AEI–2015–005, DAMTP–2015–5

A solution to the non-linear equations of D=10 super Yang–Mills theory

Carlos R. Mafra[1,*] and Oliver Schlotterer[2,†]

[1] *DAMTP, University of Cambridge, Wilberforce Road, Cambridge, CB3 0WA, UK*

[2] *Max–Planck–Institut für Gravitationsphysik, Albert–Einstein–Institut, 14476 Potsdam, Germany*

(Dated: November 5, 2018)

In this letter, we present a formal solution to the non-linear field equations of ten-dimensional super Yang–Mills theory. It is assembled from products of linearized superfields which have been introduced as multiparticle superfields in the context of superstring perturbation theory. Their explicit form follows recursively from the conformal field theory description of the gluon multiplet in the pure spinor superstring. Furthermore, superfields of higher mass dimensions are defined and their equations of motion spelled out.

arXiv:1501.05562v2 [hep-th] 19 Sep 2015

INTRODUCTION

Super Yang–Mills (SYM) theory in ten dimensions can be regarded as one of the simplest SYM theories, its spectrum contains just the gluon and gluino, related by sixteen supercharges. However, it is well-known that its dimensional reduction gives rise to various maximally supersymmetric Yang–Mills theories in lower dimensions, including the celebrated $\mathcal{N} = 4$ theory in $D = 4$ [1]. Therefore a better understanding of this theory propagates a variety of applications to any dimension $D \leq 10$.

In a recent line of research [2, 3], scattering amplitudes of ten-dimensional SYM have been determined and simplified using so-called multiparticle superfields [4]. They represent entire tree-level subdiagrams and build up in the conformal field theory (CFT) on the worldsheet of the pure spinor superstring [5] via operator product expansions (OPEs). Multiparticle superfields satisfy the linearized field equations with the addition of contact terms, i.e. inverse off-shell propagators. In this letter we demonstrate that these off-shell modifications can be resummed to capture the non-linearities in the SYM equations of motion. The generating series of multiparticle superfields as seen in (18) is shown to solve the non-linear field equations spelled out in (4).

We also define superfields of arbitrary mass dimension and reduce their non-linear expressions to the linearized superfields of lower mass dimensions. This framework simplifies the expressions of kinematic factors in higher-loop scattering amplitudes, including the D^6R^4 operator in the superstring three-loop amplitude [6].

REVIEW OF TEN-DIMENSIONAL SYM

The equations of motion of ten-dimensional SYM theory can be described covariantly in superspace by defining supercovariant derivatives [7, 8]

$$\nabla_\alpha \equiv D_\alpha - \mathbb{A}_\alpha(x,\theta), \quad \nabla_m \equiv \partial_m - \mathbb{A}_m(x,\theta). \qquad (1)$$

The connections $\mathbb{A}_\alpha$ and $\mathbb{A}_m$ take values in the Lie algebra associated with the non-abelian Yang–Mills gauge group. The derivatives are taken with respect to ten-dimensional superspace coordinates (x^m, θ^α) with vector and spinor indices $m, n = 0, \ldots, 9$ and $\alpha, \beta = 1, \ldots, 16$ of the Lorentz group. The fermionic covariant derivatives

$$D_\alpha \equiv \partial_\alpha + \tfrac{1}{2}(\gamma^m\theta)_\alpha \partial_m, \quad \{D_\alpha, D_\beta\} = \gamma^m_{\alpha\beta}\partial_m \qquad (2)$$

involve the 16×16 Pauli matrices $\gamma^m_{\alpha\beta} = \gamma^m_{\beta\alpha}$ subject to the Clifford algebra $\gamma^{(m}_{\alpha\beta}\gamma^{n)\beta\gamma} = 2\eta^{mn}\delta^\gamma_\alpha$, and the convention for (anti)symmetrizing indices does not include $\frac{1}{2}$.

The connections in (1) give rise to field-strengths

$$\mathbb{F}_{\alpha\beta} \equiv \{\nabla_\alpha, \nabla_\beta\} - \gamma^m_{\alpha\beta}\nabla_m, \quad \mathbb{F}_{mn} \equiv -[\nabla_m, \nabla_n]. \qquad (3)$$

One can show that the constraint equation $\mathbb{F}_{\alpha\beta} = 0$ puts the superfields on-shell, and Bianchi identities lead to the non-linear equations of motion [8],

$$\begin{aligned}\{\nabla_\alpha, \nabla_\beta\} &= \gamma^m_{\alpha\beta}\nabla_m\\ [\nabla_\alpha, \nabla_m] &= -(\gamma_m \mathbb{W})_\alpha\\ \{\nabla_\alpha, \mathbb{W}^\beta\} &= \tfrac{1}{4}(\gamma^{mn})_\alpha{}^\beta \mathbb{F}_{mn}\\ [\nabla_\alpha, \mathbb{F}^{mn}] &= [\nabla^{[m}, (\gamma^{n]}\mathbb{W})_\alpha]\,. \end{aligned} \qquad (4)$$

In the subsequent, we will construct an explicit solution for the superfields $\mathbb{A}_\alpha$, $\mathbb{A}_m$, $\mathbb{W}^\alpha$ and $\mathbb{F}^{mn}$ in (4). The main result is furnished by the generating series in (18) whose constituents will be introduced in the next section.

LINEARIZED MULTIPARTICLE SUPERFIELDS

In perturbation theory, it is conventional to study solutions $A_\alpha, A_m, \ldots$ of the *linearized* equations of motion

$$\begin{aligned}\{D_{(\alpha}, A_{\beta)}\} &= \gamma^m_{\alpha\beta}A_m\\ [D_\alpha, A_m] &= k_m A_\alpha + (\gamma_m W)_\alpha\\ \{D_\alpha, W^\beta\} &= \tfrac{1}{4}(\gamma^{mn})_\alpha{}^\beta F_{mn}\\ [D_\alpha, F^{mn}] &= k^{[m}(\gamma^{n]}W)_\alpha\,. \end{aligned} \qquad (5)$$

Their dependence on the bosonic coordinates x is described by plane waves $e^{k\cdot x}$ with on-shell momentum $k^2 = 0$. In a gauge where $\theta^\alpha A_\alpha = 0$, the θ dependence is known in terms of fermionic power series expansions from [9, 10] whose coefficients contain gluon polarizations and gluino wave functions.

As an efficient tool to determine and compactly represent scattering amplitudes in SYM and string theory, multiparticle versions of the linearized superfields have been constructed in [4]. They satisfy systematic modifications of the linearized equations of motion (5), and their significance for BRST invariance was pointed out in [11]. For example, their two-particle version

$$
\begin{aligned}
A^{12}_\alpha &\equiv -\tfrac{1}{2}\big[A^1_\alpha(k^1\cdot A^2)+A^1_m(\gamma^m W^2)_\alpha-(1\leftrightarrow 2)\big]\\
A^{12}_m &\equiv \tfrac{1}{2}\big[A^p_1 F^2_{pm}-A^1_m(k^1\cdot A^2)+(W^1\gamma_m W^2)-(1\leftrightarrow 2)\big]\\
W^\alpha_{12} &\equiv \tfrac{1}{4}(\gamma^{mn}W_2)^\alpha F^1_{mn}+W^\alpha_2(k^2\cdot A^1)-(1\leftrightarrow 2) \qquad (6)\\
F^{12}_{mn} &\equiv F^2_{mn}(k^2\cdot A^1)+\tfrac{1}{2}F^2_{[m}{}^p F^1_{n]p}\\
&\quad + k^1_{[m}(W^1\gamma_{n]}W^2)-(1\leftrightarrow 2)\,,
\end{aligned}
$$

can be checked via (5) to satisfy the following two-particle equations of motion:

$$
\begin{aligned}
\{D_{(\alpha},A^{12}_{\beta)}\} &= \gamma^m_{\alpha\beta}A^{12}_m+(k^1\cdot k^2)(A^1_\alpha A^2_\beta+A^1_\beta A^2_\alpha) \qquad (7)\\
[D_\alpha,A^m_{12}] &= \gamma^m_{\alpha\beta}W^\beta_{12}+k^m_{12}A^{12}_\alpha+(k^1\cdot k^2)(A^1_\alpha A^m_2-A^2_\alpha A^m_1)\\
\{D_\alpha,W^\beta_{12}\} &= \tfrac{1}{4}(\gamma^{mn})_\alpha{}^\beta F^{12}_{mn}+(k^1\cdot k^2)(A^1_\alpha W^\beta_2-A^2_\alpha W^\beta_1)\\
[D_\alpha,F^{12}_{mn}] &= k^{12}_m(\gamma_n W^{12})_\alpha-k^{12}_n(\gamma_m W^{12})_\alpha\\
&\quad +(k^1\cdot k^2)\big(A^1_\alpha F^2_{mn}+A^1_{[n}(\gamma_{m]}W^2)_\alpha-(1\leftrightarrow 2)\big)\,.
\end{aligned}
$$

The modifications as compared to the single-particle case (5) involve the overall momentum $k_{12}\equiv k_1+k_2$ whose propagator is generically off-shell, $k^2_{12}=2(k_1\cdot k_2)\neq 0$.

The construction of the two-particle superfields in (6) is guided by string theory methods. In the pure spinor formalism [5], the insertion of a gluon multiplet state on the boundary of an open string worldsheet is described by the integrated vertex operator

$$
U^i \equiv \partial\theta^\alpha A^i_\alpha+\Pi^m A^i_m+d_\alpha W^\alpha_i+\tfrac{1}{2}N^{mn}F^i_{mn}\,. \qquad (8)
$$

Worldsheet fields $[\partial\theta^\alpha,\Pi^m,d_\alpha,N^{mn}]$ with conformal weight one and well-known OPEs are combined with linearized superfields associated with particle label i. The multiplicity-two superfields in (6) are obtained from the coefficients of the conformal fields in the OPE [4]

$$
\begin{aligned}
U^{12} &\equiv -\oint (z_1-z_2)^{\alpha' k^1\cdot k^2}U^1(z_1)U^2(z_2) \qquad (9)\\
&= \partial\theta^\alpha A^{12}_\alpha+\Pi^m A^{12}_m+d_\alpha W^\alpha_{12}+\tfrac{1}{2}N^{mn}F^{12}_{mn}\,,
\end{aligned}
$$

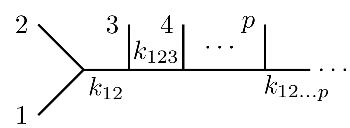
where α' denotes the inverse string tension, and total derivatives in the insertion points z_1,z_2 on the worldsheet have been discarded in the second line. The contour integral in (9) isolates the singular behaviour of the U^i w.r.t. (z_1-z_2) which translates into propagators k^{-2}_{12} of the gauge theory amplitude after performing the $\alpha'\to 0$ limit. In other words, OPEs in string theory govern the pole structure of tree-level subdiagrams in SYM theory obtained from the point-particle limit.

The CFT-inspired two-particle prescription (6) can be promoted to a recursion leading to superfields of arbitrary multiplicity, see (3.54), (3.56) and (3.59) of [4]. Their equations of motion are observed to generalize along the lines of

$$
\begin{aligned}
\{D_{(\alpha},A^{123}_{\beta)}\} &= \gamma^m_{\alpha\beta}A^{123}_m+(k^{12}\cdot k^3)\big[A^{12}_\alpha A^3_\beta-(12\leftrightarrow 3)\big]\\
&\quad +(k^1\cdot k^2)\big[A^1_\alpha A^{23}_\beta+A^{13}_\alpha A^2_\beta-(1\leftrightarrow 2)\big] \qquad (10)
\end{aligned}
$$

for suitable definitions of A^{123}_α and A^{123}_m, see (3.17), (3.19) and (3.29) of [4].

Multiparticle superfields can be arranged to satisfy kinematic analogues of the Lie algebraic Jacobi relations among structure constants, e.g. $A^{123}_\alpha+A^{231}_\alpha+A^{312}_\alpha=0$. They therefore manifest the BCJ duality [12] between color and kinematic degrees of freedom in scattering amplitudes, see [13] for a realization at tree-level. Together with the momenta $k_{12\ldots j}\equiv k_1+k_2+\ldots+k_j$ in their equations of motion, this suggests to associate them with tree-level subdiagrams shown in the subsequent figure [4]:



$$
\leftrightarrow \begin{cases} A^{123\ldots p}_\alpha \\ A^{123\ldots p}_m \\ W^\alpha_{123\ldots p} \\ F^{123\ldots p}_{mn} \end{cases}
$$

The cubic-graph organization of superfields already accounts for the quartic vertex in the bosonic Feynman rules of SYM. This ties in with the string theory origin of n-particle tree-level amplitudes where each contribution stems from $n-3$ OPEs.

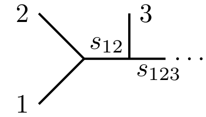
Berends–Giele currents: As a convenient basis of multiparticle fields $K_B\in\{A^B_\alpha,A^m_B,W^\alpha_B,F^{mn}_B\}$ with multiparticle label $B=12\ldots p$, we define Berends–Giele currents $\mathcal{K}_B\in\{\mathcal{A}^B_\alpha,\mathcal{A}^m_B,\mathcal{W}^\alpha_B,\mathcal{F}^{mn}_B\}$, e.g. $\mathcal{K}_1\equiv K_1$ and [4]

$$
\mathcal{K}_{12}\equiv\frac{K_{12}}{s_{12}}\,,\quad \mathcal{K}_{123}\equiv\frac{K_{123}}{s_{12}s_{123}}+\frac{K_{321}}{s_{23}s_{123}} \qquad (11)
$$

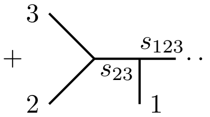
with generalized Mandelstam invariants $s_{12\ldots p}\equiv\frac{1}{2}k^2_{12\ldots p}$. Berends–Giele currents $\mathcal{K}_B$ are defined to encompass all tree subdiagrams compatible with the ordering of the external legs in B. The propagators $s^{-1}_{i\ldots j}$ absorb the appearance of explicit momenta in the contact terms of the equations of motion such as (7) and (10).

As illustrated in the following figure, the three-particle current in (11) is assembled from the s- and t-channels of a color-ordered four-point amplitude with an off-shell leg (represented by ...):

$\mathcal{K}_{123}\ =$



$+$



In contrast to the bosonic Berends–Giele currents in [14], the currents $\mathcal{K}_B$ manifest maximal supersymmetry, and their construction does not include any quartic vertices. A closed formula at arbitrary multiplicity [4, 15] involves the inverse of the momentum kernel[1] $S[\cdot|\cdot]_1$ [16],

$$\mathcal{K}_{1\sigma(23\ldots p)} \equiv \sum_{\rho\in S_{p-1}} S^{-1}[\sigma|\rho]_1 \, K_{1\rho(23\ldots p)} \ , \tag{12}$$

with permutation $\sigma \in S_{p-1}$ of legs $2, 3, \ldots, p$.

The combination of color-ordered trees as in (11) and (12) simplifies their multiparticle equations of motion [4]

$$\begin{aligned}\big\{D_{(\alpha}, \mathcal{A}^B_{\beta)}\big\} &= \gamma^m_{\alpha\beta}\mathcal{A}^B_m + \sum_{XY=B}\big(\mathcal{A}^X_\alpha\mathcal{A}^Y_\beta - \mathcal{A}^Y_\alpha\mathcal{A}^X_\beta\big) \qquad (13)\\ \big[D_\alpha, \mathcal{A}^B_m\big] &= k^B_m\mathcal{A}^B_\alpha + (\gamma_m\mathcal{W}_B)_\alpha + \sum_{XY=B}\big(\mathcal{A}^X_\alpha\mathcal{A}^Y_m - \mathcal{A}^Y_\alpha\mathcal{A}^X_m\big)\\ \big\{D_\alpha, \mathcal{W}^\beta_B\big\} &= \tfrac{1}{4}(\gamma^{mn})_\alpha{}^\beta\mathcal{F}^B_{mn} + \sum_{XY=B}\big(\mathcal{A}^X_\alpha\mathcal{W}^\beta_Y - \mathcal{A}^Y_\alpha\mathcal{W}^\beta_X\big)\\ \big[D_\alpha, \mathcal{F}^{mn}_B\big] &= k^{[m}_B(\gamma^{n]}\mathcal{W}_B)_\alpha + \sum_{XY=B}\big(\mathcal{A}^X_\alpha\mathcal{F}^{mn}_Y - \mathcal{A}^Y_\alpha\mathcal{F}^{mn}_X\big)\\ &\quad + \sum_{XY=B}\big(\mathcal{A}^{[n}_X(\gamma^{m]}\mathcal{W}_Y)_\alpha - \mathcal{A}^{[n}_Y(\gamma^{m]}\mathcal{W}_X)_\alpha\big) \ .\end{aligned}$$

Momenta $k_B \equiv k_1 + k_2 + \ldots k_p$ are associated with multiparticle labels $B = 12\ldots p$, and $\sum_{XY=B}$ instructs to sum over all their deconcatenations into non-empty $X = 12\ldots j$ and $Y = j+1\ldots p$ with $1 \leq j \leq p-1$. E.g. the three-particle equation of motion of $\mathcal{A}^{123}_\alpha$ reads

$$\begin{aligned}\big\{D_{(\alpha}, \mathcal{A}^{123}_{\beta)}\big\} &= \gamma^m_{\alpha\beta}\mathcal{A}^{123}_m \qquad (14)\\ &\quad + \mathcal{A}^1_\alpha\mathcal{A}^{23}_\beta + \mathcal{A}^{12}_\alpha\mathcal{A}^3_\beta - \mathcal{A}^{23}_\alpha\mathcal{A}^1_\beta - \mathcal{A}^3_\alpha\mathcal{A}^{12}_\beta \ ,\end{aligned}$$

and a comparison with (10) highlights the advantages of the diagram expansions in (11). Superfields up to multiplicity five satisfying (13) were explicitly constructed in [4] and there are no indications of a breakdown of (13) at higher multiplicity.

The symmetry properties of the $\mathcal{K}_B$ can be inferred from their cubic-graph expansion and summarized as [23]

$$\mathcal{K}_{A\sqcup\!\sqcup B} = 0, \quad \forall A, B \neq \emptyset \ , \tag{15}$$

where $\sqcup\!\sqcup$ denotes the shuffle product[2] [17]. For example,

$$\begin{aligned}0 &= \mathcal{K}_{1\sqcup\!\sqcup 2} = \mathcal{K}_{12} + \mathcal{K}_{21}\\ 0 &= \mathcal{K}_{1\sqcup\!\sqcup 23} = \mathcal{K}_{123} + \mathcal{K}_{213} + \mathcal{K}_{231} \qquad (16)\\ 0 &= \mathcal{K}_{12\sqcup\!\sqcup 3} - \mathcal{K}_{1\sqcup\!\sqcup 32} = \mathcal{K}_{123} - \mathcal{K}_{321} \ ,\end{aligned}$$

and symmetries (15) at higher multiplicity leave $(p-1)!$ independent permutations of $\mathcal{K}_{12\ldots p}$. Any permutation can be expanded in a basis of $\mathcal{K}_{1\sigma(23\ldots p)}$ with $\sigma \in S_{p-1}$ through the Berends–Giele symmetry

$$\mathcal{K}_{B1A} = (-1)^{|B|}\mathcal{K}_{1(A\sqcup\!\sqcup B^t)} \ , \tag{17}$$

where $|B| = p$ and $B^t = b_p \ldots b_2 b_1$ for a multi particle label $B = b_1 b_2 \ldots b_p$. Since the Berends–Giele current $\mathcal{K}_{12p}$ is composed from the cubic diagrams in a partial amplitude with an additional off-shell leg, (17) can be understood as a Kleiss–Kuijf relation among the latter [24].

GENERATING SERIES OF SYM SUPERFIELDS

In order to connect multiparticle fields and Berends–Giele currents with the non-linear field equations (4), we define generating series $\mathbb{K} \in \{\mathbb{A}_\alpha, \mathbb{A}^m, \mathbb{W}^\alpha, \mathbb{F}^{mn}\}$

$$\begin{aligned}\mathbb{K} &\equiv \sum_i \mathcal{K}_i t^i + \sum_{i,j}\mathcal{K}_{ij}t^i t^j + \sum_{i,j,k}\mathcal{K}_{ijk}t^i t^j t^k + \ldots \qquad (18)\\ &= \sum_i \mathcal{K}_i t^i + \tfrac{1}{2}\sum_{i,j}\mathcal{K}_{ij}[t^i, t^j] + \tfrac{1}{3}\sum_{i,j,k}\mathcal{K}_{ijk}[[t^i, t^j], t^k] + \cdots\end{aligned}$$

where t^i denote generators in the Lie algebra of the non-abelian gauge group. Hence, the generating series in (18) adjoin color degrees of freedom to the polarization and momentum dependence in the linearized multiparticle superfields $\mathcal{K}_B$. The second line follows from the symmetry (15), which guarantees that $\mathbb{K}$ is a Lie element [17].

As a key virtue of the series $\mathbb{K} \in \{\mathbb{A}_\alpha, \mathbb{A}^m, \mathbb{W}^\alpha, \mathbb{F}^{mn}\}$ in (18), they allow to rewrite the D_α action on Berends–Giele currents $\mathcal{K}_B \in \{\mathcal{A}^B_\alpha, \mathcal{A}^m_B, \mathcal{W}^\alpha_B, \mathcal{F}^{mn}_B\}$ in (13) as non-linear equations of motion,

$$\begin{aligned}\big\{D_{(\alpha}, \mathbb{A}_{\beta)}\big\} &= \gamma^m_{\alpha\beta}\mathbb{A}_m + \{\mathbb{A}_\alpha, \mathbb{A}_\beta\}\\ \big[D_\alpha, \mathbb{A}_m\big] &= \big[\partial_m, \mathbb{A}_\alpha\big] + (\gamma_m\mathbb{W})_\alpha + [\mathbb{A}_\alpha, \mathbb{A}_m]\\ \big\{D_\alpha, \mathbb{W}^\beta\big\} &= \tfrac{1}{4}(\gamma^{mn})_\alpha{}^\beta\mathbb{F}_{mn} + \{\mathbb{A}_\alpha, \mathbb{W}^\beta\}\\ \big[D_\alpha, \mathbb{F}^{mn}\big] &= \big[\partial^{[m}, (\gamma^{n]}\mathbb{W})_\alpha\big] + [\mathbb{A}_\alpha, \mathbb{F}^{mn}]\\ &\quad - [\mathbb{A}^{[m}, (\gamma^{n]}\mathbb{W})_\alpha] \ , \qquad (19)\end{aligned}$$

where $[\partial^m, \mathbb{K}]$ translates into components $k^m_B\mathcal{K}_B$.

Remarkably, they are equivalent to the non-linear SYM field equations (4) if the connection in (1) is defined through the representatives $\mathbb{A}_\alpha$ and $\mathbb{A}_m$ of the generating series in (18). In other words, the resummation of linearized multiparticle superfields $\{A^B_\alpha, A^m_B, W^\alpha_B, F^{mn}_B\}$ through the generating series (18) of Berends–Giele currents (12) solves the non-linear SYM equations (4).

---

[1] The momentum kernel is defined by [16]

$$S[\sigma(2,3,\ldots,p)|\rho(2,3,\ldots,p)]_1 = \prod_{j=2}^{p}\Big(s_{1,j_\sigma} + \sum_{k=2}^{j-1}\theta(j_\sigma,k_\sigma)s_{j_\sigma,k_\sigma}\Big)$$

and depends on reference leg 1 and two permutations $\sigma, \rho \in S_{p-1}$ of additional $p-1$ legs $2, 3, \ldots, p$. The symbols $\theta(j_\sigma, k_\sigma)$ keep track of labels which swap their relative positions in the two permutations σ and ρ, i.e. $\theta(j_\sigma, k_\sigma) = 1$ (=0) if the ordering of the legs j_σ, k_σ is the same (opposite) in the ordered sets $\sigma(2, 3, \ldots, p)$ and $\rho(2, 3, \ldots, p)$. The inverse in (12) is taken w.r.t. matrix multiplication which treats σ and ρ as row- and column indices.

[2] The shuffle product in $\mathcal{K}_{A\sqcup\!\sqcup B}$ is defined to sum all $\mathcal{K}_\sigma$ for permutations σ of $A \cup B$ which preserve the order of the individual elements of both sets A and B.

Given that the multiparticle superfields satisfy [4]

$$\begin{aligned}\mathcal{F}^{mn}_B &= k^{[m}_B \mathcal{A}^{n]}_B - \sum_{XY=B} (\mathcal{A}^m_X \mathcal{A}^n_Y - \mathcal{A}^m_Y \mathcal{A}^n_X) \qquad (20)\\ k^B_m (\gamma^m \mathcal{W}_B)_\alpha &= \sum_{XY=B} \big[\mathcal{A}^X_m (\gamma^m \mathcal{W}_Y)_\alpha - \mathcal{A}^Y_m (\gamma^m \mathcal{W}_X)_\alpha\big]\\ k^B_m \mathcal{F}^{mn}_B &= \sum_{XY=B} \big[2(\mathcal{W}_X \gamma^n \mathcal{W}_Y) + \mathcal{A}^X_m \mathcal{F}^{mn}_Y - \mathcal{A}^Y_m \mathcal{F}^{mn}_X\big]\,,\end{aligned}$$

the above definitions are compatible with (3) and

$$\big[\nabla_m, (\gamma^m \mathbb{W})_\alpha\big] = 0\,, \quad \big[\nabla_m, \mathbb{F}^{mn}\big] = \gamma^n_{\alpha\beta} \{\mathbb{W}^\alpha, \mathbb{W}^\beta\}\,, \tag{21}$$

whose lowest components in θ encode the Dirac and Yang-Mills equations for the gluino and gluon field.

A linearized gauge transformation in particle one,

$$\delta_1 A^1_\alpha = D_\alpha \Omega_1, \quad \delta_1 A^1_m = k^1_m \Omega_1, \tag{22}$$

can be described by a scalar superfield Ω_1, it shifts the gluon polarization by its momentum. The gluino component and the linearized field strengths are invariant under (22), $\delta_1 W^\alpha_1 = \delta_1 F^{mn}_1 = 0$, whereas Berends–Giele currents of multiparticle superfields with $B = 12\ldots p$ transform as follows [18]:

$$\begin{aligned}\delta_1 \mathcal{A}^B_\alpha &= \big[D_\alpha, \Omega_B\big] + \sum_{XY=B} \Omega_X \mathcal{A}^Y_\alpha, \quad \delta_1 \mathcal{W}^\alpha_B = \sum_{XY=B} \Omega_X \mathcal{W}^\alpha_Y\\ \delta_1 \mathcal{A}^B_m &= \big[\partial_m, \Omega_B\big] + \sum_{XY=B} \Omega_X \mathcal{A}^Y_m, \quad \delta_1 \mathcal{F}^{mn}_B = \sum_{XY=B} \Omega_X \mathcal{F}^{mn}_Y.\end{aligned} \tag{23}$$

The multiparticle gauge scalars $\Omega_{12\ldots p}$ are exemplified in appendix B of [18] and gathered in the generating series

$$\mathbb{L}_1 \equiv \Omega_1 t^1 + \sum_i \Omega_{1i}[t^1, t^i] + \sum_{j,k} \Omega_{1jk}[[t^1, t^j], t^k] + \ldots \tag{24}$$

This allows to cast (23) into the standard form of non-linear gauge transformations,

$$\begin{aligned}\delta_1 \mathbb{A}_\alpha &= \big[\nabla_\alpha, \mathbb{L}_1\big], \quad \delta_1 \mathbb{W}^\alpha = \big[\mathbb{L}_1, \mathbb{W}^\alpha\big] \qquad (25)\\ \delta_1 \mathbb{A}_m &= \big[\nabla_m, \mathbb{L}_1\big], \quad \delta_1 \mathbb{F}^{mn} = \big[\mathbb{L}_1, \mathbb{F}^{mn}\big]\,,\end{aligned}$$

such that traces w.r.t. generators t^i furnish a suitable starting point to construct gauge invariants.

HIGHER MASS DIMENSION SUPERFIELDS

The introduction of the Lie elements $\mathbb{K}$ and their associated supercovariant derivatives allow the recursive definition of superfields with higher mass dimensions,

$$\begin{aligned}\mathbb{W}^{m_1\ldots m_k \alpha} &\equiv \big[\nabla^{m_1}, \mathbb{W}^{m_2\ldots m_k \alpha}\big]\,, \qquad (26)\\ \mathbb{F}^{m_1\ldots m_k|pq} &\equiv \big[\nabla^{m_1}, \mathbb{F}^{m_2\ldots m_k|pq}\big]\,,\end{aligned}$$

where the vertical bar separates the antisymmetric pair of indices present in the recursion start $\mathbb{F}^{pq}$. Their component fields are defined by

$$\begin{aligned}\mathbb{W}^{m_1\ldots m_k\alpha} &\equiv \sum_{B\neq\emptyset} t^B \mathcal{W}^{m_1\ldots m_k\alpha}_B\,, \qquad (27)\\ \mathbb{F}^{m_1\ldots m_k|pq} &\equiv \sum_{B\neq\emptyset} t^B \mathcal{F}^{m_1\ldots m_k|pq}_B\,,\end{aligned}$$

with $t^B \equiv t^1 t^2 \ldots t^p$ for $B = 12\ldots p$. They inherit the Berends–Giele symmetries (15) and can be identified as

$$\begin{aligned}\mathcal{W}^{m_1\ldots m_k\alpha}_B &= k^{m_1}_B \mathcal{W}^{m_2\ldots m_k\alpha}_B \qquad (28)\\ &\quad + \sum_{XY=B} \big(\mathcal{W}^{m_2\ldots m_k\alpha}_X \mathcal{A}^{m_1}_Y - \mathcal{W}^{m_2\ldots m_k\alpha}_Y \mathcal{A}^{m_1}_X\big)\,,\\ \mathcal{F}^{m_1\ldots m_k|pq}_B &= k^{m_1}_B \mathcal{F}^{m_2\ldots m_k|pq}_B\\ &\quad + \sum_{XY=B} \big(\mathcal{F}^{m_2\ldots m_k|pq}_X \mathcal{A}^{m_1}_Y - \mathcal{F}^{m_2\ldots m_k|pq}_Y \mathcal{A}^{m_1}_X\big)\,.\end{aligned}$$

Note from (28) that the non-linearities in the definition of higher mass superfields do not contribute in the single-particle context with $\mathcal{W}^{m_1\ldots m_k\alpha}_i = k^{m_1}_i \mathcal{W}^{m_2\ldots m_k\alpha}_i$ whereas the simplest two-particle correction reads

$$\mathcal{W}^{m\alpha}_{12} = k^m_{12} \mathcal{W}^\alpha_{12} + \mathcal{W}^\alpha_1 \mathcal{A}^m_2 - \mathcal{W}^\alpha_2 \mathcal{A}^m_1\ . \tag{29}$$

Equations of motion at higher mass dimension: The equations of motion for the superfields of higher mass dimension (26) follow from $\big[\nabla_\alpha, \nabla_m\big] = -(\gamma_m \mathbb{W})_\alpha$ and $\big[\nabla_m, \nabla_n\big] = -\mathbb{F}_{mn}$ together with Jacobi identities among iterated brackets. The simplest examples are given by

$$\begin{aligned}\big\{\nabla_\alpha, \mathbb{W}^{m\beta}\big\} &= \tfrac{1}{4}(\gamma_{pq})_\alpha{}^\beta \mathbb{F}^{m|pq} - \big\{(\mathbb{W}\gamma^m)_\alpha, \mathbb{W}^\beta\big\}\,,\\ \big[\nabla_\alpha, \mathbb{F}^{m|pq}\big] &= (\mathbb{W}^{m[p}\gamma^{q]})_\alpha - \big[(\mathbb{W}\gamma^m)_\alpha, \mathbb{F}^{pq}\big]\,, \qquad (30)\end{aligned}$$

which translate into component equations of motion

$$\begin{aligned}D_\alpha \mathcal{W}^{m\beta}_B &= \tfrac{1}{4}(\gamma_{pq})_\alpha{}^\beta \mathcal{F}^{m|pq}_B + \sum_{XY=B} (\mathcal{A}^X_\alpha \mathcal{W}^{m\beta}_Y - \mathcal{A}^Y_\alpha \mathcal{W}^{m\beta}_X)\\ &\quad - \sum_{XY=B} \big[(\mathcal{W}_X\gamma^m)_\alpha \mathcal{W}^\beta_Y - (\mathcal{W}_Y\gamma^m)_\alpha \mathcal{W}^\beta_X\big]\,,\\ D_\alpha \mathcal{F}^{m|pq}_B &= (\mathcal{W}^{m[p}_B \gamma^{q]})_\alpha + \sum_{XY=B} (\mathcal{A}^X_\alpha \mathcal{F}^{m|pq}_Y - \mathcal{A}^Y_\alpha \mathcal{F}^{m|pq}_X)\\ &\quad - \sum_{XY=B} \big[(\mathcal{W}_X\gamma^m)_\alpha \mathcal{F}^{pq}_Y - (\mathcal{W}_Y\gamma^m)_\alpha \mathcal{F}^{pq}_X\big]. \qquad (31)\end{aligned}$$

In general, one can prove by induction that

$$\begin{aligned}\big\{\nabla_\alpha, \mathbb{W}^{N\beta}\big\} &= \tfrac{1}{4}(\gamma_{pq})_\alpha{}^\beta \mathbb{F}^{N|pq} - \sum_{\substack{M\in P(N)\\ M\neq\emptyset}} \big\{(\mathbb{W}\gamma)^M_\alpha, \mathbb{W}^{(N\setminus M)\beta}\big\}\\ \big[\nabla_\alpha, \mathbb{F}^{N|pq}\big] &= (\mathbb{W}^{N[p}\gamma^{q]})_\alpha - \sum_{\substack{M\in P(N)\\ M\neq\emptyset}} \big[(\mathbb{W}\gamma)^M_\alpha, \mathbb{F}^{(N\setminus M)pq}\big].\end{aligned} \tag{32}$$

The vector indices have been gathered to a multi-index $N \equiv n_1 n_2 \ldots n_k$. Its power set $P(N)$ consists of the 2^k ordered subsets, and $(\mathbb{W}\gamma)^N \equiv (\mathbb{W}^{n_1 \ldots n_{k-1}} \gamma^{n_k})$.

The higher-mass-dimension superfields obey further relations which can be derived from Jacobi identities of nested (anti)commutators. For example, (3) determines their antisymmetrized components

$$\begin{aligned} \mathbb{W}^{[n_1 n_2] n_3 \ldots n_k \beta} &= \big[\mathbb{W}^{n_3 \ldots n_k \beta}, \mathbb{F}^{n_1 n_2}\big] \\ \mathbb{F}^{[n_1 n_2] n_3 \ldots n_k | pq} &= \big[\mathbb{F}^{n_3 \ldots n_k | pq}, \mathbb{F}^{n_1 n_2}\big] \,. \end{aligned} \tag{33}$$

Moreover, the definitions (26) via iterated commutators imply that

$$\mathbb{F}^{[m|np]} = 0, \quad \mathbb{F}^{[mn]|pq} + \mathbb{F}^{[pq]|mn} = 0 \,, \tag{34}$$

and the gauge-variations $\delta_1 \nabla_m = [\mathbb{L}_1, \nabla_m]$ and (25) yield

$$\delta_1 \mathbb{W}^\alpha_N = \big[\mathbb{L}_1, \mathbb{W}^\alpha_N\big], \quad \delta_1 \mathbb{F}^{N|pq} = \big[\mathbb{L}_1, \mathbb{F}^{N|pq}\big] \,. \tag{35}$$

Manifold generalizations of (21), (33) and (34) can be generated using these same manipulations.

OUTLOOK AND APPLICATIONS

The representation of the non-linear superfields of ten-dimensional SYM theory described in this letter was motivated by the computation of scattering amplitudes in the pure spinor formalism. Accordingly, they give rise to generating functions for amplitudes. For example, color-dressed SYM amplitudes at tree-level $M(1,2,\ldots,n)$ involving particles $1,2,\ldots,n$ are generated by

$$\frac{1}{3}\mathrm{Tr}\langle \mathbb{V}\mathbb{V}\mathbb{V}\rangle = \sum_{n=3}^{\infty} (n-2) \sum_{i_1<i_2<\ldots<i_n} M(i_1,i_2,\ldots,i_n) \,. \tag{36}$$

As firstly pointed out in the appendix of [19], the generating series $\mathbb{V} \equiv \lambda^\alpha \mathbb{A}_\alpha$ involving the pure spinor λ^α satisfies the field equations $Q\mathbb{V} = \mathbb{V}\mathbb{V}$ of the action $\mathrm{Tr}\int d^{10}x \langle \frac{1}{2}\mathbb{V}Q\mathbb{V} - \frac{1}{3}\mathbb{V}\mathbb{V}\mathbb{V}\rangle$ [20] with BRST operator $Q \equiv \lambda^\alpha D_\alpha$. The zero mode prescription of schematic form $\langle \lambda^3 \theta^5 \rangle = 1$ extracting the gluon- and gluino components is explained in [5] and automated in [21]. With $n=3$ or $n=4$ external states, for instance, assembling the components $V^B \equiv \lambda^\alpha A^B_\alpha$ of $\mathbb{V}\mathbb{V}\mathbb{V}$ in (36) with the appropriate number of labels yields

$$M(1,2,3) = \mathrm{Tr}(t^1[t^2,t^3])\langle V^1 V^2 V^3\rangle \tag{37}$$

$$2M(1,2,3,4) = \mathrm{Tr}(t^1 t^2 t^3 t^4)\Big\langle \frac{V^{12}V^3V^4}{s_{12}} + \frac{V^{23}V^4V^1}{s_{23}} + \frac{V^{34}V^1V^2}{s_{34}} + \frac{V^{41}V^2V^3}{s_{41}}\Big\rangle + \mathrm{perm}(2,3,4) \,.$$

The pure spinor representation of ten-dimensional[3] n-point SYM amplitudes is described in [2]. Further details and generalizations to superamplitudes with a single insertion of supersymmetrized operators F^4 or D^2F^4 will be given elsewhere [22].

The multiparticle superfields of higher mass dimensions can be used to obtain simpler expressions for higher-loop kinematic factors of superstring amplitudes. For example, the complicated three-loop kinematic factors generating the matrix element of the (supersymmetrized) operator D^6R^4 [6]

$$M_{D^6R^4} = \frac{|T_{12,3,4}|^2}{s_{12}} + |T^m_{1234}|^2 + (1,2|1,2,3,4) \tag{38}$$

can be equivalently represented by

$$\begin{aligned} T_{12,3,4} &\equiv \langle (\lambda\gamma_m W^n_{12})(\lambda\gamma_n W^p_{[3})(\lambda\gamma_p W^m_{4]})\rangle \\ T^m_{1234} &\equiv \langle A^m_{(1}T_{2),3,4} + (\lambda\gamma^m W_{(1})L_{2),3,4}\rangle \\ L_{2,3,4} &\equiv \tfrac{1}{3}(\lambda\gamma^n W^q_{[2})(\lambda\gamma^q W^p_3)F^{np}_{4]}\rangle \,. \end{aligned} \tag{39}$$

In (38), the notation $(A_1,\ldots,A_p \mid A_1,\ldots,A_n)$ instructs to sum over all possible ways to choose p elements $A_1,A_2,\ldots,A_p$ out of the set $\{A_1,\ldots,A_n\}$, for a total of $\binom{n}{p}$ terms. The tensor products of left- and right-moving SYM superfields in $|T_{12,3,4}|^2 = T_{12,3,4}\tilde{T}_{12,3,4}$ are understood to yield superfields of type IIB or type IIA supergravity. Accordingly, the component polarizations of the supergravity multiplet arise from the tensor product of gluon polarizations and gluino wavefunctions within the SYM superfields.

The low-energy limit of the three-loop closed string amplitude given in (38) is proportional to the $(\alpha')^6$ correction of the corresponding tree-level amplitude which in turn defines the aforementioned D^6R^4 operator.

It would be interesting to explore the dimensional reduction [1] of the above setup and its generalization to SYM theories with less supersymmetry or to construct formal solutions to supergravity field equations along similar lines.

Acknowledgements: We thank Eduardo Casali, Evgeny Skvortsov, Massimo Taronna and Stefan Theisen for valuable discussions as well as Nathan Berkovits for helpful comments on the draft. This research has received funding from the European Research Council under the European Community's Seventh Framework Programme (FP7/2007-2013)/ERC grant agreement no. [247252].

* Electronic address: crm66@damtp.cam.ac.uk
† Electronic address: olivers@aei.mpg.de

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