All-order α'-expansion of one-loop open-string integrals

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We present a new method to evaluate the α'-expansion of genus-one integrals over open-string punctures and unravel the structure of the elliptic multiple zeta values in its coefficients. This is done by obtaining a simple differential equation of Knizhnik–Zamolodchikov–Bernard-type satisfied by generating functions of such integrals, and solving it via Picard iteration. The initial condition involves the generating functions at the cusp $\tau \to i\infty$ and can be reduced to genus-zero integrals.

INTRODUCTION

Elliptic analogues of polylogarithms [1, 2] and multiple zeta values [3] have become a driving force in higher-order computations of scattering amplitudes in quantum field theories and string theories. The study of their differential equations and their connections with modular forms turned into a vibrant research area at the interface of particle phenomenology, string theory and number theory. In the same way as a variety of Feynman integrals has been recently expressed in terms of elliptic polylogarithms and iterated integrals of modular forms [4, 5], the low-energy expansion of one-loop open-string amplitudes introduces elliptic multiple zeta values (eMZVs) [6–8].

So far, the appearance of eMZVs in one-loop open-string amplitudes arose from direct integration over the punctures on a genus-one worldsheet of cylinder or Möbius-strip topology. Although there is no conceptual bottleneck in extending the techniques of [6–8] to arbitrary multiplicities and orders in the inverse string tension α', in this letter we will present a new method to evaluate these genus-one integrals which is related to elliptic associators [9] and Tsunogai's derivations dual to Eisenstein series [10]. The results are given by eMZVs in their minimal form [3, 11] and reveal elegant structures in the α'-expansions. More details will be given in a longer companion paper [12].

OPEN-STRING INTEGRALS AT GENUS ONE

One-loop string amplitudes are described by correlation functions of vertex operators in a conformal field theory over a genus-one Riemann surface, the torus. The

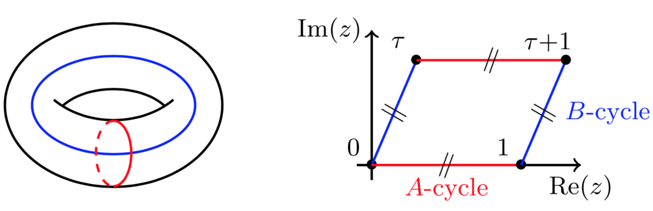

FIG. 1: We parameterize the torus through the lattice $\frac{\mathbb{C}}{\mathbb{Z}+\tau\mathbb{Z}}$ with identifications $z \cong z+1 \cong z+\tau$ along the A- and B-cycle.

location of the vertex operator associated with the $j^{\rm th}$ external string state is parameterized by the coordinates $z_j = u_j\tau + v_j$ with $u_j, v_j \in (0,1)$, where τ is the modulus with $\operatorname{Im}\tau > 0$, see figure 1, and we define $z_{ij} \equiv z_i - z_j$.

By suitable involutions of the torus [13], one obtains the surfaces describing the scattering of open-string states, the cylinder and the Möbius strip. The two boundaries of the cylinder will be parameterized by the A-cycle $z_j \in (0,1)$ and its displacement $z_j \in \frac{\tau}{2}+(0,1)$ by half a B-cycle, i.e. $u_j \in \{0, \frac{1}{2}\}$ and $\mathrm{d}z_j = \mathrm{d}v_j$. See figure 2.

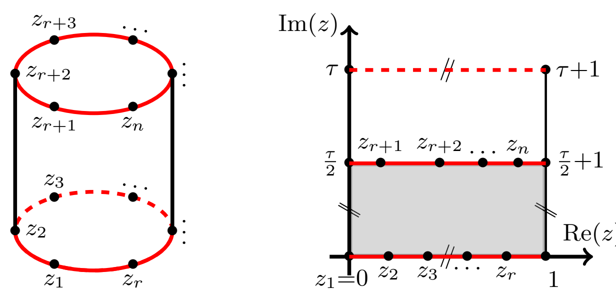

FIG. 2: The cylinder parameterization.

The massless n-point one-loop amplitudes of the open superstring give rise to integrals of the form $(z_1 = 0)$ [6]

$$\int_{\mathcal{C}(*)} \Big(\prod_{j=2}^{n} \mathrm{d}z_j \Big) \, f^{(k_1)}_{i_1 j_1} f^{(k_2)}_{i_2 j_2} \cdots \exp\Big(\sum_{i<j}^{n} s_{ij} \mathcal{G}(z_{ij}, \tau) \Big) , \quad (1)$$

with differing integration domains $\mathcal{C}(*)$ for the cylinder and the Möbius strips. For planar cylinders, we set $* \to 1, 2, \ldots, n$ and parametrize the domain as

$$\mathcal{C}(1,2,\ldots,n) = \{ z_{j=2,\ldots,n} \in \mathbb{R},\ 0 < z_2 < \ldots < z_n < 1 \} , \quad (2)$$

see figure 2 and [12] for the non-planar analogue with $* \to {}^{r+1,\ldots,n}_{1,2,3,\ldots,r}$. Furthermore, in the integrand of (1), $f^{(k)}_{ij} \equiv f^{(k)}(z_{ij}, \tau)$ denote the Laurent coefficients of the doubly-periodic Kronecker–Eisenstein series defined by [2, 14]

$$\Omega(z,\eta,\tau) = \exp\Big(2\pi i \eta \frac{\operatorname{Im} z}{\operatorname{Im}\tau} \Big) \frac{\theta_1'(0,\tau)\theta_1(z+\eta,\tau)}{\theta_1(z,\tau)\theta_1(\eta,\tau)} , \quad (3)$$

$$\Omega(z,\eta,\tau) = \sum_{k=0}^{\infty} \eta^{k-1} f^{(k)}(z,\tau) . \quad (4)$$

The simplest examples of the coefficient functions are $f^{(0)}(z,\tau)=1$ and $f^{(1)}(z,\tau)=\partial_z \log\theta_1(z,\tau)+2\pi i \frac{\operatorname{Im} z}{\operatorname{Im}\tau}$, and higher $f^{(k\geq 2)}(z,\tau)$ do not have any poles in z.

Finally, $\exp\big(\sum_{i<j}^n s_{ij}\mathcal{G}(z_{ij},\tau)\big)$ in (1) is the Koba–Nielsen factor written in terms of dimensionless Mandelstam invariants $s_{ij}=-2\alpha' k_i\cdot k_j$ and Green functions $\mathcal{G}(z,\tau)$ subject to the universal differential equation

$$\begin{aligned}\partial_{v_i}\mathcal{G}(z_{ij},\tau)&=-f^{(1)}(z_{ij},\tau) \\ 2\pi i\partial_\tau \mathcal{G}(z_{ij},\tau)&=-f^{(2)}(z_{ij},\tau)-2\zeta_2\,,\end{aligned}\tag{5}$$

where ∂_{v_i} is the derivative along the cylinder boundary, and $\zeta_n=\sum_{k=1}^\infty \frac{1}{k^n}$ with $n\geq 2$ denote Riemann zeta values.

A. Generating functions: Instead of handling the α'-expansion of the individual integrals (1) as in the method of [6–8], we will evaluate the following generating function of integrals (with $\eta_{23\ldots n}=\eta_2+\eta_3+\ldots+\eta_n$)

$$\begin{aligned}Z^\tau_{\vec\eta}(*|1,2,\ldots,n)&=\int_{\mathcal{C}(*)}\prod_{j=2}^n \mathrm{d}z_j \exp\Big(\sum_{i<j}^n s_{ij}\mathcal{G}(z_{ij},\tau)\Big) \\ &\times \Omega(z_{12},\eta_{23\ldots n},\tau)\Omega(z_{23},\eta_{3\ldots n},\tau)\ldots\Omega(z_{n-1,n},\eta_n,\tau)\,.\end{aligned}\tag{6}$$

The integrands $f^{(k_1)}_{i_1j_1}f^{(k_2)}_{i_2j_2}\cdots$ in (1) relevant to n-point open-superstring amplitudes have $k_1+k_2+\ldots=n-4$ and reside at the order of η_j^{-3} of (6). Moreover, ($n\geq 8$)-point integrands additionally involve holomorphic Eisenstein series $\mathrm{G}_{\ell\geq 4}(\tau)=-f^{(\ell)}(0,\tau)$ [6] multiplying (1) at $k_1+k_2+\ldots=n-4-\ell$ as seen at the $\eta_j^{-3-\ell}$-order of (6).

Although the cylinder contribution to one-loop open-string amplitudes is localized at purely imaginary τ as drawn in figure 2, we will define and evaluate the integrals (6) for generic τ in the upper half plane with $\operatorname{Re}\tau\neq 0$. In view of the parental torus, $Z^\tau_{\vec\eta}(1,2,\ldots,n|\cdot)$ and $Z^\tau_{\vec\eta}(\begin{smallmatrix} r+1,\ldots,n \\ 1,2,3,\ldots,r\end{smallmatrix}|\cdot)$ will be referred to as planar and non-planar *A-cycle integrals*, respectively.

Möbius-strip integrals can be reconstructed by specializing planar A-cycle integrals to $\operatorname{Re}\tau=\frac{1}{2}$, and the cancellation of tadpole divergences from one-loop open-superstring amplitudes can be analyzed as in [15].

The A-cycle integrand (6) at n points involves $n-1$ factors of the Kronecker–Eisenstein series (4) at different arguments. The second entry $Z^\tau_{\vec\eta}(*|A)$ specifies permutations $A=a_1a_2\ldots a_n\in S_n$ of these arguments, and $\Omega(\ldots)$ at different z_{a_j},η_{a_j} are related by the Fay identity

$$\begin{aligned}\Omega(z_1,\eta_1,\tau)\Omega(z_2,\eta_2,\tau)&=\Omega(z_1,\eta_1+\eta_2,\tau)\Omega(z_2-z_1,\eta_2,\tau) \\ &+\Omega(z_2,\eta_1+\eta_2,\tau)\Omega(z_1-z_2,\eta_1,\tau)\,.\end{aligned}\tag{7}$$

Repeated use of (7) and imposing $\eta_1=-\sum_{j=2}^n\eta_j$ only leaves $(n-1)!$ independent permutations of the integrand in (6), and we will use a basis of $Z^\tau_{\vec\eta}(*|1,B)$ with permutations $B\in S_{n-1}$ acting on $2,3,\ldots,n$.

B. The differential equation: As will be derived in [12], the τ-derivatives of (6) can be written as

$$2\pi i\partial_\tau Z^\tau_{\vec\eta}(A|1,B)=\sum_{C\in S_{n-1}} D^\tau_{\vec\eta}(B|C)Z^\tau_{\vec\eta}(A|1,C)\,,\tag{8}$$

where the $(n-1)!\times(n-1)!$ matrix $D^\tau_{\vec\eta}$ is a differential operator w.r.t. η_j. Its detailed form will be exemplified in the next section and follows from the properties (5) of the Green function, the vanishing of boundary terms $\int \mathrm{d}v_j\,\partial_{v_j}(\ldots)$ and the mixed heat equation ($u,v\in\mathbb{R}$)

$$2\pi i\partial_\tau\Omega(u\tau+v,\eta,\tau)=\partial_v\partial_\eta\Omega(u\tau+v,\eta,\tau)\,.\tag{9}$$

Most importantly, the form of $D^\tau_{\vec\eta}(B|C)$ does not depend on the planar or non-planar integration cycle A, and its entries are linear in the dimensionless Mandelstam invariants s_{ij} and therefore in α'.

Hence, the α'-expansion of the genus-one integrals $Z^\tau_{\vec\eta}$ follows from the solution of (8) via Picard iteration,

$$\begin{aligned}Z^\tau_{\vec\eta}(A|1,B)&=\sum_{k=0}^\infty\Big(\frac{1}{2\pi i}\Big)^k\int_{i\infty}^\tau \mathrm{d}\tau_1\int_{i\infty}^{\tau_1}\mathrm{d}\tau_2\ldots\int_{i\infty}^{\tau_{k-1}}\mathrm{d}\tau_k \\ &\times\sum_{C\in S_{n-1}}(D^{\tau_k}_{\vec\eta}\cdot\ldots\cdot D^{\tau_2}_{\vec\eta}\cdot D^{\tau_1}_{\vec\eta})(B|C)\,Z^{i\infty}_{\vec\eta}(A|1,C)\end{aligned}\tag{10}$$

with matrix products $D^{\tau_k}_{\vec\eta}\cdot\ldots\cdot D^{\tau_2}_{\vec\eta}D^{\tau_1}_{\vec\eta}$. As an initial value, the degeneration $Z^{i\infty}_{\vec\eta}$ at the cusp $\tau\to i\infty$ will be expressed in terms of disk integrals with two additional punctures from the pinching of the A-cycle in figure 1.

As will be detailed in [12], the entire τ-dependence of $D^\tau_{\vec\eta}$ is carried by Weierstrass functions (with $\mathrm{G}_0=-1$)

$$\wp(\eta,\tau)=-\frac{\mathrm{G}_0}{\eta^2}+\sum_{k=4}^\infty(k-1)\eta^{k-2}\mathrm{G}_k(\tau)\,.\tag{11}$$

This allows us to decompose

$$D^\tau_{\vec\eta}=\sum_{k=0}^\infty(1-k)\mathrm{G}_k(\tau)r_{\vec\eta}(\epsilon_k)\,,\tag{12}$$

where $r_{\vec\eta}(\epsilon_k)$ are $(n-1)!\times(n-1)!$ matrices whose entries are independent of τ, rational functions of η_j, linear in s_{ij} and may involve second derivatives $\partial_{\eta_i}\partial_{\eta_j}$. Note that $r_{\vec\eta}(\epsilon_2)=0$ and $r_{\vec\eta}(\epsilon_{2p-1})=0\ \forall\ p\in\mathbb{N}$ by (11).

C. The main result: Based on (12), the open-string integrals (10) can be expressed in terms of iterated Eisenstein integrals

$$\gamma(k_1,k_2,\ldots,k_r|\tau)=\int_\tau^{i\infty}\frac{\mathrm{d}\tau'}{2\pi i}\,\mathrm{G}_{k_r}(\tau')\,\gamma(k_1,\ldots,k_{r-1}|\tau')\tag{13}$$

subject to $\gamma(\emptyset|\tau)=1$ and tangential-base-point regularization [16], e.g. $\gamma(0|\tau)=\frac{\tau}{2\pi i}$. As the main result of this work, we can therefore bring the open-string α'-expansion into the following elegant form:

$$\begin{aligned}Z^\tau_{\vec\eta}(A|1,B)&=\sum_{r=0}^\infty\sum_{\substack{k_1,k_2,\ldots,k_r\\=0,4,6,8,\ldots}}\gamma(k_1,k_2,\ldots,k_r|\tau) \\ &\times\prod_{j=1}^r(k_j-1)\sum_{C\in S_{n-1}}r_{\vec\eta}(\epsilon_{k_r}\ldots\epsilon_{k_2}\epsilon_{k_1})_B{}^C Z^{i\infty}_{\vec\eta}(A|1,C)\,,\end{aligned}\tag{14}$$

where $r_{\vec\eta}(\epsilon_{k_r}\ldots\epsilon_{k_2}\epsilon_{k_1}) \equiv r_{\vec\eta}(\epsilon_{k_r})\cdots r_{\vec\eta}(\epsilon_{k_2}) r_{\vec\eta}(\epsilon_{k_1})$. Since each order in α' is expressible in terms of eMZVs [6–8], the $r_{\vec\eta}(\epsilon_k)$ should be matrix representations of Tsunogai's derivations ϵ_k dual to Eisenstein series [10]. In particular, (12) brings the differential equation (8) of $Z^\tau_{\vec\eta}$ into the same form as that of the elliptic Knizhnik–Zamolodchikov–Bernard associator [9], where the derivations ϵ_k act on its non-commutative arguments.

The decomposition of eMZVs into iterated Eisenstein integrals automatically incorporates all their relations over the rational numbers [11]. Moreover, the derivation of (14) does not rely on any relation among the Mandelstam invariants. The n-point results of this work are valid for $\frac{1}{2}n(n-1)$ independent s_{ij}, and one can still impose momentum conservation when applying the α'-expansion of $Z^\tau_{\vec\eta}$ to string amplitudes.

EXAMPLES FOR DIFFERENTIAL OPERATORS

In this section, we present $(n<4)$-point examples of the matrix-valued differential operators $D^\tau_{\vec\eta}$ in (8), and the four-point case is relegated to the appendix. All-multiplicity expressions as well as detailed derivations of the differential equations can be found in [12].

A. Two points allow for a single planar and non-planar A-cycle integral (6) each,

$$Z^\tau_{\eta_2}(1,2|1,2) = \int_0^1 \mathrm{d}v_2\, \Omega(v_{12},\eta_2,\tau)\, e^{s_{12}\mathcal{G}(v_{12},\tau)} \tag{15}$$
$$Z^\tau_{\eta_2}\big(\begin{smallmatrix}2\\1\end{smallmatrix}|1,2\big) = \int_0^1 \mathrm{d}v_2\, \Omega(v_{12}+\tfrac{\tau}{2},\eta_2,\tau)\, e^{s_{12}\mathcal{G}(v_{12}+\frac{\tau}{2},\tau)}\,.$$

Their τ-derivatives resulting from (5), (9) and integration by parts w.r.t. v_2 take the universal form

$$2\pi i\partial_\tau Z^\tau_{\eta_2}(*|1,2) = s_{12}\Big(\frac{1}{2}\partial^2_{\eta_2} - \wp(\eta_2,\tau) - 2\zeta_2\Big) Z^\tau_{\eta_2}(*|1,2)\,, \tag{16}$$

so one can read off the scalar differential operator in (8) and the resulting representation of the derivations,

$$D^\tau_{\eta_2}(2|2) = s_{12}\Big(\frac{1}{2}\partial^2_{\eta_2} - \wp(\eta_2,\tau) - 2\zeta_2\Big)\,, \tag{17}$$
$$r_{\eta_2}(\epsilon_0) = s_{12}\Big(\frac{1}{\eta_2^2} + 2\zeta_2 - \frac{1}{2}\partial^2_{\eta_2}\Big)\,,\quad r_{\eta_2}(\epsilon_{k\geq 4}) = s_{12}\eta_2^{k-2}\,.$$

Note that various combinations of iterated Eisenstein integrals drop out from the two-point instance of (14) since commutators $[r_{\eta_2}(\epsilon_{k_1}), r_{\eta_2}(\epsilon_{k_2})]$ with $k_1,k_2\geq 4$ vanish.

B. Three points give rise to A-cycle integrals

$$Z^\tau_{\eta_2,\eta_3}(*|1,2,3) = \int_{\mathcal{C}(*)} \mathrm{d}z_2\,\mathrm{d}z_3\, \Omega(z_{12},\eta_2+\eta_3,\tau) \tag{18}$$
$$\times\,\Omega(z_{23},\eta_3,\tau) e^{s_{12}\mathcal{G}(z_{12},\tau)+s_{13}\mathcal{G}(z_{13},\tau)+s_{23}\mathcal{G}(z_{23},\tau)}$$

that mix under τ-derivatives ($s_{12\ldots p}\equiv\sum_{1\leq i<j}^p s_{ij}$),

$$\begin{aligned}2\pi i\partial_\tau Z^\tau_{\eta_2,\eta_3}(*|1,2,3) = \Big(& -2\zeta_2 s_{123} \\ &+ s_{12}\big[\tfrac{1}{2}\partial^2_{\eta_2} - \wp(\eta_2+\eta_3,\tau)\big] + s_{13}\big[\tfrac{1}{2}\partial^2_{\eta_3} - \wp(\eta_3,\tau)\big] \\ &+ s_{23}\big[\tfrac{1}{2}(\partial_{\eta_2}-\partial_{\eta_3})^2 - \wp(\eta_3,\tau)\big]\Big) Z^\tau_{\eta_2,\eta_3}(*|1,2,3) \\ &+ s_{13}\big[\wp(\eta_2+\eta_3,\tau) - \wp(\eta_3,\tau)\big] Z^\tau_{\eta_2,\eta_3}(*|1,3,2)\,.\end{aligned} \tag{19}$$

The resulting matrix entries of the 2×2 differential operator in (8) read

$$\begin{aligned} D^\tau_{\eta_2,\eta_3}(2,3|2,3) &= -2\zeta_2 s_{123} + s_{12}\big[\tfrac{1}{2}\partial^2_{\eta_2} - \wp(\eta_2+\eta_3,\tau)\big] \\ &+ s_{23}\big[\tfrac{1}{2}(\partial_{\eta_2}-\partial_{\eta_3})^2 - \wp(\eta_3,\tau)\big] + s_{13}\big[\tfrac{1}{2}\partial^2_{\eta_3} - \wp(\eta_3,\tau)\big] \\ D^\tau_{\eta_2,\eta_3}(2,3|3,2) &= s_{13}\big[\wp(\eta_2+\eta_3,\tau) - \wp(\eta_3,\tau)\big]\,, \end{aligned} \tag{20}$$

and the first row is always sufficient to generate the remaining entries via permutations of s_{ij} and η_j, e.g.

$$\begin{aligned} D^\tau_{\eta_2,\eta_3}(3,2|3,2) &= D^\tau_{\eta_2,\eta_3}(2,3|2,3)\,\big|^{s_{12}\leftrightarrow s_{13}}_{\eta_2\leftrightarrow\eta_3} \\ D^\tau_{\eta_2,\eta_3}(3,2|2,3) &= D^\tau_{\eta_2,\eta_3}(2,3|3,2)\,\big|^{s_{12}\leftrightarrow s_{13}}_{\eta_2\leftrightarrow\eta_3}\,. \end{aligned} \tag{21}$$

One can read off the 2×2 matrix representations of the derivations ($k\neq 2$),

$$\begin{aligned} r_{\eta_2,\eta_3}(\epsilon_k) = \delta_{k,0}\Big(& 2\zeta_2 s_{123} - \frac{1}{2}s_{23}(\partial_{\eta_2}-\partial_{\eta_3})^2 - \frac{1}{2}s_{12}\partial^2_{\eta_2} \\ & - \frac{1}{2}s_{13}\partial^2_{\eta_3}\Big) 1_{2\times 2} + \eta_{23}^{k-2}\begin{pmatrix} s_{12} & -s_{13} \\ -s_{12} & s_{13}\end{pmatrix} \\ & + \eta_2^{k-2}\begin{pmatrix} 0 & 0 \\ s_{12} & s_{12}+s_{23}\end{pmatrix} + \eta_3^{k-2}\begin{pmatrix} s_{13}+s_{23} & s_{13} \\ 0 & 0\end{pmatrix}\,, \end{aligned} \tag{22}$$

where $[r_{\eta_2,\eta_3}(\epsilon_{k_1\geq 4}), r_{\eta_2,\eta_3}(\epsilon_{k_2\geq 4})]$ no longer vanish individually, and relations in the derivation algebra [10, 11, 17] hold non-trivially.

EXAMPLES FOR INITIAL VALUES

This section is dedicated to the degeneration of A-cycle integrals (6) at the cusp $\tau\to i\infty$ which enters the α'-expansion (14) as an initial value.

A. Generalities: The behaviour of A-cycle integrals at the cusp is most conveniently studied in the variables

$$\sigma_j = e^{2\pi i z_j}\,,\quad \mathrm{d}z_j = \frac{\mathrm{d}\sigma_j}{2\pi i\sigma_j}\,,\quad G_{ij} = 2\pi i\frac{\sigma_i+\sigma_j}{\sigma_i-\sigma_j}\,, \tag{23}$$

where the planar Green function and Kronecker–Eisenstein series degenerate to ($\sigma_{ji}\equiv\sigma_j-\sigma_i$)

$$\begin{aligned} \lim_{\tau\to i\infty}\Omega(v_{ij},\eta,\tau) &= \pi\cot(\pi\eta) + G_{ij} \\ \lim_{\tau\to i\infty}\mathcal{G}(v_{ij},\tau) &= \frac{1}{2}\log(\sigma_i) + \frac{1}{2}\log(\sigma_j) - \log(\sigma_{ji})\,. \end{aligned} \tag{24}$$

Their non-planar analogues take an even simpler form,

$$\lim_{\tau\to i\infty}\Omega(v_{ij}+\tfrac{\tau}{2},\eta,\tau) = \frac{\pi}{\sin(\pi\eta)}\,,\quad \lim_{\tau\to i\infty}\mathcal{G}(v_{ij}+\tfrac{\tau}{2},\tau) = 0\,. \tag{25}$$

Since string-theory applications of (14) involve the coefficients w.r.t. η_j, we will need the expansions

$$\pi\cot(\pi\eta) = \frac{1}{\eta} - 2\sum_{k=1}^{\infty}\zeta_{2k}\eta^{2k-1} \qquad (26)$$
$$\frac{\pi}{\sin(\pi\eta)} = \frac{1}{\eta} + \sum_{k=1}^{\infty}\frac{2^{2k-1}-1}{2^{2k-2}}\zeta_{2k}\eta^{2k-1}\,.$$

As will be detailed in [12], the σ_j-integration in n-point $Z^{i\infty}_{\vec\eta}$ lines up with explicitly known combinations of $N=(n+2)$-point disk integrals [18]

$$Z^{\rm tree}(a_1,a_2,\ldots,a_N|1,2,\ldots,N) = \int_{-\infty<\sigma_{a_1}<\sigma_{a_2}<\ldots<\sigma_{a_N}<\infty} \frac{\mathrm{d}\sigma_1\,\mathrm{d}\sigma_2\,\ldots\,\mathrm{d}\sigma_N}{\mathrm{vol}\,\mathrm{SL}_2(\mathbb{R})}\,\frac{\prod_{i<j}^N|\sigma_{ij}|^{-s_{ij}}}{\sigma_{12}\sigma_{23}\ldots\sigma_{N-1,N}\sigma_{N,1}}\,. \qquad (27)$$

The two extra punctures $n{+}1\to +$ and $n{+}2\to -$ are associated with Mandelstam invariants

$$s_{j+}=s_{j-}=-\frac{1}{2}\sum_{1\leq i\neq j}^{n}s_{ij}\,,\qquad s_{+,-}=\sum_{1\leq i<j}^{n}s_{ij}\,. \qquad (28)$$

The α'-expansion of (27) and therefore $Z^{i\infty}_{\vec\eta}$ involves multiple zeta values (MZVs) which can be systematically generated from the all-multiplicity methods of [19, 20].

B. Two points: Planar initial values at two points descend from four-point tree-level integrals,

$$Z^{i\infty}_{\eta_2}(1,2|1,2) = \pi\cot(\pi\eta_2)\,2i\sin\Big(\frac{\pi s_{12}}{2}\Big) \times\int_0^1\frac{\mathrm{d}\sigma_2}{2\pi i\sigma_2}\,\sigma_2^{s_{12}/2}(1-\sigma_2)^{-s_{12}} \qquad (29)$$
$$= \pi\cot(\pi\eta_2)\frac{\Gamma(1-s_{12})}{\big[\Gamma(1-\frac{s_{12}}{2})\big]^2}\,.$$

The factor of $2i\sin(\frac{\pi s_{12}}{2})$ and similar trigonometric functions below stem from contour deformations detailed in [12]. The gamma functions with standard α'-expansion

$$\frac{\Gamma(1-s_{12})}{\big[\Gamma(1-\frac{s_{12}}{2})\big]^2} = \exp\Big(\sum_{k=2}^{\infty}\frac{\zeta_k}{k}(1-2^{1-k})s_{12}^k\Big) \qquad (30)$$
$$= 1+\frac{1}{4}s_{12}^2\zeta_2+\frac{1}{4}s_{12}^3\zeta_3+\frac{19}{160}s_{12}^4\zeta_2^2+\mathcal{O}(\alpha'^5)$$

do not appear in the non-planar counterpart of (29)

$$Z^{i\infty}_{\eta_2}\big(\begin{smallmatrix}2\\1\end{smallmatrix}|1,2\big) = \frac{\pi}{\sin(\pi\eta_2)}\,. \qquad (31)$$

C. Three points: Degenerate A-cycle integrals at three points introduce five-point disk integrals,

$$Z^{i\infty}_{\eta_2,\eta_3}(1,a_2,a_3|1,2,3) \qquad (32)$$
$$= \pi^2\Big(\cot(\pi\eta_{23})\cot(\pi\eta_3)+\frac{s_{13}}{s_{123}}\Big)I^{\rm tree}(1,a_2,a_3|1)$$
$$+\pi\Big(\cot(\pi\eta_{23})+\frac{s_{23}}{s_{12}}\cot(\pi\eta_3)\Big)I^{\rm tree}(1,a_2,a_3|G_{23})\,,$$

where

$$I^{\rm tree}(1,a_2,a_3|1) = -\frac{1}{2\pi^2}\Big[\sin\Big(\frac{\pi}{2}(s_{1a_2}{+}s_{23})\Big)\sin\Big(\frac{\pi}{2}s_{1a_3}\Big) \times\Big(Z^{\rm tree}(+,a_2,a_3,1,-|+,2,3,-,1) + Z^{\rm tree}(+,a_2,a_3,1,-|+,3,2,-,1)\Big)+(2\leftrightarrow 3)\Big]$$
$$I^{\rm tree}(1,a_2,a_3|G_{23}) = \frac{1}{2\pi}\Big[\sin\Big(\frac{\pi}{2}(s_{1a_2}{+}s_{23})\Big)\cos\Big(\frac{\pi}{2}s_{1a_3}\Big) \times\Big(Z^{\rm tree}(+,a_2,a_3,1,-|+,2,3,-,1) - Z^{\rm tree}(+,a_2,a_3,1,-|+,3,2,-,1)\Big)+(2\leftrightarrow 3)\Big]\,. \qquad (33)$$

Their leading low-energy orders read [12]

$$I^{\rm tree}(1,2,3|1) = \frac{1}{2}+\frac{\zeta_2}{8}(s_{12}^2{+}s_{13}^2{+}s_{23}^2)+\mathcal{O}(\alpha'^3) \qquad (34)$$
$$I^{\rm tree}(1,2,3|G_{23}) = \frac{1}{s_{23}}+\frac{\zeta_2}{4s_{23}}(s_{12}{+}s_{13}{+}s_{23})^2+\mathcal{O}(\alpha'^2)$$

and exemplify that integrals over k factors of G_{ij} in (23) may have up to k kinematic poles.

Non-planar three-point initial values in turn boil down to four-point disk integrals with α'-expansions in (30),

$$Z^{i\infty}_{\eta_2,\eta_3}\big(\begin{smallmatrix}3\\1,2\end{smallmatrix}|1,2,3\big) = \frac{\pi^2\cot(\pi\eta_{23})}{\sin(\pi\eta_3)}\frac{\Gamma(1-s_{12})}{\big[\Gamma(1-\frac{s_{12}}{2})\big]^2} \qquad (35)$$
$$Z^{i\infty}_{\eta_2,\eta_3}\big(\begin{smallmatrix}3\\1,2\end{smallmatrix}|1,3,2\big) = \frac{\pi^2}{\sin(\pi\eta_{23})\sin(\pi\eta_3)}\frac{\Gamma(1-s_{12})}{\big[\Gamma(1-\frac{s_{12}}{2})\big]^2}\,.$$

CONCLUSIONS AND FURTHER DIRECTIONS

In this letter we presented a method to expand a generating series of genus-one integrals (6) relevant to one-loop open-string amplitudes. At each order in the inverse string tension α', our main result (14) pinpoints the accompanying eMZVs in their minimal and canonical representation via iterated Eisenstein integrals.

Genus-zero integrals relevant to open-string tree amplitudes obey Knizhnik–Zamolodchikov equations with a characteristic linear factor of α' on their right-hand side [19]. This structure is analogous to the ε-form of differential equations among Feynman integrals with dimensional-regularization parameter ε [5, 21], suggesting a correspondence between α' and ε. By the linearity of the differential operators $D^\tau_{\vec\eta}$ in $s_{ij}=-2\alpha' k_i\cdot k_j$, the Knizhnik–Zamolodchikov–Bernard-type equation (8) also becomes linear in α'. So our results generalize this intriguing correspondence to genus one and provide the string-theory analogue of the ε-form for differential equations of elliptic Feynman integrals [5].

The generating functions $Z^\tau_{\vec\eta}$ are expected to comprise any moduli-space integral in massless one-loop amplitudes of open bosonic strings and superstrings upon expansion in η_j. Accordingly, they are proposed to generalize the universal disk-integrals (27) that appear in the double-copy representation of string tree-level amplitudes [18, 22]. Hence, the study of the genus-one integrals $Z^\tau_{\vec\eta}$ is an essential step towards universal double-copy structures in one-loop amplitudes of different string theories that generalize those of the superstring [23].

The generating functions $Z^\tau_{\vec\eta}$ can be adapted to a closed-string context, encoding the integrals over torus punctures in one-loop amplitudes of type-II, heterotic and closed bosonic string theories. Closed-string analogues of $Z^\tau_{\vec\eta}$ will be shown [24] to obey similar differential equations and to shed new light on the properties of modular graph forms [25] including their relation with open-string amplitudes [26].

Moreover, the method of this work to infer moduli-space integrals from differential equations should be applicable at higher loops. In the same way as disk integrals were used as the initial value for our one-loop results, higher-genus integrals in string amplitudes are expected to obey differential equations w.r.t. complex-structure moduli such that their separating and non-separating degenerations set the initial conditions. It would be interesting to explore a differential-equation approach of this type to the higher-genus modular graph functions of [27].

In summary, our new approach to one-loop open-string amplitudes via differential equations connects with state-of-the-art techniques in particle phenomenology and provides explicit matrix representations of profound number-theoretic structures. As will be elaborated in [12], our results manifest important formal properties of string amplitudes such as uniform transcendentality, coaction formulae and the dropout of twisted eMZVs from non-planar open-string amplitudes.

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APPENDIX: FOUR-POINT EXAMPLES

This appendix provides further details on the expansion (14) of four-point A-cycle integrals (6).

A. Differential equation: The 6×6 differential operator $D^\tau_{\vec\eta} = D^\tau_{\eta_2,\eta_3,\eta_4}$ in (8) is determined by

$$\begin{aligned}
D^\tau_{\vec\eta}(2,3,4|2,3,4) &= \sum_{j=2}^4 \frac{s_{1j}}{2}\partial^2_{\eta_j} + \sum_{2\leq i<j}^4 \frac{s_{ij}}{2}(\partial_{\eta_i}-\partial_{\eta_j})^2 \\
&\quad - s_{12}\wp(\eta_{234},\tau) - (s_{13}+s_{23})\wp(\eta_{34},\tau) \\
&\quad - (s_{14}+s_{24}+s_{34})\wp(\eta_4,\tau) - 2\zeta_2 s_{1234} \qquad (36)\\
D^\tau_{\vec\eta}(2,3,4|2,4,3) &= (s_{14}+s_{24})\big[\wp(\eta_{34},\tau)-\wp(\eta_4,\tau)\big]\\
D^\tau_{\vec\eta}(2,3,4|3,2,4) &= s_{13}\big[\wp(\eta_{234},\tau)-\wp(\eta_{34},\tau)\big]\\
D^\tau_{\vec\eta}(2,3,4|3,4,2) &= s_{13}\big[\wp(\eta_{234},\tau)-\wp(\eta_{34},\tau)\big]\\
D^\tau_{\vec\eta}(2,3,4|4,2,3) &= s_{14}\big[\wp(\eta_{34},\tau)-\wp(\eta_4,\tau)\big]\\
D^\tau_{\vec\eta}(2,3,4|4,3,2) &= s_{14}\big[\wp(\eta_{34},\tau)-\wp(\eta_{234},\tau)\big]
\end{aligned}$$

with $\eta_{ij\ldots p} = \eta_i+\eta_j+\ldots+\eta_p$. The corresponding matrix representations of the derivations ($k\neq 2$)

$$\begin{aligned}
r_{\vec\eta}(\epsilon_k) &= \eta_{234}^{k-2} r_{\vec\eta}(e_{234}) + \sum_{2\leq i<j}^4 \eta_{ij}^{k-2} r_{\vec\eta}(e_{ij}) + \sum_{j=2}^4 \eta_j^{k-2} r_{\vec\eta}(e_j) \\
&\quad + \delta_{k,0}\Big(2\zeta_2 s_{1234} - \sum_{2\leq i<j}^4 \frac{s_{ij}}{2}(\partial_{\eta_i}-\partial_{\eta_j})^2 - \sum_{j=2}^4\frac{s_{1j}}{2}\partial^2_{\eta_j}\Big)1_{6\times 6} \qquad (37)
\end{aligned}$$

can be assembled from ($S_{123,4}\equiv s_{14}+s_{24}+s_{34}$)

$$r_{\vec\eta}(e_{234}) = \begin{pmatrix} s_{12} & 0 & -s_{13} & -s_{13} & 0 & s_{14}\\ 0 & s_{12} & 0 & s_{13} & -s_{14} & -s_{14}\\ -s_{12} & -s_{12} & s_{13} & 0 & s_{14} & 0\\ 0 & s_{12} & 0 & s_{13} & -s_{14} & -s_{14}\\ -s_{12} & -s_{12} & s_{13} & 0 & s_{14} & 0\\ s_{12} & 0 & -s_{13} & -s_{13} & 0 & s_{14} \end{pmatrix} \qquad (38)$$

$$r_{\vec\eta}(e_{34}) = \begin{pmatrix} s_{13}+s_{23} & -s_{14}-s_{24} & s_{13} & s_{13} & -s_{14} & -s_{14}\\ -s_{13}-s_{23} & s_{14}+s_{24} & -s_{13} & -s_{13} & s_{14} & s_{14}\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0 \end{pmatrix}$$

$$r_{\vec\eta}(e_4) = \begin{pmatrix} S_{123,4} & s_{14}+s_{24} & 0 & 0 & s_{14} & 0\\ 0&0&0&0&0&0\\ 0 & 0 & S_{123,4} & s_{14}+s_{34} & 0 & s_{14}\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0 \end{pmatrix}$$

and relabellings.

B. Initial values: The four-point integrals in massless one-loop string amplitudes descend from orders of $Z^\tau_{\vec\eta}$ with odd homogeneity degree in η_j. Since the derivations (37) do not mix odd and even functions of η_j, we only spell out the odd part of the planar initial value

$$\begin{aligned}
&Z^{i\infty}_{\vec\eta}(*|1,2,3,4)\,\big|^{\rm odd}_{{\rm planar}\,*} = \pi\cot(\pi\eta_{34}) I^{\rm tree}(*|G_{12}G_{34})\\
&\quad + \frac{\pi\cot(\pi\eta_4)}{s_{123}}\Big(s_{34}I^{\rm tree}(*|G_{12}G_{34}) - s_{14}I^{\rm tree}(*|G_{14}G_{23})\Big)\\
&\quad + \frac{\pi\cot(\pi\eta_{234})}{s_{234}}\Big(s_{12}I^{\rm tree}(*|G_{12}G_{34}) - s_{14}I^{\rm tree}(*|G_{14}G_{23})\Big)\\
&\quad + 6\zeta_2\Big(\pi\cot(\pi\eta_4)\frac{s_{13}}{s_{123}} + \pi\cot(\pi\eta_{234})\frac{s_{24}}{s_{234}}\Big)I^{\rm tree}(*|1)\\
&\quad + \pi^3\cot(\pi\eta_{234})\cot(\pi\eta_{34})\cot(\pi\eta_4)I^{\rm tree}(*|1)\,. \qquad (39)
\end{aligned}$$

Similar to (34), I^{tree} denote combinations of six-point disk integrals (27) which no longer depend on η_j, see section 5.5 of [12] for further details.

Non-planar four-point initial values reduce to four- and five-point disk integrals, e.g.

$$\begin{aligned} Z^{i\infty}_{\vec\eta}\big({}^{3,4}_{1,2} |1,2,3,4\big) &= \frac{\pi^3 \cot(\pi\eta_{234})\cot(\pi\eta_4)}{\sin(\pi\eta_{34})} \\ &\quad\times \frac{\Gamma(1-s_{12})\Gamma(1-s_{34})}{\big[\Gamma(1-\frac{s_{12}}{2})\,\Gamma(1-\frac{s_{34}}{2})\big]^2} \qquad (40) \\ Z^{i\infty}_{\vec\eta}\big({}^{2,3,4}_{1} |1,2,3,4\big) &= \frac{\pi}{\sin(\pi\eta_{234})} Z^{i\infty}_{\eta_2,\eta_3}(2,3,4|2,3,4)\,, \end{aligned}$$

see (32) for $Z^{i\infty}_{\eta_2,\eta_3}(2,3,4|2,3,4)$. By extracting the order of $\eta_{234}^{-1}\eta_{34}^{-1}\eta_4^{-1}$ from (14), we have checked (36), (39) and (40) to reproduce the α'-expansions of [6, 7] to the orders of α'^2 and α'^3 in the planar and non-planar sectors, respectively.

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