Berends–Giele recursion for double-color-ordered amplitudes

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Tree-level double-color-ordered amplitudes are computed using Berends–Giele recursion relations applied to the bi-adjoint cubic scalar theory. The standard notion of Berends–Giele currents is generalized to double-currents and their recursions are derived from a perturbiner expansion of linearized fields that solve the non-linear field equations. Two applications are given. Firstly, we prove that the entries of the inverse KLT matrix are equal to Berends–Giele double-currents (and are therefore easy to compute). And secondly, a simple formula to generate tree-level BCJ-satisfying numerators for arbitrary multiplicity is proposed by evaluating the field-theory limit of tree-level string amplitudes for various color orderings using double-color-ordered amplitudes.

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1. Introduction

As discussed in [1], the bi-adjoint cubic scalar theory with the Lagrangian[1]

$$\mathcal{L} = \frac{1}{2}\partial_m \phi_{i|a}\partial^m \phi_{i|a} + \frac{1}{3!} f_{ijk}\tilde{f}_{abc}\phi_{i|a}\phi_{j|b}\phi_{k|c} \tag{1.1}$$

gives rise to *double-color-ordered* tree amplitudes $m(A|B)$,

$$\mathcal{M}_n = \sum_{a_i, b_i \in S_n/Z_n} \mathrm{tr}(t^{a_1}t^{a_2}\ldots t^{a_n})\mathrm{tr}(\tilde{t}^{b_1}\tilde{t}^{b_2}\ldots\tilde{t}^{b_n})m(a_1,\ldots,a_n|b_1,\ldots,b_n), \tag{1.2}$$

and a diagrammatic algorithm to compute them was described. It was also demonstrated that these double-color-ordered amplitudes are related to the entries of the field-theory inverse KLT matrix [2,3,4] as well as the field-theory limit of string tree-level integrals [5,6,7]; thus providing an alternative method for their calculation which does not involve inverting a matrix nor evaluating any integrals [6].

The algorithm to compute $m(A|B)$ described in [1] involves drawing polygons and collecting the products of propagators associated to cubic graphs which are compatible with both color orderings. Their overall sign, however, requires keeping track of the polygons orientation in a process that can be challenging to automate. The connection of these double-color-ordered amplitudes with the Cachazo–He–Yuan approach [8] led to other recent proposals for their evaluation [9,10,11] (see also [12]).

Given the importance of the double-color-ordered tree amplitudes for the evaluation of the field-theory limit of string disk integrals, a fully recursive and algebraic algorithm to compute them will be given in this paper. This will be done using the *perturbiner* approach of [13] (recently emphasized in [14]) to derive recursion relations for *Berends–Giele double-currents* from a solution to the non-linear field equation of the action (1.1). The double-color-ordered tree amplitudes are then computed in the same manner as in the Berends–Giele recursive method [15].

Two immediate applications of this new method are given. In section 4, the relation between the inverse field-theory KLT matrix and double-color-ordered amplitudes observed in [1] is shown to greatly simplify when the amplitudes are written in terms of Berends–Giele double-currents. And in section 6, the efficient evaluation of the field-theory limit of string tree-level integrals for various color orderings will lead to a closed formula for BCJ-satisfying tree-level numerators [16] at arbitrary multiplicity, tremendously simplifying the case-by-case analysis of [17].

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[1] In (1.1) and (1.2), f_{ijk} and $\tilde{f}_{abc}$ are the structure constants of the color groups $U(N)$ and $U(\tilde{N})$ and t^i, $\tilde{t}^a$ are their generators satisfying $[t^i, t^j] = f^{ijk}t^k$ and $[\tilde{t}^a, \tilde{t}^b] = \tilde{f}^{abc}\tilde{t}^c$.

1.1. On notation

Multiparticle labels correspond to *words* in the alphabet $\{1, 2, 3, 4, \ldots\}$ and are denoted by capital letters (e.g., $A = 1243$) while single-particle labels are represented by lower case letters (e.g., $i = 4$). A word of length $|P|$ is given by $P \equiv p_1 p_2 \ldots p_{|P|}$ while its transpose is $\tilde{P} = p_{|P|} p_{|P|-1} \ldots p_2 p_1$. The notation $\sum_{XY=P}$ means a sum over all possible ways to deconcatenate the word P in two non-empty words X and Y. For example, $\sum_{XY=1234} T_X T_Y = T_1 T_{234} + T_{12} T_{34} + T_{123} T_4$. The shuffle product $\shuffle$ between two words A and B is defined recursively by [18]

$$\emptyset \shuffle A = A \shuffle \emptyset = A, \qquad A \shuffle B \equiv a_1(a_2 \ldots a_{|A|} \shuffle B) + b_1(b_2 \ldots b_{|B|} \shuffle A)\,, \tag{1.3}$$

and $\emptyset$ denotes the empty word. To lighten the notation and avoid summation symbols, labeled objects are considered to be linear in words; e.g., $T_{1 \shuffle 23} = T_{123} + T_{213} + T_{231}$. Finally, the Mandelstam invariants are defined by

$$s_P \equiv k_P^2 = (k_{p_1} + k_{p_2} + \cdots + k_{p_{|P|}})^2. \tag{1.4}$$

2. Review of Berends–Giele recursions for Yang–Mills theory

In this section we derive the Berends–Giele currents for Yang–Mills theory [15] from a solution to the non-linear field equations. This approach has been recently emphasized in [14] and resembles the perturbiner formalism of [13]. The same procedure will be applied in the next section to the bi-adjoint cubic scalar theory (1.1).

The Lagrangian of Yang–Mills theory is given by

$$\mathcal{L} = -\frac{1}{4}\mathrm{tr}(\mathbb{F}_{mn}\mathbb{F}^{mn}), \qquad \mathbb{F}_{mn} \equiv -[\nabla_m, \nabla_n] \tag{2.1}$$

where $\nabla_m = \partial_m - \mathbb{A}_m(x)$ and $\mathbb{A}_m(x) = \mathbb{A}_m^a(x) t^a$ is a Lie algebra-valued field with t^a the generators of a Lie group satisfying $[t^a, t^b] = f^{abc} t^c$. The non-linear field equation $[\nabla_m, \mathbb{F}^{mn}] = 0$ following from (2.1) can be rewritten in the Lorenz gauge $\partial_m \mathbb{A}^m = 0$ as

$$\Box \mathbb{A}^n(x) = [\mathbb{A}_m(x), \partial^m \mathbb{A}^n(x)] + [\mathbb{A}_m(x), \mathbb{F}^{mn}(x)]. \tag{2.2}$$

To find a solution to the equation (2.2) one writes an ansatz of the form [19,14]

$$\mathbb{A}^m(x) \equiv \sum_P \mathcal{A}_P^m(x) t^P, \qquad t^P \equiv t^{p_1} t^{p_2} \ldots t^{p_{|P|}} \tag{2.3}$$

where the sum is over all words P restricted to permutations. One can check using a plane-wave expansion $\mathcal{A}^m_P(x) = \mathcal{A}^m_P e^{k_P\cdot x}$ that the ansatz (2.3) yields the following recursion,

$$\mathcal{A}^m_P = -\frac{1}{s_P}\sum_{XY=P}\big[\mathcal{A}^X_m(k^X\cdot\mathcal{A}^Y) + \mathcal{A}^X_n\mathcal{F}^Y_{mn} - (X\leftrightarrow Y)\big], \tag{2.4}$$

where s_P is the Mandelstam invariant (1.4), the field-strength Berends–Giele current is $\mathcal{F}^{mn}_Y \equiv k^m_Y\mathcal{A}^n_Y - k^n_Y\mathcal{A}^m_Y - \sum_{RS=Y}\left(\mathcal{A}^m_R\mathcal{A}^n_S - \mathcal{A}^n_R\mathcal{A}^m_S\right)$ and $\mathcal{A}^m_i$ with a single-particle label satisfies the linearized field equation $\Box\mathcal{A}^m_i = 0$.

It can be shown [20] that the recursion (2.4) is equivalent to the recursive definition for the Berends–Giele current J^m_P derived in [15] using Feynman rules for the cubic and quartic vertices of the Lagrangian (2.1). Note however that (2.4) contains only "cubic" vertices; the quartic interactions are naturally absorbed by the non-linear terms of the field-strength. This is conceptually simpler than previous attempts for absorbing those quartic terms [21].

One can also show using either group-theory methods [22] or combinatorics of words [14] that the currents $\mathcal{A}^m_P$ satisfy

$$\mathcal{A}^m_{A\sqcup\!\sqcup B} = 0,\ \ \forall A,B\neq\emptyset \quad\iff\quad \mathcal{A}^m_{PiQ} - (-1)^{|P|}\mathcal{A}^m_{i(\tilde{P}\sqcup\!\sqcup Q)} = 0, \tag{2.5}$$

which guarantees [23] that the ansatz (2.3) is a Lie algebra-valued field (the equivalence between the two statements in (2.5) follows from the theorems proved in [23] and [24]). Finally, the color-ordered tree-level n-point amplitude is given by [15]

$$A^{\rm YM}(1,2,\ldots,n) = s_{12\ldots(n-1)}\mathcal{A}^m_{12\ldots(n-1)}\mathcal{A}^m_n. \tag{2.6}$$

As a consequence of the Berends–Giele symmetry (2.5), the amplitude (2.6) *manifestly* satisfies the Kleiss–Kuijf symmetry [25];$A^{\rm YM}(P,1,Q,n) = (-1)^{|P|}A^{\rm YM}(1,\tilde{P}\sqcup\!\sqcup Q,n)$. Alternative proofs of this statement are given in [26,27].

3. Berends–Giele recursions for the bi-adjoint cubic scalar theory

In this section we derive recursion relations for Berends–Giele double-currents using a perturbiner expansion for the solution of the non-linear field equations obtained from the bi-adjoint cubic scalar Lagrangian. These double-currents will then be used to compute the tree-level double-color-ordered amplitudes.

3.1. Berends–Giele double-currents

The field equation following from the Lagrangian (1.1) can be written as

$$\Box\Phi = [\![\Phi, \Phi]\!]\,, \tag{3.1}$$

where we defined $\Phi \equiv \phi_{i|a} t^i \tilde{t}^a$ and $[\![\Phi, \Phi]\!] \equiv (\phi_{i|a}\phi_{j|b} - \phi_{j|a}\phi_{i|b}) t^i t^j \tilde{t}^a \tilde{t}^b$. Following [19,14], a solution to the field equation (3.1) can be constructed perturbatively in terms of *Berends–Giele double-currents* $\phi_{P|Q}$ with the ansatz,

$$\Phi(x) \equiv \sum_{P,Q} \phi_{P|Q}\, t^P \tilde{t}^Q\, e^{k_P\cdot x}, \qquad t^P \equiv t^{p_1} t^{p_2} \ldots t^{p_{|P|}} \tag{3.2}$$

Since the ansatz (3.2) contains the plane-wave factor $e^{k_P\cdot x}$ (as opposed to $e^{k_Q\cdot x}$), in order to have a well-defined multiparticle interpretation $\phi_{P|Q}$ must vanish unless P is a permutation of Q, i.e. $\phi_{P|Q} \equiv 0$ if $P \setminus Q \neq \emptyset$. Plugging the ansatz (3.2) into the field equation (3.1) leads to the following recursion

$$\phi_{P|Q} = \frac{1}{s_P} \sum_{XY=P} \sum_{AB=Q} \big(\phi_{X|A}\phi_{Y|B} - (X \leftrightarrow Y)\big)\,, \quad \phi_{P|Q} \equiv 0, \text{ if } P \setminus Q \neq \emptyset, \tag{3.3}$$

where s_P is the multiparticle Mandelstam invariant (1.4) and the single-particle double-current[2] satisfies the linearized equation $\Box\phi_{i|i}(x) = 0$; therefore $\phi_{i|i}(x) = \phi_{i|i} e^{k_i\cdot x}$ with $k_i^2 = 0$ can be normalized such that $\phi_{i|i} = 1$. Since the right-hand side of (3.3) is antisymmetric in both $[XY]$ and $[AB]$, the combinatorial proof of the Berends–Giele symmetry (2.5) given in the appendix of [14] also applies to both words in the double-currents $\phi_{P|Q}$,

$$\phi_{A\sqcup\!\sqcup B|Q} = 0 \quad \iff \quad \phi_{AiB|Q} = (-1)^{|A|}\phi_{i(\tilde{A}\sqcup\!\sqcup B)|Q}, \tag{3.4}$$

and, in particular, $\phi_{Ai|Q} = (-1)^{|A|}\phi_{i\tilde{A}|Q}$ (with similar expressions for the symmetries w.r.t the word Q in $\phi_{P|Q}$). The symmetries (3.4) generalize the standard Berends–Giele symmetry (2.5) to both sets of color generators and guarantee that the ansatz (3.2) is a (double) Lie series [23], thereby preserving the Lie algebra-valued nature of $\Phi(x)$ in (3.1).

Using $\phi_{i|j} = \delta_{ij}$ a few example applications of the recursion (3.3) are given by

$$\phi_{12|12} = \frac{1}{s_{12}}\big(\phi_{1|1}\phi_{2|2} - \phi_{2|1}\phi_{1|2}\big) = \frac{1}{s_{12}}, \quad \phi_{12|21} = \frac{1}{s_{12}}\big(\phi_{1|2}\phi_{2|1} - \phi_{2|2}\phi_{1|1}\big) = -\frac{1}{s_{12}} \tag{3.5}$$

as well as

$$\begin{aligned} \phi_{123|123} &= \frac{1}{s_{123}}\big(\phi_{12|12} + \phi_{23|23}\big) = \frac{1}{s_{123}}\Big(\frac{1}{s_{12}} + \frac{1}{s_{23}}\Big), \\ \phi_{123|132} &= \frac{1}{s_{123}}\phi_{23|32} = -\frac{1}{s_{23}s_{123}}. \end{aligned} \tag{3.6}$$

In the appendix B, the Berends–Giele double-current $\phi_{P|Q}$ is given an alternative representation in terms of planar binary trees and products of epsilon tensors.

[2] In a slight abuse of notation, the single-particle double-current $\phi_{i|i}(x)$ in the ansatz (3.2) is not the same field appearing in the Lagrangian (1.1); it corresponds to its linearized truncation.

3.2. Double-color-ordered amplitudes from Berends–Giele double-currents

Without loss of generality, one can use that $m(R|S)$ is cyclically symmetric in both words R and S to rewrite an arbitrary n-point amplitude as $m(P,n|Q,n)$, where $|P| = |Q| = n-1$. Therefore, a straightforward generalization of the gluonic amplitude (2.6) using the Berends–Giele double-currents yields a formula for the double-color-ordered amplitudes[3] (recall that $\phi_{n|n} = 1$),

$$m(P,n|Q,n) = s_P \phi_{P|Q} \,. \tag{3.7}$$

It is easy to see using the symmetries (3.4) obeyed by the double-currents that the Kleiss–Kuijf relations are satisfied independently by both sets of color orderings. Since the double-currents $\phi_{P|Q}$ obey the recursion relation (3.3), the computation of double-color-ordered amplitudes is easy to automate and their overall sign requires no additional bookkeeping[4].

4. The field-theory KLT matrix and its inverse

In this section we demonstrate that the entries of the inverse field-theory KLT matrix [2,3] (also called the *momentum kernel matrix* [4]) are equal to the Berends–Giele double currents and therefore are easy to compute. This computational simplicity is important because, apart from applications related to gauge and gravity amplitudes, the field-theory KLT matrix and its inverse relate [7] the local and non-local versions of multiparticle super Yang–Mills superfields[5]

$$M_{1A} = \sum_B S^{-1}[A|B]_1 V_{1B}, \qquad V_{1A} = \sum_B S[A|B]_1 M_{1B}, \tag{4.1}$$

with manifold applications in recent developments within the pure spinor formalism applied to the computation of scattering amplitudes in both field- and string theory [5,30,31,32].

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[3] The convention for the sign of the Mandelstam invariants here is such that $m^{\rm here}(P,n|Q,n) = (-1)^{|P|} m^{\rm there}(P,n|Q,n)$ in comparison with the normalization of [1].

[4] An implementation using FORM [28] is attached to the arXiv submission.

[5] The relations (4.1) apply for all types of SYM superfields $(A_\alpha, A_m, W^\alpha, \ldots)$ [29]. The restriction to V_P in (4.1) was chosen for simplicity.

4.1. The field-theory KLT matrix

The symmetric matrix $S[P|Q]$ defined by

$$S[P|q_1 q_2 \ldots q_{|Q|}] \equiv \prod_{j=2}^{|Q|} \sum_{i=1}^{j-1} s(P|q_i, q_j), \quad s(P|q_i, q_j) \equiv \begin{cases} s_{q_i q_j}, & q_i < q_j \text{ inside } P \\ 0, & \text{otherwise} \end{cases} \tag{4.2}$$

gives rise to the KLT matrix $S[A|B]_i$ when the first letters on both words coincide

$$S[A|B]_i \equiv S(i, A|i, B)\,. \tag{4.3}$$

For example, the definition (4.3) for $i = 1$ leads to $S[2|2]_1 = s_{12}$ as well as

$$\begin{aligned} S[23|23]_1 &= s_{12}(s_{13} + s_{23}), & S[23|32]_1 &= s_{12}s_{13}, \\ S[234|234]_1 &= s_{12}(s_{13} + s_{23})(s_{14} + s_{24} + s_{34}), & S[423|234]_1 &= s_{12}(s_{13} + s_{23})s_{14}, \\ S[243|234]_1 &= s_{12}(s_{13} + s_{23})(s_{14} + s_{24}), & S[342|234]_1 &= s_{12}s_{13}(s_{14} + s_{34}), \\ S[324|234]_1 &= s_{12}s_{13}(s_{14} + s_{24} + s_{34}), & S[432|234]_1 &= s_{12}s_{13}s_{14}. \end{aligned}$$

4.2. The inverse KLT matrix

The inverse KLT matrix $S^{-1}[A|B]_i$ can be computed from the entries (4.3) using standard matrix algebra. However, this task quickly becomes tedious in practice and the direct outcome of the matrix inversion usually requires further manipulations to be simplified. Fortunately it was proven in [1] that the entries of $S^{-1}[A|B]_i$ correspond to the double-color-ordered amplitudes[6],

$$S^{-1}[A|B]_i = -m(i, A, n-1, n|i, B, n, n-1), \quad |A| = |B| = n-3, \tag{4.4}$$

completely bypassing the tedious matrix algebra necessary to invert the KLT matrix (4.2). With the Berends–Giele representation of double-color-ordered amplitudes (3.7) the computation of $S^{-1}[A|B]_i$ does not require the extra labels $n-1, n$ since (4.4) simplifies to

$$S^{-1}[A|B]_i = \phi_{iA|iB}. \tag{4.5}$$

To see this one uses the Berends–Giele amplitude formula (3.7) in (4.4) to obtain

$$\begin{aligned} S^{-1}[A|B]_i &= -s_{iA(n-1)}\phi_{iA(n-1)|(n-1)iB} = (-1)^{|A|}s_{iA(n-1)}\phi_{(n-1)\tilde{A}i|(n-1)iB} \\ &= (-1)^{|A|}\phi_{(n-1)|(n-1)}\phi_{\tilde{A}i|iB} = \phi_{iA|iB}\,. \end{aligned} \tag{4.6}$$

[6] The overall sign in (4.4) is different than in [1] due to differences in conventions.

In the first line the label $(n-1)$ has been moved to the front using (3.4)

$$\phi_{iA(n-1)|P} = (-1)^{|A|+1}\phi_{(n-1)(\widetilde{iA})|P} = -(-1)^{|A|}\phi_{(n-1)\tilde{A}i|P}, \tag{4.7}$$

and in the second line the condition $\phi_{P|Q} = 0$ unless P is a permutation of Q implies that $s_{iA(n-1)}\phi_{(n-1)\tilde{A}i|(n-1)iB} = \phi_{(n-1)|(n-1)}\phi_{\tilde{A}i|iB}$. For example,

$$\begin{aligned} S^{-1}[23|23]_1 &= \phi_{123|123} = \frac{1}{s_{12}s_{123}} + \frac{1}{s_{23}s_{123}}, \quad S^{-1}[23|32]_1 = \phi_{123|132} = -\frac{1}{s_{23}s_{123}}, \\ S^{-1}[32|32]_1 &= \phi_{132|132} = \frac{1}{s_{13}s_{123}} + \frac{1}{s_{23}s_{123}}, \end{aligned} \tag{4.8}$$

which agrees with the results of [7]. Higher-multiplicity examples follow similarly.

Using the Berends–Giele representation of the inverse KLT matrix (4.5), the first relation in (4.4) simplifies to

$$M_{1A} = \sum_B \phi_{1A|1B} V_{1B}, \tag{4.9}$$

and therefore provides an efficient algebraic alternative to the diagrammatic method to compute M_P described in the appendix of [29].

5. The field-theory limit of tree-level string integrals

The n-point open-string amplitude computed using pure spinor methods in [5] can be written in terms of (local) multiparticle vertex operators V_P [29] as

$$A(\Sigma) = {\sum_{XY=2\ldots n-2}}' \langle V_{1X} V_{(n-1)\tilde{Y}} V_n\rangle Z_\Sigma(1,X,n,Y,n-1)(-1)^{|X|} + \mathcal{P}(23\ldots n-2), \tag{5.1}$$

where the deconcatenation in ${\sum}'_{XY}$ includes empty words and $Z_\Sigma(N)$ is given by [7],

$$Z_\Sigma(1,2,3,\ldots n-1,n) \equiv \frac{1}{\mathrm{vol}(SL(2,\mathbb{R}))}\int_\Sigma \mathrm{d}z_1\mathrm{d}z_2\cdots\mathrm{d}z_n \frac{\prod_{i<j}^n |z_{ij}|^{\alpha' s_{ij}}}{z_{12}z_{23}\cdots z_{n-1,n}z_{n1}}. \tag{5.2}$$

The factor $1/\mathrm{vol}(SL(2,\mathbb{R}))$ compensates the overcounting due to the conformal Killing group of the disk[7] and the region of integration Σ is such that $z_{\sigma_i} < z_{\sigma_{i+1}}$ for all $i=1$ to $i = |M|-1$. The pure spinor bracket $\langle \ldots \rangle$ is defined in [33] but will play no role in the subsequent discussion.

[7] It amounts to fixing three coordinates z_i, z_j and z_k and inserting a Jacobian factor $|z_{ij}z_{jk}z_{ki}|$.

As pointed out in [1], the field-theory limit of the string disk integrals (5.2) is given by the double-color-ordered amplitudes,

$$\lim_{\alpha'\to 0} Z_P(Q) = (-1)^{|P|} m(P|Q)\,. \tag{5.3}$$

For example ($I = 123\ldots n$),

$$\begin{aligned}
\lim_{\alpha'\to 0} Z_I(1243) &= (-1)^4 m(1234|1243) = s_{123}\phi_{123|312} = -\frac{1}{s_{12}} \\
\lim_{\alpha'\to 0} Z_I(12354) &= (-1)^5 m(12345|12354) = -s_{1234}\phi_{1234|4123} = \frac{1}{s_{12}s_{123}} + \frac{1}{s_{23}s_{123}},
\end{aligned} \tag{5.4}$$

which agree with (C.1) and (C.5) of [7]. Higher-multiplicity examples follow similarly.

So the SYM tree amplitudes with color ordering Σ obtained from the field-theory limit of the string amplitude (5.1) are given by

$$A^{\rm SYM}(\Sigma) = \sum_{XY=2\ldots n-2}{}' \langle V_{1X} V_{(n-1)\tilde{Y}} V_n\rangle\, m(\Sigma|1,X,n,Y,n-1)(-1)^{|Y|+1} + \mathcal{P}(23\ldots n-2). \tag{5.5}$$

It was shown in [17] that a set of BCJ-satisfying numerators for SYM tree amplitudes can always be obtained from the field-theory limit of the string tree-level amplitude (5.1), and explicit expressions for numerators up to 7-points were given in that reference. Since the Berends–Giele algorithm to evaluate the double-color-ordered amplitudes is easy to automate, one can quickly obtain higher-point BCJ numerators this way. Studying their patterns leads to a proposal for a general formula giving BCJ-satisfying tree-level numerators for arbitrary multiplicities. This will be done in the next section.

6. Tree-level SYM amplitudes with manifest BCJ numerators

For the canonical ordering $\Sigma = 123\ldots n$ it is easy to see that (5.5) reproduces the pure spinor n-point SYM amplitude formula derived in [30]

$$A^{\rm SYM}(1,2,\ldots,n) = \langle E_{12\ldots n-1} V_n\rangle, \qquad E_P \equiv \sum_{XY=P} M_X M_Y\,, \tag{6.1}$$

where M_X denotes the Berends–Giele current (4.9) associated with the multiparticle vertex V_X [29]. As discussed in [17], the amplitude (6.1) is the supersymmetric generalization of the standard Berends–Giele recursions [15] and leads to an alternative proof of the Kleiss–Kuijf relations [25] (originally proven in [26]).

To prove that (5.5) reduces to (6.1) when $\Sigma = 123\ldots n$, note that $m(\Sigma|1,X,n,Y,n{-}1)$ simplifies when X and Y are also canonically ordered (which is the case for (5.5)),

$$m(12\ldots n|1,X,n,Y,n{-}1) = s_{12\ldots n-1}\phi_{12\ldots n-1|Y(n-1)1X} = -\phi_{1X|1X}\phi_{Y(n-1)|Y(n-1)}. \quad (6.2)$$

Therefore the field-theory limit of the string tree amplitude given in (5.5) becomes

$$\begin{aligned} A^{\rm SYM}(12\ldots n) &= \sum_{XY=2\ldots n-2}{}^{\prime} \langle V_{1X}V_{(n-1)\tilde{Y}}V_n\rangle\, \phi_{1X|1X}\phi_{Y(n-1)|Y(n-1)}(-1)^{|Y|} + \mathcal{P}(23\ldots n-2) \\ &= \sum_{XY=2\ldots n-2}{}^{\prime} \langle M_{1X}M_{Y(n-1)}V_n\rangle = \sum_{XY=1\ldots n-1} \langle M_X M_Y V_n\rangle = \langle E_{12\ldots n-1}V_n\rangle, \end{aligned} \quad (6.3)$$

where $\phi_{Y(n-1)|Y(n-1)} = \phi_{(n-1)\tilde{Y}|(n-1)\tilde{Y}}$ was used before applying (4.9) to identify $M_{1X} = \sum_P \phi_{1X|1P}V_{1P}$ and $M_{(n-1)\tilde{Y}} = \sum_P \phi_{(n-1)\tilde{Y}|(n-1)P}V_{(n-1)P} = (-1)^{|Y|}M_{Y(n-1)}$. Note that the permutations over $23\ldots n-2$ do not act on the labels corresponding to the canonical ordering in $\phi_{1X|1X}$ such that $\phi_{1X|1X}V_{1X} + \mathcal{P}(23\ldots n-2) = \sum_P \phi_{1X|1P}V_{1P}$.

However, for general color orderings (5.5) and (6.1) no longer manifestly coincide. For example, the field-theory limit of the string amplitude (5.5) with ordering 12435 is

$$\begin{aligned} A^{\rm SYM}(1,2,4,3,5) &= \frac{\langle (V_{12}V_{43}+V_{123}V_4)V_5\rangle}{s_{12}s_{124}} - \frac{\langle (V_1V_{423}+V_{13}V_{42})V_5\rangle}{s_{24}s_{124}} + \frac{\langle V_{12}V_{43}V_5\rangle}{s_{34}s_{12}} \\ &\quad - \frac{\langle V_1V_{432}V_5\rangle}{s_{34}s_{234}} - \frac{\langle V_1V_{423}V_5\rangle}{s_{24}s_{234}}, \end{aligned} \quad (6.4)$$

while the field-theory formula (6.1) yields

$$\begin{aligned} A^{\rm SYM}(1,2,4,3,5) &= \langle E_{1243}M_5\rangle = \langle \big(M_{124}M_3 + M_{12}M_{43} + M_1M_{243}\big)M_5\rangle \quad (6.5) \\ &= \frac{\langle V_{124}V_3V_5\rangle}{s_{12}s_{124}} + \frac{\langle V_{421}V_3V_5\rangle}{s_{24}s_{124}} + \frac{\langle V_{12}V_{43}V_5\rangle}{s_{12}s_{34}} + \frac{\langle V_1V_{243}V_5\rangle}{s_{24}s_{34}} + \frac{\langle V_1V_{342}V_5\rangle}{s_{34}s_{234}}. \end{aligned}$$

One can see from (6.5) and (6.4) that the numerators generated by the SYM amplitude formula (6.1) are mapped to the following BCJ-satisfying numerators in the string theory amplitude,

$$\begin{aligned} V_{124}V_3 &\to V_{12}V_{43} + V_{123}V_4, \qquad V_1V_{243} \to -V_1V_{423}, \\ V_{421}V_3 &\to -V_1V_{423} - V_{13}V_{42}, \quad\; V_1V_{342} \to -V_1V_{432}. \end{aligned} \quad (6.6)$$

Comparing the field-theory limit of the string amplitude (5.5) for various orderings with the outcomes of the SYM amplitude (6.1), one can check that the BCJ-satisfying numerators following from the string tree amplitude can be obtained by a mapping $\circ_{ij}$ defined by

$$\begin{aligned} V_{iAjB}\circ_{ij} V_C &\equiv \sum_{\alpha\in P(\gamma)} V_{iA\alpha}V_{j\beta}, \quad \gamma\equiv\{B,\ell(C)\}, \quad \beta\equiv\gamma\backslash\alpha, \quad (6.7) \\ V_{AiB}\circ_{ij} V_{CjD} &\equiv V_{AiB}V_{CjD} \end{aligned}$$

acting[8] on the field-theory numerators given by the SYM amplitude (6.1). In (6.7), $P(\gamma)$ denotes the powerset of γ, $\ell(C)$ is the left-to-right Dynkin bracket [18],

$$\ell(c_1 c_2 c_3 \ldots c_{|C|}) \equiv [[\ldots[c_1, c_2], c_3], \ldots,], c_{|C|}] \tag{6.8}$$

and $\ell(C)$ is considered a single letter in the definition of the powerset of $\gamma = \{B, \ell(C)\}$; the number of elements in $P(\gamma)$ is $2^{|B|+1}$.

The mapping (6.7) ensures that the labels i and j never belong to the same vertex V_A or V_B in the product $V_A \circ_{ij} V_B$. This corresponds to the label distribution in the string theory formula (6.3) if $i = 1$ and $j = n-1$ and is the result of fixing the Möbius symmetry of the disk. For example, in a five-point amplitude one chooses $i = 1$ and $j = 4$ to get,

$$\begin{aligned} V_{124} \circ_{14} V_3 &= V_{12} V_{43} + V_{123} V_4 \\ V_{142} \circ_{14} V_3 &= V_1 V_{423} + V_{12} V_{43} + V_{123} V_4 + V_{13} V_{42} \\ V_{421} \circ_{14} V_3 &= -V_1 V_{423} - V_{13} V_{42} \end{aligned} \tag{6.9}$$

Defining $M_X \circ_{ij} M_Y$ by its action on the products of $V_A \circ_{ij} V_B$ from the expansion of M_X and M_Y given by (4.9) one can check a few cases explicitly that the following superfield is BRST closed (Q is the pure spinor BRST charge [33])

$$E_P^{(ij)} \equiv \sum_{XY=P} M_X \circ_{ij} M_Y \quad \Longrightarrow \quad Q E_P^{(ij)} = 0, \quad \forall i, j \in P. \tag{6.10}$$

Assuming that $E_P^{(ij)}$ is BRST invariant to all multiplicities, one is free to use this "gauge-fixed" version of E_P in the SYM amplitude formula (6.1) to obtain

$$A^{\rm SYM}(1,2,3,\ldots,n) \equiv \langle E^{(ij)}_{123\ldots n-1} V_n \rangle, \qquad i, j \equiv (1, n-1)\,. \tag{6.11}$$

By construction, the SYM amplitudes generated by the formula (6.11) manifestly coincide with the field-theory limit of the string tree amplitude and therefore give rise to BCJ-satisfying numerators for all n-point tree amplitudes. Incidentally, the powerset appearing in the definition (6.7) naturally explains why the number of terms in BCJ-satisfying numerators is always a power of two, as firstly observed in [17].

[8] It suffices to define $\circ_{ij}$ as in (6.7) since the generalized Jacobi identity $V_{AiB} = -V_{i\ell(A)B}$ [14] can always be used to move the label i to the front.

In the appendix A the mapping (6.7) is shown to be the kinematic equivalent of the color Jacobi identity which expresses any cubic color graph in a basis where labels i and j are at the opposite ends.

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Appendix A. Proof of manifest BCJ numerators

In this appendix we prove that the rewriting of field-theory numerators given by (6.7) corresponds to the Jacobi identity obeyed by structure constants.

In a BCJ gauge of super Yang–Mills superfields, the multiparticle vertex operator V_P satisfies generalized Jacobi identities (see e.g. [18]) and therefore its symmetries correspond to a string of structure constants [29]

$$V_{AiB} = -V_{i\ell(A)B} \quad \Longleftrightarrow \quad V_{1234\ldots p} \leftrightarrow f^{12a_3} f^{a_3 3 a_4} f^{a_4 4 a_5} \cdots f^{a_p p a_{p+1}}, \tag{A.1}$$

where $\ell(A)$ denotes the Dynkin bracket (6.8). Similarly, the symmetries of three vertices are mapped to

$$V_{iAjB} V_C V_n \Longleftrightarrow (-1)^{|C|} F(i, A, j, B, n, \tilde{C}), \tag{A.2}$$

where $F(A)$ is the *multi-peripheral* color factor [26]

$$F(1, 2, 3, \ldots, (n-1), n) \equiv f^{12a_3} f^{a_3 3 a_4} f^{a_4 4 a_5} \cdots f^{a_{(n-1)}(n-1)n}. \tag{A.3}$$

Applying the generalized Jacobi identity (A.1) either once or twice, any multi-peripheral color factor can be rewritten in the Del Duca–Dixon–Maltoni (DDM) basis of [26]

$$F(A, i, B, j, C) = \begin{cases} F(i, \ell(A), B, \tilde{\ell}(\tilde{C}), j), & A \neq \emptyset,\ C \neq \emptyset \\ -F(i, B, \tilde{\ell}(\tilde{C}), j), & A = \emptyset,\ C \neq \emptyset \end{cases} \tag{A.4}$$

where $\tilde{\ell}(P) = \widetilde{\ell(P)}$. One can also derive a closed formula to arrive at the DDM basis while keeping track of the relative positions of three labels (say i, j and n),

$$\begin{aligned} F(i, A, j, B, n, C) &= -F(i, A, \tilde{\ell}(\tilde{C} n \tilde{B}), j), \quad C \neq \emptyset \qquad \text{(A.5)} \\ &= \sum_{\alpha \in P(\gamma)} (-1)^{|\beta|} F(i, A, \tilde{\alpha}, n, \beta, j), \quad \gamma \equiv \{\ell(\tilde{C}), \tilde{B}\}, \ \ \beta \equiv \gamma \backslash \alpha \end{aligned}$$

where $P(\gamma)$ is the powerset of γ and $\ell(\tilde{C})$ is to be considered a single letter in $P(\gamma)$. To arrive at the second line one uses the identity[9] (see [34])

$$\ell(PnQ) = \sum_{\alpha\in P(\gamma)} (-1)^{|\beta|+1}\tilde{\beta}\, n\, \alpha, \quad P \neq \emptyset, \;\; \gamma \equiv \{\ell(P), Q\}, \;\; \beta \equiv \gamma\backslash\alpha \tag{A.6}$$

Finally, combining the results above one gets

$$\begin{aligned} V_{iAjB}V_C V_n &\to (-1)^{|C|} F(i, A, j, B, n, \tilde{C}) \\ &= (-1)^{|C|} \sum_{\alpha\in P(\delta)} (-1)^{|\beta|} F(i, A, \tilde{\alpha}, n, \beta, j), \quad \delta \equiv \{\ell(C), \tilde{B}\}, \;\; \beta \equiv \delta\backslash\alpha \\ &\to (-1)^{|C|+1} \sum_{\alpha\in P(\delta)} V_{iA\tilde{\alpha}} V_{j\tilde{\beta}} V_n \\ &= \sum_{\alpha\in P(\gamma)} V_{iA\alpha} V_{j\beta} V_n, \quad \gamma \equiv \{B, \ell(C)\}, \;\; \beta \equiv \gamma\backslash\alpha \\ &= V_{iAjB} \circ_{ij} V_C V_n \end{aligned} \tag{A.7}$$

where in the penultimate line we transposed the set δ (while considering $\ell(C)$ a single letter) and used $\tilde{\ell}(C) = (-1)^{|C|+1}\ell(C)$ when $\ell(C)$ is part of a multiparticle label.

Therefore the expression (6.7) for the product $V_{iAjB} \circ_{ij} V_C V_n$ is the kinematic counterpart of the color identity (A.5).

Appendix B. Berends–Giele double-currents from scalar ϕ^3 theory

In this appendix an alternative derivation of the Berends–Giele double-currents is given which resembles the algorithm of [1].

The field equation $\Box\phi = \phi^2$ of the standard scalar ϕ^3 theory can be solved in a perturbiner expansion as $\phi(x) = \sum_P \phi_P e^{k\cdot x}\xi^P$, where $\xi^P = \xi^{p_1}\xi^{p_2}\ldots\xi^{p_{|P|}}$ is an auxiliary parameter and the coefficients ϕ_P obey the recursion relations of planar binary trees,

$$\phi_i = 1, \qquad \phi_P = \frac{1}{s_P}\sum_{XY=P} \phi_X\phi_Y\,, \quad X, Y \neq \emptyset. \tag{B.1}$$

It is straightforward to check that (B.1) gives rise to the recurrence relation for the Catalan numbers, $C_0 = 1$, $C_{n+1} = \sum_{i=0}^{n} C_i C_{n-i}$, where C_n refers to the number of terms in the pole expansion of $\phi_{12\ldots n+1}$. Examples of $\phi_{123\ldots n}$ up to $n = 4$ are given by,

$$\begin{aligned} &\phi_1 = 1, \qquad \phi_{12} = \frac{1}{s_{12}}, \qquad \phi_{123} = \frac{1}{s_{12}s_{123}} + \frac{1}{s_{23}s_{123}}\,, \\ &\phi_{1234} = \frac{1}{s_{1234}}\Big(\frac{1}{s_{12}s_{123}} + \frac{1}{s_{23}s_{123}} + \frac{1}{s_{12}s_{34}} + \frac{1}{s_{34}s_{234}} + \frac{1}{s_{23}s_{234}}\Big)\,. \end{aligned} \tag{B.2}$$

[9] When $P = \emptyset$, the sign factor is given by $(-1)^{|\beta|}$.

Note that the above binary trees naturally capture the kinematic pole expansion of "compatible channels" in a color-ordered tree amplitude.

The restriction of ϕ_P by an ordering given by a word A is denoted $\phi_P\big|_A$ and is defined by suppressing a term from ϕ_P if it contains any factor of $s_{abcd\ldots}$ whose letters are not adjacent in the word A. For example, if $A = 1324$ then

$$A = 1324 \Longrightarrow \begin{cases} s_{13}, s_{23}, s_{24}, s_{123}, s_{234}, s_{1234} & \text{allowed} \\ s_{12}, s_{14}, s_{34}, s_{124}, s_{134} & \text{not allowed} \end{cases} \tag{B.3}$$

and the restriction of ϕ_{1234} by $A = 1324$ yields

$$\phi_{1234}\big|_{1324} = \frac{1}{s_{1234}}\Big(\frac{1}{s_{23}s_{123}} + \frac{1}{s_{23}s_{234}}\Big)\,. \tag{B.4}$$

Now define a sign factor as follows

$$\epsilon_{A|B} \equiv \epsilon(A|b_1,b_2)\epsilon(A|b_2,b_3)\ldots\epsilon(A|b_{p-1},b_p), \quad \epsilon(A|i,j) \equiv \begin{cases} +1, & i<j \text{ inside } A \\ -1, & i>j \text{ inside } A \end{cases} \tag{B.5}$$

where "$i < j$ inside A" is true if the letter i appears before j in A. For example, $\epsilon(1324|1,4) = +1$ but $\epsilon(1324|4,1) = -1$. If $P = 123\ldots p$ is the canonical ordering, the sign factor simplifies to $\epsilon(P|Q) = \epsilon_{q_1q_2}\epsilon_{q_2q_3}\ldots\epsilon_{q_{|Q|-1}q_{|Q|}}$ where ϵ_{ij} is the standard anti-symmetric tensor; $\epsilon_{ij} = +1$ if $i < j$ and $\epsilon_{ij} = -1$ if $i > j$.

One can check that the Berends–Giele double-currents (3.3) can be written as

$$\phi_{P|Q} \equiv \epsilon(P|Q)\phi_P\big|_Q. \tag{B.6}$$

Comparing (B.6) with the algorithm of [1] one concludes that the cumbersome factor of $(-1)^{n_{\text{flip}}}$ of [1] admits a simpler representation in terms of epsilon tensors (this observation was made *en passant* in [11]).

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