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Nonlinear generalised functions on manifolds

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In this work we adopt a new approach to the construction of a global theory of algebras of generalised functions on manifolds based on the concept of smoothing operators. This produces a generalisation of previous theories in a form which is suitable for applications to differential geometry. The generalised Lie derivative is introduced and shown to extend the Lie derivative of Schwartz distributions. A new feature of this theory is the ability to define a covariant derivative of generalised scalar fields which extends the covariant derivative of distributions at the level of association. We end by sketching some applications of the theory. This work also lays the foundations for a nonlinear theory of distributional geometry that is developed in a subsequent paper [1] which is based on Colombeau algebras of tensor distributions on manifolds.

1. Introduction

The classical theory of distributions has proved a very powerful tool in the analysis of linear partial differential equations. However, the fact that in general one cannot multiply distributions makes them of limited use in situations where the equations are non-linear. Furthermore, even when dealing with linear differential equations with coefficients of low regularity the corresponding low regularity of the solution can lead to ill-defined products unless one has a theory which allows for a general multiplication of distributions and non-smooth functions.

By developing and building on an approach to distribution theory in which one thinks of generalised functions as limits of families of smooth functions (see for example [2] and [3]) Colombeau [4] showed that it is possible to construct associative, commutative differential algebras which contain the space of distributions as a linear subspace and the space of smooth functions as a differential subalgebra. Colombeau's theory of *generalised functions* has therefore increasingly had an important role to play in the theory of non-linear PDEs, enabling one to obtain generalised solutions in situations where one has ill defined products according to the classical theory, but without having to resort to ad hoc regularisation procedures. This way, genuinely new results can be obtained which are not available by classical methods. Examples of applications of Colombeau's theory to nonlinear PDEs include Burgers' equation [5], the Korteweg-de Vries equation [6], nonlinear elasticity and elastoplastic models [7] and shockwaves in quasi-linear nonconservative systems [8]. Applications to PDEs with coefficients of low regularity include [9] and [10]. For a broader overview of applications of generalised functions to PDEs we refer to the research monographs [11] and [12]. Moreover, this approach to generalised solutions of PDEs is closely related to the recent work of Ruzhansky on very weak solutions of PDEs [13]. All these applications so far restrict their attention to finding solutions to the equations on \mathbb{R}^n in a Euclidean coordinate system. However, there are many situations (see examples below) where one wants to study differential equations on manifolds or in a variety of different coordinate systems. The theory as developed by Colombeau heavily relies on the linear structure of \mathbb{R}^n so is ill-suited to such situations. In applications as for instance flows on manifolds, wave-front sets for PDEs, gauge theories, general relativity and quantum field theory on curved spacetimes it is therefore important to have a diffeomorphism invariant theory. In the present paper we derive such a theory from a new perspective based on the fundamental concept of smoothing operators. In a subsequent paper we will be particularly interested in applications of Colombeau's theory to general relativity (see [14] for a review of this topic) which will require a tensorial version of the theory developed here which we again derive using the notion of smoothing operator.

The basic idea behind the Colombeau construction is to represent generalised functions by families of smooth functions. In the special version of the theory the Colombeau algebra on \mathbb{R}^n is denoted $\mathcal{G}^s(\mathbb{R}^n)$ and the basic space used for its construction consists of nets $(f_\varepsilon)_{\varepsilon \in (0,1]}$ of smooth functions indexed by $\varepsilon \in (0, 1]$. However, this results in many different representations of what is essentially the same function so that one identifies nets which differ by something *negligible*, i.e., by a net of functions whose derivatives vanish faster than any power of ε on any compact set. This identification is realised by factoring out by the set of such functions, but this is not an ideal unless one restricts the basic space to *moderate* nets of functions whose derivatives are bounded on compact sets by some positive power of $1/\varepsilon$. The (special) algebra of generalised functions is therefore defined to be moderate nets of functions modulo negligible nets of functions, see [15] for more details.

Despite factoring out by negligible nets of functions, the notion of generalised function within Colombeau algebras is finer than that within conventional distribution theory, and it is this feature that enables one to circumvent Schwartz' result on the impossibility of multiplying distributions [16]. Although the pointwise product of smooth functions commutes with the embedding into the algebra, the pointwise product of continuous functions does not, and indeed this cannot be the case due to the Schwartz impossibility result. However, an important feature of Colombeau algebras is an equivalence relation known as *association* which coarse grains the algebra. At

the level of association the pointwise product of continuous functions does indeed commute with the embedding. Furthermore, many (but not all) elements of the algebra are associated to conventional distributions. This feature has the advantage that in many cases one may use the mathematical power of the differential algebra to perform classically ill-defined calculations but then use the notion of association to give a physical interpretation to the answer.

Unfortunately, the special algebra suffers from the disadvantage that there is no canonical embedding of distributions into it. In some situations this is not a problem because some special mathematical or physical feature of the problem may be used to define a preferred embedding. However, in general there is no such preferred embedding into the special algebra, so in section 2 we will briefly review the *full* Colombeau algebra \mathcal{G} in which the generalised functions are parameterised by elements ϕ of a space of mollifiers \mathcal{A}_k . This enables one to define an associative commutative differential algebra on \mathbb{R}^n which contains the space of smooth functions as a subalgebra and has a *canonical* embedding of the space of distributions as a linear subspace. Furthermore, the embedding commutes with (distributional) partial derivatives. Within \mathcal{G} one also has a notion of association which may be used to give a distributional interpretation to certain generalized functions.

Although the full Colombeau algebra on \mathbb{R}^n permits a canonical embedding of the space of distributions as a linear subspace, this has been bought at the price of giving up manifest coordinate invariance. Indeed, the definition of the spaces \mathcal{A}_k of mollifiers which are used to define the algebra is coordinate dependent, and the linear structure of \mathbb{R}^n is used in an essential way in the definitions of moderateness, negligibility and the embedding. However, there are many potential applications of the theory to situations where the equations are defined on *manifolds* and the corresponding equations and solutions can be defined geometrically independently of a particular coordinate system. To deal with these situations it is necessary to have a coordinate invariant generalisation of the full algebra. Such an algebra was first proposed by Colombeau and Meril [17]. Their approach was to give a local description of the algebra together with a transformation law for the generalised functions which ensures that the embedding into the algebra commutes with coordinate transformations. This work suffered from some technical problems but building on these ideas it was shown that one can construct a global Colombeau algebra of generalised functions on manifolds (see [18] for details) retaining all the distinguishing features of the local theory in the global context.

Following work by Grosser et al [19] on a local description of such a diffeomorphism invariant theory Grosser et al [18] showed how to construct a global Colombeau algebra of generalised functions on manifolds retaining all the distinguishing features of the local theory in the global context. However both these constructions were to some extent ad-hoc generalisations of previous work and although the algebra constructed in [18] is mathematically satisfactory it is not at all obvious that it is the best way of extending Colombeau's constructions from \mathbb{R}^n to manifolds. In this paper we will present a new functional analytic approach to the problem based on the more fundamental idea of using families of smoothing operators rather than convolution to construct the algebra of generalised functions. A *smoothing operator* Φ on M is a linear continuous map $\Phi: \mathcal{D}'(M) \rightarrow C^\infty(M)$. A generalised function is then roughly speaking just an object F that when smoothed by Φ gives a smooth function $F(\Phi, \cdot)$. We then consider the action of families of smoothing operators Φ_ϵ that in a certain well-defined sense tend to the identity as $\epsilon \rightarrow 0$. In order that the algebra contains distributions as a linear subspace we will require that in the limit the smoothing of a distribution u should converge in $\mathcal{D}'(M)$ to u , while in order to ensure that the algebra contains smooth functions as a differential subalgebra we will require that in the limit the smoothing operator when applied to smooth functions should be the identity in C^∞ . In addition we will require that the family of smoothing operators should be localising in the sense that (asymptotically) they do not increase the support of a smoothed distribution and that they satisfy some seminorm estimate which controls the growth of derivatives. We will see that these very natural requirements lead to an essentially unique theory with some additional features not found in [18] or [19].

As to the physical interpretation of these generalised functions, one should view them as idealized functions which can be probed by smoothings Φ_ϵ at different scales ϵ to give smooth functions. In this sense the Φ_ϵ correspond to the test functions of distribution theory, just with an additional dependence on the space variable for the purpose of localisation.

While it would have been entirely possible to define the present theory entirely using the language of smoothing operators it is convenient to make contact with the previous approach of [18] by making use of the Schwartz kernel theorem $L(\mathcal{D}'(M), C^\infty(M)) \cong C^\infty(M, \Omega_c^n(M))$ to identify our space of smoothing operators on the left with the space of smoothing kernels (as used in [18]) on the right (see details below). It then turns out that the new version of the theory described here is similar but not identical to that previously studied. We note however that there are some important subtle differences and in particular our basic space is larger than that used in [18] and this allows us to define a generalised covariant derivative. In a subsequent paper [1] we show how it is possible to use our approach applied to the smoothing of tensor distributions to define an algebra of generalised tensor fields on a manifold which contains the spaces of smooth tensor fields as a subalgebra and has a canonical coordinate independent embedding of the spaces of tensor distributions as linear subspaces. The existence of the covariant derivative given by Definition 33 in the present paper (which is not available in [18]) plays a crucial role in constructing a nonlinear theory of distributional differential geometry [1].

In order to make the presentation self-contained we begin in section 2 of this paper by briefly reviewing the Colombeau theory of generalised functions on \mathbb{R}^n emphasising the structural issues that will be important in generalising this to manifolds. Section 3 contains the new results describing how to construct an algebra of generalised functions based on the concept of smoothing operators while section 4 contains a brief discussion of applications of the theory. Note that in contrast to the theory on \mathbb{R}^n the theory of generalised functions on manifolds involves a number of technical issues involving in particular the theory of differentiation in locally convex spaces. We will not go into all the details here, but the approach will be to use the *convenient setting of global analysis* of [20].

2. The full Colombeau algebra on \mathbb{R}^n

In this section we briefly describe the construction of the full Colombeau algebra in \mathbb{R}^n (for further details and proofs see [4]). The starting point is the observation that one can smooth functions by taking the (anti)-convolution with a suitable mollifier. Let $\mathcal{D}(\mathbb{R}^n)$ denote the space of smooth functions on \mathbb{R}^n with compact support. We define $\mathcal{A}_0(\mathbb{R}^n)$ to be the set of those $\phi \in \mathcal{D}(\mathbb{R}^n)$ which satisfy the normalisation condition

$$\int_{\mathbb{R}^n} \phi(\mathbf{x}) \, d\mathbf{x} = 1.$$

Given $\epsilon > 0$ we set

$$\phi_\epsilon(\mathbf{x}) = \frac{1}{\epsilon^n} \phi\left(\frac{\mathbf{x}}{\epsilon}\right),$$

so that ϕ_ϵ has support scaled by ϵ and its amplitude adjusted so that its integral is still one.

Note that $(\phi_\epsilon)_\epsilon$ is an example of a net of smooth functions with the delta distribution as its limit in the sense that

$$\lim_{\epsilon \rightarrow 0} \int_{\mathbb{R}^n} \phi_\epsilon(\mathbf{x}) \Psi(\mathbf{x}) \, d\mathbf{x} = \Psi(0) \quad \forall \Psi \in \mathcal{D}(\mathbb{R}^n).$$

This is sometimes called a model delta net (see [11, Def. 7.9, p. 66]). Provided $f \in L^1_{\text{loc}}(\mathbb{R}^n)$ (i.e., f is a locally integrable function), for each $\phi \in \mathcal{A}_0(\mathbb{R}^n)$ we can define a 1-parameter family of smooth functions \tilde{f}_ϵ by

$$\tilde{f}_\epsilon(\mathbf{x}) = \int_{\mathbb{R}^n} f(\mathbf{y}) \phi_\epsilon(\mathbf{y} - \mathbf{x}) \, d\mathbf{x} \tag{2.1}$$

which converges to f in $\mathcal{D}'(\mathbb{R}^n)$. However, in what follows it will be important to regard ϕ as well as ϵ as a parameter so we write expression (2.1) as $\tilde{f}(\phi_\epsilon, \mathbf{x})$.

It will also be convenient to introduce the *translation operator* τ defined by

$$(\tau_{\mathbf{x}}\phi)(\mathbf{y}) = \phi(\mathbf{y} - \mathbf{x})$$

for $x, y \in \mathbb{R}^n$ and $\phi \in \mathcal{D}(\mathbb{R}^n)$. In order to match the notation of the theory on manifolds we could also write $\phi_{\mathbf{x},\varepsilon} = \tau_{\mathbf{x}}\phi_\varepsilon$, so that for fixed \mathbf{x} , $\phi_{\mathbf{x},\varepsilon}$ is a 1-parameter family of smooth functions converging in $\mathcal{D}'(\mathbb{R}^n)$ to $\delta_{\mathbf{x}}$, the delta distribution at \mathbf{x} .

A distribution $T \in \mathcal{D}'(\mathbb{R}^n)$ is a linear functional on the space of smooth test functions $\mathcal{D}(\mathbb{R}^n)$ and we may generalise equation (2.1) to distributions by defining a 1-parameter family of smooth functions \tilde{T}_ε by

$$\tilde{T}_\varepsilon = \langle T, \tau_{\mathbf{x}}\phi_\varepsilon \rangle. \quad (2.2)$$

Again, we will write this expression as $\tilde{T}(\phi_\varepsilon, \mathbf{x})$.

In order to construct the algebra of generalised functions we define a grading on the space of mollifiers in terms of moment conditions. Note that we will throughout use multi-index notation so that $\mathbf{i} = (i_1, \dots, i_n)$ and $\mathbf{x}^{\mathbf{i}} = x_1^{i_1} \dots x_n^{i_n}$.

Definition 1. For $q \in \mathbb{N}$ we define $\mathcal{A}_q(\mathbb{R}^n)$ to be the set of functions $\phi \in \mathcal{A}_0(\mathbb{R}^n)$ such that

$$\int_{\mathbb{R}^n} \mathbf{x}^{\mathbf{i}} \phi(\mathbf{x}) \, d\mathbf{x} = 0 \quad \forall \mathbf{i} \in \mathbb{N}_0^n \text{ with } 0 < |\mathbf{i}| \leq q.$$

We are now in a position to construct the full Colombeau algebra on \mathbb{R}^n . Our basic space will be the following.

Definition 2. $\mathcal{E}(\mathbb{R}^n)$ is defined to be the set of all maps

$$\begin{aligned} F: \mathcal{A}_0(\mathbb{R}^n) \times \mathbb{R}^n &\rightarrow \mathbb{R} \\ (\phi, \mathbf{x}) &\mapsto F(\phi, \mathbf{x}) \end{aligned}$$

which for fixed ϕ are smooth as functions of \mathbf{x} .

The lack of any continuity requirement with respect to ϕ reflects their role as parameters rather than test functions.

On $\mathcal{E}(\mathbb{R}^n)$ we may define the product FG by

$$(FG)(\phi, \mathbf{x}) = F(\phi, \mathbf{x})G(\phi, \mathbf{x})$$

and the derivative operation

$$(\partial_i F)(\phi, \mathbf{x}) = \frac{\partial}{\partial x^i} (F(\phi, \mathbf{x}))$$

for $i = 1 \dots n$, which together give $\mathcal{E}(\mathbb{R}^n)$ the structure of a differential algebra.

However, as it stands, the space $\mathcal{E}(\mathbb{R}^n)$ is much too large and, thinking in terms of the limit $\varepsilon \rightarrow 0$, contains many representations of what are essentially the same functions. For example, to represent a given smooth function $f \in C^\infty(\mathbb{R}^n)$ we are given $\tilde{f} \in \mathcal{E}(\mathbb{R}^n)$ as above by

$$\tilde{f}(\phi_\varepsilon, \mathbf{x}) = \int_{\mathbb{R}^n} f(\mathbf{y}) \phi_\varepsilon(\mathbf{y} - \mathbf{x}) \, d\mathbf{y} \quad (2.3)$$

but since f is smooth we can also define another family $\hat{f}(\phi, \mathbf{x})$ (which does not in fact depend on ϕ) by

$$\hat{f}(\phi_\varepsilon, \mathbf{x}) = f(\mathbf{x}). \quad (2.4)$$

Note that in the above equations (2.3) and (2.4) we have chosen to use the scaled mollifiers ϕ_ε . Strictly speaking, however, when using these equations to define elements of $\mathcal{E}(\mathbb{R}^n)$ one uses a general mollifier $\phi \in \mathcal{A}_0(\mathbb{R}^n)$.

We therefore want to introduce an equivalence relation such that \tilde{f} and \hat{f} become equivalent. Expanding $f(\mathbf{x} + \varepsilon\mathbf{y})$ in a Taylor series and using the moment conditions for $\phi \in \mathcal{A}_k(\mathbb{R}^n)$ we see that

$$\begin{aligned}\tilde{f}(\phi_\varepsilon, \mathbf{x}) &= \frac{1}{\varepsilon^n} \int_{\mathbb{R}^n} f(\mathbf{y}) \phi\left(\frac{\mathbf{y} - \mathbf{x}}{\varepsilon}\right) d\mathbf{y} \\ &= \int_{\mathbb{R}^n} f(\mathbf{x} + \varepsilon\mathbf{y}) \phi(\mathbf{y}) d\mathbf{y} \\ &= \int_{\mathbb{R}^n} \left\{ f(\mathbf{x}) + \sum_{|\mathbf{l}|=1}^q \frac{\varepsilon^{|\mathbf{l}|} \mathbf{y}^{\mathbf{l}}}{|\mathbf{l}|!} (\partial^{\mathbf{l}}) f(\mathbf{x}) \right. \\ &\quad \left. + \sum_{|\mathbf{l}|=q+1} \frac{q+1}{\mathbf{l}!} \varepsilon^{q+1} \mathbf{y}^{\mathbf{l}} \int_0^1 (1-t)^q (\partial^{\mathbf{l}}) f(\mathbf{x} + t\varepsilon\mathbf{y}) dt \right\} \phi(\mathbf{y}) d\mathbf{y} \\ &= f(\mathbf{x}) + \varepsilon^{q+1} \sum_{|\mathbf{l}|=q+1} \frac{q+1}{\mathbf{l}!} \int_{\mathbb{R}^n} \int_0^1 \mathbf{y}^{\mathbf{l}} (1-t)^q \\ &\quad \cdot \partial^{\mathbf{l}} f(\mathbf{x} + t\varepsilon\mathbf{y}) \phi(\mathbf{y}) dt d\mathbf{y} \\ &= \hat{f}(\phi_\varepsilon, \mathbf{x}) + O(\varepsilon^{q+1})\end{aligned}$$

where $\partial^{\mathbf{l}}$ is the derivative operator given by $\partial^{\mathbf{l}} = \partial_1^{l_1} \dots \partial_n^{l_n}$. Thus, by choosing ϕ to be in $\mathcal{A}_q(\mathbb{R}^n)$ for suitably large q we can make $\tilde{f} - \hat{f}$ tend to zero like an arbitrary power of ε . Requiring a similar condition for the derivatives motivates the following definition.

Definition 3 (Negligible functions). $\mathcal{N}(\mathbb{R}^n)$ is defined to be the set of functions $F \in \mathcal{E}(\mathbb{R}^n)$ such that for all compact $K \subset \mathbb{R}^n$, for all $\mathbf{k} \in \mathbb{N}_0^n$ and for all $m \in \mathbb{N}$ there is some $q \in \mathbb{N}$ such that if $\phi \in \mathcal{A}_q(\mathbb{R}^n)$ then

$$\sup_{\mathbf{x} \in K} \left| \partial^{\mathbf{k}} F(\phi_\varepsilon, \mathbf{x}) \right| = O(\varepsilon^m) \quad \text{as } \varepsilon \rightarrow 0.$$

Note that the derivative $\partial^{\mathbf{k}}$ acts only on the x -variable here, contrary to the situation later on where we also have to consider derivatives with respect to ϕ .

The key result that follows from this is that for a smooth function f we have that $\tilde{f} - \hat{f}$ is in $\mathcal{N}(\mathbb{R}^n)$. However, in order to define an algebra we would like to factor out by $\mathcal{N}(\mathbb{R}^n)$, and this requires it to be an ideal. Unfortunately, this is not the case because we can multiply elements of $\mathcal{N}(\mathbb{R}^n)$ by elements of $\mathcal{E}(\mathbb{R}^n)$ with rapid non-polynomial growth in $1/\varepsilon$ so that the conditions of Definition 3 are no longer satisfied. We therefore restrict $\mathcal{E}(\mathbb{R}^n)$ to the subalgebra of functions of moderate growth in the following sense.

Definition 4 (Moderate functions). $\mathcal{E}_M(\mathbb{R}^n)$ is defined to be the set of functions $F \in \mathcal{E}(\mathbb{R}^n)$ such that for all compact $K \subset \mathbb{R}^n$ and for all $\mathbf{k} \in \mathbb{N}_0^n$ there is some $N \in \mathbb{N}$ such that if $\phi \in \mathcal{A}_N(\mathbb{R}^n)$ then

$$\sup_{\mathbf{x} \in K} \left| \partial^{\mathbf{k}} F(\phi_\varepsilon, \mathbf{x}) \right| = O(\varepsilon^{-N}) \quad \text{as } \varepsilon \rightarrow 0.$$

Proposition 5. $\mathcal{N}(\mathbb{R}^n)$ is an ideal in $\mathcal{E}_M(\mathbb{R}^n)$.

We may therefore define the space of generalised functions $\mathcal{G}(\mathbb{R}^n)$ as a factor algebra.

Definition 6 (Generalised functions).

$$\mathcal{G}(\mathbb{R}^n) = \mathcal{E}_M(\mathbb{R}^n) / \mathcal{N}(\mathbb{R}^n).$$

Although the definition of a negligible function F requires estimates for the derivatives $|\partial^k F(\phi_\varepsilon, \mathbf{x})|$ these are in fact not needed as is shown by the following useful proposition.

Proposition 7 ([21, Theorem 1.4.8]). *Let $F \in \mathcal{E}_M(\mathbb{R}^n)$ be such that for all compact $K \subset \mathbb{R}^n$, for all $m \in \mathbb{N}$, there is some $q \in \mathbb{N}$ such that if $\phi \in \mathcal{A}_q(\mathbb{R}^n)$ then*

$$\sup_{\mathbf{x} \in K} |F(\phi_\varepsilon, \mathbf{x})| = O(\varepsilon^m) \quad \text{as } \varepsilon \rightarrow 0.$$

Then $F \in \mathcal{N}(\mathbb{R}^n)$.

Hence any moderate function which satisfies the negligibility condition, without differentiating, is negligible.

One may now show that one has an embedding

$$\begin{aligned} \iota: \mathcal{D}'(\mathbb{R}^n) &\rightarrow \mathcal{G}(\mathbb{R}^n) \\ T &\mapsto [\tilde{T}] \end{aligned} \tag{2.5}$$

where $[\tilde{T}]$ denotes the equivalence class of $\tilde{T} \in \mathcal{E}_M(\mathbb{R}^n)$ and

$$\tilde{T}(\phi, \mathbf{x}) := \langle T, \tau_{\mathbf{x}} \phi \rangle.$$

The only thing we need to establish is that \tilde{T} is moderate.

As we may assume without limitation of generality that $\tilde{T} = \partial^l f$ for some continuous function $f(\mathbf{y})$ in a neighbourhood of a given compact set, differentiating the expression for \tilde{T} with respect to \mathbf{x} we obtain

$$\partial^k \tilde{T}(\phi_\varepsilon, \mathbf{x}) = \langle \partial_{\mathbf{y}}^l f, \partial_{\mathbf{x}}^k (\tau_{\mathbf{x}} \phi_\varepsilon) \rangle = \langle f, (-1)^{|\mathbf{l}|} \partial_{\mathbf{y}}^l \partial_{\mathbf{x}}^k (\tau_{\mathbf{x}} \phi_\varepsilon) \rangle.$$

Now $\tau_{\mathbf{x}} \phi_\varepsilon(\mathbf{y}) = \frac{1}{\varepsilon^n} \phi\left(\frac{\mathbf{y}-\mathbf{x}}{\varepsilon}\right)$ so that

$$(-1)^{|\mathbf{l}|} \partial_{\mathbf{y}}^l \partial_{\mathbf{x}}^k (\tau_{\mathbf{x}} \phi_\varepsilon(\mathbf{y})) = \frac{1}{\varepsilon^{n+|\mathbf{l}|+|\mathbf{k}|}} (-1)^{|\mathbf{l}|+|\mathbf{k}|} (\partial^{\mathbf{k}+\mathbf{l}} \phi)\left(\frac{\mathbf{y}-\mathbf{x}}{\varepsilon}\right).$$

Thus, uniformly for x in the compact set we have

$$\begin{aligned} \partial^k \tilde{T}(\phi_\varepsilon, \mathbf{x}) &= \frac{1}{\varepsilon^{n+|\mathbf{l}|+|\mathbf{k}|}} \int f(\mathbf{y}) (-1)^{|\mathbf{l}|+|\mathbf{k}|} (\partial^{\mathbf{k}+\mathbf{l}} \phi)\left(\frac{\mathbf{y}-\mathbf{x}}{\varepsilon}\right) d\mathbf{y} \\ &= \frac{1}{\varepsilon^{|\mathbf{l}|+|\mathbf{k}|}} \int f(\mathbf{x} + \varepsilon \mathbf{y}) (-1)^{|\mathbf{l}|+|\mathbf{k}|} (\partial^{\mathbf{k}+\mathbf{l}} \phi)(\mathbf{y}) d\mathbf{y} = O(\varepsilon^{-|\mathbf{l}|-|\mathbf{k}|}) \end{aligned}$$

so that \tilde{T} is moderate.

Smooth functions can be embedded directly via the mapping

$$\begin{aligned} \sigma: C^\infty(\mathbb{R}^n) &\rightarrow \mathcal{G}(\mathbb{R}^n) \\ f &\mapsto [\hat{f}] \end{aligned} \tag{2.6}$$

where we set $\hat{f}(\phi, x) := f(x)$. The significance of the mapping σ lies in the fact that it is an algebra homomorphism, i.e., it preserves the product of smooth functions, which a priori is not clear for the embedding ι from above; however, it turns out that on $C^\infty(\mathbb{R}^n)$ these embeddings coincide.

More precisely, the main properties of $\mathcal{G}(\mathbb{R}^n)$ are contained in the following proposition. For proofs and further details see [4] and [11,21].

Proposition 8.

- (a) $\mathcal{G}(\mathbb{R}^n)$ is an associative commutative differential algebra.
- (b) The embedding ι defined in (2.6) embeds $\mathcal{D}'(\mathbb{R}^n)$ as a linear subspace.
- (c) For smooth functions $f \in C^\infty(\mathbb{R}^n)$ we have $\iota(f) = \sigma(f)$, so that $\mathcal{G}(\mathbb{R}^n)$ contains the space of smooth functions as a subalgebra.

(d) The embedding ι commutes with (distributional) partial differentiation so that for all $T \in \mathcal{D}'(\mathbb{R}^n)$ and $k \in \mathbb{N}_0^n$ we have

$$\iota(\partial^k T)(\phi, \mathbf{x}) = \partial^k (\iota T)(\phi, \mathbf{x}).$$

As we remarked earlier an important concept is that of association.

Definition 9 (Association). We say an element $[F]$ of $\mathcal{G}(\mathbb{R}^n)$ is associated to 0 (denoted $[F] \approx 0$) if for each $\Psi \in \mathcal{D}(\mathbb{R}^n)$ there exists some $q > 0$ with

$$\lim_{\varepsilon \rightarrow 0} \int_{\mathbf{x} \in \mathbb{R}^n} F(\phi_\varepsilon, \mathbf{x}) \Psi(\mathbf{x}) d\mathbf{x} = 0 \quad \forall \phi \in \mathcal{A}_q(\mathbb{R}^n).$$

We say two elements $[F], [G]$ of $\mathcal{G}(\mathbb{R}^n)$ are associated and write $[F] \approx [G]$ if $[F - G] \approx 0$.

Definition 10 (Associated distribution). We say $[F] \in \mathcal{G}(\mathbb{R}^n)$ admits $T \in \mathcal{D}'(\mathbb{R}^n)$ as an associated distribution if for each $\Psi \in \mathcal{D}(\mathbb{R}^n)$ there exists some $q > 0$ with

$$\lim_{\varepsilon \rightarrow 0} \int_{\mathbf{x} \in \mathbb{R}^n} F(\phi_\varepsilon, \mathbf{x}) \Psi(\mathbf{x}) d\mathbf{x} = \langle T, \Psi \rangle \quad \forall \phi \in \mathcal{A}_q(\mathbb{R}^n).$$

These definitions do not depend on the choice of representative; moreover, note that not all generalised functions are associated to a distribution.

At the level of association we regain the following compatibility results for multiplication of distributions.

Proposition 11 ([21, Thm. 1.4.26]).

(a) If $f \in C^\infty(\mathbb{R}^n)$ and $T \in \mathcal{D}'(\mathbb{R}^n)$ then

$$\iota(f)\iota(T) \approx \iota(fT).$$

(b) If $f, g \in C^0(\mathbb{R}^n)$ then

$$\iota(f)\iota(g) \approx \iota(fg).$$

Although the partial derivative commutes with the embedding this is not true of the Lie derivative. In fact, if $X \in \mathfrak{X}(\mathbb{R}^n)$ is a smooth vector field on \mathbb{R}^n and $T \in \mathcal{D}'(\mathbb{R}^n)$ then

$$\begin{aligned} (\mathcal{L}_X \tilde{T})(\phi_\varepsilon, \mathbf{x}) &= X^a(\mathbf{x}) \partial_a (\tilde{T}(\phi_\varepsilon, \mathbf{x})) \\ &= X^a(\mathbf{x}) \langle \partial_a T, \tau_{\mathbf{x}} \phi_\varepsilon \rangle. \end{aligned}$$

On the other hand,

$$(\widetilde{\mathcal{L}_X T})(\phi_\varepsilon, \mathbf{x}) = \langle X^a \partial_a T, \tau_{\mathbf{x}} \phi_\varepsilon \rangle.$$

These two expressions are not the same in general since the first only involves the value of the vector field at \mathbf{x} , while the second involves the values in a neighbourhood of \mathbf{x} . In fact, if these expressions always were the same this would mean the embedding commutes with multiplication by smooth functions, which contradicts the Schwartz impossibility result. However, by part (a) of Proposition 11 the two expressions are associated since X^a is a smooth function for $a = 1 \dots n$. We also note that if f is a smooth function then $\mathcal{L}_X \tilde{f} = \widetilde{\mathcal{L}_X f}$ since we may represent \tilde{f} by f and $\widetilde{\mathcal{L}_X f}$ by $\mathcal{L}_X f$. We therefore have the following proposition.

Proposition 12.

(a) Let $f \in C^\infty(\mathbb{R}^n)$ and X be a smooth vector field. Then,

$$\mathcal{L}_X(\iota(T)) = \iota(\mathcal{L}_X T).$$

(b) Let $T \in \mathcal{D}'(\mathbb{R}^n)$ and X be a smooth vector field. Then,

$$\mathcal{L}_X(\iota(T)) \approx \iota(\mathcal{L}_X T).$$

It is also possible to localise the entire construction to obtain $\mathcal{G}(\Omega)$ for open sets $\Omega \subset \mathbb{R}^n$ by restricting x to lie in Ω in the relevant definitions. The only technical complication relates to the embedding where one must first extend the distribution and then show that the result is independent of the extension (see [4] for details).

3. Smoothing distributions and the Colombeau algebra on manifolds

A coordinate independent description of generalised functions on open sets $\Omega \subset \mathbb{R}^n$ was proposed by Colombeau and Meril [17]. However, this suffered from a number of minor defects; in particular, their definition of a space of test objects $\mathcal{A}_k(\Omega)$ did not take into account the x -dependence of the mollifiers which meant that the definition of moderate functions was dependent on the coordinate system used. An explicit counterexample due to Jelínek [22] demonstrated that the construction was not in fact diffeomorphism invariant. In the same paper Jelínek gave an improved version of the theory which clarified a number of important issues but fell short of proving the existence of a coordinate invariant algebra. The existence of a (local) diffeomorphism invariant Colombeau algebra on open subsets Ω of \mathbb{R}^n was finally established by Grosser et al. [19]. In parallel with this, [23] proposed a definition of a global manifestly diffeomorphism invariant theory on manifolds. By making use of the characterisation results of [19] it was shown in [18] that one can construct a global Colombeau algebra $\mathcal{G}(M)$ of generalised functions on manifolds. There, it was demonstrated how to obtain a canonical linear embedding of $\mathcal{D}'(M)$ into $\mathcal{G}(M)$ that renders $C^\infty(M)$ a faithful subalgebra of $\mathcal{G}(M)$. In addition, it was shown that this embedding commutes with the generalised Lie derivative, ensuring that the theory retains all the distinguishing features of the local theory in the global context. Although this theory has a well defined generalised Lie derivative it turns out that there is no natural definition of a generalised covariant derivative. In this section we describe a new approach to Colombeau algebras [24] based on the concept of smoothing operators that is closer to the intuitive idea of a generalised function as a family of smooth functions. This results in a new basic space which allows us to define both a generalised Lie derivative and a covariant derivative. Replacing the expressions $\tau_x \phi_\varepsilon = \phi_{x,\varepsilon}$ in (2.3) with $\phi \in \mathcal{A}_q(\mathbb{R}^n)$ by suitable nets of smoothing kernels we are able to use asymptotic versions of the moment conditions and hence do not need the grading of Definition 1 anymore, which results in a quantifier less in the definitions of moderateness, negligibility and association. In contrast to [18], which made use of the local theory in a number of key places, in the current paper we give intrinsic definitions on the whole of the manifold M . As here we only outline the general theory, we refer for full proofs to [25].

On \mathbb{R}^n the space of distributions $\mathcal{D}'(\mathbb{R}^n)$ is dual to the space of smooth functions of compact support, whereas on an orientable manifold the space of distributions $\mathcal{D}'(M)$ is dual to $\Omega_c^n(M)$, the space of n -forms of compact support. On not necessarily orientable manifolds, one uses densities instead of n -forms but on an oriented manifold these are the same. In the Colombeau theory on \mathbb{R}^n smoothness of the embedded functions in x is obtained by integrating against mollifier functions $\phi(y - x)$. The obvious generalisation on manifolds is to replace the mollifier ϕ by an n -form ω . However, on a manifold it does not make sense to look at $\omega(y - x)$ since $y - x$ has no coordinate independent meaning, so instead we will look at objects $\omega_x(y)$ which are n -forms in y parameterised by $x \in M$. We therefore make the following definition.

Definition 13. A smoothing kernel ω on M is a smooth map

$$\begin{aligned} \omega: M &\rightarrow \Omega_c^n(M) \\ x &\mapsto \omega_x \end{aligned}$$

and we denote the space of such objects $SK(M)$. Thus $SK(M) = C^\infty(M, \Omega_c^n(M))$.¹

The key new idea (see [24] for more details) is not to immediately generalise the Colombeau construction of \mathbb{R}^n to a manifold M in some ad-hoc way, but to understand smoothing of distributions with the notion of smoothing operators and relate this to smoothing kernels.

Definition 14. A smoothing operator Φ on M is a linear continuous map

$$\Phi: \mathcal{D}'(M) \rightarrow C^\infty(M)$$

We denote the space of such objects by $L(\mathcal{D}'(M), C^\infty(M))$.

Given a smoothing operator Φ we may associate to it a smoothing kernel ω in the following way: if for $u \in \mathcal{D}'(M)$ and $x \in M$ we demand that

$$\Phi(u)(x) = \langle u, \omega_x \rangle \tag{3.1}$$

then setting $u = \delta_y$ implies

$$\omega_x(y) = \langle \delta_y, \omega_x \rangle = \Phi(\delta_y)(x). \tag{3.2}$$

Thus, given a smoothing operator $\Phi \in L(\mathcal{D}'(M), C^\infty(M))$, this converts a distribution $u \in \mathcal{D}'(M)$ into a smooth function $\Phi(u)$ by the action of u on the smoothing kernel ω given by (3.2). Conversely, given a smoothing kernel $\omega \in SK(M)$ we obtain a smoothing operator by (3.1). Indeed, it follows from a variant of the Schwartz kernel theorem that this correspondence is a topological isomorphism

$$L_b(\mathcal{D}'(M), C^\infty(M)) \cong C^\infty(M, \Omega_c^n(M)) = SK(M) \tag{3.3}$$

where the left hand side has the topology of bounded convergence and the right hand side has the topology of uniform convergence on compact sets in all derivatives [26].

We therefore take our basic space $\hat{\mathcal{E}}(M)$ of generalised functions to consist of (smooth) maps from the space of smoothing kernels to the space of smooth functions:

Definition 15 (Basic space).

$$\hat{\mathcal{E}}(M) := C^\infty(SK(M), C^\infty(M)).$$

Note that in this definition (and elsewhere in the paper where we consider smooth maps defined on infinite dimensional locally convex spaces) we will use the definition of smoothness based on *the convenient setting of global analysis* of [20]. The basic idea of this approach is that a map $f: E_1 \rightarrow E_2$ between locally convex spaces is smooth if it transports smooth curves in E_1 to smooth curves in E_2 (where the notion of smooth curves is straightforward via limits of difference quotients).

Actually the basic space $\hat{\mathcal{E}}(M)$ is somewhat larger than we want since for given $F \in \hat{\mathcal{E}}(M)$ it allows $F(\omega)$ to depend on ω globally, which destroys the sheaf character of the algebra. We therefore restrict to a sub-algebra $\hat{\mathcal{E}}_{loc}(M)$ consisting of *local* elements $F \in \hat{\mathcal{E}}(M)$, defined by the property that if two smoothing kernels ω and $\tilde{\omega}$ agree on some open set $U \subseteq M$ then $F(\omega)$ and $F(\tilde{\omega})$ also agree on U . Note that all embedded elements satisfy this condition so that there is no real loss of generality in restricting to this space. Therefore, for the rest of the paper we will work exclusively with $\hat{\mathcal{E}}_{loc}(M)$ as the basic space. For an in-depth exposition of this topic we refer to [27].

¹One may also view smoothing kernels as certain sections of the pullback bundle $\text{pr}_2^*(\Lambda^n T^*M)$, i.e., the pullback of top forms along the projection $\text{pr}_2: M \times M \rightarrow M$ onto the second factor. However, the definition of $SK(M)$ as it is given takes care of compact support in the second variable in the proper way and is actually necessary in the functional analytic approach, as is seen for example via its representation as the completed tensor product $C^\infty(M) \hat{\otimes} \Omega_c^n(M)$.

The basic space naturally contains both $\mathcal{D}'(M)$ and $C^\infty(M)$ via the linear embeddings ι and σ

$$\begin{aligned}\iota: \mathcal{D}'(M) &\rightarrow \hat{\mathcal{E}}_{\text{loc}}(M) & (\iota u)(\omega)(x) &:= \langle u, \omega_x \rangle \\ \sigma: C^\infty(M) &\rightarrow \hat{\mathcal{E}}_{\text{loc}}(M) & (\sigma f)(\omega)(x) &:= f(x)\end{aligned}$$

and inherits the algebra structure from $C^\infty(M)$ through the product

$$(F_1 \cdot F_2)(\omega) := F_1(\omega)F_2(\omega), \quad F_1, F_2 \in \hat{\mathcal{E}}_{\text{loc}}(M), \quad \omega \in \text{SK}(M).$$

We may regard a smooth function as a regular distribution so that one may embed it either via σ to obtain $(\sigma f)(\omega)(x) = f(x)$ or via ι to obtain $(\iota f)(\omega)(x) = \int f(y)\omega_x(y)$. In order to identify these expressions we would like to set $\omega_x = \delta_x$. Strictly speaking this is not possible, but replacing ω_x by a net $(\omega_{x,\varepsilon})_\varepsilon$ of n -forms which tends to δ_x appropriately as $\varepsilon \rightarrow 0$ and using suitable asymptotic estimates to define negligibility allows us to construct a quotient algebra in which the two embeddings of smooth functions agree.

The next key concept required is therefore that of a delta net of smoothing kernels $\omega_{x,\varepsilon}$ which will play the role of the ε dependent mollifiers $\phi_{x,\varepsilon}$ used in the embedding of distributions on \mathbb{R}^n . To make our notation clear, let $\text{SK}(M)^{(0,1]}$ be the space of nets $(\omega_\varepsilon)_{\varepsilon \in (0,1]}$ of smoothing kernels, indexed by $\varepsilon \in (0, 1]$. If $(\omega_\varepsilon)_\varepsilon \in \text{SK}(M)^{(0,1]}$ is a net of smoothing kernels then given $\varepsilon \in (0, 1]$ and $x \in M$, the n -form obtained by evaluating $\omega_\varepsilon \in \text{SK}(M) = C^\infty(M, \Omega_c^n(M))$ at x is denoted by $\omega_{x,\varepsilon}$ instead of $(\omega_\varepsilon)_x$.

Since we are working on a manifold we do not have translation and scaling operators available, so we need to consider carefully what properties are required. Again, rather than simply trying to copy the construction on \mathbb{R}^n it is useful to look at what is required from the point of view of the corresponding family of smoothing operators. The key properties are that:

- (a) the family of smoothing operators should be localising in the sense that asymptotically they do not increase the support of a smoothed distribution,
- (b) in the limit the smoothing operator when applied to smooth functions should be the identity in $C^\infty(M)$,
- (c) the family of smoothing operators should satisfy some seminorm estimates which control the growth and
- (d) in the limit the smoothing of a distribution u should converge in $\mathcal{D}'(M)$ to u .

Property (a) means that the support of the corresponding net of smoothing kernels shrinks, (b) ensures that (in the quotient algebra) the embeddings σ and ι coincide, (c) ensures that the embedding of distributions is moderate and property (d) shows that an embedded distribution is associated to the original distribution. More precisely, given a family of smoothing operators $(\Phi_\varepsilon)_{\varepsilon \in (0,1]}$ we require (with $B_r(x)$ denoting the ball of radius r and center x with respect to some fixed Riemannian metric)

- (a) on any compact $K \subset M \exists C, \varepsilon_0 > 0 \forall x \in K \forall \varepsilon \leq \varepsilon_0 \forall u \in \mathcal{D}'(M)$:

$$\left(u|_{B_{C\varepsilon}(x)} = 0 \Rightarrow \Phi_\varepsilon(u)(x) = 0 \right);$$

- (b) for any continuous seminorm p on $L_b(C^\infty(M), C^\infty(M))$ and all $m \in \mathbb{N}$ we have

$$p(\Phi_\varepsilon|_{C^\infty(M)} - \text{id}) = O(\varepsilon^m);$$

- (c) for any continuous seminorm p on $L_b(\mathcal{D}'(M), C^\infty(M))$ there is $N \in \mathbb{N}$ such that

$$p(\Phi_\varepsilon) = O(\varepsilon^{-N});$$

- (d) $\Phi_\varepsilon \rightarrow \text{id}$ in $L_b(\mathcal{D}'(M), \mathcal{D}'(M))$.

Note that in the first condition we demand linear shrinking of support for the sake of computational convenience; and in the second condition we demand convergence like $O(\varepsilon^m)$ for all m at once, which is stronger than in Colombeau's original algebra presented above.

We now use the topological isomorphism (3.3) to translate these conditions into conditions on a net $(\omega_\varepsilon)_\varepsilon$ of smoothing kernels. The first translates into the requirement that the support of the net shrinks, or more precisely that

$$\begin{aligned} &\text{on any compact } K \subset M \exists C, \varepsilon_0 > 0 \\ &\forall x \in K \forall \varepsilon \leq \varepsilon_0 : \text{supp } \omega_{x,\varepsilon} \subseteq B_{C\varepsilon}(x). \end{aligned}$$

It is not hard to see that the condition does not depend upon the particular choice of Riemannian metric.

To formulate the next condition we need the Lie derivative of a smoothing kernel ω , which we will introduce in terms of the 1-parameter family of diffeomorphisms induced by a vector field. In principle we can consider two different diffeomorphisms μ and ν which act separately on the x and y variables of ω , i.e., the pullback action on the parameter x (for fixed y) given by $(\mu^*\omega)_x := \omega_{\mu(x)}$ on the one hand and the pullback action on the form (for fixed x) given by $\nu^*(\omega_x)$ on the other hand. We will denote the combined pullback action on the smoothing kernel by $(\mu^*, \nu^*)\omega := \nu^*(\omega_{\mu(x)})$.

We can therefore also consider two different (complete) vector fields X and Y with corresponding flows Fl_t^X and Fl_t^Y acting on the x and y variables. This enables us to define the (double) Lie derivative

$$\mathcal{L}_{(X,Y)}\omega = \left. \frac{d}{dt} \right|_{t=0} \left((\text{Fl}_t^X)^*, (\text{Fl}_t^Y)^* \right) \omega.$$

Varying the x and y variables separately we have two Lie derivatives

$$(\mathcal{L}_{(X,0)}\omega)_x = \left. \frac{d}{dt} \right|_{t=0} \omega_{\text{Fl}_t^X(x)}$$

and

$$(\mathcal{L}_{(0,Y)}\omega)_x = \left. \frac{d}{dt} \right|_{t=0} (\text{Fl}_t^Y)^*(\omega_x)$$

and hence

$$\mathcal{L}_{(X,Y)}\omega = \mathcal{L}_{(X,0)}\omega + \mathcal{L}_{(0,Y)}\omega.$$

Since $\mathcal{L}_{(X,0)}\omega$ is given by the formula for the Lie derivative of a function we will denote this derivative by $\mathcal{L}_X^{C^\infty}\omega$, and since $\mathcal{L}_{(0,Y)}\omega$ is given by the Lie derivative of an n -form we will denote this derivative by $\mathcal{L}_Y^{\Omega^n}\omega$. Finally, we will often want to take the geometrically natural Lie derivative $\mathcal{L}_{(X,X)}\omega$ of a smoothing kernel which we denote $\mathcal{L}_X^{\text{SK}}\omega$. Note that $\mathcal{L}_{(X,0)}\omega$ is denoted $L'_X\omega$ and $\mathcal{L}_{(0,Y)}\omega$ is denoted $L_Y\omega$ in [18].

We may now define the convergence corresponding to the second condition above by demanding that for each $f \in C^\infty(M)$, all compact subsets $K \subset M$, all $m \in \mathbb{N}_0$ and all $X_1, \dots, X_m \in \mathfrak{X}(M)$ (the space of smooth vector fields on M) we have

$$\sup_{x \in K} \left| \left(\int_M f \mathcal{L}_{X_1}^{C^\infty} \dots \mathcal{L}_{X_m}^{C^\infty} \omega_{x,\varepsilon} \right) - \mathcal{L}_{X_1} \dots \mathcal{L}_{X_m} f(x) \right| = O(\varepsilon^m)$$

as $\varepsilon \rightarrow 0$.

It turns out that the third condition, which allows us to establish the fact that the embedding of a distribution is moderate, takes the following form. For any distribution $u \in \mathcal{D}'(M)$, on any

compact subset $K \subset M$ we require $\forall k \in \mathbb{N}_0 \forall X_1, \dots, X_k \in \mathfrak{X}(M) \exists N \in \mathbb{N}$ such that

$$\sup_{x \in K} |\mathcal{L}_{X_1} \dots \mathcal{L}_{X_k} \langle u, \omega_{x,\varepsilon} \rangle| = O(\varepsilon^{-N}).$$

Finally, the fourth condition which gives convergence in $\mathcal{D}'(M)$ is given by the condition that $\forall u \in \mathcal{D}'(M), \forall \psi \in \Omega_c^n(M)$

$$\lim_{\varepsilon \rightarrow 0} \int_{x \in M} \langle u, \omega_{x,\varepsilon} \rangle \psi(x) = \langle u, \psi \rangle.$$

We are now in a position to define a delta net of smoothing kernels (cf. [25] where the corresponding nets are called *test objects*).

Definition 16 (Delta Nets of Smoothing kernels). $(\omega_\varepsilon)_\varepsilon \in \text{SK}(M)^{(0,1]}$ is called a delta net of smoothing kernels if on any compact subset K of M it satisfies the following conditions:

- (1) $\exists C, \varepsilon_0 \forall x \in K \forall \varepsilon \leq \varepsilon_0: \text{supp } \omega_{x,\varepsilon} \subseteq B_{C\varepsilon}(x);$
 (2) $\forall f \in C^\infty(M) \forall m \in \mathbb{N}, \text{ as } \varepsilon \rightarrow 0:$

$$\sup_{x \in K} \left| \left(\int_M f \mathcal{L}_{X_1}^{C^\infty} \dots \mathcal{L}_{X_m}^{C^\infty} \omega_{x,\varepsilon} \right) - \mathcal{L}_{X_1} \dots \mathcal{L}_{X_m} f(x) \right| = O(\varepsilon^m);$$

- (3) $\forall u \in \mathcal{D}'(M) \forall k \in \mathbb{N}_0 \forall X_1, \dots, X_k \in \mathfrak{X}(M) \exists N \in \mathbb{N}:$

$$\sup_{x \in K} |\mathcal{L}_{X_1} \dots \mathcal{L}_{X_k} \langle u, \omega_{x,\varepsilon} \rangle| = O(\varepsilon^{-N});$$

- (4) $\forall u \in \mathcal{D}'(M) \forall \omega \in \Omega_c^n(M):$

$$\lim_{\varepsilon \rightarrow 0} \int_{x \in \mathbb{R}^n} \langle u, \omega_{x,\varepsilon} \rangle \omega(x) = \langle u, \omega \rangle.$$

The space of delta nets of smoothing kernels on M is denoted $\tilde{\mathcal{A}}(M)$.

Remark 17. We have seen in the previous section that the moment conditions on \mathbb{R}^n allow one to show that for a smooth function $f \in C^\infty(\mathbb{R}^n)$ and for $\phi \in \mathcal{A}_q(\mathbb{R}^n)$ we have (in the case $n = 1$)

$$\tilde{f}(\phi_\varepsilon, x) = f(x) + \frac{\varepsilon^{q+1}}{q!} \int_{\mathbb{R}} \int_0^1 y^{q+1} (1-t)^q f^{(q+1)}(x+t\varepsilon y) \phi(y) dt dy \quad (3.4)$$

so that

$$\tilde{f}(\phi_\varepsilon, x) = f(x) + O(\varepsilon^{q+1}) \quad (3.5)$$

with a similar argument giving the same estimate for the derivatives. By Proposition 7 this shows that $\tilde{f} - \hat{f}$ is negligible and hence that the two possible embeddings of a smooth function coincide in the algebra. On a manifold we have turned things round and instead used (3.5) to characterise the moment condition. As is the case in \mathbb{R}^n we will use this condition to show that the two possible embeddings of smooth functions differ by a negligible function and hence coincide in the factor algebra.

Although not necessary for the bare construction of the theory, it is beneficial for practical calculations to add L^1 -conditions on the nets of smoothing kernels. For example, if we also require that

$$\int_M |\omega_{x,\varepsilon}| \rightarrow 1 \quad \text{uniformly for } x \text{ in compact subsets of } M \quad (3.6)$$

so that (asymptotically) the L^1 -norm of the smoothing kernels is unity, one can then show that

$$\lim_{\varepsilon \rightarrow 0} \int_M f \omega_{x,\varepsilon} = f(x) \quad \forall f \in C^0(M).$$

Hence $\iota(f)(\omega_\varepsilon) = \langle f, \omega_\varepsilon \rangle$ converges to f pointwise. However, this condition is different from condition (2) which involves convergence in $C^\infty(M)$ and requires that the derivatives (of

arbitrary order) also converge to the derivatives of f . Another useful condition imitating the behaviour of scaled and translated mollifiers is

$$\forall K \subseteq M \text{ compact } \forall k \in \mathbb{N}_0 \forall X_1, \dots, X_k \in \mathfrak{X}(M) : \tag{3.7}$$

$$\sup_{x \in K} \left| \mathcal{L}_{X_1}^{C^\infty} \dots \mathcal{L}_{X_k}^{C^\infty} \omega_\varepsilon(x) \right| = O(\varepsilon^{-k}).$$

Before proceeding any further it is reasonable to ask if there are any elements in $\tilde{\mathcal{A}}(M)$ at all, or if the conditions required are so stringent that this set is empty. Elements of this set are constructed in several steps. First, we note that delta nets of smoothing kernels can be restricted to open subsets and also patched together if they are given on overlapping sets, while preserving their asymptotic properties. In other words, delta nets of smoothing kernels form a sheaf in an asymptotic sense. Because the properties defining such delta nets of smoothing kernels are invariant under the action of diffeomorphisms, it hence suffices to construct an element of $\tilde{\mathcal{A}}(\mathbb{R}^n)$. For this we choose a rapidly decreasing smooth function ϕ with integral one and all moments vanishing (note that such a function cannot have compact support). The net of smoothing kernels given by the translated scaled mollifier

$$\phi_{\mathbf{x}, \varepsilon}(\mathbf{y}) = \frac{1}{\varepsilon^n} \phi\left(\frac{\mathbf{y} - \mathbf{x}}{\varepsilon}\right)$$

has all desired properties except that it only applies to compactly supported distributions, but using a partition of unity and suitable cut-off functions one may apply it to arbitrary distributions, which gives the desired construction. Note that this is identical to how the embedding into the special algebra can be obtained ([21, (1.8)]).

Before turning to the definition of moderate and negligible functions we consider the definition of the Lie derivative for elements of the basic space. There are two different ways of thinking about the Lie derivative of an element $F \in \hat{\mathcal{E}}_{\text{loc}}(M)$. The first comes from looking at the pullback action of the diffeomorphism group on the basic space (which we call the geometrical or generalised Lie derivative) while the second comes from thinking of $F(\omega)$ for fixed ω as a smooth function. The former has the advantage that it commutes with the embedding of distributions, but on the other hand it cannot be $C^\infty(M)$ linear in X (since having *both* properties would violate the Schwartz impossibility result). The latter is simply the ordinary Lie derivative of a smooth function and therefore agrees with the directional derivative or covariant derivative of a function. This will allow us to define the covariant derivative of a generalised tensor field in [1]. Although the ordinary Lie derivative does not commute with the embedding of distributions, as is the case on \mathbb{R}^n , it does so at the level of association.

To consider the geometric Lie derivative we start by looking at the action of a diffeomorphism on a generalised function.

Definition 18 (Pullback action). *If $\psi: M \rightarrow N$ is a diffeomorphism then we define the pullback $\psi^*: \hat{\mathcal{E}}_{\text{loc}}(N) \rightarrow \hat{\mathcal{E}}_{\text{loc}}(M)$ by*

$$(\psi^*F)(\omega)(x) := F((\psi^{-1})^*, (\psi^{-1})^*\omega)(\psi(x)).$$

We are now in a position to define the Lie derivative. Let Fl_t^X be the flow generated by the (complete) smooth vector field X . Then for $F \in \hat{\mathcal{E}}_{\text{loc}}(M)$ we set

$$\hat{\mathcal{L}}_X F = \left. \frac{d}{dt} \right|_{t=0} (\text{Fl}_t^X)^* F.$$

Using the chain rule we may write this as

$$(\hat{\mathcal{L}}_X F)(\omega) = -dF(\omega)(\mathcal{L}_X^{\text{SK}} \omega) + \mathcal{L}_X(F(\omega))$$

and since this formula may also be applied to a non-complete vector field we take this as the definition in the general case.

Definition 19 (Generalised Lie Derivative). For any $F \in \hat{\mathcal{E}}_{\text{loc}}(M)$ and any $X \in \mathfrak{X}(M)$ we set

$$(\hat{\mathcal{L}}_X F)(\omega) := -dF(\omega)(\mathcal{L}_X^{\text{SK}}\omega) + \mathcal{L}_X(F(\omega)) \quad (3.8)$$

Remark 20. In the terminology of [27], the basic space of [18] is given by the (ω_x, x) -local elements of $\hat{\mathcal{E}}(M)$. On these, the formula for the generalised Lie derivative is identical to that in [18] evaluated at $\omega = \omega_x$.

The other approach is to fix the smoothing kernel $\omega \in \text{SK}(M)$ so that $x \mapsto F(\omega)(x)$ is a smooth function of x . We may then define another Lie derivative of F (which we denote $\tilde{\mathcal{L}}_X F$) by fixing ω and taking the (ordinary) Lie derivative of $F(\omega)$, so that

$$(\tilde{\mathcal{L}}_X F)(\omega) := \mathcal{L}_X(F(\omega)). \quad (3.9)$$

Having defined suitable derivatives on $\hat{\mathcal{E}}_{\text{loc}}(M)$ and established that $\tilde{\mathcal{A}}(M)$ is non-void, we turn to the definition of moderate and negligible functions on manifolds. We start with the definition of negligible functions. Consider a net Φ_ε of smoothing operators converging to the identity. Then from this point of view the natural definition of a negligible function F is one that satisfies $F(\Phi_\varepsilon) \rightarrow 0$ as $\varepsilon \rightarrow 0$ in $C^\infty(M)$ (i.e. in all derivatives). Writing this in terms of smoothing kernels we therefore require $\mathcal{L}_{X_1} \dots \mathcal{L}_{X_k}(F(\omega_\varepsilon)) \rightarrow 0$ as $\varepsilon \rightarrow 0$. Since $(\tilde{\mathcal{L}}_X F)(\omega) = \mathcal{L}_X(F(\omega))$ this automatically gives stability of the subspace of negligible functions under the ordinary Lie derivative $\tilde{\mathcal{L}}_X$. However we also require stability of negligible functions under the generalised Lie derivative $\hat{\mathcal{L}}_X$. This suggests that we require

$$(\tilde{\mathcal{L}}_{\tilde{X}_1} \dots \tilde{\mathcal{L}}_{\tilde{X}_k} \hat{\mathcal{L}}_{X_1} \dots \hat{\mathcal{L}}_{X_\ell} F)(\omega_\varepsilon) \rightarrow 0 \quad \text{as } \varepsilon \rightarrow 0.$$

However, by definition we have

$$((\tilde{\mathcal{L}}_X - \hat{\mathcal{L}}_X)F)(\omega_\varepsilon) = dF(\omega_\varepsilon)(\mathcal{L}_X^{\text{SK}}\omega_\varepsilon)$$

so that taking linear combinations of the two types of Lie derivative is equivalent to looking at dF and evaluating it on the tangent space to $\tilde{\mathcal{A}}(M)$. We therefore introduce the space

$$\tilde{\mathcal{A}}_0(M) := \{\omega_0 \in C^\infty(M, \Omega_c^n(M))^{(0,1]} : \omega \in \tilde{\mathcal{A}}(M) \Rightarrow \omega_0 + \omega \in \tilde{\mathcal{A}}(M)\}$$

and make the following definition:

Definition 21 (Negligible functions). The function $F \in \hat{\mathcal{E}}_{\text{loc}}(M)$ is negligible if for any given compact $K \subset M \forall k, j, m \in \mathbb{N}_0 \forall X_1 \dots X_k \in \mathfrak{X}(M) \forall \omega \in \tilde{\mathcal{A}}(M) \forall \omega_1 \dots \omega_j \in \tilde{\mathcal{A}}_0(M)$:

$$\sup_{x \in K} \left| (\mathcal{L}_{X_1} \dots \mathcal{L}_{X_k}(d^j F)(\omega_\varepsilon)(\omega_{1,\varepsilon} \dots \omega_{j,\varepsilon}))(x) \right| = O(\varepsilon^m) \quad \text{as } \varepsilon \rightarrow 0.$$

The set of negligible elements is denoted $\hat{\mathcal{N}}_{\text{loc}}(M)$.

In order that the space of negligible functions is an ideal we also need to restrict to the space of moderate functions.

Definition 22 (Moderate functions). The function $F \in \hat{\mathcal{E}}_{\text{loc}}(M)$ is moderate if for any given compact $K \subset M \forall k, j \in \mathbb{N}_0 \forall \omega \in \tilde{\mathcal{A}}(M) \forall \omega_1 \dots \omega_j \in \tilde{\mathcal{A}}_0(M) \exists N \in \mathbb{N}_0 \forall X_1 \dots X_k \in \mathfrak{X}(M)$:

$$\sup_{x \in K} \left| (\mathcal{L}_{X_1} \dots \mathcal{L}_{X_k}(d^j F)(\omega_\varepsilon)(\omega_{1,\varepsilon} \dots \omega_{j,\varepsilon}))(x) \right| = O(\varepsilon^{-N}) \quad \text{as } \varepsilon \rightarrow 0. \quad (3.10)$$

The set of moderate elements of $\hat{\mathcal{E}}_{\text{loc}}(M)$ is denoted $\hat{\mathcal{E}}_{M,\text{loc}}(M)$.

Remark 23. Although the above definitions require one to consider derivatives $d^j F$ of arbitrary order in practice one only needs to verify this condition is satisfied by objects that are embedded into the algebra via

σ or ι . Since σ does not depend on ω and the embedding ι is linear in ω , this leaves the cases $j = 0$ and $j = 1$.

Theorem 24.

- (a) $\hat{\mathcal{E}}_{M,\text{loc}}(M)$ is a subalgebra of $\hat{\mathcal{E}}_{\text{loc}}(M)$.
- (b) $\hat{\mathcal{N}}_{\text{loc}}(M)$ is an ideal in $\hat{\mathcal{E}}_{M,\text{loc}}(M)$.

Proof. Because of the property of derivatives it is clear from the definitions that the product of two moderate functions is moderate and the product of a negligible function with a moderate function is negligible. \square

The next result shows that one does not need derivatives to test negligibility of a moderate function.

Proposition 25. Let $F \in \hat{\mathcal{E}}_{M,\text{loc}}(M)$ be such that for all compact $K \subset M \forall m \in \mathbb{N}_0 \forall \omega \in \tilde{\mathcal{A}}(M)$

$$\sup_{x \in K} |F(\omega_\varepsilon)(x)| = O(\varepsilon^m) \quad \text{as } \varepsilon \rightarrow 0.$$

Then $F \in \hat{\mathcal{N}}_{\text{loc}}(M)$.

Proof. This follows from looking at $F(\omega_\varepsilon + \varepsilon^k \tilde{\omega}_\varepsilon)$ where $\omega \in \tilde{\mathcal{A}}(M)$, $\tilde{\omega} \in \tilde{\mathcal{A}}_0(M)$, applying the mean-value theorem and using the definition of moderateness of F with k suitably chosen. \square

Theorem 26. Let $X \in \mathfrak{X}(M)$. Then

- (a) $\hat{\mathcal{L}}_X \hat{\mathcal{E}}_{M,\text{loc}}(M) \subseteq \hat{\mathcal{E}}_{M,\text{loc}}(M)$ and $\tilde{\mathcal{L}}_X \hat{\mathcal{E}}_{M,\text{loc}}(M) \subseteq \hat{\mathcal{E}}_{M,\text{loc}}(M)$.
- (b) $\hat{\mathcal{L}}_X \hat{\mathcal{N}}_{\text{loc}}(M) \subseteq \hat{\mathcal{N}}_{\text{loc}}(M)$ and $\tilde{\mathcal{L}}_X \hat{\mathcal{N}}_{\text{loc}}(M) \subseteq \hat{\mathcal{N}}_{\text{loc}}(M)$.

This follows immediately from the definitions.

We are finally in a position to define generalised functions on manifolds.

Definition 27 (Generalised Functions). The space

$$\hat{\mathcal{G}}_{\text{loc}}(M) = \frac{\hat{\mathcal{E}}_{M,\text{loc}}(M)}{\hat{\mathcal{N}}_{\text{loc}}(M)}$$

is called the Colombeau algebra of generalised functions on M .

Theorem 28. The space of Colombeau generalised functions $\hat{\mathcal{G}}_{\text{loc}}(M)$ is a fine sheaf of associative commutative differential algebras on M .

Proof. By construction the basic space $\hat{\mathcal{E}}_{\text{loc}}(M)$ is an associative commutative differential algebra, with derivative the generalised Lie derivative $\hat{\mathcal{L}}$ given by equation (3.8). $\hat{\mathcal{E}}_{M,\text{loc}}(M)$ is a subalgebra of $\hat{\mathcal{E}}_{\text{loc}}(M)$ and $\hat{\mathcal{N}}_{\text{loc}}(M)$ is an ideal in $\hat{\mathcal{E}}_{M,\text{loc}}(M)$, hence $\hat{\mathcal{G}}_{\text{loc}}(M)$ is an algebra. Furthermore the spaces $\hat{\mathcal{E}}_{M,\text{loc}}(M)$ and $\hat{\mathcal{N}}_{\text{loc}}(M)$ are stable under both the generalised and ordinary Lie derivatives so that $\hat{\mathcal{G}}_{\text{loc}}(M)$ is a differential algebra with respect to both Lie derivatives. The sheaf properties of $\hat{\mathcal{G}}_{\text{loc}}(M)$ follow from the localisation results [27]. \square

We now want to show that we may embed the space of distributions $\mathcal{D}'(M)$ in the space of generalised functions $\hat{\mathcal{G}}_{\text{loc}}(M)$. Given a distribution T in $\mathcal{D}'(M)$ we define the function $\tilde{T} \in$

$\hat{\mathcal{E}}_{\text{loc}}(M)$ by

$$\tilde{T}(\omega)(x) = \langle T, \omega_x \rangle.$$

We now need to show that \tilde{T} is moderate. For this we need to look at Lie derivatives of $d^j \tilde{T}(\omega)$. For $j = 0$ we have $\tilde{T}(\omega_\varepsilon) = \langle T, \omega_\varepsilon \rangle$ and it then follows from property (3) of Definition 16 that

$$\mathcal{L}_{X_1} \dots \mathcal{L}_{X_k}(\tilde{T}(\omega_\varepsilon))(x) = \langle T, \mathcal{L}_{X_1}^{C^\infty} \dots \mathcal{L}_{X_k}^{C^\infty} \omega_{x,\varepsilon} \rangle = O(\varepsilon^{-N}).$$

Since the embedding is linear, we have for $j = 1$ that $(d\tilde{T})(\omega)(\omega_1) = \langle T, \omega_1 \rangle$, so the above argument gives the desired bound on the growth, while for $j \geq 2$ we have $d^j \tilde{T} = 0$. This shows that \tilde{T} is moderate and we have an embedding

$$\begin{aligned} \iota: \mathcal{D}'(M) &\rightarrow \hat{\mathcal{G}}_{\text{loc}}(M) \\ T &\mapsto [\tilde{T}] \end{aligned}$$

where $[\tilde{T}]$ is the equivalence class of \tilde{T} in $\hat{\mathcal{G}}_{\text{loc}}(M)$.

By the definition of the generalised Lie derivative we have

$$\begin{aligned} \hat{\mathcal{L}}_X(\iota T)(\omega)(x) &= -\langle T, (\mathcal{L}_X^{\text{SK}} \omega)_x \rangle + \mathcal{L}_X \langle T, \omega_x \rangle \\ &= -\langle T, \mathcal{L}_X^{\Omega^n} \omega_x \rangle \\ &= \langle \mathcal{L}_X T, \omega_x \rangle \\ &= \iota(\mathcal{L}_X T)(\omega)(x). \end{aligned}$$

Hence,

$$\hat{\mathcal{L}}_X(\iota T) = \iota(\mathcal{L}_X T) \tag{3.11}$$

and thus the embedding ι commutes with the generalised Lie derivative.

It is clear that if f is a smooth function on M then \hat{f} defined by $\hat{f}(\omega)(x) = f(x)$ is a moderate function. By passing to the equivalence class $[\hat{f}]$ we obtain the embedding $\sigma: C^\infty(M) \rightarrow \hat{\mathcal{G}}_{\text{loc}}(M)$ from above. Clearly σ gives an injective algebra homomorphism of the algebra of smooth functions on M into $\hat{\mathcal{G}}_{\text{loc}}(M)$, the algebra of generalised functions on M . Furthermore since $\sigma(f)$ has no dependence on ω we only have the second term in the formula for the definition of the generalised Lie derivative so σ also commutes with the Lie derivative. Finally, it easily follows from Definition 16 that for a smooth function the difference between \hat{f} and \hat{f} is negligible and hence on passing to the quotient ι coincides with σ on $C^\infty(M)$.

Collecting these results together we have the following theorem:

Theorem 29. $\iota: \mathcal{D}'(M) \rightarrow \hat{\mathcal{G}}_{\text{loc}}(M)$ is a linear embedding that commutes with the generalised Lie derivative and coincides with $\sigma: C^\infty(M) \rightarrow \hat{\mathcal{G}}_{\text{loc}}(M)$ on $C^\infty(M)$. Thus ι renders $\mathcal{D}'(M)$ a linear subspace and $C^\infty(M)$ a faithful subalgebra of $\hat{\mathcal{G}}_{\text{loc}}(M)$.

As explained in the introduction the concept of association is an important feature of the theory of Colombeau algebras on manifolds as in many cases it allows us to recover a description in terms of classical distributions by a method of ‘coarse graining’. We now show how this notion may be extended to generalised functions on manifolds.

Definition 30 (Association). We say an element $[F]$ of $\hat{\mathcal{G}}_{\text{loc}}(M)$ is associated to 0 (denoted $[F] \approx 0$) if for each $\mu \in \Omega_c^n(M)$ we have

$$\lim_{\varepsilon \rightarrow 0} \int_{x \in M} F(\omega_\varepsilon)(x) \mu(x) = 0 \quad \forall \omega \in \tilde{\mathcal{A}}(M).$$

We say two elements $[F], [G]$ are associated and write $[F] \approx [G]$ if $[F - G] \approx 0$.

Definition 31 (Associated distribution). We say $[F] \in \hat{\mathcal{G}}_{\text{loc}}(M)$ admits $u \in \mathcal{D}'(M)$ as an associated distribution if for each $\mu \in \Omega_c^n(M)$ we have

$$\lim_{\varepsilon \rightarrow 0} \int_{x \in M} F(\omega_\varepsilon)(x) \mu(x) = \langle u, \mu \rangle \quad \forall \omega \in \tilde{\mathcal{A}}(M).$$

Again, these definitions do not depend on the representative of the class. As in \mathbb{R}^n at the level of association we regain the usual results for multiplication of distributions, provided that suitable L^1 -conditions like (3.6) and (3.7) are used.

Proposition 32.

(a) If $f \in C^\infty(M)$ and $T \in \mathcal{D}'(M)$ then

$$\iota(f)\iota(T) \approx \iota(fT).$$

(b) If $f, g \in C^0(M)$ then

$$\iota(f)\iota(g) \approx \iota(fg).$$

The above results establish almost everything we want at the scalar level. Before going on to look at the tensor theory and develop a theory of differential geometry there is one further ingredient we will require, which is the notion of directional (or covariant) derivative $\nabla_X F$ of a generalised scalar field. Ideally this would be $C^\infty(M)$ -linear in X (so that $\nabla_{fX} F = f\nabla_X F$) and commute with the embedding. However, it is not hard to see that this is not possible since this would require that

$$\begin{aligned} \iota(f\nabla_X g) &= \iota(\nabla_{fX} g) \\ &= \nabla_{fX} \iota(g) \\ &= f\nabla_X (\iota g) \\ &= \iota(f)\iota(\nabla_X g) \end{aligned}$$

which cannot in general be true by the Schwartz impossibility result. However, in view of Proposition 32 a $C^\infty(M)$ -linear derivative that commutes with the embedding only at the level of association is not ruled out by the Schwartz result.

By thinking of $F(\omega)(x)$ for fixed ω as a function of x we may make the following definition of generalised covariant derivative

Definition 33. [Covariant derivative of a generalised scalar field] Let $F \in \hat{\mathcal{G}}_{\text{loc}}(M)$ be a generalised scalar field and X a smooth vector field. Then we define the covariant derivative $\tilde{\nabla}_X F$ by

$$(\tilde{\nabla}_X F)(\omega) = \nabla_X (F(\omega)).$$

We note that almost by definition this satisfies the requirements of a covariant derivative and for the case of a scalar field (which we are considering here) this is identical to the Lie derivative $\tilde{\mathcal{L}}_X F$ given by (3.9) and hence is well defined. Although it is $C^\infty(M)$ -linear in X this derivative does not commute with the embedding into $\hat{\mathcal{G}}_{\text{loc}}(M)$. However as we now show this derivative does commute with the embedding at the level of association.

Proposition 34. Let $T \in \mathcal{D}'(M)$ and X be a smooth vector field; then

$$\tilde{\nabla}_X \iota(T) = \tilde{\mathcal{L}}_X \iota(T) \approx \iota(\mathcal{L}_X T) = \iota(\nabla_X T) \quad (3.12)$$

Proof. In the following calculation let ω_ε be a fixed delta net of smoothing kernels. Given μ a smooth n -form of compact support then

$$\begin{aligned} \lim_{\varepsilon \rightarrow 0} \int_M (\tilde{\mathcal{L}}_X \iota(T))(\omega_\varepsilon) \mu &= \lim_{\varepsilon \rightarrow 0} \int_M (\mathcal{L}_X(\iota T(\omega_\varepsilon))) \mu \\ &= - \lim_{\varepsilon \rightarrow 0} \int_M (\iota T)(\omega_\varepsilon)(\mathcal{L}_X \mu) \\ &= - \lim_{\varepsilon \rightarrow 0} \int_M \langle T, \omega_{x,\varepsilon} \rangle (\mathcal{L}_X \mu)(x) \\ &= \langle T, -\mathcal{L}_X \mu \rangle \\ &= \langle \mathcal{L}_X T, \mu \rangle. \end{aligned} \quad \square$$

4. Applications

In this section we briefly sketch some potential applications of the theory developed above. It should be noted that variants of these results can usually be obtained also in the special algebra, which has a considerably simpler setup, but lacks coordinate invariance and a canonical embedding of distributions. In this sense, the results for special algebras should be regarded as preliminary results, whose scope is greatly extended by transferring them to the full algebra as developed in the present work.

The first application is to solutions of the wave equation $\square_g \phi = 0$ on a manifold M where g is a low-regularity Lorentzian metric. If g has regularity $C^{1,1}$ (i.e, it has Lipschitz derivatives) then one can show the existence of solutions ϕ in the Sobolev space $H_{\text{loc}}^2(M)$ [28]. This result was obtained by drawing on the methods of [10], considering generalised solutions of the wave equation on \mathbb{R}^n and showing that these converge to a weak solution in $H_{\text{loc}}^2(\mathbb{R}^n)$. One can then piece together these local solutions to obtain a weak solution on M . However, for metrics of lower regularity one would need to work with the generalised solutions directly. For example, building on the work of [29] it is shown in [30] that for the case of locally bounded Lorentzian metrics g on \mathbb{R}^{n+1} one may obtain a metric with generalised coefficients by convolution componentwise with a delta-net $\rho^{(\varepsilon)}$ so that $g_{ab}^{(\varepsilon)} = g_{ab} \star \rho^{(\varepsilon)}$. From this one obtains a unique generalised solution to the wave equation through solving $\square_{g^{(\varepsilon)}} \phi^{(\varepsilon)} = 0$ and obtaining higher order energy estimates. However, this is all done in the context of a fixed coordinate system on \mathbb{R}^{n+1} . In order to obtain a generalised solution on the manifold M , this would require the diffeomorphism invariant theory of generalised functions developed in the present paper together with the methods used in [28].

The second application we wish to consider are generalized flows and singular ODEs on manifolds. This was studied in the special version of the Colombeau algebra on \mathbb{R}^n in [31] (Theorem 3.3) where existence and uniqueness results for generalised solutions were obtained for the generalised flow equations on \mathbb{R}^n . In this context, the flow given by the generalised vector field F is a solution of the system of ODEs

$$\begin{aligned} \frac{d}{dt} \Phi(t, x) &= F(\Phi(t, x)) \quad \text{in } \mathcal{G}[\mathbb{R}^{1+n}, \mathbb{R}^n], \\ \Phi(0, \cdot) &= \text{id}_{\mathbb{R}^n} \quad \text{in } \mathcal{G}[\mathbb{R}^n, \mathbb{R}^n]. \end{aligned}$$

However, to obtain generalised solutions to (classical) singular ODEs requires a canonical embedding into the algebra as provided by the present theory. Furthermore, considering flows and generalised vector fields F on manifolds M will be possible using the theory developed in [1].

The third potential application is to the theory of generalised connections on vector bundles which was developed in [32] (see also [33] [34]) and was used to study weakly singular solutions of Yang-Mills equations. There, a singular connection is regarded as a connection in a vector bundle with coefficients that are generalised functions so that the covariant derivative of a field

V^A in the direction of a smooth vector field X is given by

$$\nabla_X V^A = X(V^A) + \Gamma_B^A(X)V^B$$

where $\Gamma_B^A(X)$ is a generalised function taking values in the Lie algebra. However, all of this was done in the special algebra, whose embedding of distributions is somewhat ad-hoc. To obtain these results in full generality one has to use the present theory, which has a canonical embedding, or rather its vector-valued extension which is developed in full in [35].

Our next example of a potential application of the present theory is the notion of generalised wavefront set. The wavefront set is a useful tool in microlocal analysis which is a geometric object describing the location and direction of the singularities of a distribution. The foundations of this theory for Colombeau generalised functions are the papers [36] and [37] which develop the theory within the special Colombeau algebra. However, if the aim of wavefront analysis is to apply this to the study of solutions to singular PDEs one needs a canonical and diffeomorphism invariant way of embedding these into the Colombeau algebra. This is provided by the theory developed here.

The final example of an application of this theory (which in fact motivated the present work) is distributional differential geometry. In particular we are interested in solutions to Einstein's equations for metrics of low differentiability. These metrics are tensorial rather than scalar objects. Because the embedding into the algebra does not commute with multiplication (except on the subalgebra of smooth functions) one cannot in this case simply work with the coordinate components of a tensor and use the theory of generalised scalars. In a subsequent paper [1] we show how it is possible to define an algebra of generalised tensor fields on a manifold which contains the spaces of smooth tensor fields as a subalgebra and has a canonical coordinate independent embedding of the spaces of tensor distributions as linear subspaces.

5. Conclusion

In this paper we have reviewed the construction of the Colombeau algebra on \mathbb{R}^n and adapted it to define the Colombeau algebra on a manifold M . The key idea has been to look at the construction on manifolds first of all in terms of smoothing operators and then translate this into the language of smoothing kernels. The result of this is to replace the mollifiers $\phi(\mathbf{y} - \mathbf{x})$ by smoothing kernels $\omega_x(y)$ and the scaled mollifiers $\phi_\varepsilon(\mathbf{y} - \mathbf{x})$ by delta nets of smoothing kernels $\omega_{x,\varepsilon}(y)$. In this way, given a locally integrable function f we may approximate it by a 1-parameter family of smooth functions (depending on ω) according to

$$\tilde{f}_\varepsilon(x) = \int_{y \in M} f(y) \omega_{x,\varepsilon}(y).$$

For fixed $\omega \in \tilde{\mathcal{A}}(M)$ these may be treated just like smooth functions on manifolds so all the standard operations that may be carried out on smooth functions extend to the smoothed functions \tilde{f}_ε . The embedding extends to distributions $T \in \mathcal{D}'(M)$ by defining $\tilde{T}_\varepsilon(x) = \langle T, \omega_{x,\varepsilon} \rangle$. The nets of smoothing kernels tend to δ_x as $\varepsilon \rightarrow 0$ and by using the rate at which this happens we have a condition which corresponds to the vanishing moment condition on \mathbb{R}^n . We can therefore define the spaces of moderate and negligible functions which allows us to define $\hat{\mathcal{G}}_{\text{loc}}(M)$ as the quotient $\hat{\mathcal{G}}_{\text{loc}}(M) = \hat{\mathcal{E}}_{M,\text{loc}}(M) / \hat{\mathcal{N}}_{\text{loc}}(M)$. The algebra of generalised functions $\hat{\mathcal{G}}_{\text{loc}}(M)$ contains the space of smooth functions as a subalgebra and has the space of distributions as a canonically embedded linear subspace. We also introduced the generalised Lie derivative which commutes with the embedding and makes $\hat{\mathcal{G}}_{\text{loc}}(M)$ into a differential algebra. Finally we defined the covariant derivative of generalised scalar fields on the manifold M and showed that this commutes with the distributional (covariant) derivative at the level of association. We have briefly outlined a number of potential applications of the theory and in a subsequent paper [1] we will consider one of these in detail. In particular we will show how the theory can be extended to a nonlinear theory of tensor distributions on a manifold M and use this to develop a theory of nonlinear distributional geometry.

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