}\end{aligned} \tag{A.2}$$

[7] This identity was suggested by Pierre Vanhove during discussions of [8].

where

$$f^{mnpqrs}_{abcdetuv} = 18(\delta^{rs}_{uv}\delta^{abcde}_{mnpqt} - \delta^{np}_{uv}\delta^{abcde}_{qrsmt} + \delta^{mn}_{qr}\delta^{abcde}_{pstuv}) \tag{A.3}$$

$$+54(\delta^{nv}_{rs}\delta^{abcde}_{mpqtu} - \delta^{rv}_{np}\delta^{abcde}_{qsmtu} + \delta^{ps}_{tv}\delta^{abcde}_{mnqru}) + [mnp] + [qrs] + [tuv].$$

Using the gamma matrix identities

$$(\lambda\gamma^m\gamma^{np}\theta) = (\lambda\gamma^{mnp}\theta) + \eta^{mn}(\lambda\gamma^p\theta) - \eta^{mp}(\lambda\gamma^n\theta),$$

$$(\lambda\gamma^{abc}\gamma^{de}\theta) = +(\lambda\gamma^{abcde}\theta) - 2\delta^{bc}_{de}(\lambda\gamma^a\theta) + 2\delta^{ac}_{de}(\lambda\gamma^b\theta) - 2\delta^{ab}_{de}(\lambda\gamma^c\theta)$$

$$-\delta^c_e(\lambda\gamma^{abd}\theta) + \delta^c_d(\lambda\gamma^{abe}\theta) + \delta^b_e(\lambda\gamma^{acd}\theta) - \delta^b_d(\lambda\gamma^{ace}\theta) - \delta^a_e(\lambda\gamma^{bcd}\theta) + \delta^a_d(\lambda\gamma^{bce}\theta)$$

and the definitions above, all correlations considered in this paper turn into a linear combination of the following building-blocks:

$$\langle(\lambda\gamma^m\theta)(\lambda\gamma^n\theta)(\lambda\gamma^p\theta)(\theta\gamma_{ijk}\theta)\rangle = \frac{1}{120}\delta^{mnp}_{ijk} \tag{A.4}$$

$$\langle(\lambda\gamma^{mnp}\theta)(\lambda\gamma_q\theta)(\lambda\gamma_t\theta)(\theta\gamma_{ijk}\theta)\rangle = \frac{1}{70}\delta^{[m}_{[q}\eta_{t][i}\delta^n_j\delta^{p]}_{k]} \tag{A.5}$$

$$\langle(\lambda\gamma_t\theta)(\lambda\gamma^{mnp}\theta)(\lambda\gamma^{qrs}\theta)(\theta\gamma_{ijk}\theta)\rangle = \frac{1}{8400}\epsilon^{ijkmnpqrst} + \tag{A.6}$$

$$+\frac{1}{140}\left[\delta^{[m}_t\delta^n_{[i}\eta^{p][q}\delta^r_j\delta^{s]}_{k]} - \delta^{[q}_t\delta^r_{[i}\eta^{s][m}\delta^n_j\delta^{p]}_{k]}\right] - \frac{1}{280}\left[\eta_{t[i}\eta^{v[q}\delta^r_j\eta^{s][m}\delta^n_{k]}\delta^{p]}_v - \eta_{t[i}\eta^{v[m}\delta^n_j\eta^{p][q}\delta^r_{k]}\delta^{s]}_v\right].$$

$$\langle(\lambda\gamma^{mnpqr}\theta)(\lambda\gamma_{stu}\theta)(\lambda\gamma^v\theta)(\theta\gamma_{fgh}\theta) = \frac{1}{35}\eta^{v[m}\delta^n_{[s}\delta^p_t\eta_{u][f}\delta^q_g\delta^{r]}_{h]} - \frac{2}{35}\delta^{[m}_{[s}\delta^n_t\delta^p_{u]}\delta^q_{[f}\delta^{r]}_g\delta^v_{h]} \tag{A.7}$$

$$+\frac{1}{120}\epsilon^{mnpqr}{}_{abcde}\left(\frac{1}{35}\eta^{v[a}\delta^b_{[s}\delta^c_t\eta_{u][f}\delta^d_g\delta^{e]}_{h]} - \frac{2}{35}\delta^{[a}_{[s}\delta^b_t\delta^c_{u]}\delta^d_{[f}\delta^{e]}_g\delta^v_{h]}\right)$$

$$\langle(\lambda\gamma^{mnpqr}\lambda)(\lambda\gamma^u\theta)(\theta\gamma_{fgh}\theta)(\theta\gamma_{jkl}\theta)\rangle = \tag{A.8}$$

$$-\frac{4}{35}\left[\delta^{[m}_{[j}\delta^n_k\delta^p_{l]}\delta^q_{[f}\delta^{r]}_g\delta^u_{h]} + \delta^{[m}_{[f}\delta^n_g\delta^p_{h]}\delta^q_{[j}\delta^{r]}_k\delta^u_{l]} - \frac{1}{2}\delta^{[m}_{[j}\delta^n_k\eta_{l][f}\delta^p_g\delta^q_{h]}\eta^{r]u} - \frac{1}{2}\delta^{[m}_{[f}\delta^n_g\eta_{h][j}\delta^p_k\delta^q_{l]}\eta^{r]u}\right]$$

$$-\frac{1}{1050}\epsilon^{mnpqr}{}_{abcde}\Big[\delta^{[a}_{[j}\delta^b_k\delta^c_{l]}\delta^d_{[f}\delta^{e]}_g\delta^u_{h]} + \delta^{[a}_{[f}\delta^b_g\delta^c_{h]}\delta^d_{[j}\delta^{e]}_k\delta^u_{l]}$$

$$-\frac{1}{2}\delta^{[a}_{[j}\delta^b_k\eta_{l][f}\delta^c_g\delta^d_{h]}\eta^{e]u} - \frac{1}{2}\delta^{[a}_{[f}\delta^b_g\eta_{h][j}\delta^c_k\delta^d_{l]}\eta^{e]u}\Big]$$

$$\langle(\lambda\gamma^{mnpqr}\theta)(\lambda\gamma_d\theta)(\lambda\gamma_e\theta)(\theta\gamma_{fgh}\theta) = -\frac{1}{42}\delta^{mnpqr}_{defgh} - \frac{1}{5040}\epsilon^{mnpqr}{}_{defgh} \tag{A.9}$$

$$\langle(\lambda\gamma^{mnpqr}\lambda)(\lambda\gamma^{stu}\theta)(\theta\gamma_{fgh}\theta)(\theta\gamma_{jkl}\theta)\rangle = \tag{A.10}$$

$$-\frac{12}{35}\Big[\delta^{[s}_{[f}\delta^t_g\eta^{u][m}\delta^n_{h]}\delta^p_{[j}\delta^q_k\delta^{r]}_{l]} + \delta^{[s}_{[j}\delta^t_k\eta^{u][m}\delta^n_{l]}\delta^p_{[f}\delta^q_g\delta^{r]}_{h]}$$

$$
-\eta^{v[s}\delta^{t}_{[f}\eta^{u][m}\delta^{n}_{g}\eta_{h][j}\delta^{p}_{k}\delta^{q}_{l]}\delta^{r]}_{v} - \eta^{v[s}\delta^{t}_{[j}\eta^{u][m}\delta^{n}_{k}\eta_{l][f}\delta^{p}_{g}\delta^{q}_{h]}\delta^{r]}_{v}\Big]
$$

$$
-\frac{1}{350}\epsilon^{mnpqr}{}_{abcde}\Big[\delta^{[s}_{[f}\delta^{t}_{g}\eta^{u][a}\delta^{b}_{h]}\delta^{c}_{[j}\delta^{d}_{k}\delta^{e]}_{l]} + \delta^{[s}_{[j}\delta^{t}_{k}\eta^{u][a}\delta^{b}_{l]}\delta^{c}_{[f}\delta^{d}_{g}\delta^{e]}_{h]}
$$

$$
-\eta^{v[s}\delta^{t}_{[f}\eta^{u][a}\delta^{b}_{g}\eta_{h][j}\delta^{c}_{k}\delta^{d}_{l]}\delta^{e]}_{v} - \eta^{v[s}\delta^{t}_{[j}\eta^{u][a}\delta^{b}_{k}\eta_{l][f}\delta^{c}_{g}\delta^{d}_{h]}\delta^{e]}_{v}\Big].
$$

$$
\langle(\lambda\gamma^{mnp}\theta)(\lambda\gamma^{qrs}\theta)(\lambda\gamma_{tuv}\theta)(\theta\gamma_{ijk}\theta)\rangle = \tag{A.11}
$$

$$
-\frac{3}{175}\Big[-\delta^{[i}_{a}\delta^{j}_{[q}\delta^{k]}_{r}\delta^{[m}_{s]}\delta^{n}_{[t}\delta^{p]}_{u}\delta^{a}_{v]} + \delta^{[i}_{a}\delta^{j}_{[t}\delta^{k]}_{u}\delta^{[m}_{v]}\delta^{n}_{[q}\delta^{p]}_{r}\delta^{a}_{s]} + \delta^{[i}_{[q}\delta^{j}_{r}\eta^{k][m}\eta_{s][t}\delta^{n}_{u}\delta^{p]}_{v]}
$$

$$
+\delta^{a}_{[t}\eta^{b[i}\delta^{j}_{u}\eta^{k][m}\eta_{v][q}\delta^{n}_{r}\eta_{s]a}\delta^{p]}_{b} - \delta^{a}_{[q}\eta^{b[i}\delta^{j}_{r}\eta^{k][m}\eta_{s][t}\delta^{n}_{u}\eta_{v]a}\delta^{p]}_{b} - \delta^{[i}_{[t}\delta^{j}_{u}\eta^{k][m}\eta_{v][q}\delta^{n}_{r}\delta^{p]}_{s]}\Big]
$$

$$
+\frac{1}{33600}\epsilon^{abcde}{}_{a_1a_2a_3a_4a_5}f^{mnpqrs}_{abcdefgh}\Big[\delta^{[t}_{[f}\delta^{u}_{g}\eta^{v][a_1}\delta^{a_2}_{h]}\delta^{a_3}_{[i}\delta^{a_4}_{j}\delta^{a_5]}_{k]} + \delta^{[t}_{[i}\delta^{u}_{j}\eta^{v][a_1}\delta^{a_2}_{k]}\delta^{a_3}_{[f}\delta^{a_4}_{g}\delta^{a_5]}_{h]}
$$

$$
-\eta^{z[t}\delta^{u}_{[f}\eta^{v][a_1}\delta^{a_2}_{g}\eta_{h][i}\delta^{a_3}_{j}\delta^{a_4}_{k]}\delta^{a_5]}_{z} - \eta^{z[t}\delta^{u}_{[i}\eta^{v][a_1}\delta^{a_2}_{j}\eta_{k][f}\delta^{a_3}_{g}\delta^{a_4}_{h]}\delta^{a_5]}_{z}\Big].
$$

These identities can be straightforwardly derived. The recipe is the following. One writes the most general tensor containing Kronecker deltas with the same symmetry properties as the left hand side and then contracts some appropriate indices to find the coefficients which satisfy the normalization $\langle\lambda^3\theta^5\rangle = 1$. After obtaining all terms containing only Kronecker deltas one can find terms with ϵ_{10} tensors considering the duality properties of the gamma matrices:

$$
(\gamma^{m_1m_2m_3m_4m_5})_{\alpha\beta} = +\frac{1}{5!}\epsilon^{m_1m_2m_3m_4m_5n_1n_2n_3n_4n_5}(\gamma_{n_1n_2n_3n_4n_5})_{\alpha\beta},
$$

$$
(\gamma^{m_1m_2m_3m_4m_5m_6})_{\alpha}{}^{\beta} = +\frac{1}{4!}\epsilon^{m_1m_2m_3m_4m_5m_6n_1n_2n_3n_4}(\gamma_{n_1n_2n_3n_4})_{\alpha}{}^{\beta},
$$

$$
(\gamma^{m_1m_2m_3m_4m_5m_6m_7})_{\alpha\beta} = -\frac{1}{3!}\epsilon^{m_1m_2m_3m_4m_5m_6m_7n_1n_2n_3}(\gamma_{n_1n_2n_3})_{\alpha\beta},
$$

$$
(\gamma^{m_1m_2m_3m_4m_5m_6m_7m_8})_{\alpha}{}^{\beta} = -\frac{1}{2!}\epsilon^{m_1m_2m_3m_4m_5m_6m_7m_8n_1n_2}(\gamma_{n_1n_2})_{\alpha}{}^{\beta}.
$$

The following identities turn out to be useful when doing all these manipulations and can be derived using the properties of pure spinors and gamma matrices:

$$
(\gamma^{mnp})_{\alpha\beta}(\gamma_{mnp})^{\gamma\delta} = 48\left(\delta^{\gamma}_{\alpha}\delta^{\delta}_{\beta} - \delta^{\gamma}_{\beta}\delta^{\delta}_{\alpha}\right), \qquad (\lambda\gamma_m\psi)(\lambda\gamma^m\xi) = 0 \quad \forall\psi^\alpha,\ \xi^\alpha \tag{A.12}
$$

$$
(\lambda\gamma^{mnpqr}\lambda)(\lambda\gamma_{mna}\theta) = 0, \qquad (\lambda\gamma^{mnpqr}\lambda)(\lambda\gamma_m\theta) = 0 \tag{A.13}
$$

$$
(\lambda\gamma^{amn}\theta)(\lambda\gamma_a\theta) = 2(\lambda\gamma^m\theta)(\lambda\gamma^n\theta), \qquad (\lambda\gamma^{abm}\theta)(\lambda\gamma^{abn}\theta) = -4(\lambda\gamma^m\theta)(\lambda\gamma^n\theta) \tag{A.14}
$$

$$(\lambda\gamma^{mabcn}\lambda)(\theta\gamma_{abc}\theta) = 96(\lambda\gamma^m\theta)(\lambda\gamma^n\theta), \tag{A.15}$$

$$(\lambda\gamma^{abcmn}\theta)(\lambda\gamma_{abc}\theta) = -36(\lambda\gamma^m\theta)(\lambda\gamma^n\theta), \tag{A.16}$$

$$(\lambda\gamma^a\gamma^{bcmn}\theta)(\lambda\gamma_{abc}\theta) = -28(\lambda\gamma^m\theta)(\lambda\gamma^n\theta), \tag{A.17}$$

$$\begin{aligned}(\lambda\gamma^{abc}\theta)(\lambda\gamma^{ade}\theta) =& -(\lambda\gamma^{cde}\theta)(\lambda\gamma^b\theta) + (\lambda\gamma^{bde}\theta)(\lambda\gamma^c\theta) + (\lambda\gamma^{bce}\theta)(\lambda\gamma^d\theta)\\ &-(\lambda\gamma^{bcd}\theta)(\lambda\gamma^e\theta) - \eta^{ce}(\lambda\gamma^b\theta)(\lambda\gamma^d\theta) + \eta^{cd}(\lambda\gamma^b\theta)(\lambda\gamma^e\theta)\\ &+\eta^{be}(\lambda\gamma^c\theta)(\lambda\gamma^d\theta) - \eta^{bd}(\lambda\gamma^c\theta)(\lambda\gamma^e\theta)\end{aligned} \tag{A.18}$$

$$\begin{aligned}(\lambda\gamma^{abcde}\theta)(\lambda\gamma^{agh}\theta) =& +(\lambda\gamma^{hbcde}\theta)(\lambda\gamma^g\theta) - (\lambda\gamma^{gbcde}\theta)(\lambda\gamma^h\theta) + (\lambda\gamma^{bcd}\theta)(\lambda\gamma^{egh}\theta)\\ &-(\lambda\gamma^{bce}\theta)(\lambda\gamma^{dgh}\theta) + (\lambda\gamma^{bde}\theta)(\lambda\gamma^{cgh}\theta) - (\lambda\gamma^{cde}\theta)(\lambda\gamma^{bgh}\theta)\\ &-\eta^{he}(\lambda\gamma^{bcd}\theta)(\lambda\gamma^g\theta) + \eta^{hd}(\lambda\gamma^{bce}\theta)(\lambda\gamma^g\theta) - \eta^{hc}(\lambda\gamma^{bde}\theta)(\lambda\gamma^g\theta)\\ &+\eta^{hb}(\lambda\gamma^{cde}\theta)(\lambda\gamma^g\theta) + \eta^{ge}(\lambda\gamma^{bcd}\theta)(\lambda\gamma^h\theta) - \eta^{gd}(\lambda\gamma^{bce}\theta)(\lambda\gamma^h\theta)\\ &+\eta^{gc}(\lambda\gamma^{bde}\theta)(\lambda\gamma^h\theta) - \eta^{gb}(\lambda\gamma^{cde}\theta)(\lambda\gamma^h\theta)\end{aligned}$$

$$\begin{aligned}(\lambda\gamma^{abcde}\theta)(\lambda\gamma^{abh}\theta) =& -4\eta^{eh}(\lambda\gamma^c\theta)(\lambda\gamma^d\theta) + 4\eta^{dh}(\lambda\gamma^c\theta)(\lambda\gamma^e\theta) - 4\eta^{hc}(\lambda\gamma^d\theta)(\lambda\gamma^e\theta)\\ &-2(\lambda\gamma^{cde}\theta)(\lambda\gamma^h\theta)\end{aligned}$$

A.2. Other pure spinor representations for t_8 and ϵ_{10}

The following correlations also give rise to identities for t_8 and ϵ_{10},

$$\langle(\lambda\gamma^m\theta)(\lambda\gamma_a W^1)(\lambda\gamma_b W^2)(W^3\gamma^{abn}W^4)\rangle + \mathrm{perm}(1234),$$

$$\langle(\lambda\gamma^a W^1)(\lambda\gamma^b W^2)(\lambda\gamma^n W^3)(\theta\gamma^m\gamma_{ab}W^4)\rangle + \mathrm{perm}(1234).$$

Indeed one can show that

$$\langle(\lambda\gamma^{[m|}\theta)(\lambda\gamma_a\gamma^{m_1n_1}\theta)(\lambda\gamma_b\gamma^{m_2n_2}\theta)(\theta\gamma^{m_3n_3}\gamma^{ab|n]}\gamma^{m_4n_4}\theta)\rangle + \mathrm{p}(1234) = -\frac{116}{525}\epsilon^{mnm_1n_1\ldots m_4n_4}$$

$$\eta_{mn}\langle(\lambda\gamma^m\theta)(\lambda\gamma_a\gamma^{m_1n_1}\theta)(\lambda\gamma_b\gamma^{m_2n_2}\theta)(\theta\gamma^{m_3n_3}\gamma^{abn}\gamma^{m_4n_4}\theta)\rangle + \mathrm{p}(1234) = \frac{16}{15}t_8^{m_1n_1\ldots m_4n_4}$$

$$\langle(\lambda\gamma^a\gamma^{m_1n_1}\theta)(\lambda\gamma^b\gamma^{m_2n_2}\theta)(\lambda\gamma^{[n}\gamma^{m_3n_3}\theta)(\theta\gamma^{m]}\gamma_{ab}\gamma^{m_4n_4}\theta)\rangle + \mathrm{p}(1234) = \frac{2}{175}\epsilon^{mnm_1n_1\ldots m_4n_4}$$

$$\eta_{mn}\langle(\lambda\gamma^a\gamma^{m_1n_1}\theta)(\lambda\gamma^b\gamma^{m_2n_2}\theta)(\lambda\gamma^n\gamma^{m_3n_3}\theta)(\theta\gamma^m\gamma_{ab}\gamma^{m_4n_4}\theta)\rangle + \mathrm{p}(1234) = -\frac{16}{15}t_8^{m_1n_1\ldots m_4n_4}.$$

References

[1] K. Lee and W. Siegel, "Conquest of the ghost pyramid of the superstring," arXiv:hep-th/0506198.

[2] K. Lee and W. Siegel, "Simpler superstring scattering," arXiv:hep-th/0603218.

[3] N. Berkovits, "Super-Poincare covariant quantization of the superstring," JHEP **0004**, 018 (2000) [arXiv:hep-th/0001035].

[4] N. Berkovits, "Multiloop amplitudes and vanishing theorems using the pure spinor formalism for the superstring," JHEP **0409**, 047 (2004) [arXiv:hep-th/0406055].

[5] N. Berkovits, "Super-Poincare covariant two-loop superstring amplitudes," JHEP **0601**, 005 (2006) [arXiv:hep-th/0503197].

[6] L. Anguelova, P. A. Grassi and P. Vanhove, "Covariant one-loop amplitudes in D = 11," Nucl. Phys. B **702**, 269 (2004) [arXiv:hep-th/0408171].

[7] B. C. Vallilo and N. Berkovits, "Consistency of super-Poincare covariant superstring tree amplitudes," JHEP **0007**, 015 (2000) [arXiv:hep-th/0004171].

[8] C. R. Mafra, "Four-point one-loop amplitude computation in the pure spinor formalism," JHEP **0601**, 075 (2006) [arXiv:hep-th/0512052].

[9] N. Berkovits and C. R. Mafra, "Equivalence of two-loop superstring amplitudes in the pure spinor and RNS formalisms," Phys. Rev. Lett. **96**, 011602 (2006) [arXiv:hep-th/0509234].

[10] N. Berkovits, "Pure spinor formalism as an N = 2 topological string," JHEP **0510**, 089 (2005) [arXiv:hep-th/0509120].

[11] E. Witten, "Twistor - Like Transform In Ten-Dimensions," Nucl. Phys. B **266**, 245 (1986). ;
W. Siegel, "Superfields In Higher Dimensional Space-Time," Phys. Lett. B **80**, 220 (1979).

[12] H. Ooguri, J. Rahmfeld, H. Robins and J. Tannenhauser, "Holography in superspace," JHEP **0007**, 045 (2000) [arXiv:hep-th/0007104]. ;
P. A. Grassi and L. Tamassia, "Vertex operators for closed superstrings," JHEP **0407**, 071 (2004) [arXiv:hep-th/0405072]. ;
G. Policastro and D. Tsimpis, "R**4, purified," arXiv:hep-th/0603165.

[13] M. B. Green and J. H. Schwarz, "The Hexagon Gauge Anomaly In Type I Superstring Theory," Nucl. Phys. B **255**, 93 (1985).

[14] D. Friedan, E. J. Martinec and S. H. Shenker, "Conformal Invariance, Supersymmetry And String Theory," Nucl. Phys. B **271**, 93 (1986).

[15] J. Polchinski and Y. Cai, "Consistency Of Open Superstring Theories," Nucl. Phys. B **296**, 91 (1988).

[16] J. Liu, "Gauge And Gravitational Anomaly Cancellation In Type I SO(32) Superstring Theory," Nucl. Phys. B **362**, 141 (1991).

[17] Green, M. B., Schwarz, J. H., Witten, E. *Superstring Theory: 2. Loop Amplitudes, Anomalies & Phenomenology.* Cambridge University Press (1987)

[18] E. D'Hoker and D. H. Phong, "The Geometry Of String Perturbation Theory," Rev. Mod. Phys. **60**, 917 (1988).

[19] U. Gran, "GAMMA: A Mathematica package for performing Gamma-matrix algebra and Fierz transformations in arbitrary dimensions," arXiv:hep-th/0105086.