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Four-Point One-Loop Amplitude Computation in the Pure Spinor Formalism

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The massless 4-point one-loop amplitude computation in the pure spinor formalism is shown to agree with the computation in the RNS formalism.

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1. Introduction

In the year 2000, Berkovits proposed a new formalism for the superstring with manifest space-time supersymmetry that can be covariantly quantized[1]. Since then, the formalism has evolved to a point where multiloop superstring amplitudes are computed in a manifestly super-Poincaré manner [2] with relative easy when compared with the RNS formalism. In the last five years there have been lots of consistency checks, and up to now the pure spinor formalism has bravely survived. The last one of these checks was the agreement with the RNS result for massless 4-point two-loop amplitudes [3][4][5][6] (see also [7]).

The one-loop agreement has already been considered in [8], where it was argued that the pure spinor amplitude coincides with the RNS result of [9] for constant field-strength. However, we will show that there are subtleties in the computation at zero momentum and that the naive computation of [8] gives the wrong answer. In this paper we will perform this computation for non-constant field-strength and will obtain complete agreement with the RNS computation.

2. Massless 4-point one-loop amplitude in the pure spinor formalism

In [2] Berkovits obtained the following formula for the massless 4-point one-loop amplitude for the type-IIB superstring, which we rewrite in a slightly different fashion as,

$$\mathcal{A} = K\overline{K} \int \frac{d^2\tau}{(\mathrm{Im}\tau)^2} F_c(\tau), \tag{2.1}$$

where $F_c(\tau)$ is a modular invariant function defined by [10]

$$F_c(\tau) = \frac{1}{(\mathrm{Im}\tau)^3} \int d^2 z_2 \int d^2 z_3 \int d^2 z_4 \prod_{i<j} G(z_i, z_j)^{k_i \cdot k_j},$$

$G(z_i, z_j)$ is the scalar Green's function and K is a kinematic factor which reads

$$K = \int d^{16}\theta (\epsilon T^{-1})^{((\alpha\beta\gamma))}_{[\rho_1 \dots \rho_{11}]} \theta^{\rho_1} \dots \theta^{\rho_{11}} (\gamma_{mnpqr})_{\beta\gamma} A_{1\alpha}(\theta)(W_2(\theta)\gamma^{mnp} W^3(\theta))\mathcal{F}_4^{qr}(\theta) + \mathrm{perm}(234).$$

Using the same trick of [4], where $\int d^{16}\theta (\epsilon T^{-1})^{((\alpha\beta\gamma))}_{[\rho_1 \dots \rho_{11}]} \theta^{\rho_1} \dots \theta^{\rho_{11}} f_{\alpha\beta\gamma}$ is expressed as the tree-level pure spinor correlator $\langle \lambda^\alpha \lambda^\beta \lambda^\gamma (\theta)^5 D^5 f_{\alpha\beta\gamma} \rangle$, K can be rewritten as

$$K = \langle (\theta)^5 D^5 (\lambda A^1)(\lambda\gamma^{[m} W^2)(\lambda\gamma^{n]} W^3) \mathcal{F}^4_{mn} \rangle + \mathrm{perm}(234). \tag{2.2}$$

When all external states are in the Neveu-Schwarz sector (2.2) will be shown to coincide with the well-known RNS result, i.e., $K^{4-\mathrm{NS}} \propto t_8 F^1 F^2 F^3 F^4$, where the t_8-tensor is defined in [10].

3. Equivalence with the RNS formalism

Since $A_\alpha(\theta)$ and $W^\alpha(\theta)$ are fermionic while $\mathcal{F}(\theta)$ is bosonic, the contributions when all external states are NS come from terms in which an odd (even) number of covariant derivatives act upon the fermionic (bosonic) superfields. One therefore has

$$K^{4-\mathrm{NS}} = (\theta)^5 \Big[20D^3(\lambda A^1)(\lambda\gamma^{[m}DW^2)(\lambda\gamma^{n]}DW^3)\mathcal{F}^4_{mn} + \tag{3.1}$$

$$+60D(\lambda A^1)(\lambda\gamma^{[m}DW^2)(\lambda\gamma^{n]}DW^3)D^2\mathcal{F}^4_{mn} +$$

$$+20D(\lambda A^1)(\lambda\gamma^{[m}D^3W^2)(\lambda\gamma^{n]}DW^3)\mathcal{F}^4_{mn} + 20D(\lambda A^1)(\lambda\gamma^{[m}DW^2)(\lambda\gamma^{n]}D^3W^3)\mathcal{F}^4_{mn}\Big],$$

where the spinor indices of $(\theta)^5$ are contracted with the covariant derivatives and the combinatoric factors in (3.1) come from the different ways of splitting up these five indices.

Using the following relations,

$$\theta^\alpha D_\alpha W^\beta = -\frac{1}{4}(\gamma^{mn}\theta)^\beta \mathcal{F}_{mn},$$

$$\theta^\alpha\theta^\beta D_\alpha D_\beta \mathcal{F}^{mn} = \frac{1}{4}k^{[m}(\theta\gamma^{n]tu}\theta)\mathcal{F}_{tu},$$

$$\theta^{\alpha_1}\theta^{\alpha_2}\theta^{\alpha_3}D_{\alpha_1}D_{\alpha_2}D_{\alpha_3}W^\beta = -\frac{1}{8}(\gamma^{mn}\theta)^\beta k_m(\theta\gamma_n\gamma^{pq}\theta)\mathcal{F}_{pq}$$

$$\theta^{\alpha_1}\theta^{\alpha_2}\theta^{\alpha_3}D_{\alpha_1}D_{\alpha_2}D_{\alpha_3}(\lambda A) = \frac{3}{16}F_{mn}(\lambda\gamma_p\theta)(\theta\gamma^{mnp}\theta),$$

where [3] $D_\alpha = \partial_\alpha + \frac{1}{2}k_m(\gamma^m\theta)_\alpha$ and

$$\lambda^\alpha A_\alpha(\theta) = \frac{1}{2}e_m(\lambda\gamma^m\theta) - \frac{1}{3}(\xi\gamma_m\theta)(\lambda\gamma^m\theta) - \frac{1}{32}F_{mn}(\lambda\gamma_p\theta)(\theta\gamma^{mnp}\theta) + \dots$$

equation (3.1) becomes,

$$K_1^{NS} = +\frac{15}{64}F^1_{mn}F^2_{pq}F^3_{rs}F^4_{tu}\langle(\lambda\gamma^{[t}\gamma^{pq}\theta)(\lambda\gamma^{u]}\gamma^{rs}\theta)(\lambda\gamma_a\theta)(\theta\gamma^{mna}\theta)\rangle + \tag{3.2}$$

$$+\frac{15}{16}(k^4_m e^1_n)F^2_{pq}F^3_{rs}F^4_{tu}\langle(\lambda\gamma^{[m}\gamma^{pq}\theta)(\lambda\gamma^{a]}\gamma^{rs}\theta)(\lambda\gamma^n\theta)(\theta\gamma_a\gamma^{tu}\theta)\rangle +$$

$$+\frac{5}{16}(k^2_m e^1_n)F^2_{pq}F^3_{rs}F^4_{tu}\langle(\lambda\gamma^{[t}\gamma^{ma}\theta)(\lambda\gamma^{u]}\gamma^{rs}\theta)(\lambda\gamma^n\theta)(\theta\gamma_a\gamma^{pq}\theta)\rangle +$$

$$+\frac{5}{16}(k^3_m e^1_n)F^2_{pq}F^3_{rs}F^4_{tu}\langle(\lambda\gamma^{[t}\gamma^{pq}\theta)(\lambda\gamma^{u]}\gamma^{ma}\theta)(\lambda\gamma^n\theta)(\theta\gamma_a\gamma^{rs}\theta)\rangle.$$

In [8] the authors ignored the last three lines of (3.2) by considering a constant field-strength and have reported to obtain the correct RNS result. However, we will show that this does not happen. Agreement with the RNS formalism is only obtained after summing up all contributions in (3.2), and the inability to get the correct result supposing F_{mn} constant may be related to subtleties in amplitude computations at zero momentum, as will be commented in the last section.

Using the identity $\gamma^m\gamma^{np} = \gamma^{mnp} + \eta^{m[n}\gamma^{p]}$ one can check that three types of correlation functions[2] will be needed to evaluate (3.2)

$$\langle(\lambda\gamma_t\theta)(\lambda\gamma^{mnp}\theta)(\lambda\gamma^{qrs}\theta)(\theta\gamma_{ijk}\theta)\rangle = C\epsilon^{ijkmnpqrst}+ \tag{3.3}$$

$$+A\Big[\delta_t^{[m}\delta_{[i}^{n}\eta^{p][q}\delta_j^r\delta_{k]}^{s]} - \delta_t^{[q}\delta_{[i}^{r}\eta^{s][m}\delta_j^n\delta_{k]}^{p]}\Big] + B\Big[\eta_{t[i}\eta^{v[q}\delta_j^r\eta^{s][m}\delta_{k]}^n\delta_v^{p]} - \eta_{t[i}\eta^{v[m}\delta_j^n\eta^{p][q}\delta_{k]}^r\delta_v^{s]}\Big]$$

$$\langle(\lambda\gamma^{mnp}\theta)(\lambda\gamma_q\theta)(\lambda\gamma_t\theta)(\theta\gamma_{ijk}\theta)\rangle = \frac{1}{70}\delta_{[q}^{[m}\eta_{t][i}\delta_j^n\delta_{k]}^{p]} \tag{3.4}$$

$$\langle(\lambda\gamma^m\theta)(\lambda\gamma^n\theta)(\lambda\gamma^p\theta)(\theta\gamma_{ijk}\theta)\rangle = \frac{1}{120}\delta_{ijk}^{mnp}, \tag{3.5}$$

where $A = -2B = \frac{1}{140}$, $C = \frac{1}{8400}$, as will be shown in the sequence. Furthermore, it is not difficult to justify (3.4) and (3.5) by noting that they are the only possible linear combinations of η_{mn} tensors that have the appropriate symmetries and are compatible with the properties of the pure spinor λ^α. Moreover, they are normalized such that

$$\langle(\lambda\gamma_m\theta)(\lambda\gamma_n\theta)(\lambda\gamma_p\theta)(\theta\gamma^{mnp}\theta)\rangle = 1. \tag{3.6}$$

The following identity

$$(\lambda\gamma^{mnp}\theta)(\lambda\gamma^{qrs}\theta) = -\frac{1}{32\cdot 5!}(\lambda\gamma^{abcde}\lambda)(\theta\gamma^{mnp}\gamma_{abcde}\gamma^{qrs}\theta)$$

$$\equiv -\frac{1}{32\cdot 5!}(\lambda\gamma^{abcde}\lambda)(\theta\gamma^{tuv}\theta)f^{mnpqrs}_{abcdetuv}, \tag{3.7}$$

will allow one to determine both coefficients A and B. From (3.7) it follows that

$$\langle(\lambda\gamma_t\theta)(\lambda\gamma^{mnp}\theta)(\lambda\gamma^{qrs}\theta)(\theta\gamma_{ijk}\theta)\rangle = -\frac{1}{3840}\langle(\lambda\gamma^{abcde}\lambda)(\lambda\gamma_t\theta)(\theta\gamma_{ijk}\theta)(\theta\gamma^{uvx}\theta)\rangle f^{mnpqrs}_{abcdeuvx}, \tag{3.8}$$

[2] In version 3 of [8], equation (3.3) was not correctly obtained since all deltas in the right hand side were ignored. Their identity for (3.4) was also wrong. After being informed of these facts, Pierre Vanhove has independently obtained a much simpler way to obtain the coefficients A and B than the one presented here [11].

where the correlation function in the right hand side of (3.8) has already been determined, up to terms involving Levi-Civita's epsilons, to be [4]

$$\langle(\lambda\gamma^{mnpqr}\lambda)(\lambda\gamma^{u}\theta)(\theta\gamma_{fgh}\theta)(\theta\gamma_{jkl}\theta)\rangle = \tag{3.9}$$

$$= -\frac{4}{35}\Big[\delta^{[m}_{[j}\delta^{n}_{k}\delta^{p}_{l]}\delta^{q}_{[f}\delta^{r]}_{g}\delta^{u}_{h]} + \delta^{[m}_{[f}\delta^{n}_{g}\delta^{p}_{h]}\delta^{q}_{[j}\delta^{r]}_{k}\delta^{u}_{l]} - \frac{1}{2}\delta^{[m}_{[j}\delta^{n}_{k}\eta_{l][f}\delta^{p}_{g}\delta^{q}_{h]}\eta^{r]u} - \frac{1}{2}\delta^{[m}_{[f}\delta^{n}_{g}\eta_{h][j}\delta^{p}_{k}\delta^{q}_{l]}\eta^{r]u}\Big].$$

In [4] it was argued that all terms containing Levi-Civita's epsilons in the correlation function (3.9) would not contribute to the two-loop amplitude under consideration and were safely ignored. However, in the present application of (3.9) these epsilon-terms will contribute non-epsilon terms to the left hand side of (3.8) when they are contracted with epsilons contained in $f^{mnpqrs}_{abcdetuv}$. One therefore needs to determine them, which can be easily done by considering the self-duality condition

$$\gamma^{mnpqr}_{\alpha\beta} = \frac{1}{120}\epsilon^{mnpqrstuvx}\left(\gamma_{stuvx}\right)_{\alpha\beta}\,, \tag{3.10}$$

because it will relate non-epsilon with epsilon terms in (3.9). Using (3.10) one can obtain the complete correlation function (3.9) that, when written out explicitly, is given by

$$\langle(\lambda\gamma^{mnpqr}\lambda)\,(\lambda\gamma^{u}\theta)\,(\theta\gamma_{fgh}\theta)\,(\theta\gamma_{jkl}\theta)\rangle = \tag{3.11}$$

$$\begin{aligned}
&+\frac{4}{105}\Big[-\delta^{l}_{u}\delta^{fghjk}_{mnpqr} + \delta^{k}_{u}\delta^{fghjl}_{mnpqr} - \delta^{j}_{u}\delta^{fghkl}_{mnpqr} - \delta^{h}_{u}\delta^{fgjkl}_{mnpqr} + \delta^{g}_{u}\delta^{fhjkl}_{mnpqr}\\
&-\delta^{f}_{u}\delta^{ghjkl}_{mnpqr} + \frac{1}{3}\delta^{h}_{l}\delta^{fgjku}_{mnpqr} - \frac{1}{3}\delta^{h}_{k}\delta^{fgjlu}_{mnpqr} + \frac{1}{3}\delta^{h}_{j}\delta^{fgklu}_{mnpqr} - \frac{1}{3}\delta^{g}_{l}\delta^{fhjku}_{mnpqr}\\
&+\frac{1}{3}\delta^{g}_{k}\delta^{fhjlu}_{mnpqr} - \frac{1}{3}\delta^{g}_{j}\delta^{fhklu}_{mnpqr} + \frac{1}{3}\delta^{f}_{l}\delta^{ghjku}_{mnpqr} - \frac{1}{3}\delta^{f}_{k}\delta^{ghjlu}_{mnpqr} + \frac{1}{3}\delta^{f}_{j}\delta^{ghklu}_{mnpqr}\Big]\\
&+\frac{1}{3150}\Big[+\delta^{l}_{u}\epsilon^{fghjkmnpqr} - \delta^{k}_{u}\epsilon^{fghjlmnpqr} + \delta^{j}_{u}\epsilon^{fghklmnpqr} + \delta^{h}_{u}\epsilon^{fgjklmnpqr} - \delta^{g}_{u}\epsilon^{fhjklmnpqr}\\
&+\delta^{f}_{u}\epsilon^{ghjklmnpqr} + \frac{1}{3}\delta^{h}_{l}\epsilon^{fgjkmnpqru} - \frac{1}{3}\delta^{h}_{k}\epsilon^{fgjlmnpqru} + \frac{1}{3}\delta^{h}_{j}\epsilon^{fgklmnpqru} - \frac{1}{3}\delta^{g}_{l}\epsilon^{fhjkmnpqru}\\
&-\frac{1}{3}\delta^{g}_{k}\epsilon^{fhjlmnpqru} + \frac{1}{3}\delta^{g}_{j}\epsilon^{fhklmnpqru} + \frac{1}{3}\delta^{f}_{l}\epsilon^{ghjkmnpqru} - \frac{1}{3}\delta^{f}_{k}\epsilon^{ghjlmnpqru} + \frac{1}{3}\delta^{f}_{j}\epsilon^{ghklmnpqru}\Big].
\end{aligned}$$

After finding the above identity one must obtain the explicit form of the f-tensor (3.7), which is straightforward in principle, but tedious in practice. This task was done with the Mathematica package GAMMA [12], along with some custom-made functions to handle

Levi-Civita's epsilons and duality relations for the gamma matrices. In particular, the following identities were used,

$$(\gamma^{m_1m_2m_3m_4m_5m_6})_\alpha^{\ \beta} = +\frac{1}{4!}\epsilon^{m_1m_2m_3m_4m_5m_6n_1n_2n_3n_4}\,(\gamma_{n_1n_2n_3n_4})_\alpha^{\ \beta}$$

$$(\gamma^{m_1m_2m_3m_4m_5m_6m_7})_{\alpha\beta} = -\frac{1}{3!}\epsilon^{m_1m_2m_3m_4m_5m_6m_7n_1n_2n_3}\,(\gamma_{n_1n_2n_3})_{\alpha\beta}$$

$$(\gamma^{m_1m_2m_3m_4m_5m_6m_7m_8})_\alpha^{\ \beta} = -\frac{1}{2!}\epsilon^{m_1m_2m_3m_4m_5m_6m_7m_8n_1n_2}\,(\gamma_{n_1n_2})_\alpha^{\ \beta}.$$

After determining the f-tensor one can obtain the correlation function (3.3) using equation (3.8). The coefficients of (3.3) are then found to be $A = -2B = \frac{1}{140}$, $C = \frac{1}{8400}$. With these coefficients one can check that the following consistency condition between (3.3) and (3.4) is indeed satisfied,

$$\langle(\lambda\gamma_t\theta)(\lambda\gamma^{tnp}\theta)(\lambda\gamma^{qrs}\theta)(\theta\gamma_{ijk}\theta)\rangle = 2\langle(\lambda\gamma^{qrs}\theta)(\lambda\gamma^p\theta)(\lambda\gamma^n\theta)(\theta\gamma_{ijk}\theta)\rangle.$$

Using the identities (3.3),(3.4) and (3.5) the kinematic factor (3.2) can be straightforwardly computed. After summing over the permutations, using momentum conservation, $(k^R\cdot e^R) = 0$ and expressing everything in terms of the Mandelstam variables $u = -2(k^1\cdot k^3)$ and $t = -2(k^2\cdot k^3)$ only, the first line of (3.2) gives the following result:

$$\begin{aligned}
&-\frac{1}{56}(k^2\cdot e^3)(k^2\cdot e^4)(k^3\cdot e^2)(k^4\cdot e^1) - \frac{1}{56}(k^2\cdot e^3)(k^3\cdot e^2)(k^3\cdot e^4)(k^4\cdot e^1)\\
&-\frac{1}{56}(k^2\cdot e^3)(k^2\cdot e^4)(k^3\cdot e^1)(k^4\cdot e^2) + \frac{1}{56}(k^3\cdot e^1)(k^3\cdot e^2)(k^3\cdot e^4)(k^4\cdot e^3)\\
&+\frac{1}{56}(k^3\cdot e^2)(k^3\cdot e^4)(k^4\cdot e^1)(k^4\cdot e^3) - \frac{1}{56}(k^2\cdot e^4)(k^3\cdot e^1)(k^4\cdot e^2)(k^4\cdot e^3)\\
&+\frac{1}{56}(k^3\cdot e^1)(k^3\cdot e^4)(k^4\cdot e^2)(k^4\cdot e^3) + \frac{1}{56}(k^3\cdot e^4)(k^4\cdot e^1)(k^4\cdot e^2)(k^4\cdot e^3) - \frac{11}{168}(k^2\cdot e^3)(k^2\cdot e^4)(e^1\cdot e^2)t\\
&-\frac{11}{168}(k^2\cdot e^4)(k^4\cdot e^3)(e^1\cdot e^2)t - \frac{1}{112}(k^3\cdot e^4)(k^4\cdot e^3)(e^1\cdot e^2)t + \frac{11}{168}(k^2\cdot e^4)(k^3\cdot e^2)(e^1\cdot e^3)t\\
&-\frac{11}{168}(k^3\cdot e^4)(k^4\cdot e^2)(e^1\cdot e^3)t + \frac{1}{112}(k^2\cdot e^3)(k^3\cdot e^2)(e^1\cdot e^4)t + \frac{11}{168}(k^2\cdot e^3)(k^4\cdot e^2)(e^1\cdot e^4)t\\
&+\frac{11}{168}(k^4\cdot e^2)(k^4\cdot e^3)(e^1\cdot e^4)t - \frac{11}{168}(k^2\cdot e^4)(k^3\cdot e^1)(e^2\cdot e^3)t - \frac{1}{112}(k^2\cdot e^4)(k^4\cdot e^1)(e^2\cdot e^3)t\\
&-\frac{1}{112}(k^3\cdot e^4)(k^4\cdot e^1)(e^2\cdot e^3)t - \frac{11}{168}(k^2\cdot e^3)(k^4\cdot e^1)(e^2\cdot e^4)t - \frac{11}{168}(k^3\cdot e^1)(k^4\cdot e^3)(e^2\cdot e^4)t\\
&-\frac{11}{168}(k^4\cdot e^1)(k^4\cdot e^3)(e^2\cdot e^4)t - \frac{1}{112}(k^3\cdot e^1)(k^3\cdot e^2)(e^3\cdot e^4)t - \frac{1}{112}(k^3\cdot e^2)(k^4\cdot e^1)(e^3\cdot e^4)t
\end{aligned} \tag{3.12}$$

$$+\frac{19}{336}(k^3\cdot e^1)(k^4\cdot e^2)(e^3\cdot e^4)t-\frac{1}{112}(k^4\cdot e^1)(k^4\cdot e^2)(e^3\cdot e^4)t+\frac{1}{224}(e^1\cdot e^4)(e^2\cdot e^3)t^2$$
$$+\frac{11}{336}(e^1\cdot e^3)(e^2\cdot e^4)t^2+\frac{1}{224}(e^1\cdot e^2)(e^3\cdot e^4)t^2-\frac{11}{168}(k^2\cdot e^3)(k^2\cdot e^4)(e^1\cdot e^2)u$$
$$-\frac{11}{168}(k^2\cdot e^3)(k^3\cdot e^4)(e^1\cdot e^2)u-\frac{1}{112}(k^3\cdot e^4)(k^4\cdot e^3)(e^1\cdot e^2)u+\frac{11}{168}(k^2\cdot e^4)(k^3\cdot e^2)(e^1\cdot e^3)u$$
$$+\frac{11}{168}(k^3\cdot e^2)(k^3\cdot e^4)(e^1\cdot e^3)u+\frac{1}{112}(k^2\cdot e^4)(k^4\cdot e^2)(e^1\cdot e^3)u+\frac{11}{168}(k^2\cdot e^3)(k^4\cdot e^2)(e^1\cdot e^4)u$$
$$-\frac{11}{168}(k^3\cdot e^2)(k^4\cdot e^3)(e^1\cdot e^4)u-\frac{11}{168}(k^2\cdot e^4)(k^3\cdot e^1)(e^2\cdot e^3)u-\frac{11}{168}(k^3\cdot e^1)(k^3\cdot e^4)(e^2\cdot e^3)u$$
$$-\frac{11}{168}(k^3\cdot e^4)(k^4\cdot e^1)(e^2\cdot e^3)u-\frac{1}{112}(k^2\cdot e^3)(k^3\cdot e^1)(e^2\cdot e^4)u-\frac{11}{168}(k^2\cdot e^3)(k^4\cdot e^1)(e^2\cdot e^4)u$$
$$-\frac{1}{112}(k^3\cdot e^1)(k^4\cdot e^3)(e^2\cdot e^4)u-\frac{1}{112}(k^3\cdot e^1)(k^3\cdot e^2)(e^3\cdot e^4)u+\frac{19}{336}(k^3\cdot e^2)(k^4\cdot e^1)(e^3\cdot e^4)u$$
$$-\frac{1}{112}(k^3\cdot e^1)(k^4\cdot e^2)(e^3\cdot e^4)u-\frac{1}{112}(k^4\cdot e^1)(k^4\cdot e^2)(e^3\cdot e^4)u+\frac{11}{336}(e^1\cdot e^4)(e^2\cdot e^3)tu$$
$$+\frac{11}{336}(e^1\cdot e^3)(e^2\cdot e^4)tu-\frac{1}{42}(e^1\cdot e^2)(e^3\cdot e^4)tu+\frac{11}{336}(e^1\cdot e^4)(e^2\cdot e^3)u^2$$
$$+\frac{1}{224}(e^1\cdot e^3)(e^2\cdot e^4)u^2+\frac{1}{224}(e^1\cdot e^2)(e^3\cdot e^4)u^2$$

which is clearly seen not to be proportional to $t_8F^1F^2F^3F^4$, as incorrectly claimed in [8]. Repeating the same procedure for the second line of (3.2) one obtains:

$$+\frac{1}{56}(k^2\cdot e^3)(k^2\cdot e^4)(k^3\cdot e^2)(k^4\cdot e^1)+\frac{1}{56}(k^2\cdot e^3)(k^3\cdot e^2)(k^3\cdot e^4)(k^4\cdot e^1) \tag{3.13}$$
$$+\frac{1}{56}(k^2\cdot e^3)(k^2\cdot e^4)(k^3\cdot e^1)(k^4\cdot e^2)-\frac{1}{56}(k^3\cdot e^1)(k^3\cdot e^2)(k^3\cdot e^4)(k^4\cdot e^3)$$
$$-\frac{1}{56}(k^3\cdot e^2)(k^3\cdot e^4)(k^4\cdot e^1)(k^4\cdot e^3)+\frac{1}{56}(k^2\cdot e^4)(k^3\cdot e^1)(k^4\cdot e^2)(k^4\cdot e^3)$$
$$-\frac{1}{56}(k^3\cdot e^1)(k^3\cdot e^4)(k^4\cdot e^2)(k^4\cdot e^3)-\frac{1}{56}(k^3\cdot e^4)(k^4\cdot e^1)(k^4\cdot e^2)(k^4\cdot e^3)-\frac{19}{672}(k^2\cdot e^3)(k^2\cdot e^4)(e^1\cdot e^2)t$$
$$-\frac{19}{672}(k^2\cdot e^4)(k^4\cdot e^3)(e^1\cdot e^2)t+\frac{1}{112}(k^3\cdot e^4)(k^4\cdot e^3)(e^1\cdot e^2)t+\frac{19}{672}(k^2\cdot e^4)(k^3\cdot e^2)(e^1\cdot e^3)t$$
$$-\frac{19}{672}(k^3\cdot e^4)(k^4\cdot e^2)(e^1\cdot e^3)t-\frac{1}{112}(k^2\cdot e^3)(k^3\cdot e^2)(e^1\cdot e^4)t+\frac{19}{672}(k^2\cdot e^3)(k^4\cdot e^2)(e^1\cdot e^4)t$$
$$+\frac{19}{672}(k^4\cdot e^2)(k^4\cdot e^3)(e^1\cdot e^4)t-\frac{19}{672}(k^2\cdot e^4)(k^3\cdot e^1)(e^2\cdot e^3)t+\frac{1}{112}(k^2\cdot e^4)(k^4\cdot e^1)(e^2\cdot e^3)t$$
$$+\frac{1}{112}(k^3\cdot e^4)(k^4\cdot e^1)(e^2\cdot e^3)t-\frac{19}{672}(k^2\cdot e^3)(k^4\cdot e^1)(e^2\cdot e^4)t-\frac{19}{672}(k^3\cdot e^1)(k^4\cdot e^3)(e^2\cdot e^4)t$$

$$
\begin{aligned}
&-\frac{19}{672}(k^4\cdot e^1)(k^4\cdot e^3)(e^2\cdot e^4)t+\frac{1}{112}(k^3\cdot e^1)(k^3\cdot e^2)(e^3\cdot e^4)t+\frac{1}{112}(k^3\cdot e^2)(k^4\cdot e^1)(e^3\cdot e^4)t\\
&+\frac{25}{672}(k^3\cdot e^1)(k^4\cdot e^2)(e^3\cdot e^4)t+\frac{1}{112}(k^4\cdot e^1)(k^4\cdot e^2)(e^3\cdot e^4)t-\frac{1}{224}(e^1\cdot e^4)(e^2\cdot e^3)t^2\\
&+\frac{19}{1344}(e^1\cdot e^3)(e^2\cdot e^4)t^2-\frac{1}{224}(e^1\cdot e^2)(e^3\cdot e^4)t^2-\frac{19}{672}(k^2\cdot e^3)(k^2\cdot e^4)(e^1\cdot e^2)u\\
&-\frac{19}{672}(k^2\cdot e^3)(k^3\cdot e^4)(e^1\cdot e^2)u+\frac{1}{112}(k^3\cdot e^4)(k^4\cdot e^3)(e^1\cdot e^2)u+\frac{19}{672}(k^2\cdot e^4)(k^3\cdot e^2)(e^1\cdot e^3)u\\
&+\frac{19}{672}(k^3\cdot e^2)(k^3\cdot e^4)(e^1\cdot e^3)u-\frac{1}{112}(k^2\cdot e^4)(k^4\cdot e^2)(e^1\cdot e^3)u+\frac{19}{672}(k^2\cdot e^3)(k^4\cdot e^2)(e^1\cdot e^4)u\\
&-\frac{19}{672}(k^3\cdot e^2)(k^4\cdot e^3)(e^1\cdot e^4)u-\frac{19}{672}(k^2\cdot e^4)(k^3\cdot e^1)(e^2\cdot e^3)u-\frac{19}{672}(k^3\cdot e^1)(k^3\cdot e^4)(e^2\cdot e^3)u\\
&-\frac{19}{672}(k^3\cdot e^4)(k^4\cdot e^1)(e^2\cdot e^3)u+\frac{1}{112}(k^2\cdot e^3)(k^3\cdot e^1)(e^2\cdot e^4)u-\frac{19}{672}(k^2\cdot e^3)(k^4\cdot e^1)(e^2\cdot e^4)u\\
&+\frac{1}{112}(k^3\cdot e^1)(k^4\cdot e^3)(e^2\cdot e^4)u+\frac{1}{112}(k^3\cdot e^1)(k^3\cdot e^2)(e^3\cdot e^4)u+\frac{25}{672}(k^3\cdot e^2)(k^4\cdot e^1)(e^3\cdot e^4)u\\
&+\frac{1}{112}(k^3\cdot e^1)(k^4\cdot e^2)(e^3\cdot e^4)u+\frac{1}{112}(k^4\cdot e^1)(k^4\cdot e^2)(e^3\cdot e^4)u+\frac{19}{1344}(e^1\cdot e^4)(e^2\cdot e^3)tu\\
&+\frac{19}{1344}(e^1\cdot e^3)(e^2\cdot e^4)tu-\frac{31}{1344}(e^1\cdot e^2)(e^3\cdot e^4)tu+\frac{19}{1344}(e^1\cdot e^4)(e^2\cdot e^3)u^2\\
&-\frac{1}{224}(e^1\cdot e^3)(e^2\cdot e^4)u^2-\frac{1}{224}(e^1\cdot e^2)(e^3\cdot e^4)u^2
\end{aligned}
$$

which is again not proportional to $t_8F^1F^2F^3F^4$. Note, however, that the sum of (3.12) and (3.13) is:

$$
\begin{aligned}
\frac{3}{32}\Big[&-(k^2\cdot e^3)(k^2\cdot e^4)(e^1\cdot e^2)t-(k^2\cdot e^4)(k^4\cdot e^3)(e^1\cdot e^2)t+(k^2\cdot e^4)(k^3\cdot e^2)(e^1\cdot e^3)t\\
&-(k^3\cdot e^4)(k^4\cdot e^2)(e^1\cdot e^3)t+(k^2\cdot e^3)(k^4\cdot e^2)(e^1\cdot e^4)t+(k^4\cdot e^2)(k^4\cdot e^3)(e^1\cdot e^4)t\\
&-(k^2\cdot e^4)(k^3\cdot e^1)(e^2\cdot e^3)t-(k^2\cdot e^3)(k^4\cdot e^1)(e^2\cdot e^4)t-(k^3\cdot e^1)(k^4\cdot e^3)(e^2\cdot e^4)t\\
&-(k^4\cdot e^1)(k^4\cdot e^3)(e^2\cdot e^4)t+(k^3\cdot e^1)(k^4\cdot e^2)(e^3\cdot e^4)t+(k^3\cdot e^2)(k^4\cdot e^1)(e^3\cdot e^4)u\\
&-(k^2\cdot e^3)(k^2\cdot e^4)(e^1\cdot e^2)u-(k^2\cdot e^3)(k^3\cdot e^4)(e^1\cdot e^2)u+(k^2\cdot e^4)(k^3\cdot e^2)(e^1\cdot e^3)u\\
&+(k^3\cdot e^2)(k^3\cdot e^4)(e^1\cdot e^3)u+(k^2\cdot e^3)(k^4\cdot e^2)(e^1\cdot e^4)u-(k^3\cdot e^2)(k^4\cdot e^3)(e^1\cdot e^4)u\\
&-(k^2\cdot e^4)(k^3\cdot e^1)(e^2\cdot e^3)u-(k^3\cdot e^1)(k^3\cdot e^4)(e^2\cdot e^3)u-(k^3\cdot e^4)(k^4\cdot e^1)(e^2\cdot e^3)u\\
&-(k^2\cdot e^3)(k^4\cdot e^1)(e^2\cdot e^4)u+\frac{1}{2}(e^1\cdot e^3)(e^2\cdot e^4)t^2+\frac{1}{2}(e^1\cdot e^4)(e^2\cdot e^3)u^2\\
&+\frac{1}{2}(e^1\cdot e^4)(e^2\cdot e^3)tu+\frac{1}{2}(e^1\cdot e^3)(e^2\cdot e^4)tu-\frac{1}{2}(e^1\cdot e^2)(e^3\cdot e^4)tu\Big]
\end{aligned}
$$

which can be checked to be proportional to $t_8F^1F^2F^3F^4$.

Repeating these same steps one can check that the last 2 lines of (3.2), after summing over the permutations, will also independently add up to a combination proportional to the RNS result. The equivalence with the RNS formalism, after summing up *all* contributions in (3.2), is then established.

4. On the result

A few comments regarding the calculations done here can be made. There should be no doubt that to obtain equivalence with the RNS result, all terms in (3.2) must be considered. If the assumption of a constant field-strength is made and only the first term in (3.2) is computed, one will obtain the wrong answer (3.12).

There are some possible explanations for this odd-looking fact, which certainly deserve further investigation. The discussion in [13] emphasizes the subtleties related to amplitude computations at zero momentum and explains that naive computations give incorrect results because of contact terms, and that the correct procedure is to analyticaly continue computations at non-zero momentum.

There may be another possible subtlety that was overlooked in the computation of [8]. The polarization vector of a constant field-strength is given by $e_m(X) = F_{mn}X^n$, so one is explicitly introducing the center-of-mass mode of X^m in the vertex operator. However, as explained in [14], the BRST cohomology is modified if one allows vertex operators and gauge parameters involving the center-of-mass mode of X^m. For example, one of the central tenets of the pure spinor formalism is that the cohomology of the BRST operator $Q_{BRST} = \oint \lambda^\alpha d_\alpha$ at ghost number three is given by $(\lambda^3\theta^5) \equiv (\lambda\gamma^m\theta)(\lambda\gamma^n\theta)(\lambda\gamma^p\theta)(\theta\gamma_{mnp}\theta)$. However, when the center-of-mass mode of X^m is present, the ghost number three cohomology of Q_{BRST} turns out to be trivial because [15],

$$(\lambda^3\theta^5) = Q_{BRST}\Big[X^m(\lambda\gamma^n\theta)(\lambda\gamma^p\theta)(\theta\gamma_{mnp}\theta)\Big]. \tag{4.1}$$

So to avoid the above subtleties, in this paper the kinematic factor of the massless 4-point one-loop amplitude in the pure spinor formalism was computed with a non-constant field-strength. Equivalence with the RNS formalism computation of [9] was correctly obtained when all external particles are in the Neveu-Schwarz sector.

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