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Zassenhaus varieties of general linear Lie algebras

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Abstract

Let $\mathfrak{g}$ be a Lie algebra over an algebraically closed field of characteristic $p > 0$ and let $U(\mathfrak{g})$ be the universal enveloping algebra of $\mathfrak{g}$. We prove in this paper for $\mathfrak{g} = \mathfrak{gl}_n$ and $\mathfrak{g} = \mathfrak{sl}_n$ that the centre of $U(\mathfrak{g})$ is a unique factorisation domain and its field of fractions is rational. For $\mathfrak{g} = \mathfrak{sl}_n$ our argument requires the assumption that $p \nmid n$ while for $\mathfrak{g} = \mathfrak{gl}_n$ it works for any p. It turned out that our two main results are closely related to each other. The first one confirms in type A a recent conjecture of A. Braun and C. Hajarnavis while the second answers a question of J. Alev.

1. Introduction

Let K be an algebraically closed field of characteristic $p > 0$. In this note G denotes a connected reductive K-group with Lie algebra $\mathfrak{g}$. Mostly we will be in the situation where $G = \mathrm{GL}_n(K)$ or $G = \mathrm{SL}_n(K)$ and $p \nmid n$. Let $x \mapsto x^{[p]}$ denote the canonical pth power map on $\mathfrak{g}$ equivariant under the adjoint action of G.

Let $U = U(\mathfrak{g})$ denote the universal enveloping algebra of $\mathfrak{g}$. The group G acts on U as algebra automorphisms. This action extends the adjoint action of G on $\mathfrak{g}$, hence preserves the standard filtration $(U_i)_{i \geqslant 0}$ of U. The associated graded algebra $\mathrm{gr}(U) = S(\mathfrak{g})$ is

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a domain, so U has no zero divisors. The centre Z of U is therefore a filtered K-algebra, a domain, and a filtered G-module.

Let $\mathcal{Q} = \mathcal{Q}(\mathfrak{g})$ be the field of fractions of Z. By a classical result of Zassenhaus, Z is Noetherian and integrally closed in $\mathcal{Q}$; see [23]. Moreover, $\operatorname{tr.deg}_K \mathcal{Q} = \dim \mathfrak{g}$ and the localisation $\mathcal{D}(\mathfrak{g}) := \mathcal{Q} \otimes_Z U(\mathfrak{g})$ is a central division algebra over $\mathcal{Q}$ of dimension N^2 where N is the maximal dimension of irreducible $\mathfrak{g}$-modules. When $G = \mathrm{GL}_n(K)$ or $G = \mathrm{SL}_n(K)$ we have $N = p^{n(n-1)/2}$; see [14], for example. The maximal spectrum $\mathcal{Z}$ of the algebra Z is called the *Zassenhaus variety* of $\mathfrak{g}$. By the above discussion, the variety $\mathcal{Z}$ is affine, irreducible and normal. Furthermore, $\dim \mathcal{Z} = \dim \mathfrak{g}$. It is proved in [4] that under rather mild assumptions on p the singular points of $\mathcal{Z}$ are exactly the maximal ideals $\mathfrak{m}$ for which $(Z/\mathfrak{m}Z) \otimes_Z U$ is not isomorphic to the matrix algebra $\mathrm{Mat}_N(K)$.

At present very little is known about the division algebra $\mathcal{D}(\mathfrak{g})$ and its class in the Brauer group of $\mathcal{Q}$. In order to get started here it will be important to address the following question posed to the first author by Jacques Alev.

Question (J. Alev). *Is it true that $\mathcal{Q}$ is K-isomorphic to the field of rational functions $K(X_1, \dots, X_m)$ with $m = \dim \mathfrak{g}$? In other words, is it true that the Zassenhaus variety $\mathcal{Z}$ is rational*?

This is known as the *commutative Gelfand–Kirillov conjecture*, see below. Until now the answer to this question was known only in the simplest case $\mathfrak{g} = \mathfrak{sl}_2$. Another interesting question related to $\mathcal{Z}$ was recently raised in [3] and answered positively for $\mathfrak{g} = \mathfrak{sl}_2$ (mild characteristic restrictions may apply).

Conjecture (A. Braun and C. Hajarnavis). *The centre of $U(\mathfrak{g})$ is a unique factorisation domain.*

Similar problems can be raised in the characteristic zero case as well. Here one has to replace $U(\mathfrak{g})$ by the quantised enveloping algebra $U_\epsilon(\mathfrak{g}_\mathbb{C})$ without divided powers at a root of unity $\epsilon \in \mathbb{C}$; see [3] for more detail.

The main result of this paper is the following theorem which solves both problems in the modular case for $\mathfrak{g} = \mathfrak{gl}_n$ and for $\mathfrak{g} = \mathfrak{sl}_n$ with $p \nmid n$.

Theorem. *If $\mathfrak{g} = \mathfrak{gl}_n$ or $\mathfrak{g} = \mathfrak{sl}_n$ and $p \nmid n$, then the centre of $U(\mathfrak{g})$ is a unique factorisation domain and its field of fractions is rational.*

One expects this result to extend to the Lie algebras $\mathfrak{g}$ isomorphic to $\mathfrak{sl}_n$, $\mathfrak{pgl}_n$ and $\mathfrak{psl}_n$ with $p \mid n$. However, to obtain such an extension by our methods one would need an explicit description of the invariant algebra $S(\mathfrak{g})^\mathfrak{g}$, which is currently unavailable. As for the Lie algebras of other types, both problems remain open and new ideas are required here.

Our proof of the unique factorisation property of the centre of $U(\mathfrak{gl}_n)$ relies on the irreducibility of a certain polynomial function $d \in K[\mathfrak{gl}_n]$ semiinvariant relative to a maximal parabolic subgroup P of $\mathrm{GL}_n(K)$. In Section 5, we use the irreducibility of d to describe *all* semiinvariants of P in $K[\mathfrak{gl}_n]$. In Section 6, we establish an infinitesimal version of this result making use of a Jacobian criterion obtained by Skryabin in [20]. It is worth remarking that all results of Section 5 are valid in the characteristic zero case as well (the proofs are essentially the same).

For the moment we drop the assumptions on K and $\mathfrak{g}$. The *Gelfand–Kirillov conjecture* for $\mathfrak{g}$ states that the fraction field of $U(\mathfrak{g})$ is isomorphic to a Weyl skew field $D_n(L)$ over a purely transcendental extension L of K. The centre of the fraction field of $U(\mathfrak{g})$ is the fraction field $\mathcal{Q}$ of Z. In characteristic 0 this is proved in [6], for instance. In positive characteristic this follows from the fact that the fraction field of $U(\mathfrak{g})$ is nothing but the division algebra $\mathcal{D}(\mathfrak{g})$ introduced above. The centre of $D_n(L)$ equals L in characteristic 0 and in characteristic p it is generated over L by the pth powers of the standard generators of $D_n(L)$ over L. So in both cases it is rational (a purely transcendental extension of K). Therefore the original GK-conjecture implies the 'commutative' GK-conjecture which states that $\mathcal{Q}$ is rational.

Jacques Alev has informed us that some results of this note can be used to prove the GK-conjecture for $\mathfrak{g} = \mathfrak{gl}_n$ in characteristic p. It is worth mentioning here that the original GK-conjecture for finite-dimensional simple Lie algebras over $\mathbb{C}$ remains open in all cases except in type A where it was proved by Gelfand and Kirillov themselves; see [11]. It seems that proving the rationality of $\mathcal{Q}$ for all reductive Lie algebras might shed more light into this area of Lie Theory.

2. Preliminaries

2.1. Given an element x of a commutative ring S we denote by (x) the ideal of S generated by x. Recall that x is called *prime* if (x) is a prime ideal of S.

Let A be an associative ring with an ascending filtration $(A_i)_{i\in\mathbb{Z}}$. If I is a two-sided ideal of A, then the abelian group I and the ring A/I inherit an ascending filtration from A and we have an embedding $\mathrm{gr}(I) \hookrightarrow \mathrm{gr}(A)$ of graded abelian groups. If we identify $\mathrm{gr}(I)$ with a graded subgroup of the graded additive group $\mathrm{gr}(A)$ by means of this embedding, then $\mathrm{gr}(I)$ is a two-sided ideal of $\mathrm{gr}(A)$ and there is an isomorphism $\mathrm{gr}(A/I) \cong \mathrm{gr}(A)/\mathrm{gr}(I)$; see [1, Chapter 3, Section 2.4].

Now assume that $\bigcup_i A_i = A$ and $\bigcap_i A_i = \{0\}$. For a nonzero $x \in A$ we define $\deg(x) := \min\{i \in \mathbb{Z} \mid x \in A_i\}$ and $\mathrm{gr}(x) := x + A_{k-1} \in \mathrm{gr}(A)^k = A_k/A_{k-1}$ where $k = \deg(x)$. If $\mathrm{gr}(A)$ has no zero divisors, then the same holds for A and we have for $x, y \in A \setminus \{0\}$ that $\deg(xy) = \deg(x) + \deg(y)$, $\mathrm{gr}(xy) = \mathrm{gr}(x)\,\mathrm{gr}(y)$, and $\mathrm{gr}((x)) = (\mathrm{gr}(x))$. We mention for completeness that if $A = \bigoplus_{n\in\mathbb{Z}} A^n$ is a graded ring, then $(A_n)_{n\in\mathbb{Z}} = (\sum_{k\leqslant n} A^k)_{n\in\mathbb{Z}}$ defines an ascending filtration of A with the two properties mentioned above and $A \cong \mathrm{gr}(A)$ as algebras.

2.2. The *p-centre* Z_p of U is defined as the subalgebra of U generated by all elements $x^p - x^{[p]}$ with $x \in \mathfrak{g}$. It is well known (and easily seen) that $Z_p \subseteq Z$ is a polynomial algebra in $x_i^p - x_i^{[p]}$ where $\{x_i\}$ is any basis of $\mathfrak{g}$. For a vector space V over K the *Frobenius twist* $V^{(1)}$ of V is defined as the vector space over K with the same additive group as V and with scalar multiplication given by $\lambda \cdot x = \lambda^{1/p} x$. Note that the linear functionals and the polynomial functions on $V^{(1)}$ are the pth powers of those of V. The Frobenius twist of a

K-algebra is defined similarly (only the scalar multiplication is modified). Following [18] we define $\eta : S(\mathfrak{g})^{(1)} \to Z_p$ by setting $\eta(x) = x^p - x^{[p]}$ for all $x \in \mathfrak{g}$; see also [19]. This is a G-equivariant algebra isomorphism, hence it restricts to an algebra isomorphism

$$\eta : \big(S(\mathfrak{g})^G\big)^{(1)} = \big(S(\mathfrak{g})^{(1)}\big)^G \xrightarrow{\sim} Z_p^G.$$

We have $\operatorname{gr}(\eta(x)) = x^p$ for all $x \in \mathfrak{g} \setminus \{0\}$. Furthermore the associated graded algebra of the filtered algebra $Z_p \subset U$ is G-equivariantly isomorphic to the graded subalgebra $S(\mathfrak{g})^p$ of $S(\mathfrak{g})$.

2.3. In the remainder of this note we assume that $G = \mathrm{GL}_n(K)$ or $G = \mathrm{SL}_n(K)$ and $p \nmid n$. In this case Theorem 1.4 in [10] shows that the filtered G-modules $U(\mathfrak{g})$ and $S(\mathfrak{g})$ are isomorphic (the isomorphism in [10] is obtained by composing the Mil'ner map $\phi : U \to S(U)$ with a G-equivariant projection from U onto $\mathfrak{g}$). Consequently, each G-module U_{n-1} has a G-invariant direct complement in U_n. This implies that the associated graded algebras of U^G and Z are isomorphic to $S(\mathfrak{g})^G$ and $S(\mathfrak{g})^{\mathfrak{g}}$, respectively.

The trace form $\beta : \mathfrak{gl}_n \times \mathfrak{gl}_n \to K$ associated with the vector representation of $\mathrm{GL}_n(K)$ is nondegenerate and the same holds for its restriction to $\mathfrak{sl}_n$ as $p \nmid n$. Let $\theta : S(\mathfrak{g}^*) \to S(\mathfrak{g})$ denote the G-equivariant algebra isomorphism induced by β (it takes $f \in \mathfrak{g}^*$ to a unique $x \in \mathfrak{g}$ such that $f(y) = \beta(x, y)$ for all $y \in \mathfrak{g}$).

Let $\mathfrak{h}$ be the subalgebra of all diagonal matrices in $\mathfrak{gl}_n$ and $\mathfrak{h}' = \mathfrak{h} \cap \mathfrak{sl}_n$. Let $\mathfrak{n}^+$ (respectively $\mathfrak{n}^-$) be the subalgebra of all strictly upper (respectively lower) triangular matrices in $\mathfrak{g}$. To unify notation we set $\mathfrak{t} = \mathfrak{h}$ if $\mathfrak{g} = \mathfrak{gl}_n$ and $\mathfrak{t} = \mathfrak{h}'$ if $\mathfrak{g} = \mathfrak{sl}_n$. Then we have $\mathfrak{g} = \mathfrak{n}^- \oplus \mathfrak{t} \oplus \mathfrak{n}^+$. Also, $\mathfrak{t} = \operatorname{Lie} T$ where T is the group of all diagonal matrices in G. Furthermore, $\mathfrak{t}$ is the orthogonal complement to $\mathfrak{n}^- \oplus \mathfrak{n}^+$ with respect to β.

2.4. The Weyl group action induced by the adjoint action of the normaliser $N_G(T)$ on $\mathfrak{t}$ is nothing but the restriction to $\mathfrak{t}$ of the permutation action of the symmetric group $\mathfrak{S}_n$ on the space of diagonal matrices $\mathfrak{h}$. In [16, Theorem 4], Kac and Weisfeiler proved that a modular version of the Chevalley restriction theorem holds for the coadjoint action of any simple, simply connected algebraic K-group. Their arguments are known to work for all connected reductive K-groups with simply connected derived subgroups. In particular, they apply to our group G. Since $\theta : K[\mathfrak{g}] \to K[\mathfrak{g}^*]$ is a G-equivariant algebra isomorphism, Theorem 4 in [16] implies that the restriction map $K[\mathfrak{g}] \to K[\mathfrak{t}]$ induces an algebra isomorphism $K[\mathfrak{g}]^G \xrightarrow{\sim} K[\mathfrak{t}]^{\mathfrak{S}_n}$.

For $1 \leqslant i \leqslant n$ define $s_i \in K[\mathfrak{gl}_n]^{\mathrm{GL}_n}$ by setting $s_i(x) = \operatorname{tr}(\bigwedge^i x)$ for all $x \in \mathfrak{gl}_n$, where $\bigwedge^i x$ is the ith exterior power of x. Then

$$\chi_x(X) = X^n + \sum_{i=1}^{n} (-1)^i s_i(x) X^{n-i}$$

is the characteristic polynomial of x. Let $\{e_{i,j} \mid 1 \leqslant i, j \leqslant n\}$ be the basis of $\mathfrak{gl}_n$ consisting of the matrix units and let $\{\xi_{ij} \mid 1 \leqslant i, j \leqslant n\}$ be the corresponding dual basis in $\mathfrak{gl}_n^*$. To ease notation identify each ξ_{ii} with its restriction to the diagonal subalgebra $\mathfrak{h}$. For $1 \leqslant i \leqslant n$ the restriction of s_i to $\mathfrak{h}$ is then the ith elementary symmetric function σ_i in $\xi_{11}, \xi_{22}, \ldots, \xi_{nn}$. By the theorem on symmetric functions, $\sigma_1, \ldots, \sigma_n$ are algebraically independent and generate the invariant algebra $K[\mathfrak{h}]^{\mathfrak{S}_n}$. Our discussion in Section 2.3 now shows that the s_i's are algebraically independent and generate the invariant algebra $K[\mathfrak{gl}_n]^{\mathrm{GL}_n}$.

Suppose $p \nmid n$. Given a polynomial function f on $\mathfrak{gl}_n$ we denote by f' its restriction to $\mathfrak{sl}_n$. The span of all $\xi_{ii} - \xi_{jj}$ is an $\mathfrak{S}_n$-invariant direct complement to the line $K\sigma_1$ in $\mathfrak{h}^*$, hence the K-subalgebra generated by all $\xi_{ii} - \xi_{jj}$ is an $\mathfrak{S}_n$-invariant direct complement to the ideal of $K[\mathfrak{h}]$ generated by σ_1. From this it is immediate that the restriction map $K[\mathfrak{h}] \to K[\mathfrak{h}']$ induces an epimorphism $K[\mathfrak{h}]^{\mathfrak{S}_n} \twoheadrightarrow K[\mathfrak{h}']^{\mathfrak{S}_n}$ whose kernel is the ideal of $K[\mathfrak{h}]^{\mathfrak{S}_n}$ generated by σ_1. Since the subalgebra of $K[\mathfrak{h}]^{\mathfrak{S}_n}$ generated by $\sigma_2, \ldots, \sigma_n$ is a direct complement in $K[\mathfrak{h}]^{\mathfrak{S}_n}$ to this ideal, we deduce that the restrictions $s'_2|_{\mathfrak{h}'}, \ldots, s'_n|_{\mathfrak{h}'}$ are algebraically independent and generate $K[\mathfrak{h}']^{\mathfrak{S}_n}$. But then $s'_2, \ldots, s'_n$ are algebraically independent and generate the invariant algebra $K[\mathfrak{sl}_n]^{\mathrm{SL}_n}$ by our discussion in Section 2.3.

Under the G-equivariant isomorphism $\theta : S(\mathfrak{g}^*) \xrightarrow{\sim} S(\mathfrak{g})$ and the induced $\mathfrak{S}_n$-equivariant isomorphism $S(\mathfrak{t}^*) \xrightarrow{\sim} S(\mathfrak{t})$, the restriction map $S(\mathfrak{g}^*) \to S(\mathfrak{t}^*)$ corresponds to the projection homomorphism $\Phi : S(\mathfrak{g}) \to S(\mathfrak{t})$ defined as follows: if we identify $S(\mathfrak{g})$ with $S(\mathfrak{n}^-) \otimes S(\mathfrak{t}) \otimes S(\mathfrak{n}^+)$, then $\Phi(x \otimes h \otimes y) = x^0 h y^0$ where f^0 denotes the zero degree part of $f \in S(\mathfrak{g})$. By the above, Φ induces an algebra isomorphism $S(\mathfrak{g})^G \cong S(\mathfrak{t})^{\mathfrak{S}_n}$.

2.5. In [16], Kac and Weisfeiler also proved a noncommutative version of the Chevalley restriction theorem. Again the arguments in [16] are known to generalise to all connected reductive K-groups with simply connected derived subgroups; see [14, Section 9]. In particular, they apply to our group G.

Let $\Psi : U = U(\mathfrak{n}^-) \otimes U(\mathfrak{t}) \otimes U(\mathfrak{n}^+) \to U(\mathfrak{t}) = S(\mathfrak{t})$ be the linear map taking $x \otimes h \otimes y$ to $x^0 h y^0$, where u^0 denotes the scalar part of $u \in U$ with respect to the decomposition $U = K1 \oplus U_+$ where U_+ is the augmentation ideal of U. The restriction of Ψ to $U^{N_G(T)}$ is an algebra homomorphism.

For $G = \mathrm{GL}_n$ define $\rho \in \mathfrak{h}^*$ as $\sum_{i=1}^{n-1} (n-i)\xi_{ii}$, where ξ_{ii} is the functional $A \mapsto A_{ii}$ and for $G = \mathrm{SL}_n$ let ρ denote the corresponding restriction. In the latter case ρ is the differential of the character of T that equals the half sum of the positive roots. Then ρ is as in [14, Section 9.2]. Define the shift homomorphism $\gamma : S(\mathfrak{t}) \to S(\mathfrak{t})$ by setting $\gamma(h) = h - \rho(h)$ for all $h \in \mathfrak{t}$. In [16, Section 8] there was defined an action of the Weyl group W on $S(\mathfrak{t}) = K[\mathfrak{t}^*]$ which is called the dot action in [14]. The dot action of W on $S(\mathfrak{t})$ is related to the natural action as follows: $w_{\bullet} = \gamma^{-1} \circ w \circ \gamma$. It follows from [14, Theorem 9.3] that $\gamma \circ \Psi$ induces an algebra isomorphism between U^G and $S(\mathfrak{t})^{\mathfrak{S}_n}$. See also [16, Theorem 1]. As a consequence, U^G is a polynomial algebra in $\dim \mathfrak{t}$ variables.

Using the descriptions of Φ and Ψ and a PBW-basis it follows that for $x \in U \setminus \{0\}$ with $\Phi(\mathrm{gr}(x)) \neq 0$ we have $\Psi(x) \neq 0$ and

$$\mathrm{gr}\big(\gamma\big(\Psi(x)\big)\big) = \mathrm{gr}\big(\Psi(x)\big) = \Phi\big(\mathrm{gr}(x)\big).$$

By the injectivity of the restriction of Φ to $S(\mathfrak{g})^G$, the displayed equalities hold for all $x \in U^G$. Thus we can deduce the injectivity of $\gamma \circ \Psi : U^G \to S(\mathfrak{t})^{\mathfrak{S}_n}$ from that of $\Phi : S(\mathfrak{g})^G \to S(\mathfrak{t})^{\mathfrak{S}_n}$. The same applies to the surjectivity; see the proof of Proposition 2.1 in [22].

3. Invariants for the Lie algebra

3.1. The aim of this section is to put together all results on Lie algebra invariants that will be needed later on. The results in Sections 3.1, 3.2 and 3.5 are known for reductive groups satisfying certain standard hypotheses, but their proofs are spread over the literature (and folklore); see [5,8,10,15, Section 7,16,22], and the references therein.

Given $x \in \mathfrak{g}$ we denote by $\mathfrak{z}_{\mathfrak{g}}(x)$ the centraliser of x in $\mathfrak{g}$. An element $x \in \mathfrak{g}$ is called *regular* if $\dim \mathfrak{z}_{\mathfrak{g}}(x) = \dim \mathfrak{t}$. It is well known and not hard to see that $\dim \mathfrak{z}_{\mathfrak{g}}(x) \geqslant \dim \mathfrak{t}$ for all $x \in \mathfrak{g}$.[1] Moreover, the set $\mathfrak{g}_{\mathrm{reg}}$ of all regular elements in $\mathfrak{g}$ is nonempty and Zariski open in $\mathfrak{g}$. Furthermore, Linear Algebra shows that x is regular in $\mathfrak{gl}_n$ if and only if the minimal polynomial of x equals $\chi_x(X)$, which happens if and only if the column space K^n is a cyclic $K[x]$-module.

The first result we need is a modular version of Kostant's differential criterion of regularity [17]. It is essentially due to Veldkamp [22].

Lemma 1. *For $x \in \mathfrak{gl}_n$ the following are equivalent*:

(1) *the element x is regular*;
(2) *the differentials $\mathrm{d}_x s_1, \ldots, \mathrm{d}_x s_n$ are linearly independent.*

Proof. That the independence of $\mathrm{d}_x s_1, \ldots, \mathrm{d}_x s_n$ implies the regularity of x is proved in [22, Section 7]. The proof requires a lemma on the invariant algebra $K[\mathfrak{g}]^G$ [22, Lemma 7.2], the fact that the semisimple irregular elements of $\mathfrak{g}$ form a dense subset in $\mathfrak{g} \setminus \mathfrak{g}_{\mathrm{reg}}$ [22, Proposition 4.9], and a result from [2, Proposition 6, Chapter 5, Section 5.5]. All these are valid for $\mathfrak{g} = \mathfrak{gl}_n$.

That the regularity of x implies the independence of $\mathrm{d}_x s_1, \ldots, \mathrm{d}_x s_n$ is much easier to prove. Given $\mathbf{a} = (a_1, \ldots, a_n) \in K^n$ we set

$$x_{\mathbf{a}} = \begin{bmatrix} a_1 & a_2 & \cdots & a_{n-1} & a_n \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \end{bmatrix}.$$

Each $x_{\mathbf{a}}$ is regular in $\mathfrak{gl}_n$ as the minimal polynomial of $x_{\mathbf{a}}$ equals $X^n - \sum_{i=1}^n a_i X^{n-i}$. The set $\mathcal{S} = \{x_{\mathbf{a}} \mid \mathbf{a} \in K^n\}$ is an n-dimensional affine subspace in $\mathfrak{gl}_n$ through the point $x_{\mathbf{0}}$. The restriction to $\mathcal{S}$ of the morphism $x \mapsto (s_1(x), \ldots, s_n(x))$ is an isomorphism of $\mathcal{S}$ onto $\mathbb{A}^n$. From this it is immediate that the differentials $\mathrm{d}_x s_1, \ldots, \mathrm{d}_x s_n$ are linearly independent for all $x \in \mathcal{S}$. On the other hand, every matrix x whose minimal polynomial equals $\chi_x(X)$ is similar to a matrix from $\mathcal{S}$. Hence these differentials are independent for all regular x. □

[1] As in the group case, take a Borel subgroup B of G with $x \in \mathrm{Lie}(B)$ and consider the morphism $B \to \mathrm{Lie}(B, B)$ sending $g \in B$ to $(\mathrm{Ad}\, g)(x) - x \in \mathrm{Lie}(B, B)$; see [21, p. 1]

3.2. Now we look at the regular elements in $\mathfrak{sl}_n$. Recall the notational conventions of Section 2.4. It is immediate from the definition that $x \in \mathfrak{gl}_n$ is regular if and only so is $x + \lambda I_n$ for any $\lambda \in K$.

Corollary. *Suppose $p \nmid n$. For $x \in \mathfrak{sl}_n$ the following are equivalent*:

(1) *the element x is regular in $\mathfrak{sl}_n$;*
(2) *the element x is regular in $\mathfrak{gl}_n$;*
(3) *the differentials $\mathrm{d}_x s_2', \dots, \mathrm{d}_x s_n'$ are linearly independent.*

Proof. We have $\mathfrak{z}_{\mathfrak{gl}_n}(x) = \mathfrak{z}_{\mathfrak{sl}_n}(x) \oplus K I_n$. This shows that (1) and (2) are equivalent. The differentials $\mathrm{d}_x s_1, \dots, \mathrm{d}_x s_n$ are independent if and only so are the restrictions of $\mathrm{d}_x s_2, \dots, \mathrm{d}_x s_n$ to $\mathfrak{sl}_n$, the kernel of $\mathrm{d}_x s_1 = s_1$. The equivalence of (2) and (3) now follows from Lemma 1. $\square$

3.3. As mentioned in the introduction, our proof of the main theorem will rely on the following proposition communicated to us by S. Skryabin. We were unable to trace this result in the literature. Although it resembles strongly one of the basic facts of the invariant theory of groups, it also captures some essential features of the invariant theory of restricted Lie algebras.

Recall that the coordinate algebra $K[V]$ of a finite-dimensional vector space V over K is a unique factorisation domain. The algebra $K[V] \cong \bigoplus_{i \geqslant 0} S^i(V^*)$ is graded and $\mathfrak{gl}(V)$ acts on $K[V]$ as homogeneous derivations of degree 0. Therefore, $K[V]^p \subseteq K[V]^{\mathfrak{gl}(V)}$.

Proposition 1. *Let L be a Lie algebra with $L = [L, L]$ and let V be a finite-dimensional L-module. Then the invariant algebra $K[V]^L$ is a unique factorisation domain and the irreducible elements of $K[V]^L$ are the pth powers of the irreducible elements of $K[V]$ not invariant under L and the irreducible elements of $K[V]$ contained in $K[V]^L$.*

Proof. Let f be a nonzero element in $K[V]^L$ and suppose $f = f_1 f_2$ where $f_1, f_2 \in K[V]$ are coprime of positive degree. Let x be any element in L. Since $(x \cdot f_1) f_2 = -f_1(x \cdot f_2)$, the uniqueness of prime factorisation in $K[V]$ implies that f_2 divides $x \cdot f_2$. As $\deg(x \cdot f_2) \leqslant \deg f_2$ it must be that $x \cdot f_2 = \chi(x) f_2$ for some $\chi(x) \in K$. The map $\chi : L \to K$ is a character of L. As $L = [L, L]$, it must be that $\chi = 0$. This shows that $f_1, f_2 \in K[V]^L$. Now suppose $f = g^n$ for some $n \in \mathbb{N}$. Write $n = sp + r$ with $s, r \in \mathbb{Z}_+$ and $0 \leqslant r < p$. Then $0 = x \cdot f = n g^{n-1}(x \cdot g)$. For $r \neq 0$ this yields $g \in K[V]^L$, while for $r = 0$ we have $f = (g^p)^s$ with $g^p \in K[V]^L$.

This shows that any irreducible element in $K[V]^L$ is either an irreducible element of $K[V]$ invariant under L or a pth power of an irreducible element in $K[V] \setminus K[V]^L$. Now the unique factorisation property of $K[V]^L$ follows from that of $K[V]$. $\square$

3.4. Let X be an affine algebraic variety defined over K, and let $\mathcal{L}$ be a finite-dimensional restricted Lie algebra together with a restricted homomorphism $\mathcal{L} \to \mathrm{Der}_K K[X]$.

Define $\mathcal{L}_x$ to be the stabiliser of the maximal ideal $\mathfrak{m}_x$ of $K[X]$ corresponding to a point $x \in X$. Following [20, Section 5], we put

$$c_{\mathcal{L}}(X) := \max_{x \in X} \operatorname{codim}_{\mathcal{L}} \mathcal{L}_x.$$

In the situation of Section 3.3, where $X = V$ is a finite-dimensional restricted $\mathcal{L}$-module, it is easy to see that $\mathcal{L}_x = \{l \in \mathcal{L} \mid l(x) = 0\}$ for every $x \in V$.

Lemma 2. *We have $K[\mathfrak{gl}_n]^{\mathfrak{gl}_n} = K[\mathfrak{gl}_n]^{\mathfrak{sl}_n}$ for all $n \in \mathbb{N}$. Moreover, $K[\mathfrak{gl}_n]^{\mathfrak{gl}_n}$ is a unique factorisation domain and the irreducible elements of $K[\mathfrak{gl}_n]^{\mathfrak{gl}_n}$ are the pth powers of the irreducible elements of $K[\mathfrak{gl}_n]$ not invariant under $\mathfrak{gl}_n$ and the irreducible elements of $K[\mathfrak{gl}_n]$ contained in $K[\mathfrak{gl}_n]^{\mathfrak{gl}_n}$.*

Proof. 1. For $p \nmid n$ the first part of the statement is obvious as $\mathfrak{gl}_n = \mathfrak{sl}_n \oplus K I_n$. To tackle it in the general case we recall our notation in Section 2.3 and set $V = \mathfrak{gl}_n$. It follows from our remarks above that $(\mathfrak{gl}_n)_x = \mathfrak{z}_{\mathfrak{gl}_n}(x)$ for all $x \in V$. So the discussion in Section 3.1 yields that $c_{\mathfrak{gl}_n}(V) = n^2 - n$. Let h be a regular element of $\mathfrak{gl}_n$ contained in $\mathfrak{h}$. Then we have $(\mathfrak{gl}_n)_h = \mathfrak{z}_{\mathfrak{gl}_n}(h) = \mathfrak{h}$ and $\mathfrak{gl}_n = \mathfrak{sl}_n + (\mathfrak{gl}_n)_h$. But then $K[\mathfrak{gl}_n]^{\mathfrak{gl}_n} = K[\mathfrak{gl}_n]^{\mathfrak{sl}_n}$ in view of [20, Corollary 5.3].

2. The second part of the statement follows immediately from Proposition 1 if $(p, n) \neq (2, 2)$, since then, as is well known, $\mathfrak{sl}_n$ is perfect. To establish it in general we will slightly modify our arguments in the proof of Proposition 1. If for $f \in K[V]^{\mathfrak{gl}_n}$ we have $f = f_1 f_2$ with $f_1, f_2 \in K[V]$ coprime, then as in that proof $x \cdot f_2 = \chi(x) f_2$ for all $x \in \mathfrak{gl}_n$. The character $\chi : \mathfrak{gl}_n \to K$ must vanish on $[\mathfrak{gl}_n, \mathfrak{gl}_n] = \mathfrak{sl}_n$. But then $f_1, f_2 \in K[V]^{\mathfrak{gl}_n}$, by part 1 of this proof. The rest of the proof of Proposition 1 applies in our present situation, and the result follows. □

3.5. The statement below is known but we wanted to streamline its proof by employing the relationship between filtered and graded algebras in a more systematic way. Assertion (iv) is often referred to as Veldkamp's theorem; see [22, Theorem 3.1].

Proposition 2. *Let m be the rank of $\mathfrak{g}$, i.e. $m = \dim \mathfrak{t}$, and put $(t_1, \ldots, t_m) = (s_1, \ldots, s_n)$ for $\mathfrak{g} = \mathfrak{gl}_n$ and $(t_1, \ldots, t_m) = (s'_2, \ldots, s'_n)$ for $\mathfrak{g} = \mathfrak{sl}_n$. Define $u_i \in U^G$ by $u_i = ((\gamma \circ \Psi)^{-1} \circ \Phi)(\theta(t_i)) = (\gamma \circ \Psi)^{-1}(\theta(t_i|_{\mathfrak{t}}))$. Then the following hold*:

(i) *The set $\mathfrak{g} \setminus \mathfrak{g}_{\mathrm{reg}}$ is Zariski closed of pure codimension* 3 *in* $\mathfrak{g}$.
(ii) *$K[\mathfrak{g}]^{\mathfrak{g}}$ is a free $K[\mathfrak{g}]^p$-module with basis $\{t_1^{k_1} \cdots t_m^{k_m} \mid 0 \leqslant k_i < p\}$.*
(iii) *$S(\mathfrak{g})^{\mathfrak{g}}$ is a free $S(\mathfrak{g})^p$-module with basis $\{\theta(t_1)^{k_1} \cdots \theta(t_m)^{k_m} \mid 0 \leqslant k_i < p\}$.*
(iv) *Z is a free Z_p-module with basis $\{u_1^{k_1} \cdots u_m^{k_m} \mid 0 \leqslant k_i < p\}$.*

Proof. (i) The first assertion is proved in [22, Theorem 4.12]. The arguments there also apply to $\mathfrak{g} = \mathfrak{gl}_n$.

(ii) By Lemma 1, its Corollary and (i), the Zariski closed subset of $\mathfrak{g}$ consisting of all x for which the differentials $\mathrm{d}_x t_1, \ldots, \mathrm{d}_x t_m$ are linearly dependent has codimension 3 in $\mathfrak{g}$. The second assertion now follows from [20, Theorem 5.4] applied to the variety $X = \mathfrak{g}$. Arguing as in the proof of Lemma 2 one observes that $c_{\mathfrak{g}}(X) = n^2 - n$ in our case. Therefore, $\dim X - c_{\mathfrak{g}}(X) = m$.

(iii) The third assertion follows immediately from part (ii) in view of the isomorphism $\theta : K[\mathfrak{g}] \xrightarrow{\sim} S(\mathfrak{g})$.

(iv) Recall from Sections 2.2 and 2.3 that the associated graded algebras of Z, U^G and Z_p are $S(\mathfrak{g})^{\mathfrak{g}}$, $S(\mathfrak{g})^G$ and $S(\mathfrak{g})^p$, respectively. By our remarks in Sections 2.3 and 2.5 we have $\theta(t_i) = \operatorname{gr}(u_i)$. The fourth assertion now follows from part (iii) by a standard induction argument; see the proof of Theorem 3.1 in [22] for more details. □

Remark 1. It follows from Proposition 2 that the bases in (ii), (iii), (iv) are also bases of $K[\mathfrak{g}]^G$, $S(\mathfrak{g})^G$ and U^G over $(K[\mathfrak{g}]^p)^G$, $(S(\mathfrak{g})^p)^G$ and Z_p^G, respectively. This implies that $K[\mathfrak{g}]^{\mathfrak{g}} \cong K[\mathfrak{g}]^p \otimes_{(K[\mathfrak{g}]^p)^G} K[\mathfrak{g}]^G$, $S(\mathfrak{g})^{\mathfrak{g}} \cong S(\mathfrak{g})^p \otimes_{(S(\mathfrak{g})^p)^G} S(\mathfrak{g})^G$ and $Z \cong Z_p \otimes_{Z_p^G} U^G$ as algebras. The first two of these isomorphisms are known as Friedlander–Parshall factorisations; see [10, Theorem 4.1].

Remark 2. It also follows from Proposition 2 that $\mathcal{Q}(\mathfrak{g})$ is a finite extension of the field of fractions of $Z_p \cong S(\mathfrak{g})^{(1)}$ and hence $\operatorname{tr.deg}_K \mathcal{Q}(\mathfrak{g}) = \dim \mathfrak{g}$.[2] The analogous statements for the fields of fractions of $K[\mathfrak{g}]^{\mathfrak{g}}$ and $S(\mathfrak{g})^{\mathfrak{g}}$ are obvious.

4. Proof of the main theorems

4.1. Define $\partial_{ij} \in \operatorname{Der}_K K[\mathfrak{gl}_n]$ be setting $\partial_{ij}(\xi_{rs}) = 1$ if $(r, s) = (i, j)$ and 0, otherwise. It is immediate from our discussion in Section 2.4 that s_k is the sum of all diagonal minors of order k of the matrix $\sum_{i,j} \xi_{ij} e_{i,j}$ with entries in $K[\mathfrak{gl}_n]$. If we write each s_k as a polynomial in the ξ_{ij}, then we obtain n equations in the ξ_{ij} and the s_k. By the above, ξ_{ij} with one fixed row or column index are not multiplied among each other in these equations. In particular these equations are *linear* in $\xi_{11}, \xi_{12}, \ldots, \xi_{1n}$.

Let R denote the $\mathbb{F}_p$-subalgebra of $K[\mathfrak{gl}_n]$ generated by all ξ_{ij} with $i > 1$. Set

$$M = \begin{bmatrix} \partial_{11}(s_1) & \partial_{12}(s_1) & \ldots & \partial_{1n}(s_1) \\ \partial_{11}(s_2) & \partial_{12}(s_2) & \ldots & \partial_{1n}(s_2) \\ \vdots & \vdots & \ddots & \vdots \\ \partial_{11}(s_n) & \partial_{12}(s_n) & \ldots & \partial_{1n}(s_n) \end{bmatrix}, \qquad \mathbf{c} = \begin{bmatrix} \xi_{11} \\ \xi_{12} \\ \vdots \\ \xi_{1n} \end{bmatrix}, \qquad \mathbf{s} = \begin{bmatrix} s_1 \\ s_2 \\ \vdots \\ s_n \end{bmatrix}.$$

By the preceding paragraph the matrix M has entries in R and the following vector equation holds:

$$M \cdot \mathbf{c} = \mathbf{s} + \mathbf{r}, \quad \text{where } M \in \mathfrak{gl}_n(R) \text{ and } \mathbf{r} \in R^n. \tag{1}$$

[2] This also follows from a version of the PWB theorem; see [12, Chapter 5, Section 7, Lemma 4].

Clearly, M is a matrix with functional entries. Hence $M(x) \in \mathfrak{gl}_n$ is well defined for any $x \in \mathfrak{gl}_n$. Let $d = \det M$, a regular function on $\mathfrak{g}$. Recall from Section 3.1 the definition of the affine subspace $\mathcal{S} = \{x_{\mathbf{a}} \mid \mathbf{a} \in K^n\}$ of $\mathfrak{gl}_n$.

Lemma 3. *For all $\mathbf{a} \in K^n$ we have $d(x_{\mathbf{a}}) = (-1)^{\lfloor n/2 \rfloor}$. In particular, $d \neq 0$.*

Proof. Let $\xi_1, \dots, \xi_n$ be the coordinate functions on K^n and let ∂_i be the derivation of the coordinate ring of K^n such that $\partial_i(\xi_j) = 1$ when $i = j$ and 0 otherwise. Then it is easy to see that $\partial_{1j}(f)(x_{\mathbf{a}}) = \partial_j(\mathbf{b} \mapsto f(x_{\mathbf{b}}))(\mathbf{a})$ for all $f \in K[\mathfrak{gl}_n]$. Furthermore, it follows from the formula displayed in Section 2.4 and our remarks in the proof of Lemma 1 that $s_i(x_{\mathbf{a}}) = (-1)^{i-1} a_i$. So the (i, j)-th entry of $M(x_{\mathbf{a}})$ equals $(-1)^{i-1} \partial_j(\xi_i)$. But then $M(x_{\mathbf{a}}) = \operatorname{diag}(1, -1, \dots, (-1)^{n-1})$ and $(\det M)(x_{\mathbf{a}}) = (-1)^{\lfloor n/2 \rfloor}$. $\square$

4.2. Let Q denote the field of fractions of $K[\mathfrak{g}]^{\mathfrak{g}}$. It follows from Proposition 2 that Q is generated by $m + \dim \mathfrak{g}$ elements. Using Lemma 3 we will show that m generators can be made redundant here. Since $\operatorname{tr.deg}_K Q = \dim \mathfrak{g}$, this will imply that Q is rational. We will then use a very similar method to establish the rationality of $\mathcal{Q}$.

Let $F : f \mapsto f^p$ denote the Frobenius endomorphism of $K[\mathfrak{gl}_n]$. It acts componentwise on $\mathfrak{gl}_n(K[\mathfrak{gl}_n])$ and $K[\mathfrak{gl}_n]^n$. Note that $R^F \subset R$.

Theorem 1. *Both $S(\mathfrak{g})^{\mathfrak{g}}$ and Z have rational fields of fractions.*

Proof. 1. First we assume that $\mathfrak{g} = \mathfrak{gl}_n$. Applying F to both sides of (1) we get

$$M^F \cdot \mathbf{c}^F = \mathbf{s}^F + \mathbf{r}^F, \quad \text{where } M \in \mathfrak{gl}_n(R^p) \text{ and } \mathbf{r} \in (R^p)^n. \tag{2}$$

By Lemma 3, $\det(M^F) = d^p \neq 0$. Therefore, $\mathbf{c}^F$ has components in the $\mathbb{F}_p$-subalgebra of Q generated by $s_1^p, \dots, s_n^p$, $(d^p)^{-1}$ and ξ_{ij}^p with $i > 1$. As a result, Q is generated by the n^2 elements $s_1, \dots, s_n$ and ξ_{ij}^p with $i > 1$. These elements must be algebraically independent because $\operatorname{tr.deg}_K Q = n^2$; see Remark 2. Thus Q is rational over K. The same assertion then holds for the field of fractions of $S(\mathfrak{g})^{\mathfrak{g}}$ in view of the G-equivariant algebra isomorphism $\theta : K[\mathfrak{g}] \xrightarrow{\sim} S(\mathfrak{g})$.

2. Recall from Sections 2.2 and 2.4 that $\eta \circ \theta : K[\mathfrak{g}]^{(1)} \to Z_p$ is a G-equivariant algebra isomorphism. Observe that $\theta(\xi_{ij}) = e_{j,i}$ and that $\mathcal{R} := \eta(\theta(R))$ is the $\mathbb{F}_p$-subalgebra of Z_p generated by all $e_{i,j}^p - e_{i,j}^{[p]}$ with $j > 1$. Let $\mathbf{e} \in Z_p^n$ denote the column vector whose ith component equals $e_{i,1}^p - e_{i,1}^{[p]}$. Applying $\eta \circ \theta$ to both sides of (1) yields

$$\mathcal{M} \cdot \mathbf{e} = \eta\big(\theta(\mathbf{s})\big) + \tilde{\mathbf{r}}, \quad \text{where } \mathcal{M} \in \mathfrak{gl}_n(\mathcal{R}) \text{ and } \tilde{\mathbf{r}} \in \mathcal{R}^n. \tag{3}$$

By Proposition 2, $\mathcal{Q}$ is generated over K by the elements $e_{i,j}^p - e_{i,j}^{[p]}$ and n algebraically independent elements generating Z^G. Besides, $\operatorname{tr.deg}_K \mathcal{Q} = n^2$; see Remark 2. Since $\eta(\theta(s_i)) \in Z_p^G$ and $\det \mathcal{M} = \eta(\theta(d)) \neq 0$, we now argue as in part 1 of this proof to deduce that $\mathcal{Q}$ is rational over K.

3. Now assume that $\mathfrak{g} = \mathfrak{sl}_n$ and $p \nmid n$. Recall our notation from Section 2.4. Applying the restriction homomorphism $K[\mathfrak{gl}_n] \to K[\mathfrak{sl}_n]$, $f \mapsto f'$, to both sides of (1) we obtain the following equations in the ξ'_{ij} and $s'_2, \dots, s'_n$:

$$M' \cdot \mathbf{c}' = \mathbf{s}' + \mathbf{r}', \quad \text{where } M' \in \mathfrak{gl}_n(R') \text{ and } \mathbf{r}' \in (R')^n.$$

Here M', $\mathbf{c}'$, $\mathbf{s}'$, $\mathbf{r}'$ have the obvious meaning and R' is the $\mathbb{F}_p$-subalgebra of $K[\mathfrak{sl}_n]$ generated by all ξ'_{ij} with $i > 1$. Note that we now have $\theta(\xi'_{ij}) = e_{j,i}$ for $i \neq j$ and $\theta(\xi'_{ii}) = e_{i,i} - (1/n)I_n$. Since $\mathcal{S} \cap \mathfrak{sl}_n \neq \emptyset$, Lemma 3 shows that $d' = \det(M') \neq 0$. We can thus repeat our arguments from parts 1 and 2 of this proof to deduce that the generators $(\xi'_{11})^p, \dots, (\xi'_{1n})^p$ and $(e_{1,1} - (1/n)I_n)^p - (e_{1,1} - (1/n)I_n), e_{2,1}^p, \dots, e_{n,1}^p$ of Q and $\mathcal{Q}$, respectively, are redundant. This proves that both Q and $\mathcal{Q}$ are K-rational in the present case (recall that we now have one generator less and $\mathrm{tr.deg}_K Q = \mathrm{tr.deg}_K \mathcal{Q} = n^2 - 1$; see Proposition 2 and Remark 2). $\square$

4.3. We now turn our attention to the second problem: the unique factorisation property. The determinant d will play a prominent rôle here.

Proposition 3. *The polynomial function d is irreducible in $K[\mathfrak{gl}_n]$.*

Proof. 1. Let $\mathfrak{g} = \mathfrak{gl}_n$ and let P be the maximal parabolic subgroup of $G = \mathrm{GL}_n(K)$ consisting of all invertible matrices (λ_{ij}) with $\lambda_{i1} = 0$ for all $i > 1$. As a first step, we are going to show that d is a semiinvariant for P. We have

$$d = \sum_{\pi \in \mathfrak{S}_n} \mathrm{sgn}(\pi) \partial_{1,\pi(1)}(s_1) \cdots \partial_{1,\pi(n)}(s_n). \tag{4}$$

The adjoint action of G on $\mathfrak{g}$ induces a natural action of G on the Lie algebra $\mathrm{Der}_K K[\mathfrak{g}]$. The subspace $\mathfrak{D}$ of $\mathrm{Der}_K K[\mathfrak{g}]$ consisting of all homogeneous derivations of degree -1 is G-stable and has $\{\partial_{ij} \mid 1 \leqslant i, j \leqslant n\}$ as a basis. We define $\mathfrak{D}_0$ to be the subspace of $\mathfrak{D}$ spanned by all ∂_{1i} with $1 \leqslant i \leqslant n$.

Let $\mathfrak{g}_0^*$ denote the subspace of $\mathfrak{g}^*$ spanned by all $\xi_{i,j}$ with $i > 1$. It is easy to see that $\mathfrak{g}_0^*$ consists of all linear functions ψ on $\mathfrak{g}^*$ with $\psi(e_{1,i}) = 0$ for all i. As the linear span of all $e_{1,i}$ is $(\mathrm{Ad}\, P)$-invariant, $\mathfrak{g}_0^*$ is invariant under the coadjoint action of P on $\mathfrak{g}^*$. As $\mathfrak{D}_0 = \{D \in \mathfrak{D} \mid \mathfrak{g}_0^* \subset \mathrm{Ker}\, D\}$, it follows that $g \circ D \circ g^{-1} \in \mathfrak{D}_0$ for all $D \in \mathfrak{D}_0$ and $g \in P$. Thus P acts on $\mathfrak{D}_0$. We denote by τ the corresponding representation of P.

Let g be any element in P and denote by $A = (a_{ij})$ the matrix of $\tau(g)$ relative to the basis $\{\partial_{1i} \mid 1 \leqslant i \leqslant n\}$ of $\mathfrak{D}_0$. Since each s_i is G-invariant, we have

$$g(\partial_{1j}(s_i)) = (g \circ \partial_{1j} \circ g^{-1})(s_i) = (\tau(g)(\partial_{1j}))(s_i) \quad (1 \leqslant i, j \leqslant n).$$

Combining this with (4) we now obtain

$$g(d) = \sum_{\pi \in \mathfrak{S}_n} \mathrm{sgn}(\pi) (\tau(g)(\partial_{1,\pi(1)}))(s_1) \cdots (\tau(g)(\partial_{1,\pi(n)}))(s_n)$$

$$= \sum_{\pi \in \mathfrak{S}_n} \operatorname{sgn}(\pi) \Big(\sum_k a_{k,\pi(1)} \partial_{1,k}(s_1) \Big) \cdots \Big(\sum_k a_{k,\pi(n)} \partial_{1,k}(s_n) \Big)$$

$$= \det\Big(\Big(\sum_k a_{kj} \partial_{1k}(s_i) \Big)_{ij} \Big) = \det(M \cdot A) = (\det A) d.$$

2. Let B be the Borel subgroup of G consisting of all invertible upper triangular matrices. Clearly, $B \subset P$. Since d is a semiinvariant for P, the Borel subgroup B acts on the line Kd through a rational character, say χ. Let T be as in Section 2.3, a maximal torus of G contained in B. We need to determine the weight of d with respect to T. Note that the maximal unipotent subgroup U^+ of B acts trivially on Kd.

Let $X(T)$ denote the lattice of rational characters of T. For $i \in \{1, \dots, n\}$ we denote by ε_i the rational character $\operatorname{diag}(\lambda_1, \dots, \lambda_n) \mapsto \lambda_i$ of T. It is well known that $X(T)$ is a free $\mathbb{Z}$-module with $\varepsilon_1, \dots, \varepsilon_n$ as a basis, and $\Sigma = \{\varepsilon_i - \varepsilon_j \mid i \neq j\}$ is the set of roots of G with respect to T. For $1 \leqslant i \leqslant n-1$ put $\alpha_i = \varepsilon_i - \varepsilon_{i+1}$. It is well known that Σ is a root system of type A_{n-1} in its $\mathbb{R}$-span in $\mathbb{R} \otimes_{\mathbb{Z}} X(T)$ and, moreover, $\alpha_1, \dots, \alpha_{n-1}$ form the basis of simple roots of Σ relative to B. We denote the corresponding fundamental weights by $\varpi_1, \dots, \varpi_{n-1}$.

From the fact that ξ_{ij} has weight $\varepsilon_j - \varepsilon_i$ relative to T it follows that ∂_{ij} has weight $\varepsilon_i - \varepsilon_j$. This implies that all summands in (4) have the same T-weight $\sum_{i=1}^n (\varepsilon_1 - \varepsilon_i) = n\varepsilon_1 - \sum_{i=1}^n \varepsilon_i$ which is therefore also the T-weight of d. Using Bourbaki's tables it is now easy to observe that $\chi|_T = n\varpi_1$; see [2].

3. Now we will show that d is irreducible. Let $d = f_1^{m_1} \cdots f_r^{m_r}$ be the prime factorisation of d in the factorial ring $K[\mathfrak{g}]$. Since d is homogeneous, so are all f_i. By the uniqueness of prime factorisation, the group B permutes the lines $Kf_1, \dots, Kf_r$. Since B is connected, each f_i is a semiinvariant for B. Let χ_i denote the character of B through which B acts on Kf_i.

Observe that all weights of the G-module $K[\mathfrak{g}]$ are in the root lattice of Σ. Since U^+ fixes f_i, it must be that $\chi_i|_T = \sum_{j=1}^{n-1} k_{i,j} \varpi_j$ where all $k_{i,j}$ are nonnegative integers; see, e.g., [13, Proposition II.2.6]. The prime factorisation of d and the concluding remark in part 2 of this proof yield

$$n\varpi_1 = \sum_{i=1}^r m_i \Big(\sum_{j=1}^{n-1} k_{i,j} \varpi_j \Big) = \sum_{j=1}^{n-1} \Big(\sum_{i=1}^r m_i k_{i,j} \Big) \varpi_j.$$

Since all m_i are strictly positive, we obtain that $n = \sum_{i=1}^r m_i k_{i,1}$ and $k_{i,j} = 0$ for all $j > 1$. Since all $\chi_i|_T = k_{i,1}\varpi_1$ are in the root lattice of Σ, it must be that $n \mid k_{i,1}$ for all i. So there is j such that $k_{j,1} = n$, $m_j = 1$ and $k_{i,1} = 0$ for $i \neq j$. In other words, $d = d_1 d_2$ where d_1 is an *irreducible* semiinvariant for B and d_2 is a *homogeneous* polynomial function on $\mathfrak{g}$ invariant under $TU^+ = B$.

On the other hand, it is well known that $K[\mathfrak{g}]^B = K[\mathfrak{g}]^G$ (this is immediate from the completeness of the flag variety G/B). Hence $d_2 \in K[s_1, \dots, s_n]$. Since $s_i(x_{\mathbf{0}}) = 0$ for all i, Lemma 3 shows that d_2 is a nonzero scalar. We conclude that d is irreducible as desired. $\square$

Corollary. *If $p \nmid n$, then the polynomial function d' is irreducible in $K[\mathfrak{sl}_n]$.*

Proof. Let $G = \mathrm{GL}_n(K)$. The restriction map $K[\mathfrak{gl}_n] \to K[\mathfrak{sl}_n]$ is G-equivariant. As in parts 1 and 2 of the previous proof one proves that d' is a semiinvariant for P of weight $n\varpi_1$. The argument in part 3 then shows that d' is irreducible. □

4.4. We will need a result from Commutative Algebra often referred to as *Nagata's lemma*; see [9, Lemma 19.20], for example. It asserts the following: if x is a prime element of a Noetherian integral domain S such that $S[x^{-1}]$ is factorial, then S is factorial.

Theorem 2. *The centre of $U(\mathfrak{g})$ is a unique factorisation domain.*

Proof. 1. Suppose $\mathfrak{g} = \mathfrak{gl}_n$, where $n \geqslant 2$, and set $d_0 = \eta(\theta(d))$. It is immediate from (3) that $Z[d_0^{-1}]$ is isomorphic to a localisation of a polynomial algebra in $\dim \mathfrak{g}$ variables. Since any localisation of a factorial ring is again factorial, $Z[d_0^{-1}]$ is a unique factorisation domain. We claim that d_0 is a prime element of Z. Our remarks in Sections 2.1 and 2.2 show that $\mathrm{gr}(d_0) = \theta(d^p)$ and that

$$\mathrm{gr}\big(Z/(d_0)\big) \cong S(\mathfrak{g})^{\mathfrak{g}}/\big(\theta\big(d^p\big)\big) \cong K[\mathfrak{g}]^{\mathfrak{g}}/\big(d^p\big).$$

Hence the claim will follow if we establish that $K[\mathfrak{g}]^{\mathfrak{g}}/(d^p)$ has no zero divisors; see Section 2.1 for more detail.

By our remarks in the proof of Proposition 3, the semiinvariant d has weight $\chi|_T = n\varpi_1$ relative to T. So $\chi|_T \notin pX(T)$, for $n\varpi_1$ is indivisible in $X(T)$. It follows that the Lie algebra $\mathfrak{h} = \mathrm{Lie}\, T$ does not annihilate d. As a result, $d \notin K[\mathfrak{g}]^{\mathfrak{g}}$. So Proposition 3 and Lemma 2 yield that d^p is an irreducible element of the factorial ring $K[\mathfrak{g}]^{\mathfrak{g}}$. But then $K[\mathfrak{g}]^{\mathfrak{g}}/(d^p)$ has no zero divisors, as wanted. Thus d_0 is a prime element of Z. Applying Nagata's lemma we finally deduce that Z is factorial in the present case.

2. Suppose $\mathfrak{g} = \mathfrak{sl}_n$ and $p \nmid n$. Then $\mathfrak{g} = [\mathfrak{g}, \mathfrak{g}]$. For the moment we let T denote the group of all diagonal matrices in GL_n and we put $T_0 = T \cap \mathrm{SL}_n(K)$. The restriction homomorphism $X(T) \to X(T_0)$ induces an isomorphism of root systems. We denote the images of the α_i and ϖ_i under this isomorphism by the same symbols. Now the weight lattice of the root system Σ coincides with the character group $X(T_0)$. By the proof of Proposition 3 d' has weight $n\varpi_1$ relative to T_0. Since $p \nmid n$, we have $n\varpi_1 \notin pX(T_0)$. So the Lie algebra $\mathfrak{h}' = \mathrm{Lie}\, T_0$ does not annihilate d', forcing $d' \notin K[\mathfrak{g}]^{\mathfrak{g}}$. In view of Corollary to Proposition 1 this shows that $(d')^p$ is a prime element of the factorial ring $K[\mathfrak{g}]^{\mathfrak{g}}$.

Now set $d_0' = \eta(\theta(d'))$. Repeating the argument from the beginning of part 1 of this proof we now see that d_0' is a prime element of Z. A version of (3) for $\mathfrak{g} = \mathfrak{sl}_n$ with $p \nmid n$ implies that $Z[(d_0')^{-1}]$ is a unique factorisation domain. But then so is Z, by Nagata's lemma, completing the proof. □

5. P-semiinvariants of the coordinate ring of $\mathfrak{gl}_n$

For $G = \mathrm{GL}_n(K)$ we let T, B and P be as in Section 4.3, and we denote by $X^+(T)$ the set of dominant characters of T with respect to B. Let B^- be the Borel subgroup of G that consists of all invertible lower triangular matrices. Given $\lambda \in X^+(T)$ we let $\mathrm{ind}_{B^-}^G(\lambda)$ and $V(\lambda)$ denote the induced module and the Weyl module for G corresponding to λ; see [13] for more detail. In this section we investigate the following problem:

what is the smallest m *for which* $\mathrm{Hom}_{\mathrm{GL}_n}(V(n\varpi_1), S^m(\mathfrak{gl}_n^*)) \neq 0$?

We will solve this problem by using some results of Section 4 and the following result of Donkin which is a modular version of Theorem 11 in [17].

Theorem 3 [7, Theorem 2.2]. *Assume that either* $G = \mathrm{GL}_n(K)$ *for some* $n \geqslant 1$ *or that* G *is almost simple, simply connected and* p *is good. Then* $K[\mathfrak{g}]$ *has a* $(K[\mathfrak{g}]^G, G)$*-module filtration* $0 = A_0 \subseteq A_1 \subseteq \cdots$ *such that for some labelling* $\lambda_1, \lambda_2, \ldots$ *of* $X^+(T)$ *with* $\lambda_i < \lambda_j$ *for* $i < j$*, we have*

$$A_i/A_{i-1} \cong E_i \otimes K[\mathfrak{g}]^G, \quad i \geqslant 1,$$

as $(K[\mathfrak{g}]^G, G)$*-modules, where* E_i *is the direct sum of* $\dim \mathrm{ind}_{B^-}^G(\lambda_i)^T$ *copies of* $\mathrm{ind}_{B^-}^G(\lambda_i)$.

Remark. It is known that $\mathrm{ind}_{B^-}^G(\lambda)^T \neq 0$ if and only if λ is dominant and in the root lattice of G relative to T.

Proposition 4. *Let* G *be as in Theorem* 3. *For every* $\mu \in X^+(T)$*, the weight space* $K[\mathfrak{g}]_\mu^{U^+}$ *of the invariant algebra* $K[\mathfrak{g}]^{U^+}$ *is a free module of rank* $\dim \mathrm{ind}_{B^-}^G(\mu)^T$ *over* $K[\mathfrak{g}]^G$.

Proof. First we make two general observations.

1. Let F be a functor between abelian categories such that the exactness of $0 \to M \to N \to P \to 0$ implies that of $F(M) \to F(N) \to F(P)$. Then F has the following property: if for an object M and a sub-object N we have $F(N) = 0$ and $F(M/N) = 0$, then $F(M) = 0$. For example, the right-derived functors of a left-exact functor have this property. This follows by looking at the long exact homology sequence. If F is left-exact, then F has the following stronger property: if, for M and N as above, $F(M/N) = 0$, then $F(M) = F(N)$.

2. Let M be a finite-dimensional rational G-module. The functors $(M^* \otimes -)$ and $H^m(G, -)$ commute with taking direct limits (over a right-directed preordered index set), and hence so does the functor

$$\mathrm{Ext}_G^m(M, -) \cong \big(N \mapsto H^m(G, M^* \otimes N)\big).$$

In particular this functor commutes with taking direct sums.

Let $0 = A_0 \subseteq A_1 \subseteq \cdots$ and $\lambda_1, \lambda_2, \ldots$ be a filtration of $K[\mathfrak{g}]$ and the indexing of the dominant characters of T as given by Theorem 3. Let j be the index with $\lambda_j = \mu$. We have

$$\mathrm{Hom}_G\big(V(\mu), \mathrm{ind}_{B^-}^G(\lambda)\big) \cong \mathrm{Hom}_G\big(L(\mu), L(\lambda)\big) = 0 \quad \text{for } \lambda \neq \mu.$$

Since the G-module A_i/A_{i-1} is a direct sum of copies of $\mathrm{ind}_{B^-}^G(\lambda_i)$ and $M \mapsto M_\mu^{U^+} \cong \mathrm{Hom}_G(V(\mu), -)$ commutes with taking direct sums we have that $(A_i/A_{i-1})_\mu^{U^+} = 0$ for all $i \neq j$. Using the left-exactness of $M \mapsto M_\mu^{U^+}$ we obtain that $(A_i)_\mu^{U^+} = 0$ for all i with $0 \leqslant i < j$. Furthermore, we obtain that $(A_i)_\mu^{U^+} = (A_j)_\mu^{U^+}$ for all i with $i > j$ and therefore, since the functor $M \mapsto M_\mu^{U^+}$ commutes with direct limits, that $K[\mathfrak{g}]_\mu^{U^+} = (A_j)_\mu^{U^+}$.

By [13, Proposition II.4.13] we have $\mathrm{Ext}_G^1(V(\mu), \mathrm{ind}_{B^-}^G(\lambda)) = 0$ for all $\lambda \in X^+(T)$. It now follows, similarly as above, that $\mathrm{Ext}_G^1(V(\mu), A_i) = 0$ for all $i \geqslant 0$.

Using $(A_{j-1})_\mu^{U^+} = 0$, $\mathrm{Ext}_G^1(V(\mu), A_{j-1}) = 0$ and the exactness of the long homology sequence, we obtain that $K[\mathfrak{g}]_\mu^{U^+} = (A_j)_\mu^{U^+}$ is isomorphic to $(A_j/A_{j-1})_\mu^{U^+} \cong (E_i)_\mu^{U^+} \otimes K[\mathfrak{g}]^G$, and hence is a free $K[\mathfrak{g}]^G$-module of rank $\dim(\mathrm{ind}_{B^-}^G(\mu)^T)$. □

Recall the definition of the element d from Section 4.1.

Corollary 1. *Let $G = \mathrm{GL}_n(K)$ and let r be any nonnegative integer. Then the weight space $K[\mathfrak{g}]_{rn\varpi_1}^{U^+}$ is a free $K[\mathfrak{g}]^G$-module of rank 1 with generator d^r.*

Proof. By the tensor identity we have $\mathrm{ind}_{B^-}^G(rn\varpi_1) \cong \mathrm{ind}_{B^-}^G(rn\varepsilon_1) \otimes K_{\det^{-r}}$, where K_λ denotes K considered as a G-module via λ. Denote for $M = \mathrm{ind}_{B^-}^G(\lambda)$ the formal character $\sum_\mu \dim(M_\mu)e(\mu)$ of M by $\mathrm{ch}(\lambda)$. Then $\mathrm{ch}(rn\varpi_1)e(\det|_T)^r = \mathrm{ch}(rn\varepsilon_1)$.

By [13, II.2.16], all weight spaces of $\mathrm{ind}_{B^-}^G(rn\varepsilon_1)$ are one-dimensional and the weights are the elements $\sum_{i=1}^n a_i\varepsilon_i$ with $a_i \in \mathbb{Z}_+$ and $\sum_{i=1}^n a_i = rn$. Therefore, all weight spaces of $\mathrm{ind}_{B^-}^G(n\varpi_1)$ are one-dimensional and the weights are of the form $\sum_{i=1}^n a_i\varepsilon_i$ where the a_i are integers $\geqslant -r$ and $\sum_{i=1}^n a_i = 0$. So, by the preceding proposition, $K[\mathfrak{gl}_n]_{rn\varpi_1}^{U^+}$ is a free $K[\mathfrak{gl}_n]^{\mathrm{GL}_n}$-module of rank 1. Let f be a generator of this module. Since G acts on $\mathfrak{gl}_n$ by homogeneous automorphisms, f must be homogeneous. Write $d^r = uf$ for some $u \in K[\mathfrak{gl}_n]^G = K[s_1, \ldots, s_n]$. Clearly, u is homogeneous. Applying both sides of this equation to the matrix $x_\mathbf{0}$ from Section 3.1 we see that u must be a nonzero scalar. This shows that d^r generates the module $K[\mathfrak{gl}_n]^G$-module $K[\mathfrak{gl}_n]_{rn\varpi_1}^{U^+}$. □

Corollary 2. *The smallest m for which* $\mathrm{Hom}_{\mathrm{GL}_n}(V(n\varpi_1), S^m(\mathfrak{gl}_n^*)) \neq 0$ *equals* $\deg(d) = n(n-1)/2$.

Corollary 3. *If $f \in K[\mathfrak{gl}_n]$ is a semiinvariant for P, then $f = gd^r$ for some $g \in K[\mathfrak{gl}_n]^G$ and $r \in \mathbb{Z}_+$.*

Proof. Let ψ be the character of P through which P acts on f. Then $\psi|_T$ is dominant, lies in the root lattice of G relative to T, and vanishes on $T \cap (P, P)$. From this it is immediate that $\psi|_T = rn\varpi_1$ for some $r \in \mathbb{Z}_+$. But then f lies in $K[\mathfrak{gl}_n]_{rn\varpi_1}^{U^+}$, and the result follows from Corollary 1 to Proposition 4. □

6. Infinitesimal invariants

Let G and P be as in Section 5, and let χ be the character of P through which P acts on Kd. Put $P_0 = \mathrm{Ker}(\chi)$ and $\mathfrak{p}_0 = \mathrm{Lie}(P_0)$. In this section we will determine the invariant algebra $K[\mathfrak{gl}_n]^{\mathfrak{p}_0}$. For this we will need results from the previous two sections.

Theorem 4. *The invariant algebra $K[\mathfrak{gl}_n]^{\mathfrak{p}_0}$ is a free $K[\mathfrak{gl}_n]^p$-module with basis $\{s_1^{k_1}\cdots s_n^{k_n}\cdot d^{k_{n+1}} \mid 1 \leqslant k_i < p\}$.*

Proof. We have $\dim(\mathfrak{p}_0) = \dim(P) - 1 = n^2 - n$. As in Section 3.4, we are going to apply [20, Corollary 5.3], and we expect that $n+1 = n^2 - c_{\mathfrak{p}_0}(\mathfrak{gl}_n)$, i.e. that

$$n^2 - n - \min_{x\in\mathfrak{g}} \dim \mathfrak{z}_{\mathfrak{p}_0}(x) = c_{\mathfrak{p}_0}(\mathfrak{g}) = n^2 - n - 1.$$

Put differently, we expect that $\min_{x\in\mathfrak{g}} \dim \mathfrak{z}_{\mathfrak{p}_0}(x) = 1$. Now $K\,\mathrm{id} \subseteq \mathfrak{p}_0$, so we need to find an $x \in \mathfrak{gl}_n$ with $\mathfrak{z}_{\mathfrak{p}_0}(x) = K\,\mathrm{id}$. By Section 3.1, the element $x_{\mathbf{0}}$ is regular in $\mathfrak{gl}_n$, and hence $\mathfrak{z}_{\mathfrak{gl}_n}(x_{\mathbf{0}})$ is spanned by $\mathrm{id}, x_{\mathbf{0}}, \ldots, x_{\mathbf{0}}^{n-1}$. Note that the vectors $\{x_{\mathbf{0}}^i(\mathbf{e}_1) \mid 0 \leqslant i < n\}$ form a basis of the column space K^n. Now let $y \in \mathfrak{z}_{\mathfrak{p}_0}(x_{\mathbf{0}})$. Then $y = f(x_{\mathbf{0}})$ for some polynomial $f \in K[X]$ of degree $< n$. Since $y \in \mathfrak{p}_0 \subseteq \mathrm{Lie}(P)$ we must have $f(x_{\mathbf{0}})(\mathbf{e}_1) \in K\mathbf{e}_1$. But then, by the independence of $\mathbf{e}_1, x(\mathbf{e}_1), \ldots, x^{n-1}(\mathbf{e}_1)$, we have $y \in K\,\mathrm{id}$, as wanted.

Let M_J be the Jacobian matrix of $s_1, \ldots, s_n, d$, and let J denote the Jacobian ideal of $s_1, \ldots, s_n, d$. The ideal J is generated by all $(n+1)$-minors of M_J. To apply [20, Corollary 5.3], we need to check that the variety $V(J)$ of common zeros of J has codimension $\geqslant 2$ in $\mathfrak{gl}_n$. Let J_0 be the ideal generated by the elements $\partial_{ij}(d)$. Clearly, $V(J_0) \neq \mathfrak{gl}_n$, for otherwise d would be a pth power contrary to Proposition 3. Also, $V(J_0) \subseteq V(J)$. Furthermore, $V(J) \setminus V(J_0) \subseteq V(d)$, since $d \cdot \partial_{ij}(d) \in J$ for all i, j (to see this one should bear in mind that $\partial_{1i}(d) = 0$ for all $i \leqslant n$). So $V(J) \subseteq V(J_0) \cup V(J + (d))$ and it suffices to prove that $V(J_0)$ and $V(J+(d))$ are of codimension $\geqslant 2$ in $\mathfrak{gl}_n$. Note that, by Euler's formula, we have $\sum_{i,j} \xi_{ij}\partial_{ij}(d) = (n(n-1)/2)d$. So if $n(n-1)/2$ is nonzero in K, then we have $V(J) \subseteq V(d)$. This will not be used in the proof.

First we will show that $V(J_0)$ has no irreducible components of codimension 1. Indeed, suppose the contrary. Then there exists an irreducible regular function f on $\mathfrak{gl}_n$ such that $V(f)$ is an irreducible component of $V(J_0)$. Since the variety $V(J_0)$ is a cone, so is $V(f)$. Since the K-span of all $\partial_{ij}(d)$ is P-stable, the connected group P must stabilise all irreducible components of $V(J_0)$. From this it follows that f is a nonzero homogeneous semiinvariant of P. By Corollary 3 to Proposition 4, $f = gd^r$ for some $r \in \mathbb{Z}_+$ and $g \in K[\mathfrak{gl}_n]^{\mathrm{GL}_n}$. Since f divides all $\partial_{ij}(d)$ we have $\deg f < \deg d$. This yields $r = 0$, implying $f \in K[\mathfrak{gl}_n]^{\mathrm{GL}_n}$. On the other hand, it follows from the Chevalley restriction theorem that f contains a monomial in $\xi_{11}, \ldots, \xi_{nn}$ which contains ξ_{11}. This is a contradiction, since d and all $\partial_{ij}(d)$ are polynomials in ξ_{ij} with $i > 1$.

Now we will prove that $V(J+(d))$ has codimension $\geqslant 2$ in $\mathfrak{gl}_n$. As $V(J+(d)) \subseteq V(d)$ and d is irreducible, it suffices to find a matrix $A \in \mathfrak{gl}_n$ with $d(A) = 0$ and $A \notin V(J)$.

First assume $n = 2$. Then $s_1 = \xi_{11} + \xi_{22}$, $s_2 = \xi_{11}\xi_{22} - \xi_{12}\xi_{21}$,

$$M = \begin{bmatrix} 1 & 0 \\ \xi_{22} & -\xi_{21} \end{bmatrix}, \qquad d = -\xi_{12}, \qquad M_J = \begin{bmatrix} 1 & 0 & 0 & 1 \\ \xi_{22} & -\xi_{21} & -\xi_{12} & \xi_{11} \\ 0 & 0 & -1 & 0 \end{bmatrix},$$

where the variables are taken in the following order: $\xi_{11}, \xi_{12}, \xi_{21}, \xi_{22}$. Then $J = (\xi_{21}, \xi_{11} - \xi_{22})$, hence we can choose A to be the matrix unit $e_{1,1}$ in this case.

Now assume $n \geqslant 3$. Put $\alpha = ((11), \ldots, (n1), (n2))$, and let α_i denote the ith component of α. Set

$$A = e_{n-1,1} + \sum_{i=1}^{n-1} e_{i,i+1}.$$

The columns of M_J are indexed by the pairs (i, j) with $1 \leqslant i, j \leqslant n$. Let M_α be the $(n+1)$-square submatrix of M_J consisting of the columns with indices from α. We will show that $d(A) = 0$ and that the minor $d_\alpha := \det(M_\alpha)$ of M_J is nonzero at A.

Set $\mathcal{X} = \sum_{i,j} \xi_{ij} e_{i,j}$. From the Laplace expansion formulae for the determinant we can deduce the following fact. Let Λ_1 and Λ_2 be subsets of $\{1, \ldots, n\}$ with the same number of elements. Then

$$\partial_{ij}\big(\det(\mathcal{X}_{\Lambda_1,\Lambda_2})\big) = \begin{cases} \pm\det\big(\mathcal{X}_{\Lambda_1\setminus\{i\},\Lambda_2\setminus\{j\}}\big) & \text{when } (i,j) \in (\Lambda_1 \times \Lambda_2), \\ 0 & \text{when } (i,j) \notin (\Lambda_1 \times \Lambda_2). \end{cases}$$

For $k \leqslant n$ we have $s_k = \sum_\Lambda \det(\mathcal{X}_{\Lambda,\Lambda})$ where the sum ranges over all k-subsets Λ of $\{1, \ldots, n\}$. Let $i \in \{1, \ldots, n\}$ and $\Lambda \subseteq \{1, \ldots, n\}$.

Assume $\partial_{i1}(\det(\mathcal{X}_{\Lambda,\Lambda}))$ is nonzero at A. Then we have:

- $1, i \in \Lambda$, and $i \neq n \Rightarrow n \notin \Lambda$;
- $j \in \Lambda \Rightarrow j + 1 \in \Lambda$ for all j with $1 \leqslant j \leqslant n - 1$ and $j \neq i$;
- $j \in \Lambda \Rightarrow j - 1 \in \Lambda$ for all j with $2 \leqslant j \leqslant n$.

It follows that $\Lambda = \{1, \ldots, i\}$. But then $(\partial_{i1} s_k)(A) = \pm\delta_{ik}$ for all $i, k \in \{1, \ldots, n\}$.

Now assume $\partial_{1i}(\det(\mathcal{X}_{\Lambda\Lambda}))$ is nonzero at A. Then we have:

- $1, i \in \Lambda$ and $n \notin \Lambda$;
- $j \in \Lambda \Rightarrow j + 1 \in \Lambda$ for all j with $2 \leqslant j \leqslant n - 2$ and also for $j = n - 1$ if $i = 1$;
- $j \in \Lambda \Rightarrow j - 1 \in \Lambda$ for all j with $2 \leqslant j \leqslant n$ and $j \neq i$.

It follows that $i \neq n$, that Λ is the $(n + 1 - i)$-subset $\{1, i, i + 1, \ldots, n - 1\}$ if $i \neq 1$, and that $\Lambda = \{1\}$ if $i = 1$. But then we have for $i, k \in \{1, \ldots, n\}$ that $(\partial_{1i} s_k)(A) = 0$ if $k = n$ and $(\partial_{1,n+1-i} s_k)(A) = \pm\delta_{ik}$ if $k \neq n$. In view of Eq. (4) in Section 4.3 this shows that $d(A) = 0$. Furthermore,

$$(\partial_{\alpha_i} d)(A) = \left(\sum_{\pi \in \mathfrak{S}_n} \operatorname{sgn}(\pi)\partial_{1,\pi(1)}(s_1) \cdots \partial_{1,\pi(n-1)}(s_{n-1})\partial_{\alpha_i}\partial_{1,\pi(n)}(s_n)\right)(A)$$

$$= \pm\partial_{1,1}(s_1)\cdot\partial_{1,n-1}(s_2)\cdots\partial_{1,n+1-i}(s_i)\cdots\partial_{1,2}(s_{n-1})\cdot\partial_{\alpha_i}\partial_{1,n}(s_n)(A).$$

So $(\partial_{\alpha_i}d)(A) = \pm(\partial_{\alpha_i}\partial_{1,n}s_n)(A) = \pm(\partial_{\alpha_i}\det(\mathcal{X}_{\underline{n}\setminus\{1\},\underline{n}\setminus\{n\}}))(A)$ where $\underline{n}$ denotes the set $\{1,\dots,n\}$. This is 0 if $i=1$, as $\mathcal{X}_{\underline{n}\setminus\{1\},\underline{n}\setminus\{n\}}$ does not contain ξ_{11}. For $i\in\{2,\dots,n\}$ the RHS equals $\pm\det(\mathcal{X}_{\underline{n}\setminus\{1,i\},\underline{n}\setminus\{1,n\}})(A)$, which is 0 as the first column of $\mathcal{X}_{\underline{n}\setminus\{1,i\},\underline{n}\setminus\{1,n\}}(A)$ is zero (we are assuming that $n\geqslant 3$). Finally,

$$(\partial_{\alpha_{n+1}}d)(A) = (\partial_{2,n}d)(A) = \pm\big(\det\big(\mathcal{X}_{\underline{n}\setminus\{1,n\},\underline{n}\setminus\{2,n\}}\big)\big)(A) = \pm 1.$$

We conclude that for $1\leqslant j\leqslant n$ the jth column of $M_\alpha(A)$ has ±1 at its jth position and zeros elsewhere, and that the last column of $M_\alpha(A)$ has ±1 at its last position. Hence $d_\alpha(A)=\pm1$, implying that $V(J)$ has codimension $\geqslant 2$ in $\mathfrak{gl}_n$. We have thus checked that the action of $\mathfrak{p}_0$ on $K[\mathfrak{gl}_n]$ satisfies the conditions of Corollary 5.3 in [20]. The result follows. □

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