Algorithmic construction of SYM multiparticle superfields in the BCJ gauge

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We write down closed formulas for all necessary steps to obtain multiparticle super Yang–Mills superfields in the so-called BCJ gauge. The superfields in this gauge have obvious applications in the quest for finding BCJ-satisfying representations of amplitudes. As a benefit of having these closed formulas, we identify the explicit *finite* gauge transformation responsible for attaining the BCJ gauge. To do this, several combinatorial maps on words are introduced and associated identities rigorously proven.

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1. Introduction

The definition and usage of multiparticle superfields [1,2] of supersymmetric Yang–Mills (SYM) theory [3] has proved to be an essential feature in obtaining compact expressions for high-multiplicity amplitudes in superstring [4] and field theories [5] using the pure spinor formalism [6].

In the simplest formulation of multiparticle superfields in the *Lorenz gauge*, their definition is given by a straightforward recursion over the particle labels [2]. While this recursive definition has its own merits and is certainly useful in relating the new expressions for tree-level amplitudes [7] to the standard Berends–Giele recursions [8], there is an alternative formulation related by a non-linear gauge transformation whose properties have more appeal, the *BCJ-gauge* representation [1]. As will be reviewed in section 2.3, the superfields in this gauge satisfy *generalized Jacobi identities* [9] in their particle labels, for example $A^m_{12} = -A^m_{21}$, $A^m_{123} + A^m_{231} + A^m_{312} = 0$, and so forth. In this gauge, they constitute the natural building blocks used in the expressions of local SYM numerators satisfying the Bern–Carrasco–Johansson numerator identities [10] at tree- [11] and loop-level [12,13].

As explained in [2], the gauge transformations required to go to the BCJ gauge are encoded in so-called *redefining* superfields $H_{[P,Q]}$ to be reviewed below. Until now, the explicit expressions of these superfields were known only up to multiplicity five [2]. In section 4.2.1 of this paper this restriction will be lifted when we propose a recursive formula for $H_{[P,Q]}$, namely

$$H_{[P,Q]} = (-1)^{|Q|}\frac{|P|}{|P|+|Q|}\sum_{XjY=\dot{p}\tilde{Q}}(-1)^{|Y|}H'_{\tilde{Y},j,X} - (P\leftrightarrow Q)\,, \qquad H_{[i,j]} = 0\,, \quad (1.1)$$

where the auxiliary superfields $H'_{A,B,C}$ are defined by

$$\begin{aligned}H'_{P,Q,R} &\equiv H_{P,Q,R} + \Big[\frac{1}{2}H_{[P,Q]}(k_{PQ}\cdot A_R) + \text{cyclic}(P,Q,R)\Big]\\ &\quad - \Big[\sum_{\substack{XjY=P\\ \delta(Y)=R\otimes S}}(k^X\cdot k^j)\big[H_{[XR,Q]}H_{[jS,R]} - (X\leftrightarrow j)\big] + \text{cyclic}(P,Q,R)\Big]\,,\\ H_{P,Q,R} &\equiv -\frac{1}{4}A^m_P A^n_Q F^{mn}_R + \frac{1}{2}(W_P\gamma_m W_Q)A^m_R + \text{cyclic}(P,Q,R)\,.\end{aligned}$$

As a consequence of the quadratic corrections H^2 in these formulas, we will show in section 5.3 that the superfields satisfying the generalized Jacobi identities follow from a *standard* gauge transformation of SYM theory in its *finite* form,

$$\mathbb{A}^{\text{BCJ}}_m = U\mathbb{A}^{\text{L}}_m U^{-1} + \partial_m U U^{-1} \text{ with } U = \exp(-\mathbb{H})\,, \qquad (1.3)$$

whose series representation is given by

$$\mathbb{A}_m^{\rm BCJ} = \mathbb{A}_m^{\rm L} + [\mathbb{H}, \partial_m] - [\mathbb{H}, \mathbb{A}_m^{\rm L}] - \frac{1}{2}[\mathbb{H}, [\mathbb{H}, \partial_m]] + \frac{1}{2}[\mathbb{H}, [\mathbb{H}, \mathbb{A}_m^{\rm L}]] + \frac{1}{3!}[\mathbb{H}, [\mathbb{H}, [\mathbb{H}, \partial_m]]] + \cdots \tag{1.4}$$

We note that in [2] only the first three terms of (1.4) were identified.

While in pursuit of finding these formulas we also filled some gaps of the previous discussions. These mostly concern writing down closed formulas for expressing contact terms (in a multitude of different situations) where the multiparticle labels are given in terms of an arbitrary configuration of nested Lie brackets. As will be explained in section 3, we found a novel recursive description of such terms which is *universal* and whose backbone is given by the solution to a purely combinatorial problem. Several equations relevant to the framework of multiparticle superfields can be written down using this newly found recursion and we prove several associated results.

Finally, in the appendices we write down some longer examples of applications of several recursive maps from the main text, among other things.

2. Review

In this section we review some aspects of the construction of 10d supersymmetric Yang–Mills superfields following the recent discussions of [2,1] using the framework of perturbiners [14]. For the original references on the covariant description of super Yang–Mills in ten dimensions, see [15,16]

2.1. Notation and conventions

2.1.1. Ten-dimensional superspace

The ten-dimensional superspace coordinates are denoted $\{x^m, \theta^\alpha\}$, where $m = 0, \ldots, 9$ are the vector indices and $\alpha = 1, \ldots, 16$ denote the spinor indices of the Lorentz group. The spinor representation is based on the 16×16 Pauli matrices $\gamma^m_{\alpha\beta} = \gamma^m_{\beta\alpha}$ satisfying the Clifford algebra $\gamma^{(m}_{\alpha\beta}\gamma^{n)\beta\gamma} = 2\eta^{mn}\delta^\gamma_\alpha$. In this paper the (anti)symmetrization of n indices does not include a factor of $\frac{1}{n!}$.

2.1.2. Multiparticle index notation

In the following discussions we will use a notation based on "words" composed of "letters" from the alphabet of natural numbers. Capital letters from the Latin alphabet are used to represent words (e.g. $P = 1423$) while their composing letters are represented by lower case letters (e.g. $i = 3$). The length of a word P is denoted $|P|$ and it is given by the number of its letters. The reversal of a word $P = p_1 p_2 \ldots p_{|P|}$ is $\tilde{P} = p_{|P|} \ldots p_2 p_1$. The word notation is also used in place of arbitrary commutators, such as $P = [1,2] \equiv 12 - 21$; the context will disambiguate whether a word denotes a sequence of letters or a bracketing structure. In addition, when the bracketing structure is nested from left to right such as $P = [[[[1,2],3],4],5]$ we will often write it as $P = 12345$. Such structures may be referred to as (left-to-right) "Dynkin brackets"

The multiparticle momentum for a word with letters (labels) from massless particles $(k_i \cdot k_i) = 0$ and its associated Mandelstam invariant are given by

$$k_P^m \equiv k_{p_1}^m + \cdots + k_{p_{|P|}}^m \,, \qquad s_P \equiv \frac{1}{2}(k_P \cdot k_P)\,. \tag{2.1}$$

For example $k_{123}^m \equiv k_1^m + k_2^m + k_3^m$ and $s_{123} = s_{12} + s_{13} + s_{23}$.

2.2. Non-linear supersymmetric Yang–Mills

To describe ten-dimensional SYM one introduces Lie algebra-valued superfield connections $\mathbb{A}_\alpha = \mathbb{A}_\alpha(x,\theta)$ and $\mathbb{A}_m = \mathbb{A}_m(x,\theta)$ and the supercovariant derivatives [16,15],

$$\nabla_\alpha \equiv D_\alpha - \mathbb{A}_\alpha \,, \qquad \nabla_m \equiv \partial_m - \mathbb{A}_m \,, \tag{2.2}$$

where the superspace derivative $D_\alpha \equiv \frac{\partial}{\partial\theta^\alpha} + \frac{1}{2}(\gamma^m\theta)_\alpha \partial_m$ satisfies $\{D_\alpha, D_\beta\} = \gamma^m_{\alpha\beta}\partial_m$. The constraint $\big\{\nabla_\alpha, \nabla_\beta\big\} = \gamma^m_{\alpha\beta}\nabla_m$ and the associated Bianchi identities imply the following non-linear equations of motion [15],

$$\begin{aligned}
\big\{\nabla_\alpha, \nabla_\beta\big\} &= \gamma^m_{\alpha\beta}\nabla_m \,, & \big\{\nabla_\alpha, \mathbb{W}^\beta\big\} &= \frac{1}{4}(\gamma^{mn})_\alpha{}^\beta \mathbb{F}_{mn} \,, \\
\big[\nabla_\alpha, \nabla_m\big] &= -(\gamma_m \mathbb{W})_\alpha \,, & \big[\nabla_\alpha, \mathbb{F}^{mn}\big] &= (\mathbb{W}^{[m}\gamma^{n]})_\alpha \,,
\end{aligned} \tag{2.3}$$

where

$$\mathbb{F}_{mn} \equiv -\big[\nabla_m, \nabla_n\big] \,, \qquad \mathbb{W}^\alpha_m \equiv \big[\nabla_m, \mathbb{W}^\alpha\big]\,. \tag{2.4}$$

These equations are invariant under the gauge transformations of the superpotentials

$$\delta_\Omega \mathbb{A}_\alpha = \big[\nabla_\alpha, \Omega\big] \,, \qquad \delta_\Omega \mathbb{A}_m = \big[\nabla_m, \Omega\big] \tag{2.5}$$

which in turn induce the gauge transformations of their field-strengths $\delta_\Omega \mathbb{W}^\alpha = \big[\Omega, \mathbb{W}^\alpha\big]$, $\delta_\Omega \mathbb{F}^{mn} = \big[\Omega, \mathbb{F}^{mn}\big]$, and $\delta_\Omega \mathbb{W}^\alpha_m = \big[\Omega, \mathbb{W}^\alpha_m\big]$ where $\Omega \equiv \Omega(x,\theta)$ is a Lie algebra-valued gauge parameter superfield. The equations of motion (2.3) can also be rewritten as

$$
\begin{aligned}
\{\nabla_{(\alpha}, \mathbb{A}_{\beta)}\} &= \gamma^m_{\alpha\beta}\mathbb{A}_m & \{\nabla_\alpha, \mathbb{W}^\beta\} &= \frac{1}{4}(\gamma^{mn})_\alpha{}^\beta \mathbb{F}_{mn} \\
[\nabla_\alpha, \mathbb{A}_m] &= [\partial_m, \mathbb{A}_\alpha] + (\gamma_m \mathbb{W})_\alpha\,, & [\nabla_\alpha, \mathbb{F}^{mn}] &= (\mathbb{W}^{[m}\gamma^{n]})_\alpha\,.
\end{aligned}
\tag{2.6}
$$

2.2.1. Non-linear wave equations and Berends–Giele supercurrents

Alternatively, in the *Lorenz gauge* (defined by the constraint $[\partial_m, \mathbb{A}^m] = 0$), the equations of motion (2.3) are equivalent to the non-linear wave equations [2],

$$
\begin{aligned}
\Box \mathbb{A}_\alpha &= \big[\mathbb{A}_m, [\partial^m, \mathbb{A}_\alpha]\big] + \big[(\gamma^m \mathbb{W})_\alpha, \mathbb{A}_m\big] \\
\Box \mathbb{A}_m &= \big[\mathbb{A}_p, [\partial^p, \mathbb{A}^m]\big] + \big[\mathbb{F}^{mp}, \mathbb{A}_p\big] + \gamma^m_{\alpha\beta}\{\mathbb{W}^\alpha, \mathbb{W}^\beta\} \\
\Box \mathbb{W}^\alpha &= \big[\mathbb{A}_m, [\partial^m, \mathbb{W}^\alpha]\big] + \big[\mathbb{A}^m, \mathbb{W}^\alpha_m\big] + \frac{1}{2}\big[\mathbb{F}_{mn}, (\gamma^{mn}\mathbb{W})^\alpha\big] \\
\Box \mathbb{F}^{mn} &= [\mathbb{A}_p, [\partial^p, \mathbb{F}^{mn}]] + [\mathbb{A}_p, \mathbb{F}^{p|mn}] + 2[\mathbb{F}^{mp}, \mathbb{F}_p{}^n] + 4\{(\mathbb{W}^{[m}\gamma^{n]})_\alpha, \mathbb{W}^\alpha\}\,,
\end{aligned}
\tag{2.7}
$$

where $\Box \mathbb{K} \equiv [\partial^m, [\partial_m, \mathbb{K}]]$ and $\mathbb{F}^{p|mn} \equiv [\nabla^p, \mathbb{F}^{mn}]$.

To solve the wave equations (2.7) we use the perturbiner method of Selivanov [14]. In this approach, one expands the superfields $\mathbb{K} \in \{\mathbb{A}_\alpha, \mathbb{A}^m, \mathbb{W}^\alpha, \mathbb{F}^{mn}\}$ as a series with respect to the generators t^{i_j} of a Lie algebra summed over all possible non-empty words P as

$$
\mathbb{K} \equiv \sum_P \mathcal{K}_P t^P\,, \qquad t^P \equiv t^{p_1}t^{p_2}\cdots t^{p_{|P|}}\,.
\tag{2.8}
$$

After plugging these series in (2.7) one learns that the expansion coefficients $\mathcal{K}_P \in \{\mathcal{A}^P_\alpha, \mathcal{A}^m_P, \mathcal{W}^\alpha_P, \mathcal{F}^{mn}_P\}$ turn out to be the Berends–Giele currents,

$$
\mathcal{K}_P = \frac{1}{s_P}\sum_{XY=P}\mathcal{K}_{[X,Y]}\,,
\tag{2.9}
$$

where $s_P = \frac{1}{2}k_P^2$ arises from the $\Box$ operator acting on plane waves of momentum k^m_P and

$$
\begin{aligned}
\mathcal{A}^{[P,Q]}_\alpha &= -\frac{1}{2}\big[\mathcal{A}^P_\alpha(k^P\cdot\mathcal{A}^Q) + \mathcal{A}^P_m(\gamma^m\mathcal{W}^Q)_\alpha - (P\leftrightarrow Q)\big]\,, \\
\mathcal{A}^{[P,Q]}_m &= -\frac{1}{2}\big[\mathcal{A}^P_m(k^P\cdot\mathcal{A}^Q) + \mathcal{A}^P_n\mathcal{F}^Q_{mn} - (\mathcal{W}^P\gamma_m\mathcal{W}^Q) - (P\leftrightarrow Q)\big]\,, \\
\mathcal{W}^\alpha_{[P,Q]} &= -\frac{1}{2}\big[\mathcal{W}^\alpha_P(k_P\cdot\mathcal{A}_Q) + \mathcal{W}^{m\alpha}_P\mathcal{A}^m_Q + \frac{1}{2}(\gamma_{rs}\mathcal{W}_P)^\alpha\mathcal{F}^{rs}_Q - (P\leftrightarrow Q)\big]\,, \\
\mathcal{F}^{mn}_{[P,Q]} &= -\frac{1}{2}\big[\mathcal{F}^{mn}_P(k_P\cdot\mathcal{A}_Q) + \mathcal{F}^{p|mn}_P\mathcal{A}^Q_p + 2\mathcal{F}^{mp}_P\mathcal{F}^n_{Q\,p} + 4\gamma^{[m}_{\alpha\beta}\mathcal{W}^{n]\alpha}_P\mathcal{W}^\beta_Q - (P\leftrightarrow Q)\big]\,.
\end{aligned}
\tag{2.10}
$$

Notice that the above Berends–Giele currents are non-local superfields as they contain inverse factors of Mandelstams variables.

2.2.2. Linearized description of 10d SYM

The *linearized* description of ten-dimensional super-Yang–Mills is obtained by discarding the quadratic terms from the equations of motion (2.6) and yields

$$
\begin{aligned}
D_\alpha A^i_\beta + D_\beta A^i_\alpha &= \gamma^m_{\alpha\beta} A^i_m\,, & D_\alpha F^i_{mn} &= \partial_m(\gamma_n W_i)_\alpha - \partial_n(\gamma_m W_i)_\alpha\,,\\
D_\alpha A^i_m &= (\gamma_m W_i)_\alpha + \partial_m A^i_\alpha\,, & D_\alpha W^\beta_i &= \frac{1}{4}(\gamma^{mn})_\alpha{}^\beta F^i_{mn}\,.
\end{aligned}
\tag{2.11}
$$

In the context of scattering amplitudes, the superfields are labelled with a distinct natural number i to associate them with the i-th particle taking part in the scattering process. This association will be generalized below.

2.3. Generalized Jacobi identities

As we will discuss below in the context of multiparticle superfields, there is the notion of a superfield satisfying certain symmetries dubbed *BCJ symmetries* in [2]. These symmetries can be given a precise mathematical characterization in terms of what is called *generalized Jacobi identities* in the mathematics literature [9,17].

Let A be a word and $\ell(A)$ its left-to-right bracketing defined in (A.1). The *generalized Jacobi identities* correspond to the elements in the kernel of ℓ. For example

$$
\ell(12+21)=0\,, \qquad \ell(123+231+312)=0\,, \tag{2.12}
$$

which correspond with the antisymmetry and Jacobi identity of the Lie bracket.

Using the identity $\ell(P\ell(Q)) = [\ell(P),\ell(Q)]$ it is easy to see that $\ell(A\ell(B)+B\ell(A))=0$ for any words A and B. In addition, due to the recursive definition of ℓ if $\ell(P)=0$ it also follows that $\ell(PQ)=0$ for any word Q. Therefore, for objects labelled by words, the generalized Jacobi identities can be characterized by an abstract operator $\pounds_k$

$$
\pounds_k \circ K_{ABC} \equiv K_{A\ell(B)C} + K_{B\ell(A)C}, \quad \forall A,B\neq\emptyset \text{ and } \forall C \text{ such that } |A|+|B|=k. \tag{2.13}
$$

We emphasize the arbitrary partition of non-empty words A and B in the above definition (while C can be empty), leading to a non-unique operator $\pounds$. For instance

$$
\begin{aligned}
\pounds_3\circ K_{123} &= K_{123}-K_{132}+K_{231}\,, \quad \text{for } A=1,\ B=23 \text{ and } C=\emptyset\\
\pounds_3\circ K_{123} &= K_{123}+K_{312}-K_{321}\,, \quad \text{for } A=12,\ B=3 \text{ and } C=\emptyset\,.
\end{aligned}
\tag{2.14}
$$

Note that if $\pounds_2\circ K_{123}=0$ then the right-hand side of the expressions in (2.14) agree and can be written as the cyclic sum $K_{123}+K_{231}+K_{312}$.

Definition 1. *The objects K_P are said to satisfy generalized Jacobi identities iff*

$$\pounds_k \circ K_P = 0\,, \quad \forall k \leq |P|\,. \tag{2.15}$$

The generalized Jacobi identities are also called BCJ symmetries.

The defining identities for objects K_P of increasing multiplicities can be written as

$$\begin{aligned} K_{12C} + K_{21C} = 0, &\quad \forall\, C, \\ K_{123C} + K_{231C} + K_{312C} = 0, &\quad \forall\, C, \\ K_{1234C} + K_{2143C} + K_{3412C} + K_{4321C} = 0, &\quad \forall\, C. \end{aligned} \tag{2.16}$$

where we have already used the fact that K_P satisfies the BCJ symmetries $\pounds_k \circ K_P = 0$ for all $k \leq |P|$ to simplify the appearance of the above. This fact in general can be used to show the equivalence of the BCJ symmetries for the various partitions of $P = ABC$ as mentioned after the example (2.14).

It is not hard to be convinced that the BCJ symmetries are equivalent to the symmetries of a concatenated string of structure constants, $K_{12\ldots p} \leftrightarrow f^{12a_2} f^{a_2 3a_3} f^{a_3 4a_4} \ldots f^{a_{p-1} p a_p}$.

If K_P satisfies BCJ symmetries then it is convenient to use the notation $K_{\ell(P)} \equiv K_P$. In particular, this implies that for superfields in the BCJ gauge we have [18],

$$K_{[P,Q]} = K_{P\ell(Q)}\,. \tag{2.17}$$

For example, $K_{[12,34]} = K_{1234} - K_{1243}$. In addition, it follows from the definitions (2.13) and (2.15) that if K_P with $|P| = n$ satisfies generalized Jacobi identities then

$$K_{AiB} = -K_{i\ell(A)B}\,, \quad A \neq \emptyset\,, \forall B\,, \tag{2.18}$$

which implies that there is an $(n-1)!$ basis of K_P.

3. Contact terms for general Lie polynomials

For the purpose of this paper, P is a *Lie polynomial* if it is a linear combination of words written in terms of (nested) Lie brackets $[x, y] \equiv xy - yx$. For example $P = [[1, 2], 3] = 123 - 213 - 312 + 321$ is a Lie polynomial while $Q = 123$ is not[1].

In this section we will introduce mathematical maps acting on words and Lie polynomials that will play a central role in later discussions about several aspects of local and non-local multiparticle superfields.

3.1. Planar binary tree map on words

A nested Lie bracket can be interpreted as a planar binary tree and vice versa [20]. In the context of tree-level scattering amplitudes one can map each planar binary tree to a product of inverse Mandelstam invariants. For example the two binary trees with three leaves are mapped to

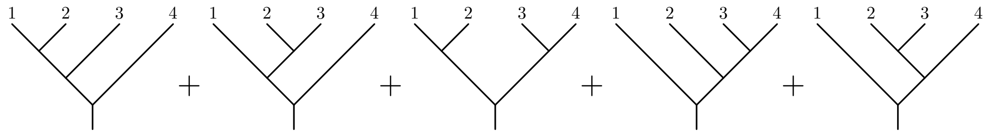
Mapping the sum over all binary trees with a given number of leaves will be related to Berends–Giele currents later on, and the explicit expansions can be generated from the following recursion.

Definition 2 (Binary tree map). *A word P of length $|P|$ is recursively mapped to a Lie polynomial built from a sum over all planar binary trees with $|P|$ leaves as*

$$b(i) = i, \qquad b(P) = \frac{1}{s_P} \sum_{XY=P} [b(X), b(Y)]\,, \tag{3.1}$$

where s_P is the Mandelstam invariant (2.1).

[1] It may not be immediately obvious that a given linear combination of words is a Lie polynomial. For $P = 12-21$ this is clear, but it is harder to see that $P = 1324+1423-1432-2134+2341-3124+3214-3241-4123+4213-4231+4312$ is the Lie polynomial $P = [[[1, 2], 3], 4]+[[[2, 3], 4], 1]$. A theorem by Dynkin–Specht-Wever states that if $\ell(P) = |P|P$ then P is a Lie polynomial [17], and this fact can be used to find the expression written in terms of nested Lie brackets [19].



$$b(1234) = \frac{[[[1,2],3],4]}{s_{12}s_{123}s_{1234}} + \frac{[[1,[2,3]],4]}{s_{23}s_{123}s_{1234}} + \frac{[[1,2],[3,4]]}{s_{12}s_{34}s_{1234}} + \frac{[1,[2,[3,4]]]}{s_{34}s_{234}s_{1234}} + \frac{[1,[[2,3],4]]}{s_{23}s_{234}s_{1234}}$$

Fig. 1 The sum generated by the recursion (3.1) of $b(1234)$.

The number of terms in the recursion above is given by the Catalan numbers $1, 2, 5, 14, \ldots$ and one gets, for example,

$$b(1) = 1, \quad b(12) = \frac{[1,2]}{s_{12}}, \quad b(123) = \frac{[[1,2],3]}{s_{12}s_{123}} + \frac{[1,[2,3]]}{s_{23}s_{123}}, \tag{3.2}$$
$$b(1234) = \frac{[[[1,2],3],4]}{s_{12}s_{123}s_{1234}} + \frac{[[1,[2,3]],4]}{s_{123}s_{1234}s_{23}} + \frac{[[1,2],[3,4]]}{s_{12}s_{1234}s_{34}} + \frac{[1,[[2,3],4]]}{s_{1234}s_{23}s_{234}} + \frac{[1,[2,[3,4]]]}{s_{1234}s_{234}s_{34}}.$$

These expansions are easily seen to be examples of *Lie polynomials* [17], see figure fig. 1 for the diagrammatic representation of $b(1234)$.

3.2. Contact terms associated to Lie polynomials

Given the Lie polynomial $[1,2]$ we can associate to it the following contact terms proportional to $(k_1 \cdot k_2) = s_{12}$; $C \circ [1,2] \equiv (k_1 \cdot k_2)(1 \otimes 2 - 2 \otimes 1)$. It is easy to see that this definition leads to a deconcatenation of $b(12)$,

$$C \circ b(12) = b(1) \otimes b(2) - b(2) \otimes b(1) = \sum_{XY=12} \big(b(X) \otimes b(Y) - (X \leftrightarrow Y)\big). \tag{3.3}$$

We would like to extend this action to an arbitrary Lie polynomial $C \circ [P, Q]$ such that

$$C \circ b(P) = \sum_{XY=P} \big(b(X) \otimes b(Y) - (X \leftrightarrow Y)\big). \tag{3.4}$$

The following definition does the job, as will be proven below.

Definition 3 (Contact term map). *Let C be the coproduct $C : \mathrm{Lie} \to \mathrm{Lie} \otimes \mathrm{Lie}$ that maps a Lie polynomial into the tensor product of two Lie polynomials recursively by*

$$C \circ i \equiv 0 \tag{3.5}$$
$$C \circ [P, Q] \equiv \big(C \circ P\big) \wedge Q + P \wedge \big(C \circ Q\big) + (k_P \cdot k_Q)\big(P \otimes Q - Q \otimes P\big),$$

where $\wedge$ *is defined by*[2]

$$(A \otimes B) \wedge C \equiv [A, C] \otimes B + A \otimes [B, C] \qquad (3.6)$$
$$A \wedge (B \otimes C) \equiv [A, B] \otimes C + B \otimes [A, C] \,,$$

and $k_P^m \equiv k^m_{p_1 p_2 \dots p_{|P|}}$, *where* p_i *for* $i = 1$ *to* $i = |P|$ *are the letters of* P.

As an immediate consistency check, we note that the definitions given in (3.6) imply that $C \circ [Q, P] = -C \circ [P, Q]$. Note that when the contact term map is used to generate combinations of superfields, the notation described in (C.5) and (5.1) may be used. For example applications of the C map, see the appendix C.

Proposition 1. *The* C *map satisfies*

$$C \circ b(P) = \sum_{XY=P} \big(b(X) \otimes b(Y) - (X \leftrightarrow Y) \big) \,. \qquad (3.7)$$

Proof. The proof is inductive in nature. When the word P has length two the statement has been verified explicitly in (3.3). We now assume that the relation (3.7) is satisfied for any word P of length less than n, and let Q be a word of length n. Then we get

$$s_Q C \circ b(Q) = C \circ \sum_{XY=Q} [b(X), b(Y)] \qquad (3.8)$$
$$= \sum_{XY=Q} \Big[(C \circ b(X)) \wedge b(Y) + b(X) \wedge (C \circ b(Y)) + (k^X \cdot k^Y)(b(X) \otimes b(Y) - b(Y) \otimes b(X)) \Big]$$

where we have used the definition of the contact term algorithm (3.5). Now we separate the above into the three possible cases; both of $|X|$ and $|Y|$ being greater than 1, $|X| = 1$, and $|Y| = 1$. We then use that $C \circ b(i) = 0$ for i a letter, and that the induction hypothesis (3.7) holds for all $C \circ b(P)$ such that $|P| < |Q|$, so that every application of the map C can be removed from this equation. This leaves us with

$$s_Q C \circ b(Q) = \sum_{XY=Q} (k^X \cdot k^Y) \Big(b(X) \otimes b(Y) - b(Y) \otimes b(X) \Big) \qquad (3.9)$$
$$\sum_{\substack{XY=Q \\ |X|>1, |Y|>1}} \sum_{AB=X} \big(b(A) \otimes b(B) - b(B) \otimes b(A) \big) \wedge b(Y) + \sum_{\substack{XY=Q \\ |Y|=1}} \sum_{CD=X} \big(b(C) \otimes b(D) - b(D) \otimes b(C) \big) \wedge b(Y)$$
$$+ \sum_{\substack{XY=Q \\ |X|>1, |Y|>1}} b(X) \wedge \sum_{CD=Y} \big(b(C) \otimes b(D) - b(D) \otimes b(C) \big) + \sum_{\substack{XY=Q \\ |X|=1}} b(X) \wedge \sum_{AB=Y} \big(b(A) \otimes b(B) - b(B) \otimes b(A) \big)$$

[2] Note the relations (3.6) should be used to remove $\wedge$ operations in the reverse order to that which they are introduced. Without such a criterion ambiguities can arise when objects of the form $A \wedge [B, C] \wedge D$ are considered.

Absorbing the $|X| = 1$ and $|Y| = 1$ summations into the $|X| > 1$, $|Y| > 1$ cases we get

$$s_Q C \circ b(Q) = \sum_{XY=Q} (k^X \cdot k^Y)\Big(b(X) \otimes b(Y) - b(Y) \otimes b(X)\Big) \tag{3.10}$$

$$+\sum_{\substack{XY=Q\\|X|>1}} \sum_{AB=X} \Big(b(A)\otimes b(B) - b(B)\otimes b(A)\Big)\wedge b(Y) + \sum_{\substack{XY=Q\\|Y|>1}} \sum_{CD=Y} b(X)\wedge\Big(b(C)\otimes b(D) - b(D)\otimes b(C)\Big)$$

Now we shall consider the two double sums. First of all we merge them using that, for example, $\sum_{XY=Q,|X|>1} \sum_{AB=X}$ is the same as $\sum_{ABY=Q}$. Then we remove the $\wedge$ using the definition (3.6) to get

$$\sum_{ABY=Q} \Big(b(A) \otimes b(B) - b(B) \otimes b(A)\Big) \wedge b(Y) + \sum_{XCD=Q} b(X) \wedge \Big(b(C) \otimes b(D) - b(D) \otimes b(C)\Big)$$

$$= \sum_{ABY=Q} \big([b(A), b(Y)] \otimes b(B) + b(A) \otimes [b(B), b(Y)] - [b(B), b(Y)] \otimes b(A) - b(B) \otimes [b(A), b(Y)]\big)$$

$$+ \sum_{XCD=Q} \big([b(X), b(C)] \otimes b(D) + b(C) \otimes [b(X), b(D)] - [b(X), b(D)] \otimes b(C) - b(D) \otimes [b(X), b(C)]\big)$$

We can now group the terms into two sets of four in a convenient way

$$= \Bigg(\sum_{ABY=Q} \Big([b(A), b(Y)] \otimes b(B) - b(B) \otimes [b(A), b(Y)]\Big) \tag{3.11}$$

$$+ \sum_{XCD=Q} \Big(b(C) \otimes [b(X), b(D)] - [b(X), b(D)] \otimes b(C)\Big)\Bigg)$$

$$+ \Bigg(\sum_{ABY=Q} \Big(b(A) \otimes [b(B), b(Y)] - [b(B), b(Y)] \otimes b(A)\Big)$$

$$+ \sum_{XCD=Q} \Big([b(X), b(C)] \otimes b(D) - b(D) \otimes [b(X), b(C)]\Big)\Bigg)$$

which we will now look at separately. With the first set of terms, it is clear from relabeling the second sum that it is just

$$\sum_{ABY=Q}\Big([b(A), b(Y)] \otimes b(B) - b(B) \otimes [b(A), b(Y)] + b(B) \otimes [b(A), b(Y)] - [b(A), b(Y)] \otimes b(B)\Big)$$

which is identically zero. The second set of terms in (3.11) can be simplified using the definition of the b map (3.1) leading to

$$\sum_{ABY=Q}\Big(b(A) \otimes b(BY)s_{BY} - s_{BY}b(BY) \otimes b(A)\Big) \tag{3.12}$$

$$+ \sum_{XCD=Q}\Big(s_{XC}b(XC) \otimes b(D) - b(D) \otimes b(XC)s_{XC}\Big).$$

Then, since B and Y are adjacent everywhere they appear in the first sum, we can condense them into a single word, and likewise for X and C in the second sum. This leaves us with[3]

$$\sum_{XY=Q} s_Y \Big(b(X)\otimes b(Y) - b(Y)\otimes b(X)\Big) + \sum_{XY=Q} s_X \Big(b(X)\otimes b(Y) - b(Y)\otimes b(X)\Big)\,. \quad (3.13)$$

We now return to (3.8) and, using that the double sum terms are given by (3.13), we finally obtain

$$\begin{aligned} C\circ b(Q) &= \frac{1}{s_Q}\sum_{XY=Q}\Big[(s_X + s_Y + (k^X\cdot k^Y))\Big(b(X)\otimes b(Y) - b(Y)\otimes b(X)\Big)\Big] \\ &= \sum_{XY=Q}\Big(b(X)\otimes b(Y) - b(Y)\otimes b(X)\Big) \end{aligned} \quad (3.14)$$

since $s_X + s_Y + (k^X\cdot k^Y) = s_{XY}$. Hence the result is proved. □

Lemma 1. *If P has the form a left-to-right Dynkin bracket $P = [[...[p_1, p_2], p_3], ...], p_{|P|}]$,*

$$C\circ P = \sum_{\substack{XjY=P \\ \delta(Y)=R\otimes S}} (k^X\cdot k^j)\big[XR\otimes jS - (X\leftrightarrow j)\big]\,, \quad (3.15)$$

where the deshuffle map $\delta(Y)$ is defined in (A.2).

Proof. We use induction. From (3.5) it follows that $C\circ[1,2] = (k^1\cdot k^2)(1\otimes 2 - 2\otimes 1)$. We then suppose that the relation (3.15) is satisfied for the bracket P, and consider $C\circ[P,q]$, where q is a single letter.

$$\begin{aligned} C\circ[P,q] &= (C\circ P)\wedge q + P\wedge(C\circ q) + (k^P\cdot k^q)(P\otimes q - q\otimes P) \quad (3.16)\\ &= \sum_{\substack{XjY=P \\ \delta(Y)=R\otimes S}} (k^X\cdot k^j)\big(XR\otimes jS - (X\leftrightarrow j)\big)\wedge q + (k^P\cdot k^q)(P\otimes q - q\otimes P) \\ &= \sum_{\substack{XjY=P \\ \delta(Y)=R\otimes S}} (k^X\cdot k^j)\big(XRq\otimes jS + XR\otimes jSq - (X\leftrightarrow j)\big) + (k^P\cdot k^q)(P\otimes q - q\otimes P) \\ &= \sum_{\substack{XjY=P \\ \delta(Yq)=R\otimes S}} (k^X\cdot k^j)\big(XR\otimes jS - (X\leftrightarrow j)\big) + (k^P\cdot k^q)(P\otimes q - q\otimes P) \\ &= \sum_{\substack{XjY=Pq \\ \delta(Y)=R\otimes S}} (k^X\cdot k^j)\big(XR\otimes jS - (X\leftrightarrow j)\big) \end{aligned}$$

where δ is the deshuffle map (A.2). Hence if (3.15) is true for the Dynkin bracket P, it is true for the Dynkin bracket $[P,q]$, and so by induction the result is proved. □

This result is important, as it shows that the general redefinition formulae of this paper reduce to those previously found in [2] when the multiplicity is less than six.

[3] There should be a $|Y|>1$ in the first sum and a $|X|>1$ in the second, as these words come from combining two words of non-zero length. This can be left implicit since $s_P = 0$ if $|P| = 1$.

3.2.1. Contact term-like algorithms for simplifying redefinition terms

In this subsection a further pair of algorithms based around that of contact terms (3.5) will be defined, which will be useful when simplifying the redefinition terms (4.25) in the next section. The first of these will be denoted $\tilde{C}$, and is defined by

$$\tilde{C} \circ i \equiv 0, \qquad \tilde{C} \circ [A, B] \equiv \big(C \circ A\big) \,\tilde{\wedge}\, B + A \,\tilde{\wedge}\, \big(C \circ B\big)\,, \tag{3.17}$$

(note the C map (3.5) on the right-hand side) where $\tilde{\wedge}$ is defined by

$$(A \otimes B) \,\tilde{\wedge}\, C \equiv [A, C] \otimes B\,, \qquad A \,\tilde{\wedge}(B \otimes C) \equiv [A, B] \otimes C\,. \tag{3.18}$$

In addition we define a related algorithm $\tilde{C}'$ in terms of $\tilde{C}$,

$$\tilde{C}' \circ i \equiv 0, \quad \tilde{C}' \circ [A, B] \equiv \tilde{C} \circ [A, B] - \frac{1}{2}(k^A \cdot k^B)(A \otimes B - B \otimes A)\,. \tag{3.19}$$

The following notation, similar to that of (C.5), will be used with these maps

$$\tilde{C}[\![K, S]\!] \circ [P, Q] \equiv [\![K, S]\!] \circ \big(\tilde{C} \circ [P, Q]\big)\,, \quad \tilde{C}'[\![K, S]\!] \circ [P, Q] \equiv [\![K, S]\!] \circ \big(\tilde{C}' \circ [P, Q]\big) \tag{3.20}$$

where the double bracket $[\![\cdot, \cdot]\!]$ is defined in (5.1).

Lemma 2. *The map $\tilde{C}$ satisfies*

$$\tilde{C} \circ [P, Q] = \sum_{\substack{XjY=P \\ \delta(Y)=R\otimes S}} (k^X \cdot k^j)\big([XR, Q] \otimes jS - (X \leftrightarrow j)\big) - (P \leftrightarrow Q)\,, \tag{3.21}$$

for any Dynkin brackets P and Q.

Proof. To see this we use the identity (3.15) as follows,

$$\begin{aligned} \tilde{C} \circ [P, Q] &= (C \circ P) \,\tilde{\wedge}\, Q + P \,\tilde{\wedge}(C \circ Q) \qquad (3.22) \\ &= \sum_{\substack{XjY=P \\ \delta(Y)=R\otimes S}} (k^X \cdot k^j)\big(XR \otimes jS - (X \leftrightarrow j)\big) \,\tilde{\wedge}\, Q + P \,\tilde{\wedge} \sum_{\substack{XjY=Q \\ \delta(Y)=R\otimes S}} (k^X \cdot k^j)\big(XR \otimes jS - (X \leftrightarrow j)\big) \\ &= \sum_{\substack{XjY=P \\ \delta(Y)=R\otimes S}} (k^X \cdot k^j)\big([XR, Q] \otimes jS - (X \leftrightarrow j)\big) + \sum_{\substack{XjY=Q \\ \delta(Y)=R\otimes S}} (k^X \cdot k^j)\big([P, XR] \otimes jS - (X \leftrightarrow j)\big), \end{aligned}$$

the second equality coming from the definition (3.18). The result follows after using the antisymmetry $[P, XR] = -[XR, P]$ in the final line. ☐

For illustrative examples of the $\tilde{C}$ map, see the appendix D.1.2.

4. Redefinitions of local multiparticle superfields

In this section we write down the redefinition algorithms to obtain multiparticle superfields in the so-called BCJ gauge starting from both the Lorenz and hybrid gauges with the most general bracketing configurations. The characterization of these redefinitions as a gauge transformation was identified in [2] and it will be reviewed and expanded in the next section.

4.1. Multiparticle superfields

It was shown in [1,2] that the single-particle description admits a generalization in terms of *multiparticle* superfields $A^P_\alpha(x,\theta)$, $A^P_m(x,\theta)$, $W^\alpha_P(x,\theta)$ and $F^P_{mn}(x,\theta)$, which, for convenience, are collected in the set K_P

$$K_P \in \{A^P_\alpha(x,\theta),\ A^m_P(x,\theta),\ W^\alpha_P(x,\theta),\ F^{mn}_P(x,\theta)\}\,. \tag{4.1}$$

We will review two different ways to construct them below. At the same time we will seamlessly fill some gaps in the discussions of [1,2] by utilizing the framework developed in the previous section.

4.1.1. Multiparticle superfield in the Lorenz gauge

The generalization of the single-particle linearized superfields of (2.11) to an arbitrary number of labels follows from the local version of the recursive solution to the non-linear wave equations (2.7) and can be summarized by the following definition[4]:

Definition 6 (Lorenz gauge). *Multiparticle super-Yang–Mills superfields in the Lorenz gauge are defined starting with the multiplicity-one superfields* $\hat A^i_\alpha$, $\hat A^i_m$, $\hat W^\alpha_i$ *and* $\hat F^{mn}_i$ *and recursively for arbitrary nested bracketings via*

$$\begin{aligned}
\hat A^{[P,Q]}_\alpha &= -\frac{1}{2}\big[\hat A^P_\alpha(k^P\cdot\hat A^Q) + \hat A^P_m(\gamma^m\hat W^Q)_\alpha - (P\leftrightarrow Q)\big] \\
\hat A^{[P,Q]}_m &= -\frac{1}{2}\big[\hat A^P_m(k^P\cdot\hat A^Q) + \hat A^P_n\hat F^Q_{mn} - (\hat W^P\gamma_m\hat W^Q) - (P\leftrightarrow Q)\big] \\
\hat W^\alpha_{[P,Q]} &= \frac{1}{4}\hat F^P_{rs}(\gamma^{rs}\hat W^Q)^\alpha - \frac{1}{2}(k^P\cdot\hat A^Q)\hat W^\alpha_P - \frac{1}{2}\hat W^{m\alpha}_P A^m_Q - (P\leftrightarrow Q) \\
\hat F^{[P,Q]}_{mn} &= -\frac{1}{2}\big[\hat F^{mn}_P(k_P\cdot\hat A_Q) + \hat F^{p|mn}_P\hat A^Q_p + 2\hat F^{mp}_P\hat F^n_{Q\,p} + 4\gamma^{[m}_{\alpha\beta}\hat W^{n]\alpha}_P\hat W^\beta_Q - (P\leftrightarrow Q)\big]
\end{aligned} \tag{4.2}$$

[4] The Lorenz gauge discussion in [2] is missing the definition of the general field-strength $\hat F^{mn}_{[P,Q]}$ while the definition of $\hat W^\alpha_{[P,Q]}$ is misleading as $\pounds_3\circ\hat W^\alpha_{[12,3]}\neq 0$ if one does not use momentum conservation.

where

$$\hat{W}^{m\alpha}_{[P,Q]} = k^m_{PQ}\hat{W}^{\alpha}_{[P,Q]} - C[\![\hat{A}^m, \hat{W}^{\alpha}]\!] \circ [P,Q] \tag{4.3}$$
$$\hat{F}^{m|pq}_{[P,Q]} = k^m_{PQ}\hat{F}^{pq}_{[P,Q]} - C[\![\hat{A}^m, \hat{F}^{pq}]\!] \circ [P,Q]\,,$$

and the map $C\circ$ *is defined in (3.5). Alternatively, the field-strength can be written as*

$$\hat{F}^{mn}_{[P,Q]} = k^m_{PQ}\hat{A}^n_{[P,Q]} - k^m_{PQ}\hat{A}^m_{[P,Q]} - C[\![\hat{A}^m, \hat{A}^n]\!] \circ [P,Q]\,. \tag{4.4}$$

These recursions apply to arbitrary bracketing structures encompassed by P and Q. For example $\hat{A}^m_{[[1,2],[[3,4],5]]}$ implies that $P = [1,2]$ and $Q = [[3,4],5]$ and leads to

$$\hat{A}^m_{[[1,2],[[3,4],5]]} = -\frac{1}{2}\Big[\hat{A}^{[1,2]}_m(k^{12}\cdot\hat{A}^{[[3,4],5]}) + \hat{A}^{[1,2]}_n\hat{F}^{[[3,4],5]}_{mn} \tag{4.5}$$
$$- (\hat{W}^{[1,2]}\gamma_m\hat{W}^{[[3,4],5]}) - ([1,2]\leftrightarrow[[3,4],5])\Big]\,.$$

In addition, from the example for $C\circ[[1,2],[3,4]]$ in (D.1) we have for (4.4),

$$\begin{aligned}\hat{F}^{mn}_{[[1,2],[3,4]]} &= k^m_{1234}\hat{A}^n_{[[1,2],[3,4]]} - k^n_{1234}\hat{A}^m_{[[1,2],[3,4]]} \\ &- (k^1\cdot k^2)\big(\hat{A}^m_{[1,[3,4]]}\hat{A}^n_2 + \hat{A}^m_1\hat{A}^n_{[2,[3,4]]} - (1\leftrightarrow 2)\big) \\ &- (k^3\cdot k^4)\big(\hat{A}^m_{[[1,2],3]}\hat{A}^n_4 + \hat{A}^m_3\hat{A}_{[[1,2],4]} - (3\leftrightarrow 4)\big) \\ &- (k^{12}\cdot k^{34})\big(\hat{A}^m_{[1,2]}\hat{A}^n_{[3,4]} - \hat{A}^m_{[3,4]}\hat{A}^n_{[1,2]}\big)\,.\end{aligned} \tag{4.6}$$

Identifying the pair of words P and Q for the superfields on the right-hand side of (4.5) leads to further applications of the recursions in (4.2) until eventually all superfields are of single-particle nature.

4.1.2. Multiparticle superfields in the hybrid gauge

Let us assume that all superfields of multiplicities P and Q in K_P and K_Q have been redefined to satisfy all the BCJ symmetries (2.15) (we will explain how to do this below). Since multiparticle superfields K_P in the BCJ gauge satisfy the same symmetries as the Dynkin bracket $P = [[...[p_1,p_2],p_3],...],p_{|P|}]$ their multiparticle labels will be written as plain words $P = p_1p_2\ldots p_{|P|}$. One then defines higher-multiplicity superfields in $\check{K}_{[P,Q]}$ as follows:

Definition 7 (Hybrid gauge). *Multiparticle super-Yang–Mills superfields in the hybrid gauge are distinguished by a check accent* $\check{K}_{[P,Q]}$ *and are defined by*

$$\begin{aligned}
\check{A}^{[P,Q]}_\alpha &= -\frac{1}{2}[A^P_\alpha(k^P\cdot A^Q) + A^P_m(\gamma^m W^Q)_\alpha - (P\leftrightarrow Q)] \\
\check{A}^{[P,Q]}_m &= -\frac{1}{2}[A^P_m(k^P\cdot A^Q) + A^P_n F^Q_{mn} - (W^P\gamma_m W^Q) - (P\leftrightarrow Q)] \\
\check{W}^\alpha_{[P,Q]} &= \frac{1}{4}F^P_{rs}(\gamma^{rs}W^Q)^\alpha - \frac{1}{2}(k^P\cdot A^Q)W^\alpha_P - \frac{1}{2}W^{m\alpha}_P A^m_Q - (P\leftrightarrow Q) \\
\check{F}^{[P,Q]}_{mn} &= -\frac{1}{2}\big[F^{mn}_P(k_P\cdot A_Q) + F^{p|mn}_P A^Q_p + 2F^{mp}_P F^n_{Q\,p} + 4\gamma^{[m}_{\alpha\beta}W^{n]\alpha}_P W^\beta_Q - (P\leftrightarrow Q)\big]
\end{aligned} \tag{4.7}$$

where the superfields in K_P *and* K_Q *on the right-hand side satisfy the generalized Jacobi identities (2.15) and*

$$\begin{aligned}
W^{m\alpha}_{[P,Q]} &= k^m_{PQ}W^\alpha_{[P,Q]} - C[\![A^m, W^\alpha]\!]\circ[P,Q] \\
F^{m|pq}_{[P,Q]} &= k^m_{PQ}F^{pq}_{[P,Q]} - C[\![A^m, F^{pq}]\!]\circ[P,Q]\,,
\end{aligned} \tag{4.8}$$

are the local form of the superfields of higher-mass dimension defined in [2] with the map $C[\![\cdot,\cdot]\!]$ *as in (C.5).*

Note an important difference with respect to the definitions of superfields $\hat{K}_{[P,Q]}$ in the Lorenz gauge (4.2). The definitions in the Lorenz gauge are recursive while in the hybrid gauge they are *not* – the superfields $\check{K}_{[P,Q]}$ on the left-hand side of (4.7) have to be redefined before they can be used as the input on the right-hand side at the next step. However, from a purely practical perspective, to obtain the explicit expressions of the superfields in the BCJ gauge it is more convenient to use the hybrid gauge.

4.2. From hybrid gauge to BCJ gauge

The general formula to redefine the superfields $\check{K}_{[P,Q]}\in\{\check{A}_\alpha,\check{A}^m,\check{W}^\alpha\}$ from the hybrid gauge (4.7) to superfields $K_{[P,Q]}\in\{A_\alpha,A^m,W^\alpha\}$ in the BCJ gauge is given by

$$\begin{aligned}
K_{[P,Q]} \equiv \check{K}_{[P,Q]} &- \sum_{\substack{P=XjY\\ \delta(Y)=R\otimes S}}(k_X\cdot k_j)\big[H_{[XR,Q]}\,K_{jS} - (X\leftrightarrow j)\big] \\
&+ \sum_{\substack{Q=XjY\\ \delta(Y)=R\otimes S}}(k_X\cdot k_j)\big[H_{[XR,P]}\,K_{jS} - (X\leftrightarrow j)\big] - \begin{cases} D_\alpha H_{[P,Q]} & : \; K=A_\alpha \\ k^m_{PQ}H_{[P,Q]} & : \; K=A^m \\ 0 & : \; K=W^\alpha \end{cases}\,.
\end{aligned} \tag{4.9}$$

Alternatively, the identity (3.21) can be used to rewrite (4.9) more succinctly as[5]

$$K^{[P,Q]} = \check{K}^{[P,Q]} - \tilde{C}[\![H,K]\!] \circ [P,Q] - \begin{cases} D_\alpha H_{[P,Q]} & : \; K = A_\alpha \\ k^m_{PQ} H_{[P,Q]} & : \; K = A^m \\ 0 & : \; K = W^\alpha \end{cases} . \tag{4.10}$$

These redefinitions introduce new superfields $H_{[P,Q]}$ whose purpose is to make the resulting linear combinations satisfy the BCJ symmetries. For example, the first instances of the redefinition (4.9) for $A^m_{[P,Q]}$ up to multiplicity $|P| + |Q| = 5$ are given by (recall that $\check{A}^m_i \equiv A^m_i$ and $\check{A}^m_{[i,j]} \equiv A^m_{ij}$)

$$\begin{aligned}
A^m_{[1,2]} &= \check{A}^m_{[1,2]} \\
A^m_{[12,3]} &= \check{A}^m_{[12,3]} - k^m_{123} H_{[12,3]} \\
A^m_{[12,34]} &= \check{A}^m_{[12,34]} - (k^1 \cdot k^2)\Big[H_{[1,34]} A^m_2 - H_{[2,34]} A^m_1\Big] \\
&\quad + (k^3 \cdot k^4)\Big[H_{[3,12]} A^m_4 - H_{[4,12]} A^m_3\Big] - k^m_{1234} H_{[12,34]} \\
A^m_{[123,4]} &= \check{A}^m_{[123,4]} - (k^1 \cdot k^2)\Big[H_{[13,4]} A^m_2 - H_{[23,4]} A^m_1\Big] \\
&\quad - (k^{12} \cdot k^3) H_{[12,4]} A^m_3 - k^m_{1234} H_{[123,4]} \\
A^m_{[1234,5]} &= \check{A}^m_{[1234,5]} - (k_1 \cdot k_2)\big[H_{[134,5]} A^m_2 + H_{[14,5]} A^m_{23} + H_{[13,5]} A^m_{24} - (1 \leftrightarrow 2)\big] \\
&\quad - (k_{12} \cdot k_3)\big[H_{[124,5]} A^m_3 + H_{[12,5]} A^m_{34} - (12 \leftrightarrow 3)\big] \\
&\quad - (k_{123} \cdot k_4) H_{[123,5]} A^m_4 - k^m_{12345} H_{[1234,5]} \\
A^m_{[123,45]} &= \check{A}^m_{[123,45]} - (k^1 \cdot k^2)\Big[H_{[13,45]} A^m_2 + H_{[1,45]} A^m_{23} - (1 \leftrightarrow 2)\Big] \\
&\quad - (k^{12} \cdot k^3)\Big[H_{[12,45]} A^m_3 - (12 \leftrightarrow 3)\Big] \\
&\quad + (k^4 \cdot k^5)\Big[H_{[4,123]} A^m_5 - (4 \leftrightarrow 5)\Big] - k^m_{12345} H_{[123,45]}
\end{aligned} \tag{4.11}$$

To help in elucidating the outcome of the above redefinitions we note that, for suitable $H_{[P,Q]}$ to be given below, the superfields $K_{[P,Q]}$ on the left-hand side satisfy all the identities implied by the bracket structure. For example,

$$A^m_{[12,3]} = -A^m_{[21,3]} = -A^m_{[3,12]} , \qquad A^m_{[12,3]} + A^m_{[23,1]} + A^m_{[31,2]} = 0 . \tag{4.12}$$

The above means that $A^m_{[[1,2],3]}$ satisfies the same symmetries as $[[1,2],3]$ and can be represented via the shorthand $A^m_{123} \equiv A^m_{[[1,2],3]}$. In general, the effect of the above redefinitions is such that $K_{[P,Q]} = K_{P\ell(Q)}$, as shown in (2.17).

[5] It should be noted that, despite (4.10) being defined for general bracketing structures, it has only been verified for P and Q Dynkin brackets in accordance with (4.9).

We have not yet discussed how the field strength $F^{mn}_{[P,Q]}$ superfields in the BCJ gauge are found. These are most easily described by constructing them in terms of the above redefined BCJ gauge superfields and using the contact-term map (3.5),

$$F^{mn}_{[P,Q]} = k^{PQ}_m A^{[P,Q]}_n - k^{PQ}_n A^{[P,Q]}_m - C[\![A_m, A_n]\!] \circ [P,Q]\,. \tag{4.13}$$

4.2.1. The explicit expression of $H_{[A,B]}$

In [2] the explicit form of the superfields $H_{[A,B]}$ was only given up to multiplicity five. We now propose the following recursive solution for general multiplicities[6]

$$H_{[i,j]} = 0\,, \qquad H_{[A,B]} = (-1)^{|B|}\frac{|A|}{|A|+|B|}\sum_{XjY=\dot{a}\tilde{B}}(-1)^{|Y|}H'_{\tilde{Y},j,X} - (A\leftrightarrow B)\,, \tag{4.14}$$

where $\dot{a}$ and $\dot{b}$ denote the letterifications of A and B as defined in the appendix A and

$$\begin{aligned} H'_{A,B,C} \equiv{}& H_{A,B,C} + \Big[\frac{1}{2}H_{[A,B]}(k_{AB}\cdot A_C) + \text{cyclic}(A,B,C)\Big] \\ &- \Big[\sum_{\substack{XjY=A\\ \delta(Y)=R\otimes S}}(k^X\cdot k^j)\big[H_{[XR,B]}H_{[jS,C]} - (X\leftrightarrow j)\big] + \text{cyclic}(A,B,C)\Big]\,, \end{aligned} \tag{4.15}$$

$$H_{A,B,C} \equiv -\frac{1}{4}A^m_A A^n_B F^{mn}_C + \frac{1}{2}(W_A\gamma_m W_B)A^m_C + \text{cyclic}(A,B,C)\,. \tag{4.16}$$

Given that $H_{[A,B]}$ of multiplicities less than three vanish, it is easy to see that the second line of (4.15) can only be probed when the superfields have multiplicity six or higher. Furthermore, note that $H_{[A,B]}$ satisfies generalized Jacobi identities within A and B and therefore will be written using plain[7] words.

The superfields $H_{[P,Q]}$ up to multiplicity seven are given by

$$\begin{aligned} H_{[12,3]} &= \frac{1}{3}\big(H'_{1,2,3}\big) \\ H_{[123,4]} &= \frac{1}{4}\big(H'_{12,3,4} - H'_{1,2,43}\big) \\ H_{[12,34]} &= \frac{1}{4}\big(2H'_{1,2,34} - 2H'_{3,4,12}\big) \\ H_{[1234,5]} &= \frac{1}{5}\big(H'_{123,4,5} - H'_{12,3,54} + H'_{1,2,543}\big) \end{aligned} \tag{4.17}$$

---

[6] We acknowledge the invaluable usage of `FORM` [21] in these calculations.

[7] By convention, a plain word in a BCJ-gauge superfield is a shorthand for the left-to-right nested bracketing, e.g $P = 1234 \leftrightarrow P = [[[1,2],3],4]$.

$$
\begin{aligned}
H_{[123,45]} &= \frac{1}{5}\big(2H'_{12,3,45} - 2H'_{1,2,453} - 3H'_{4,5,123}\big)\\
H_{[12345,6]} &= \frac{1}{6}\big(H'_{1234,5,6} - H'_{123,4,65} + H'_{12,3,654} - H'_{1,2,6543}\big)\\
H_{[1234,56]} &= \frac{1}{6}\big(2H'_{123,4,56} - 2H'_{12,3,564} + 2H'_{1,2,5643} - 4H'_{5,6,1234}\big)\\
H_{[123,456]} &= \frac{1}{6}\big(3H'_{12,3,456} - 3H'_{1,2,4563} - 3H'_{45,6,123} + 3H'_{4,5,1236}\big)\\
H_{[123456,7]} &= \frac{1}{7}\big(H'_{12345,6,7} - H'_{1234,5,76} + H'_{123,4,765} - H'_{12,3,7654} + H'_{1,2,76543}\big)\\
H_{[12345,67]} &= \frac{1}{7}\big(2H'_{1234,5,67} - 2H'_{123,4,675} + 2H'_{12,3,6754} - 2H'_{1,2,67543} - 5H'_{6,7,12345}\big)\\
H_{[1234,567]} &= \frac{1}{7}\big(3H'_{123,4,567} - 3H'_{12,3,5674} + 3H'_{1,2,56743} - 4H'_{56,7,1234} + 4H'_{5,6,12347}\big)\,,
\end{aligned}
$$

while higher multiplicity examples can be easily generated using the general formula (4.14). We have explicitly tested that the superfields up to and including multiplicity nine following from the formulas (4.9) and (4.14) satisfy the generalized Jacobi identities[8]. Since new corrections cubic in $H_{[A,B]}$ could be present at multiplicity nine, the fact that these formulas lead to superfields satisfying the BCJ symmetries suggest that (4.14) is correct for arbitrary multiplicity.

4.3. From Lorenz gauge to BCJ gauge

Alternatively, one can generate superfields in the BCJ gauge by starting from the superfields in the Lorenz gauge obtained through the recursions (4.2). The redefinitions are more involved in this case and one can show that to obtain their BCJ gauge counterparts requires the following iterated redefinition,

$$
K^{[P,Q]} = L_1 \circ \hat{K}^{[P,Q]}\,, \tag{4.18}
$$

where the operator L_j is defined by

$$
L_j \circ \hat{K}^{[P,Q]} \equiv \hat{K}^{[P,Q]} - \frac{1}{j} C[\![\hat{H}, L_{(j+1)} \circ \hat{K}]\!] \circ [P,Q] - \frac{1}{j}\begin{cases} D_\alpha \hat{H}_{[P,Q]} & : \ K = A_\alpha \\ k^m_{PQ} \hat{H}_{[P,Q]} & : \ K = A^m \\ 0 & : \ K = W^\alpha \end{cases} \,, \tag{4.19}
$$

[8] To simplify the algebra we tested the bosonic components. Since the backbone of the recursion (4.14) is given by the supersymmetric $H_{A,B,C}$ we believe that (4.14) also leads to correct fermionic components.

while $C \circ [\![\cdot,\cdot]\!]$ is defined in (C.5). Notice that $L_j \circ \hat{K}^{[P,Q]}$ gives rise to the action of the operator $L_{(j+1)} \circ \hat{K}^{[A,B]}$ on the right-hand side with $|A|+|B|<|P|+|Q|$. Therefore this is a iteration over the index j which eventually stops. As we will see below, the iteration built into the redefinition (4.18) yields the infinite series of non-linear terms present in the finite gauge transformation (5.11).

The examples (4.11) of redefinitions from the hybrid to BCJ gauge have the following Lorenz to BCJ counterparts, using (4.18) and keeping all the nested Lie brackets explicit

$$\begin{aligned}
A^m_{[1,2]} &= \hat{A}^m_{[1,2]}\,, \\
A^m_{[[1,2],3]} &= \hat{A}^m_{[[1,2],3]} - k^m_{123}\hat{H}_{[[1,2],3]}\,, \\
A^m_{[[1,2],[3,4]]} &= \hat{A}^m_{[[1,2],[3,4]]} - (k_1\cdot k_2)\Big(\hat{H}_{[1,[3,4]]}\hat{A}^m_2 - \hat{H}_{[2,[3,4]]}\hat{A}^m_1\Big) \\
&\quad + (k_3\cdot k_4)\Big(\hat{H}_{[[1,2],4]}\hat{A}^m_3 - \hat{H}_{[[1,2],3]}\hat{A}^m_4\Big) - k^m_{1234}\hat{H}_{[[1,2],[3,4]]}\,, \\
A^m_{[[[1,2],3],4]} &= \hat{A}^m_{[[[1,2],3],4]} - (k_1\cdot k_2)\Big(\hat{H}_{[[1,3],4]}\hat{A}^m_2 - \hat{H}_{[[2,3],4]}\hat{A}^m_1\Big) \\
&\quad - (k_{12}\cdot k_3)\Big(\hat{H}_{[[1,2],4]}\hat{A}^m_3\Big) - (k_{123}\cdot k_4)\Big(\hat{H}_{[[1,2],3]}\hat{A}^m_4\Big) - k^m_{1234}\hat{H}_{[[[1,2],3],4]}\,, \\
A^m_{[[[[1,2],3],4],5]} &= \hat{A}^m_{[[[[1,2],3],4],5]} - (k_1\cdot k_2)\Big(\hat{H}_{[[1,3],4]}\hat{A}^m_{[2,5]} + \hat{H}_{[[1,3],5]}\hat{A}^m_{[2,4]} + \hat{H}_{[[1,4],5]}\hat{A}^m_{[2,3]} \\
&\qquad + \hat{H}_{[[[1,3],4],5]}\hat{A}^m_2 - (1\leftrightarrow 2)\Big) \\
&\quad - (k_{12}\cdot k_3)\Big(\hat{H}_{[[1,2],4]}\hat{A}^m_{[3,5]} + \hat{H}_{[[1,2],5]}\hat{A}^m_{[3,4]} + \hat{H}_{[[[1,2],4],5]}\hat{A}^m_3 - ([1,2]\leftrightarrow 3)\Big) \\
&\quad - (k_{123}\cdot k_4)\Big(\hat{H}_{[[1,2],3]}\hat{A}^m_{[4,5]} + \hat{H}_{[[[1,2],3],5]}\hat{A}^m_4\Big) \\
&\quad - (k_{1234}\cdot k_5)\Big(\hat{H}_{[[[1,2],3],4]}\hat{A}^m_5\Big) - \hat{H}_{[[[[1,2],3],4],5]}k^m_{12345}\,, \\
A^m_{[[[1,2],3],[4,5]]} &= \hat{A}^m_{[[[1,2],3],[4,5]]} - (k_1\cdot k_2)\Big(\hat{H}_{[1,[4,5]]}\hat{A}^m_{[2,3]} + \hat{H}_{[[1,3],[4,5]]}\hat{A}^m_2 - (1\leftrightarrow 2)\Big) \\
&\quad - (k_{12}\cdot k_3)\Big(\hat{H}_{[[1,2],[4,5]]}\hat{A}^m_3 - \hat{H}_{[3,[4,5]]}\hat{A}^m_{[1,2]}\Big) \\
&\quad - (k_{123}\cdot k_{45})\Big(\hat{H}_{[[1,2],3]}\hat{A}^m_{[4,5]}\Big) \\
&\quad + (k_4\cdot k_5)\Big(\hat{H}_{[[[1,2],3],5]}\hat{A}^m_4 - \hat{H}_{[[[1,2],3],4]}\hat{A}^m_5\Big) - k^m_{12345}\hat{H}_{[[[1,2],3],[4,5]]}\,.
\end{aligned} \tag{4.20}$$

To illustrate (4.18) when there is more than one iteration, consider the redefinition of the superfield $\hat{A}_m^{[[12,34],56]}$ to the BCJ gauge. It starts as

$$\begin{aligned}
A_m^{[[12,34],56]} &= L_1 \circ \hat{A}_m^{[[12,34],56]} \\
&= \hat{A}_m^{[[12,34],56]} - k_m^{123456}\hat{H}_{[[12,34],56]} - C[\![\hat{H}, L_2\circ\hat{A}^m]\!]\circ[[12,34],56]
\end{aligned} \tag{4.21}$$

Using the definition of the $C\circ$ map from (3.5) leads to

$$\begin{aligned}
A_m^{[[12,34],56]} &= \hat{A}_m^{[[12,34],56]} - k_m^{123456}\hat{H}_{[[12,34],56]} \qquad (4.22)\\
&- (k^1\cdot k^2)\Big(\big(L_2\circ \hat{A}_m^{2}\big)\hat{H}_{[[1,34],56]} + \big(L_2\circ \hat{A}_m^{[2,34]}\big)\hat{H}_{[1,56]}\\
&\qquad + \big(L_2\circ \hat{A}_m^{[2,56]}\big)\hat{H}_{[1,34]} - (1\leftrightarrow 2)\Big)\\
&- (k^{12}\cdot k^{34})\Big(\big(L_2\circ \hat{A}_m^{34}\big)\hat{H}_{[12,56]} - (12\leftrightarrow 34)\Big)\\
&- (k^{1234}\cdot k^{56})\big(L_2\circ \hat{A}_m^{56}\big)\hat{H}_{[12,34]}\\
&- (k^3\cdot k^4)\Big(\big(L_2\circ \hat{A}_m^{4}\big)\hat{H}_{[123,56]} + \big(L_2\circ \hat{A}_m^{[12,4]}\big)\hat{H}_{[3,56]}\\
&\qquad + \big(L_2\circ \hat{A}_m^{[4,56]}\big)\hat{H}_{[12,3]} - (3\leftrightarrow 4)\Big)\\
&- (k^5\cdot k^6)\Big(\big(L_2\circ \hat{A}_m^{6}\big)\hat{H}_{[[12,34],5]} - (5\leftrightarrow 6)\Big)\,.
\end{aligned}$$

Note that on most of the terms the iteration stops since $L_2\circ\hat{A}_m^i = \hat{A}_m^i$ and $L_2\circ\hat{A}_m^{ij} = \hat{A}_m^{ij}$. The only remaining non-trivial action $L_2\circ\hat{A}_m^P$ are on terms are of multiplicity three. From (4.18) we obtain,

$$L_2\circ\hat{A}_m^{[12,3]} = \hat{A}_m^{[12,3]} - \frac{1}{2}k_{123}^m\hat{H}_{[12,3]}\,, \qquad L_2\circ\hat{A}_m^{[1,23]} = \hat{A}_m^{[1,23]} - \frac{1}{2}k_{123}^m\hat{H}_{[1,23]}. \qquad (4.23)$$

Plugging all of this into (4.22) yields

$$\begin{aligned}
A_m^{[[12,34],56]} &= \hat{A}_m^{[[12,34],56]} - k_m^{123456}\hat{H}_{[[12,34],56]} \qquad (4.24)\\
&- (k^1\cdot k^2)\Big(\hat{A}_m^{2}\hat{H}_{[[1,34],56]} + \hat{A}_m^{[2,34]}\hat{H}_{[1,56]} + \hat{A}_m^{[2,56]}\hat{H}_{[1,34]}\\
&\qquad - \frac{1}{2}k_m^{234}\hat{H}_{[2,34]}\hat{H}_{[1,56]} - \frac{1}{2}k_m^{256}\hat{H}_{[2,56]}\hat{H}_{[1,34]} - (1\leftrightarrow 2)\Big)\\
&- (k^{12}\cdot k^{34})\Big(\hat{A}_m^{34}\hat{H}_{[12,56]} - (12\leftrightarrow 34)\Big)\\
&- (k^{1234}\cdot k^{56})\hat{A}_m^{56}\hat{H}_{[12,34]}\\
&- (k^3\cdot k^4)\Big(\hat{A}_m^{4}\hat{H}_{[123,56]} + \hat{A}_m^{[12,4]}\hat{H}_{[3,56]} + \hat{A}_m^{[4,56]}\hat{H}_{[12,3]}\\
&\qquad - \frac{1}{2}k_m^{124}\hat{H}_{[12,4]}\hat{H}_{[3,56]} - \frac{1}{2}k_m^{456}\hat{H}_{[4,56]}\hat{H}_{[12,3]} - (3\leftrightarrow 4)\Big)\\
&- (k^5\cdot k^6)\Big(\hat{A}_m^{6}\hat{H}_{[[12,34],5]} - (5\leftrightarrow 6)\Big)\,.
\end{aligned}$$

Higher-rank examples can be similarly generated from the recursion (4.19).

4.3.1. Explicit form of $\hat{H}_{[P,Q]}$ for the Lorenz to BCJ gauge redefinition

Each $\hat{H}_{[P,Q]}$ is defined by enforcing the BCJ symmetry on the corresponding superfield $K_{[P,Q]}$. It has been found that up to multiplicity eight that these can be simplified as

$$\begin{aligned} \hat{H}_{[A,B]} &= \hat{H}'_{[A,B]} - \frac{1}{2}\tilde{C}[\![\hat{H},\hat{H}]\!] \circ [A,B]\,, \\ \hat{H}'_{[A,B]} &= H_{[A,B]} - \frac{1}{2}\Big[(\hat{H}'_A k^m_A - \tilde{C}'[\![\hat{H}^m,\hat{H}]\!] \circ A)A^B_m - (A \leftrightarrow B)\Big]\,, \\ \hat{H}'_i &= \hat{H}'_{[i,j]} = 0\,, \end{aligned} \tag{4.25}$$

where the $H_{[A,B]}$ are defined as they were in (4.14) - (4.16), and $\hat{H}^m_A \equiv k^m_A \hat{H}_A$. Furthermore, the maps $\tilde{C}$ and $\tilde{C}'$ are the variants of the contact-term map C defined in the section 3.2.1.

To demonstrate the meaning of these maps we will now provide examples. First of all note that the $\tilde{C}$ and $\tilde{C}'$ maps in (4.25) are both associated with pairs of $\hat{H}$ superfields, each of which requires three indices, and so these terms will only be non-zero when $|A|+|B| \geq 6$. Thus at lower multiplicities these relations reduce to equation (3.15) of [2], as the $\tilde{C}$ and $\tilde{C}'$ terms only start contributing at multiplicity 6+. An example of the relations in this case is as follows:

$$\begin{aligned} \hat{H}_{[[[1,2],3],[4,5]]} &= \hat{H}'_{[[[1,2],3],[4,5]]} \\ &= H_{[[[1,2],3],[4,5]]} - \frac{1}{2}k^m_{123}\hat{H}'_{[[1,2],3]}A^{[4,5]}_m \\ &= H_{[[[1,2],3],[4,5]]} - \frac{1}{2}H_{[[1,2],3]}(k_{123} \cdot A^{[4,5]})\,. \end{aligned} \tag{4.26}$$

We will now outline an example of (4.25) for the multiplicity six redefinition term $\hat{H}_{[[[[1,2],3],[4,5]],6]}$, which should demonstrate the formulae more clearly.

$$\hat{H}_{[[[[1,2],3],[4,5]],6]} = \hat{H}'_{[[[[1,2],3],[4,5]],6]} - \frac{1}{2}\tilde{C}[\![\hat{H},\hat{H}]\!] \circ [[[[1,2],3],[4,5]],6]\,. \tag{4.27}$$

The expansion of the $\tilde{C}$ term above is given as the example (D.4) in appendix D.1.2, and from it we see that

$$\begin{aligned} \tilde{C}[\![\hat{H},\hat{H}]\!] \circ [[[[1,2],3],[4,5]],6] &= (k^1 \cdot k^2)\big(\hat{H}_{[[1,3],6]}\hat{H}_{[2,[4,5]]} - \hat{H}_{[[2,3],6]}\hat{H}_{[1,[4,5]]}\big) \\ &+ (k^{12} \cdot k^3)\big(\hat{H}_{[[1,2],6]}\hat{H}_{[3,[4,5]]}\big) + (k^{123} \cdot k^{45})\big(\hat{H}_{[[4,5],6]}\hat{H}_{[[1,2],3]}\big) \\ &= (k^1 \cdot k^2)\big(H_{[[1,3],6]}H_{[2,[4,5]]} - H_{[[2,3],6]}H_{[1,[4,5]]}\big) \\ &+ (k^{12} \cdot k^3)\big(H_{[[1,2],6]}H_{[3,[4,5]]}\big) + (k^{123} \cdot k^{45})\big(H_{[[4,5],6]}H_{[[1,2],3]}\big) \end{aligned} \tag{4.28}$$

As for the $\hat{H}'_{[[[[1,2],3],[4,5]],6]}$ term, this piece is given by

$$\begin{aligned}
\hat{H}'_{[[[[1,2],3],[4,5]],6]} &= H_{[[[[1,2],3],[4,5]],6]} - \frac{1}{2}\big[(\hat{H}'_{[[[1,2],3],[4,5]]}k^m_{12345} - \tilde{C}'[\![\hat{H},\hat{H}]\!] \circ [[[1,2],3],[4,5]])A^6_m\big] \\
&= H_{[[[[1,2],3],[4,5]],6]} - \frac{1}{2}H_{[[[1,2],3],[4,5]]}(k^m_{12345} \cdot A^6) \\
&\quad + \frac{1}{4}H_{[[1,2],3]}(k_{123} \cdot A^{45})(k^{12345} \cdot A^6)\,, \qquad (4.29)
\end{aligned}$$

where we have used (4.26) and that the action of $\tilde{C}'[\![\hat{H},\hat{H}]\!]$ on any Lie polynomial with less than six letters is zero. Putting this all together we thus have that

$$\begin{aligned}
\hat{H}_{[[[[1,2],3],[4,5]],6]} &= H_{[[[[1,2],3],[4,5]],6]} \qquad (4.30)\\
&- \frac{1}{2}H_{[[[1,2],3],[4,5]]}(k^m_{12345} \cdot A^6) + \frac{1}{4}H_{[[1,2],3]}(k_{123} \cdot A^{45})(k^{12345} \cdot A^6) \\
&- \frac{1}{2}(k_1 \cdot k_2)\big(H_{[[1,3],6]}H_{[2,[4,5]]} - H_{[[2,3],6]}H_{[1,[4,5]]}\big) \\
&- \frac{1}{2}(k_{12} \cdot k_3)\big(H_{[[1,2],6]}H_{[3,[4,5]]}\big) - \frac{1}{2}(k_{123} \cdot k_{45})\big(H_{[[4,5],6]}H_{[[1,2],3]}\big)\,.
\end{aligned}$$

Unfortunately to see an example where the $\tilde{C}'$ map in the definition of $\hat{H}'$ comes into affect requires going to multiplicity seven, which considerably increases the number of terms involved and makes any such example less easy to follow. The process is not terribly different from the one just outlined though, there are just more terms involved.

It might raise some concerns that (4.25) and (4.14) - (4.16) are in some places defined in terms of BCJ gauge superfields, and so this might not represent a true gauge transformation. This is however not an issue, as a purely Lorenz gauge version of (4.25) can be found by just replacing the BCJ superfields with their Lorenz gauge expansions (4.18). Some difficulty may arise doing this for $H_{A,B,C}$ due to the presence of F^{mn}_P terms. However, we do the same thing, and plug the Lorenz gauge expansions into (4.13) to get

$$F^{mn}_{[P,Q]} = k^{PQ}_m(L_1 \circ \hat{A}^{[P,Q]}_n) - k^{PQ}_m(L_1 \circ \hat{A}^{[P,Q]}_m) - C[\![(L_1 \circ \hat{A}_m),(L_1 \circ \hat{A}_n)]\!] \circ [P,Q]\,. \quad (4.31)$$

The notation of (4.25) has just been chosen for its compactness and clarity.

5. BCJ symmetries and standard gauge transformations

In this section we will briefly review the result of [2] that the redefinitions of a local superfield K from the Lorenz to the BCJ gauge amount to a standard gauge transformation of the corresponding non-linear superfield $\mathbb{K}$ introduced in section 2.2. However, the discussion of [2] was based on examples up to multiplicity five and consequently missed an infinite number of correction terms. As a result, the gauge transformations were identified only in infinitesimal form. We will prove that the iterative redefinitions (3.5) lead to a *finite* gauge transformation instead.

To show this one uses the perturbiner series expansion $\mathbb{K}$ as given in (2.8) in terms of its Berends–Giele currents. Before proceeding, we review the definition of the Berends–Giele currents using a formulation based on the b map (5.2).

5.1. Berends–Giele currents and contact terms from maps on words

We will define the notion of a *Berends–Giele current* from a purely combinatorial point of view based on the map $b(P)$ acting on words. In order to do this for arbitrary labelled objects such as multiparticle superfields, let us define the a replacement of words by arbitrary superfields as

$$[\![K]\!] \circ P \equiv K_P\,, \qquad [\![K,S]\!] \circ P \otimes Q \equiv K_P S_Q\,. \tag{5.1}$$

In turn, this definition can be used to define the Berends–Giele currents and related concepts through the b and C maps.

Definition 4 (Berends–Giele map). *If $K_P \in \{A^P_\alpha, A^m_P, W^\alpha_P, F^{mn}_P\}$ is a local multiparticle superfield, its associated Berends–Giele current is represented by a calligraphic letter $\mathcal{K}_P \in \{\mathcal{A}^P_\alpha, \mathcal{A}^m_P, \mathcal{W}^\alpha_P, \mathcal{F}^{mn}_P\}$ and is given by*

$$\mathcal{K}_P \equiv [\![K]\!] \circ b(P)\,, \tag{5.2}$$

where $[\![\,\cdot\,]\!]$ is defined in (5.1).

For example, the Berends–Giele currents up to multiplicity five associated to the vector potential A^m_P following from the definition $\mathcal{A}^m_P = [\![A^m]\!] \circ b(P)$ are given by $\mathcal{A}^m_1 = A^m_1$ and

$$\begin{aligned}
\mathcal{A}^m_{12} &= \frac{A^m_{[1,2]}}{s_{12}}\,, \\
\mathcal{A}^m_{123} &= \frac{A^m_{[[1,2],3]}}{s_{12}s_{123}} + \frac{A^m_{[1,[2,3]]}}{s_{123}s_{23}}\,,
\end{aligned} \tag{5.3}$$

$$
\begin{aligned}
\mathcal{A}^m_{1234} &= \frac{A^m_{[[[1,2],3],4]}}{s_{12}s_{123}s_{1234}} + \frac{A^m_{[[1,[2,3]],4]}}{s_{123}s_{1234}s_{23}} + \frac{A^m_{[[1,2],[3,4]]}}{s_{12}s_{1234}s_{34}} + \frac{A^m_{[1,[[2,3],4]]}}{s_{1234}s_{23}s_{234}} + \frac{A^m_{[1,[2,[3,4]]]}}{s_{1234}s_{234}s_{34}}\,,\\
\mathcal{A}^m_{12345} &= \frac{A^m_{[[[[1,2],3],4],5]}}{s_{12}s_{123}s_{1234}s_{12345}} + \frac{A^m_{[[[1,[2,3]],4],5]}}{s_{123}s_{1234}s_{12345}s_{23}} + \frac{A^m_{[[[1,2],[3,4]],5]}}{s_{12}s_{1234}s_{12345}s_{34}} + \frac{A^m_{[[[1,2],3],[4,5]]}}{s_{12}s_{123}s_{12345}s_{45}}\\
&+ \frac{A^m_{[[1,[[2,3],4]],5]}}{s_{1234}s_{12345}s_{23}s_{234}} + \frac{A^m_{[[1,[2,[3,4]]],5]}}{s_{1234}s_{12345}s_{234}s_{34}} + \frac{A^m_{[[1,[2,3]],[4,5]]}}{s_{123}s_{12345}s_{23}s_{45}} + \frac{A^m_{[[1,2],[[3,4],5]]}}{s_{12}s_{12345}s_{34}s_{345}}\\
&+ \frac{A^m_{[[1,2],[3,[4,5]]]}}{s_{12}s_{12345}s_{345}s_{45}} + \frac{A^m_{[1,[[[2,3],4],5]]}}{s_{12345}s_{23}s_{234}s_{2345}} + \frac{A^m_{[1,[[2,[3,4]],5]]}}{s_{12345}s_{234}s_{2345}s_{34}} + \frac{A^m_{[1,[[2,3],[4,5]]]}}{s_{12345}s_{23}s_{2345}s_{45}}\\
&+ \frac{A^m_{[1,[2,[[3,4],5]]]}}{s_{12345}s_{2345}s_{34}s_{345}} + \frac{A^m_{[1,[2,[3,[4,5]]]]}}{s_{12345}s_{2345}s_{345}s_{45}}\,.
\end{aligned}
$$

The multiplicity six case is given in equation (F.7) of the appendix. Moreover, one can show that $M_P = [\![V]\!] \circ b(P)$ reproduces the intuitive Berends–Giele definition given in the appendix of [1]. See fig. 2.

5.2. BCJ symmetries of local superfields as a gauge transformation

It was already pointed out in [2] that the redefinitions of the local multiparticle superfields in the Lorenz gauge correspond to a gauge transformation of the corresponding Berends–Giele current.

Indeed, if we define the Berends–Giele currents using (5.2)

$$
\mathcal{A}^m_P \equiv [\![A^m]\!] \circ b(P)\,, \qquad \mathcal{H} \equiv [\![H]\!] \circ b(P)\,, \tag{5.4}
$$

one can show using the relations (4.20) and (5.3) up to multiplicity five that [2],

$$
\begin{aligned}
\mathcal{A}^{m,\mathrm{BCJ}}_{123} &= \mathcal{A}^{m,\mathrm{L}}_{123} - k^m_{123}\mathcal{H}_{123}\,, \qquad (5.5)\\
\mathcal{A}^{m,\mathrm{BCJ}}_{1234} &= \mathcal{A}^{m,\mathrm{L}}_{1234} - k^m_{1234}\mathcal{H}_{1234} + \mathcal{A}^{m,\mathrm{L}}_{1}\mathcal{H}_{234} - \mathcal{A}^{m,\mathrm{L}}_{4}\mathcal{H}_{123}\,,\\
\mathcal{A}^{m,\mathrm{BCJ}}_{12345} &= \mathcal{A}^{m,\mathrm{L}}_{12345} - k^m_{12345}\mathcal{H}_{12345} + \mathcal{A}^{m,\mathrm{L}}_{1}\mathcal{H}_{2345} + \mathcal{A}^{m,\mathrm{L}}_{12}\mathcal{H}_{345} - \mathcal{A}^{m,\mathrm{L}}_{5}\mathcal{H}_{1234} - \mathcal{A}^{m,\mathrm{L}}_{45}\mathcal{H}_{123}\,.
\end{aligned}
$$

Therefore, in terms of the perturbiner series

$$
\mathbb{H} \equiv \sum_P \mathcal{H}_P t^P\,, \tag{5.6}
$$

the equations (5.5) correspond to the infinitesimal non-linear gauge transformation (2.5) with $\Omega = -\mathbb{H}$

$$
\mathbb{A}^{\mathrm{BCJ}}_m = \mathbb{A}^{\mathrm{L}}_m - [\partial_m, \mathbb{H}] + [\mathbb{A}^{\mathrm{L}}_m, \mathbb{H}]\,. \tag{5.7}
$$

However, the identification of (5.7) as the gauge transformation relating the superfields in the different gauges is not complete. This is because the analysis of [2] was restricted to multiplicity five, whereas we know from (4.14) and (4.15) that there are non-linear corrections to the superfields $H_{[A,B]}$ that start at multiplicity six – see for instance the quadratic terms $\sim \frac{1}{2}k^m H^2$ in the redefinition of $\hat{A}_m^{[[12,34],56]}$ (4.24).

In fact, using the general formulas for the redefinitions and the Berends–Giele currents one can show, after considerable effort,

$$
\begin{aligned}
\mathcal{A}_m^{m,\mathrm{BCJ}} &= \mathcal{A}_{123456}^{m,\mathrm{L}} - k_{123456}^m \mathcal{H}_{123456} \\
&+ \mathcal{A}_1^{m,\mathrm{L}}\mathcal{H}_{23456} + \mathcal{A}_{12}^{m,\mathrm{L}}\mathcal{H}_{3456} + \mathcal{A}_{123}^{m,\mathrm{L}}\mathcal{H}_{456} - \mathcal{A}_6^{m,\mathrm{L}}\mathcal{H}_{12345} - \mathcal{A}_{56}^{m,\mathrm{L}}\mathcal{H}_{1234} - \mathcal{A}_{456}^{m,\mathrm{L}}\mathcal{H}_{123} \\
&- \frac{1}{2}k_{123}^m\mathcal{H}_{123}\mathcal{H}_{456} + \frac{1}{2}k_{456}^m\mathcal{H}_{456}\mathcal{H}_{123}\,.
\end{aligned}
\tag{5.8}
$$

Therefore, at multiplicity six the transformation between Lorenz and BCJ gauge follows from

$$
\mathbb{A}_m^{\mathrm{BCJ}} = \mathbb{A}_m^{\mathrm{L}} - [\partial_m, \mathbb{H}] + [\mathbb{A}_m^{\mathrm{L}}, \mathbb{H}] - \frac{1}{2}[[\partial_m, \mathbb{H}], \mathbb{H}]\,. \tag{5.9}
$$

We will now demonstrate that there is an infinite series of non-linear corrections to (5.9) which generate a *finite* gauge variation.

5.3. BCJ symmetries from finite gauge transformations

If $\mathbb{H}$ represents a generating series of Berends–Giele superfields $\mathcal{H}_{\mathcal{P}}$ (5.6), one can show that the series representation of the recursive iterations (4.19) for the gauge superpotential $\mathbb{A}_m$ is given by

$$
\mathbb{L}_j \circ \mathbb{A}_m = \mathbb{A}_m - \frac{1}{j}[\partial_m, \mathbb{H}] - \frac{1}{j}[\mathbb{H}, \mathbb{L}_{j+1} \circ \mathbb{A}_m]\,. \tag{5.10}
$$

Iterating the series representation of the transformation $\mathbb{A}_m^{\mathrm{BCJ}} = \mathbb{L}_1 \circ \mathbb{A}_m^{\mathrm{L}}$ from Lorenz to BCJ gauge leads to ($\nabla_m^{\mathrm{L}} \equiv \partial_m - \mathbb{A}_m^{\mathrm{L}}$)

$$
\begin{aligned}
\mathbb{A}_m^{\mathrm{BCJ}} &= \mathbb{A}_m^{\mathrm{L}} + [\mathbb{H}, \partial_m] - [\mathbb{H}, \mathbb{A}_m^{\mathrm{L}}] - \frac{1}{2}[\mathbb{H}, [\mathbb{H}, \partial_m]] + \frac{1}{2}[\mathbb{H}, [\mathbb{H}, \mathbb{A}_m^{\mathrm{L}}]] + \frac{1}{3!}[\mathbb{H}, [\mathbb{H}, [\mathbb{H}, \partial_m]]] + \cdots \\
&= \mathbb{A}_m^{\mathrm{L}} + [\mathbb{H}, \nabla_m^{\mathrm{L}}] - \frac{1}{2}[\mathbb{H}, [\mathbb{H}, \nabla_m^{\mathrm{L}}]] + \frac{1}{3!}[\mathbb{H}, [\mathbb{H}, [\mathbb{H}, \nabla_m^{\mathrm{L}}]]] + \cdots
\end{aligned}
\tag{5.11}
$$

Unsurprisingly, the expression (5.11) is nothing more than the series expansion of the *finite* gauge transformation given by

$$
\mathbb{A}_m^{\mathrm{BCJ}} = U\mathbb{A}_m^{\mathrm{L}}U^{-1} + \partial_m U U^{-1}\,, \quad U = \exp(-\mathbb{H})\,. \tag{5.12}
$$

Alternatively (5.11) can be rewritten as $\nabla_m^{\mathrm{BCJ}} = e^{-\mathrm{ad}_{\mathbb{H}}}(\nabla_m^{\mathrm{L}})$, where $\mathrm{ad}_{\mathbb{H}}(X) \equiv [\mathbb{H}, X]$.

6. Conclusions and outlook

One of the main achievements of this paper is the recursive solution to the redefinition superfields $H_{[A,B]}$ given in (4.14). These superfields encode the non-linear gauge variations required to obtain local multiparticle superfields in the BCJ gauge. The pursuit of this formula led to improvements to and clarifications of earlier discussions given in [1,2]. In particular, in going beyond the multiplicity-five examples of [2], we found an infinite set of higher-order corrections leading to the perturbiner representation of a *finite* gauge transformation (5.11).

We also introduced new combinatorial maps on words and rigorously proved key statements that address some natural although not crucial questions previously left unanswered. For instance, we found closed formulas for the gauge redefinition of $K_{[P,Q]}$ for arbitrary nested bracketings as well as the field-strength form of $F^{mn}_{[P,Q]}$ and related superfields at higher-mass dimension. Several other formulas along these lines can now be written down, such as the local equations of motion (B.1) for the Lorenz-gauge superfields $\hat{K}_{[P,Q]}$, again for arbitrary Lie bracket structure. The precise definition of maps in section 3 ultimately related to the definition of Berends–Giele currents also lead to explanations of *why* some patterns are ubiquitous when discussing BRST variations of various superfields in the pure spinor formalism as seen in the discussions of [18].

We will end this paper with some observations that could lead to further investigations.

6.1.1. Tree-level amplitudes using redefinition superfields

The gauge transformations responsible for the BCJ gauge require redefinitions by superfields of ghost-number zero $H_{[A,B]}$ determined recursively through (4.14). Customarily, after performing the redefinitions using the redefining superfields one writes down the tree amplitudes of SYM using the newly obtained superfields [7]. For example, using the compact language of the pure spinor superspace [22] one gets

$$A^{\rm SYM}(1,2,3,4,5) = \frac{\langle V_{123}V_4V_5\rangle}{s_{12}s_{123}} + \frac{\langle V_{321}V_4V_5\rangle}{s_{23}s_{123}} + \frac{\langle V_{12}V_{34}V_5\rangle}{s_{12}s_{34}} + \frac{\langle V_1V_{432}V_5\rangle}{s_{34}s_{234}} + \frac{\langle V_1V_{234}V_5\rangle}{s_{23}s_{234}}\,, \tag{6.1}$$

where $V_P \equiv \lambda^\alpha A^P_\alpha$ is a BCJ-satisfying superfield whose explicit expression contains the redefinition superfields $H'_{A,B,C}$ in various combinations.

So, in the usual formulation, we see that the superfields in the BCJ gauge are used to write down the local numerators of tree-level SYM amplitudes. These numerators have ghost number three [6] and, if one wishes to produce expressions written in terms of particle polarizations and momenta, require the standard pure spinor zero-mode rule $\langle(\lambda\gamma^m\theta)(\lambda\gamma^n\theta)(\lambda\gamma^p\theta)(\theta\gamma_{mnp}\theta)\rangle = 1$ [6] to integrate out the pure spinors. Somewhat surprisingly, it turns out that the redefinition superfields themselves give rise to numerators of the tree amplitudes of SYM.

6.1.2. Tree-level amplitudes as a map on planar binary trees

The observation above can be made more intuitive and intriguing if we frame it in terms of the b map (3.1). The SYM tree amplitudes can be viewed as a map $A^{\rm SYM}\circ$ acting on the Lie polynomials in the expansion of (3.1). More precisely,

$$A^{\rm SYM}(P,n) = s_P A^{\rm SYM}\circ\big(b(P)b(n)\big), \tag{6.2}$$

where the map $A^{\rm SYM}\circ$ admits two formulations

$$A^{\rm SYM}\circ[P,Q]n \equiv \begin{cases}\langle V_P V_Q V_n\rangle \\ H'_{P,Q,n}\end{cases} \tag{6.3}$$

For example, using the Lie bracket expansion from fig. 1 and the top line of the map (6.3) gives rise to amplitude expression (6.1). Using the bottom line of the map yields instead

$$\begin{aligned} A^{\rm SYM}(1,2,3,4,5) &= s_{1234}A^{\rm SYM}\circ b(1234)b(5) \\ &= \frac{H'_{123,4,5}}{s_{12}s_{123}} + \frac{H'_{321,4,5}}{s_{23}s_{123}} + \frac{H'_{12,34,5}}{s_{12}s_{34}} + \frac{H'_{1,432,5}}{s_{34}s_{234}} + \frac{H'_{1,234,5}}{s_{23}s_{234}} \end{aligned} \tag{6.4}$$

In hindsight, the statement that tree-level amplitudes can be written using the definition of $H_{A,B,C}$ could be made when putting together the results of [7] and [2]. But now we have explicitly checked up to multiplicity nine that all the new corrections introduced in (4.15) that lead to the definition of $H'_{A,B,C}$ do not affect the final results of the amplitudes.

These observations give rise to the speculation that the new prescription to compute tree level amplitudes from [23] naturally gives rise to the amplitudes written in terms of $H'_{A,B,C}$. After all the prescription in [23] does not involve unintegrated vertices (so no pure spinors) and the end result will have to involve the double poles in the OPEs among integrated vertices. This agrees with the mechanism in the usual formulation [1] where the double poles are distributed among the simple poles using integration by parts, and it is after this step that the superfields in the numerators satisfy BCJ symmetries. This may give rise to a systematic derivation of the $H'_{A,B,C}$ redefinitions via OPE calculations and it is an interesting question left to the future.

BCJ numerators were constructed for gauge theories deformed by $\alpha' F^3$ and ${\alpha'}^2 F^4$ interactions by finding appropriate α' corrections to the H_P fields [24]. Since low-multiplicity examples show that these corrections can also be written in terms of α'-corrected $H_{A,B,C}$ in a similar manner as discussed in this paper, one may wonder whether the all-multiplicity formulas found here can be applied with minimal changes to the setup of [24].

The color-kinematics duality has given reasons to speculate about the existence of a "kinematic algebra" [25] in the same way as the color factors are related to standard Lie algebras. It will be interesting to connect this line of thought with the gauge variation approach pursued here. See [26] for a recent account on the quest for the kinematic algebra.

Finally, the Berends–Giele recursion relations have been recently derived using the technology of an L_∞-algebra in [27]. It would be interesting to find a new derivation of the recursions for the gauge parameter $H_{[A,B]}$ using the methods of [27].

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Appendix A. Some common operations on words

In this appendix we list some of the operations on words used in this paper. With the exception of the letterification introduced below, the following definitions are standard and can be found in [17].

The left-to-right bracketing map $\ell(A)$ is defined recursively by

$$\ell(123..n) \equiv \ell(123...n-1)n - n\ell(123...n-1), \quad \ell(i) = i, \quad \ell(\emptyset) = 0\,. \tag{A.1}$$

The deshuffle map is defined by

$$\delta(P) = \sum_{X,Y} \langle P, X \sqcup\!\sqcup Y \rangle \, X \otimes Y\,, \tag{A.2}$$

where $\langle \cdot, \cdot \rangle$ denotes the scalar product on words

$$\langle A, B \rangle \equiv \delta_{A,B}, \qquad \delta_{A,B} = \begin{cases} 1, & \text{if } A = B \\ 0, & \text{otherwise} \end{cases}. \tag{A.3}$$

The shuffle product $\sqcup\!\sqcup$ between $A = a_1 a_2 \ldots a_{|A|}$ and $B = b_1 b_2 \ldots b_{|B|}$ is given by

$$\emptyset \sqcup\!\sqcup A = A \sqcup\!\sqcup \emptyset = A, \qquad A \sqcup\!\sqcup B \equiv a_1(a_2 \ldots a_{|A|} \sqcup\!\sqcup B) + b_1(b_2 \ldots b_{|B|} \sqcup\!\sqcup A)\,, \tag{A.4}$$

where $\emptyset$ represents the empty word.

In certain formulas such as (4.14) it is necessary to handle a word as if it were a single letter to avoid it being split by other maps. To deal with these situations we introduce a *letterfication* operation whereby a *word* Q is mapped to a *letter* $\dot{q}$,

$$Q \to \dot{q}\,. \tag{A.5}$$

Since a letter can not be deconcatenated this freezes the individual letters within Q. In the end $\dot{q}$ is restored by its original word Q. For example, suppose that the word $Q = 12$ has been letterified to $\dot{q} = 12$ – as may be the case in a formula such as (4.14)– and that $P = 3$. Then deconcatenating QP is different than deconcatenating $\dot{q}P$. For example, one gets only one term

$$Q = 12, P = 3 \to \sum_{XY=\dot{q}P} S_X T_Y = S_{\dot{q}} T_3 = S_{12} T_3 \tag{A.6}$$

instead of the usual two $(S_1 T_{23} + S_{12} T_3)$ if Q is not letterified.

Appendix B. Equations of motion for local $\hat{K}_{[P,Q]}$

In this appendix we will write down the equations of motion satisfied by the multiparticle superfields in the Lorenz gauge for general nested Lie brackets.

The equations of motion satisfied by the local multiparticle superfields (4.2) can be written as a local counterpart of the non-linear equations (2.6)

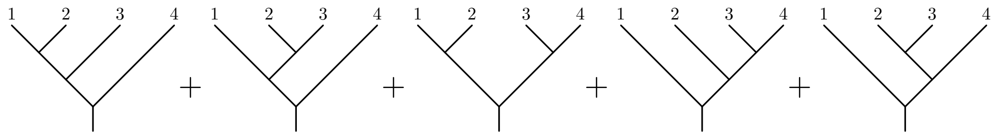
$$\begin{aligned}
&\nabla^{(L)}_{(\alpha} \hat{A}^{[P,Q]}_{\beta)} = \gamma^m_{\alpha\beta} \hat{A}^m_{[P,Q]} && \nabla^{(L)}_{\alpha} \hat{W}^{\beta}_{[P,Q]} = \frac{1}{4} (\gamma^{mn})_\alpha{}^\beta \hat{F}^{[P,Q]}_{mn} \\
&\nabla^{(L)}_{\alpha} \hat{A}^m_{[P,Q]} = (\gamma^m \hat{W}_{[P,Q]})_\alpha + k^m_{PQ} \hat{A}_{\alpha\,[P,Q]} && \nabla^{(L)}_{\alpha} \hat{F}^{mn}_{[P,Q]} = \big(\hat{W}^{[m}_{[P,Q]} \gamma^{n]}\big)_\alpha
\end{aligned} \tag{B.1}$$

where $\nabla^{(L)}_\alpha$ is the local counterpart of $\nabla_\alpha \equiv D_\alpha - \mathbb{A}_\alpha$ and is defined by

$$\nabla^{(L)}_\alpha \equiv D_\alpha - C[\![\hat{A}_\alpha, \cdot\,]\!]\,, \qquad C[\![\hat{A}_\alpha, \cdot\,]\!] K_{[P,Q]} \equiv C[\![\hat{A}_\alpha, K]\!] \circ [P,Q]\,. \tag{B.2}$$

where $C[\![\cdot,\cdot]\!]$ is the contact-term coproduct map on words defined in (3.5) and (C.5). To illustrate the above equations, consider $\nabla^{(L)}_\alpha \hat{A}^m_{[1,2]} = (\gamma^m \hat{W}_{[1,2]})_\alpha + k^m_{12} \hat{A}_{\alpha\,[1,2]}$ where

$$\begin{aligned}
\nabla^{(L)}_\alpha \hat{A}^m_{[1,2]} &= D_\alpha \hat{A}^m_{[1,2]} - C[\![\hat{A}_\alpha, \hat{A}^m]\!] \circ [1,2] \\
&= D_\alpha \hat{A}^m_{[1,2]} - (k_1 \cdot k_2)(\hat{A}^1_\alpha \hat{A}^m_2 - \hat{A}^2_\alpha \hat{A}^m_1)
\end{aligned} \tag{B.3}$$



$$M_{1234} = \frac{V_{[[[1,2],3],4]}}{s_{12}s_{123}s_{1234}} + \frac{V_{[[1,[2,3]],4]}}{s_{23}s_{123}s_{1234}} + \frac{V_{[[1,2],[3,4]]}}{s_{12}s_{34}s_{1234}} + \frac{V_{[1,[2,[3,4]]]}}{s_{34}s_{234}s_{1234}} + \frac{V_{[1,[[2,3],4]]}}{s_{23}s_{234}s_{1234}}$$

Fig. 2 The Berends–Giele current $M_{1234} = [\![V]\!] \circ b(1234)$ according to the map (5.2).

where we used the first example in (C.7). Therefore the equation of motion of $\hat{A}^m_{[1,2]}$ reads

$$D_\alpha \hat{A}^m_{[1,2]} = (\gamma^m \hat{W}_{[1,2]})_\alpha + k^m_{12} \hat{A}_{\alpha\,[1,2]} + (k_1 \cdot k_2)(\hat{A}^1_\alpha \hat{A}^m_2 - \hat{A}^2_\alpha \hat{A}^m_1)\,. \tag{B.4}$$

Appendix C. Symmetries and deconcatenations of Berends–Giele currents

C.1. Symmetries of Berends–Giele currents

We have seen on section 3.1 that $b(P)$ is a Lie polynomial. A standard result in the theory of free Lie algebras states that any Lie polynomial is orthogonal to non-trivial shuffles [17]. This implies that

$$\langle A \sqcup\!\sqcup B, b(P) \rangle = 0\,, \quad \forall A, B \neq \emptyset \quad |A| + |B| = |P|\,, \tag{C.1}$$

where $\langle \cdot, \cdot \rangle$ is the scalar product of words and $\sqcup\!\sqcup$ is the shuffle product defined in (A.3) and (A.4), respectively. A more compact way of stating (C.1) is through the shorthand $b(A \sqcup\!\sqcup B) = 0$.

Using the property (C.1) it follows that every Berends–Giele current defined via (5.2) is annihilated by proper shuffles, i.e. (note $\mathcal{K}_{A \sqcup\!\sqcup B} \equiv \sum_{\sigma \in A \sqcup\!\sqcup B} \mathcal{K}_\sigma$)

$$\mathcal{K}_{A \sqcup\!\sqcup B} = 0\,, \quad \forall A, B \neq \emptyset. \tag{C.2}$$

Note that the original currents J^m_P defined by Berends and Giele in [8] were argued to satisfy $J^m_{A \sqcup\!\sqcup B} = 0$ in [28]. One can show that, in our conventions, $J^m_P = \mathcal{A}^m_P$ [7].

C.2. Deconcatenation terms in the equations of motion

The equations of motion of local multiparticle superfields (see the appendix B) contain contact-term corrections with respect to their single-particle counterparts. When expressed in terms of Berends–Giele currents, these contact terms corrections are translated to a deconcatenation structure. For example, the Berends–Giele counterpart of the local equation of motion

$$D_\alpha \hat{A}^m_{[1,2]} = (\gamma^m \hat{W}_{[1,2]})_\alpha + k^m_{12} \hat{A}_{\alpha\,[1,2]} + (k_1 \cdot k_2)(\hat{A}^1_\alpha \hat{A}^m_2 - \hat{A}^2_\alpha \hat{A}^m_1)\,, \tag{C.3}$$

is given by

$$D_\alpha \hat{\mathcal{A}}^m_{12} = (\gamma^m \hat{\mathcal{W}}_{12})_\alpha + k^m_{12} \hat{\mathcal{A}}^{12}_\alpha + \sum_{XY=12} \big(\hat{\mathcal{A}}^X_\alpha \hat{\mathcal{A}}^m_Y - (X \leftrightarrow Y)\big)\,. \tag{C.4}$$

These observations can now be given a universal justification as follows. If one assigns the superfields K and S to the contact terms of a Lie polynomial $[P, Q]$ as

$$C[\![K, S]\!] \circ [P, Q] \equiv [\![K, S]\!] \circ \big(C \circ [P, Q]\big)\,, \tag{C.5}$$

it follows from (3.7) that

$$C[\![K, S]\!] \circ b(P) = \sum_{XY=P} \big(\mathcal{K}_X \mathcal{S}_Y - (X \leftrightarrow Y)\big)\,, \tag{C.6}$$

which demonstrates several deconcatenation formulas of this kind from a local superfield perspective. Using the contact-term map C displayed in (D.1), the simplest example applications of (C.5) read

$$\begin{aligned} C[\![\hat{A}_\alpha, \hat{A}^m]\!] \circ [1,2] &= (k_1 \cdot k_2)(\hat{A}^1_\alpha \hat{A}^m_2 - \hat{A}^2_\alpha \hat{A}^m_1)\,, \\ C[\![V, T]\!] \circ [[1,2],3] &= (k_1 \cdot k_2)\big(V_{[1,3]} T_2 + V_1 T_{[2,3]} - V_{[2,3]} T_1 - V_2 T_{[1,3]}\big) \\ &\quad + (k_{12} \cdot k_3)\big(V_{[1,2]} T_3 - V_3 T_{[1,2]}\big)\,. \end{aligned} \tag{C.7}$$

In addition, the contact terms generated with the formula (C.5) can be used to write down the BRST variations of the multiparticle unintegrated V_P for arbitrary nested Lie bracketings. This generalizes the previous formula valid for the left-to-right nesting [1]. More precisely, the BRST variation can be written as

$$QV_{[P,Q]} = \frac{1}{2} C[\![V, V]\!] \circ [P, Q]\,. \tag{C.8}$$

For example, using (C.8) one can write down the BRST variation of $V_{[1,[2,3]]}$ directly,

$$QV_{[1,[2,3]]} = (k_2 \cdot k_3)\big(V_{[1,2]} V_3 + V_2 V_{[1,3]}\big) + (k_1 \cdot k_{23}) V_1 V_{[2,3]}\,. \tag{C.9}$$

Previously one would need to use $V_{[1,[2,3]]} = V_{123} - V_{132}$ before applying the formula for QV_P for $P = [[...[p_1, p_2], p_3], ...], p_{|P|}] \equiv p_1 p_2 \ldots p_{|P|}$ given in [18],

$$QV_P = \sum_{\substack{P = XjY \\ \delta(Y) = R \otimes S}} (k_X \cdot k_j)\, V_{XR} V_{jS}\,. \tag{C.10}$$

It is worth mentioning that (3.15) shows the equivalence between (C.8) and (C.10).

Appendix D. Example applications of the C and $\tilde{C}$ maps

In this appendix we display some example applications of the C and $\tilde{C}$ maps acting over some simple Lie polynomials. These examples help to elucidate how the algorithms are used, and can be used to verify that the redefinition formulas arising from the general formulas match the formulas for the simplest cases that were previously known.

D.1.1. Examples of the C map

To demonstrate the (3.5) algorithm, the first few expansions generated from it are

$$\begin{aligned}
C \circ 1 &= 0 \qquad\qquad (\text{D.1})\\
C \circ [1,2] &= (k_1 \cdot k_2)(1 \otimes 2 - 2 \otimes 1)\\
C \circ [[1,2],3] &= (k_1 \cdot k_2)\big([1,3] \otimes 2 + 1 \otimes [2,3] - [2,3] \otimes 1 - 2 \otimes [1,3]\big)\\
&\quad + (k_{12} \cdot k_3)\big([1,2] \otimes 3 - 3 \otimes [1,2]\big)\\
C \circ [1,[2,3]] &= (k_2 \cdot k_3)\big([1,2] \otimes 3 + 2 \otimes [1,3] - [1,3] \otimes 2 - 3 \otimes [1,2]\big)\\
&\quad + (k_1 \cdot k_{23})\big(1 \otimes [2,3] - [2,3] \otimes 1\big)\\
C \circ [[[1,2],3],4] &= (k_1 \cdot k_2)\big([[1,3],4] \otimes 2 + [1,3] \otimes [2,4] + [1,4] \otimes [2,3] + 1 \otimes [[2,3],4]\\
&\qquad - [[2,3],4] \otimes 1 - [2,3] \otimes [1,4] - [2,4] \otimes [1,3] - 2 \otimes [[1,3],4]\big)\\
&\quad + (k_{12} \cdot k_3)\big([[1,2],4] \otimes 3 + [1,2] \otimes [3,4] - [3,4] \otimes [1,2] - 3 \otimes [[1,2],4]\big)\\
&\quad + (k_{123} \cdot k_4)\big([[1,2],3] \otimes 4 - 4 \otimes [[1,2],3]\big)\\
C \circ [[1,[2,3]],4] &= (k_2 \cdot k_3)\big([[1,2],4] \otimes 3 + [1,2] \otimes [3,4] + [2,4] \otimes [1,3] + 2 \otimes [[1,3],4]\\
&\qquad - [[1,3],4] \otimes 2 - [1,3] \otimes [2,4] - [3,4] \otimes [1,2] - 3 \otimes [[1,2],4]\big)\\
&\quad + (k_1 \cdot k_{23})\big([1,4] \otimes [2,3] + 1 \otimes [[2,3],4] - [[2,3],4] \otimes 1 - [2,3] \otimes [1,4]\big)\\
&\quad + (k_{123} \cdot k_4)\big([1,[2,3]] \otimes 4 - 4 \otimes [1,[2,3]]\big)\\
C \circ [[1,2],[3,4]] &= (k_1 \cdot k_2)\big([1,[3,4]] \otimes 2 + 1 \otimes [2,[3,4]] - [2,[3,4]] \otimes 1 - 2 \otimes [1,[3,4]]\big)\\
&\quad + (k_3 \cdot k_4)\big([[1,2],3] \otimes 4 + 3 \otimes [[1,2],4] - [[1,2],4] \otimes 3 - 4 \otimes [[1,2],3]\big)\\
&\quad + (k_{12} \cdot k_{34})\big([1,2] \otimes [3,4] - [3,4] \otimes [1,2]\big)\\
C \circ [1,[2,[3,4]]] &= (k_3 \cdot k_4)\big([1,[2,3]] \otimes 4 + [2,3] \otimes [1,4] + [1,3] \otimes [2,4] + 3 \otimes [1,[2,4]]\\
&\qquad - [1,[2,4]] \otimes 3 - [2,4] \otimes [1,3] - [1,4] \otimes [2,3] - 4 \otimes [1,[2,3]]\big)\\
&\quad + (k_2 \cdot k_{34})\big([1,2] \otimes [3,4] + 2 \otimes [1,[3,4]] - [1,[3,4]] \otimes 2 - [3,4] \otimes [1,2]\big)\\
&\quad + (k_1 \cdot k_{234})\big(1 \otimes [2,[3,4]] - [2,[3,4]] \otimes 1\big)
\end{aligned}$$

$$\begin{aligned}C\circ[1,[[2,3],4]] = (k_2\cdot k_3)\big([1,[2,4]]\otimes 3+[1,2]\otimes[3,4]+[2,4]\otimes[1,3]+2\otimes[1,[3,4]]\\ -[1,[3,4]]\otimes 2-[1,3]\otimes[2,4]-[3,4]\otimes[1,2]-3\otimes[1,[2,4]]\big)\\ +(k_{23}\cdot k_4)\big([1,[2,3]]\otimes 4+[2,3]\otimes[1,4]-[1,4]\otimes[2,3]-4\otimes[1,[2,3]]\big)\\ +(k_1\cdot k_{234})\big(1\otimes[[2,3],4]-[[2,3],4]\otimes 1\big)\,.\end{aligned}$$

One application at multiplicity five is given by

$$\begin{aligned}C\circ[[[1,2],3],[4,5]] = (k_1\cdot k_2)\big(1\otimes[[2,3],[4,5]]+[1,3]\otimes[2,[4,5]]\\ +[1,[4,5]]\otimes[2,3]+[[1,3],[4,5]]\otimes 2-(1\leftrightarrow 2)\big)\\ +(k_{12}\cdot k_3)\big([1,2]\otimes[3,[4,5]]+[[1,2],[4,5]]\otimes 3-([1,2]\leftrightarrow 3)\big)\\ +(k_{123}\cdot k_{45})\big([[1,2],3]\otimes[4,5]-([[1,2],3]\leftrightarrow[4,5])\big)\\ +(k_4\cdot k_5)\big(4\otimes[[[1,2],3],5]+[[[1,2],3],4]\otimes 5-(4\leftrightarrow 5)\big)\,,\end{aligned} \tag{D.2}$$

which, after using the formula (4.18), reproduces the redefinition (B.2) from [2] which was written down without justification.

D.1.2. Examples of the $\tilde{C}$ map

As an illustration of the $\tilde{C}$ map, we get

$$\begin{aligned}\tilde{C}\circ 1 &= 0\\ \tilde{C}\circ[1,2] &= 0\\ \tilde{C}\circ[[1,2],3] &= (k_1\cdot k_2)\big([1,3]\otimes 2-[2,3]\otimes 1\big)\\ \tilde{C}\circ[1,[2,3]] &= (k_2\cdot k_3)\big([1,2]\otimes 3-[1,3]\otimes 2\big)\\ \tilde{C}\circ[[[1,2],3],4] &= (k_1\cdot k_2)\big([[1,3],4]\otimes 2+[1,4]\otimes[2,3]-[[2,3],4]\otimes 1-[2,4]\otimes[1,3]\big)\\ &\quad +(k_{12}\cdot k_3)\big([[1,2],4]\otimes 3-[3,4]\otimes[1,2]\big)\\ \tilde{C}\circ[[1,[2,3]],4] &= (k_2\cdot k_3)\big([[1,2],4]\otimes 3+[2,4]\otimes[1,3]-[[1,3],4]]\otimes 2-[3,4]\otimes[1,2]\big)\\ &\quad +(k_1\cdot k_{23})\big([1,4]\otimes[2,3]-[[2,3],4]\otimes 1\big)\\ \tilde{C}\circ[[1,2],[3,4]] &= (k_1\cdot k_2)\big([1,[3,4]]\otimes 2-[2,[3,4]]\otimes 1\big)\\ &\quad +(k_3\cdot k_4)\big([[1,2],3]\otimes 4-[[1,2],4]\otimes 3\big)\\ \tilde{C}\circ[1,[2,[3,4]]] &= (k_3\cdot k_4)\big([1,[2,3]]\otimes 4+[1,3]\otimes[2,4]-[1,[2,4]]\otimes 3-[1,4]\otimes[2,3]\big)\\ &\quad +(k_2\cdot k_{34})\big([1,2]\otimes[3,4]-[1,[3,4]]\otimes 2\big)\\ \tilde{C}\circ[1,[[2,3],4]] &= (k_2\cdot k_3)\big([1,[2,4]]\otimes 3+[1,2]\otimes[3,4]-[1,[3,4]]\otimes 2-[1,3]\otimes[2,4]\big)\\ &\quad +(k_{23}\cdot k_4)\big([1,[2,3]]\otimes 4-[1,4]\otimes[2,3]\big)\end{aligned} \tag{D.3}$$

One application at multiplicity six is given by

$$\tilde{C} \circ [[[[1,2],3],[4,5]],6] = (k_1 \cdot k_2)\big([[[1,3],[4,5]],6] \otimes 2 + [[1,3],6] \otimes [2,[4,5]] \\ + [[1,[4,5]],6] \otimes [2,3] + [1,6] \otimes [[2,3],[4,5]] - (1 \leftrightarrow 2)\big) \\ + (k_{12} \cdot k_3)\big([[[1,2],[4,5]],6] \otimes 3 + [[1,2],6] \otimes [3,[4,5]] - ([1,2] \leftrightarrow 3)\big) \\ + (k_4 \cdot k_5)\big([[[[1,2],3],4],6] \otimes 5 + [4,6] \otimes [[[1,2],3],5] - (4 \leftrightarrow 5)\big) \\ + (k_{123} \cdot k_{45})\big([[[1,2],3],6] \otimes [4,5] - ([[1,2],3] \leftrightarrow [4,5])\big)\,. \tag{D.4}$$

This will be of particular use in the example discussed in section 4.3.1.

Appendix E. Freedom in defining Hs

There is considerable freedom in defining the Hs, arising from the symmetries within the $H'_{A,B,C}$ terms. These are by construction antisymmetric in A, B, and C. Furthermore each of the sets of indices will satisfy generalized Jacobi identities, for instance

$$H'_{123,B,C} + H'_{213,B,C} = 0, \\ H'_{123,B,C} + H'_{231,B,C} + H'_{312,B,C} = 0. \tag{E.1}$$

Also there are a number of other more complex relations between some H' terms, which can be identified from the condition that $H_{[A,B]}$ satisfies generalized Jacobi identities in each of A and B. For example, we must have that $\pounds_3 \circ H_{[123,4]} = 0$, $\pounds_3 \circ H_{[1234,5]} = 0$, and $\pounds_4 \circ H_{[1234,5]} = 0$, and so writing these relations in terms of their H' expansions, we see that we must have

$$\pounds_3 \circ \big(H'_{12,3,4} + H'_{34,1,2}\big) = 0, \\ \pounds_3 \circ \big(H'_{123,4,5} - H'_{543,2,1} + H'_{54,3,12}\big) = 0, \\ \pounds_4 \circ \big(H'_{123,4,5} - H'_{543,2,1} + H'_{54,3,12}\big) = 0. \tag{E.2}$$

These identities can be described in general with the formula (4.14) for $H_{[A,B]}$. Consider $\pounds_n \circ H_{[A,B]}$, with $n \leq |A|$. One half of (4.14) will disappear under the action of the $\pounds$, as

$$\pounds_n \circ \left(\sum_{XjY = \dot{a}\tilde{B}} (-1)^{|Y|} H'_{\tilde{Y},j,X} \right) = \pounds_n \circ \left(\sum_{XjY = \tilde{B}} (-1)^{|Y|} H'_{\tilde{Y},j,\dot{a}X} \right) = 0, \tag{E.3}$$

where in the second sum X is not constrained to be non-empty. The final equality then just comes from the fact that $H'_{A,B,C}$ is constructed so as to satisfy generalized Jacobi identities in each of A, B, and C. Using this and (4.14) it then just follows that, if $\pounds_n \circ H_{[A,B]} = 0$ for $n \leq |A|$, then

$$\pounds_n \circ \left(\sum_{XjY=\dot{b}\tilde{A}} (-1)^{|Y|} H'_{\tilde{Y},j,X} \right) = 0, \qquad n < |A| \tag{E.4}$$

for any word A and letterification $\dot{b}$.

Appendix F. BCJ gauge versus Lorenz gauge at multiplicity six

The redefinitions for moving from the Lorenz to the BCJ gauge for all possible topologies at rank six are identified with the usual formula (4.18), and are stated below for convenience. We emphasize the typographical convention of representing a left-to-right nested bracket by its composing letters, e.g. $\hat{H}_{[[[1,2],3],4]} \equiv \hat{H}_{1234}$, even though the parent superfields do not obey BCJ symmetries.

$$\begin{aligned}
A^m_{[12345,6]} = \hat{A}^m_{[12345,6]} &- k^m_{123456}\hat{H}_{[12345,6]} \\
&- (k^1 \cdot k^2)\Big(\hat{H}_{13456}\hat{A}^m_2 + \hat{H}_{1345}\hat{A}^m_{26} + \hat{H}_{1346}\hat{A}^m_{25} + \hat{H}_{1356}\hat{A}^m_{24} \\
&\qquad + \hat{H}_{1456}\hat{A}^m_{23} + \hat{H}_{134}\hat{A}^m_{256} + \hat{H}_{135}\hat{A}^m_{246} + \hat{H}_{136}\hat{A}^m_{245} \\
&\qquad + \hat{H}_{145}\hat{A}^m_{236} + \hat{H}_{146}\hat{A}^m_{235} + \hat{H}_{156}\hat{A}^m_{234} \\
&\qquad - \frac{1}{2}\hat{H}_{134}\hat{H}_{256}k^m_{256} - \frac{1}{2}\hat{H}_{135}\hat{H}_{246}k^m_{246} - \frac{1}{2}\hat{H}_{136}\hat{H}_{245}k^m_{245} \\
&\qquad - \frac{1}{2}\hat{H}_{145}\hat{H}_{236}k^m_{236} - \frac{1}{2}\hat{H}_{146}\hat{H}_{235}k^m_{235} - \frac{1}{2}\hat{H}_{156}\hat{H}_{234}k^m_{234} \\
&\qquad - (1 \leftrightarrow 2)\Big) \\
&- (k^{12} \cdot k^3)\Big(\hat{H}_{12456}\hat{A}^m_3 + \hat{H}_{1245}\hat{A}^m_{36} + \hat{H}_{1246}\hat{A}^m_{35} + \hat{H}_{1256}\hat{A}^m_{34} \\
&\qquad + \hat{H}_{124}\hat{A}^m_{356} + \hat{H}_{125}\hat{A}^m_{346} + \hat{H}_{126}\hat{A}^m_{345} \\
&\qquad - \frac{1}{2}\hat{H}_{124}\hat{H}_{356}k^m_{356} - \frac{1}{2}\hat{H}_{125}\hat{H}_{346}k^m_{346} - \frac{1}{2}\hat{H}_{126}\hat{H}_{345}k^m_{345} \\
&\qquad - \hat{H}_{3456}\hat{A}^m_{12} - \hat{H}_{345}\hat{A}^m_{126} - \hat{H}_{346}\hat{A}^m_{125} - \hat{H}_{356}\hat{A}^m_{124} \\
&\qquad + \frac{1}{2}\hat{H}_{345}\hat{H}_{126}k^m_{126} + \frac{1}{2}\hat{H}_{346}\hat{H}_{125}k^m_{125} + \frac{1}{2}\hat{H}_{356}\hat{H}_{124}k^m_{124}\Big) \\
&- (k^{123} \cdot k^4)\Big(\hat{H}_{12356}\hat{A}^m_4 + \hat{H}_{1235}\hat{A}^m_{46} + \hat{H}_{1236}\hat{A}^m_{45} \\
&\qquad + \hat{H}_{123}\hat{A}^m_{456} - \hat{H}_{456}\hat{A}^m_{123}
\end{aligned}$$

$$
\begin{aligned}
&\quad - \frac{1}{2}\hat{H}_{123}\hat{H}_{456}k^m_{456} + \frac{1}{2}\hat{H}_{456}\hat{H}_{123}k^m_{123}\Big) \\
&- (k^{1234}\cdot k^5)\Big(\hat{H}_{12346}\hat{A}^m_5 + \hat{H}_{1234}\hat{A}^m_{56}\Big) \\
&- (k^{12345}\cdot k^6)\hat{H}_{12345}\hat{A}^m_6
\end{aligned} \tag{F.1}
$$

$$
\begin{aligned}
A^m_{[1234,56]} &= \hat{A}^m_{[1234,56]} - k^m_{123456}\hat{H}_{[1234,56]} \\
&- (k^1\cdot k^2)\Big(\hat{H}_{[1,56]}\hat{A}^m_{234} + \hat{H}_{[13,56]}\hat{A}^m_{24} + \hat{H}_{[14,56]}\hat{A}^m_{23} \\
&\quad + \hat{H}_{[134,56]}\hat{A}^m_2 + \hat{H}_{134}\hat{A}^m_{[2,56]} \\
&\quad - \frac{1}{2}\hat{H}_{134}\hat{H}_{[2,56]}k^m_{256} - \frac{1}{2}\hat{H}_{[1,56]}\hat{H}_{234}k^m_{234} - (1\leftrightarrow 2)\Big) \\
&- (k^{12}\cdot k^3)\Big(\hat{H}_{[12,56]}\hat{A}^m_{34} + \hat{H}_{[124,56]}\hat{A}^m_3 + \hat{H}_{124}\hat{A}^m_{[3,56]} \\
&\quad - \hat{H}_{[3,56]}\hat{A}^m_{124} - \hat{H}_{[34,56]}\hat{A}^m_{12} \\
&\quad - \frac{1}{2}\hat{H}_{124}\hat{H}_{[3,56]}k^m_{356} + \frac{1}{2}\hat{H}_{[3,56]}\hat{H}_{124}k^m_{124}\Big) \\
&- (k^{123}\cdot k^4)\Big(\hat{H}_{[123,56]}\hat{A}^m_4 + \hat{H}_{123}\hat{A}^m_{[4,56]} - \hat{H}_{[4,56]}\hat{A}^m_{123} \\
&\quad - \frac{1}{2}\hat{H}_{123}\hat{H}_{[4,56]}k^m_{456} + \frac{1}{2}\hat{H}_{[4,56]}\hat{H}_{123}k^m_{123}) \\
&- (k^{1234}\cdot k^{56})(\hat{H}_{1234}\hat{A}^m_{56}\Big) \\
&+ (k^5\cdot k^6)\Big(\hat{H}_{[5,1234]}\hat{A}^m_6 - (5\leftrightarrow 6)\Big)
\end{aligned} \tag{F.2}
$$

$$
\begin{aligned}
A^m_{[123,456]} &= \hat{A}^m_{[123,45,6]} - k^m_{123456}\hat{H}_{[123,456]} \\
&- (k^1\cdot k^2)\Big(\hat{H}_{[1,456]}\hat{A}^m_{23} + \hat{H}_{[13,456]}\hat{A}^m_2 - (1\leftrightarrow 2)\Big) \\
&- (k^{12}\cdot k^3)\Big(\hat{H}_{[12,456]}\hat{A}^m_3 - \hat{H}_{[3,456]}\hat{A}^m_{12}\Big) \\
&- (k^{123}\cdot k^{456})\Big(\hat{H}_{123}\hat{A}^m_{456} - \hat{H}_{456}\hat{A}^m_{123} \\
&\quad - \frac{1}{2}\hat{H}_{123}\hat{H}_{456}k^m_{456} + \frac{1}{2}\hat{H}_{456}\hat{H}_{123}k^m_{123}\Big) \\
&+ (k^4\cdot k^5)\Big(\hat{H}_{[4,123]}\hat{A}^m_{56} + \hat{H}_{[46,123]}\hat{A}^m_5 - (4\leftrightarrow 5)\Big) \\
&+ (k^{45}\cdot k^6)\Big(\hat{H}_{[45,123]}\hat{A}^m_6 - \hat{H}_{[6,123]}\hat{A}^m_{45}\Big)
\end{aligned} \tag{F.3}
$$

$$
\begin{aligned}
A^m_{[[12,34],56]} &= \hat{A}^m_{[[12,34],56]} - k^m_{123456}\hat{H}_{[[12,34],56]} \\
&- (k^1\cdot k^2)\Big(\hat{H}_{[1,34]}\hat{A}^m_{[2,56]} + \hat{H}_{[1,56]}\hat{A}^m_{[2,34]} + \hat{H}_{[[1,34],56]}\hat{A}^m_2 \\
&\quad - \frac{1}{2}\hat{H}_{[1,34]}\hat{H}_{[2,56]}k^m_{256} - \frac{1}{2}\hat{H}_{[1,56]}\hat{H}_{[2,34]}k^m_{234} - (1\leftrightarrow 2)\Big) \\
&+ (k^3\cdot k^4)\Big(\hat{H}_{[3,12]}\hat{A}^m_{[4,56]} + \hat{H}_{[3,56]}\hat{A}^m_{[4,12]} + \hat{H}_{[[3,12],56]}\hat{A}^m_4
\end{aligned}
$$

$$
\begin{aligned}
&\quad - \frac{1}{2}\hat{H}_{[3,12]}\hat{H}_{[4,56]}k^m_{456} - \frac{1}{2}\hat{H}_{[3,56]}\hat{H}_{[4,12]}k^m_{124} - (3 \leftrightarrow 4)\Big) \\
&- (k^{12}\cdot k^{34})\Big(\hat{H}_{[12,56]}\hat{A}^m_{34} - \hat{H}_{[34,56]}\hat{A}^m_{12}\Big) \\
&- (k^{1234}\cdot k^{56})\Big(\hat{H}_{[12,34]}\hat{A}^m_{56}\Big) \\
&+ (k^{5}\cdot k^{6})\Big(\hat{H}_{[[12,34],6]}\hat{A}^m_{5} - \hat{H}_{[[12,34],5]}\hat{A}^m_{6}\Big)
\end{aligned} \tag{F.4}
$$

$$
\begin{aligned}
A^m_{[[123,45],6]} &= \hat{A}^m_{[[123,45],6]} - k^m_{123456}\hat{H}_{[[123,45],6]} \\
&- (k^1\cdot k^2)\Big(\hat{H}_{[1,45]}\hat{A}^m_{236} + \hat{H}_{136}\hat{A}^m_{[2,45]} + \hat{H}_{[[1,45],6]}\hat{A}^m_{23} \\
&\quad + \hat{H}_{[13,45]}\hat{A}^m_{26} + \hat{H}_{[[13,45],6]}\hat{A}^m_{2} \\
&\quad - \frac{1}{2}\hat{H}_{[1,45]}\hat{H}_{236}k^m_{236} - \frac{1}{2}\hat{H}_{136}\hat{H}_{[2,45]}k^m_{245} - (1 \leftrightarrow 2)\Big) \\
&- (k^{12}\cdot k^3)\Big(\hat{H}_{126}\hat{A}^m_{[3,45]} + \hat{H}_{[12,45]}\hat{A}^m_{36} + \hat{H}_{[[12,45],6]}\hat{A}^m_{3} \\
&\quad - \hat{H}_{[3,45]}\hat{A}^m_{126} - \hat{H}_{[[3,45],6]}\hat{A}^m_{12} \\
&\quad - \frac{1}{2}\hat{H}_{126}\hat{H}_{[3,45]}k^m_{345} + \frac{1}{2}\hat{H}_{[3,45]}\hat{H}_{126}k^m_{126}\Big) \\
&+ (k^4\cdot k^5)\Big(\hat{H}_{[4,123]}\hat{A}^m_{56} + \hat{H}_{[[4,123],6]}\hat{A}^m_{5} - (4 \leftrightarrow 5)\Big) \\
&- (k^{123}\cdot k^{45})\Big(\hat{H}_{1236}\hat{A}^m_{45} + \hat{H}_{123}\hat{A}^m_{456} - \hat{H}_{456}\hat{A}^m_{123} \\
&\quad - \frac{1}{2}\hat{H}_{123}\hat{H}_{456}k^m_{456} + \frac{1}{2}\hat{H}_{456}\hat{H}_{123}k^m_{123}\Big) \\
&- (k^{12345}\cdot k^6)\Big(\hat{H}_{[123,45]}\hat{A}^m_{6}\Big)
\end{aligned} \tag{F.5}
$$

$$
\begin{aligned}
A^m_{[[[12,34],5],6]} &= \hat{A}^m_{[[[12,34],5],6]} - k^m_{123456}\hat{H}_{[[[12,34],5],6]} \\
&- (k^1\cdot k^2)\Big(\hat{H}_{156}\hat{A}^m_{[2,34]} + \hat{H}_{[1,34]}\hat{A}^m_{256} + \hat{H}_{[[1,34],6]}\hat{A}^m_{25} \\
&\quad + \hat{H}_{[[1,34],5]}\hat{A}^m_{26} + \hat{H}_{[[[1,34],5],6]}\hat{A}^m_{2} \\
&\quad - \frac{1}{2}\hat{H}_{156}\hat{H}_{[2,34]}k^m_{234} - \frac{1}{2}\hat{H}_{[1,34]}\hat{H}_{256}k^m_{256} - (1 \leftrightarrow 2)\Big) \\
&+ (k^3\cdot k^4)\Big(\hat{H}_{356}\hat{A}^m_{[4,12]} + \hat{H}_{[3,12]}\hat{A}^m_{456} + \hat{H}_{[[3,12],6]}\hat{A}^m_{45} \\
&\quad + \hat{H}_{[[3,12],5]}\hat{A}^m_{46} + \hat{H}_{[[[3,12],5],6]}\hat{A}^m_{4} \\
&\quad - \frac{1}{2}\hat{H}_{356}\hat{H}_{[4,12]}k^m_{124} - \frac{1}{2}\hat{H}_{[3,12]}\hat{H}_{456}k^m_{456} - (3 \leftrightarrow 4)\Big) \\
&- (k^{12}\cdot k^{34})\Big(\hat{H}_{1256}\hat{A}^m_{34} + \hat{H}_{126}\hat{A}^m_{345} + \hat{H}_{125}\hat{A}^m_{346} \\
&\quad - \frac{1}{2}\hat{H}_{126}\hat{H}_{345}k^m_{345} - \frac{1}{2}\hat{H}_{125}\hat{H}_{346}k^m_{346} - (12 \leftrightarrow 34)\Big) \\
&- (k^{1234}\cdot k^5)\Big(\hat{H}_{[12,34]}\hat{A}^m_{56} + \hat{H}_{[[12,34],6]}\hat{A}^m_{5}\Big) \\
&- (k^{12345}\cdot k^6)\Big(\hat{H}_{[[12,34],5]}\hat{A}^m_{6}\Big)
\end{aligned} \tag{F.6}
$$

where the redefinition terms $\hat{H}_P$ are defined so as to enforce generalized Jacobi identities upon superfields, and can be identified most easily with repeated use of (4.25) and (4.14) - (4.16).

The rank six Berends-Giele currents are found using the Berends-Giele map (5.2), and for a general superfield K is

$$
\begin{aligned}
s_{123456}\mathcal{K}_{123456} = {} & \frac{K_{[[[[[1,2],3],4],5],6]}}{s_{12}s_{123}s_{1234}s_{12345}} + \frac{K_{[[[[1,[2,3]],4],5],6]}}{s_{123}s_{1234}s_{12345}s_{23}} + \frac{K_{[[[[1,2],[3,4]],5],6]}}{s_{12}s_{1234}s_{12345}s_{34}} + \frac{K_{[[[[1,2],3],[4,5]],6]}}{s_{12}s_{123}s_{12345}s_{45}} \\
& + \frac{K_{[[[[1,2],3],4],[5,6]]}}{s_{12}s_{123}s_{1234}s_{56}} + \frac{K_{[[[1,[[2,3],4]],5],6]}}{s_{1234}s_{12345}s_{23}s_{234}} + \frac{K_{[[[1,[2,[3,4]]],5],6]}}{s_{1234}s_{12345}s_{234}s_{34}} + \frac{K_{[[[1,[2,3]],[4,5]],6]}}{s_{123}s_{12345}s_{23}s_{45}} \\
& + \frac{K_{[[[1,[2,3]],4],[5,6]]}}{s_{123}s_{1234}s_{23}s_{56}} + \frac{K_{[[[1,2],[[3,4],5]],6]}}{s_{12}s_{12345}s_{34}s_{345}} + \frac{K_{[[[1,2],[3,[4,5]]],6]}}{s_{12}s_{12345}s_{345}s_{45}} + \frac{K_{[[[1,2],[3,4]],[5,6]]}}{s_{12}s_{1234}s_{34}s_{56}} \\
& + \frac{K_{[[[1,2],3],[[4,5],6]]}}{s_{12}s_{123}s_{45}s_{456}} + \frac{K_{[[[1,2],3],[4,[5,6]]]}}{s_{12}s_{123}s_{456}s_{56}} + \frac{K_{[[1,[[[2,3],4],5]],6]}}{s_{12345}s_{23}s_{234}s_{2345}} + \frac{K_{[[1,[[2,[3,4]],5]],6]}}{s_{12345}s_{234}s_{2345}s_{34}} \\
& + \frac{K_{[[1,[[2,3],[4,5]]],6]}}{s_{12345}s_{23}s_{2345}s_{45}} + \frac{K_{[[1,[[2,3],4]],[5,6]]}}{s_{1234}s_{23}s_{234}s_{56}} + \frac{K_{[[1,[2,[[3,4],5]]],6]}}{s_{12345}s_{2345}s_{34}s_{345}} + \frac{K_{[[1,[2,[3,[4,5]]]],6]}}{s_{12345}s_{2345}s_{345}s_{45}} \\
& + \frac{K_{[[1,[2,[3,4]]],[5,6]]}}{s_{1234}s_{234}s_{34}s_{56}} + \frac{K_{[[1,[2,3]],[[4,5],6]]}}{s_{123}s_{23}s_{45}s_{456}} + \frac{K_{[[1,[2,3]],[4,[5,6]]]}}{s_{123}s_{23}s_{456}s_{56}} + \frac{K_{[[1,2],[[[3,4],5],6]]}}{s_{12}s_{34}s_{345}s_{3456}} \\
& + \frac{K_{[[1,2],[[3,[4,5]],6]]}}{s_{12}s_{345}s_{3456}s_{45}} + \frac{K_{[[1,2],[[3,4],[5,6]]]}}{s_{12}s_{34}s_{3456}s_{56}} + \frac{K_{[[1,2],[3,[[4,5],6]]]}}{s_{12}s_{3456}s_{45}s_{456}} + \frac{K_{[[1,2],[3,[4,[5,6]]]]}}{s_{12}s_{3456}s_{456}s_{56}} \\
& + \frac{K_{[1,[[[[2,3],4],5],6]]}}{s_{23}s_{234}s_{2345}s_{23456}} + \frac{K_{[1,[[[2,[3,4]],5],6]]}}{s_{234}s_{2345}s_{23456}s_{34}} + \frac{K_{[1,[[[2,3],[4,5]],6]]}}{s_{23}s_{2345}s_{23456}s_{45}} + \frac{K_{[1,[[[2,3],4],[5,6]]]}}{s_{23}s_{234}s_{23456}s_{56}} \\
& + \frac{K_{[1,[[2,[[3,4],5]],6]]}}{s_{2345}s_{23456}s_{34}s_{345}} + \frac{K_{[1,[[2,[3,[4,5]]],6]]}}{s_{2345}s_{23456}s_{345}s_{45}} + \frac{K_{[1,[[2,[3,4]],[5,6]]]}}{s_{234}s_{23456}s_{34}s_{56}} + \frac{K_{[1,[[2,3],[[4,5],6]]]}}{s_{23}s_{23456}s_{45}s_{456}} \\
& + \frac{K_{[1,[[2,3],[4,[5,6]]]]}}{s_{23}s_{23456}s_{456}s_{56}} + \frac{K_{[1,[2,[[[3,4],5],6]]]}}{s_{23456}s_{34}s_{345}s_{3456}} + \frac{K_{[1,[2,[[3,[4,5]],6]]]}}{s_{23456}s_{345}s_{3456}s_{45}} + \frac{K_{[1,[2,[[3,4],[5,6]]]]}}{s_{23456}s_{34}s_{3456}s_{56}} \\
& + \frac{K_{[1,[2,[3,[[4,5],6]]]]}}{s_{23456}s_{3456}s_{45}s_{456}} + \frac{K_{[1,[2,[3,[4,[5,6]]]]]}}{s_{23456}s_{3456}s_{456}s_{56}} . \qquad \text{(F.7)}
\end{aligned}
$$

Verifying that the redefinitions (F.1) - (F.6) amount to a gauge transformation in the Berends-Giele currents means plugging them into the above, and checking that in the resulting expression the Mandelstams cancel perfectly and the formula (5.8), which has the form of a gauge transformation, is produced.

Clearly this calculation requires considerable effort, but it has been performed and the result works as it should. A more efficient alternative approach based on (3.7) of Proposition 1 is possible though, and works as follows. We begin with the definition of the BG current, $\mathcal{A}^{m,BCJ}_{123456} = [\![A^m]\!] \circ b(123456)$. Using the general form of the gauge transformation (4.18) we see that this is just

$$\mathcal{A}^{m,BCJ}_{123456} = [\![\hat{A}^m]\!] \circ b(123456) - C[\![\hat{H}, L_2 \circ \hat{A}^m]\!] \circ b(123456) - [\![\hat{H}^m]\!] \circ b(123456), \qquad \text{(F.8)}$$

which by (5.2) and (3.7) is just

$$\mathcal{A}^{m,BCJ}_{123456} = \mathcal{A}^{m,L}_{123456} - k^{123456}_m \mathcal{H}_{123456} - [\![\hat{H}, L_2 \circ \hat{A}^m]\!] \circ \sum_{XY=12\ldots6} \Big(b(X) \otimes b(Y) - b(Y) \otimes b(X) \Big)$$
$$= \mathcal{A}^{m,L}_{123456} - k^{123456}_m \mathcal{H}_{123456} - \sum_{XY=12\ldots6} \Big(\mathcal{H}_X [\![L_2 \circ \hat{A}^m]\!] \circ b(Y) - \mathcal{H}_Y [\![L_2 \circ \hat{A}^m]\!] \circ b(X) \Big).$$

Completing another round of the same sort of calculation on the $[\![L_2 \circ \hat{A}^m]\!]$ terms yields[9]

$$\mathcal{A}^{m,BCJ}_{123456} = \mathcal{A}^{m,L}_{123456} - k^{123456}_m \mathcal{H}_{123456} - \sum_{XY=12\ldots6} \Big(\mathcal{H}_X \mathcal{A}^{m,L}_Y - \mathcal{H}_Y \mathcal{A}^{m,L}_X \Big) \qquad \text{(F.9)}$$
$$+ \frac{1}{2} \sum_{XY=12\ldots6} \Big(\mathcal{H}_X \mathcal{H}_Y k^m_Y - \mathcal{H}_Y \mathcal{H}_X k^m_X \Big)\,,.$$

This is then just (5.8), as was desired. By a similar argument it could be shown that all redefinitions produced by (4.18) have the form of a gauge transformation.

[9] Note there are no L_3 terms in the below. These have been omitted intentionally as any such terms would be of the form $\sum_{XYZ=12\ldots6} H_X H_Y A_Z$, and since each H requires at least three indices to be non-zero all terms of this form will be zero.

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