UNIVERSITY OF SOUTHAMPTON

# Solution sets of equations in infinite discrete groups

by


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ABSTRACT

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Solution sets of equations in groups can be thought of as group-theoretic analogues of algebraic varieties. In this thesis we apply methods of geometric group theory to study such solution sets in infinite groups from probabilistic and algebraic points of view.

This is a 'three paper thesis', the main body of which consists of the following papers:

[1] M. Valiunas, *Rational growth and degree of commutativity of graph products*, J. Algebra **522** (2019), 309–331.

[2] A. Martino, M. Tointon, M. Valiunas, and E. Ventura, *Probabilistic nilpotence in infinite groups*, preprint, available at arXiv:1805.11520 [math.GR], 2018.

[3] M. Valiunas, *Acylindrical hyperbolicity of groups acting on quasi-median graphs and equations in graph products*, preprint, available at arXiv:1811.02975 [math.GR], 2018.

In [1], we study graph products of groups – a generalisation of direct and free products. We use regularity of growth of certain graph products to establish bounds on sizes of spheres in Cayley graphs of such groups $G$. We use these bounds to show that in a graph product that is not virtually abelian, two randomly chosen elements inside a large ball in a Cayley graph of $G$ will 'almost never' commute – that is, the solution set of the equation $[X_1, X_2] = 1$ is negligible.

In [2], we use similar methods to study higher commutators, that is, the equations $[X_1, \ldots, X_{k+1}] = 1$. In particular, we show that for most classes of groups that are not virtually $k$-step nilpotent, the $(k+1)$-fold simple commutator of randomly chosen elements will almost never be trivial. Here, 'randomly chosen' refers either to using sequences of measures on a finitely generated group that are well-behaved with respect to finite-index subgroups, or to looking at finite quotients of a residually finite group. We also analyse regularity of the solution set of such an equation in virtually $k$-step nilpotent groups, and produce examples of groups showing necessity of our assumptions on finite generation or residual finiteness.

In [3], we study negative curvature in graph products of groups. We use quasi-median graphs – a class of 'nonpositively curved' graphs generalising the notion of $CAT(0)$ cube

complexes – to construct explicit acylindrical actions of graph products on spaces quasi-isometric to trees. We use this action to show that, given a finite collection of groups $G_i$ with the property that any system of equations in $G_i$ is equivalent to a finite subsystem, certain graph products of the $G_i$ will also satisfy this property.

# Contents

## Research Thesis: Declaration of Authorship

| | |
|---|---|
| Name: | Motiejus Valiunas |

| | |
|---|---|
| Title of thesis: | Solution sets of equations in infinite discrete groups |

I declare that this thesis and the work presented in it is my own and has been generated by me as the result of my own original research.

I confirm that:

1. This work was done wholly or mainly while in candidature for a research degree at this University;
2. Where any part of this thesis has previously been submitted for a degree or any other qualification at this University or any other institution, this has been clearly stated;
3. Where I have consulted the published works of others, this is always clearly attributed;
4. Where I have quoted from the work of others, the source is always given. With the exception of such quotations, this thesis is entirely my own work;
5. I have acknowledged all main sources of help;
6. Where the thesis is based on work done by myself or jointly with others, I have made clear exactly what was done by others and what I have contributed myself;
7. Parts of this work have been published as:

[1] M. Valiunas, *Rational growth and degree of commutativity of graph products*, J. Algebra **522** (2019), 309–331.

[2] A. Martino, M. Tointon, M. Valiunas, and E. Ventura, *Probabilistic nilpotence in infinite groups*, preprint, available at arXiv:1805.11520 [math.GR], 2018.

[3] M. Valiunas, *Acylindrical hyperbolicity of groups acting on quasi-median graphs and equations in graph products*, preprint, available at arXiv:1811.02975 [math.GR], 2018.

Signature: ______________________ Date: ______________________

# Notation

| | |
|---|---|
| $\mathbb{N}$ | Natural numbers $= \{0, 1, 2, \ldots\}$. |
| $C_n$ | The cyclic group of order $n$. |
| $F_n$ | The free group of rank $n$. |
| $1$ | The identity element of a group. |
| $G$ is virtually $\mathcal{P}$ | The group $G$ has a finite-index subgroup that has the property $\mathcal{P}$. |
| $[g, h]$ | $g^{-1}h^{-1}gh$. |
| $[g_0, \ldots, g_r]$ | left-normed $(r+1)$-fold simple commutator: inductively, for $r \geq 2$, $[g_0, \ldots, g_r] := [[g_0, \ldots, g_{r-1}], g_r]$. |
| $\mathrm{Cay}(G, X)$ | The Cayley graph of a group $G$ with respect to a generating set $X$. |
| $B_{G,X}(n)$ | The ball of radius $n$ in $\mathrm{Cay}(G, X)$. |
| $S_{G,X}(n)$ | The sphere of radius $n$ in $\mathrm{Cay}(G, X)$. |
| $\lvert g \rvert_X$ | The word length of a group element $g \in G$ with respect to a generating set $X$. |
| $s_{G,X}(z)$ | The spherical growth series of a group $G$ with respect to a finite generating set $X$, $s_{G,X}(z) := \sum_{g \in G} z^{\lvert g \rvert_X} = \sum_{n=1}^{\infty} \lvert S_{G,X}(n) \rvert z^n$. |
| $V(\Gamma)$ | The vertex set of a graph $\Gamma$. |
| $E(\Gamma)$ | The edge set of a graph $\Gamma$. |
| $v \sim w$ | The vertices $v, w$ of a graph $\Gamma$ are adjacent. |
| $\Gamma\mathcal{G}$ | The graph product of non-trivial groups $\mathcal{G} = \{G_v \mid v \in V\Gamma\}$ over a simplicial graph $\Gamma$. |

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# Chapter 0

# Background

This chapter provides background on three papers included in this thesis: we will refer to them as Paper 1 [Val19b], Paper 2 [MTVV18] and Paper 3 [Val18a]. Of these, Papers 1 and 3 are single-author papers; Paper 2 is a four-author paper, to which all four named authors have contributed equally. Other preprints produced by the author during his PhD studies – namely, [Val17], [Val18b] and [Val19a] – can be found on the arXiv.

## 0 Introduction

The overarching theme of this PhD thesis is equations over groups. We consider the solution set of an equation or a system of equations, as introduced in Section 0.1, and use several methods to analyse it.

In Papers 1 and 2, we take a probabilistic approach by tackling the following question: given an equation over a group, what is the 'probability' that randomly chosen elements of the group will form a solution to this equation? As almost all of the groups we consider in this thesis are infinite, there are various ways one can define 'probability' here. In Paper 1, we do this by counting elements in a Cayley graph of a (finitely generated) group $G$, and try to answer the question on when a randomly chosen pair $(g,h) \in G^2$ will be a solution of the equation $[X_1,X_2] \in F_2$ with positive probability. In Paper 2, we consider instead notions of probability that are well-behaved with respect to finite-index subgroups, and ask the same question for the equation $[X_1,\ldots,X_{k+1}] \in F_{k+1}$. Material in Papers 1 and 2 builds on the study of *degree of commutativity* of a finite group, introduced by Erdős and Turán [ET68] and Gustafson [Gus73] 50 years ago and studied ever since, as well as generalisations to infinite groups, introduced by Antolín, Martino and Ventura in [AMV17], and generalisations to higher commutators.

In Paper 3 we take a more algebraic approach. Namely, we ask when a solution set of a system of equations in a group $G$ will coincide with the solution set of some finite subsystem; $G$ is said to be equationally Noetherian if this is always the case: see Section 0.2. The class of equationally Noetherian groups is easily seen to be closed under several standard group-theoretic constructions (such as direct products and subgroups), but the relationship with 'negatively curved' constructions (such as hyperbolic groups and free products) is much more complicated – although the results one would naively expect in this setting turn out to be true. We study the latter relationship, building on the work of Sela – see [Sel10, Sel09], for instance – as well as recent work of Groves and Hull [GH17].

While the classes or groups we study in Paper 2 are very broad – all finitely generated or all residually finite groups, for instance – in Papers 1 and 3 we restrict our attention to graph products of groups (see Section 0.3), a class of groups that interpolates between direct and free products. We also introduce quasi-isometries (see Section 0.4), and use them to describe hyperbolic spaces and hyperbolic groups (see Section 0.5) – the former are crucial for our argument in Paper 3, while the latter appear as important examples of groups that we study in Papers 1 and 2.

## 0.1 Equations in groups

**Definition 0.1** (Equations, Solution sets)**.** Let $n \geq 1$ be an integer and let $F_n$ be a free group of rank $n$ with free basis $X_1, \ldots, X_n$. An *equation* $\varphi$ on $n$ variables in a group $G$ is an element $\varphi \in F_n * G$. Given an element $g = (g_1, \ldots, g_n) \in G^n$, we may (uniquely) define a homomorphism $\bar{g} : F_n * G \to G$ by setting $\bar{g}(X_i) = g_i$ for $1 \leq i \leq n$ and $\bar{g}(h) = h$ for $h \in G$; throughout this thesis, we will write $\varphi(g_1, \ldots, g_n)$ for $\bar{g}(\varphi)$. We then say that $g = (g_1, \ldots, g_n) \in G^n$ is a *solution* of $\varphi$ if $\varphi(g_1, \ldots, g_n) = 1$, and define the *solution set* of an equation $\varphi$ as

$$V_G(\varphi) = \{g \in G^n \mid \varphi(g_1, \ldots, g_n) = 1\}.$$

More generally, we may call a subset $\Phi \subseteq F_n * G$ a *system of equations*, and define the *solution set* of $\Phi$ to be

$$V_G(\Phi) = \{g \in G^n \mid \varphi(g_1, \ldots, g_n) = 1 \text{ for all } \varphi \in \Phi\}.$$

We will only use Definition 0.1 in (almost) full generality for Paper 3. For Papers 1 and 2, we will concentrate on the case of a single equation. Most of the time we will be interested when an equation is a simple commutator, defined as follows.

**Definition 0.2** (Commutator)**.** Let $G$ be a group. For elements $g_0, g_1 \in G$, we define the *commutator* of $g_0$ and $g_1$ to be $[g_0, g_1] = g_0^{-1} g_1^{-1} g_0 g_1$. Inductively, for $k \geq 2$ we define the (*left-normed*) $(k+1)$*-fold simple commutator* of elements $g_0, \ldots, g_k \in G$ to be $[g_0, \ldots, g_k] = [[g_0, \ldots, g_{k-1}], g_k]$.

We use simple commutators for studying nilpotent groups. Recall that a group $G$ is said to be *$k$-step nilpotent* (for some $k \in \mathbb{N}$) if $G$ has a *central series* of length $k$: that is, a chain of normal subgroups

$$\{1\} = G_0 \leq G_1 \leq \cdots \leq G_k = G$$

where each $G_{i+1}/G_i$ is central in $G/G_i$. In particular, if

$$G = \gamma_1(G) \geq \gamma_2(G) \geq \cdots$$

is the *lower central series* of a group $G$, then $G$ is $k$-step nilpotent if and only if $\gamma_{k+1}(G) = \{1\}$. Here, for $i \in \mathbb{N}$, $\gamma_i(G)$ is the subgroup of $G$ generated by commutators $[g, h]$ for $g \in \gamma_{i-1}(G)$ and $h \in G$; it follows that $\gamma_i(G)$ is generated by all $i$-fold simple commutators of elements of $G$. In particular, a group $G$ is $k$-step nilpotent if and only if any $(k+1)$-fold simple commutator of elements of $G$ is trivial.

Given the discussion above, we expect, therefore, that if the solution set $V_G(c^{(k)})$ of the simple commutator $c^{(k)} = [X_1, \ldots, X_{k+1}] \in F_{k+1}(X_1, \ldots, X_{k+1})$ is 'large', then the group $G$ contains a $k$-step nilpotent subgroup of finite index. We study this claim in Paper 1 for $k = 1$ and in Paper 2 for arbitrary $k$.

## 0.2 Equationally Noetherian groups

**Definition 0.3** (Equationally Noetherian groups)**.** A group $G$ is said to be *equationally Noetherian* if for all $n \in \mathbb{N}$ and all $\Phi \subseteq F_n$, there exists a finite subset $\Phi_0 \subseteq \Phi$ such that $V_{\Phi_0}(G) = V_\Phi(G)$. We say that $G$ is *strongly equationally Noetherian* if this holds, in addition, for all $\Phi \subseteq F_n * G$.

The usual notion of an equationally Noetherian group – see, for instance, [BMR99, §2.2] – coincides with the notion of a *strongly* equationally Noetherian group as introduced here. However, we use the (weaker) notion of an equationally Noetherian group since it is more susceptible to the methods of Groves and Hull used in [GH17]. It is worth noting, however, that any finitely generated equationally Noetherian group is also strongly equationally Noetherian [BMR99, §2.2, Proposition 3], and so the two concepts agree in the class of finitely generated groups.

**Example 0.4.**

(i) *Any abelian group is strongly equationally Noetherian [BMR99, §2.2, Theorem 1]: this follows by an application of the Euclidean algorithm.*

(ii) *Any group that is linear over a field is strongly equationally Noetherian [BMR99, §2.2, Theorem B1]: this is a consequence of the Hilbert Basis Theorem. In partic-*

*ular, finitely generated free groups and, more generally, right-angled Artin groups (see Section 0.3) are strongly equationally Noetherian.*

(iii) *A result of Sela states that torsion-free hyperbolic groups are strongly equationally Noetherian [Sel09, Theorem 1.22].*

(iv) *It is easy to see that the classes of equationally Noetherian and strongly equationally Noetherian groups are closed under taking subgroups and finite direct products. These classes are also preserved under taking finite extensions [BMR97, Theorem 1 and its proof] and finite free products ([Sel10, Theorem 9.1], [GH17, Corollary C]). The most general result in this direction is probably a theorem of Groves and Hull [GH17, Theorem D], stating that a group hyperbolic relative to equationally Noetherian subgroups is equationally Noetherian.*

Thus, equationally Noetherian groups cover a large class of examples. An example of a group that is *not* equationally Noetherian is a (permutational) wreath product $W = G \wr H$, where $G$ is non-abelian and $H$ is not torsion. Indeed, if we consider the system $\Phi = \{[X_1, X_3^{-n} X_2 X_3^n] \mid n \in \mathbb{Z}\} \subset F_3$ and if $\Phi_0 \subset \Phi$ is a proper subset, then $[X_1, X_3^{-N} X_2 X_3^N] \notin \Phi_0$ for some $N \in \mathbb{Z}$ and we have $(h^{-N}kh^N, g, h) \in V_W(\Phi_0) \setminus V_W(\Phi)$ for any $g, k \in G$ with $[g, k] \neq 1$ and any element $h \in H$ of infinite order. Consequently, $V_W(\Phi_0) \neq V_W(\Phi)$ for any finite $\Phi_0 \subset \Phi$.

Another class of examples of finitely generated non-equationally Noetherian groups is given by certain Baumslag–Solitar groups. Indeed, it is known that a finitely generated equationally Noetherian group $G$ must be *hopfian* [GH17, Corollary 3.13] – that is, any surjective homomorphism $G \to G$ is an isomorphism. Thus, in many cases (for instance, if $|m| \neq 1$ and some prime divisor of $n$ does not divide $m$) the Baumslag–Solitar group $BS(m, n)$ is not equationally Noetherian.

## 0.3 Graph products of groups

This section describes a construction to build new groups out of a given collection of groups; it generalises the construction of direct sums and free products of groups.

**Definition 0.5** (Graph product)**.** Let $\Gamma$ be a simplicial graph, and $\mathcal{G} = \{G_v \mid v \in V(\Gamma)\}$ be a collection of non-trivial groups indexed by the vertices of $\Gamma$. We define the *graph product* of $\mathcal{G}$ over $\Gamma$ to be the group

$$\Gamma\mathcal{G} = \mathop{\ast}_{v \in V(\Gamma)} G_v \Big/ \left\langle\!\!\left\langle \bigcup_{v \sim w} [G_v, G_w] \right\rangle\!\!\right\rangle,$$

where $v \sim w$ denotes adjacency of two vertices $v, w \in V(\Gamma)$, and $\langle\!\langle \mathcal{R} \rangle\!\rangle$ denotes the normal closure of a subset $\mathcal{R} \subseteq \ast_{v \in V(\Gamma)} G_v$.

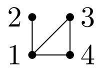
Given a subset $A \subseteq V(\Gamma)$ for a simplicial graph $\Gamma$, we denote by $\Gamma_A$ the full subgraph of $\Gamma$ spanned by $A$, and we set $\mathcal{G}_A := \{G_v \mid v \in A\}$. It is easy to see that the graph product $\Gamma_A\mathcal{G}_A$ is canonically a subgroup of $\Gamma\mathcal{G}$.

*Remark* 0.6. In the definition above, we lose no generality in requiring the groups $G_v$ to be non-trivial. Indeed, even if some of the groups in $\mathcal{G}$ were trivial, then $\Gamma\mathcal{G}$ would be isomorphic to $\Gamma_A\mathcal{G}_A$, where $A = \{v \in V(\Gamma) \mid G_v \ncong \{1\}\}$. Thus, the assumption of non-triviality of the $G_v$ is taken only for convenience.

**Example 0.7.**

(i) *If $\Gamma$ is a discrete graph (that is, $v \nsim w$ for any $v, w \in V(\Gamma)$), then $\Gamma\mathcal{G}$ is isomorphic to the free product $*_{v\in V(\Gamma)} G_v$.*

(ii) *If $\Gamma$ is a complete graph (that is, $v \sim w$ for any distinct $v, w \in V(\Gamma)$), then $\Gamma\mathcal{G}$ is isomorphic to the direct sum $\bigoplus_{v\in V(\Gamma)} G_v$.*

(iii) *More generally, any group 'built from' a given collection of groups by taking direct sums and free products can be described as a graph product. For instance, for the graph $\Gamma =$* 

*we have $\Gamma\mathcal{G} \cong G_1 \times (G_2 * (G_3 \times G_4))$. However, if both $\Gamma$ and its complement $\Gamma^C$ are connected – for instance, $\Gamma =$*  *– then, in general, $\Gamma\mathcal{G}$ need not be directly or freely decomposable.*

(iv) *If $\Gamma$ is finite and $G_v \cong \mathbb{Z}$ for all $v \in V(\Gamma)$, then $\Gamma\mathcal{G}$ is the* right-angled Artin group *(RAAG) on $\Gamma$. These groups have been used to produce many examples of groups with various finiteness properties. Specifically, in [BB97] Bestvina and Brady related the homology and homotopy groups of the flag simplicial complex whose 1-skeleton is $\Gamma$ to the finiteness properties of the* Bestvina–Brady group *$BB_\Gamma$ – the kernel of the map $\Gamma\mathcal{G} \to \mathbb{Z}$ defined by sending the generator of each $G_v$ to $1 \in \mathbb{Z}$. This partially motivated expansive study of RAAGs over the past two decades.*

(v) *If $\Gamma$ is finite and $G_v \cong C_2$ for all $v \in V(\Gamma)$, then $\Gamma\mathcal{G}$ is the* right-angled Coxeter group *(RACG) on $\Gamma$. This is a special case of* Coxeter groups *– groups generated by reflections through hyperplanes in $\mathbb{R}^n$ passing through the origin. In particular, Coxeter groups – and so RACGs – are linear.*

An important consequence of the last example is that RAAGs are linear. This follows from the fact that a RAAG can be embedded into a RACG, via the following procedure. Let $\Gamma\mathcal{G}$ be a RAAG – that is, $\mathcal{G} = \{G_v \cong \mathbb{Z} \mid v \in V(\Gamma)\}$ – and let $\hat{\Gamma}$ be the finite simplicial graph obtained by *doubling the vertices* of $\Gamma$. That is, let $V(\hat{\Gamma}) = V(\Gamma) \times \{0,1\}$, and let $(v,\alpha), (w,\beta) \in V(\Gamma) \times \{0,1\}$ be adjacent in $\hat{\Gamma}$ if and only if $v$ and $w$ are adjacent in $\Gamma$. Then $\Gamma\mathcal{G}$ embeds in $\hat{\Gamma}\mathcal{H}$, where $\mathcal{H} = \{H_{\mathbf{v}} \cong C_2 \mid \mathbf{v} \in V(\hat{\Gamma})\}$: indeed, it is easy to see that $\hat{\Gamma}\mathcal{H} \cong \Gamma\mathcal{K}$, where $\mathcal{K} = \{K_v \cong C_2 * C_2 \mid v \in V(\Gamma)\}$, and the existence of an injective homomorphism $\Gamma\mathcal{G} \hookrightarrow \Gamma\mathcal{K}$ follows from the existence of injective homomorphisms $G_v \cong \mathbb{Z} \hookrightarrow C_2 * C_2 \cong K_v$ for every $v \in V(\Gamma)$. In fact, we may even do better: in [DJ00], Davis and Januszkiewitz give an embedding of any RAAG into a RACG as a *finite-index* subgroup.

## 0.4 Quasi-isometries

An important notion in the study of hyperbolic spaces and groups (defined in Section 0.5 below), as well as finitely generated groups in general, is that of a quasi-isometry, defined as follows.

**Definition 0.8** (Quasi-isometry, Quasi-geodesic)**.** Let $(X, d_X)$ and $(Y, d_Y)$ be metric spaces, and fix constants $\lambda \geq 1$ and $K, C \geq 0$. A map $f : X \to Y$ is called a $(\lambda, K)$-*quasi-isometric embedding* if

$$\lambda^{-1} d_X(x, y) - K \leq d_Y(f(x), f(y)) \leq \lambda d_X(x, y) + K$$

for all $x, y \in X$, and a $(\lambda, K, C)$-*quasi-isometry* if furthermore for any point $y \in Y$, there exists some $x \in X$ with $d_Y(y, f(x)) \leq C$. We also say $f : X \to Y$ is a *quasi-isometric embedding* (respectively, a *quasi-isometry*) if $f$ is a $(\lambda, K)$-quasi-isometric embedding (respectively, a $(\lambda, K, C)$-quasi-isometry) for some $\lambda$, $K$ and $C$. A $(\lambda, K)$-*quasi-geodesic* in a metric space $X$ is (the image of) a $(\lambda, K)$-quasi-isometric embedding $\gamma : ([0, \ell], d_{\mathbb{R}}) \to X$, for some $\ell \geq 0$.

*Remark* 0.9. In the literature, a '$(\lambda, K)$-quasi-geodesic' in $X$ often refers to a $(\lambda, K)$-quasi-isometric embedding of a ray, $\gamma : ([0, \infty), d_{\mathbb{R}}) \to X$, or of a line, $\gamma : (\mathbb{R}, d_{\mathbb{R}}) \to X$. Although we do not use this viewpoint here, this (more general) notion of a $(\lambda, K)$-quasi-geodesic is useful as well. This also allows one to say that $\gamma$ is a *quasi-geodesic* (without refering to $\lambda$ and $K$); note that, in our context, any map $\gamma : ([0, \ell], d_{\mathbb{R}}) \to X$ with bounded image is a $(\lambda, K)$-quasi-geodesic for some $\lambda$ and $K$, so the reference to $(\lambda, K)$ is essentially unavoidable.

It is easy to see that quasi-isometries define an equivalence relation between metric spaces. In particular, if $f : X \to Y$ is a quasi-isometry, then there exists a quasi-isometry $\bar{f} : Y \to X$, and if $f : X \to Y$ and $g : Y \to Z$ are quasi-isometries, then so is $g \circ f : X \to Z$. Two metric spaces are said to be *quasi-isometric* if there is a quasi-isometry between them. In geometric group theory, probably the most important examples of quasi-isometries are as follows.

**Example 0.10.**

(i) *Any group $G$ with a generating set $S \subseteq G$ can be equipped with a metric (called a* word metric*) $d_S : G \times G \to \mathbb{Z}$, where $d_S(g, h)$ is the smallest integer $n$ such* *that $gh^{-1} = s_1 \cdots s_n$ for some $s_i \in S \cup S^{-1}$. If the group $G$ is finitely generated and $S$, $T$ are two finite generating set for $G$, then it is easy to see that the map $(G, d_S) \to (G, d_T), g \mapsto g$ is a $(\lambda, 0, 0)$-quasi-isometry, where*

$$\lambda = \max(\{d_S(1, t) \mid t \in T\} \cup \{d_T(1, s) \mid s \in S\}).$$

*In particular, given two finitely generated groups $G$ and $H$, it makes sense to say that $G$ is* quasi-isometric *to $H$ without refering to specific generating sets.*

(ii) *More generally, suppose $G$ is a finitely generated group (with finite generating set $S$, say) and $H \leq G$ a subgroup of finite index. Then $H$ is also finitely generated by Schreier's Lemma (see [Ser02, Lemma 4.2.1]), by a finite set $T$, say, and the map $(H, d_T) \to (G, d_S), h \mapsto h$ is a quasi-isometry. In particular, if finitely generated groups $G_1$ and $G_2$ are* commensurable *– that is, there exist finite-index subgroups $H_i \leq G_i$ such that $H_1 \cong H_2$ – then $G_1$ and $G_2$ are quasi-isometric.*

The following result is usually useful for dealing with quasi-isometries, and signifies the importance of properly discontinuous and cocompact group actions.

**Proposition 0.11** (Švarc–Milnor Lemma; see [BH99, Proposition I.8.19])**.** *Let a group $G$ act on a geodesic metric space $(Y, d)$. Suppose that:*

(i) *$Y$ is* proper*: for any $r \geq 0$ and any $y \in Y$, the closed ball $B_Y(r; y) = \{x \in Y \mid d(x, y) \leq r\}$ is compact;*

(ii) *$G \curvearrowright Y$ is* properly discontinuous*: for any compact subset $K \subseteq Y$, the set $\{g \in G \mid K^g \cap K \neq \varnothing\}$ is finite; and*

(iii) *$G \curvearrowright Y$ is* cocompact*: there exists a compact subset $K \subseteq Y$ such that we have $Y = \bigcup_{g \in G} K^g$.*

*Then $G$ is finitely generated, and given any finite generating set $S$ for $G$ and any $y \in Y$, the map $(G, d_S) \to Y, g \mapsto y^g$ is a quasi-isometry.*

## 0.5 Hyperbolic spaces and groups

Let $Y$ be a geodesic metric space. Given three points $x, y, z \in Y$, we may construct a geodesic triangle $\Delta$ by picking geodesics $\gamma_{x,y}$, $\gamma_{x,z}$ and $\gamma_{y,z}$ between $x$ and $y$, between $x$ and $z$ and between $y$ and $z$, respectively. Given $\delta \geq 0$, we say the triangle $\Delta$ is $\delta$-slim if $\gamma_{x,y}$ (respectively $\gamma_{x,z}$, $\gamma_{y,z}$) belongs to the $\delta$-neighbourhood of $\gamma_{x,z} \cup \gamma_{y,z}$ (respectively $\gamma_{x,y} \cup \gamma_{y,z}$, $\gamma_{x,y} \cup \gamma_{x,z}$): see Figure 1.

**Definition 0.12** (Hyperbolic space)**.** Given $\delta \geq 0$, a geodesic metric space $Y$ is said to be $\delta$-*hyperbolic* if all geodesic triangles in $Y$ are $\delta$-slim. $Y$ is said to be *hyperbolic* (or

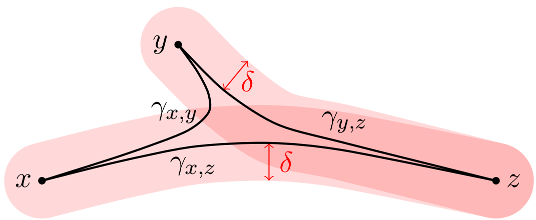



FIGURE 1: A $\delta$-slim triangle: $\gamma_{x,y}$ is in the $\delta$-neighbourhood of $\gamma_{x,z} \cup \gamma_{y,z}$.

*Gromov-hyperbolic*) if it is $\delta$-hyperbolic for some $\delta$. A group $G$ is said to be *hyperbolic* if it has a finite generating set $S$ such that the Cayley graph $\mathrm{Cay}(G, S)$ is hyperbolic.

There are many alternative definitions of a hyperbolic metric space (and even more of a hyperbolic group), all of them equivalent to Definition 0.12 – although the constant $\delta \geq 0$ that makes a given space $\delta$-hyperbolic might depend on the definition chosen. For instance, given a geodesic metric space $(Y, d)$ with a fixed basepoint $w \in Y$, we could define an 'inner product' on $Y$ by setting

$$(x \cdot y)_w = \frac{d(w,x) + d(w,y) - d(x,y)}{2}$$

for $x, y \in Y$. Then $Y$ is hyperbolic if and only if there exists a constant $\delta' \geq 0$ such that

$$(x \cdot y)_w \geq \min\{(x \cdot z)_w, (y \cdot z)_w\} - \delta'$$

for all $x, y, z \in Y$. These and several other definitions, as well as proofs of their equivalence, can be found in [ABC+90].

Given any $\lambda \geq 1$ and $K \geq 0$, it is not difficult to show that there exists a constant $D$ such that any $(\lambda, K)$-quasi-geodesic in a hyperbolic metric space will be in the $D$-neighbourhood of a geodesic (see, for instance, [Gro87, Proposition 7.2.A]). As a consequence, any space quasi-isometric to a hyperbolic metric space will be hyperbolic itself. In particular, given two finitely generated commensurable groups $G$ and $H$, it follows from Example 0.10 that $G$ is hyperbolic if and only if $H$ is. Notable examples of hyperbolic spaces and groups are as follows.

**Example 0.13.**

(i) *It is easy to see that any simplicial tree will be* 0*-hyperbolic. Thus, any* quasi-tree *– a space quasi-isometric to a simplicial tree – is hyperbolic as well. As a consequence, a finitely generated free group, or a finitely generated virtually free group (such as $SL_2(\mathbb{Z})$), is hyperbolic.*

(ii) *A particular case of the previous example says that a point $\{*\}$ and the real line $\mathbb{R}$ are hyperbolic. Thus, any virtually cyclic (finite or virtually $\mathbb{Z}$) group is hyperbolic. Such groups are called* elementary *hyperbolic groups, and usually have properties that are vastly different from the ones enjoyed by non-elementary hyperbolic groups.*

(iii) *The hyperbolic plane $\mathbb{H}^2$ is, as the name suggests, hyperbolic. Since $\mathbb{H}^2$ is the universal cover of a compact orientable closed surface $\Sigma_g$ of genus $g \geq 2$, it follows that $\pi_1(\Sigma_g)$ is hyperbolic. Indeed, a* Cayley complex *of $\pi_1(\Sigma_g)$ – a 2-complex whose 1-skeleton is a Cayley graph and 2-cells correspond to relators – can be visualised as the order-4g 4g-gonal tiling of $\mathbb{H}^2$. Alternatively, we could use the fact that $\pi_1(\Sigma_g)$ acts properly discontinuously and cocompactly on $\mathbb{H}^2$, together with the Švarc–Milnor Lemma (Proposition 0.11), to see that $\pi_1(\Sigma_g)$ is hyperbolic.*

In general, it is known that 'most' groups are hyperbolic: with a certain definition of 'random', Gromov showed in [Gro93, Section 9.B] that a random group will be infinite hyperbolic with overwhelming probability.

# 1 Background for Paper 1

In Paper 1 we study statistical properties of Cayley graphs of certain graph products. Our first main result, Theorem 1.4, considers groups that have rational growth (see Section 1.1) and gives bounds on sphere sizes in Cayley graphs of such groups (see Section 1.2). In Section 1.3, we use a Cayley graph of a finitely generated group $G$ to define the density of a subset $A \subseteq G^r$. Finally, as described in Section 1.4, we apply Theorem 1.4 to show that in certain graph products $G$ – for instance, right-angled Artin or Coxeter groups – the density of the pairs $(g, h) \in G^2$ such that $gh = hg$ is non-zero if and only if $G$ is virtually abelian, verifying Conjecture 1.9 for these groups (with respect to certain generating sets).

## 1.1 Growth series

Let $G$ be a finitely generated group, and let $X$ be a finite generating set for $G$; for simplicity, we will always assume that $X$ is *symmetric* – that is, $t^{-1} \in X$ for all $t \in X$ – and that $1 \in X$. We would like to study how 'regular' the growth of $G$ with respect to $X$ is. In particular, for any $n \in \mathbb{N}$ we may define the *ball* $B_{G,X}(n) = X^n \subseteq G$ of radius $n$ in $G$ with respect to $X$ – that is, the set of elements $g \in G$ that are products of $\leq n$ elements of $X$; note that this coincides with the ball in the Cayley graph $\mathrm{Cay}(G, X)$ of radius $n$ centered at $1 \in G$, if this Cayley graph is given the combinatorial metric. We may also define the *sphere* $S_{G,X}(n) \subseteq G$ of radius $n$ in $G$ with respect to $X$ by setting $S_{G,X}(0) = \{1\}$ and $S_{G,X}(n) = B_{G,X}(n) \setminus B_{G,X}(n-1)$ for $n \geq 1$.

**Definition 1.1** (Rational growth)**.** Let $G$ be a finitely generated group, and let $X$ be a finite generating set for $G$. The *spherical growth series* for $G$ with respect to $X$ is the formal power series $s_{G,X}(z) = \sum_{n=0}^{\infty} |S_{G,X}(n)| z^n \in \mathbb{Z}[[z]]$. Similarly, the *volume growth series* for $G$ with respect to $X$ is $b_{G,X}(z) = \sum_{n=0}^{\infty} |B_{G,X}(n)| z^n \in \mathbb{Z}[[z]]$; note that $s_{G,X}(z) = (1-z)b_{G,X}(z)$. We say that $G$ has *rational growth* with respect to $X$ is $s_{G,X}(z)$ (equivalently, $b_{G,X}(z)$) is a rational function, that is, a ratio of two polynomials: $s_{G,X}(z) = \frac{P(z)}{Q(z)}$ for some $P(z), Q(z) \in \mathbb{Z}[z]$ with $Q(z) \not\equiv 0$.

*Remark* 1.2. Note that we lose no generality in requiring the polynomials $P(z)$ and $Q(z)$ in Definition 1.1 to have integer coefficients. Indeed, suppose $s_{G,X}(z) = \frac{P(z)}{Q(z)}$ where $P(z) \in \mathbb{C}[z]$ has degree $m \in \mathbb{N}$ and $Q(z) = \sum_{i=0}^{r} q_i z^i$ for some $q_0, \ldots, q_r \in \mathbb{C}$. Since $s_{G,X}(z)Q(z) = P(z)$, by comparing coefficients of $z^j$ for $j$ large enough we see that $(x_0, \ldots, x_r) = (q_0, \ldots, q_r) \in \mathbb{C}^{r+1}$ is a solution to the system of equations

$$\sum_{i=0}^{r} |S_{G,X}(j-i)| x_i = 0 \qquad \text{for all } j \geq \max\{m+1, r\}; \tag{1}$$

and conversely, given any solution $(\hat{q}_0, \ldots, \hat{q}_r) \in \mathbb{C}^{r+1}$ of (1), the series $s_{G,X}(z)\widehat{Q}(z)$ is a polynomial, where $\widehat{Q}(z) = \sum_{i=0}^{r} \hat{q}_i z^i$. Thus, given a polynomial $Q(z) \in \mathbb{C}[z]$, we have $s_{G,X}(z)Q(z) \in \mathbb{C}[z]$ if and only if the coefficients of $Q(z)$ form a solution of (1).

But as (1) is a system of homogeneous linear equations that has a non-zero solution, it follows that its solution space $V \subseteq \mathbb{C}^{r+1}$ is a (linear) subspace of dimension $\geq 1$. As the equations in (1) have integer coefficients, it follows that $V \cap \mathbb{Q}^{r+1}$ is a subspace of $\mathbb{Q}^{r+1}$ of dimension $\geq 1$, and so $V \cap \mathbb{Z}^{r+1} \neq \{0\}$. Thus, without loss of generality we may assume that $Q(z) \in \mathbb{Z}[z]$; but this then implies that $P(z) = s_{G,X}(z)Q(z)$ has integer coefficients as well, as claimed. This result also appears as [Sto96, Lemma 3.1].

**Example 1.3.**

(i) *Any hyperbolic group $G$ is known to have rational growth with respect to any generating set $X$ [ECH+92, Theorem 3.4.5]; this can be attributed to Cannon, Gromov and Thurston. This follows from the fact that given any finite generating set $X$, there exists a* regular language *(that is, a language recognised by a finite state automaton) that maps bijectively to $G$ and consists only of geodesics in the Cayley graph* $\mathrm{Cay}(G, X)$.

(ii) *Any finitely generated abelian group $G$ has rational growth vith any generating set as well: this is due to Benson [Ben83, Theorem 1.2].*

(iii) *The only other group that is known to have rational growth with respect to* any *generating set is the integral Heisenberg group*

$$H_3(\mathbb{Z}) = \left\{ \begin{pmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix} \middle| \, a, b, c \in \mathbb{Z} \right\} \leq SL_3(\mathbb{Z})$$

*(Duchin and Shapiro [DS14, Theorem 1]). In contrast with hyperbolic groups, $H_3(\mathbb{Z})$ cannot have a regular language consisting of exactly one geodesic word for*

*each element of $H_3(\mathbb{Z})$ with respect to any generating set [Sto95, Theorem 1], and so a different approach is needed to prove rationality of its growth.*

(iv) *There are also several classes of groups that cannot have rational growth with respect to any generators. For instance, it is well-known that having a rational growth series implies that the group has either polynomial or exponential growth, and so groups of intermediate growth cannot have rational growth (see also Theorem 1.4 below). Also, it is easy to see that if $s_{G,X}(z)$ is rational then the numbers $|S_{G,X}(z)|$ are computable, and so $G$ has solvable word problem – hence groups with unsolvable word problem cannot have rational growth.*

(v) *Some groups are known to have rational growth with certain generating sets – for instance, right-angled Artin / Coxeter groups [Chi94, Proposition 1] and soluble Baumslag-Solitar groups $BS(1,n)$ [CEG94] have rational growth with respect to the standard generating sets.*

(vi) *The 5-dimensional integral Heisenberg group,*

$$H_5(\mathbb{Z}) = \left\{ \begin{pmatrix} 1 & a_1 & a_2 & c \\ 0 & 1 & 0 & b_1 \\ 0 & 0 & 1 & b_2 \\ 0 & 0 & 0 & 1 \end{pmatrix} \middle| \, a_1, a_2, b_1, b_2, c \in \mathbb{Z} \right\} \leq SL_4(\mathbb{Z}),$$

*has two generating sets ($X$ and $X'$, say) such that $s_{H_5(\mathbb{Z}),X}(z)$ is rational but $s_{H_5(\mathbb{Z}),X'}(z)$ is not [Sto96, Theorems A and B]. Thus, in general, rationality of the growth series depends on the choice of a generating set.*

## 1.2 Estimates of sphere sizes

Having rational growth allows us to estimate the sphere sizes $|S_{G,X}(n)|$. In particular, if a group $G$ has rational growth with respect to a finite generating set $X$, then the numbers $|S_{G,X}(n)|$ satisfy a homogeneous linear recurrence relation with constant coefficients – cf (1). It follows that there exist $c_1, \dots, c_r \in \mathbb{C} \setminus \{0\}$, $\lambda_1, \dots, \lambda_r \in \mathbb{C}$ and $\alpha_1, \dots, \alpha_r \in \mathbb{N}$ such that

$$|S_{G,X}(n)| = \sum_{i=1}^{r} c_i n^{\alpha_i} \lambda_i^n \tag{2}$$

for all sufficiently large $n$, and hence

$$\limsup_{n \to \infty} \frac{|S_{G,X}(n)|}{n^\alpha \lambda^n} \leq \sum_{i=1}^{r} |c_i|, \tag{3}$$

where $\lambda = \max\{|\lambda_i| \mid 1 \leq i \leq r\}$ and $\alpha = \max\{\alpha_i \mid 1 \leq i \leq r, |\lambda_i| = \lambda\}$. The following result says that, in addition to the upper bound for $|S_{G,X}(n)|/(n^\alpha \lambda^n)$ given by (3), we have a lower bound as well.

**Theorem 1.4** (Paper 1 [Val19b, Theorem 1])**.** *Let $G$ be an infinite finitely generated group with a finite generating set $X$. Suppose that $G$ has rational growth with respect to $X$. Then there exist constants $\lambda \in [1,\infty)$, $\alpha \in \mathbb{N}$ and $D > C > 0$ such that*

$$Cn^\alpha \lambda^n \leq |S_{G,X}(n)| \leq Dn^\alpha \lambda^n$$

*for all $n \geq 1$.*

*Remark* 1.5. An important ingredient to Theorem 1.4 is *submultiplicativity* of sphere sizes in $G$: that is, we have $|S_{G,X}(m+n)| \leq |S_{G,X}(m)| \times |S_{G,X}(n)|$ for every $m,n \in \mathbb{N}$. This follows from the fact that an element $g \in S_{G,X}(m+n)$ can be expressed as a product $g_1g_2$ for $g_1 \in S_{G,X}(m)$ and $g_2 \in S_{G,X}(n)$, and so there is an injection $S_{G,X}(m+n) \hookrightarrow S_{G,X}(m) \times S_{G,X}(n)$ given by $g \mapsto (g_1,g_2)$. In fact, submultiplicativity and rationality are enough: given any rational series $\sum_{n=0}^\infty a_n z^n$ such that $a_n \in \mathbb{Z}_{\geq 1}$ and $a_{m+n} \leq a_m a_n$ for all $m,n \in \mathbb{N}$, there exist constants $\alpha$, $\lambda$, $C$ and $D$ as in Theorem 1.4 such that $Cn^\alpha\lambda^n \leq a_n \leq Dn^\alpha\lambda^n$ for all $n \geq 1$: see Theorem 11 in Paper 1.

In particular, instead of considering sizes of the spheres $S_{G,X}(n)$ in Theorem 1.4, one may equally well consider the sizes of balls $B_{G,X}(n) = \bigcup_{i=0}^n S_{G,X}(i)$, as they form a submultiplicative sequence of rational growth. However, Theorem 1.4 is written in its current form as the bounds on sizes of spheres are a stronger result than corresponding bounds on sizes of balls. Indeed, if the conclusion of Theorem 1.4 holds, then there exist constants $\widehat{D} > \widehat{C} > 0$ such that

$$\widehat{C}n^{\widehat{\alpha}}\lambda^n \leq |B_{G,X}(n)| \leq \widehat{D}n^{\widehat{\alpha}}\lambda^n \tag{4}$$

for all $n \geq 1$, where $\widehat{\alpha} = \alpha+1$ if $\lambda = 1$ and $\widehat{\alpha} = \alpha$ otherwise. The reverse implication can be seen to hold if $\widehat{\alpha} = 0$.

The bounds given by Theorem 1.4 appear in the literature for several classes of groups. In particular, a result of Coornaert [Coo93, Théorème 7.2] says that if $G$ is hyperbolic and not virtually cyclic then the bounds on Theorem 1.4 hold, and in addition we have $\alpha = 0$. As hyperbolic groups have rational growth with respect to any generating set [ECH+92, Theorem 3.4.5], one may deduce Coornaert's result from Theorem 1.4: see [Val19a]; this eliminates the need of Patterson–Sullivan measures on hyperbolic groups to give such bounds. Similar bounds (with $\alpha = 0$ as well) have also been shown to hold for relatively hyperbolic groups [Yan13, Theorem 1.9] and right-angled Artin/Coxeter groups that do not split as direct products, when $X$ is the standard generating set [GTT17, Theorem 2.2].

For groups of polynomial growth (that is, when $\lambda = 1$) – which, by Gromov's theorem [Gro81], coincide with the class of finitely generated virtually nilpotent groups – we have a slightly different situation. For such groups $G$, Pansu [Pan83] showed that the limit $\lim_{n\to\infty} |B_{G,X}(n)|/n^{\widehat{\alpha}}$ exists for some $\widehat{\alpha} \in \mathbb{Z}_{\geq 1}$, and therefore (4) holds. However, this does not imply that the conclusion of Theorem 1.4 holds for $(\lambda, \alpha) = (1, \widehat{\alpha} - 1)$, and indeed it is not known if this conclusion holds in general [Man12, Chapter 18, Problem 4]. The 'upper bound' part of this conclusion – that is, $|S_{G,X}(n)| \leq Dn^{\widehat{\alpha}-1}$ – holds for 2-step nilpotent groups by a result of Stoll [Sto98, Theorem 5.3].

If $s_{G,X}(z)$ is rational, we may also extract some information about the numbers $c_i$ and $\lambda_i$ as above. For instance, if we write $s_{G,X}(z) = P(z)/Q(z)$ for polynomials $P(z), Q(z) \in \mathbb{C}[z]$ which have no common roots, then the numbers $\lambda_1, \dots, \lambda_r \in \mathbb{C}$ appearing in (2) may be taken to be the roots of $Q(z)$, counted with multiplicity. In particular, as we may take $Q(z) \in \mathbb{Z}[z]$ (see Remark 1.2), it follows that the $\lambda_i$ are algebraic numbers. But then $(c_1, \dots, c_r) \in \mathbb{C}^r$ is just a solution of a system (2) of linear equations with coefficients in $K = \mathbb{Q}(\lambda_1, \dots, \lambda_r)$, and so we have $c_i \in K$: in particular, the $c_i$ are algebraic. In fact, if $\lambda := \max\{|\lambda_i| \mid 1 \leq i \leq r\} = 1$, then $\lambda_j = 1$ for some $j$ and we may even show that $c_j \in \mathbb{Q}$ [Sto96, Proposition 3.3]. This is used to show (see [Sto96, Theorem B]) that the growth of the 5-dimensional integral Heisenberg group $H_5(\mathbb{Z})$ with respect to its standard generators is not rational.

We may deduce an even stronger conclusion when $s_{G,X}(z)$ is a *positive* rational function: that is, $s_{G,X}(z) \in \mathbb{C}_+(z)$ where $\mathbb{C}_+(z)$ is the smallest sub-semiring of $\mathbb{C}(z)$ containing $\mathbb{N}[z]$ and closed under the operation $f(z) \mapsto (1 - zf(z))^{-1}$; for instance, this holds if a language $\mathcal{L} \subseteq X^*$ consisting of a single geodesic representing each element of $G$ is recognised by a finite state automaton. In this case, it follows from a result of Berstel [Ber71, Propriété] that $\lambda_i/\lambda$ is a root of unity whenever $|\lambda_i| = \lambda$. It then follows that the numbers $d_n = |S_{G,X}(n)|/(n^\alpha \lambda^n)$ are *asymptotically periodic*: that is, there exist constants $k \in \mathbb{Z}_{\geq 1}$ and $D_0, \dots, D_{k-1}$ such that $d_{kn+i} \to D_i$ as $n \to \infty$ for all $i \in \{0, \dots, k-1\}$. However, it is not clear why we could expect $s_{G,X}(z)$ to be in $\mathbb{C}_+(z)$ whenever it is rational; in particular, the language $\mathcal{L}$ as above does not always exist even if the growth is rational: see Example 1.3 (iii).

## 1.3 Statistical properties of Cayley graphs

Recall that a *Cayley graph* $\Gamma = \mathrm{Cay}(G, X)$ is a (directed) graph with vertices $V(\Gamma) = G$ and edges $E(\Gamma) = \{(g, xg) \mid g \in G, x \in X\}$. We may furthermore assume, if necessary, that $X$ is *symmetric* (that is, $X = X^{-1}$), in which case we view $\mathrm{Cay}(G, X)$ as an *undirected* graph by replacing each pair of directed edges $\left( g \bullet \rightleftarrows \bullet\, xg \right)$ by an indirected one $\left( g \bullet\!\!\longrightarrow\!\!\bullet\, xg \right)$, and replacing directed loops with undirected ones. We may also give $\mathrm{Cay}(G, X)$ a metric by setting each edge to have length one.

We would like to study which subsets of $G$ (or, more generally, of $G^r$ for some $r \geq 1$) are 'small' and which ones are 'large' in a sense of geometry of $\mathrm{Cay}(G, X)$. To do this, we introduce the following definition, which generalises the definition given by Burillo and Ventura in [BV02] to finite direct powers of a given group.

**Definition 1.6** (Negligible, Generic)**.** Given $n \in \mathbb{N}$, let $B_{G,X}(n)$ be the (vertices in the) *ball* of radius $n$ centered at $1 \in G$ in the Cayley graph $\mathrm{Cay}(G, X)$: that is, $B_{G,X}(n)$ consists of elements of $G$ of the form $x_1 \cdots x_m$ for some $m \leq n$ and $x_i \in X \cup X^{-1}$. Given $r \in \mathbb{N}$ and a subset $A \subseteq G^r$, we define the *natural density* of $A$ with respect to $X$ to be

$$\delta_X(A) = \limsup_{n\to\infty} \frac{|A \cap B_{G,X}(n)^r|}{|B_{G,X}(n)|^r}.$$

We say that $A$ is *negligible* with respect to $X$ if $\delta_X(A) = 0$, and we say that $A$ is *generic* if $G \setminus A$ is negligible.

*Remark* 1.7. Instead of taking lim sup in Definition 1.6, one might alternatively take lim inf or the $\omega$-limit for a non-principal ultrafilter $\omega$ on $\mathbb{N}$. We chose to work with lim sup as being 'negligible' (or 'generic') in the sense of Definition 1.6 implies being such in the sense of such alternative definitions. In particular, if $\delta_X(A) = 0$ for some $A \subseteq G^r$, then actually $\frac{|A \cap B_{G,X}(n)^r|}{|B_{G,X}(n)|^r} \to 0$ as $n \to \infty$.

## 1.4 Degree of commutativity

Given a finite group $G$, one may ask what is the probability that two elements, chosen from $G$ uniformly and independently at random, commute. In particular, we may define the *degree of commutativity* of $G$ to be

$$\mathrm{dc}(G) = \frac{|\{(g, h) \in G \times G \mid gh = hg\}|}{|G|^2}.$$

This was first introduced by Erdős and Turán [ET68] and widely studied ever since. For instance, Gustafson has shown that if $\mathrm{dc}(G) > 5/8$ then $G$ is abelian [Gus73], while Peter Neumann has shown that if, for a given $\alpha > 0$, we have $\mathrm{dc}(G) \geq \alpha$, then $G$ is (finite of order $\leq N_\alpha$)-by-abelian-by-(finite of order $\leq N_\alpha$) for some constant $N_\alpha \in \mathbb{N}$ [Neu89, Theorem 1].

One may ask if we could generalise this definition to infinite groups $G$. If $G$ is finitely generated, then one of the most natural-sounding generalisations is to consider the probability that two elements in a large ball of a Cayley graph of $G$ commute. Thus, the following definition was introduced by Antolín, Martino and Ventura in [AMV17].

**Definition 1.8** (Degree of commutativity)**.** Let $G$ be a finitely generated group and let $X$ be a finite generating set for $G$. Then the *degree of commutativity* of $G$ with respect to $X$ is

$$\begin{aligned}\mathrm{dc}_X(G) &= \limsup_{n\to\infty} \frac{|\{(g, h) \in B_{G,X}(n) \times B_{G,X}(n) \mid gh = hg\}|}{|B_{G,X}(n)|^2} \\ &= \delta_X(\{(g, h) \in G^2 \mid gh = hg\}).\end{aligned}$$

Note that if $G$ is finite then $B_{G,X}(n) = G$ for all sufficiently large $n$ and so we have $\mathrm{dc}_X(G) = \mathrm{dc}(G)$ for any generating set $X$. Generalisations of Definition 1.8 also exist: for instance, in [Toi17], Tointon considers degree of commutativity of a group $G$ with respect to sequences of measures on $G$.

The motivating question in this area is the following: if we have a finitely generated group $G$ that is not virtually abelian, does it follow that $\mathrm{dc}_X(G) = 0$ for any finite generating set $X$? This is believed to be true.

**Conjecture 1.9** (Antolín–Martino–Ventura [AMV17, Conjecture 1.8]). *Let $G$ be a finitely generated group, and $X$ a finite generating set for $G$. Then,*

(i) $\mathrm{dc}_X(G) > 0$ *if and only if $G$ is virtually abelian;*

(ii) $\mathrm{dc}_X(G) > 5/8$ *if and only if $G$ is abelian.*

This conjecture is true for all finitely generated residually finite groups of subexponential growth [AMV17, Theorem 1.5], hyperbolic [AMV17, Theorem 1.9] and relatively hyperbolic [Val17, Corollary 1.5] groups, and ascending HNN-extensions of $\mathbb{Z}^m$ (including soluble Baumslag–Solitar groups and fundamental groups of orientable prime 3-dimensional *Nil*- and *Sol*-manifolds) [Val18b, Theorem 1.12]. Moreover, it follows from [AMV17, Theorem 1.5] and Theorem 1.16 in Paper 2 that part (i) of Conjecture 1.9, if it is true, implies both part (ii) and independence of $\mathrm{dc}_X(G)$ from the choice of $X$.

In Paper 1 we prove Conjecture 1.9 for graph products $\Gamma\mathcal{G}$ over finite graphs with a suitable choice of a generating set. For each group $G_v$ in $\mathcal{G} = \{G_v \mid v \in V(\Gamma)\}$, we choose a finite generating set $X_v$ and assume that each $(G_v, X_v)$ satisfies the following two conditions:

(i) $G_v$ has rational growth with respect to $X_v$: in this case, the graph product $\Gamma\mathcal{G}$ has rational growth with respect to $X = \bigcup_{v\in V(\Gamma)} X_v$ (due to Chiswell [Chi94, Corollary 1]), and so we may apply Theorem 1.4 to $\Gamma\mathcal{G}$ and its full subgroups.

(ii) There exist constants $P, \beta \in \mathbb{N}$ such that $|C_{G_v}(g) \cap B_{G_v,X_v}(n)| \leq Pn^\beta$ for all $n \geq 1$ and all non-trivial $g \in G_v$: this allows us to control growth of centralisers in $\Gamma\mathcal{G}$, which were explicitly described by Barkauskas [Bar07].

Note that these two conditions are satisfied for any finitely generated virtually abelian group (with respect to any generating set), and in particular for the cyclic groups $\mathbb{Z}$ and $C_2$. In particular, we may apply the following result to right-angled Artin / Coxeter groups and their standard generating sets.

**Theorem 1.10** (Paper 1 [Val19b, Theorem 6]). *Let $\Gamma$ be a finite simple graph, and let $\mathcal{G} = \{G_v \mid v \in V(\Gamma)\}$ be a collection of finitely generated groups. Suppose that for each*

*$v \in V(\Gamma)$, there exists a finite generating set $X_v$ for $G_v$ such that $(G_v, X_v)$ satisfies the conditions (i) and (ii) above, and let $X = \bigcup_{v \in V(\Gamma)} X_v \subseteq \Gamma\mathcal{G}$. Then $\mathrm{dc}_X(\Gamma\mathcal{G}) > 0$ if and only if $\Gamma\mathcal{G}$ is virtually abelian.*

# 2 Background for Paper 2

Paper 2 aims to characterise groups $G$ by their degree of $k$-nilpotence – the probability that the $(k+1)$-step commutator of 'randomly chosen' elements of $G$ is trivial. We employ different notions of such a probability. In Section 2.1 we define what it means for a sequence of measures to *measure index uniformly* – a condition that is satisfied for measures coming from random walks or Følner sequences in amenable groups. In Section 2.2, we introduce the *degree of $k$-nilpotence* of a finitely generated group $G$ with respect to such sequences of measures, which turns out to be non-zero if and only if $G$ is virtually $k$-step nilpotent (Theorem 2.4). In Section 2.3, we explain why even if $G$ is virtually $k$-step nilpotent, its degree of $k$-nilpotence is independent of the sequence of measures chosen (as long as this sequence is reasonable enough): see Theorem 2.7. In Section 2.4, we consider residually finite groups $G$, and obtain the analogous conclusion for $G$ if we define its degree of $k$-nilpotence to be the infimum of degrees of $k$-nilpotence of finite quotients of $G$ (Theorem 2.11).

## 2.1 Measures on groups

In Paper 2 we consider various ways to measure subsets of groups. To make our arguments work, we usually require our measures to behave well with respect to finite index subgroups: in particular, we expect the 'density' of a subgroup of index $d$ (and any its coset) to be $1/d$. This motivates the following definition.

**Definition 2.1** (Uniform measurement of index)**.** For a group $G$, let $\mu_n : G \to [0,1]$, $n \in \mathbb{N}$, be a sequence of measures on $G$. We say that $(\mu_n)_{n=0}^{\infty}$ *measures index uniformly* if $\mu_n(xH) \to [G:H]^{-1}$ as $n \to \infty$ uniformly over all $x \in G$ and all subgroups $H \leq G$ (here we define $[G:H]^{-1} = 0$ if $H$ has infinite index in $G$). That is, $(\mu_n)$ measures index uniformly if for every $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that $\left|\mu_n(xH) - [G:H]^{-1}\right| < \varepsilon$ for all $x \in G$, $H \leq G$ and $n \geq N$.

The following examples of sequences of measures measuring index uniformly are due to Tointon [Toi17].

**Example 2.2.**

(i) *Let $G$ be a countable* amenable group*: that is, there exists a function $\mu : G \to [0,1]$ that is a* finitely additive *$(\mu(A \sqcup B) = \mu(A) + \mu(B)$ for all disjoint $A, B \subseteq G)$*

left-invariant *($\mu(gA) = \mu(A)$ for all $g \in G$ and $A \subseteq G$)* probability measure *($\mu(G) = 1$); note that we do note require $\mu$ to be countably additive. Examples of amenable groups include virtually solvable groups and finitely generated groups of subexponential growth. If $G$ is countable and amenable, then $G$ has a sequence of measures that measures index uniformly [Toi17, Theorem 1.13].*

(ii) *As a particular case of the example (i) above, note that if $G$ is a finitely generated group (with a finite generating set $X$, say) of subexponential growth, then the sequence $(\mu_n)_{n=0}^{\infty}$, defined by $\mu_n(A) = |B_{G,X}(m_n) \cap A| / |B_{G,X}(m_n)|$ for some strictly increasing sequence $(m_n)_{n=0}^{\infty} \subseteq \mathbb{N}$, measures index uniformly on $G$. This comes from the fact that for $F_n = B_{G,X}(m_n)$, the sequence $(F_n)_{n=0}^{\infty}$ is a* Følner sequence *in $G$: that is, $F_n \subseteq F_{n+1}$ for all $n$, $G = \bigcup_{n \in \mathbb{N}} F_n$, and $\frac{|gF_n \setminus F_n|}{|F_n|} \to 0$ as $n \to \infty$ for all $g \in G$.*

(iii) *Let $G$ be a finitely generated group (with a finite symmetric generating set $X$, say), and suppose $1 \in X$. For each $n \in \mathbb{N}$ and $A \subseteq G$, let $\mu_n(A)$ be the probability that a random walk on* $\mathrm{Cay}(G, X)$ *starting at $1 \in G$ will end up in $A$ after $n$ steps. Then $(\mu_n)_{n=0}^{\infty}$ measures index uniformly [Toi17, Theorem 1.14].*

Notice that for any sequence $(\mu_n)$ of measures on $G$ that measure index uniformly, we must have $\mu_n(G) \to 1$ as $n \to \infty$: in particular, the measures $\mu_n$ 'converge to' a probability measure. Indeed, in most cases the measures on groups we consider will be probability measures. Moreover, given any sequence $(\mu_n)$ of non-zero measures on $G$ we may 'rescale' these measures to make them probability measures; if these measures measure index uniformly on $G$, then this will not affect the results in Paper 2. Thus, we may assume without loss of generality that all the measures that we consider are probability measures.

### 2.2 Degree of nilpotence

Given a finite group, we may generalise its degree of commutativity, introduced in Section 1.4, to higher commutators. In particular, given $k \in \mathbb{Z}_{\geq 1}$ and a finite group $G$ we may define the *degree of $k$-nilpotence* of $G$ to be

$$\mathrm{dc}^k(G) = \frac{|\{(g_0, \ldots, g_k) \in G^{k+1} \mid [g_0, \ldots, g_k] = 1\}|}{|G|^{k+1}},$$

where $[g_0, \ldots, g_k]$ is the $(k+1)$-fold simple commutator of $g_0, \ldots, g_k$: see Definition 0.2. Notice that, for any finite group $G$, we have $\mathrm{dc}^1(G) = \mathrm{dc}(G)$, and $\mathrm{dc}^k(G) = 1$ if and only if $G$ is $k$-step nilpotent.

Given a sequence of measures on an infinite group, we may generalise the definition of degree of $k$-nilpotence as follows.

**Definition 2.3** (Degree of $k$-nilpotence)**.** Let $k \in \mathbb{Z}_{\geq 1}$, let $G$ be a group and let $M = (\mu_n)_{n=0}^{\infty}$ be a sequence of probability measures on $G$. Then each $\mu_n$ induces a product measure $\mu_n^{k+1} = \mu_n \times \cdots \times \mu_n$ on $G^{k+1}$. The *degree of $k$-nilpotence* of $G$ with respect to $M$ is

$$\mathrm{dc}_M^k(G) = \limsup_{n\to\infty} \mu_n^{k+1}(\{(g_0, \ldots, g_k) \in G^{k+1} \mid [g_0, \ldots, g_k] = 1\}).$$

We may expect a result of a similar flavour to Conjecture 1.9 hold in this setting as well: if $\mathrm{dc}_M^k(G) > 0$ then we may expect that $G$ contains a $k$-step nilpotent subgroup of finite index. If the sequence $M$ 'behaves well' with subgroups – that is, $M$ measures index uniformly – then such a naive conjecture is true, and even quantitative estimates can be given.

**Theorem 2.4** (Paper 2 [MTVV18, Theorem 1.8])**.** *Let $r, k \in \mathbb{Z}_{\geq 1}$ and let $\alpha > 0$. Then there exists a constant $m = m(r, k, \alpha) > 0$ such that the following holds. Let $G$ be a group generated by $r$ elements, let $M$ be a sequence of measures that measures index uniformly, and suppose that $\mathrm{dc}_M^k(G) \geq \alpha$. Then $G$ has a $k$-step nilpotent subgroup of index $\leq m$.*

In Paper 2, we also show that for all $m, k, d \in \mathbb{Z}_{\geq 1}$, if $G$ is finitely generated, $\Gamma \leq G$ is a subgroup of index $\leq m$, and $H \lhd \Gamma$ is a subgroup of cardinality $\leq d$ such that $\Gamma/H$ is $k$-step nilpotent, then $\mathrm{dc}_M^k(G) \geq \frac{1}{m^{k+1}d}$: see Paper 2, Proposition 1.12. Combining this with Theorem 2.4 gives the following result (although we expect this result to be known).

**Corollary 2.5** (Paper 2 [MTVV18, Corollary 1.13])**.** *For any $r, k, d \in \mathbb{Z}_{\geq 1}$, there exists a constant $m = m(r, k, d) > 0$ such that the following holds. Let $G$ be a group generated by $r$ elements, and let $H \lhd G$ be a subgroup of cardinality $\leq d$ such that $G/H$ is $k$-step nilpotent. Then $G$ has a $k$-step nilpotent subgroup of index $\leq m$.*

In Section 7 of Paper 2, we construct a sequence of finite groups showing that for any $k \in \mathbb{Z}_{\geq 1}$, any sufficiently large $d \in \mathbb{Z}_{\geq 1}$ and any sufficiently small $\alpha > 0$, the constants $m$ in Theorem 2.4 and Corollary 2.5 cannot be chosen to be independent of $r$. In fact, by taking a direct limit of these finite groups, we can show that there exists a group $G$ that is not virtually $k$-step nilpotent but satisfies all the assumptions in Theorem 2.4 and Corollary 2.5 apart from being generated by $r$ elements: see Proposition 1.14 in Paper 2.

## 2.3 Finitely generated virtually nilpotent groups

Most of our results in Paper 2 are related to degree of $k$-nilpotence of infinite groups. For finitely generated virtually nilpotent groups, our results apply in more generality. In particular, given a group $G$ with a sequence $M = (\mu_n)_{n=0}^{\infty}$ of measures on $G$, and given any equation $\varphi \in F_k * G$, we define the *degree of satisfiability* of $\varphi$ in $G$ with respect to $M$ as

$$\mathrm{d}\varphi_M(G) = \limsup_{n\to\infty} \mu_n^k(\{(g_1,\ldots,g_k) \in G^k \mid \varphi(g_1,\ldots,g_k) = 1\}).$$

Thus $\mathrm{dc}_M^k(G)$ is a special case of degree of satisfiability, where we consider the equation $c^{(k)} = [X_0,\ldots,X_k] \in F_{k+1}(X_0,\ldots,X_k) * G$.

*Remark* 2.6. Since (among finitely generated groups) virtually nilpotent groups are precisely the ones of polynomial growth [Gro81], it follows that a subsequence of balls in a Cayley graph of such a group $G$ can be used to define a sequence of measures that measure index uniformly: see Example 2.2 (ii). In fact, more is true in this case – it follows from a theorem of Pansu [Pan83] that $\frac{|B_{G,X}(n+1)|}{|B_{G,X}(n)|} \to 1$ as $n \to \infty$, which implies that the sequence of *all* balls, $(B_{G,X}(n))_{n=0}^{\infty}$, is a Følner sequence in $G$. In particular, for virtually nilpotent groups $G$, our results in Paper 2 on sequences of measures that measure index uniformly fit in well with the setting of [AMV17] and of Paper 1, where only 'ball counting measures' are considered. In particular, we are able to answer Question 1.33 in Paper 2 positively when $G$ is a virtually nilpotent group.

For the next result, note that a finitely generated nilpotent group is virtually torsion free. To see this, note that such a group $G$ is polycyclic (by [Hal79, Theorem 1.8], for instance), and so a finite index subgroup $H \leq G$ has a subnormal series $\{1\} = H_0 \lhd H_1 \lhd \cdots \lhd H_r = H$ with $H_i/H_{i-1} \cong \mathbb{Z}$ for each $i$ (see [Seg83, Chapter 1, Proposition 2]) – in particular, $H$ is torsion free. It follows that any finitely generated virtually nilpotent group $G$ has a normal torsion-free nilpotent subgroup $N \lhd G$ of finite index.

**Theorem 2.7** (Paper 2 [MTVV18, Theorem 1.20 and Corollary 1.21])**.** *Let $G$ be a finitely generated virtually nilpotent group, and let $N \lhd G$ be a normal torsion-free nilpotent subgroup of finite index. Let $\varphi \in F_k * G$. Then we have a partition*

$$\{(g_1,\ldots,g_k) \in G^k \mid \varphi(g_1,\ldots,g_k) = 1\} = \mathcal{A} \sqcup \mathcal{B},$$

*where*

(i) *$\mathcal{A}$ is a union of cosets of $N^k$ in $G^k$, and*

(ii) *for every $\varepsilon > 0$, there exists a finite-index normal subgroup $K \lhd G$ such that $|\mathcal{B}K^k/K^k| \leq \varepsilon |G^k/K^k|$.*

*In particular, $\mathrm{d}\varphi_M(G) = a|G/N|^{-k}$ where $a$ is the number of cosets of $N^k$ contained in $\mathcal{A}$, for any sequence $M$ of measures that measure index uniformly on $G$.*

*Remark* 2.8. The main conclusion of Theorem 2.7 is algebraic, although it allows us to deduce the probabilistic result on $\mathrm{d}\varphi_M(G)$. In particular, the set $\mathcal{A}$ is a finite union of cosets of a finite-index subgroup $N^k \lhd G^k$, and so it has the 'expected' measure with respect to any sequence of measures that measure index uniformly, whereas the set $\mathcal{B}$ has to have zero measure: see Proposition 1.18 in Paper 2. This allows us to deduce the last statement of the Theorem.

To prove Theorem 2.7, we use the idea of polynomial mappings, introduced by Leibman in [Lei02]. In particular, given two groups $G$, $H$ and a function $\varphi : G \to H$, we may define the *$u$-derivative* (for $u \in G$) $\partial_u \varphi : G \to H$ by setting $\partial_u \varphi(x) = \varphi(x)^{-1}\varphi(ux)$. We then say that $\varphi$ is a *polynomial of degree* $\leq d$ (for some $d \in \mathbb{N}$) if $\partial_{u_0} \cdots \partial_{u_d} \varphi \equiv 1$ for all $u_0, \ldots, u_d \in G$.

If in addition $G$ and $H$ are finitely generated torsion-free nilpotent groups, then there is an alternative description of polynomial mappings from $G$ to $H$. In this case, $G$ and $H$ have central series with each factor infinite cyclic: see, for instance, [KM79, Theorem 17.2.2]. In particular, we may choose central series

$$\{1\} = G_0 < G_1 < \cdots < G_r = G \qquad \text{and} \qquad \{1\} = H_0 < H_1 < \cdots < H_s = H$$

where $G_{i+1}/G_i = \langle g_i G_i \rangle \cong \mathbb{Z}$ and $H_{j+1}/H_j = \langle h_j H_j \rangle \cong \mathbb{Z}$ for each $i$ and $j$. Then there exist bijective *coordinate mappings* $\alpha : \mathbb{Z}^r \to G$ and $\beta : \mathbb{Z}^s \to H$, defined by $\alpha(x_0, \ldots, x_{r-1}) = g_{r-1}^{x_{r-1}} \cdots g_0^{x_0}$ and $\beta(y_0, \ldots, y_{s-1}) = h_{s-1}^{y_{s-1}} \cdots h_0^{y_0}$, and the following is true.

**Proposition 2.9** (Leibman [Lei02, Proposition 3.12])**.** *A mapping $\varphi : G \to H$ is polynomial if and only if the mapping $\beta^{-1} \circ \varphi \circ \alpha : \mathbb{Z}^r \to \mathbb{Z}^s$ is polynomial.*

Here, the notion of a polynomial mapping from $\mathbb{Z}^r$ to $\mathbb{Z}^s$ agrees with the usual notion of a polynomial: that is, a mapping $\psi : \mathbb{Z}^r \to \mathbb{Z}^s$ is polynomial if and only if it is the 'evaluation' of an element $\tilde{\psi} \in (\mathbb{Q}[X_1, \ldots, X_r])^s$. Note that such a polynomial $\tilde{\psi}$ in general does not need to have integer coefficients, as long as it attains values in $\mathbb{Z}^s$ at any point of $\mathbb{Z}^r$: for instance, the mapping $\mathbb{Z} \to \mathbb{Z}, x \mapsto \frac{x^2+x}{2}$ is polynomial.

The key idea in the proof of Theorem 2.7 is that polynomial mappings $H \to G$ for $G$ nilpotent form a group under pointwise multiplication [Lei02, Theorem 3.2]. Since the maps $G^k \to G$ defined by $(g_1, \ldots, g_k) \to g_i$ (for $1 \leq i \leq k$) and $(g_1, \ldots, g_k) \mapsto h$ (for some constant $h \in G$) are clearly polynomial, it follows that, given an equation $\varphi \in F_k * G$, the map $(g_1, \ldots, g_k) \mapsto \varphi(g_1, \ldots, g_k)$ is also a polynomial. This can alternatively be seen, modulo Proposition 2.9, using a result of Hall [Hal79, Theorem 6.5], which says that the maps $G^2 \to G, (g, h) \mapsto gh$ and $G \to G, g \mapsto g^{-1}$ are polynomial mappings if $G$ is a finitely generated torsion-free nilpotent group.

If a group $G$ is merely finitely generated virtually nilpotent, then it has a torsion-free normal nilpotent subgroup $N \lhd G$ of finite index. Given $\varphi \in F_k * G$ and elements $n_1, \ldots, n_k \in N$ and $g_1, \ldots, g_k \in G$, note that $\varphi(n_1 g_1, \ldots, n_k g_k) \in N\varphi(g_1, \ldots, g_k)$, and so we may define a map

$$\begin{aligned}\varphi_{g_1,\dots,g_k} &: N^k \to N, \\ (n_1,\dots,n_k) &\mapsto \varphi(n_1g_1,\dots,n_kg_k)\varphi(g_1,\dots,g_k)^{-1}.\end{aligned}$$

We then show that

(i) for each $g_1,\dots,g_k \in G$, the mapping $\varphi_{g_1,\dots,g_k}$ is polynomial (Paper 2, Lemma 4.3); and

(ii) given a polynomial mapping $\psi : H \to N$, where $H$ is finitely generated and $N$ is nilpotent, and an element $x \in N$, either we have $\psi^{-1}(x) = H$, or, for each $\varepsilon > 0$, there exists a subgroup $K_\varepsilon \lhd H$ such that $|\psi^{-1}(x)K_\varepsilon/K_\varepsilon| \leq \varepsilon|H/K_\varepsilon|$ (Paper 2, Theorem 1.29).

After noticing that

$$\begin{aligned}&\{(g_1,\dots,g_k) \in G^k \mid \varphi(g_1,\dots,g_k) = 1\} \\ &\quad = \bigsqcup_{\substack{(t_1,\dots,t_k)\in\mathcal{T}^k \\ \varphi(t_1,\dots,t_k)\in N}} \varphi_{t_1,\dots,t_k}^{-1}\left(\varphi(t_1,\dots,t_k)^{-1}\right) \cdot (t_1,\dots,t_k),\end{aligned}$$

where $\mathcal{T}$ is a transversal for $N$ in $G$, Theorem 2.7 can be deduced from the facts (i) and (ii) above.

*Remark* 2.10. It is known that finitely generated virtually nilpotent groups are linear [Jen55], and therefore strongly equationally Noetherian. However, it is easy to see how one could use the fact (i) above to give a perhaps more direct proof that such groups are strongly equationally Noetherian. Indeed, this conclusion follows from the fact that there are no strictly descending infinite chains of intersections of kernels of polynomial mappings $\mathbb{Z}^r \to \mathbb{Z}^s$; to see this, one could use the Hilbert Basis Theorem to show that there are no infinite strictly descending chains of algebraic varieties in $\mathbb{Q}^r$.

## 2.4 Residually finite groups

Another way to consider degree of nilpotence of an infinite group is by looking at its finite quotients; this makes sense to consider mostly for residually finite groups. Recall that a group $G$ is said to be *residually finite* if for any non-trivial $g \in G$, there exists a quotient $\pi : G \to Q$ with $Q$ finite such that $\pi(g) \neq 1$.

**Theorem 2.11** (Paper 2 [MTVV18, Theorem 1.31]). *Let $G$ be a residually finite group, let $k \geq 1$, and suppose that there exists a constant $\alpha > 0$ such that $\mathrm{dc}^k(G/N) \geq \alpha$ for every finite-index normal subgroup $N \trianglelefteq G$. Then $G$ is virtually $k$-step nilpotent.*

This theorem generalises results of Lévai and Pyber, who prove it for $k = 1$ [LP00, Theorem 1.1 (iii)], and of Shalev, who proves it when $G$ is finitely generated [Sha18, Theorem 1.1]. It also answers a question posed by Shalev [Sha18, Problem 3.1].

For the proof of Theorem 2.11, we use the following auxilliary result, which might be of interest to finite group theorists. This theorem generalises results of Gallagher, who proves it for $k = 1$ [Gal70], and of Moghaddam, Salemkar and Chiti [MSC05, Theorem A], who prove it when the centraliser of each element of $G$ is normal.

**Theorem 2.12** (Paper 2 [MTVV18, Theorem 1.32]). *Let $G$ be a finite group and let $N \trianglelefteq G$. Then $\mathrm{dc}^k(G) \leq \mathrm{dc}^k(N)\,\mathrm{dc}^k(G/N)$ for every $k \geq 1$.*

To prove Theorem 2.11, we assume that $G$ is not virtually $k$-step nilpotent and construct a chain $G = G_0 > G_1 > G_2 > \cdots$ of finite-index normal subgroups of $G$ such that $G_i/G_{i+1}$ is not $k$-step nilpotent. We then use a result of Erfanian, Rezaei and Lescot [ERL07, Theorem 5.1], stating that there exists a constant $\gamma_k < 1$ such that $\mathrm{dc}^k(H) \leq \gamma_k$ whenever $H$ is a finite group that is not $k$-step nilpotent. Using Theorem 2.12, this allows us to get the estimate

$$\mathrm{dc}^k(G/G_n) \leq \prod_{i=0}^{n-1} \mathrm{dc}^k(G_i/G_{i+1}) \leq \gamma_k^n,$$

and so $\mathrm{dc}^k(G/G_n) \to 0$ as $n \to \infty$. In particular, no constant $\alpha > 0$ as in Theorem 2.11 can exist.

## 3 Background for Paper 3

In Paper 3 we study a specific action of a graph product on a hyperbolic metric space which we use to show that certain graph products are equationally Noetherian. In Section 3.1 we introduce *quasi-median graphs*, generalising the notion of a CAT(0) cube complex – there is such a graph canonically associated to any graph product. In Section 3.2, we study combinatorics of hyperplanes in a quasi-median graph (explored by Genevois in his thesis [Gen17]), akin to the combinatorics of hyperplanes in CAT(0) cube complex (explored by Sageev in [Sag95]). This allows us to define the *contact graph* of a quasi-median graph, which turns out to be a quasi-tree (Theorem 3.7). In Section 3.3, we introduce *acylindrical* group actions on metric spaces; it turns out that for many groups acting on quasi-median graphs, including graph products, the induced action on the contact graph is acylindrical (Theorem 3.13). In Section 3.4, we introduce *$\mathcal{AH}$-accessibility* of groups – existence of the 'largest' cobounded acylindrical action on a hyperbolic space for a given group; the action of a graph product on the contact graph can be used to show that many graph products are $\mathcal{AH}$-accessible (Corollary 3.17). In

Section 3.5, we give our main application of the action of a graph product on the contact graph: we explain how, using the methods of Groves and Hull [GH17], this action can be used to show that many graph products of equationally Noetherian groups are equationally Noetherian (Theorem 3.21).

## 3.1 Quasi-median graphs

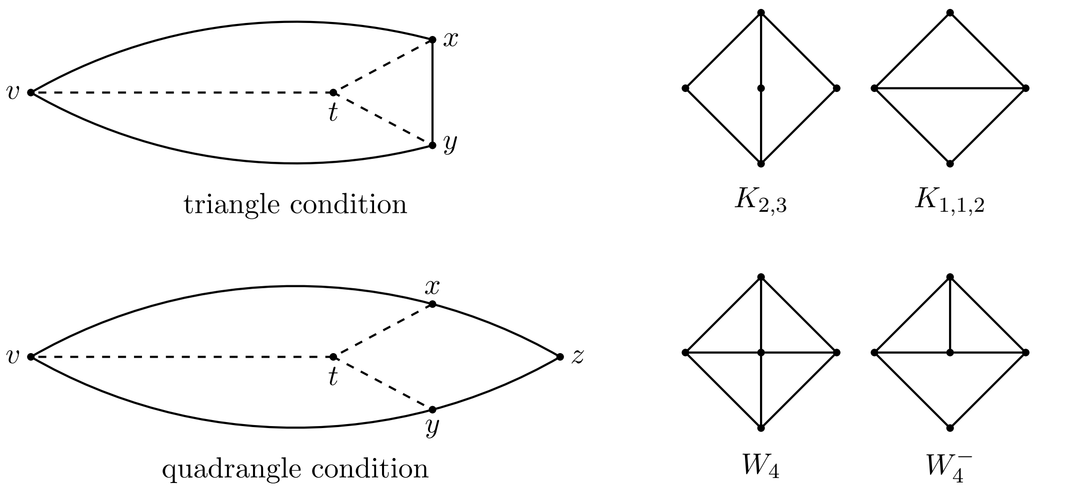
Recall that a connected graph $X$ is called *median* if for any three vertices $x, y, z \in V(X)$, there exists a *unique* vertex $m \in V(X)$ such that there are geodesics in $X$ between $x$ and $y$, between $x$ and $z$ and between $y$ and $z$, all passing through $m$. The class of quasi-median graphs, introduced here, is a generalisation of median graphs.

**Definition 3.1** (Quasi-median graph)**.** We say a connected graph $X$ is *weakly modular* if $X$ satisfies:

**triangle condition:** for any $k \geq 1$ and any vertices $v, x, y \in V(X)$ such that $d_X(v,x) = d_X(v,y) = k$ and $d_X(x,y) = 1$, there exists $t \in V(X)$ such that $d_X(v,t) = k-1$ and $d_X(x,t) = d_X(y,t) = 1$; and

**quadrangle condition:** for any $k \geq 1$ and any vertices $v, x, y, z \in V(X)$ such that $d_X(v,z) = k+1$, $d_X(v,x) = d_X(v,y) = k$ and $d_X(x,z) = d_X(y,z) = 1$, there exists $t \in V(X)$ such that $d_X(v,t) = k-1$ and $d_X(x,t) = d_X(y,t) = 1$.

We say $X$ is a *quasi-median graph* if it is weakly modular and does not contain induced subgraphs isomorphic to $K_{2,3}$ or $K_{1,1,2}$.




(a) Triangle and quadrangle conditions. (b) The graphs $K_{2,3}$, $K_{1,1,2}$, $W_4$, $W_4^-$.

FIGURE 2: Arrangements appearing in Definition 3.1 and Remark 3.3.

It is not hard to show that any median graph is quasi-median. More precisely, it is known that a (simplicial) graph $X$ is median if and only if $X$ is quasi-median and triangle-free: see, for instance, [Gen17, Corollary 2.92].

We are interested in quasi-median graphs mostly due to their applications in studying graph products. In particular, we have the following result.

**Theorem 3.2** (Genevois [Gen17, Proposition 8.2])**.** *Let $\Gamma$ be a simplicial graph, and let $\mathcal{G} = \{G_v \mid v \in V(\Gamma)\}$ be a collection of non-trivial groups. Let $S = \bigcup_{v \in V(\Gamma)}(G_v \setminus \{1\})$ be a generating set for $\Gamma\mathcal{G}$. Then the Cayley graph* $\mathrm{Cay}(\Gamma\mathcal{G}, S)$ *is quasi-median.*

It was shown by Chepoi [Che00, Theorem 6.1] that the class of CAT(0) cube complexes coincides with median graphs: in particular, a graph is median if and only if it coincides with the 1-skeleton of a CAT(0) cube complex. Thus, quasi-median graphs can be viewed as generalisations of CAT(0) cube complexes.

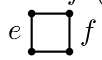
Indeed, a useful way to think of a quasi-median graph $X$ is as of a *CAT(0) prism complex*: given a maximal induced Hadamard subgraph of $X$ – that is, a cartesian product (in the graph theory sense) of a collection of complete subgraphs – we may 'fill it in' with a prism – a cartesian product (in the set theory sense) of simplices (possibly infinite-dimensional) – in the obvious way. If we then give a metric to the resulting CW-complex by giving each prism the usual cartesian metric, the resulting complex turns out to be CAT(0) [Gen17, Theorem 2.120]. We also have a description of when the 1-skeleton of a prism complex is a quasi-median graph [Gen17, Theorem 2.127], akin to the result saying that a cube complex is CAT(0) if and only if it is simply connected and the link of each vertex is a flag simplicial complex [Gro87, p. 122, 4.2.C].

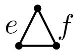
*Remark* 3.3. An alternative generalisation of CAT(0) cube complexes (and their 1-skeleta, median graphs) is given by bucolic complexes (and their 1-skeleta, bucolic graphs), introduced in [BCC$^+$13]. A graph is said to be *bucolic* if it is weakly modular and does not contain $K_{2,3}$, $W_4$, $W_4^-$ and infinite complete graphs as induced subgraphs (see Figure 2(b)); similarly, a prism complex is *bucolic* if its 1-skeleton is a bucolic graph not containing infinite hypercubes (see [BCC$^+$13, Theorem 1]). Although both classes – bucolic graphs and quasi-median graphs – generalise the class of median graphs, neither of the former two classes includes the other. Indeed, it is easy to check that the graph $K_{1,1,2}$ is bucolic but not quasi-median. Conversely, the complete graph on infinitely many vertices, $K_\infty$, is quasi-median but not bucolic. However, it is clear that a bucolic graph that does not have $K_{1,1,2}$ as an induced subgraph is quasi-median, whereas a locally finite quasi-median graph is bucolic (as $K_{1,1,2}$ is an induced subgraph of both $W_4$ and $W_4^-$).

For our purposes, we found it more suitable to work with quasi-median graphs, rather than bucolic graphs. The main reason for this is that bucolic quasi-median graphs are, in most cases, locally finite. For example, the Cayley graph $\mathrm{Cay}(\Gamma\mathcal{G}, S)$ introduced in Theorem 3.2 is bucolic if and only if $\Gamma$ is locally finite and all the groups in $\mathcal{G}$ are finite, and so bucolic graphs would only allow us to study graph products of *finite* groups in this way.

### 3.2 Hyperplanes in quasi-median graphs

In [Sag95], Sageev introduced a rich combinatorial theory for CAT(0) cube complexes, notably, hyperplanes – see below. In his PhD thesis [Gen17], Genevois generalised this theory to quasi-median graphs, and did in-depth analysis of applications of quasi-median graphs to geometric group theory.

**Definition 3.4** (Hyperplanes)**.** Let $X$ be a quasi-median graph. Let $\sim$ be an equivalence relation on the edge set $E(X)$ generated by relations $e \sim f$ (where $e, f \in E(X)$) whenever $e$ and $f$ are either opposite sides of a square ( 

) or two sides of a triangle ( 

). A *hyperplane* in $X$ is just an equivalence class of edges under $\sim$: that is, an element $[e] \in E(X)/\sim$. If $H = [e]$ for some $e \in E(X)$, we say that $H$ and $e$ are *dual* to each other.

Hyperplanes have proven to be a powerful tool in the study of CAT(0) cube complexes. In particular, each hyperplane $H = [e]$ can be thought to 'separate' a CAT(0) cube complex $X$ into two connected components, called *halfspaces*, obtained by removing $[e] \subseteq E(X)$ from the 1-skeleton of $X$. It is clear that the set of half-spaces can be equipped with the structure of a *pocset* – a poset with an order-reversing involution – where the order relation and the involution are inclusion and complementation, respectively. And conversely, given a pocset one may build a median graph (and so a CAT(0) cube complex) out of it – this is called *cubulation of a pocset*. See [Sag14] for details.

An analogous theory has been developed for quasi-median graphs $X$ by Genevois [Gen17, Section 2.4]. The main difference from CAT(0) cube complexes here is that a hyperplane does not (in general) separate $X$ into exactly two halfspaces, but rather into $N \geq 2$ (possibly $N = \infty$) *sectors*.

*Remark* 3.5. Given a quasi-median graph $X$, we can construct a CAT(0) cube complex $\widetilde{X}$ out of $X$ as follows. Consider a pocset consisting of all sectors delimited by hyperplanes in $X$ and their complements, with the order relation and the involution given by inclusion and complementation, as before. We may then cubulate this pocset to obtain a CAT(0) cube complex $\widetilde{X}$. This complex can be also thought of as a 'binary subdivision' of $X$, as follows. Viewing $X$ as a prism complex, where each prism is a cartesian product of simplices, we may replace each of these simplices by a star (the graph ) by adding an extra vertex and joining it to each vertex of the original simplex. Prisms are then replaced by cartesian products of these stars in the obvious way.

Now let $\Gamma$ be a simplicial graph, let $\mathcal{G} = \{G_v \mid v \in V(\Gamma)\}$ be a collection of non-trivial groups, and let $X$ be the quasi-median graph associated to $\Gamma\mathcal{G}$, as given in Theorem 3.2. Then the CAT(0) cube complex $\widetilde{X}$ built from $X$ as above is precisely the *Davis complex* associated to the graph product $\Gamma\mathcal{G}$: see [Dav98]. However, we find the action of $\Gamma\mathcal{G}$ on $X$ to be easier to study than that on $\widetilde{X}$: for instance, unless $|G_v| < \infty$ for every $v \in V(\Gamma)$, the Davis complex $\widetilde{X}$ will not be locally compact and will contain vertices with infinite stabilisers. It is therefore not immediately clear how we could weaken the conditions in Theorem 3.13 below, say, if we require $X$ to be a CAT(0) cube complex, so that this result is still applicable to all graph products over finite graphs. We thus chose to work with arbitrary quasi-median graphs instead.

It is worth noting, however, that there is a lot of similarity between $X$ and $\widetilde{X}$. In many cases of interest – for instance, when $X$ is the quasi-median graph associated to a graph product $\Gamma\mathcal{G}$ over a finite graph $\Gamma$ – $X$, $\mathcal{C}X$ and $\Delta X$ turn out to be equivariantly quasi-isometric to $\widetilde{X}$, $\mathcal{C}\widetilde{X}$ and $\Delta\widetilde{X}$, respectively. We may therefore expect that there should be no intrinsic advantage in studying $\widetilde{X}$ instead of $X$ (or vice versa).

Two other objects we construct for quasi-median graphs are associated contact and crossing graphs. Roughly speaking, vertices in the crossing graph are hyperplanes, and two vertices are adjacent if thee corresponding hyperplanes are not separated by a third hyperplane.

**Definition 3.6** (Contact graph, Crossing graph)**.** Let $X$ be a quasi-median graph, let $e, e' \in E(X)$, and let $H = [e]$, $H' = [e']$ be hyperplanes. Suppose that $e$ and $e'$ are distinct but share an endpoint. Then we say that $H$ and $H'$ *intersect* if $e$ and $e'$ are adjacent edges in a square, and we say that $H$ and $H'$ *osculate* otherwise.

We now define the *crossing graph* $\Delta X$ and the *contact graph* $\mathcal{C}X$ of $X$, as follows. We let $V(\Delta X) = V(\mathcal{C}X) = E(X)/{\sim}$, the hyperplanes of $X$. We let vertices $H$ and $H'$ be adjacent in $\Delta X$ (respectively $\mathcal{C}X$) if $H$ and $H'$ intersect (respectively either intersect or osculate).

In fact, the crossing graph will in many cases be quasi-isometric to the contact graph, as shown in Paper 3, Theorem B (i). We have decided to study the contact graph instead of the crossing graph for two reasons: because $\mathcal{C}X$, unlike $\Delta X$, is always connected, and (more specifically) because of the following theorem.

**Theorem 3.7** (Hagen [Hag14, Theorem 4.1]; Paper 3 [Val18a, Theorem A])**.** *Let $X$ be a CAT(0) cube complex, or a quasi-median graph. Then $\mathcal{C}X$ is quasi-isometric to a tree.*

## 3.3 Acylindrical actions

We will be interested in group actions on hyperbolic metric spaces. In particular, we may define acylindrical actions, generalising the notion of a properly discontinuous action on a metric space.

**Definition 3.8** (Acylidrically hyperbolic group)**.** An action of a group $G$ on a metric space $(Y, d)$ by isometries is said to be *acylindrical* if for any $\varepsilon > 0$, there exist constants $D_\varepsilon, N_\varepsilon > 0$ such that for any $x, y \in Y$ with $d(x, y) \geq D_\varepsilon$, there are at most $N_\varepsilon$ elements $g \in G$ such that $d(x, x^g) \leq \varepsilon$ and $d(y, y^g) \leq \varepsilon$: see Figure 3. A group $G$ that is not virtually cyclic is said to be *acylindrically hyperbolic* if $G$ acts on a hyperbolic metric space acylindrically and with unbounded orbits.

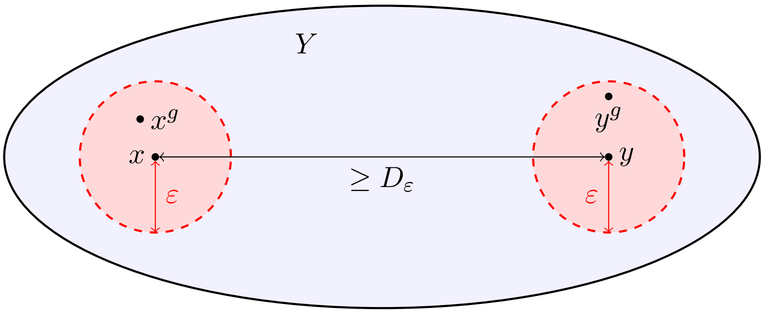



FIGURE 3: Acylindricity condition.

If a group $G$ acts on a hyperbolic metric space $Y$, we may introduce the following terminology. We say that an element $g \in G$ is *elliptic* with respect to $G \curvearrowright Y$ if some (equivalently, any) $\langle g \rangle$-orbit in $X$ is bounded. We say that $g \in G$ is *loxodromic* with respect to $G \curvearrowright Y$ if for some (equivalently, any) $x \in Y$, the map $\mathbb{Z} \to Y, n \mapsto x^{g^n}$ is a quasi-isometric embedding. It is clear that an elliptic element cannot be loxodromic; thus, the following result allows us to partition all elements of $G$ into elliptic and loxodromic.

**Lemma 3.9** (Bowditch [Bow08, Lemma 2.2])**.** *If a group $G$ acts on a hyperbolic metric space $Y$ acylindrically, then every element of $G$ is either elliptic or loxodromic with respect to $G \curvearrowright Y$.*

**Example 3.10.**

(i) *Any hyperbolic group $G$ (that is not virtually cyclic) is acylindrically hyperbolic, as it acts properly discontinuously, and therefore acylindrically, on the (locally finite) hyperbolic Cayley graph $Y = \mathrm{Cay}(G, S)$, where $S$ is any finite generating set for $G$.*

(ii) *Let $G$ be a group which is* relatively hyperbolic *with respect to a collection of subgroups $\{H_\lambda \mid \lambda \in \Lambda\}$: roughly speaking, this means that $G$ 'looks hyperbolic' after we cone-off cosets of the subgroups $H_\lambda$. Then, given a finite subset $S \subseteq G$ such that $S \cup \bigcup_{\lambda\in\Lambda} H_\lambda$ generates $G$, the Cayley graph $Y = \mathrm{Cay}\left(G, S \cup \bigcup_{\lambda\in\Lambda} H_\lambda\right)$ is hyperbolic [Osi06, Theorem 1.7] and the action of $G$ on $Y$ is acylindrical [Osi16, Proposition 5.2]. In particular, $G$ is either virtually cyclic or acylindrically hyperbolic.*

(iii) *Let $\Sigma$ be a compact orientable surface and suppose that $3g + p \geq 5$, where $g$ and $p$ are the genus and number of boundary components of $\Sigma$, respectively. Let $Y$ be the* curve complex *of $\Sigma$ – a simplicial complex whose vertices are essential non-peripheral simple closed curves on $\Sigma$, and a collection of such curves spans a simplex in $Y$ if they can be realised disjointly. Then $Y$ is hyperbolic [MM99, Theorem 1.1], and the* mapping class group *$G$ of $\Sigma$ – the group of homotopy classes of* *orientation-preserving homeomorphisms $\Sigma \to \Sigma$ – acts on $Y$ acylindrically [Bow08, Theorem 1.3]. In particular, $G$ is acylindrically hyperbolic.*

(iv) *Let $X$ be a CAT(0) cube complex. Let $G$ be a group acting properly, cocompactly and cellularly on $X$ with a $G$-invariant* factor system *(see [BHS17, Definition 8.1]): the latter is a technical condition which is satisfied for all known examples of such actions $G \curvearrowright X$. Then, by a result of Behrstock, Hagen and Sisto [BHS17, Corollary 14.5], the induced action of $G$ on the contact graph $\mathcal{C}X$ is acylindrical. Combining this with Theorem 3.7 implies that if $\mathcal{C}X$ is unbounded, then $G$ is either virtually cyclic or acylindrically hyperbolic.*

A condition seemingly significantly weaker than acylindrical hyperbolicity was also introduced by Bestvina and Fujiwara in [BF02]. In particular, let a group $G$ act on a hyperbolic metric space $(Y, d)$, and let $h \in G$ be a loxodromic element with respect to this action. We say that $h$ satisfies the *weak proper discontinuity condition* (or $h$ is a *WPD element* for short) if for every $\varepsilon \geq 0$ and $x \in Y$, there exists $n \in \mathbb{N}$ such that there are only finitely many elements $g \in G$ for which $d(x, x^g) \leq \varepsilon$ and $d(x^{h^n}, x^{h^n g}) \leq \varepsilon$.

It is clear that if $G$ acts on a hyperbolic metric space acylindrically, then every loxodromic element of $G$ (with respect to this action) will be a WPD element. Perhaps surprisingly, we can also go the other way: given an action with at least one loxodromic WPD element of $G$, we may construct an acylindrical action of $G$ on a (possibly different) hyperbolic metric space. In particular, this leads to the following result.

**Theorem 3.11** (Osin [Osi16, Theorem 1.2])**.** *Let $G$ be a group that is not virtually cyclic. Then $G$ is acylindrically hyperbolic if and only if $G$ admits an action on a hyperbolic metric space with at least one loxodromic WPD element.*

This allows us to find new examples of acylindrically hyperbolic groups.

**Example 3.12.**

(i) *Let $G = \mathrm{Out}(F_n)$ be the outer automorphism group of a free group of rank $n \geq 2$, and let $g \in G$ be (represented by) a* fully irreducible *automorphism (also called an* irreducible with irreducible powers*, or* IWIP*, automorphism): that is, an automorphism $g$ such that $g^k(F)$ is not conjugate to $F$ for any proper non-trivial free factor $F < F_n$ and any $k \geq 1$. In [BF10], Bestvina and Feighn constructed a hyperbolic graph on which $G$ acts acylindrically, and such that $g$ is a loxodromic WPD element with respect to this action. In particular, by Theorem 3.11, $G$ is acylindrically hyperbolic. It is worth noting that an explicit example of a hyperbolic metric space on which $G$ acts acylindrically is not known.*

(ii) *Let $\Gamma$ be a finite simplicial graph with $\geq 2$ vertices and connected complement, and let $\mathcal{G} = \{G_v \mid v \in V(\Gamma)\}$ be a collection of non-trivial groups. Then the graph* *product $\Gamma\mathcal{G}$ splits as an amalgamated free product, $\Gamma\mathcal{G} \cong \Gamma_A\mathcal{G}_A *_{\Gamma_C\mathcal{G}_C} \Gamma_B\mathcal{G}_B$, where $A = V(\Gamma) \setminus \{v\}$, $B = \mathrm{star}(v)$ and $C = \mathrm{link}(v)$ for some vertex $v \in V(\Gamma)$. In [MO15], Minasyan and Osin showed that the action of $\Gamma\mathcal{G}$ on the associated Bass-Serre tree has loxodromic WPD elements. In particular, if either $|V(\Gamma)| \geq 3$ or $|G_v| \geq 3$ for some $v \in V(\Gamma)$, then $\Gamma\mathcal{G}$ is acylindrically hyperbolic by Theorem 3.11.*

In Paper 3 we study acylindrical hyperbolicity of graph products. One of the aims of this paper was to find an explicit description of an acylindrical action of a graph product on a hyperbolic metric space, whose existence is given by Example 3.12 (ii) and Theorem 3.11. Our description is analogous to the description given in 3.10 (iv); in particular, we prove the following theorem. Here, by a *special* action of a group $G$ on a quasi-median graph $X$ we mean that the quotient $X/G$ is a quasi-median analogue of a special cube complex in the sense of Haglund and Wise [HW07]; see Definition 2.5 in Paper 3 or [Gen17, Definition 4.4] for a precise definition.

**Theorem 3.13** (Paper 3 [Val18a, Theorem B (ii)]). *Let a group $G$ act specially on a quasi-median graph $X$. Suppose there exists a constant $D \in \mathbb{N}$ such that each stabiliser $\mathrm{Stab}_G(x)$, for $x \in V(X)$, is a finite group of cardinality $\leq D$, and each vertex of $\Delta X/G$ has $\leq D$ neighbours. Then the induced action $G \curvearrowright \mathcal{C}X$ is acylindrical.*

We may thus apply this result to the quasi-median graph $X$ associated to a graph product $\Gamma\mathcal{G}$, described in Theorem 3.2. Indeed, Genevois showed in [Gen17, Chapter 8] that the action $\Gamma\mathcal{G} \curvearrowright X$ is special [Gen17, Proposition 8.11] and that the quotient $\Delta X/G$ is isomorphic to $\Gamma$ [Gen17, Lemmas 8.6 and 8.12]; moreover, as $X$ is a Cayley graph of $\Gamma\mathcal{G}$, vertex stabilisers under this action are trivial. Thus, the following result follows from Theorem 3.13.

**Corollary 3.14** (Paper 3 [Val18a, Corollary C]). *Let $\Gamma$ be a simplicial graph of bounded degree with $|V(\Gamma)| \geq 2$, and let $\mathcal{G} = \{G_v \mid v \in V(\Gamma)\}$ be a collection of non-trivial groups. Let $X$ be the quasi-median graph associated to $\Gamma\mathcal{G}$, as given in Theorem 3.2. Then the action $\Gamma\mathcal{G} \curvearrowright \mathcal{C}X$ is acylindrical. In particular, if the complement of $\Gamma$ is connected and either $|V(\Gamma)| \geq 3$ or $|G_v| \geq 3$ for some $v \in V(\Gamma)$, then $\Gamma\mathcal{G}$ is acylindrically hyperbolic.*

### 3.4 $\mathcal{AH}$-accessibility

Another notion we use is the poset of actions of groups on (usually hyperbolic) metric spaces, explored by Abbott, Balasubramanya and Osin in [ABO17]. Given a group $G$, we may equip the set of (equivalence classes of) actions of $G$ on metric spaces with a partial order, as follows.

**Definition 3.15** (Dominates, Weakly equivalent). Let $G$ be a group, and let $X$ and $Y$ be two metric spaces on which $G$ acts by isometries. We say the action $G \curvearrowright X$

*dominates* $G \curvearrowright Y$, written $G \curvearrowright Y \preceq G \curvearrowright X$, if for some (equivalently, every) $x \in X$ and $y \in Y$, there exists a constant $C \in \mathbb{N}$ such that

$$d_Y(y, y^g) \leq C d_X(x, x^g) + C$$

for every $g \in G$. We say two actions $G \curvearrowright X$ and $G \curvearrowright Y$ are *weakly equivalent* if $G \curvearrowright Y \preceq G \curvearrowright X$ and $G \curvearrowright X \preceq G \curvearrowright Y$. As $\preceq$ is clearly reflexive and transitive, this defines an equvalence relation; we denote the equivalence class of $G \curvearrowright X$ by $[G \curvearrowright X]$.

Note that, given any two actions $G \curvearrowright X$ and $G \curvearrowright Y$, if there exists a coarsely $G$-equivariant quasi-isometry $X \to Y$ then the actions $G \curvearrowright X$ and $G \curvearrowright Y$ are weakly equivalent. The converse is not true in general: for instance, the action $\mathbb{Z} \curvearrowright \mathbb{R}$ by translations is weakly equivalent to the action $\mathbb{Z} \cong \langle x \rangle \curvearrowright X = \mathrm{Cay}(F_2(x,y), \{x,y\})$, obtained by restricting to $\langle x \rangle$ the usual action of $F_2(x,y)$ on its Cayley graph $X$, but $\mathbb{R}$ is not quasi-isometric to $X$. However, if we restrict to cobounded actions, then weak equivalence of $G \curvearrowright X$ and $G \curvearrowright Y$ is equivalent to existence of a coarsely $G$-equivariant quasi-isometry [ABO17, Lemma 3.8].

Definition 3.15 allows us to give the set of equivalence classes of isometric actions of $G$ on metric spaces the structure of a poset $\mathfrak{P}(G)$. One may then study the structure of $\mathfrak{P}(G)$: in particular, one may ask if this poset or a particular induced sub-poset has a largest element.

We are in particular interested in the sub-poset $\mathcal{AH}(G) \subseteq \mathfrak{P}(G)$ of cobounded acylindrical actions of a group $G$ on hyperbolic metric spaces. We say the group $G$ is *$\mathcal{AH}$-accessible* if this poset $\mathcal{AH}(G)$ has a largest element $[G \curvearrowright X]$, and we say $G$ is *strongly $\mathcal{AH}$-accessible* if in addition $G \curvearrowright Y \preceq G \curvearrowright X$ for all (not necessarily cobounded) acylindrical actions $G \curvearrowright Y$ with $Y$ hyperbolic.

**Example 3.16.**

(i) *([ABO17, Example 7.7]) If $G$ is hyperbolic, then the action $G \curvearrowright \mathrm{Cay}(G,S)$ dominates any other action of $G$, where $S$ is a finite generating set for $G$: see, for instance, [ABO17, Lemma 3.10]. In particular, as $\mathrm{Cay}(G,S)$ is hyperbolic and this action is proper and cocompact (hence, acylindrical and cobounded), $G$ is strongly $\mathcal{AH}$-accessible.*

(ii) *([ABO17, Example 7.7]) If $G$ is a group that is not acylindrically hyperbolic, then either $G$ is virtually cyclic (in which case $G$ is strongly $\mathcal{AH}$-accessible by the previous example), or every acylindrical action of $G$ on a hyperbolic space has bounded orbits, and so is weakly equivalent to the action of $G$ on a point. Thus $G$ is strongly $\mathcal{AH}$-accessible.*

(iii) *Many acylindrically hyperbolic groups are also $\mathcal{AH}$-accessible: for instance, it was shown in [ABO17, Theorem 2.18] that finitely generated relatively hyperbolic groups*

> *with non-acylindrically hyperbolic parabolics and right-angled Artin groups are $\mathcal{AH}$-accessible. Moreover, hierarchically hyperbolic groups – for instance, mapping class groups, or groups acting properly and cocompactly on a CAT(0) cube complex with an invariant factor system – are $\mathcal{AH}$-accessible by [ABD17, Theorem A].*

It is harder to think of examples of groups which are not $\mathcal{AH}$-accessible. The first such example was given by Osin in [Osi16, Example 6.10]: it is the group $*_{n\geq 1}(\mathbb{Z}\times\mathbb{Z}/n\mathbb{Z})$. In particular, this example tells us that we cannot expect all graph products (over not necessarily finite graphs) of infinite $\mathcal{AH}$-accessible groups to be $\mathcal{AH}$-accessible in general. However, if we restrict to graph products over finite graphs then such a result is true, as the following consequence of Corollary 3.14 shows.

**Corollary 3.17** (Paper 3 [Val18a, Corollary D])**.** *Let $\Gamma$ be a finite simplicial graph, and let $\mathcal{G} = \{G_v \mid v \in V(\Gamma)\}$ be a collection of infinite groups. Suppose that for each isolated vertex $v \in V(\Gamma)$, the group $G_v$ is strongly $\mathcal{AH}$-accessible. Then $\Gamma\mathcal{G}$ is strongly $\mathcal{AH}$-accessible, and if $\Gamma$ has no isolated vertices then $[\Gamma\mathcal{G} \curvearrowright \mathcal{C}X]$ is the largest element of $\mathcal{AH}(\Gamma\mathcal{G})$.*

It is also known that there also exist finitely presented groups which are not $\mathcal{AH}$-accessible. Indeed, it was shown in [ABO17, Theorem 7.3] that if $N$ is a non-hyperbolic normal subgroup of a hyperbolic group $G$ with $G/N \cong \mathbb{Z}$, then $N$ cannot be $\mathcal{AH}$-accessible. Furthermore, such a subgroup $N$ can be taken to be finitely presented by [Bra99, Theorem 1.1].

## 3.5 Relation to equationally Noetherian groups

We here describe a result due to Groves and Hull [GH17, Theorem B] giving a criterion for an acylindrically hyperbolic group to be equationally Noetherian. The methods used in [GH17] are inspired by the work of Sela, who showed that free products of equationally Noetherian groups [Sel10, Theorem 9.1] and torsion-free hyperbolic groups [Sel09, Theorem 1.22] are equationally Noetherian.

In [GH17], an equivalent characterisation of equationally Noetherian groups is used, defined as follows. Let $\omega$ be a *non-principal ultrafilter* on $\mathbb{N}$ – that is, a partition of the power set of $\mathbb{N}$ into '$\omega$-large' and '$\omega$-small' subsets in such a way that each $\omega$-large subset is infinite, and the collection of $\omega$-small subsets is closed under inclusions and finite unions; existence of such an $\omega$ follows by the axiom of choice. Given a collection of properties $\{\mathcal{P}(i) \mid i \in \mathbb{N}\}$, we say that $\mathcal{P}(i)$ holds *$\omega$-almost surely* if the subset $\{i \in \mathbb{N} \mid \mathcal{P}(i) \text{ holds}\} \subseteq \mathbb{N}$ is $\omega$-large. Given two groups $G$, $H$ and a sequence of homomorphisms $(\varphi_i : H \to G)_{i=0}^{\infty}$, we define the $\omega$-kernel of $(\varphi_i)$ to be

$$H_\omega = \{h \in H \mid h \in \ker(\varphi_i)\ \omega\text{-almost surely}\},$$

which is easily seen to be a normal subgroup of $H$. We then have the following equivalent characterisation of equationally Noetherian groups.

**Theorem 3.18** (Groves–Hull [GH17, Theorem 3.5]). *A group $G$ is equationally Noetherian if and only if for any finitely generated group $H$, any non-principal ultrafilter $\omega$ and any sequence of homomorphisms $\varphi_i : H \to G$, we have $H_\omega \subseteq \ker(\varphi_i)$ $\omega$-almost surely.*

This characterisation provides an easy proof that any finitely generated equationally Noetherian group $G$ must be hopfian – that is, any surjective homomorphism $\varphi : G \to G$ is an isomorphism. Indeed, if $G$ is finitely generated and non-hopfian (thus admits a surjective non-injective homomorphism $\varphi : G \to G$), then by setting $H = G$ and $\varphi_i = \varphi^i : G \to G$ we see that $\{1\} = \ker(\varphi_0) \subsetneq \ker(\varphi_1) \subsetneq \ker(\varphi_2) \subsetneq \cdots$ and so $H_\omega = \bigcup_{j\in\mathbb{N}} \ker(\varphi_j) \nsubseteq \ker(\varphi_i)$ for any $i \in \mathbb{N}$; it then follows from Theorem 3.18 that $G$ cannot be equationally Noetherian. Thus, for example, the Baumslag–Solitar group $BS(2,3) = \langle a, t \mid t^{-1}a^2t = a^3\rangle$ is not equationally Noetherian.

To describe the relationship between acylindrically hyperbolic and equationally Noetherian groups, we need the following definition.

**Definition 3.19** (Divergent sequence of homomorphisms). Let $G$ be a group acting on a metric space $(Y, d)$ by isometries, let $H$ be a finitely generated group (with a finite generating set $S$, say), and let $\omega$ be a non-principal ultrafilter. We say that a sequence of homomorphisms $(\varphi_i : H \to G)_{i=0}^{\infty}$ is *divergent* with respect to $G \curvearrowright Y$ if, for any $K \in \mathbb{N}$, we $\omega$-almost surely have $\max_{s\in S} d(x, x^{\varphi_i(s)}) \geq K$ for all $x \in Y$.

This allows us to state the main technical theorem of [GH17]. In particular, the following theorem states that any sequence of homomorphisms $\varphi_i : H \to G$ as above can be reduced to a non-divergent sequence (cf Theorem 3.18).

**Theorem 3.20** (Groves–Hull [GH17, Theorem B]). *Let $G$ be a group that is not virtually cyclic and acts on a hyperbolic metric space $Y$ acylindrically with unbounded orbits. Suppose that for any finitely generated group $H$, any non-principal ultrafilter $\omega$ and any sequence of homomorphisms $\varphi_i : H \to G$ that is not divergent with respect to $G \curvearrowright Y$, we have $H_\omega \subseteq \ker(\varphi_i)$ $\omega$-almost surely. Then $G$ is equationally Noetherian.*

We apply Theorem 3.20 in the case when $G = \Gamma\mathcal{G}$ is a graph product over a finite graph $\Gamma$. In particular, given $H$, $\omega$ and $(\varphi_i)$ as above, using Theorem 3.20 and Corollary 3.14 we are able to reduce to the case where $H = \Gamma\mathcal{F}$ for some collection $\mathcal{F} = \{F_v \mid v \in V(\Gamma)\}$ of finitely generated groups and $\varphi_i(F_v) \subseteq \Gamma_{\mathrm{link}(v)}\mathcal{G}_{\mathrm{link}(v)}$ for every $i \in \mathbb{N}$ and $v \in V(\Gamma)$. If $\Gamma$ is triangle-free and square-free and if all groups in $\mathcal{G}$ are equationally Noetherian, we are then able to deduce that in such a setting we have $H_\omega \subseteq \ker(\varphi_i)$ $\omega$-almost surely. In particular, we prove the following result.

**Theorem 3.21** (Paper 3 [Val18a, Theorem E]). *Let $\Gamma$ be a finite simplicial triangle-free and square-free graph, and let $\mathcal{G} = \{G_v \mid v \in V(\Gamma)\}$ be a collection of equationally Noetherian groups. Then the graph product $\Gamma\mathcal{G}$ is equationally Noetherian.*

We were not able to decide if the condition on $\Gamma$ to be triangle-free and square-free in Theorem 3.21 is necessary. In particular, the author is unaware of any graph products of equationally Noetherian groups (over finite graphs) that can be shown not to be equationally Noetherian.

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# Paper 1

## RATIONAL GROWTH AND DEGREE OF COMMUTATIVITY OF GRAPH PRODUCTS

MOTIEJUS VALIUNAS

ABSTRACT. Let $G$ be an infinite group and let $X$ be a finite generating set for $G$ such that the growth series of $G$ with respect to $X$ is a rational function; in this case $G$ is said to have rational growth with respect to $X$. In this paper a result on sizes of spheres (or balls) in the Cayley graph $\Gamma(G, X)$ is obtained: namely, the size of the sphere of radius $n$ is bounded above and below by positive constant multiples of $n^\alpha \lambda^n$ for some integer $\alpha \geq 0$ and some $\lambda \geq 1$.

As an application of this result, a calculation of degree of commutativity (d. c.) is provided: for a finite group $F$, its d. c. is defined as the probability that two randomly chosen elements in $F$ commute, and Antolín, Martino and Ventura have recently generalised this concept to all finitely generated groups. It has been conjectured that the d. c. of a group $G$ of exponential growth is zero. This paper verifies the conjecture (for certain generating sets) when $G$ is a right-angled Artin group or, more generally, a graph product of groups of rational growth in which centralisers of non-trivial elements are "uniformly small".

## 1 Introduction

Let $G$ be a group which has a finite generating set $X$. For any element $g \in G$, let $|g| = |g|_X$ be the word length of $g$ with respect to $X$. For any $n \in \mathbb{Z}_{\geq 0}$, let

$$B_{G,X}(n) := \{g \in G \mid |g|_X \leq n\}$$

be the *ball* in $G$ with respect to $X$ of radius $n$, and let

$$S_{G,X}(n) := \{g \in G \mid |g|_X = n\}$$

be the *sphere* in $G$ with respect to $X$ of radius $n$. One writes $B_G(n)$ or $B(n)$ for the ball (and $S_G(n)$ or $S(n)$ for the sphere) if the generating set or the group itself is clear. A group $G$ is said to have *exponential growth* if

$$\liminf_{n\to\infty} \frac{\log |B_{G,X}(n)|}{n} > 0 \tag{1.1}$$

and *subexponential growth* otherwise; note that as there are at most $(2|X|)^n$ words over $X^{\pm 1}$ of length $n$, the limit in (1.1) is finite, so the group cannot have 'superexponential' growth. A group $G$ is said to have *polynomial growth of degree $d$* if

$$d := \limsup_{n\to\infty} \frac{\log |B_{G,X}(n)|}{\log n} < \infty$$

and *superpolynomial growth* otherwise. It is well-known that having exponential growth or polynomial growth of degree $d$ is independent of the generating set $X$.

The pairs $(G, X)$ as above considered in this paper will have some special properties. In particular, consider the (*spherical*) *growth series* $s_{G,X}(t)$ of a finitely generated group $G$ with a finite generating set $X$, defined by

$$s_{G,X}(t) = \sum_{g \in G} t^{|g|_X} = \sum_{n=0}^{\infty} |S_{G,X}(n)| t^n.$$

Cases of particular interest includes pairs $(G, X)$ for which $s_{G,X}(t)$ is a rational function, i.e. a ratio two polynomials; in this case $G$ is said to have *rational growth* with respect to $X$. In general, this property depends on the chosen generating set: for instance, the higher Heisenberg group $G = H_2(\mathbb{Z})$ has two finite generating sets $X_1$, $X_2$ such that $s_{G,X_1}(t)$ is rational but $s_{G,X_2}(t)$ is not [19].

Rational growth series implies some nice properties on the growth of a group. In particular, one can obtain the first main result of this paper:

**Theorem 1.** *Let $G$ be an infinite group with a finite generating set $X$ such that $s_{G,X}(t)$ is a rational function. Then there exist constants $\alpha \in \mathbb{Z}_{\geq 0}$, $\lambda \in [1, \infty)$ and $D > C > 0$ such that*

$$Cn^{\alpha}\lambda^n \leq |S_{G,X}(n)| \leq Dn^{\alpha}\lambda^n$$

*for all $n \geq 1$.*

Some of the ideas that go into the proof of Theorem 1 appear in the work of Stoll [19], where asymptotics of ball sizes are used to show that the higher Heisenberg group $G = H_2(\mathbb{Z})$ has a finite generating set $X$ such that the series $s_{G,X}(t)$ is transcendental.

*Remark* 2. It is clear that, with the assumptions and notation as above, Theorem 1 implies

$$\liminf_{n\to\infty} \frac{|S_{G,X}(n)|}{n^{\alpha}\lambda^n} \geq C > 0 \qquad \text{and} \qquad \limsup_{n\to\infty} \frac{|S_{G,X}(n)|}{n^{\alpha}\lambda^n} \leq D < \infty.$$

It is easy to check that the converse implication is also true. In particular, the conclusion of Theorem 1 is equivalent to the statement that there exist $\alpha \in \mathbb{Z}_{\geq 0}$ and $\lambda \in [1, \infty)$ such that

$$\liminf_{n\to\infty} \frac{|S_{G,X}(n)|}{n^{\alpha}\lambda^n} > 0 \qquad \text{and} \qquad \limsup_{n\to\infty} \frac{|S_{G,X}(n)|}{n^{\alpha}\lambda^n} < \infty.$$

Theorem 1 agrees with the result for hyperbolic groups. Indeed, it is known that if $G$ is a hyperbolic group and $X$ is a finite generating set, then $s_{G,X}(t)$ is rational [14, Theorem 8.5.N]. In this case the Theorem gives a weaker version of [8, Théorème 7.2], which states that the conclusion of Theorem 1 holds with $\alpha = 0$.

As an application of Theorem 1 a calculation of degree of commutativity is provided. For a finite group $F$, the *degree of commutativity* of $F$ was defined by Erdős and Turán [10] and Gustafson [15] as

$$\mathrm{dc}(F) := \frac{|\{(x, y) \in F^2 \mid [x, y] = 1\}|}{|F|^2}, \tag{1.2}$$

i.e. the probability that two elements of $F$ chosen uniformly at random commute. In [1], Antolín, Martino and Ventura generalise this definition to infinite finitely generated groups:

**Definition 3.** Let $G$ be a finitely generated group and let $X$ be a finite generating set for $G$. The *degree of commutativity* for $G$ with respect to $X$ is

$$\begin{aligned}\mathrm{dc}_X(G) &:= \limsup_{n\to\infty} \frac{|\{(x, y) \in B_{G,X}(n)^2 \mid [x, y] = 1\}|}{|B_{G,X}(n)|^2} \\ &= \limsup_{n\to\infty} \frac{\sum_{x\in B_{G,X}(n)} |C_G(x) \cap B_{G,X}(n)|}{|B_{G,X}(n)|^2},\end{aligned}$$

where $C_G(x)$ is the centraliser of $x$ in $G$.

Note that if $G$ is finite then for any generating set $X$ one has $B_{G,X}(N) = G$ for all sufficiently large $N$, so this definition agrees with (1.2).

It is known that $\mathrm{dc}_X(G) = 0$ when $G$ is either a non-virtually-abelian residually finite group of subexponential growth [1, Theorem 1.3] or a non-elementary hyperbolic group [1, Theorem 1.7], independently of the generating set $X$. It has been conjectured that indeed $\mathrm{dc}_X(G) = 0$ whenever $G$ has superpolynomial growth [1, Conjecture 1.6].

The interest of this paper is the degree of commutativity of graph products of groups.

**Definition 4.** Let $\Gamma$ be a finite simple (undirected) graph, and let $\mathbf{H} : V(\Gamma) \to \mathcal{G}$ be a map from the vertex set of $\Gamma$ to the category $\mathcal{G}$ of groups; suppose that $\mathbf{H}(v) \ncong \{1\}$ for each $v \in V(\Gamma)$. Let

$$\tilde{G}(\Gamma, \mathbf{H}) := *_{v\in V(\Gamma)}\mathbf{H}(v)$$

be a free product of groups, and let

$$R(\Gamma, \mathbf{H}) := \{[g, h] \mid g \in \mathbf{H}(v), h \in \mathbf{H}(w), \{v, w\} \in E(\Gamma)\}.$$

Then the *graph product* associated with $\Gamma$ and $\mathbf{H}$ is defined to be the group

$$G(\Gamma, \mathbf{H}) := \tilde{G}(\Gamma, \mathbf{H})/\langle\langle R(\Gamma, \mathbf{H})\rangle\rangle^{\tilde{G}(\Gamma,\mathbf{H})}.$$

In particular, this is the construction of *right-angled Artin* (respectively *Coxeter*) *groups* if $\mathbf{H}(v) \cong \mathbb{Z}$ (respectively $\mathbf{H}(v) \cong C_2$) for all $v \in \Gamma$.

This paper considers groups $G$ which, together with their finite generating sets $X$, belong to a certain class, defined as follows.

**Definition 5.** Say a pair $(G,X)$ with a group $G$ and a finite generating set $X$ of $G$ is a *rational pair with small centralisers* if the following two conditions hold:

(i) $s_{G,X}(t)$ is a rational function;

(ii) there exist constants $P,\beta \in \mathbb{Z}_{\geq 1}$ such that $|C_G(g) \cap B_{G,X}(n)| \leq Pn^\beta$ for all $n \geq 1$ and all non-trivial elements $g \in G$.

Note that condition (ii) is independent of the choice of a generating set $X$: indeed, as any word metrics on $G$ associated with generating sets $X$ and $\hat{X}$ are bi-Lipschitz equivalent, the inequality $|C_G(g)\cap B_{G,X}(n)| \leq Pn^\beta$ implies the inequality $|C_G(g)\cap B_{G,\hat{X}}(n)| \leq \hat{P}n^\beta$ for some $\hat{P} \in \mathbb{Z}_{\geq 1}$ depending only on $\hat{X}$ and $P$.

It was shown in [7] that, given a finite simple graph $\Gamma$ with a group $\mathbf{H}(v)$ and a finite generating set $X(v) \subseteq \mathbf{H}(v)$ associated to every vertex $v \in V(\Gamma)$, if $s_{\mathbf{H}(v),X(v)}(t)$ is rational for each $v \in V(\Gamma)$ then so is $s_{G(\Gamma,\mathbf{H}),X(\Gamma,\mathbf{H})}(t)$, where $X(\Gamma,\mathbf{H}) = \bigsqcup_{v\in V(\Gamma)} X(v)$.

If $G(\Gamma,\mathbf{H})$ has exponential growth, then, together with an explicit form of centralisers in $G(\Gamma,\mathbf{H})$, described in [2], Theorem 1 can be used to compute the degree of commutativity of $G(\Gamma,\mathbf{H})$:

**Theorem 6.** *Let $\Gamma$ be a finite simple graph, and for each $v \in V(\Gamma)$, let $(\mathbf{H}(v),X(v))$ be a rational pair with small centralisers. Suppose that $G(\Gamma,\mathbf{H})$ has exponential growth, and let $X = \bigsqcup_{v\in V(\Gamma)} X(v)$. Then*

$$\mathrm{dc}_X(G(\Gamma,\mathbf{H})) = 0.$$

*Remark* 7. Theorem 6 is enough to confirm [1, Conjecture 1.6] in this setting: that is, either $G = G(\Gamma,\mathbf{H})$ is virtually abelian, or $\mathrm{dc}_X(G) = 0$. Indeed, $G(\Gamma,\mathbf{H})$ has subexponential growth if and only if all the $\mathbf{H}(v)$ have subexponential growth, the complement $\Gamma^C$ of $\Gamma$ contains no length 2 paths, and $\mathbf{H}(v) \cong C_2$ for every non-isolated vertex $v$ of $\Gamma^C$. In this case, rationality of $s_{\mathbf{H}(v),X(v)}(t)$ implies that the $\mathbf{H}(v)$ all have polynomial growth (by Theorem 1, for instance). Thus $G(\Gamma,\mathbf{H})$ is a direct product of groups of polynomial growth: namely, the group $\mathbf{H}(v)$ for each isolated vertex $v$ of $\Gamma^C$, and an infinite dihedral group for each edge in $\Gamma^C$. Consequently, $G(\Gamma,\mathbf{H})$ itself has polynomial growth, and so [1, Corollary 1.5] implies that either $G(\Gamma,\mathbf{H})$ is virtually abelian, or $\mathrm{dc}_X(G(\Gamma,\mathbf{H})) = 0$.

Cases of particular interest of Theorem 6 include right-angled Artin groups and graph products of finite groups. More generally, let us note two special cases of pairs of $(G,X)$ satisfying Definition 5:

(i) Let $G$ be virtually nilpotent, and $X$ be a finite generating set with $s_{G,X}(t)$ rational: in particular, this holds whenever $G$ is virtually abelian [3] and for $G = H_1$, the integral Heisenberg group [9]. It was shown that by Wolf [20] that if $G$ is virtually nilpotent then it has polynomial growth (by Gromov's Theorem [13], the converse is also true), and so part (ii) of Definition 5 holds trivially by bounding growth of centralisers by the growth of $G$ itself.

(ii) Let $G$ be a torsion-free hyperbolic group, and $X$ be any finite generating set. Cannon [6] and Gromov [14, Theorem 8.5.N] have shown that hyperbolic groups have rational growth with respect to any generating set, and all infinite-order elements have virtually cyclic centralisers. Moreover, for any torsion-free hyperbolic group $G$ with a finite generating set $X$, there is a constant $P > 0$ such that $|C_G(g) \cap B_{G,X}(n)| \leq Pn$ for all $n \geq 1$ and all non-trivial $g \in G$: see the proof of Theorem 1.7 in [1] for details and references.

The paper is structured as follows. Section 2 applies to all infinite groups with rational spherical growth series and is dedicated to a proof of Theorem 1. Section 3 is used to prove Theorem 6.

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## 2 Groups with rational growth series

This section provides a proof of Theorem 1. Let $G$ be an infinite group, and suppose that the growth series of $G$ with respect to a finite generating set $X$ is a rational function. In particular, the spherical growth series is

$$s(t) = s_{G,X}(t) = \sum_{n=0}^{\infty} \mathfrak{S}(n) t^n = \frac{p(t)}{q(t)}$$

where $\mathfrak{S}(n) = \mathfrak{S}_G(n) = \mathfrak{S}_{G,X}(n) := |S_{G,X}(n)|$, and

$$q(t) = q_0 t^c \prod_{i=1}^{r} (1 - \lambda_i t)^{\alpha_i + 1} \quad \text{and} \quad p(t) = p_0 t^{\tilde{c}} \prod_{i=1}^{\tilde{r}} (1 - \tilde{\lambda}_i t)^{\tilde{\alpha}_i + 1}$$

are non-zero polynomials with no common roots (and so either $c = 0$ or $\tilde{c} = 0$), with $\alpha_i, \tilde{\alpha}_i \in \mathbb{Z}_{\geq 0}$ for all $i$. Since the series $(\mathfrak{S}(n))_{n=0}^{\infty}$ grows at most exponentially, $s(t)$ is analytic (and so continuous) at 0, hence one has

$$1 = \mathfrak{S}(0) = \lim_{t \to 0} s(t) = \frac{p_0}{q_0} \lim_{t \to 0} t^{\tilde{c}-c}$$

and so $c = \tilde{c}$ and $p_0 = q_0$. Thus $c = \tilde{c} = 0$ and, without loss of generality, $q_0 = p_0 = 1$.

Coefficients of such a series are described in [16, Lemma 1]; in particular, it follows that

$$\mathfrak{S}(n) = \sum_{i=1}^{r} \sum_{j=0}^{\alpha_i} b_{i,j} n^j \lambda_i^n \tag{1.3}$$

for $n$ large enough, with $b_{i,\alpha_i} \neq 0$ for all $i$.

Now consider the terms of (1.3) that give a non-negligible contribution to $\mathfrak{S}(n)$ for large $n$. In particular, one may assume without loss of generality that

$$\lambda := |\lambda_1| = |\lambda_2| = \cdots = |\lambda_{\tilde{k}}| > |\lambda_{\tilde{k}+1}| \geq |\lambda_{\tilde{k}+2}| \geq \cdots \geq |\lambda_r|$$

for some $\tilde{k} \leq r$ and that

$$\alpha := \alpha_1 = \alpha_2 = \cdots = \alpha_k > \alpha_{k+1} \geq \alpha_{k+2} \geq \cdots \geq \alpha_{\tilde{k}}$$

for some $k \leq \tilde{k}$. Note that one must have $\lambda \geq 1$: otherwise the radius of convergence of $s(t)$ is $\lambda^{-1} > 1$ and so the series $\sum_n \mathfrak{S}(n)$ converges, contradicting the fact that $G$ is infinite.

For $n \in \mathbb{Z}_{\geq 0}$, define

$$c_n = \sum_{j=1}^{k} b_{j,\alpha} \exp(i\varphi_j n)$$

where $\lambda_j = \lambda \exp(i\varphi_j)$ for some $\varphi_j \in (-\pi, \pi]$, for $1 \leq j \leq k$. It follows that

$$\mathfrak{S}(n) = n^\alpha \lambda^n (c_n + o(1)) \tag{1.4}$$

as $n \to \infty$. In particular, since $\mathfrak{S}(n) \in (0, \infty) \subseteq \mathbb{R}$ for all $n$, it follows that

$$\liminf_{n \to \infty} \mathrm{Re}(c_n) \geq 0 \quad \text{and} \quad \lim_{n \to \infty} \mathrm{Im}(c_n) = 0. \tag{1.5}$$

It is clear that

$$\limsup_{n \to \infty} \frac{\mathfrak{S}(n)}{n^\alpha \lambda^n} \leq \sum_{j=1}^{k} |b_{j,\alpha}|,$$

which shows existence of the constant $D$ in Theorem 1; in order to prove the Proposition, it is enough to show that $\liminf_{n \to \infty} \mathfrak{S}(n)/(n^\alpha \lambda^n) > 0$. However, this bound does not follow solely from the fact that $s(t)$ is a rational function: see Example 12 (i) at the end of this section.

*Remark* 8. Clearly, for any $n_1, n_2 \geq 0$, if $g \in G$ has $|g|_X = n_1 + n_2$ (respectively $|g|_X \leq n_1 + n_2$), then one can write $g = g_1 g_2$ where $|g_j|_X = n_j$ (respectively $|g_j|_X \leq n_j$) for $j \in \{1, 2\}$. This gives injections $S(n_1 + n_2) \to S(n_1) \times S(n_2)$ and $B(n_1 + n_2) \to B(n_1) \times B(n_2)$ by mapping $g \mapsto (g_1, g_2)$. In particular, it follows that

$$\mathfrak{S}(n_1 + n_2) \leq \mathfrak{S}(n_1)\mathfrak{S}(n_2) \quad \text{and} \quad \mathfrak{B}(n_1 + n_2) \leq \mathfrak{B}(n_1)\mathfrak{B}(n_2)$$

for any $n_1, n_2 \in \mathbb{Z}_{\geq 0}$. This property is called *submultiplicativity* of sphere and ball sizes in $G$.

The aim is now to show that submultiplicativity of the sequence $(\mathfrak{S}(n))_{n=0}^{\infty}$, together with rationality of $s(t)$, implies the conclusion of Theorem 1. As the $b_{j,\alpha}$ are non-zero and the $\varphi_j$ are distinct, given (1.5) the following result seems highly likely:

**Lemma 9.** *The numbers $c_n$ are real, and for some constant $\delta > 0$, the set*

$$E_\delta := \{n \in \mathbb{Z}_{\geq 0} \mid c_n \geq \delta\}$$

*is relatively dense in $[0, \infty)$, i.e. the inclusion $E_\delta \hookrightarrow [0, \infty)$ is a $(1, K)$-quasi-isometry for some $K \geq 0$.*

However, the author has been unable to come up with a straightforward proof of Lemma 9 without using some additional theory on 'quasi-periodicity' of the sequence $(c_n)_{n=0}^{\infty}$. Before giving a proof, let us deduce Theorem 1 from Lemma 9.

Assuming Lemma 9, one can find $N \in \mathbb{Z}_{\geq 1}$ such that for all $n$, there exists a constant $\beta = \beta_n \in \{0, \ldots, N\}$ such that $c_{n+\beta} \geq \delta$. Define

$$R := \max\{\lambda^{-\beta}\mathfrak{S}(\beta) \mid 0 \leq \beta \leq N\},$$

and let $M \in \mathbb{Z}_{\geq 1}$ be such that for all $n \geq M$, one has

$$\mathfrak{S}(n) \geq n^\alpha \lambda^n \left(c_n - \frac{\delta}{2}\right)$$

(such an $M$ exists by (1.4)). Then submultiplicativity of sphere sizes implies that for all $n \geq M$,

$$\begin{aligned}
\frac{\delta}{2}(n + \beta_n)^\alpha \lambda^{n+\beta_n} &\leq \left(c_{n+\beta_n} - \frac{\delta}{2}\right)(n + \beta_n)^\alpha \lambda^{n+\beta_n} \\
&\leq \mathfrak{S}(n + \beta_n) \leq \mathfrak{S}(n)\mathfrak{S}(\beta_n) \leq \mathfrak{S}(n) R \lambda^{\beta_n}.
\end{aligned}$$

It follows that

$$\mathfrak{S}(n) \geq \frac{\delta}{2R}(n + \beta_n)^\alpha \lambda^n \geq \frac{\delta}{2R} n^\alpha \lambda^n$$

for $n \geq M$, showing that

$$\liminf_{n\to\infty} \frac{\mathfrak{S}(n)}{n^\alpha \lambda^n} \geq \frac{\delta}{2R} > 0,$$

which shows existence of the constant $C > 0$ in Theorem 1. Thus in order to prove Theorem 1 it is now enough to prove Lemma 9.

*Proof of Lemma 9.* To prove the Lemma, one may employ a digression into a certain class of functions from $\mathbb{R}$ to $\mathbb{C}$, called 'uniformly almost periodic functions'. The theory for these functions is presented in a book by Besicovitch [5].

Let $f : \mathbb{R} \to \mathbb{C}$ be a function. Given $\varepsilon > 0$, define the set $E(f, \varepsilon) \subseteq \mathbb{R}$ to be the set of all numbers $\tau \in \mathbb{R}$ (called the *translation numbers* for $f$ belonging to $\varepsilon$) such that

$$\sup_{x\in\mathbb{R}} |f(x+\tau) - f(x)| \leq \varepsilon.$$

The function $f$ is said to be *uniformly almost periodic* (*u. a. p.*) if, for any $\varepsilon > 0$, the set $E(f, \varepsilon)$ is relatively dense in $\mathbb{R}$, i.e. the inclusion $E(f, \varepsilon) \hookrightarrow \mathbb{R}$ is a $(1, K)$-quasi-isometry for some $K \geq 0$. It is easy to see that any periodic function is u. a. p., and that every continuous u. a. p. function is bounded.

Now note that the function

$$\begin{aligned} c : \mathbb{R} &\to \mathbb{C} \\ t &\mapsto \sum_{j=1}^{k} b_{j,\alpha} \exp(i\varphi_j t) \end{aligned}$$

is a sum of continuous periodic functions, and so is a continuous u. a. p. function by [5, Section 1.1, Theorem 12]. By definition, $c_n = c(n)$ for any $n \in \mathbb{Z}_{\geq 0}$.

The aim is to show that the function $\bar{c} : t \mapsto c(\lfloor t \rfloor)$ is also u. a. p. For this, note that $c$ is everywhere differentiable and the derivative $c'(t)$ is a sum of continuous periodic functions, so is continuous and u. a. p. – in particular, it is bounded, by some $R > 0$, say. For a given $\varepsilon \in (0, R)$, set a constant $M := \varepsilon / \left(2 \sin\left(\frac{\pi\varepsilon}{2R}\right)\right)$ and define $f : \mathbb{R} \to \mathbb{R}$ by $f(t) = M \sin(\pi t)$. It is easy to check that

$$E\left(f, \frac{\varepsilon}{2}\right) \subseteq \bigcup_{n\in\mathbb{Z}} \left[n - \frac{\varepsilon}{2R}, n + \frac{\varepsilon}{2R}\right]. \tag{1.6}$$

For any $\tau \in \mathbb{R}$, define $n_\tau = \left\lfloor \tau + \frac{1}{2} \right\rfloor \in \mathbb{Z}$ to be the nearest integer to $\tau$. Pick $\tau \in E\left(f, \frac{\varepsilon}{2}\right) \cap E\left(c, \frac{\varepsilon}{2}\right)$ – then $|c(x+\tau) - c(x)| \leq \frac{\varepsilon}{2}$ for all $x \in \mathbb{R}$, and, by (1.6), $|\tau - n_\tau| \leq \frac{\varepsilon}{2R}$, so in particular $|c(x+\tau) - c(x+n_\tau)| \leq \frac{\varepsilon}{2}$ for all $x \in \mathbb{R}$ by the choice of $R$. Thus $|c(x+n_\tau) - c(x)| \leq \varepsilon$ for all $x \in \mathbb{R}$, i.e. $n_\tau \in E(c, \varepsilon)$.

But by [5, Section 1.1, Theorem 11], the set $E\left(f, \frac{\varepsilon}{2}\right) \cap E\left(c, \frac{\varepsilon}{2}\right)$ is relatively dense, hence (by the previous paragraph) so is the set $E(c,\varepsilon)\cap\mathbb{Z}$. However, for any $n\in E(c,\varepsilon)\cap\mathbb{Z}$ and any $x\in\mathbb{R}$ one has

$$|\bar{c}(x+n)-\bar{c}(x)|=|c(\lfloor x+n\rfloor)-c(\lfloor x\rfloor)|=|c(\lfloor x\rfloor+n)-c(\lfloor x\rfloor)|\leq\varepsilon$$

and so $E(c,\varepsilon)\cap\mathbb{Z}\subseteq E(\bar{c},\varepsilon)\cap\mathbb{Z}$. It follows that $E(\bar{c},\varepsilon)\cap\mathbb{Z}$ is relatively dense (and so the function $\bar{c}:t\mapsto c(\lfloor t\rfloor)$ is u. a. p.).

Now recall that (1.5) provides constraints for limits of sequences $(\mathrm{Re}(c_n))$ and $(\mathrm{Im}(c_n))$: namely,

$$\liminf_{n\to\infty}\mathrm{Re}(c_n)\geq 0\quad\text{and}\quad\lim_{n\to\infty}\mathrm{Im}(c_n)=0.\tag{1.7}$$

It is easy to see that $c_n\in\mathbb{R}_{\geq 0}$ for all $n$: indeed, if either $\mathrm{Re}(c_n)=-\delta<0$ or $|\mathrm{Im}(c_n)|=\delta>0$ for some $n$ then the fact that the set $E(\bar{c},\delta/2)\cap\mathbb{Z}$ is relatively dense contradicts (1.7). Similarly, if $c_N>0$ for some $N$ then the set $E(\bar{c},\delta)\cap\mathbb{Z}$ is a relatively dense set contained in the set $\{n\in\mathbb{Z}\mid c(n)\geq\delta\}$, where $\delta=c_N/2$. To prove Lemma 9 it is therefore enough to show that the sequence $(c_n)_{n=0}^{\infty}$ is not identically zero.

Now recall that the sequence $(c_n)$ is defined by

$$c_n=\sum_{j=1}^{k}b_{j,\alpha}\exp(i\varphi_j n),$$

and suppose for contradiction that $c_n=0$ for all $n\in\mathbb{Z}_{\geq 0}$, and in particular for $0\leq n\leq k-1$. This is the same as saying that $Mv=0$, where

$$M=\begin{pmatrix}1 & 1 & \cdots & 1\\ \exp(i\varphi_1) & \exp(i\varphi_2) & \cdots & \exp(i\varphi_k)\\ \vdots & \vdots & \ddots & \vdots\\ \exp(i\varphi_1)^{k-1} & \exp(i\varphi_2)^{k-1} & \cdots & \exp(i\varphi_k)^{k-1}\end{pmatrix}$$

and

$$v=\begin{pmatrix}b_{1,\alpha-1}\\ b_{2,\alpha-1}\\ \vdots\\ b_{k,\alpha-1}\end{pmatrix}.$$

Thus $M$ has a zero eigenvalue and so $\det M=0$. But $M^t$ is a Vandermonde matrix with pairwise distinct rows, so $\det M\neq 0$. This gives a contradiction which completes the proof. $\square$

*Remark* 10. A stronger conclusion of Theorem 1 holds if in addition $s_{G,X}(t)$ is a *positive* rational function, i.e. it is contained in the smallest sub-semiring of $\mathbb{C}(t)$ containing the semiring $\mathbb{Z}_{\geq 0}[t]$ and closed under quasi-inversion, $f(t)\mapsto(1-f(t))^{-1}$ (for $f(t)\in\mathbb{C}(t)$ with $f(0)=0$). This is the case in particular if there exists a language $\mathcal{L}$ in $(X\cup X^{-1})^*$

that is regular (i.e. recognised by a finite state automaton), the monoid homomorphism $\Phi : \mathcal{L} \to G$ extending the inclusion $X \cup X^{-1} \hookrightarrow G$ is a bijection, and $\mathcal{L}$ consists only of geodesic words in the Cayley graph of $G$ with respect to $X$, i.e. the length of any word $l \in \mathcal{L}$ is $|\Phi(l)|_X$. If $s_{G,X}(t)$ is a positive rational function, then the numbers $\varphi_j$ above are in fact rational multiples of $\pi$ [4], and as a consequence the sequence $(c_n)$ is periodic.

However, the author has not been able to find a reason why the function $s_{G,X}(t)$, in case it is rational, must also be positive. In particular, one can find pairs $(G, X)$ such that $s_{G,X}(t)$ is rational but there are no regular languages $\mathcal{L}$ as above, and one can even find groups $G$ such that this holds for $(G, X)$ for any generating set $X$. For instance, it can be shown that growth of the 2-step nilpotent Heisenberg group

$$G = H_3 = \langle a, b, c \mid [a, b] = c, [a, c] = [b, c] = 1\rangle$$

is rational with respect to any generating set [9, Theorem 1], but there are no languages $\mathcal{L}$ as above when $G$ is a 2-step nilpotent group that is not virtually abelian [18, Corollary 3].

It is easy to check that the conclusion of Theorem 1 implies that

$$\liminf_{n\to\infty} \frac{|B_{G,X}(n)|}{n^{\hat{\alpha}}\lambda^n} > 0 \qquad \text{and} \qquad \limsup_{n\to\infty} \frac{|B_{G,X}(n)|}{n^{\hat{\alpha}}\lambda^n} < \infty, \tag{1.8}$$

where $\hat{\alpha} = \alpha + 1$ if $\lambda = 1$ and $\hat{\alpha} = \alpha$ otherwise. Asymptotics similar to these have been obtained for nilpotent groups, even without the condition on rational growth. In particular, in [17] Pansu showed that given a nilpotent group $G$ with a finite generating set $X$, there exists $\hat{\alpha} \in \mathbb{Z}_{\geq 0}$ such that $\frac{|B_{G,X}(n)|}{n^{\hat{\alpha}}} \to C$ as $n \to \infty$ for some $C > 0$. Moreover, in [19] Stoll calculates the constant $C$ for certain 2-step nilpotent groups $G$ explicitly to show that the corresponding growth series $s_{G,X}(t)$ *cannot* be rational. However, in general – for groups that are not virtually nilpotent – one cannot expect lim sup and lim inf in (1.8) to be equal, as the hyperbolic group $C_2 * C_3$ shows: see [12, §3].

Finally, note that the same proof indeed shows a more general result:

**Theorem 11.** *Let $(a_n)_{n=0}^{\infty}$ be a submultiplicative sequence of numbers in $\mathbb{Z}_{\geq 1}$ such that $s(t) = \sum a_n t^n$ is a rational function. Then there exist constants $\alpha \in \mathbb{Z}_{\geq 0}$, $\lambda \in [1, \infty)$ and $D > C > 0$ such that for all $n \geq 1$,*

$$Cn^{\alpha}\lambda^n \leq a_n \leq Dn^{\alpha}\lambda^n.$$

The example below shows that both submultiplicativity and rationality are necessary requirements.

**Example 12.** (i) *Let*

$$p(t) = 1 + 12t^2 - 16t^3$$

*and*

$$q(t) = (1-t)(1-2t)(1-2\omega t)(1-2\bar{\omega}t),$$

*where $\omega$ is a $6^{th}$ primitive root of unity. Let $s(t)$, $(a_n)$, $\lambda$, $\alpha$ and $(c_n)$ be as above. Then $\lambda = 2$ and $\alpha = 0$, and [16, Lemma 1] can be used to calculate*

$$a_n = c_n 2^n + 1$$

*where*

$$c_n = 4 - 2\omega^n - 2\bar{\omega}^n = \begin{cases} 0, & n \equiv 0 \pmod 6, \\ 2, & n \equiv \pm 1 \pmod 6, \\ 6, & n \equiv \pm 2 \pmod 6, \\ 8, & n \equiv 3 \pmod 6. \end{cases}$$

*But as $c_n = 0$ for infinitely many values of $n$, one has*

$$\liminf_{n\to\infty} a_n/(n^\alpha \lambda^n) = 0.$$

*Note that in this case $a_7 = 257 > 5 = a_1 a_6$, so the sequence $(a_n)$ is not submultiplicative.*

(ii) *For $n \geq 0$, let $a_n = 2^{b(n)}$, where $b(n)$ is the sum of digits in the binary representation of $n$. Then $(a_n)$ is a submultiplicative sequence, but $\sum a_n t^n$ is not a rational function. For each $n \geq 0$, one has $a_{2^n-1} = 2^n$ and $a_{2^n} = 2$. Thus*

$$\liminf_{n\to\infty} \frac{a_n}{n} \leq \liminf_{n\to\infty} \frac{2}{2^n} = 0$$

*and*

$$\limsup_{n\to\infty} a_n \geq \limsup_{n\to\infty} 2^n = \infty,$$

*so $(a_n)$ does not satisfy the conclusion of Theorem 11 for any $\lambda \geq 1$ and $\alpha \in \mathbb{Z}_{\geq 0}$.*

## 3 Degree of commutativity

The aim of this section is to prove Theorem 6. For this, let $\Gamma$ be a finite simple graph and for each $v \in V(\Gamma)$, let $(\mathbf{H}(v), X(v))$ be a rational pair with small centralisers (see Definition 5). To simplify notation, suppose in addition that the sets $X(v)$ are symmetric and do *not* contain the identity $1 \in \mathbf{H}(v)$: clearly this does not affect the results. Suppose in addition that $G = G(\Gamma, \mathbf{H})$ is a group of exponential growth. One thus aims to show that $\mathrm{dc}_X(G) = 0$, where $X = \bigsqcup_{v \in V(\Gamma)} X(v)$.

### 3.1 Preliminaries

This subsection collects the terminology and preliminary results used in the proof of Theorem 6.

Let $\ell_n : X^* \to \mathbb{Z}_{\geq 0}$ be the *normal form length* function ($n$ in $\ell_n$ stands for 'normal'): for $w \in X^*$, set $\ell_n(w) := m$ where $m$ is the minimal integer for which $w \equiv w_1 w_2 \cdots w_m$ as words, where $w_i \in X(v_i)^*$ for some $v_i \in V(\Gamma)$. Moreover, let $\ell_w : X^* \to \mathbb{Z}_{\geq 0}$ be the *word length* function ($w$ in $\ell_w$ stands for 'word'), i.e. let $\ell_w(w)$ be the number of letters in $w \in X^*$.

The following result says that given any word $w \in X^*$ representing $g \in G$, there is a simple algorithm to transform it into a word $\hat{w}$ representing $g$ with $\ell_n(\hat{w})$ or $\ell_w(\hat{w})$ small. This follows quite easily from a result of Green [11].

**Proposition 13.** *Let $\ell : X^* \to \mathbb{Z}_{\geq 0}$ be either $\ell = \ell_n$ or $\ell = \ell_w$. Let $w \in X^*$ be a word representing an element $g \in G$, and let $\hat{w}$ be a word representing $g$ with $(\ell(\hat{w}), \ell_w(\hat{w}))$ minimal (in the lexicographical ordering) among such words. Then $\hat{w}$ can be obtained from $w$ by applying a sequence of moves of two types:*

(i) *for some $w_u \in X(u)^*$ and $w_v \in X(v)^*$ with $\{u, v\} \in E(\Gamma)$, replacing a subword $w_u w_v$ with $w_v w_u$;*

(ii) *for some $v \in V(\Gamma)$ and some subword $w_1 \in X(v)^*$, replacing the subword $w_1$ with a word $w_0 \in X(v)^*$ representing the same element in $\mathbf{H}(v)$, such that we have $\ell_w(w_0) \leq \ell_w(w_1)$.*

*Proof.* Suppose first that $\ell = \ell_n$, and let $\hat{w} \equiv w_1 \cdots w_m$, where $w_i \in X(v_i)^*$ for some $v_i \in V(\Gamma)$ and $m = \ell_n(w)$. In [11, Theorem 3.9], Green showed that by using moves (i) and (ii) we can transform $w$ into a word $\hat{w}' \equiv w_1' \cdots w_m'$ where $w_i' \in X(v_i)^*$ and $w_i$, $w_i'$ represent the same element of $\mathbf{H}(v)$. Notice that we have $\ell_w(w_i) \leq \ell_w(w_i')$ for each $i$: otherwise, existence of the word $w_1 \cdots w_{i-1} w_i' w_{i+1} \cdots w_m$ would contradict the minimality of $\hat{w}$. Thus a sequence of moves (ii) allows us to transform $\hat{w}'$ into $\hat{w}$, as required.

Suppose now that $\ell = \ell_w$. Let $\hat{w}_n \in X^*$ be a word representing $g$ with $(\ell_n(\hat{w}_n), \ell_w(\hat{w}_n))$ minimal among all such words. Then the result for $\ell = \ell_n$ says that $\hat{w}$ can be transformed into $\hat{w}_n$ by using the moves (i)–(ii). Notice that if $w' \in X^*$ is obtained from $w \in X^*$ by applying move (i) or (ii), then $\ell_w(w') \leq \ell_w(w)$, and if the equality holds then there exists a move that transforms $w'$ back into $w$. By definition of $\hat{w}$, no moves strictly decreasing the word length are used when transforming $\hat{w}$ to $\hat{w}_n$, and so there exists a sequence of moves transforming $\hat{w}_n$ into $\hat{w}$ as well. Thus we may apply moves (i)–(ii) to obtain $\hat{w}_n$ from $w$ and subsequently $\hat{w}$ from $\hat{w}_n$, as required. □

Note that it follows from the proof of Proposition 13 that minimal values of $\ell_n(w)$ and $\ell_w(w)$ can be obtained simultaneously. This justifies the following:

**Definition 14.** For $g \in G$, define a *normal form* of $g$ to be a word $w \in X^*$ with both $\ell_n(w)$ and $\ell_w(w)$ minimal (so that $\ell_w(w) = |g|_X$). Write $w = w_1 w_2 \cdots w_n$ for $w_i \in X$, and define the *support* of $g$ as

$$\operatorname{supp}(g) := \{v \in V(\Gamma) \mid w_i \in X(v) \text{ for some } i\};$$

by Proposition 13 this does not depend on the choice of $w$.

Now suppose for contradiction that $\mathrm{dc}_X(G) > 0$. That means that for some constant $\varepsilon > 0$, one has

$$\sum_{g \in B(n)} \frac{|C_G(g) \cap B(n)|}{\mathfrak{B}(n)^2} \geq \varepsilon \tag{1.9}$$

for infinitely many values of $n$, where $C_G(g)$ denotes the centraliser of an element $g \in G$, and $\mathfrak{B}(n) = \mathfrak{B}_G(n) = \mathfrak{B}_{G,X}(n) := |B_{G,X}(n)|$.

In the proof certain conjugates of elements in $G$ will be considered. In particular, let $g \in G$, and pick a conjugate $\tilde{g} \in G$ of $g$ such that $g = p_g^{-1} \tilde{g} p_g$ with $|g| = 2|p_g| + |\tilde{g}|$ and such that $|\tilde{g}|$ is minimal subject to this. If $p_g = 1$, then $g$ is called *cyclically reduced*; hence $\tilde{g}$ is cyclically reduced. Note that being cyclically reduced is a weaker condition than being cyclically normal in the sense of [2].

For any subset $A \subseteq V(\Gamma)$, let $G_A$ denote $G(\Gamma(A), \mathbf{H}|_A)$, where $\Gamma(A)$ is the full subgraph of $\Gamma$ spanned by $A$. These will be viewed as subgroups (called the *special subgroups*) of $G$. One may also define the *link* of $A$ to be

$$\operatorname{link} A = \{u \in V(\Gamma) \mid (u, v) \in E(\Gamma) \text{ for all } v \in A\}.$$

Before carrying on with the proof, consider the sequence $(d_n)_{n=0}^{\infty}$ where

$$d_n := \frac{|\{(x, y) \in B_{G,X}(n)^2 \mid [x, y] = 1\}|}{\mathfrak{B}_{G,X}(n)^2}.$$

One aims to show that $d_n \to 0$ as $n \to \infty$. Note that for many groups of exponential growth, including all the non-elementary hyperbolic groups [1], the sequence $(d_n)_{n=0}^{\infty}$ converges to zero exponentially fast. However, the following example shows that this is not always the case for graph products. The result of Theorem 6 may be therefore more delicate than one might think.

**Example 15.** *Suppose $\Gamma$ is a complete bipartite graph $K_{k,k}$, i.e. $\Gamma$ has vertex set*

$$V(\Gamma) = \{u_1, \ldots, u_k, v_1, \ldots, v_k\}$$

*and edge set*

$$E(\Gamma) = \{\{u_i, v_j\} \mid 1 \leq i, j \leq k\},$$

*and let* $\mathbf{H}(u) \cong \mathbb{Z}$ *with generators* $X(u) = \{x_u, x_u^{-1}\}$ *for each* $u \in V(\Gamma)$*. In this case one has* $G(\Gamma, \mathbf{H}) \cong F_k \times F_k$ *(direct product of two free groups of rank* $k$*) and so one can calculate sphere sizes in* $G(\Gamma, \mathbf{H})$ *and its special subgroups easily. Note that clearly (by the definition of link) every element of* $G_A \leq G$ *commutes with every element of* $G_{\operatorname{link} A} \leq G$*. Now consider the case where* $A = \{u_1, \ldots, u_k\}$ *and so* $\operatorname{link} A = \{v_1, \ldots, v_k\}$*. It follows that*

$$\{(x, y) \in B(n)^2 \mid [x, y] = 1\} \supseteq B_{G_A}(n) \times B_{G_{\operatorname{link} A}}(n).$$

*An explicit computation shows that*

$$\mathfrak{B}_{G_A}(n) = \mathfrak{B}_{G_{\operatorname{link} A}}(n) = \frac{k(2k-1)^n - 1}{k-1}$$

*and*

$$\mathfrak{B}_G(n) = \frac{2k^2 n(2k-1)^n}{(k-1)(2k-1)} + e_1(2k-1)^n + e_2$$

*where* $e_1 = e_1(k)$ *and* $e_2 = e_2(k)$ *are some constants. It follows that*

$$d_n \geq \frac{\mathfrak{B}_{G_A}(n)\mathfrak{B}_{G_{\operatorname{link} A}}(n)}{\mathfrak{B}_G(n)^2} \sim \left(\frac{2k-1}{2kn}\right)^2$$

*as* $n \to \infty$*. In particular, the sequence* $(d_n)_{n=0}^{\infty}$ *converges to zero only at a polynomial rate for* $G = G(\Gamma, \mathbf{H})$*.*

The proof of Theorem 6 is based on the fact that if (1.9) held for infinitely many $n$ then there would exist a subset $A \subseteq V(\Gamma)$ such that the growth of both $G_A$ and $G_{\operatorname{link} A}$ would be comparable to that of $G$. More precisely, the outline of the proof is as follows:

(i) finding such a subset $A \subseteq V(\Gamma)$ and showing that $G_A$ is not negligible in $G$, i.e. $\frac{\mathfrak{B}_{G_A}(n)}{\mathfrak{B}_G(n)} \not\to 0$ as $n \to \infty$ (subsection 3.2);

(ii) finding a collection $\mathcal{H}$ of subgroups of $G$ having (uniformly) polynomial growth such that, for all $H \in \mathcal{H}$, $G_{\operatorname{link} A} \times H$ is a subgroup of $G$ and $\frac{|(G_{\operatorname{link} A} \times H) \cap B_G(n)|}{\mathfrak{B}_G(n)}$ is uniformly bounded below as $n \to \infty$ (subsection 3.3);

(iii) using the embedding $G_A \times G_{\operatorname{link} A} \subseteq G$ and Theorem 1 to obtain a contradiction (subsection 3.4).

### 3.2 A non-negligible special subgroup

Note that (1.9) can be rewritten as

$$\sum_{A \subseteq V(\Gamma)} \sum_{\substack{g \in B(n) \\ \mathrm{supp}(\tilde{g}) = A}} \frac{|C_G(g) \cap B(n)|}{\mathfrak{B}(n)^2} \geq \varepsilon \tag{1.10}$$

and so (1.10) holds for infinitely many $n$. But as $\Gamma$ is finite, there are only $2^{|V(\Gamma)|} < \infty$ subsets of $V(\Gamma)$, thus in particular there exists a subset $A \subseteq V(\Gamma)$ such that

$$\sum_{\substack{g \in B(n) \\ \mathrm{supp}(\tilde{g}) = A}} \frac{|C_G(g) \cap B(n)|}{\mathfrak{B}(n)^2} \geq 2^{-|V(\Gamma)|} \varepsilon \tag{1.11}$$

holds for infinitely many $n$. One may restrict the subset of elements $g \in G$ considered even further:

**Lemma 16.** *There exist constants $\tilde{\varepsilon} > 0$ and $s \in \mathbb{Z}_{\geq 0}$ such that*

$$\sum_{\substack{g \in B(n) \\ \mathrm{supp}(\tilde{g}) = A \\ |p_g| \leq s}} \frac{|C_G(g) \cap B(n)|}{\mathfrak{B}(n)^2} \geq \tilde{\varepsilon}$$

*for infinitely many $n$.*

*Proof.* As $G$ has rational spherical growth series by [7], Theorem 1 says that there exist constants $\alpha \in \mathbb{Z}_{\geq 0}$, $\lambda \geq 1$, $C = C_G > 0$ and $D = D_G > C$ such that

$$Cn^\alpha \lambda^n \leq \mathfrak{S}(n) \leq Dn^\alpha \lambda^n \tag{1.12}$$

for all $n \geq 1$. As it is also assumed that $G$ has exponential growth, one has $\lambda > 1$. It is easy to show that in this case

$$Cn^\alpha \lambda^n < \mathfrak{B}(n) < \frac{D\lambda}{\lambda - 1} n^\alpha \lambda^n \tag{1.13}$$

for all $n \geq 1$.

Now one can bound the number of terms in (1.11) corresponding to elements $g \in G$ with $|p_g|$ large (even without requiring $\mathrm{supp}(\tilde{g}) = A$). Indeed, as any $g \in G$ can be written as $g = p_g^{-1}\tilde{g}p_g$ with $|g| = 2|p_g| + |\tilde{g}|$, (1.12) and (1.13) imply

$$\begin{aligned}\frac{1}{\mathfrak{B}(n)} \sum_{\substack{g\in B(n)\\|p_g|>s}} \frac{|C_G(g)\cap B(n)|}{\mathfrak{B}(n)} &\leq \frac{|\{g\in B(n)\mid |p_g|>s\}|}{\mathfrak{B}(n)} \leq \sum_{i=s+1}^{\lfloor\frac{n}{2}\rfloor} \frac{\mathfrak{S}(i)\mathfrak{B}(n-2i)}{\mathfrak{B}(n)} \\ &\leq \frac{D}{C}\left(\frac{1}{2}\right)^{\alpha}\lambda^{-\frac{n}{2}} + \frac{D^2\lambda}{C(\lambda-1)}\sum_{i=s+1}^{\lfloor\frac{n-1}{2}\rfloor}\left(\frac{i(n-2i)}{n}\right)^{\alpha}\lambda^{-i}.\end{aligned} \tag{1.14}$$

The first term of the sum above clearly tends to zero as $n\to\infty$, and the second term is bounded above by the infinite sum $\sum_{i=s+1}^{\infty} i^{\alpha}\lambda^{-i}$, which tends to zero as $s\to\infty$ since the series $\sum_i i^{\alpha}\lambda^{-i}$ converges. Hence there exists a value of $s\in\mathbb{Z}_{\geq 0}$ which ensures that the right hand side in (1.14) is less than $2^{-|V(\Gamma)|-1}\varepsilon$ for $n$ large enough. This means that

$$\sum_{\substack{g\in B(n)\\ \operatorname{supp}(\tilde{g})=A\\ |p_g|\leq s}} \frac{|C_G(g)\cap B(n)|}{\mathfrak{B}(n)^2} \geq 2^{-|V(\Gamma)|-1}\varepsilon$$

for infinitely many $n$, so setting $\tilde{\varepsilon} := 2^{-|V(\Gamma)|-1}\varepsilon$ completes the proof. ☐

Now note that one may write

$$\begin{aligned}\sum_{\substack{g\in B(n)\\ \operatorname{supp}(\tilde{g})=A\\ |p_g|\leq s}} \frac{|C_G(g)\cap B(n)|}{\mathfrak{B}(n)^2} &\leq \frac{|\{g\in B(n)\mid \operatorname{supp}(\tilde{g})=A, |p_g|\leq s\}|}{\mathfrak{B}(n)} \\ &\times \max\left\{\frac{|C_G(g)\cap B(n)|}{\mathfrak{B}(n)} \,\middle|\, g\in B(n), \operatorname{supp}(\tilde{g})=A, |p_g|\leq s\right\}\end{aligned}$$

where both terms in the product are bounded above by 1. It follows by Lemma 16 that both

$$\frac{|\{g\in B(n)\mid \operatorname{supp}(\tilde{g})=A, |p_g|\leq s\}|}{\mathfrak{B}(n)} \geq \tilde{\varepsilon} \tag{$*$}$$

and

$$\max\left\{\frac{|C_G(g)\cap B(n)|}{\mathfrak{B}(n)} \,\middle|\, g\in B(n), \operatorname{supp}(\tilde{g})=A, |p_g|\leq s\right\} \geq \tilde{\varepsilon} \tag{$\dagger$}$$

hold for infinitely many $n$.

The aim is now to show that ($*$) and ($\dagger$) imply that the special subgroups $G_A$ and $G_{\operatorname{link} A}$ (respectively) are non-negligible in $G$. For the latter, one may consider explicit forms of centralisers of $G$: see the next subsection. For the former, note that the set in the numerator consists of elements $g\in B_G(n)$ which have an expression $g = p_g^{-1}\tilde{g}p_g$ with $p_g\in B_G(s)$ and $\tilde{g}\in B_{G_A}(n)$. It follows that

$$|\{g\in B(n)\mid \operatorname{supp}(\tilde{g})=A, |p_g|\leq s\}| \leq \mathfrak{B}_G(s)\mathfrak{B}_{G_A}(n)$$

and so ($*$) implies that

$$\frac{\tilde{\varepsilon}}{\mathfrak{B}_G(s)} \leq \frac{\mathfrak{B}_{G_A}(n)}{\mathfrak{B}_G(n)} \leq 1 \tag{$**$}$$

for infinitely many $n$, where the second inequality comes from the fact that $B_{G_A}(n) \subseteq B_G(n)$.

### 3.3 Centralisers in $G$

In order to use ($\dagger$), one needs to consider forms of centralisers of elements $g \in G$ with $\mathrm{supp}(\tilde{g}) = A$. Fix an element $g \in G$ with $\mathrm{supp}(\tilde{g}) = A$ and note that one clearly has $C_G(g) = p_g^{-1} C_G(\tilde{g}) p_g$, so if $|p_g| \leq s$ then one has

$$|C_G(g) \cap B(n)| \leq |C_G(\tilde{g}) \cap B(n + 2s)|. \tag{1.15}$$

In particular, it follows from ($\dagger$) that for infinitely many $n$, there exists an element $g \in B(n)$ with $\mathrm{supp}(\widetilde{g}) = A$ and $|p_g| \leq s$ such that

$$\tilde{\varepsilon} \leq \frac{|C_G(\tilde{g}) \cap B_G(n + 2s)|}{\mathfrak{B}_G(n)} \leq \mathfrak{B}_G(2s); \tag{$\dagger\dagger$}$$

here the second inequality comes from the fact that $|C_G(\tilde{g}) \cap B(n+2s)| \leq \mathfrak{B}(n+2s) \leq \mathfrak{B}(n)\mathfrak{B}(2s)$.

Now define an element $g \in G$ to be *cyclically normal* (in the sense of [2]) if either $\ell_n(g) \leq 1$, or $n := \ell_n(g) \geq 2$ and for any normal form $w = w_1 \cdots w_n \in X^*$ of $g$, where $w_i \in X(v_i)^*$ for some $v_i \in V(\Gamma)$, one has $v_1 \neq v_n$. Then one has

**Lemma 17.** *For any $g \in G$ with $\mathrm{supp}(\tilde{g}) = A$, there exists an element $\tilde{p}_g \in G_A$ such that $\hat{g} := \tilde{p}_g \tilde{g} \tilde{p}_g^{-1}$ is cyclically normal and $\mathrm{supp}(\hat{g}) = A$.*

*Proof.* If $\ell_n(\tilde{g}) \leq 1$ then $\tilde{p}_g = 1$ does the job. Thus suppose that $n := \ell_n(\tilde{g}) \geq 2$. Let

$$\begin{aligned} E(\tilde{g}) := \{g_0 \mid\ & w = w_1 \cdots w_n \in X^* \text{ is a normal form for } \tilde{g} \text{ where} \\ & w_i \in X(v_i)^* \text{ for some } v_i \in V(\Gamma), \text{ and } w_n \text{ represents } g_0\} \end{aligned}$$

be a finite subset of $G_A$. By Proposition 13, any two elements in $E(\tilde{g})$ commute, and so, for any two distinct elements $g_1 \in \mathbf{H}(v_1)$ and $g_2 \in \mathbf{H}(v_2)$ of $E(\tilde{g})$, one has $v_1 \neq v_2$. Now define $\tilde{p}_g := \prod_{g_n \in E(\tilde{g})} g_n$. Then $\tilde{p}_g \in G_A$, and following the proof of [2, Lemma 23] one can see that $\hat{g} := \tilde{p}_g \tilde{g} \tilde{p}_g^{-1}$ is cyclically normal. Since $\mathrm{supp}(\tilde{p}_g) \subseteq A$ and $\mathrm{supp}(\tilde{g}) = A$, it is clear that $\mathrm{supp}(\hat{g}) \subseteq A$. It also follows by [2, Lemma 18] that $\mathrm{supp}(\hat{g}) \cup \mathrm{supp}(\tilde{p}_g) \supseteq A$. Thus one only needs to check that $\mathrm{supp}(\tilde{p}_g) \subseteq \mathrm{supp}(\hat{g})$.

Suppose for contradiction that there exists some vertex $v \in \mathrm{supp}(\tilde{p}_g) \setminus \mathrm{supp}(\hat{g})$, and let $g_v \in E(\tilde{g}) \cap \mathbf{H}(v)$ be the (unique) element. It is easy to see that $v \notin \mathrm{link}(A \setminus \{v\})$:

otherwise any normal form of $\hat{g}$ would contain a subword in $X(v)^*$ representing $g_v$ and so $v \in \operatorname{supp}(\hat{g})$. Then, following again the proof of [2, Lemma 23], one has $\tilde{n} := \ell_n(g_v \tilde{g} g_v^{-1}) \leq n-1$, with $\tilde{n} = n-1$ if and only if $\tilde{g}$ has no normal form $w_1 \cdots w_n$, where $w_i \in X(v_i)^*$ for some $v_i \in V(\Gamma)$, with $w_1$ and $w_n$ representing $g_v^{-1}$ and $g_v$, respectively. Thus, by minimality of $|\tilde{g}|$, clearly $\tilde{n} = n-1$; but this cannot happen by [2, Lemma 18], since by assumption $v \notin \operatorname{supp}(\hat{g})$. Hence $\operatorname{supp}(\tilde{p}_g) \subseteq \operatorname{supp}(\hat{g})$, as required. □

The following Proposition describes growth of centralisers in $G$.

**Proposition 18.** *Let $g, \tilde{g} \in G$ and $A \subseteq V(\Gamma)$ be as above. Then*

$$C_G(\tilde{g}) = H_1 \times \cdots \times H_k \times G_{\operatorname{link} A}$$

*for some subgroups $H_1, \ldots, H_k \leq G$, and the following hold:*

(i) *for any $h_1 \in H_1, \ldots, h_k \in H_k$ and $c \in G_{\operatorname{link} A}$,*

$$|h_1 \cdots h_k c|_X = |h_1|_X + \cdots + |h_k|_X + |c|_X;$$

(ii) *there exist constants $D_1, \ldots, D_k, \alpha_1, \ldots, \alpha_k \in \mathbb{Z}_{\geq 1}$ such that*

$$|H_i \cap B_{G,X}(n)| \leq D_i n^{\alpha_i}$$

*for all $n \geq 1$.*

*Furthermore, the number $k \in \mathbb{Z}_{\geq 1}$, the $D_i$ and the $\alpha_i$ only depend on $A$ and not on $g$.*

*Proof.* Let $A_1, \ldots, A_k \subseteq A$ form a partition of $A$ such that the graphs $\Gamma(A_i)^C$ are precisely the connected components of the graph $\Gamma(A)^C$, where $\Delta^C$ denotes the complement of a graph $\Delta$. Let $\tilde{p}_g, \hat{g} \in G_A$ be as in Lemma 17. Then $\operatorname{supp}(\hat{g}) = A$ and so $\hat{g}$ can be expressed as

$$\hat{g} = \hat{g}_1 \cdots \hat{g}_k$$

where $\operatorname{supp}(\hat{g}_i) = A_i$.

Now suppose without loss of generality that for some $m$, the sets $A_i = \{v_i\}$ are singletons for $1 \leq i \leq m$, and $|A_i| \geq 2$ for $m+1 \leq i \leq k$. Then Proposition 25, Theorem 32 and Theorem 52 in [2] state that the centraliser of $\hat{g}$ in $G$ is

$$C_G(\hat{g}) = C_{\mathbf{H}(v_1)}(\hat{g}_1) \times \cdots \times C_{\mathbf{H}(v_m)}(\hat{g}_m) \times \langle h_{m+1} \rangle \times \cdots \times \langle h_k \rangle \times G_{\operatorname{link} A}$$

where $h_{m+1}, \ldots, h_k \in G$ are some infinite order elements with $\operatorname{supp}(h_i) = A_i$ (in fact, one has $\hat{g}_i = h_i^{\beta_i}$ for some $\beta_i \in \mathbb{Z} \setminus \{0\}$).

In particular, since $\tilde{p}_g \in G_A$, one has $\tilde{p}_g = p_1 \cdots p_k$ for some $p_i \in G_{A_i}$. Thus one has $\tilde{p}_g^{-1} q_i \tilde{p}_g = p_i^{-1} q_i p_i$ for any $q_i \in G_{A_i}$, and $\tilde{p}_g^{-1}(G_{\mathrm{link}\,A})\tilde{p}_g = G_{\mathrm{link}\,A}$, hence

$$\begin{aligned} C_G(\tilde{g}) = \tilde{p}_g^{-1} C_G(\hat{g}) \tilde{p}_g = \; & C_{\mathbf{H}(v_1)}(\tilde{g}_1) \times \cdots \times C_{\mathbf{H}(v_m)}(\tilde{g}_m) \\ & \times \langle \tilde{g}_{m+1} \rangle \times \cdots \times \langle \tilde{g}_k \rangle \times G_{\mathrm{link}\,A} \end{aligned}$$

where $\tilde{g}_i := p_i^{-1} \hat{g}_i p_i$ for $1 \leq i \leq m$, and $\tilde{g}_i := p_i^{-1} h_i p_i$ for $m+1 \leq i \leq k$. Therefore, by setting $H_i := C_{\mathbf{H}(v_i)}(\tilde{g}_i)$ for $1 \leq i \leq m$ and $H_i := \langle \tilde{g}_i \rangle \cong \mathbb{Z}$ for $m+1 \leq i \leq k$ one obtains the required expression. By construction, $k$ depends only on $A$ (and not on $g$).

To show (i), it is enough to note that $H_i \leq G_{A_i}$ for each $i$, and that by construction the subsets $A_i$ are pairwise disjoint and disjoint from $\mathrm{link}\,A$. Indeed, then it follows from Proposition 13 that if $w_i$ (respectively $u$) is a normal form for an element $h_i \in G_{A_i}$ (respectively $c \in G_{\mathrm{link}\,A}$), then $w_1 \cdots w_k u$ is a normal form for the element $h_1 \cdots h_k c$. This implies (i).

To show (ii) and the last part of the Proposition, one may consider cases $1 \leq i \leq m$ and $m+1 \leq i \leq k$ separately. For $1 \leq i \leq m$, note that, as a consequence of Proposition 13, $|h|_X = |h|_{X(v_i)}$ for all $h \in H_i$, and therefore $|H_i \cap B_{G,X}(n)| = |H_i \cap B_{\mathbf{H}(v_i),X(v_i)}(n)|$ for all $n \geq 1$. Thus, (ii) follows from the facts that $\tilde{g}_i \neq 1$ and that $(\mathbf{H}(v_i), X(v_i))$ is a rational pair with small centralisers; it also follows that $D_i, \alpha_i$ do not depend on $g$. For $m+1 \leq i \leq k$, it follows from the proof of [2, Lemma 37] that since $\hat{g}_i$ is cyclically normal and since $\Gamma(\mathrm{supp}(\hat{g}_i))^C = \Gamma(A_i)^C$ is connected, one has $\ell_n(\tilde{g}_i^\gamma) \geq \ell_n(\hat{g}_i^\gamma) = |\gamma| \ell_n(\hat{g}_i)$ for all $\gamma \in \mathbb{Z}$. In particular, $|\tilde{g}_i^\gamma|_X \geq \ell_n(\tilde{g}_i^\gamma) \geq |\gamma|$ for any $\gamma \in \mathbb{Z}$ and so $|H_i \cap B_{G,X}(n)| \leq 2n+1 \leq 3n$ for all $n \geq 1$. Thus taking $D_i = 3$ and $\alpha_i = 1$ shows (ii); independence from $g$ is clear. $\square$

### 3.4 Products of special subgroups

To finalise the proof, one employs the following general result:

**Lemma 19.** *Let $G$ be a group with a finite generating set $X$. Let $H, K \leq G$ be subgroups such that $H \times K$ is also a subgroup of $G$, i.e. the map $H \times K \to G, (h,k) \mapsto hk$ is an injective group homomorphism. Suppose that there exist constants $\alpha_H, \alpha_K \in \mathbb{Z}_{\geq 0}$, $\lambda_H, \lambda_K \in [1, \infty)$ and $D > C \geq 0$ such that*

$$\begin{aligned} & C n^{\alpha_H} \lambda_H^n \leq |H \cap S_{G,X}(n)| \leq D n^{\alpha_H} \lambda_H^n \\ \text{and} \qquad & C n^{\alpha_K} \lambda_K^n \leq |K \cap S_{G,X}(n)| \leq D n^{\alpha_K} \lambda_K^n \end{aligned}$$

*for all $n \geq 1$. Furthermore, suppose that $|hk|_X = |h|_X + |k|_X$ for all $h \in H^{(n)}$, $k \in K^{(n)}$, and that $\lambda_H \geq \lambda_K$. If $\lambda_H > \lambda_K$, then there exists some $\widetilde{D} = \widetilde{D}(D, \alpha_H, \alpha_K, \lambda_H, \lambda_K) > 0$,* *which does not depend on $H$ or $K$, such that*

$$|(H \times K) \cap S_{G,X}(n)| \leq \widetilde{D} n^{\alpha_H} \lambda_H^n$$

*for all $n \geq 1$. Furthermore, if $\lambda_H = \lambda_K$ and $C > 0$, then no such constant $\widetilde{D}$ exists.*

*Proof.* Suppose first that $\lambda_H > \lambda_K$. Clearly it is enough to show that

$$\limsup_{n\to\infty} \frac{|(H \times K) \cap S_{G,X}(n)|}{n^{\alpha_H} \lambda_H^n} < \infty.$$

Fix $n \geq 1$. As $|hk|_X = |h|_X + |k|_X$ for any $h \in H$, $k \in K$, one has

$$\begin{aligned} \frac{|(H \times K) \cap S_{G,X}(n)|}{n^{\alpha_H} \lambda_H^n} &= \frac{1}{n^{\alpha_H} \lambda_H^n} \sum_{i=0}^{n} |H \cap S_{G,X}(n-i)| \times |K \cap S_{G,X}(i)| \\ &\leq D^2 \left( 1 + \sum_{i=1}^{n-1} \left(\frac{\lambda_K}{\lambda_H}\right)^i \left(\frac{n-i}{n}\right)^{\alpha_H} i^{\alpha_K} + \left(\frac{\lambda_K}{\lambda_H}\right)^n n^{\alpha_K - \alpha_H} \right). \end{aligned}$$

As $\lambda_K/\lambda_H < 1$, limits of the first and third term above as $n \to \infty$ are $D^2$ and $0$, respectively. The second term can be bounded above by an upper bound for the series $D^2 \sum_i (\lambda_K/\lambda_H)^i i^{\alpha_K}$, which converges by the ratio test. Hence one indeed has $\limsup_{n\to\infty} |(H \times K) \cap S_{G,X}(n)|/(n^{\alpha_H}\lambda_H^n) < \infty$, which implies the result. It is also clear from the inequality above that $\widetilde{D}$ depends only on $D$, $\alpha_H$, $\alpha_K$, $\lambda_H$ and $\lambda_K$.

Conversely, suppose that $C > 0$ and $\lambda_H = \lambda_K =: \lambda$. Let $n \geq 20$, so that $\lceil \sqrt{n} \rceil \leq n/4$. Then

$$\begin{aligned} \frac{|(H \times K) \cap S_{G,X}(n)|}{n^{\alpha_H} \lambda^n} &= \frac{1}{n^{\alpha_H} \lambda^n} \sum_{i=0}^{n} |H \cap S_{G,X}(n-i)| \times |K \cap S_{G,X}(i)| \\ &\geq C^2 \sum_{i=1}^{n-1} \left(\frac{n-i}{n}\right)^{\alpha_H} i^{\alpha_K} \geq C^2 \sum_{i=\lceil\sqrt{n}\rceil}^{\lfloor n/2 \rfloor} \left(\frac{1}{2}\right)^{\alpha_H} (\sqrt{n})^{\alpha_K} \\ &\geq C^2 2^{-(\alpha_H+2)} n^{\frac{\alpha_K}{2}+1}. \end{aligned}$$

In particular, one has $|(H \times K) \cap S_{G,X}(n)|/(n^{\alpha_H}\lambda_H^n) \to \infty$ as $n \to \infty$, implying the result. $\square$

Given this Lemma, the proof can be finalised as follows. Recall (see (1.12) and (1.13)) that one has constants $\alpha \in \mathbb{Z}_{\geq 0}$, $\lambda > 1$ and $D_{V(\Gamma)} > C_{V(\Gamma)} > 0$ such that

$$\begin{aligned} C_{V(\Gamma)} n^\alpha \lambda^n \leq \mathfrak{S}_G(n) \leq D_{V(\Gamma)} n^\alpha \lambda^n \\ \text{and} \quad C_{V(\Gamma)} n^\alpha \lambda^n < \mathfrak{B}_G(n) < \frac{D_{V(\Gamma)} \lambda}{\lambda - 1} n^\alpha \lambda^n \end{aligned} \tag{1.16}$$

for all $n \geq 1$. Now (∗∗) implies that, for infinitely many $n$,

$$\widetilde{C}_A n^\alpha \lambda^n \leq \mathfrak{B}_{G_A}(n) \leq \widetilde{D}_A n^\alpha \lambda^n \tag{1.17}$$

for some $\widetilde{D}_A > \widetilde{C}_A > 0$. But as $G_A$ has rational growth with respect to $\bigsqcup_{v\in A} X(v)$, it follows from Theorem 1 that in fact, after modifying the constants $\widetilde{D}_A$ and $\widetilde{C}_A$ if necessary, (1.17) holds for all $n \geq 1$, and since $\lambda > 1$, after further modifying $\widetilde{C}_A$, one has

$$\widetilde{C}_A n^\alpha \lambda^n \leq \mathfrak{S}_{G_A}(n) \leq \widetilde{D}_A n^\alpha \lambda^n \tag{$***$}$$

for all $n \geq 1$.

Moreover, (††) implies that for infinitely many $n \geq 2s+1$ there exists $g \in B(n)$ such that

$$\widetilde{C}'(n-2s)^\alpha \lambda^{n-2s} \leq |C_G(\tilde{g}) \cap B_{G,X}(n)| \leq \widetilde{D}'(n-2s)^\alpha \lambda^{n-2s}$$

for some $\widetilde{D}' > \widetilde{C}' > 0$. After decreasing the constant $\widetilde{C}' > 0$ if necessary, one may therefore assume that, for infinitely many $n$,

$$\widetilde{C}' n^\alpha \lambda^n \leq |C_G(\tilde{g}) \cap B_{G,X}(n)| \leq \widetilde{D}' n^\alpha \lambda^n \tag{1.18}$$

for some $g \in B(n)$ with $\mathrm{supp}(\widetilde{g}) = A$ and $|p_g| \leq s$.

Note that $G_{\mathrm{link}\,A}$ has rational growth with respect to $\bigsqcup_{v\in \mathrm{link}\,A} X(v)$ as it is a special subgroup of $G$, and so by Theorem 1 it follows that, for all $n \geq 1$,

$$\widetilde{C}_{\mathrm{link}\,A} n^{\alpha_0} \lambda_0^n \leq \mathfrak{S}_{G_{\mathrm{link}\,A}}(n) \leq \widetilde{D}_{\mathrm{link}\,A} n^{\alpha_0} \lambda_0^n \tag{1.19}$$

for some $\widetilde{D}_{\mathrm{link}\,A} > \widetilde{C}_{\mathrm{link}\,A} > 0$ and some $\alpha_0 \in \mathbb{Z}_{\geq 0}$, $\lambda_0 \geq 1$.

One may now show that $(\lambda_0, \alpha_0) = (\lambda, \alpha)$. Indeed, as $\mathfrak{S}_{G_{\mathrm{link}\,A}}(n) \subseteq \mathfrak{S}_G(n)$, it follows from (1.16) that either $\lambda_0 < \lambda$ or $\lambda_0 = \lambda$ and $\alpha_0 \leq \alpha$. Let $g \in G$ be such that $\mathrm{supp}(\widetilde{g}) = A$ for all $n$. By Proposition 18, one has an expression

$$C_G(\widetilde{g}) = H_1 \times \cdots \times H_k \times G_{\mathrm{link}\,A}.$$

One now applies Lemma 19 $k$ times. In particular, for each $i = k, k-1, \ldots, 1$ in order, it follows from Proposition 18 that Lemma 19 can be applied for

$$\begin{aligned} H &:= H_{i+1} \times \cdots \times H_k \times G_{\mathrm{link}\,A}, \\ K &:= H_i, \end{aligned}$$

$$(\alpha_H, \lambda_H) := \begin{cases} (\alpha_0, \lambda_0) & \text{if } \lambda_0 > 1, \\ (0, \frac{\lambda+1}{2}) & \text{if } \lambda_0 = 1, \end{cases}$$
$$(\alpha_K, \lambda_K) := \left(0, \frac{\lambda_H + 1}{2}\right),$$
$$C := 0,$$
$$\text{and} \qquad D = \overline{D}_i := \max\{\widetilde{D}_i, \widetilde{D}'_i\}.$$

Here $\widetilde{D}'_i > 0$ is such that $D_i n^{\alpha_i} \leq \widetilde{D}'_i \lambda_K^n$ for each $n \geq 1$, where $D_i$ and $\alpha_i$ are as in Proposition 18, $\widetilde{D}_k$ is such that $\mathfrak{S}_{G_{\mathrm{link}\,A}}(n) \leq \widetilde{D}_k n^{\alpha_H} \lambda_H^n$ for all $n \geq 1$, and, for each $i = k-1, k-2, \ldots, 1$, $\widetilde{D}_i = \widetilde{D}(\overline{D}_{i+1}, \alpha_H, \alpha_K, \lambda_H, \lambda_K)$ is the constant given by Lemma 19.

It then follows that, for all $g \in G$ with $\mathrm{supp}(\widetilde{g}) = A$ and $|p_g| \leq s$,

$$|C_G(\widetilde{g}) \cap S_{G,X}(n)| \leq \widetilde{D} n^{\alpha_H} \lambda_H^n \tag{1.20}$$

for all $n \geq 1$, where $\widetilde{D} = \widetilde{D}(\overline{D}_1, \alpha_H, \alpha_K, \lambda_H, \lambda_K)$ is the constant, independent from $g$, given by Lemma 19. Since $\lambda_H > 1$, by further increasing $\widetilde{D}$ we may replace $S_{G,X}(n)$ with $B_{G,X}(n)$ in (1.20). But by construction, one has either $\lambda_H < \lambda$ or $\lambda_H = \lambda$ and $\alpha_H \leq \alpha$, and so together with (1.18) this implies that $(\lambda_H, \alpha_H) = (\lambda, \alpha)$. Thus, by the choice of $(\lambda_H, \alpha_H)$, one has $(\lambda_0, \alpha_0) = (\lambda, \alpha)$, as claimed. In particular, (1.19) can be rewritten as

$$\widetilde{C}_{\mathrm{link}\,A} n^\alpha \lambda^n \leq \mathfrak{S}_{G_{\mathrm{link}\,A}}(n) \leq \widetilde{D}_{\mathrm{link}\,A} n^\alpha \lambda^n. \tag{†††}$$

Finally, note that the group $G_{A \cup \mathrm{link}\,A} = G_A \times G_{\mathrm{link}\,A}$ is a special subgroup of $G$ and so one has $S_{G_{A\cup \mathrm{link}\,A}}(n) \subseteq S_G(n)$. It then follows from (∗∗∗), (†††) and the last sentence in Lemma 19 that for any $\widetilde{D} > 0$ one has

$$\mathfrak{S}_G(n) \geq \mathfrak{S}_{G_{A\cup \mathrm{link}\,A}}(n) > \widetilde{D} n^\alpha \lambda^n$$

for some $n$, which contradicts (1.16). This completes the proof of Theorem 6.

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# Paper 2

## PROBABILISTIC NILPOTENCE IN INFINITE GROUPS

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Abstract.The 'degree of $k$-step nilpotence' of a finite group $G$ is the proportion of the tuples $(x_1, \ldots, x_{k+1}) \in G^{k+1}$ for which the simple commutator $[x_1, \ldots, x_{k+1}]$ is equal to the identity. In this paper we study versions of this for an infinite group $G$, with the degree of nilpotence defined by sampling $G$ in various natural ways, such as with a random walk, or with a Følner sequence if $G$ is amenable. In our first main result we show that if $G$ is finitely generated then the degree of $k$-step nilpotence is positive if and only if $G$ is virtually $k$-step nilpotent (Theorem 1.8). This generalises both an earlier result of the second author treating the case $k = 1$ and a result of Shalev for finite groups, and uses techniques from both of these earlier results. We also show, using the notion of polynomial mappings of groups developed by Leibman and others, that to a large extent the degree of nilpotence does not depend on the method of sampling (Theorem 1.16). As part of our argument we generalise a result of Leibman by showing that if $\varphi$ is a polynomial mapping into a torsion-free nilpotent group then the set of roots of $\varphi$ is sparse in a certain sense (Theorem 1.29). In our second main result we consider the case where $G$ is residually finite but not necessarily finitely generated. Here we show that if the degree of $k$-step nilpotence of the finite quotients of $G$ is uniformly bounded from below then $G$ is virtually $k$-step nilpotent (Theorem 1.31), answering a question of Shalev. As part of our proof we show that degree of nilpotence of finite groups is sub-multiplicative with respect to quotients (Theorem 1.32), generalising a result of Gallagher.

# 1 Introduction

If two elements $x, y$ are chosen independently uniformly at random from a finite group $G$, we define the probability that they commute to be the *commuting probability* or *degree of commutativity* of $G$, and denote it by $\mathrm{dc}(G)$. Peter Neumann proved the following structure theorem for groups with a high degree of commutativity.

**Theorem 1.1** (P. M. Neumann [19, Theorem 1]). *Let $G$ be a finite group such that $\mathrm{dc}(G) \geq \alpha > 0$. Then $G$ has a normal subgroup $\Gamma$ of index at most $\alpha^{-1} + 1$ and a normal subgroup $H$ of cardinality at most $\exp(O(\alpha^{-O(1)}))$ such that $H \subset \Gamma$ and $\Gamma/H$ is abelian.*

There are many natural ways in which one might seek to generalise this result. Here we seek to generalise it in two ways. The first is to higher-degree commutators. Given elements $x_i$ in a group $G$, we define the *simple commutators* $[x_1, \ldots, x_k]$ inductively by setting $[x_1, x_2] = x_1^{-1}x_2^{-1}x_1x_2$ and setting $[x_1, \ldots, x_k] = [[x_1, \ldots, x_{k-1}], x_k]$. If $G$ is finite, we define $\mathrm{dc}^k(G)$ to be the probability that $[x_1, \dots, x_{k+1}] = 1$ if $x_1, \dots x_{k+1}$ are chosen independently uniformly at random from $G$.

Shalev [20] recently considered higher-order commutators in residually finite groups, proving the following results.

**Theorem 1.2** (Shalev; see the proof of [20, Theorem 1.1])**.** *Let $G$ be a finite group of rank at most $r$, and let $k \in \mathbb{N}$. Suppose that $\mathrm{dc}^k(G) \geq \alpha > 0$. Then $G$ has a $k$-step nilpotent subgroup of index at most $O_{r,k,\alpha}(1)$.*

**Corollary 1.3** (Shalev [20, Theorem 1.1])**.** *Let $G$ be a finitely generated residually finite group of rank at most $r$, and let $k \in \mathbb{N}$. Suppose that $\mathrm{dc}^k(G/H) \geq \alpha > 0$ for every finite-index normal subgroup $H$ of $G$. Then $G$ has a $k$-step nilpotent subgroup of index at most $O_{r,k,\alpha}(1)$.*

The second way in which we seek to generalise results of this type is by considering groups that are not necessarily finite or even residually finite. The first question in this setting is how to define the probability that two group elements commute. In [1] Antolín and the first and fourth authors approach this issue by considering sequences of finitely supported probability measures whose supports converge to the whole of $G$. Given a probability measure $\mu$ on $G$, define the *degree of commutativity* $\mathrm{dc}_\mu(G)$ of $G$ with respect to $\mu$ via

$$\mathrm{dc}_\mu(G) = \mu(\{(x, y) \in G \times G : xy = yx\})$$

(here, and throughout, we abuse notation slightly by writing $\mu(X)$ for $(\mu \times \cdots \times \mu)(X)$ when $X \subset G^k$). Then, given a sequence $M = (\mu_n)_{n=1}^{\infty}$ of probability measures on $G$, define the *degree of commutativity* $\mathrm{dc}_M(G)$ of $G$ with respect to $M$ via

$$\mathrm{dc}_M(G) = \limsup_{n\to\infty} \mathrm{dc}_{\mu_n}(G).$$

Here we extend this notion to more general equations. For each $k \in \mathbb{N}$, write $F_k$ for the free group on $k$ generators, denoted $x_1, \dots, x_k$.

**Definition 1.4.** Let $G$ be a group.

(i) An *equation* in $k$ variables over $G$ is a word $\varphi \in F_k * G$. Abusing notation slightly, we may view $\varphi$ as a function $G^k \to G$ by defining $\varphi(g_1, \dots, g_k)$ to be the element of $G$ resulting from replacing each instance of $x_i$ in the word $\varphi$ by $g_i$.

(ii) Given a probability measure $\mu$ on $G$ and an equation $\varphi$ in $k$ variables over $G$, define the *degree of satisfiability* $\mathrm{d}\varphi_\mu(G)$ of $\varphi$ in $G$ with respect to $\mu$ via

$$\mathrm{d}\varphi_\mu(G) = \mu(\{(g_1, \dots, g_k) \in G^k \mid \varphi(g_1, \dots, g_k) = 1\}).$$

Then, given a sequence $M = (\mu_n)_{n=1}^{\infty}$ of probability measures on $G$, define the *degree of satisfiability* $\mathrm{d}\varphi_M(G)$ of $\varphi$ in $G$ with respect to $M$ via

$$\mathrm{d}\varphi_M(G) = \limsup_{n\to\infty} \mathrm{d}\varphi_{\mu_n}(G).$$

When $G$ is finite and $\mu$ is the uniform probability measure on $G$ we write simply $\mathrm{d}\varphi(G) = \mathrm{d}\varphi_\mu(G)$.

In particular, if $c = [x_1, x_2] \in F_2$ is a commutator, then we obtain the usual definitions of $\mathrm{dc}_\mu(G)$ and $\mathrm{dc}_M(G)$, as above. More generally, here and throughout we denote by $c^{(k)}$ the $(k+1)$-fold simple commutator, $c^{(k)} = [x_1, \ldots, x_{k+1}] \in F_{k+1}$, so that $c = c^{(1)}$. We call the resulting number $\mathrm{dc}^k_\mu(G)$ (respectively $\mathrm{dc}^k_M(G)$) the *degree of $k$-nilpotence* of $G$ with respect to $\mu$ (respectively $M$). For notational convenience in the inductive proof of Theorem 1.8, below, we also define $c^{(0)} = x_1 \in F_1$, so that $\mathrm{dc}^0_M(G) = \limsup_{n\to\infty} \mu_n(\{1\})$.

In [1] Antolín and the first and fourth authors suggest that for any 'reasonable' sequence $M = (\mu_n)_{n=1}^{\infty}$ of probability measures on $G$ we should have $\mathrm{dc}^k_M(G) > 0$ if and only if $G$ is virtually $k$-step nilpotent. They further suggest that 'reasonable' might mean that the measures $\mu_n$ cover $G$ with 'enough homogeneity' as $n \to \infty$. A specific example they give of what should be a 'reasonable' sequence is where $\mu$ is some finite probability measure on $G$, and $\mu_n = \mu^{*n}$ is defined by letting $\mu^{*n}(x)$ be the probability that a random walk of length $n$ on $G$ with respect to $\mu$ ends at $x$. If $G$ is amenable, another natural sequence of measures to consider is the sequence of uniform probability measures on a Følner sequence, or more generally an *almost-invariant sequence of measures*, which is to say a sequence $(\mu_n)_{n=1}^{\infty}$ of probability measures satisfying

$$\|x \cdot \mu_n - \mu_n\|_1 \to 0$$

for every $x \in G$ (here $x \cdot \mu$ is defined by setting $x \cdot \mu(A) = \mu(x^{-1}A)$).

In [21] the second author gave some fairly general conditions on a sequence $(\mu_n)_{n=1}^{\infty}$ of measures under which such a theorem holds in the case $k = 1$. Two specific cases of this are as follows.

**Theorem 1.5** ([21, Theorems 1.13–1.15])**.** *Let $G$ be a finitely generated group. Suppose that either*

(i) *$\mu$ is a symmetric, finitely supported generating probability measure on $G$ with $\mu(\{1\}) > 0$, and $M = (\mu^{*n})_{n=1}^{\infty}$ is the sequence of measures corresponding to the steps of the random walk on $G$ with respect to $\mu$; or*

(ii) *$G$ is amenable and $M = (\mu_n)_{n=1}^{\infty}$ is an almost-invariant sequence of probability measures on $G$.*

*Suppose that* $\mathrm{dc}_M(G) \geq \alpha > 0$*. Then* $G$ *has a normal subgroup* $\Gamma$ *of index at most* $\lceil \alpha^{-1} \rceil$ *and a normal subgroup* $H$ *of cardinality at most* $\exp(O(\alpha^{-O(1)}))$ *such that* $H \subset \Gamma$ *and* $\Gamma/H$ *is abelian. In particular, if the rank of* $G$ *is at most* $r$ *then* $G$ *has an abelian subgroup of index at most* $O_{r,\alpha}(1)$*.*

One of the main aims of [21] was to provide a concrete but more-general set of hypotheses on $M$ under which Theorem 1.5 holds. This led to the following definitions.

**Definition 1.6** (Uniform detection of index)**.** Let $\pi : (0, 1] \to (0, 1]$ be a non-decreasing function such that $\pi(\gamma) \to 0$ as $\gamma \to 0$. We say that a sequence $M = (\mu_n)_{n=1}^{\infty}$ of probability measures on a group $G$ *detects index uniformly at rate* $\pi$ if for every $\varepsilon > 0$ there exists $N = N(\varepsilon) \in \mathbb{N}$ such that for every $m \in \mathbb{N}$ if $[G : H] \geq m$ then $\mu_n(H) \leq \pi(\frac{1}{m}) + \varepsilon$ for every $n \geq N$. We also say simply that $M$ *detects index uniformly* to mean that there exists some $\pi$ such that $M$ detects index uniformly at rate $\pi$.

The word 'uniform' in Definition 1.6 refers to the requirement that the definition be satisfied by the same $N(\varepsilon)$ for all subgroups $H$.

**Definition 1.7** (Uniform measurement of index)**.** We say that a sequence $M = (\mu_n)_{n=1}^{\infty}$ of probability measures on a group $G$ *measures index uniformly* if $\mu_n(xH) \to 1/[G : H]$ uniformly over all $x \in G$ and all subgroups $H$ of $G$ (here we define $1/[G : H] = 0$ if $[G : H] = \infty$).

Note that if a sequence of probability measures on a group measures index uniformly then it also detects index uniformly with rate $\iota : (0, 1] \to (0, 1]$ defined by $\iota(x) = x$.

The second author shows in [21, Theorems 1.13 & 1.14] that on a finitely generated group every sequence of measures corresponding to the steps of a random walk measures index uniformly, as does every almost-invariant sequence of measures. This is a key ingredient in the proof of Theorem 1.5.

In the present paper we combine Shalev's techniques with those of [21] to generalise Theorem 1.2 similarly to arbitrary finitely generated groups, as follows.

**Theorem 1.8.** *Let* $G$ *be a finitely generated group of rank at most* $r$*, and let* $M = (\mu_n)_{n=1}^{\infty}$ *be a sequence of measures that detects index uniformly at rate* $\pi$*. Suppose that* $\mathrm{dc}^k_M(G) \geq \alpha > 0$*. Then* $G$ *has a* $k$*-step nilpotent subgroup of index at most* $O_{r,k,\pi,\alpha}(1)$*.*

The following specific cases of interest of Theorem 1.8 then follow from [21, Theorems 1.13 & 1.14].

**Theorem 1.9.** *Let* $G$ *be a finitely generated group of rank at most* $r$*, and let* $k \in \mathbb{N}$*. Suppose that either*

(i) *$\mu$ is a symmetric, finitely supported generating probability measure on $G$ with $\mu(\{1\}) > 0$, and $M = (\mu^{*n})_{n=1}^{\infty}$ is the sequence of measures corresponding to the steps of the random walk on $G$ with respect to $\mu$; or*

(ii) *$G$ is amenable and $M = (\mu_n)_{n=1}^{\infty}$ is an almost-invariant sequence of probability measures on $G$.*

*Suppose that $\mathrm{dc}_M^k(G) \geq \alpha > 0$. Then $G$ has a $k$-step nilpotent subgroup of index at most $O_{r,k,\alpha}(1)$.*

We prove Theorem 1.8 in Section 3.

Shalev actually proves a slightly more general result than Theorem 1.2. Given a finite group $G$ and an element $g \in G$, write

$$P^k(G, g) = \mu(\{(x_1, \ldots, x_{k+1}) \in G^{k+1} : [x_1, \ldots, x_{k+1}] = g\}),$$

noting that $\mathrm{dc}^k(G) = P^k(G, 1)$. What Shalev shows is that Theorem 1.2 remains true if the assumption that $\mathrm{dc}^k(G) \geq \alpha > 0$ is replaced by the weaker assumption that $P^k(G, g) \geq \alpha > 0$ for some $g \in G$.

We can adapt the statement of Theorem 1.8 similarly. First, given a probability measure $\mu$ on a group $G$, define

$$P_\mu^k(G, g) = \mu(\{(x_1, \ldots, x_{k+1}) \in G^{k+1} : [x_1, \ldots, x_{k+1}] = g\}).$$

Then, given a sequence $M = (\mu_n)_{n=1}^{\infty}$ of probability measures on $G$, define

$$P_M^k(G, g) = \limsup_{n\to\infty} P_{\mu_n}^k(G, g).$$

**Proposition 1.10.** *Let $G$ be a group, and let $M = (\mu_n)_{n=1}^{\infty}$ be a sequence of probability measures on $G$ that measures index uniformly. Then $P_M^k(G, 1) \geq P_M^k(G, g)$ for every $g \in G$.*

Combined with Theorem 1.8 this immediately gives the following.

**Corollary 1.11.** *Let $G$ be a finitely generated group of rank at most $r$, and let $M = (\mu_n)_{n=1}^{\infty}$ be a sequence of measures that measures index uniformly on $G$. Let $g \in G$, and suppose that $P_M^k(G, g) \geq \alpha > 0$. Then $G$ has a $k$-step nilpotent subgroup of index at most $O_{r,k,\alpha}(1)$.*

We prove Proposition 1.10 in Section 3.

It is easy to see that if a finitely generated group $G$ has a nilpotent subgroup of finite index then $\mathrm{dc}^k_M(G) > 0$ for every sequence $M$ of measures measuring index uniformly on $G$. The conclusion of Theorem 1.8 is therefore qualitatively optimal. Note, however, that Theorem 1.5 shows that in the case $k = 1$ Theorem 1.8 can be improved quantitatively—in the sense that the bounds can be made independent of the rank of $G$—at the expense of concluding that $G$ is bounded-by-abelian-by-bounded as in Theorem 1.1, rather than virtually abelian.

The following result suggests that a quantitatively optimal result for $\mathrm{dc}^k_M(G)$ might also allow for bounded-by-nilpotent-by-bounded groups in its conclusion.

**Proposition 1.12.** *Let $m, d, k \in \mathbb{N}$. Let $G$ be a finitely generated group, let $\Gamma$ be a subgroup of $G$ of index at most $m$, and let $H$ be a subgroup of cardinality at most $d$ such that $\Gamma/H$ is $k$-step nilpotent. Let $M = (\mu_n)_{n=1}^\infty$ be a sequence of measures that measures index uniformly on $G$. Then $\mathrm{dc}^k_M(G) \geq \frac{1}{m^{k+1}d}$.*

We prove Proposition 1.12 in Section 3.

Note that Theorem 1.8 and Proposition 1.12 combine to give a new proof of the following folklore result, which we have been unable to find in the literature (although after the statement we reference two published arguments that give results in a similar direction).

**Corollary 1.13** (Finite-by-($k$-step nilpotent) groups are virtually $k$-step nilpotent)**.** *Let $r, d, k \in \mathbb{N}$. Let $G$ be a finitely generated group of rank at most $r$, and let $H$ be a subgroup of cardinality at most $d$ such that $G/H$ is $k$-step nilpotent. Then $G$ contains a $k$-step nilpotent subgroup of index at most $O_{r,d,k}(1)$.*

It is shown in the proof of [3, Corollary 11.7] that the group $G$ contains a $(k + 1)$-step nilpotent subgroup of index at most $O_d(1)$. The proof of [5, Proposition 3.4] shows how to pass to a $k$-step nilpotent subgroup of finite index as in Corollary 1.13, but without any obvious control over the index.

On the other hand, in Section 7 we give examples for all $k \geq 1$ to show that the dependence of the bound on the rank is necessary in Theorem 1.8 and Corollary 1.13 as stated. Moreover, the following result states that without the assumption on $G$ to be finitely generated, the (qualitative) conclusion of Theorem 1.8 and Corollary 1.13 on $G$ being virtually $k$-step nilpotent need not be true. However, such a conclusion in Corollary 1.13 is still true if we replace 'finitely generated of rank at most $r$' with 'residually finite': see Remark 7.5.

**Proposition 1.14.** *For any $k \geq 1$ and any odd prime $p$, there exists a group $G$ and a finite normal subgroup $H \lhd G$ of order $p$ such that $G/H$ is $k$-step nilpotent, but $G$ is not virtually $k$-step nilpotent.*

We prove Proposition 1.14 in Section 7.

**Question 1.15.** *Can one make the bounds in Theorem 1.8 independent of the rank of $G$ at the expense of broadening the qualitative conclusion?*

Equations over virtually nilpotent groups. The second author shows in [21, Theorem 1.19] that if $G$ is a finitely generated group and $M = (\mu_n)_{n=1}^{\infty}$ is a sequence of measures that measures index uniformly on $G$ then the lim sup in the definition of $\mathrm{dc}_M$ is actually a limit, and that this limit does not depend on the choice of $M$. In the present work we extend this to $\mathrm{dc}_M^k$ for $k \geq 2$, as follows.

**Theorem 1.16.** *Let $G$ be a finitely generated group. Then $\mathrm{dc}_M^k(G)$ takes the same value for all sequences $M$ of measures that measure index uniformly on $G$, and for every such sequence $M$ the* lim sup *in the definition of $\mathrm{dc}_M^k(G)$ actually a limit.*

In view of Theorem 1.8, in proving Theorem 1.16 it is enough to consider virtually nilpotent groups, and in that context we actually prove something more general: we show that for *any* equation $\varphi$ over a finitely generated virtually nilpotent group $G$ the numbers $\mathrm{d}\varphi_M(G)$ are well behaved in the sense of Theorem 1.16.

To do this we use a notion of sparsity that is independent of any particular sequence of measures, as follows.

**Definition 1.17.** Given a group $G$, a set $V \subset G$ is said to be *negligible by finite quotients of $G$* if for every $\varepsilon > 0$ there exists a finite-index normal subgroup $N \lhd G$ such that $|VN/N| \leq \varepsilon|G/N|$.

The utility of this definition lies in the following proposition, which we prove in Section 2.

**Proposition 1.18.** *Let $(\mu_n)_{n=1}^{\infty}$ be a sequence of measures that measure index uniformly on a group $G$, and suppose that $V \subset G^k$ is negligible by finite quotients of $G^k$. Then $\mu_n(V) \to 0$ as $n \to \infty$.*

*Remark* 1.19. To see that being negligible by finite quotients is *strictly* stronger than having zero density with respect to a sequence of measures measuring index uniformly, consider the example in which $\mu_n$ is the uniform probability measure on the Følner set $\{-n, \dots, n\} \subset \mathbb{Z}$, and the set $A$ is defined as

$$A = \bigcup_{k=1}^{\infty} \{2^k + 1, \dots, 2^k + k\}.$$

Then $A$ satisfies $\mu_n(A) \to 0$ as $n \to \infty$, but is not negligible by finite quotients of $\mathbb{Z}$.

**Theorem 1.20.** *Let $G$ be a finitely generated virtually nilpotent group, and let $N$ be a torsion-free nilpotent normal subgroup of finite index in $G$. Let $\varphi$ be an equation in $k$ variables over $G$. Then the set*

$$G_\varphi = \{(g_1, \dots, g_k) \in G^k : \varphi(g_1, \dots, g_k) = 1\}$$

*of solutions to $\varphi$ is the union of a set of cosets of $N^k$ and a set that is negligible by finite quotients of $G^k$.*

Recall that virtually nilpotent groups are always virtually torsion-free, so by defining the subgroup $N$ in Theorem 1.20 we are merely fixing notation, rather than imposing an additional hypothesis. In particular, Proposition 1.18 and Theorem 1.20 immediately imply the following result.

**Corollary 1.21.** *Let $G$ be a finitely generated virtually nilpotent group, and let $\varphi$ be an equation over $G$. Then $\mathrm{d}\varphi_M(G)$ is the same for all sequences $M$ of measures that measure index uniformly on $G$, and for every such sequence $M$ the* $\limsup$ *in Definition 1.4 is actually a limit.*

In particular, combined with Theorem 1.8 this implies Theorem 1.16. Indeed, in the case of a residually finite group Theorems 1.8 and 1.20 even give the value of $\mathrm{dc}^k_M(G)$ in terms of $\mathrm{dc}^k$ of the finite quotients of $G$, as follows.

**Corollary 1.22.** *Let $G$ be a residually finite group and let $M$ be a sequence of measures on $G$ that measures index uniformly. Let $H_1 > H_2 > \cdots$ be a sequence of finite-index normal subgroups of $G$ such that $\bigcap_{m=1}^{\infty} H_m = \{1\}$. Then $\mathrm{dc}^k(G/H_m) \to \mathrm{dc}^k_M(G)$ as $m \to \infty$.*

Note also that if $G$ is assumed a priori to be virtually nilpotent then Theorem 1.20 similarly gives the value of $\mathrm{d}\varphi_M(G)$ in Corollary 1.21.

**Corollary 1.23.** *Let $G$ be a finitely generated virtually nilpotent group, and let $\varphi$ be an equation over $G$. Let $H_1 > H_2 > \cdots$ be a sequence of finite-index normal subgroups of $G$ such that $\bigcap_{m=1}^{\infty} H_m = \{1\}$. Then $\mathrm{d}\varphi(G/H_m) \to \mathrm{d}\varphi_M(G)$ as $m \to \infty$.*

*Remark* 1.24. It is easy to see, for $G$, $M$ and $(H_i)$ as in Corollary 1.22 and $\varphi$ an arbitrary equation over $G$, that the sequence $(\mathrm{d}\varphi(G/H_m))_{m=1}^{\infty}$ is decreasing and bounded below by 0—and hence converges to some limit—and that

$$\mathrm{d}\varphi_M(G) \leq \lim_{m\to\infty} \mathrm{d}\varphi(G/H_m). \tag{1.1}$$

Corollaries 1.22 and 1.23 say that if $\varphi = c^{(k)}$, or if $G$ is virtually nilpotent, then we have equality in (1.1).

An equation $\varphi$ in $k$ variables over a group $G$ is a *probabilistic identity* with respect to a sequence $M$ of measures if $\mathrm{d}\varphi_M(G) > 0$; it is a *coset identity* if there exists a finite-index subgroup $H < G$ and elements $g_1, \dots, g_k \in G$ such that $\varphi(g_1H, \dots, g_kH) = 1$. Shalev [20, Corollary 1.2] notes that Theorem 1.2 implies that if $[x_1, \dots, x_{k+1}]$ is a coset identity in a finitely generated residually finite group $G$ then $G$ is virtually $k$-step nilpotent. The following is immediate from Theorem 1.20.

**Corollary 1.25.** *Let $G$ be a finitely generated virtually nilpotent group, and let $M$ be a sequence of measures on $G$ that measures index uniformly. Then an equation $\varphi$ in $G$ is a probabilistic identity with respect to $M$ if and only if it is a coset identity.*

We prove Theorem 1.20 in Section 4. An important tool in the proof is the notion of a *polynomial mapping* of a group. These have been studied extensively by Leibman [15], and have found applications to finding prime solutions to linear systems of equations [13] and to the study of harmonic functions on groups [17].

A *polynomial mapping* of a group is defined as follows.

**Definition 1.26** (Derivatives, Polynomial mappings)**.** Let $G, H$ be groups and let $\varphi : G \to H$. Given $u \in G$, we define the *$u$-derivative* $\partial_u\varphi : G \to H$ of $\varphi$ via $\partial_u\varphi(x) = \varphi(x)^{-1}\varphi(xu)$. Given $d \in \mathbb{N}$, we say that $\varphi$ is *polynomial of degree $d$* if $\partial_{u_1} \cdots \partial_{u_{d+1}}\varphi \equiv 1$ for all $u_1, \ldots, u_{d+1} \in G$.

*Remark.* Leibman actually defines the more refined notion of being polynomial *relative to a generating set $S$* for $G$; Definition 1.26 corresponds to being polynomial relative to $G$. Nonetheless, in the present paper the range of every mapping we consider will be nilpotent, and Leibman shows that a mapping of $G$ into a nilpotent group is polynomial relative to some generating set for $G$ if and only if it is polynomial relative to $G$ [15, Proposition 3.5], so we lose no generality by using Definition 1.26.

The basic scheme of the proof of Theorem 1.20 is to show that equations over virtually nilpotent groups are polynomial mappings, so that the set of solutions to an arbitrary equation can be viewed as the set of roots of some polynomial. The idea is then to use the familiar notion that the set of roots of a polynomial is 'sparse' in some sense.

Leibman has already shown that the set of roots of a polynomial mapping into a torsion-free nilpotent group is sparse with respect to Følner sequences, as follows.

**Definition 1.27** (Closed subgroup)**.** A subgroup $\Gamma$ of a group $G$ is said to be *closed* in $G$ if for every $x \in G$ and $n \in \mathbb{Z}$ we have $x^n \in \Gamma$ if and only if $x \in \Gamma$.

**Theorem 1.28** (Leibman [15, Proposition 4.3])**.** *Let $G$ be a countable amenable group, and let $(\mu_n)_{n=1}^{\infty}$ be the sequence of uniform probability measures on some Følner sequence on $G$. Let $N$ be a nilpotent group, let $\Gamma$ be a closed subgroup of $N$, and let $\varphi : G \to N$ be polynomial. Then for every $x \in N$ such that $\varphi(G) \not\subset x\Gamma$ we have $\mu_n(\varphi^{-1}(x\Gamma)) \to 0$ as $n \to \infty$.*

In Section 5 we strengthen this theorem in the finitely generated case to show that set of roots of a polynomial mapping into a torsion-free nilpotent groups is sparse in the stronger sense of Definition 1.17, as follows.

**Theorem 1.29.** *Let $G$ be a finitely generated group, let $N$ be a nilpotent group with a closed subgroup $\Gamma$, and let $\varphi : G \to N$ be polynomial. Then for every $x \in N$ such that $\varphi(G) \not\subset x\Gamma$ the set $\varphi^{-1}(x\Gamma)$ is negligible by finite quotients of $G$.*

This is a strengthening of Theorem 1.28 thanks to Proposition 1.18 and Remark 1.19; indeed, it implies the following result, which by [21, Theorem 1.13] directly implies Theorem 1.28 in the case where $G$ is finitely generated.

**Corollary 1.30.** *Let $G$ be a finitely generated group, and let $(\mu_n)_{n=1}^{\infty}$ be a sequence of measures that measures index uniformly on $G$. Let $N$ be a nilpotent group with a closed subgroup $\Gamma$, and let $\varphi : G \to N$ be polynomial. Then for every $x \in N$ such that $\varphi(G) \not\subset x\Gamma$ we have $\mu_n(\varphi^{-1}(x\Gamma)) \to 0$ as $n \to \infty$.*

It is also worth noting that, unlike Theorem 1.28, Theorem 1.29 and Corollary 1.30 are valid when $G$ is not amenable. That said, we should emphasise that it follows from [15, Proposition 3.21] that every polynomial mapping of $G$ into a nilpotent group factors through some amenable quotient of $G$. Theorem 1.28 therefore has some implicit content even when $G$ is not amenable.

Finite quotients. In the last of our main results we remove the 'finitely generated' assumption from Corollary 1.3, answering a question posed by Shalev [20, Problem 3.1].

**Theorem 1.31.** *Let $G$ be a residually finite group, and let $\mathcal{N}$ be a family of finite-index normal subgroups of $G$ that is closed under finite intersections and such that $\bigcap_{N \in \mathcal{N}} N = \{1\}$. Let $k \in \mathbb{N}$, and suppose that there exists a constant $\alpha > 0$ such that $\mathrm{dc}^k(G/N) \geq \alpha$ for every $N \in \mathcal{N}$. Then $G$ has a $k$-step nilpotent subgroup of finite index.*

This generalises a result of Lévai and Pyber [16, Theorem 1.1 (iii)], who prove it in the case $k = 1$. They also note that the finite index of the abelian subgroup in that case need not be bounded in terms of $\alpha$, citing the examples of direct products of abelian groups and extra-special groups [16, §1]. In Section 7 we generalise these examples to show that, for any $k \geq 1$, the index of the $k$-step nilpotent subgroup coming from Theorem 1.31 need not be bounded in terms of $k$ and $\alpha$.

The key ingredient in the proof of Theorem 1.31 is the following result of independent interest on finite groups.

**Theorem 1.32.** *Let $G$ be a finite group and let $N \trianglelefteq G$ be a normal subgroup. Then $\mathrm{dc}^k(G) \leq \mathrm{dc}^k(N)\,\mathrm{dc}^k(G/N)$ for all $k \in \mathbb{N}$.*

Theorem 1.32 generalises the main result of Gallagher [11], who proves it for $k = 1$. It also generalises a theorem by Moghaddam, Salemkar and Chiti [18, Theorem A], who prove Theorem 1.32 when the centraliser of every element of $G$ is normal. As is noted in [10, Section 2], the centraliser of every element of $G$ being normal implies that $G$ is 3-step nilpotent; this is an extremely strong hypothesis for Theorem 1.32, rendering it trivial for $k \geq 3$, for example.

We prove Theorems 1.31 and 1.32 in Section 6.

Degree of nilpotence with respect to uniform measures on balls. If $G$ is generated by a finite set $X$, one may naturally define the degree of nilpotence of $G$ using the sequence of measures $(\mu_n)_{n=1}^{\infty}$ defined by taking $\mu_n$ to be the uniform probability measure on the ball of radius $n$ in $G$ with respect to $X$. The main problem with adapting our results to this setting is that, in general, the sequence $\mu_n$ does not measure index uniformly (see the remark immediately after Theorem 1.13 of [21]).

**Question 1.33.** *Let $G$ be a group with a finite generating set $X$, let $\mu_n$ be the uniform probability measure on the ball of radius $n$ in $G$ with respect to $X$, and write $M = (\mu_n)_{n=1}^{\infty}$. Suppose that $G$ is not virtually $k$-step nilpotent. Do we have $\mathrm{dc}_M^k(G) = 0$?*

If $M$ is defined using the uniform probability measures on the balls with respect to a finite generating set as in Question 1.33, and if $G$ is virtually nilpotent, then it is well known and easy to check that $M$ is an almost-invariant sequence of probability measures on $G$. It therefore follows from [21, Theorem 1.13] that $M$ measures index uniformly. In light of Theorem 1.8, a positive answer to Question 1.33 would therefore extend both Theorems 1.8 and 1.16 to the sequence of uniform probability measures on the balls with respect to a finite generating set.

Question 1.33 seems to be difficult in general, although the answer is positive in some cases. For example, in Appendix B we present an argument that was communicated to us by Yago Antolín answering Question 1.33 for hyperbolic groups (see Corollary B.2).

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## 2 Products of measures that measure index uniformly

In this short section we prove the following result and use it to deduce Proposition 1.18.

**Lemma 2.1.** *Let $G_1, \ldots, G_k$ be groups, and for each $i$ let $(\mu_n^{(i)})_{n=1}^{\infty}$ be a sequence of measures such that for every $x_i \in G_i$ and every finite-index subgroup $H_i < G_i$ we have $\mu_n^{(i)}(x_i H_i) \to 1/[G_i : H_i]$ as $n \to \infty$. Then*

$$\mu_n^{(1)} \times \cdots \times \mu_n^{(k)}(xH) \to \frac{1}{[G_1 \times \cdots \times G_k : H]}$$

*for every finite-index subgroup $H < G_1 \times \cdots \times G_k$ and every element $x \in G_1 \times \cdots \times G_k$.*

*Proof.* Note first that

$$\mu_n^{(1)} \times \cdots \times \mu_n^{(k)} \left( x \prod_{i=1}^k H_i \right) \to \frac{1}{[\prod_{i=1}^k G_i : \prod_{i=1}^k H_i]}$$

for all $x \in \prod_{i=1}^k G_i$ and all finite-index subgroups of the form $\prod_{i=1}^k H_i < \prod_{i=1}^k G_i$ with $H_i < G_i$ for each $i$. It therefore suffices to show that an arbitrary finite-index subgroup $H < \prod_{i=1}^k G_i$ has a finite-index subgroup of the form $\prod_{i=1}^k H_i$. However, if $H$ has index $d$ in $G$ then $H \cap G_i$ has index at most $d$ in $G_i$ for each $i$. The product subgroup $\prod_{i=1}^k (H \cap G_i)$ therefore has index at most $d^k$ in $\prod_{i=1}^k G_i$, and hence in $H$, as required. □

*Proof of Proposition 1.18.* Let $\varepsilon > 0$, and let $H \lhd G^k$ be a finite-index normal subgroup such that $|VH/H| \leq \frac{1}{2}\varepsilon[G^k : H]$. As $G^k$ contains only finitely many cosets of $H$, it follows from Lemma 2.1 that there exists $N \in \mathbb{N}$ such that for every $x \in G^k$ and every $n \geq N$ we have $\mu_n(xH) \leq 2/[G^k : H]$, and hence $\mu_n(V) \leq \mu_n(VH) \leq \varepsilon$. We therefore have $\mu_n(V) \to 0$, as required. □

## 3 The algebraic structure of probabilistically nilpotent groups

In this section we study the relation between $\mathrm{dc}_M^k(G)$ and the existence of finite-index $k$-step nilpotent subgroups of $G$, proving Theorem 1.8, Proposition 1.10 and Proposition 1.12. We start our proof of Theorem 1.8 with the following version of [21, Proposition 2.1], which was itself based on an argument of Neumann [19].

**Proposition 3.1.** *Let $k \in \mathbb{N}$. Let $G$ be a group and let $M = (\mu_n)_{n=1}^\infty$ be a sequence of measures on $G$ that detects index uniformly at rate $\pi$. Let $\alpha \in (0, 1]$, and suppose that $\mathrm{dc}_M^k(G) \geq \alpha$. Let $\gamma \in (0, 1)$ be such that $\pi(\gamma) < \alpha$, and write*

$$X = \{(x_1, \ldots, x_k) \in G^k : [G : C_G([x_1, \ldots, x_k])] \leq \tfrac{1}{\gamma}\}.$$

*Then* $\limsup_{n\to\infty} \mu_n(X) \geq \alpha - \pi(\gamma)$.

*Proof.* By definition of $\mathrm{dc}_M$ there exists a sequence $n_1 < n_2 < \cdots$ such that $\mathrm{dc}_{\mu_{n_i}}^k(G) \geq \alpha - o(1)$. Writing $\mathbb{E}^{(n)}$ for expectation with respect to $\mu_n$, this means precisely that

$$\mathbb{E}^{(n_i)}_{(x_1,\ldots,x_k)\in G^k}(\mu_{n_i}(C_G([x_1, \ldots, x_k]))) \geq \alpha - o(1).$$

Following Neumann [19], we note that therefore

$$\alpha \leq \mu_{n_i}(X)\mathbb{E}^{(n_i)}_{(x_1,\dots,x_k)\in X}(\mu_{n_i}(C_G([x_1,\dots,x_k]))) \\ +\mu_{n_i}(G\backslash X)\mathbb{E}^{(n_i)}_{(x_1,\dots,x_k)\in G^k\backslash X}(\mu_{n_i}(C_G([x_1,\dots,x_k]))) + o(1),$$

and hence, by uniform detection of index,

$$\alpha \leq \mu_{n_i}(X) + \pi(\gamma) + o(1).$$

The result follows. □

**Lemma 3.2.** *Let $m, r \in \mathbb{N}$, and let $G$ be a group generated by $r$ elements. Then $G$ has at most $O_{m,r}(1)$ subgroups of index $m$.*

*Proof.* A subgroup $H$ of index $m$ is the stabiliser of an action of $G$ on the set $G/H$, which has size $m$. There at most $|\operatorname{Sym}(m)|^r$ possible actions of $G$ on a set of size $m$, and so there are at most $|\operatorname{Sym}(m)|^r$ possibilities for this stabiliser. □

*Proof of Theorem 1.8.* We combine an induction used by Shalev in [20, Proposition 2.2] with the proof of [21, Theorem 1.10]. If $k = 0$ then $\limsup_{n\to\infty}\mu_n(\{1\}) \geq \alpha$, and so the order of $G$ is $O_{\pi,\alpha}(1)$, and the theorem holds. For $k > 0$, let $\gamma = \frac{1}{2}\inf\{\beta \in (0,1] : \pi(\beta) \geq \frac{\alpha}{2}\}$, noting that therefore $\pi(\gamma) < \frac{\alpha}{2}$. Proposition 3.1 therefore gives

$$\limsup_{n\to\infty}\mu_n(\{(x_1,\dots,x_k) \in G^k : [G : C_G([x_1,\dots,x_k])] \leq \tfrac{1}{\gamma}\}) \geq \tfrac{\alpha}{2}. \tag{3.1}$$

Write $\Gamma$ for the intersection of all subgroups of $G$ of index at most $\frac{1}{\gamma}$, noting that $\Gamma$ is normal and has index $O_{r,\pi,\alpha}(1)$ by Lemma 3.2. It follows from (3.1) that

$$\limsup_{n\to\infty}\mu_n(\{(x_1,\dots,x_k) \in G^k : [x_1,\dots,x_k] \in C_G(\Gamma)\}) \geq \tfrac{\alpha}{2},$$

or equivalently that

$$\mathrm{dc}^{k-1}_M(G/C_G(\Gamma)) \geq \tfrac{\alpha}{2}.$$

By induction, $G/C_G(\Gamma)$ has a $(k-1)$-step nilpotent subgroup $N_0$ of index at most $O_{r,\pi,k,\alpha}(1)$. Writing $N$ for the pullback of $N_0$ to $G$, the intersection $N \cap \Gamma$ is $k$-step nilpotent and has index at most $O_{r,\pi,k,\alpha}(1)$ in $G$, and so the theorem is proved. □

Proposition 1.10 is essentially based on the following lemma.

**Lemma 3.3.** *Let $G$ be a group, and $u, g \in G$. Then $\{x \in G : [u,x] = g\}$ is either empty or a coset of $C_G(u)$.*

*Proof.* If $\{x \in G : [u,x] = g\}$ is not empty then $[u,x_0] = g$ for some $x_0 \in G$, in which case we have $\{x \in G : [u,x] = g\} = \{x \in G : u^x = u^{x_0}\} = C_G(u)x_0$. □

The point of the following lemma is that sequences of measures that measure index uniformly give the same measure to right-cosets of a subgroup that they give to left-cosets of that subgroup.

**Lemma 3.4.** *Let $M = (\mu_n)_{n=1}^{\infty}$ be a sequence of measures that measures index uniformly. Then $\mu_n(Hx) \to 1/[G : H]$ uniformly over all $x \in G$ and all subgroups $H$ of $G$.*

*Proof.* This is because $Hx = x(H^x)$ and $H^x$ has the same index as $H$. □

*Proof of Proposition 1.10.* Lemma 3.3, Lemma 3.4 and the definition of uniform measurement of index imply that for every $x_1, \ldots, x_k \in G$ we have either

$$\begin{aligned}\mu_n(\{x \in G : [[x_1, \ldots, x_k], x] = g\}) &\to \frac{1}{[G : C_G([x_1, \ldots, x_k])]} \\ &= \lim_{n\to\infty} \mu_n(C_G([x_1, \ldots, x_k]))\end{aligned}$$

or $\mu_n(\{x \in G : [[x_1, \ldots, x_k], x] = g\}) \to 0$, and that this convergence is uniform over all $x_1, \ldots, x_k$. Writing $\mathbb{E}^{(n)}$ for expectation with respect to $\mu_n$, it follows that

$$\begin{aligned}\limsup_{n\to\infty} P^k_{\mu_n}(G, g) &= \limsup_{n\to\infty} \mathbb{E}^{(n)}_{(x_1,\ldots,x_k)\in G^k}(\mu_n(\{x \in G : [[x_1, \ldots, x_k], x] = g\})) \\ &\leq \limsup_{n\to\infty} \mathbb{E}^{(n)}_{(x_1,\ldots,x_k)\in G^k}(\mu_n(C_G([x_1, \ldots, x_k]))) \\ &= P^k_M(G, 1),\end{aligned}$$

as required. □

We close this section by proving Proposition 1.12.

*Proof of Proposition 1.12.* We use a similar argument to that of [21, Proposition 1.17]. Fix elements $x_1, \ldots, x_k \in \Gamma$. The fact that $\Gamma/H$ is $k$-step nilpotent implies that $[x_1, \ldots, x_k, y] \in H$ for every $y \in \Gamma$. Thus $[x_1, \ldots, x_k, y] = [x_1, \ldots, x_k]^{-1}[x_1, \ldots, x_k]^y$ takes at most $d$ distinct values as $y$ ranges over $\Gamma$, and hence that the conjugacy class of $[x_1, \ldots, x_k]$ has size at most $d$. The orbit-stabiliser theorem therefore implies that $C_\Gamma([x_1, \ldots, x_k])$ has index at most $d$ in $\Gamma$, and hence at most $dm$ in $G$. Since this holds for every $x_1, \ldots, x_k \in \Gamma$, the result follows by uniform measurement of index. □

# 4 Equations over virtually nilpotent groups in terms of polynomial mappings

In this section we prove Theorem 1.20. The rough idea is to express an equation over the group $G$ in terms of polynomial mapping, and then apply Theorem 1.29. Recall that, given a group $G$ and an equation $\varphi \in F_k * G$ over $G$, we denote the set of solutions to $\varphi$ in $G^k$ by

$$G_\varphi = \{(g_1, \dots, g_k) \in G^k : \varphi(g_1, \dots, g_k) = 1\}.$$

The proof of Theorem 1.20 is particularly straightforward in the case where $G$ itself is torsion-free nilpotent, thanks to the following two results of Leibman, the first of which shows that if $\varphi, \psi : H \to N$ are two polynomial mappings into a nilpotent group $N$ then the *pointwise product* $\varphi\psi : H \to N$ defined by setting $(\varphi\psi)(h) = \varphi(h)\psi(h)$, and the *pointwise inverse* $\varphi^{(-1)} : H \to N$ defined by setting $\varphi^{(-1)}(h) = \varphi(h)^{-1}$, are also polynomial.

**Theorem 4.1** (Leibman [15, Theorem 3.2])**.** *If $H$ is a group and $N$ is a nilpotent group then the polynomial mappings $H \to N$ form a group under the operations of taking pointwise products and pointwise inverses.*

Since constant maps $G^k \to G$ are trivially polynomial of degree 0, and the maps $G^k \to G$ sending $(x_1, \dots, x_k)$ to $x_i$ are trivially polynomial of degree 1, it follows immediately from Theorem 4.1 that an equation over a nilpotent group $G$ is a polynomial $G^k \to G$. Thus, if $G$ itself is torsion-free nilpotent then Theorem 1.20 follows from Theorem 1.29. We spend the rest of this section explaining how to generalise this argument to the case in which $G$ is merely *virtually* nilpotent.

Let $G$ be a group, let $H \lhd G$ be a normal subgroup, let $\varphi \in F_k * G$ be an equation over $G$ and let $g \in G^k$. Given $h \in H^k$, note that $\varphi(hg) \in H\varphi(g)$, so that we may define a mapping $\varphi_{H,g} : H^k \to H$ via

$$\varphi_{H,g}(h) = \varphi(hg)\varphi(g)^{-1}.$$

We may then describe the set of solutions to $\varphi = 1$ in the coset $H^k g$ as

$$G_\varphi \cap H^k g = \{h \in H^k : \varphi_{H,g}(h) = \varphi(g)^{-1}\}g. \tag{4.1}$$

**Lemma 4.2.** *Let $G$ be a group, let $H \lhd G$ be a finite-index normal subgroup, and let $g \in G$. Let $V \subset H$ be negligible by finite quotients in $H$. Then $Vg$ is negligible by finite quotients in $G$.*

*Proof.* Let $\varepsilon > 0$. Then there exists a normal subgroup $K \lhd H$ of finite index such that $|VK/K| \le \varepsilon |G/K|$. Since $K$ has finite index in $G$, there exists a finite-index subgroup $L < K$ such that $L \lhd G$, and then we have $|VgL/L| = |VL/L| \le |VK/K||K/L| \le \varepsilon |G/K||K/L| = \varepsilon |G/L|$. □

**Lemma 4.3.** *Let $G$ be a group, let $N \lhd G$ be a nilpotent normal subgroup, let $\varphi \in F_k * G$ be an equation over $G$, and let $g \in G^k$. Then the map $\varphi_{N,g} : N^k \to N$ is polynomial.*

*Proof.* We can view the equation $\varphi$ as a concatenation of *variables* $x_i^{\pm 1} \in F_k$ and *constants* $c \in G$. Write $g = (g_1, \ldots, g_k)$, and let $h = (h_1, \ldots, h_k) \in N^k$. Moving the elements $g_i^{\pm 1} \in G$ and constants $c \in G$ one by one to the right of the word $\varphi(h_1 g_1, \ldots, h_k g_k)$, conjugating the elements $h_i^{\pm 1}$ as we go, we see that $\varphi_{N,g}(h)$ is a product of elements of the form $(h_i^{\pm 1})^x$, with $x \in G$ depending only on $g_1, \ldots, g_k$, not on $h_1, \ldots, h_k$. Given any fixed $x \in G$ the maps $N^k \to N$ defined by $(h_1, \ldots, h_k) \mapsto h_i^x$ are polynomial of degree 1, and so Theorem 4.1 implies that $\varphi_{N,g}$ is polynomial, as required. □

*Proof of Theorem 1.20.* Since $N^k$ has finite index in $G^k$, the theorem may be restated as saying that for every $g \in G^k$ with $G_\varphi \cap N^k g \neq N^k g$ we have $G_\varphi \cap N^k g$ negligible by finite quotients of $G^k$. This follows readily from (4.1), Theorem 1.29, Lemma 4.2 and Lemma 4.3. □

## 5 Sparsity of roots of polynomial mappings

In this section we prove Theorem 1.29. We divide the proof into two parts. The first part reduces to the case where $N = \mathbb{Z}$, as follows.

**Proposition 5.1.** *Let $G$ be a finitely generated group, let $N$ be a nilpotent group with a closed subgroup $\Gamma$, let $\varphi : G \to N$ be polynomial, and let $x \in N$ be such that $\varphi(G) \not\subset x\Gamma$. Then there is a non-constant polynomial mapping $\psi : G \to \mathbb{Z}$ such that $\varphi^{-1}(x\Gamma) \subset \psi^{-1}(0)$.*

The second part proves the theorem in this case, as follows.

**Proposition 5.2.** *Let $G$ be a finitely generated group and let $\varphi : G \to \mathbb{Z}$ be a non-constant polynomial mapping. Then $\varphi^{-1}(0)$ is negligible by finite quotients.*

In proving Proposition 5.1 we use the following characterisation of closed subgroups of nilpotent groups.

**Proposition 5.3** (Bergelson–Leibman [2, Proposition 1.19])**.** *Let $N$ be a finitely generated nilpotent group. Then a subgroup $\Gamma < N$ is closed in $N$ if and only if there exists a series $\Gamma = \Gamma_0 \lhd \Gamma_1 \lhd \ldots \lhd \Gamma_r = N$ with $\Gamma_i / \Gamma_{i-1} \cong \mathbb{Z}$ for every $i$.*

We also use the following trivial lemma.

**Lemma 5.4** (Leibman [15, Proposition 1.10])**.** *Let $G$, $H$ and $H'$ be groups, let $\varphi : G \to H$ be polynomial of degree $d$, and let $\pi : H \to H'$ be a homomorphism. Then the composition $\pi \circ \varphi : G \to H'$ is also polynomial of degree $d$.*

*Proof of Proposition 5.1.* Since identity map is polynomial of degree 1, it follows from Theorem 4.1 that $x^{-1}\varphi$ is polynomial. Since $(x^{-1}\varphi)(g) \in \Gamma$ precisely when $\varphi(g) \in x\Gamma$, upon replacing $\varphi$ by $x^{-1}\varphi$ we may therefore assume that $x = 1$.

Let $\Gamma = \Gamma_0 \lhd \Gamma_1 \lhd \ldots \lhd \Gamma_r = N$ be the series given by Proposition 5.3, so that $\Gamma_i/\Gamma_{i-1} \cong \mathbb{Z}$ for every $i$, and let $k$ be minimal such that $\varphi(G) \subset \Gamma_k$, noting that $k \geq 1$ by assumption. Write $\pi : \Gamma_k \to \Gamma_k/\Gamma_{k-1} \cong \mathbb{Z}$ for the quotient homomorphism, and define $\psi = \pi \circ \varphi : G \to \Gamma_k/\Gamma_{k-1} \cong \mathbb{Z}$, noting that $\psi$ is polynomial by Lemma 5.4. We then have $\psi^{-1}(0) = \varphi^{-1}(\Gamma_{k-1}) \supset \varphi^{-1}(\Gamma)$, as required. □

The first step in our proof of Proposition 5.2 is to reduce to the case where $G$ is torsion-free nilpotent, via the following result of Meyerovitch, Perl, Yadin and the second author [17].

**Lemma 5.5.** *Let $G$ be a group, and let $\varphi : G \to \mathbb{Z}$ be a polynomial mapping of degree $d$. Then there is a torsion-free $d$-step nilpotent quotient $G'$ of $G$ and a polynomial mapping $\hat{\varphi} : G' \to \mathbb{Z}$ of degree $d$ such that, writing $\pi : G \to G'$ for the quotient homomorphism, we have $\varphi = \hat{\varphi} \circ \pi$.*

*Proof.* This is immediate from [17, Lemmas 2.5 & 4.4]. □

In fact, although Lemma 5.5 is sufficient for our purposes in the present paper, in Appendix A we take the opportunity to deduce from it a similar result for polynomial mappings into arbitrary torsion-free nilpotent groups.

An important benefit of Lemma 5.5 is that it allows us in the proof of Proposition 5.2 to exploit the existence of certain coordinate systems on torsion-free nilpotent groups. We give a basic description of coordinate systems here; see [15, 3.8–3.19] for a more detailed description of coordinate systems and their relationship to polynomial mappings, and [17, §4.2] for details on a particularly natural coordinate system to use when studying polynomial mappings to nilpotent groups.

Given a finitely generated torsion-free nilpotent group $G$, there exists a central series $\{1\} = G_0 \lhd G_1 \lhd \ldots \lhd G_m = G$ such that $G_i/G_{i-1} \cong \mathbb{Z}$ for every $i$. Picking $e_i \in G_i$ for each $i$ in such a way that $G_{i-1}e_i$ is a generator for $G_i/G_{i-1}$, every element $g \in G$ then has a unique expression

$$g = e_1^{v_1} \cdots e_m^{v_m} \tag{5.1}$$

for some $v_1, \ldots, v_m \in \mathbb{Z}$; we call $(e_1, \ldots, e_m)$ a *basis* for $G$. We call the $v_i$ in the expression (5.1) the *coordinates* of $g$ with respect to $(e_1, \ldots, e_m)$, and call the map $G \to \mathbb{Z}^m$ taking an element of $G$ to its coordinates the *coordinate mapping* of $G$ with respect to $(e_1, \ldots, e_m)$. We often abbreviate the expression $e_1^{v_1} \cdots e_m^{v_m}$ as $\mathbf{e}^{\mathbf{v}}$.

**Proposition 5.6** (Leibman [15, Proposition 3.12])**.** *Let $G$ and $N$ be finitely generated torsion-free nilpotent groups with bases $(e_1,\ldots,e_m)$ and $(f_1,\ldots,f_n)$, respectively, and let $\alpha:G\to\mathbb{Z}^m$ and $\beta:N\to\mathbb{Z}^n$ be the corresponding coordinate mappings. Then a mapping $\varphi:G\to N$ is polynomial if and only if $\beta\circ\varphi\circ\alpha^{-1}:\mathbb{Z}^m\to\mathbb{Z}^n$ is polynomial.*

Polynomial mappings $\mathbb{Z}^m\to\mathbb{Z}^n$ are just standard polynomials in $m$ variables, although we caution, as Leibman does in [15, 1.8], that these polynomials can have non-integer rational coefficients: the polynomial $\frac{1}{2}n^2+\frac{1}{2}n$ maps $\mathbb{Z}\to\mathbb{Z}$, for example.

Leibman [15, Corollary 3.7] shows that in a nilpotent group $G$ the operations of multiplication $G\times G\to G$ defined by $(g_1,g_2)\mapsto g_1g_2$, and raising to a power $G\times\mathbb{Z}\to G$ defined by $(g,n)\mapsto g^n$, are polynomial mappings. Given a finitely generated torsion-free nilpotent group $G$ with basis $(e_1,\ldots,e_m)$, it therefore follows from Proposition 5.6 that there exist polynomials $\mu_1,\ldots,\mu_m:\mathbb{Z}^{2m}\to\mathbb{Z}$ and $\epsilon_1,\ldots,\epsilon_m:\mathbb{Z}^{m+1}\to\mathbb{Z}$ such that

$$\mathbf{e}^{\mathbf{v}}\cdot\mathbf{e}^{\mathbf{w}}=e_1^{\mu_1(\mathbf{v},\mathbf{w})}\cdots e_m^{\mu_m(\mathbf{v},\mathbf{w})}\tag{5.2}$$

and

$$(\mathbf{e}^{\mathbf{v}})^n=e_1^{\epsilon_1(\mathbf{v},n)}\cdots e_m^{\epsilon_m(\mathbf{v},n)}\tag{5.3}$$

for every $\mathbf{v},\mathbf{w}\in\mathbb{Z}^m$ and every $n\in\mathbb{Z}$. Leibman notes this in [15, Corollary 3.13]. It recovers a result of Hall [14, Theorem 6.5].

In light of Proposition 5.6, if $G$ is torsion-free nilpotent then Proposition 5.2 follows from the following result.

**Proposition 5.7.** *Let $G$ be a torsion-free nilpotent group with basis $(e_1,\ldots,e_m)$, and let $\alpha:G\to\mathbb{Z}^m$ be the corresponding coordinate mapping. Let $p:\mathbb{Z}^m\to\mathbb{Z}$ be a non-zero polynomial, and let*

$$\mathcal{N}_p:=\{g\in G:p\circ\alpha(g)=0\}.$$

*Then $\mathcal{N}_p$ is negligible in $G$ by finite quotients.*

The first step in the proof of Proposition 5.7 is to construct the quotients that we will use to show that $\mathcal{N}_p$ is negligible by finite quotients. Given a group $G$ we write $G^{(n)}$ is the subgroup generated by all $n^{\text{th}}$ powers of elements of $G$, and $G(n)$ for the quotient $G/G^{(n)}$. If $G$ is finitely generated and torsion-free nilpotent with basis $(e_1,\ldots,e_m)$ then we write $G_i(n)$ for the image of $G_i$ under the quotient map $G\to G(n)$. The precise statement that we prove in order to deduce Proposition 5.7 is then as follows.

**Proposition 5.8.** *Let $G$ be a torsion-free nilpotent group with basis $(e_1,\ldots,e_m)$, and let $\alpha:G\to\mathbb{Z}^m$ be the corresponding coordinate mapping. Let $p:\mathbb{Z}^m\to\mathbb{Z}$ be a non-zero polynomial, and let*

$$\mathcal{N}_p:=\{g\in G:p\circ\alpha(g)=0\}.$$

*Then*

$$\frac{|\mathcal{N}_p G^{(n)}/G^{(n)}|}{|G/G^{(n)}|} \to 0 \tag{5.4}$$

*as $n \to \infty$ through the primes.*

*Remark* 5.9. An inspection of the argument shows that there exists some integer $n_0 = n_0(G, e_1, \ldots, e_m)$ such that (5.4) holds as $n \to \infty$ through those positive integers coprime to $n_0$.

**Lemma 5.10.** *Let $G$ be a finitely generated torsion-free nilpotent group with basis $(e_1, \ldots, e_m)$. Then there exists an integer $n_0 = n_0(G, e_1, \ldots, e_m)$ such that for every positive integer $n$ coprime to $n_0$ and every $i = 1, \ldots, m$ we have $G_i(n)/G_{i-1}(n) \cong C_n$.*

*Proof.* Pick $n_0$ so that the coefficients of the polynomials $\mu_i, \epsilon_i$ given in (5.2) and (5.3) all lie in $\frac{1}{n_0}\mathbb{Z}$. Fix $n$ coprime to $n_0$, and write $\Phi_n : G \to G(n)$ for the quotient homomorphism.

Note that $\epsilon_i(\mathbf{v}, 0) = 0$ for each $i$ and each $\mathbf{v} \in \mathbb{Z}^m$, so that the polynomials $\epsilon_i(\mathbf{v}, -) : \mathbb{Z} \to \mathbb{Z}$ have no constant term. By the definition of $n_0$, for each $i$ and each $\mathbf{v} \in \mathbb{Z}^m$ we have $\epsilon_i(\mathbf{v}, n) = c_{i,\mathbf{v},n} n/n_0 \in \mathbb{Z}$ for some $c_{i,\mathbf{v},n} \in \mathbb{Z}$. As $n$ is coprime to $n_0$, it follows that $n$ divides $\epsilon_i(\mathbf{v}, n)$, and so

$$G^{(n)} = \langle e_1^n, \ldots, e_m^n \rangle. \tag{5.5}$$

The polynomials $\mu_i : \mathbb{Z}^{2m} \to \mathbb{Z}$ similarly have no constant term, and so there exist polynomials $\bar{\mu}_1, \ldots, \bar{\mu}_m : \mathbb{Z}^{2m} \to \mathbb{Z}$ such that

$$\mathbf{e}^{\mathbf{v}n} \cdot \mathbf{e}^{\mathbf{w}n} = e_1^{\bar{\mu}_1(\mathbf{v},\mathbf{w})n} \cdots e_m^{\bar{\mu}_m(\mathbf{v},\mathbf{w})n}. \tag{5.6}$$

By (5.5) and successive applications of (5.6), it therefore follows that

$$G^{(n)} = \{\mathbf{e}^{\mathbf{v}n} : \mathbf{v} \in \mathbb{Z}^m\}. \tag{5.7}$$

It is clear that $G_i(n)/G_{i-1}(n)$ is generated by $G_{i-1}(n)\Phi_n(e_i)$ and is a quotient of $C_n$, so it is enough to show that $\Phi_n(e_i^r) \notin G_{i-1}(n)$ for $r = 1, \ldots, n-1$. If, on the contrary, $\Phi_n(e_i^r) \in G_{i-1}(n)$ for some such $r$, then if would follow from (5.7) that

$$e_i^r = \mathbf{e}^{\mathbf{v}n} \cdot e_1^{w_1} \cdots e_{i-1}^{w_{i-1}}$$

for some $\mathbf{v} \in \mathbb{Z}^m$ and some $w_1, \ldots, w_{i-1} \in \mathbb{Z}$, which would give

$$e_i^{v_i n - r} e_{i+1}^{v_{i+1}n} \cdots e_m^{v_m n} \in G_{i-1}.$$

Since $v_i n - r \neq 0$ whenever $1 \leq r \leq n-1$, this would contradict the uniqueness of coordinates, and so we indeed have $\Phi_n(e_i^r) \notin G_{i-1}(n)$, as required. □

*Remark* 5.11. Note that the conclusion of Lemma 5.10 does not necessarily hold for an arbitrary $n \in \mathbb{N}$. For instance, if $G = \left(\begin{smallmatrix} 1 & \mathbb{Z} & \mathbb{Z} \\ 0 & 1 & \mathbb{Z} \\ 0 & 0 & 1 \end{smallmatrix}\right)$ is the integral Heisenberg group then $m = 3$ and $G_1(2)/G_0(2)$ is trivial.

*Proof of Proposition 5.8.* Throughout this proof $n$ is a prime. Write $d$ for the degree of $p$, and for each prime $n$ write $\Phi_n : G \to G(n)$ for the quotient homomorphism. By Lemma 5.10 the desired conclusion (5.4) is equivalent to the statement that

$$\frac{|\Phi_n(\mathcal{N}_p)|}{n^m} \to 0 \tag{5.8}$$

as $n \to \infty$ through the primes. If $m = 1$ then $|\mathcal{N}_p| \leq d$, and so for every $n$ we also have $|\Phi_n(\mathcal{N}_p)| \leq d$, which certainly implies (5.8). We may therefore assume that $m \geq 2$ and proceed by induction on $m$.

We can view $p$ as an element of $\mathbb{Q}[X_1, \ldots, X_m]$. If $p \in \mathbb{Q}[X_2, \ldots, X_m]$ then it follows by applying the induction hypothesis to $G/G_1$ that

$$\frac{|\Phi_n(\mathcal{N}_p)G_1(n)/G_1(n)|}{n^{m-1}} \to 0$$

as $n \to \infty$, which implies (5.8) by Lemma 5.10. We may therefore assume that

$$p(X_1, \ldots, X_m) = \sum_{i=0}^{d} X_1^i p_i(X_2, \ldots, X_m)$$

for some $p_0, \ldots, p_d \in \mathbb{Q}[X_2, \ldots, X_m]$ with $p_j \neq 0$ for some $j \geq 1$. Writing

$$\mathcal{P} = \{g \in G : p_j(\alpha_2(g), \ldots, \alpha_m(g)) = 0\},$$

we have

$$\frac{|\Phi_n(\mathcal{P})/G_1(n)|}{n^{m-1}} \to 0$$

as $n \to \infty$ by induction, and hence

$$\frac{|\Phi_n(\mathcal{P})|}{n^m} \to 0. \tag{5.9}$$

For $g \in \mathcal{N}_p \setminus \mathcal{P}$, on the other hand, $\alpha_1(g)$ is a root of the non-zero polynomial

$$p(X, \alpha_2(g), \ldots, \alpha_m(g)) \in \mathbb{Q}[X]$$

of degree at most $d$, and so $|(\mathcal{N}_p \setminus \mathcal{P}) \cap G_1 x| \leq d$ for all $x \in G$. By taking images in $G(n)$ this implies that $|\Phi_n(\mathcal{N}_p \setminus \mathcal{P}) \cap G_1(n)x| \leq d$ for all $x \in G(n)$, and so

$$|\Phi_n(\mathcal{N}_p \setminus \mathcal{P})| \leq d|G(n)/G_1(n)| = dn^{m-1}$$

for every large enough prime $n$. Combined with (5.9), this implies (5.8), as required. □

*Proof of Proposition 5.2.* Let $G'$, $\pi$ and $\hat{\varphi}$ be as given by Lemma 5.5. It follows from Propositions 5.6 and 5.8 that $\hat{\varphi}^{-1}(0)$ is negligible by finite quotients of $G'$, and hence that $\varphi^{-1}(0) = \pi^{-1}(\hat{\varphi}^{-1}(0))$ is negligible by finite quotients of $G$, as required. □

# 6 Finite quotients

In this section we prove Theorem 1.32 and deduce Theorem 1.31 from it. We fix $k \in \mathbb{N}$ throughout. In Sections 6.1 to 6.4 we also fix a finite group $G$ and a normal subgroup $N \lhd G$, and define

$$\mathcal{N}_k(G) := \{(x_1, \ldots, x_{k+1}) \in G^{k+1} \mid [x_1, \ldots, x_{k+1}] = 1\}.$$

Note that Theorem 1.32 is equivalent to the following result.

**Theorem 6.1.** *We have* $|\mathcal{N}_k(G)| \le |\mathcal{N}_k(N)| \times |\mathcal{N}_k(G/N)|$.

We prove Theorem 6.1 in Sections 6.1 to 6.4. In Section 6.5 we prove Theorem 1.31.

## 6.1 Submultiplicativity of degree of nilpotence

Here we sketch the proof of Theorem 6.1, omitting the proof of a technical result—Proposition 6.3—that we give in Sections 6.2 to 6.4.

For subsets $A_1, \ldots, A_{k+1} \subseteq G$, define

$$f_k(A_1, \ldots, A_{k+1}) = |\mathcal{N}_k(G) \cap (A_1 \times \cdots \times A_{k+1})|.$$

If $A_i = \{a_i\}$ is a singleton for some $i$, we will abuse the notation slightly by writing $f_k(\ldots, \{a_i\}, \ldots)$ as $f_k(\ldots, a_i, \ldots)$.

Given cosets $x_1N, \ldots, x_{k+1}N \in G/N$, it is clear that if $f_k(x_1N, \ldots, x_{k+1}N) \neq 0$ then the element $[x_1N, \ldots, x_{k+1}N]$ is trivial in $G/N$. Thus the number of $(x_1N, \ldots, x_{k+1}N) \in (G/N)^{k+1}$ with $f_k(x_1N, \ldots, x_{k+1}N) \neq 0$ is at most $|\mathcal{N}_k(G/N)|$, and we obtain

$$|\mathcal{N}_k(G)| \le |\mathcal{N}_k(G/N)| \times \max\{f_k(x_1N, \ldots, x_{k+1}N) \mid (x_1N, \ldots, x_{k+1}N) \in (G/N)^{k+1}\}.$$

Since $f_k(N, \ldots, N) = |\mathcal{N}_k(N)|$, Theorem 6.1 therefore follows from the following Lemma.

**Lemma 6.2.** *For every* $(x_1N, \ldots, x_{k+1}N) \in (G/N)^{k+1}$ *we have*

$$f_k(x_1N, \ldots, x_{k+1}N) \le f_k(N, \ldots, N).$$

The proof of Lemma 6.2 uses the following proposition, to be proved in Sections 6.2 to 6.4.

**Proposition 6.3.** *For any $g \in G$ and $xN \in G/N$, we have*

$$f_k(xN, g, N, N, \dots, N) \leq f_k(N, g, N, N, \dots, N).$$

*Proof of Lemma 6.2.* We will show that for each $i \in \{0, \dots, k\}$, we have

$$f_k(x_1N, \dots, x_iN, x_{i+1}N, N, \dots, N) \leq f_k(x_1N, \dots, x_iN, N, N, \dots, N),$$

which will imply the result. Note that for $i = 0$, this follows immediately from Proposition 6.3 (by summing over $g \in N$), hence we can without loss of generality assume that $i \geq 1$.

Let $s \in \mathbb{N}$. Since $[y^{-1}, g] = [g, y]^{y^{-1}}$ for all $y, g \in G$, we have an identity

$$[(xn)^{-1}, g, n_3, \dots, n_{s+1}] = [g, xn, n_3^{xn}, \dots, n_{s+1}^{xn}]^{(xn)^{-1}}.$$

As $N$ is normal in $G$, this induces a bijection

$$\mathcal{N}_s(G) \cap \left(x^{-1}N \times \{g\} \times N^{s-1}\right) \leftrightarrow \mathcal{N}_s(G) \cap \left(\{g\} \times xN \times N^{s-1}\right),$$

and so we have

$$f_s(x^{-1}N, g, N, \dots, N) = f_s(g, xN, N, \dots, N)$$

for all $g \in G$ and $xN \in G/N$. Thus Proposition 6.3 implies that

$$\begin{aligned} f_s(g, xN, N, \dots, N) &= f_s(x^{-1}N, g, N, \dots, N) \\ &\leq f_s(N, g, N, \dots, N) = f_s(g, N, N, \dots, N). \end{aligned}$$

Now fix $i \in \{1, \dots, k\}$. Then this last inequality implies

$$\begin{aligned} f_k(x_1N, \dots, x_iN, x_{i+1}N, N, \dots, N) &= \sum_{n_1, \dots, n_i \in N} f_{k-i+1}([x_1n_1, \dots, x_in_i], x_{i+1}N, N, \dots, N) \\ &\leq \sum_{n_1, \dots, n_i \in N} f_{k-i+1}([x_1n_1, \dots, x_in_i], N, N, \dots, N) \\ &= f_k(x_1N, \dots, x_iN, N, N, \dots, N), \end{aligned}$$

as required. □

## 6.2 Sketch of the proof of Proposition 6.3

Here we give a proof of Proposition 6.3, omitting proofs of two equalities to be proved in Section 6.4. Throughout this section, fix $g \in G$ and $xN \in G/N$.

We aim to show that

$$f_k(xN, g, N, \dots, N) \leq f_k(N, g, N, \dots, N),$$

or in other words,

$$\sum_{(n_3,\dots,n_{k+1}) \in N^{k-1}} f_k(xN, g, n_3, \dots, n_{k+1}) \leq \sum_{(n_3,\dots,n_{k+1}) \in N^{k-1}} f_k(N, g, n_3, \dots, n_{k+1}).$$

The idea of the proof is to split this up into smaller parts: that is, to find a partition $N_1 \sqcup \cdots \sqcup N_p$ of $N^{k-1}$ such that

$$\sum_{(n_3,\dots,n_{k+1}) \in N_q} f_k(xN, g, n_3, \dots, n_{k+1}) \leq \sum_{(n_3,\dots,n_{k+1}) \in N_q} f_k(N, g, n_3, \dots, n_{k+1}) \tag{6.1}$$

for each $q \in \{1, \dots, p\}$. The proof relies on periodic behaviour (in a certain sense) of the numbers $f_k(x^i N, g, n_3, \dots, n_{k+1})$ where $i \in \mathbb{Z}$ and $(n_3, \dots, n_{k+1}) \in N_q$. In particular, each part $N_q$ will be subdivided further: in Section 6.3 we will define a function

$$L : N_q \to \mathbb{Z}/d\mathbb{Z}$$

for some $d = d(q) \in \mathbb{N}$, with the property that, for any $i \in \mathbb{Z}$ and $(n_3, \dots, n_{k+1}) \in N_q$, the number $f_k(x^i N, g, n_3, \dots, n_{k+1})$ depends only on the value of $L(n_3, \dots, n_{k+1}) + i$ in $\mathbb{Z}/d\mathbb{Z}$. That is, given any $(n_3, \dots, n_{k+1}), (\tilde{n}_3, \dots, \tilde{n}_{k+1}) \in N_q$ and $i, \tilde{i} \in \mathbb{Z}$, we have

$$\begin{aligned} \text{if} \quad & L(n_3, \dots, n_{k+1}) + i = L(\tilde{n}_3, \dots, \tilde{n}_{k+1}) + \tilde{i} \quad (\text{in } \mathbb{Z}/d\mathbb{Z}), \\ \text{then} \quad & f_k(x^i N, g, n_3, \dots, n_{k+1}) = f_k(x^{\tilde{i}} N, g, \tilde{n}_3, \dots, \tilde{n}_{k+1}); \end{aligned} \tag{6.2}$$

we will prove (6.2) in Section 6.4.

This implies that there exist some integers $h_j$ (where $j \in \mathbb{Z}/d\mathbb{Z}$) such that

$$f_k(x^i N, g, n_3, \dots, n_{k+1}) = h_{L(n_3,\dots,n_{k+1})+i}$$

for all $i \in \mathbb{Z}$ and $(n_3, \dots, n_{k+1}) \in N_q$. Thus

$$\sum_{(n_3,\dots,n_{k+1}) \in N_q} f_k(N, g, n_3, \dots, n_{k+1}) = \sum_{j \in \mathbb{Z}/d\mathbb{Z}} |L^{-1}(j)| h_j$$

and

$$\sum_{(n_3,\dots,n_{k+1}) \in N_q} f_k(xN, g, n_3, \dots, n_{k+1}) = \sum_{j \in \mathbb{Z}/d\mathbb{Z}} |L^{-1}(j)| h_{j+1},$$

and so (6.1) becomes

$$\sum_{j \in \mathbb{Z}/d\mathbb{Z}} |L^{-1}(j)| h_{j+1} \leq \sum_{j \in \mathbb{Z}/d\mathbb{Z}} |L^{-1}(j)| h_j.$$

Moreover, in Section 6.4 we will show that

$$|L^{-1}(j)|h_{j+1} = |L^{-1}(j+1)|h_j \tag{6.3}$$

for each $j \in \mathbb{Z}/d\mathbb{Z}$. Proposition 6.3 then follows from the following Lemma:

**Lemma 6.4.** *Let $d \in \mathbb{N}$ and for each $j \in \mathbb{Z}/d\mathbb{Z}$, let $r_j$ and $h_j$ be non-negative integers such that $r_j h_{j+1} = r_{j+1} h_j$ for each $j$. Then*

$$\sum_{j\in\mathbb{Z}/d\mathbb{Z}} r_j h_{j+1} \leq \sum_{j\in\mathbb{Z}/d\mathbb{Z}} r_j h_j.$$

*Proof.* For a fixed $j \in \mathbb{Z}/d\mathbb{Z}$, since $r_j h_{j+1} = r_{j+1} h_j$, we have either $r_j \leq r_{j+1}$ and $h_j \leq h_{j+1}$, or $r_j \geq r_{j+1}$ and $h_j \geq h_{j+1}$. This implies that

$$\begin{aligned} 0 &\leq \sum_{j\in\mathbb{Z}/d\mathbb{Z}} (r_j - r_{j+1})(h_j - h_{j+1}) \\ &= \sum_{j\in\mathbb{Z}/d\mathbb{Z}} r_j h_j - \sum_{j\in\mathbb{Z}/d\mathbb{Z}} r_j h_{j+1} - \sum_{j\in\mathbb{Z}/d\mathbb{Z}} r_{j+1} h_j + \sum_{j\in\mathbb{Z}/d\mathbb{Z}} r_{j+1} h_{j+1} \\ &= 2\left(\sum_{j\in\mathbb{Z}/d\mathbb{Z}} r_j h_j - \sum_{j\in\mathbb{Z}/d\mathbb{Z}} r_j h_{j+1}\right), \end{aligned}$$

as required. □

## 6.3 Combinatorial structure of $\mathcal{N}_k(G)$

Here we clarify the notation used in Section 6.2. In particular, we define the subsets $N_q \subseteq N^{k-1}$, and for a given $q \in \{1, \dots, p\}$, the number $d \in \mathbb{N}$ and the function $L : N_q \to \mathbb{Z}/d\mathbb{Z}$ used in Section 6.2.

A key fact used in the argument is the following commutator identity:

**Lemma 6.5.** *For any $z, y \in G$ and $n_3, \dots, n_{k+1} \in N$,*

$$[zy, g, n_3, \dots, n_{k+1}] = \left[z, g, n_3^{\alpha_3^{-1}}, \dots, n_{k+1}^{\alpha_{k+1}^{-1}}\right]^{\alpha_{k+2}} [y, g, n_3, \dots, n_{k+1}],$$

*where $\alpha_i = y \prod_{j=2}^{i-2} [y, g, n_3, \dots, n_j]$ for $3 \leq i \leq k+2$ (for the avoidance of doubt, $\alpha_3 = y$ and $\alpha_4 = y[y, g]$).*

*Proof.* We proceed by induction on $k$. We make repeated use of the commutator identity

$$[ab, c] = [a, c]^b [b, c]. \tag{6.4}$$

For the base case $k = 1$, we use (6.4) with $a = z$, $b = y$ and $c = g$, by noting that $\alpha_3 = y$.

For $k \geq 2$, the inductive hypothesis gives

$$[zy, g, n_3, \ldots, n_{k+1}] = \left[ \overbrace{\left[z, g, n_3^{\alpha_3^{-1}}, \ldots, n_k^{\alpha_k^{-1}}\right]^{\alpha_{k+1}}}^{a_{k+1}} \overbrace{[y, g, n_3, \ldots, n_k]}^{b_{k+1}}, n_{k+1} \right].$$

Since $\alpha_{k+2} = \alpha_{k+1} b_{k+1}$, the result follows by applying (6.4) with $a = a_{k+1}$, $b = b_{k+1}$ and $c = n_{k+1}$. □

This Lemma motivates the following construction. Let $\Gamma$ be a directed labelled multigraph (that is, a directed labelled graph in which loops and multiple edges are allowed) with vertex set

$$V(\Gamma) = N^{k-1}$$

and edge set

$$E(\Gamma) = \left\{ (n_3, \ldots, n_{k+1}) \xrightarrow{y} \left(n_3^{\alpha_3^{-1}}, \ldots, n_{k+1}^{\alpha_{k+1}^{-1}}\right) \;\middle|\; y \in xN, n_3, \ldots, n_{k+1} \in N, \right.$$
$$\left. [y, g, n_3, \ldots, n_{k+1}] = 1 \right\},$$

where $\alpha_3 = \alpha_3(y), \alpha_4 = \alpha_4(y), \alpha_5 = \alpha_5(y, n_3), \ldots, \alpha_{k+1} = \alpha_{k+1}(y, n_3, \ldots, n_{k-2}) \in G$ are as in Lemma 6.5.

Now write $\Gamma$ as a union of its connected components,

$$\Gamma = \Gamma_1 \sqcup \cdots \sqcup \Gamma_p,$$

and write $N_q$ for $V(\Gamma_q)$, where $1 \leq q \leq p$. This defines a partition

$$N^{k-1} = N_1 \sqcup \cdots \sqcup N_p,$$

as above. Fix $q \in \{1, \ldots, p\}$. In what follows, a walk in $\Gamma_q$ is not required to follow directions of the edges, but does have a choice of orientation associated with it.

**Definition 6.6.** (i) For a walk $\gamma$ of length $s_+ + s_-$ from $v \in N_q$ to $w \in N_q$, define the *directed length* $\ell(\gamma)$ of $\gamma$ to be $s_+ - s_-$, where $s_+$ (respectively $s_-$) is the number of edges in $\gamma$ with directions coincident with (respectively opposite to) the direction of $\gamma$. Note that given any walk $\gamma$ in $\Gamma_q$ we have $\ell(\gamma^{-1}) = -\ell(\gamma)$.

(ii) Define the *period* of $\Gamma_q$ to be

$$d = \gcd(\{o\} \cup \{|\ell(c)| \mid c \text{ is a closed walk in } \Gamma_q\}),$$

where $o$ is the order of $xN$ in $G/N$.

(iii) Choose a base vertex $v_q$ of $\Gamma_q$. For any vertex $v \in \Gamma_q$, define the *level* of $v$ to be

$$L(v) = \ell(\gamma_v) + d\mathbb{Z} \in \mathbb{Z}/d\mathbb{Z}$$

where $\gamma_v$ is a walk in $\Gamma_q$ from $v_q$ to $v$. Note that if $\gamma_v$, $\tilde{\gamma}_v$ are two such walks, then $c = \gamma_v^{-1}\tilde{\gamma}_v$ is a *closed* walk, and so $d$ divides $\ell(c) = -\ell(\gamma_v) + \ell(\tilde{\gamma}_v)$ by the choice of $d$. Thus $L(v)$ does not depend on the choice of $\gamma_v$.

*Remark.* The set $\mathcal{W}(\Gamma_q)$ of walks in $\Gamma_q$ forms a group under concatenation, with inverses given by changing orientation, and in this setting $\ell : \mathcal{W}(\Gamma_q) \to \mathbb{Z}$ is a homomorphism.

## 6.4 Completing the proof of Proposition 6.3

We now prove (6.2) and (6.3) from Section 6.2, which will complete the proof of Proposition 6.3.

The last part of the following Lemma shows (6.2) is true:

**Lemma 6.7.** (i) *For any walk $\gamma$ from $(n_3, \dots, n_{k+1}) \in N_q$ to $(\tilde{n}_3, \dots, \tilde{n}_{k+1}) \in N_q$ and any $i \in \mathbb{Z}$, we have*

$$f_k(x^i N, g, n_3, \dots, n_{k+1}) = f_k(x^{i-\ell(\gamma)} N, g, \tilde{n}_3, \dots, \tilde{n}_{k+1}).$$

(ii) *For any $(n_3, \dots, n_{k+1}) \in N_q$ and $i \in \mathbb{Z}$, we have*

$$f_k(x^i N, g, n_3, \dots, n_{k+1}) = f_k(x^{i-d} N, g, n_3, \dots, n_{k+1}).$$

(iii) *For any $(n_3, \dots, n_{k+1}), (\tilde{n}_3, \dots, \tilde{n}_{k+1}) \in N_q$ and $i, \tilde{i} \in \mathbb{Z}$, if*

$$L(n_3, \dots, n_{k+1}) + i = L(\tilde{n}_3, \dots, \tilde{n}_{k+1}) + \tilde{i} \quad (\textit{in } \mathbb{Z}/d\mathbb{Z}),$$

*then*

$$f_k(x^i N, g, n_3, \dots, n_{k+1}) = f_k(x^{\tilde{i}} N, g, \tilde{n}_3, \dots, \tilde{n}_{k+1}).$$

*Proof.* (i) We proceed by induction on the length of $\gamma$. For the base case (when $\gamma$ is an edge), note that by Lemma 6.5, an edge $(n_3, \dots, n_{k+1}) \xrightarrow{y} (\tilde{n}_3, \dots, \tilde{n}_{k+1})$ in $\Gamma_q$ defines a bijection between elements $zy \in x^i N$ with $[zy, g, n_3, \dots, n_{k+1}] = 1$ and elements $z \in x^{i-1}N$ with $[z, g, \tilde{n}_3, \dots, \tilde{n}_{k+1}] = 1$, and hence

$$f_k((xN)^i, g, n_3, \dots, n_{k+1}) = f_k((xN)^{i-1}, g, \tilde{n}_3, \dots, \tilde{n}_{k+1}). \tag{6.5}$$

For the inductive step (when the length of $\gamma$ is at least 2), note that we can write $\gamma = \tilde{\gamma} e^{\varepsilon}$ for some $e \in E(\Gamma_q)$, $\varepsilon \in \{\pm 1\}$, and a walk $\tilde{\gamma}$ that is strictly shorter than $\gamma$. Thus, applying the inductive hypothesis to $\tilde{\gamma}$ and (6.5) to $e$ yields the result.

(ii) Fix a vertex $v = (n_3, \dots, n_{k+1}) \in N_q$ and $i \in \mathbb{Z}$. By definition of $d$, there exist closed walks $c_1, \dots, c_r$ and integers $m, m_1, \dots, m_r \in \mathbb{Z}$ such that

$$d = mo + m_1\ell(c_1) + \dots + m_r\ell(c_r).$$

Note that we may transform the closed walks $c_j$ to ones that start and end at $v$: indeed, if $\gamma_j$ is a walk from $v$ to the starting (and ending) vertex of $c_j$, then $\tilde{c}_j = \gamma_j c_j \gamma_j^{-1}$ is a closed walk starting and ending at $v$, and $\ell(\tilde{c}_j) = \ell(c_j)$. This allows us to construct a closed walk

$$\tilde{c} = \tilde{c}_1^{m_1} \cdots \tilde{c}_r^{m_r}$$

and we have

$$\ell(\tilde{c}) = m_1\ell(\tilde{c}_1) + \dots + m_r\ell(\tilde{c}_r) = d - mo.$$

Substituting $\gamma = \tilde{c}$ to part ((i)) yields

$$f(x^i N, g, n_3, \dots, n_{k+1}) = f_k(x^{i-d+mo} N, g, n_3, \dots, n_{k+1}).$$

But since $o$ is the order of $xN$ in $G/N$, we get $x^{i-d+mo}N = (x^{i-d}N)((xN)^o)^m = x^{i-d}N$, which gives the result.

(iii) Let $\gamma$ (respectively $\tilde{\gamma}$) be a walk in $\Gamma_q$ from the base vertex $v_q$ to $(n_3, \dots, n_{k+1})$ (respectively $(\tilde{n}_3, \dots, \tilde{n}_{k+1})$). By definition of level, we have

$$\ell(\gamma^{-1}\tilde{\gamma}) + d\mathbb{Z} = -\ell(\gamma) + \ell(\tilde{\gamma}) + d\mathbb{Z} = -L(n_3, \dots, n_{k+1}) + L(\tilde{n}_3, \dots, \tilde{n}_{k+1}) = i - \tilde{i} + d\mathbb{Z},$$

and so $\ell(\gamma^{-1}\tilde{\gamma}) = i - \tilde{i} + md$ for some $m \in \mathbb{Z}$. By part ((i)), we have

$$\begin{aligned} f_k(x^i N, g, n_3, \dots, n_{k+1}) &= f_k(x^{i-(i-\tilde{i}+md)} N, g, \tilde{n}_3, \dots, \tilde{n}_{k+1}) \\ &= f_k(x^{\tilde{i}-md} N, g, \tilde{n}_3, \dots, \tilde{n}_{k+1}), \end{aligned}$$

and so $|m|$ applications of part ((ii)) to the right hand side gives the result. □

Finally, we prove (6.3):

**Lemma 6.8.** *For each $j \in \mathbb{Z}/d\mathbb{Z}$, we have $|L^{-1}(j)|h_{j+1} = |L^{-1}(j+1)|h_j$.*

*Proof.* We will give a bijection between the set

$$\mathcal{A} = \bigsqcup_{\substack{(n_3,\dots,n_{k+1}) \in N_q \\ L(n_3,\dots,n_{k+1}) = j}} \mathcal{N}_k(G) \cap (xN \times \{g\} \times \{n_3\} \times \dots \times \{n_{k+1}\})$$

and the set

$$\mathcal{B} = \bigsqcup_{\substack{(\tilde{n}_3,\ldots,\tilde{n}_{k+1})\in N_q \\ L(\tilde{n}_3,\ldots,\tilde{n}_{k+1})=j+1}} \mathcal{N}_k(G) \cap \left(x^{-1}N \times \{g\} \times \{\tilde{n}_3\} \times \ldots \times \{\tilde{n}_{k+1}\}\right).$$

Since $\mathcal{A}$ is a disjoint union of $|L^{-1}(j)|$ sets, each of cardinality $h_{j+1}$, and $\mathcal{B}$ is a disjoint union of $|L^{-1}(j+1)|$ sets, each of cardinality $h_j$, this will imply the result.

Now consider

$$\begin{aligned} \theta : xN \times \{g\} \times N^{k-1} &\to x^{-1}N \times \{g\} \times N^{k-1}, \\ (y, g, n_3, \ldots, n_{k+1}) &\mapsto \left(y^{-1}, g, n_3^{\alpha_3^{-1}}, \ldots, n_{k+1}^{\alpha_{k+1}^{-1}}\right), \end{aligned}$$

where $\alpha_3 = \alpha_3(y)$, $\alpha_4 = \alpha_4(y)$, $\alpha_5 = \alpha_5(y, n_3)$, $\ldots$, $\alpha_{k+1} = \alpha_{k+1}(y, n_3, \ldots, n_{k-1})$ are as in Lemma 6.5. First, we claim that $\theta$ is a bijection. Indeed, for each $i$, the element $\alpha_i$ does not depend on $n_i, \ldots, n_{k+1}$, and so it follows (by induction on $k - i$) that the restriction of $\theta$ given by

$$\theta_i : \{y\}\times\{g\}\times\{n_3\}\times\cdots\times\{n_{i+1}\}\times N^{k-i} \to \left\{y^{-1}\right\}\times\{g\}\times\left\{n_3^{\alpha_3^{-1}}\right\}\times\cdots\times\left\{n_{i+1}^{\alpha_{i+1}^{-1}}\right\}\times N^{k-i}$$

is a bijection for each $y \in xN$ and $(n_3, \ldots, n_{i+1}) \in N^{i-1}$. In particular,

$$\theta_1 : \{y\} \times \{g\} \times N^{k-1} \to \left\{y^{-1}\right\} \times \{g\} \times N^{k-1}$$

is a bijection for each $y \in xN$, and hence $\theta$ is a bijection as well.

It is now enough to show that $\theta(\mathcal{A}) = \mathcal{B}$. By substituting $z = y^{-1}$ in Lemma 6.5, it follows that $[y, g, n_3, \ldots, n_{k+1}] = 1$ if and only if $\left[y^{-1}, g, n_3^{\alpha_3^{-1}}, \ldots, n_{k+1}^{\alpha_{k+1}^{-1}}\right] = 1$, and hence that

$$\theta\left(\mathcal{N}_k(G) \cap (xN \times \{g\} \times N^{k-1})\right) = \mathcal{N}_k(G) \cap (x^{-1}N \times \{g\} \times N^{k-1}).$$

Furthermore, for an arbitrary edge $(n_3, \ldots, n_{k+1}) \xrightarrow{y} (\tilde{n}_3, \ldots, \tilde{n}_{k+1})$ in $\Gamma$ (note that we have $(\tilde{n}_3, \ldots, \tilde{n}_{k+1}) = \left(n_3^{\alpha_3^{-1}}, \ldots, n_{k+1}^{\alpha_{k+1}^{-1}}\right)$ in this case), its endpoints are in the same connected component of $\Gamma$, that is, $(n_3, \ldots, n_{k+1}) \in N_q$ if and only if $(\tilde{n}_3, \ldots, \tilde{n}_{k+1}) \in N_q$. Moreover, if it is the case that $(n_3, \ldots, n_{k+1}), (\tilde{n}_3, \ldots, \tilde{n}_{k+1}) \in N_q$ then by definition of level we have

$$L(\tilde{n}_3, \ldots, \tilde{n}_{k+1}) = L(n_3, \ldots, n_{k+1}) + 1.$$

Hence $\theta(\mathcal{A}) = \mathcal{B}$, as required. $\square$

### 6.5 Residually finite groups

Finally, we prove Theorem 1.31. We follow the argument of Antolín and the first and fourth authors in [1, Theorem 1.3]. The proof uses the following result.

**Theorem 6.9** (Erfanian, Rezaei, Lescot [10, Theorem 5.1])**.** *Let $G$ be a finite group that is not $k$-step nilpotent. Then*

$$\mathrm{dc}^k(G) \leq \frac{2^{k+2}-3}{2^{k+2}}.$$

*Proof of Theorem 1.31.* We describe a recursive process outputting a (possibly finite) sequence $G_0 > G_1 > G_2 > \ldots$ of members of $\mathcal{N}$ as follows. We may assume without loss of generality that $G \in \mathcal{N}$ and set $G_0 = G$. Once $G_{i-1}$ is defined, if it is $k$-step nilpotent we terminate the process. If not, there exist $x_1, \ldots, x_{k+1} \in G_{i-1}$ such that $[x_1, \ldots, x_{k+1}] \neq 1$. Since $\bigcap_{N \in \mathcal{N}} N = \{1\}$, there therefore exists $N_i \in \mathcal{N}$ such that $[x_1, \ldots, x_{k+1}] \notin N_i$. Set $G_i = G_{i-1} \cap N_i$, noting that $G_i \in \mathcal{N}$ by the finite-intersection property, and that $G_{i-1}/G_i$ is not $k$-step nilpotent.

Writing $\gamma_k = (2^{k+2}-3)/2^{k+2}$, it follows from Theorem 6.9 that $\mathrm{dc}^k(G_{i-1}/G_i) \leq \gamma_k$ for every $i$, and hence from Theorem 1.32 that for every $n$ with $G_n$ defined we have

$$\mathrm{dc}^k(G/G_n) \leq \prod_{i=1}^{n} \mathrm{dc}^k(G_{i-1}/G_i) \leq \gamma_k^n.$$

The process must therefore terminate for some $n \leq \log \alpha / \log \gamma_k$, meaning that $G_n$ is a $k$-step nilpotent subgroup of finite index in $G$. □

## 7 Dependence on rank

Here we give an example, for any odd prime $p$ and any $k \in \mathbb{N}$, of a family $\left(G^{(n)}\right)_{n=1}^{\infty}$ of finite $p$-groups that are $(k+1)$-step nilpotent but not $k$-step nilpotent, and such that the centre $Z\left(G^{(n)}\right)$ of $G^{(n)}$ has order $p$. Moreover, we will show that any $k$-step nilpotent subgroup $K^{(n)}$ of $G^{(n)}$ has index at least $p^n$. As $G^{(n)}/Z\left(G^{(n)}\right)$ is $k$-step nilpotent, this will show that the bound on the index of a $k$-step nilpotent subgroup of $G$ in Corollary 1.13 has to depend on the rank of $G$. By Proposition 1.12, the same can be said about the bound in Theorem 1.8.

Furthermore, note that this example will show that the index of a $k$-step nilpotent subgroup in Theorem 1.31 cannot be bounded in terms of $k$ and $\alpha$. To see this, it is enough to apply Proposition 1.12 and to note that if $\mathrm{dc}^k\left(G^{(n)}\right) \geq \alpha$ then also $\mathrm{dc}^k\left(G^{(n)}/N\right) \geq \alpha$ for any normal subgroup $N \lhd G^{(n)}$.

Finally, we may show that the direct limit of the groups $G^{(n)}$, $G = \varinjlim G^{(n)}$, is not $k$-step nilpotent, but has a finite normal subgroup $H$ (whose cardinality can be chosen independently of $k$) such that $G/H$ is $k$-step nilpotent; this is described in Section 7.3, where we prove Proposition 1.14. Such a group cannot be finitely generated (by Corollary 1.13) or residually finite (see Remark 7.5). Thus, this demonstrates the necessity of the assumptions on finite generation in Theorem 1.8 and Corollary 1.13 and on residual finiteness in Theorem 1.31.

Throughout this section, we fix an odd prime $p$, and denote the finite field of cardinality $p$ by $\mathbb{F}_p$. For $r, s \in \mathbb{N}$, we denote by $\mathrm{Mat}_{r\times s}(\mathbb{F}_p)$ the $\mathbb{F}_p$-vector space of $r \times s$ matrices with entries in $\mathbb{F}_p$.

## 7.1 The group $G_k(n, r, s)$

Let $k \in \mathbb{Z}_{\geq 0}$ and let $n, r, s \in \mathbb{N}$. We consider the following subgroup of $GL_{r+kn+s}(\mathbb{F}_p)$ consisting of block upper unitriangular matrices:

$$G_k(n,r,s) = \left\{ \begin{pmatrix} I_r & A_0 & A_1 & \cdots & A_{k-1} & C \\ & I_n & D_{1,1} & \cdots & D_{1,k-1} & B_1 \\ & & I_n & \ddots & \vdots & \vdots \\ & & & \ddots & D_{k-1,k-1} & B_{k-1} \\ & & & & I_n & B_k \\ & & & & & I_s \end{pmatrix} \left| \begin{array}{l} A_i \in \mathrm{Mat}_{r\times n}(\mathbb{F}_p) \\ \qquad \text{for } 0 \leq i \leq k-1, \\ B_i \in \mathrm{Mat}_{n\times s}(\mathbb{F}_p) \\ \qquad \text{for } 1 \leq i \leq k, \\ C \in \mathrm{Mat}_{r\times s}(\mathbb{F}_p), \\ D_{i,j} \in \mathrm{Mat}_{n\times n}(\mathbb{F}_p) \\ \qquad \text{for } 1 \leq i \leq j \leq k-1 \end{array} \right. \right\}.$$

For a matrix $X \in G_k(n, r, s)$, we will write $A_j(X)$, $B_i(X)$, $C(X)$ and $D_{i,j}(X)$ for the corresponding blocks of $X$. For a subset $U \subseteq G_k(n, r, s)$ we will similarly write $A_j(U) = \{A_j(X) \mid X \in U\}$, etc.

Note that for $k = 0$, the group $G_0(n, r, s) = \left\{ \begin{pmatrix} I_r & C \\ 0 & I_s \end{pmatrix} \middle| C \in \mathrm{Mat}_{r\times s}(\mathbb{F}_p) \right\}$ is just the elementary abelian group of order $p^{rs}$. For $k = r = s = 1$, the group $G_1(n, 1, 1)$ is the extraspecial group of exponent $p$. It is well-known that such a group is 2-step nilpotent, has centre of order $p$, but no abelian subgroups of index $< p^n$ (see, for instance, Lemma 7.1 and [22, Theorem 1.8]). We aim to generalise this example; in particular, for the sequence $\left(G^{(n)}\right)$ of groups described above we will take $G^{(n)} = G_k(n, 1, 1)$. We thus need to show that $G_k(n, 1, 1)$ is $(k + 1)$-step nilpotent, has centre of order $p$ and has no $k$-step nilpotent subgroups of index $< p^n$.

The first two of these statements follow from the following Lemma, whose proof is easy and left as an exercise for the reader.

**Lemma 7.1.** *Let $k \in \mathbb{Z}_{\geq 0}$ and $n, r, s \in \mathbb{N}$. Let $G = G_k(n, r, s)$, and let $G = \gamma_1(G) \geq \gamma_2(G) \geq \cdots$ and $\{1\} = Z_0(G) \leq Z_1(G) \leq \cdots$ be the lower and upper central series of*

*$G$, respectively. Then*

$$\gamma_{\ell+1}(G) = Z_{k+1-\ell}(G) = \{X \in G \mid A_j(X) = 0 \text{ for } j < \ell,\ B_i(X) = 0 \text{ for } k - i < \ell, \\ D_{i,j}(X) = 0 \text{ for } j - i < \ell\}$$

*for all $\ell \in \{0, \dots, k\}$.* □

We are therefore left to show that $G_k(n, 1, 1)$ has no $k$-step nilpotent subgroups of index $< p^n$. In Section 7.2 we will prove the following proposition, which is slightly more general.

**Proposition 7.2.** *Let $k, n, r, s \in \mathbb{N}$. If a subgroup $K \leq G_k(n, r, s)$ has index $< p^n$, then $K$ is not $k$-step nilpotent.*

*Remark* 7.3. Note that in the case $r = s = 1$, the bound in Proposition 7.2 is sharp: indeed, $\{X \in G_k(n, 1, 1) \mid A_0(X) = 0\}$ is a subgroup of $G_k(n, 1, 1)$ of index $p^n$, and it is not hard to verify that it is $k$-step nilpotent.

### 7.2 Non-existence of large $k$-step nilpotent subgroups

Let $G = G_k(n, r, s)$. By Lemma 7.1, the abelianisation map $\rho : G \to G^{ab}$ is given by mapping a matrix in $G$ to the set of its superdiagonal blocks:

$$\rho : G \to \mathrm{Mat}_{r\times n}(\mathbb{F}_p) \oplus \left( \bigoplus_{i=1}^{k-1} \mathrm{Mat}_{n\times n}(\mathbb{F}_p) \right) \oplus \mathrm{Mat}_{n\times s}(\mathbb{F}_p) \cong \mathbb{F}_p^{n(r+(k-1)n+s)},$$
$$X \mapsto (A_0(X), D_{1,1}(X), \dots, D_{k-1,k-1}(X), B_k(X))$$

for $k \geq 1$, and $\rho : G \to \mathrm{Mat}_{r\times s}(\mathbb{F}_p), X \mapsto C(X)$ for $k = 0$. For a subgroup $K \leq G$, we define the *quasi-rank* (respectively *quasi-corank*) of $K$ in $G$ to be the dimension (respectively codimension) of $\rho(K)$ in the $\mathbb{F}_p$-vector space $G^{ab}$. Note that if $K$ has quasi-corank $q$ then we have $[G : K\gamma_2(G)] = p^q$. We thus aim to show that the quasi-corank of a $k$-step nilpotent subgroup of $G_k(n, r, s)$ will be at least $n$.

The inductive proof of Proposition 7.2 is based on the surjective homomorphism $\pi = \pi_{k,n,r,s}$, obtained by taking the bottom-right $(kn + s) \times (kn + s)$ submatrix:

$$\pi : G_k(n, r, s) \to G_{k-1}(n, n, s),$$

$$\begin{pmatrix} I_r & A_0 & A_1 & \cdots & A_{k-1} & C \\ & I_n & D_{1,1} & \cdots & D_{1,k-1} & B_1 \\ & & I_n & \ddots & \vdots & \vdots \\ & & & \ddots & D_{k-1,k-1} & B_{k-1} \\ & & & & I_n & B_k \\ & & & & & I_s \end{pmatrix} \mapsto \begin{pmatrix} I_n & D_{1,1} & \cdots & D_{1,k-1} & B_1 \\ & I_n & \ddots & \vdots & \vdots \\ & & \ddots & D_{k-1,k-1} & B_{k-1} \\ & & & I_n & B_k \\ & & & & I_s \end{pmatrix}.$$

Note that if $K \leq G_k(n,r,s)$ has quasi-corank $q$, then $\pi(K) \leq G_{k-1}(n,n,s)$ will have quasi-corank at most $q$.

For any $X \in \gamma_k(G_k(n,r,s))$ we have $B_2(X) = \cdots = B_k(X) = 0$ by Lemma 7.1, and for any $Y \in \ker \pi_{k,n,r,s}$ we have $B_1(Y) = \cdots = B_k(Y) = 0$ by the definition of $\pi_{k,n,r,s}$. Therefore,

$$\begin{aligned} C([X,Y]) &= (C(Y) + A_0(X)B_1(Y) + \cdots + A_{k-1}(X)B_k(Y) + C(X)) \\ &\quad - (C(X) + A_0(Y)B_1(X) + \cdots + A_{k-1}(Y)B_k(X) + C(Y)) \\ &= -A_0(Y)B_1(X) \qquad \text{for all } X \in \gamma_k(G_k(n,r,s)) \text{ and } Y \in \ker \pi_{k,n,r,s}. \end{aligned} \tag{7.1}$$

Thus, in order to prove Proposition 7.2, given a subgroup $K \leq G_k(n,r,s)$ of quasi-corank $< n$ we need to find matrices $X \in \gamma_k(K)$ and $Y \in K \cap \ker \pi$ such that $A_0(X)B_1(Y) \neq 0$.

We first prove a slightly stronger version of Proposition 7.2 under the additional assumption that $r = n$.

**Lemma 7.4.** *Let $k \in \mathbb{Z}_{\geq 0}$ and $n, s \in \mathbb{N}$. Let $K$ be a subgroup of $G_k(n,n,s)$ of quasi-corank $q < n$. Then the subspace*

$$C(\gamma_{k+1}(K)) = \{C(X) \mid X \in \gamma_{k+1}(K)\} \leq \mathrm{Mat}_{n\times s}(\mathbb{F}_p)$$

*has codimension at most $q$.*

*Proof.* By induction on $k$. For $k = 0$, we have $G_0(n,n,s) \cong \mathrm{Mat}_{n\times s}(\mathbb{F}_p)$ and $C(\gamma_1(K)) = C(K) \cong K$, hence the result is clear.

Now suppose $k \geq 1$, and let $\pi = \pi_{k,n,n,s}$. As $K$ has quasi-corank $q$ in $G_k(n,n,s)$, the subgroup $\pi(K) \leq G_{k-1}(n,n,s)$ will have quasi-corank at most $q$. Therefore, by induction hypothesis, the subspace

$$C(\gamma_k(\pi(K))) = B_1(\gamma_k(K)) = \{B_1(X) \mid X \in \gamma_k(K)\} \leq \mathrm{Mat}_{n\times s}(\mathbb{F}_p)$$

will have codimension at most $q$.

Moreover, it is clear by the definition of the quasi-corank that the subspace

$$A_0(K \cap \ker \pi) := \{A_0(X) \mid X \in K \cap \ker \pi\} \leq \mathrm{Mat}_{n\times n}(\mathbb{F}_p)$$

will have codimension at most $q$, so in particular

$$\dim A_0(K \cap \ker \pi) \geq n^2 - q > n^2 - n.$$

It follows by [8, Corollary 13] that $A_0(K \cap \ker \pi)$ is generated by matrices of rank $n$, so in particular there exists a matrix $Y \in K \cap \ker \pi$ such that $A_0(Y)$ is invertible. But now, as $C(\gamma_{k+1}(K))$ contains $C([X,Y]) = -A_0(Y)B_1(X)$ for any $X \in \gamma_k(K)$ (see (7.1)), it follows that

$$\operatorname{codim} C(\gamma_{k+1}(K)) \leq \operatorname{codim} B_1(\gamma_k(K)) \leq q,$$

as required. □

*Proof of Proposition 7.2.* Let $q$ be the quasi-corank of $K$ in $G = G_k(n,r,s)$. Then we have

$$p^q = [G : K\gamma_2(G)] \leq [G : K] < p^n$$

and so $q < n$. Consider again the map $\pi = \pi_{k,n,r,s}$, and let $q_1$ be the quasi-corank of $\pi(K)$ in $G_{k-1}(n,n,s)$. By Lemma 7.4, the subspace $B_1(\gamma_k(K)) = C(\gamma_k(\pi(K)))$ will have codimension at most $q_1$ in $\mathrm{Mat}_{n\times s}(\mathbb{F}_p)$. By the rank-nullity theorem, the subspace $A_0(K \cap \ker \pi) \leq \mathrm{Mat}_{r\times n}(\mathbb{F}_p)$ will have codimension $q - q_1 =: q_2$.

Now consider the projections $\tau_1 : \mathrm{Mat}_{r\times n}(\mathbb{F}_p) \to \mathbb{F}_p^n$ and $\tau_2 : \mathrm{Mat}_{n\times s}(\mathbb{F}_p) \to \mathbb{F}_p^n$ of matrices to the top row and to the right column, respectively. By (7.1), for any $X \in \gamma_k(K)$ and $Y \in K \cap \ker \pi$, the top right entry of $[X,Y]$ will be $-\langle \tau_1(A_0(Y)), \tau_2(B_1(X))\rangle$, where $\langle -,- \rangle$ is the standard bilinear form on $\mathbb{F}_p^n$. Furthermore, it is clear that $T_1 := \tau_1(A_0(K \cap \ker \pi))$ and $T_2 := \tau_2(B_1(\gamma_k(K)))$ will have codimensions (in $\mathbb{F}_p^n$) at most $q_1$ and at most $q_2$, respectively. Thus, as $q < n$, we have

$$\dim T_1 + \dim T_2 \geq (n - q_1) + (n - q_2) = 2n - q > n,$$

and so, as $\langle -,- \rangle$ is non-degenerate,

$$\dim T_1 > n - \dim T_2 = \dim T_2^{\perp}.$$

This implies that $T_1 \nleq T_2^{\perp}$, that is, $\langle T_1, T_2 \rangle \neq 0$. Therefore, there exist matrices $X \in \gamma_k(K)$ and $Y \in K \cap \ker \pi$ such that the top right entry of $[X,Y]$ is non-zero, so $K$ is not $k$-step nilpotent. □

### 7.3 Qualitative conclusions

Apart from the rank-dependence of quantitative conclusions of Theorem 1.8 and Corollary 1.13, we may use the groups $G_k(n,1,1)$ to give counterexamples to qualitative conclusions as well for groups that are not finitely generated. In particular, we will prove Proposition 1.14, which shows that the assumption for $G$ to be finitely generated in Corollary 1.13 is necessary. Throughout the rest of this section, fix a prime $p$ and, for each $n \geq 1$, let $G_k(n) := G_k(n,1,1)$ be the finite groups defined in Section 7.1.

*Proof of Proposition 1.14.* Our proof relies on the observation that $G_k(n)$ can be seen as a subgroup of $G_k(n+1)$. In particular, it is easy to see that

$$\overline{G_k(n)} = \left\{ \left. \begin{pmatrix} 1 & \overline{\mathbf{a}_0}^T & \overline{\mathbf{a}_1}^T & \cdots & \overline{\mathbf{a}_{k-1}}^T & c \\ & I_{n+1} & \overline{D_{1,1}} & \cdots & \overline{D_{1,k-1}} & \overline{\mathbf{b}_1} \\ & & I_{n+1} & \ddots & \vdots & \vdots \\ & & & \ddots & \overline{D_{k-1,k-1}} & \overline{\mathbf{b}_{k-1}} \\ & & & & I_{n+1} & \overline{\mathbf{b}_k} \\ & & & & & 1 \end{pmatrix} \right| \begin{array}{l} \mathbf{a}_i, \mathbf{b}_i \in \mathbb{F}_p^n \\ \qquad \text{for } 0 \leq i \leq k-1, \\ c \in \mathbb{F}_p, \\ D_{i,j} \in \mathrm{Mat}_{n\times n}(\mathbb{F}_p) \\ \qquad \text{for } 1 \leq i \leq j \leq k-1 \end{array} \right\}$$

is a subgroup of $G_k(n+1)$ isomorphic to $G_k(n)$, where given any $A \in \mathrm{Mat}_{n\times n}(\mathbb{F}_p)$ and $\mathbf{a} \in \mathbb{F}_p^n$ we define $\overline{A} = \begin{pmatrix} A & \mathbf{0} \\ \mathbf{0}^T & 0 \end{pmatrix} \in \mathrm{Mat}_{(n+1)\times(n+1)}(\mathbb{F}_p)$ and $\overline{\mathbf{a}} = \begin{pmatrix} \mathbf{a} \\ 0 \end{pmatrix} \in \mathbb{F}_p^{n+1}$. This allows us to define the direct limit

$$G_k := \varinjlim G_k(n).$$

Given $n \geq 1$, let $f_n : G_k(n) \to G_k$ be the canonical inclusion. It follows from Lemma 7.1 that for each $n$, the image $f_n(Z(G_k(n)))$ of the centre of $G_k(n)$ in $G_k$ is the same subgroup ($H_k$, say) of $G_k$ of order $p$. Hence we have $Z(G_k) = \bigcup_{n\geq 1} f_n(Z(G_k(n))) = H_k$, and in particular, $H_k$ is normal in $G_k$. We will show that $G = G_k$ and $H = H_k$ satisfy the conclusion of the Proposition.

To show that $G_k/H_k$ is $k$-step nilpotent, let $g_0, \ldots, g_k \in G_k$ be arbitrary elements. Then, for any sufficiently large $n$ and all $i \in \{0, \ldots, k\}$ we have $g_i = f_n(h_i)$ for some $h_i \in G_k(n)$, and so

$$[h_0, \ldots, h_k] \in \gamma_{k+1}(G_k(n)) \leq Z(G_k(n))$$

as $G_k(n)$ is $(k+1)$-step nilpotent. In particular,

$$[g_0, \ldots, g_k] = f_n([h_0, \ldots, h_k]) \in f_n(Z(G_k(n))) = H_k,$$

and so $G_k/H_k$ is $k$-step nilpotent, as required.

Finally, to show that $G_k$ is not virtually $k$-step nilpotent, let $N \leq G_k$ be a subgroup of index $m < \infty$. Let $n \in \mathbb{Z}_{\geq 1}$ be such that $p^n > m$, and note that $[G_k(n) : f_n^{-1}(N)] \leq [G_k : N] = m < p^n$. Thus, by Proposition 7.2, $f_n^{-1}(N)$ cannot be $k$-step nilpotent. But as $f_n$ is injective, $f_n^{-1}(N)$ is isomorphic to a subgroup of $N$, and so $N$ cannot be $k$-step nilpotent either. □

As the group $G_k$ constructed in Proposition 1.14 is a direct limit of finite groups, $\varinjlim G_k(n)$, it is amenable, and in particular the finite subgroups $G_k(n)$ form a Følner sequence for $G_k$. We may thus define measures $\mu_n$ on $G_k$ by setting

$$\mu_n(A) = \frac{|A \cap f_n(G_k(n))|}{|G_k(n)|}$$

for any $A \subseteq G_k$, where $f_n : G_k(n) \to G_k$ is the canonical inclusion. It follows from a result of the second author [21, Theorem 1.13] that the sequence $M = (\mu_n)_{n=1}^{\infty}$ measures index uniformly on $G$. Moreover, we know that $|\gamma_{k+1}(G_k(n))| = p$ which, when combined with Proposition 1.10, implies that $\mathrm{dc}^k(G_k(n)) \geq 1/p$ and therefore $\mathrm{dc}^k_M(G_k) = \limsup_{n\to\infty} \mathrm{dc}^k(G_k(n)) \geq 1/p$. This shows that the assumption for $G$ to be finitely generated is necessary in Theorem 1.8 as well.

*Remark* 7.5. Note that the group $G$ in Proposition 1.14 cannot be residually finite. Indeed, any finite-index subgroup $N \leq G$ cannot be $k$-step nilpotent, and therefore

$$\{1\} \neq \gamma_{k+1}(N) \leq \gamma_{k+1}(G) \cap N \leq H \cap N, \tag{7.2}$$

where the last inclusion comes from the fact that $G/H$ is $k$-step nilpotent. Hence, $H \cap N \neq \{1\}$. But now if $G$ was residually finite, then for each non-trivial $g \in H$ we could pick a finite-index subgroup $N_g \leq G$ such that $g \notin N_g$. Then $N := \bigcap_{g\in H} N_g$ is a finite-index subgroup of $G$ (as $H$ is finite) and $N \cap H = \{1\}$, contradicting (7.2). Thus $G$ cannot be residually finite, as claimed.

# Appendix

## A Polynomial mappings into torsion-free nilpotent groups

In this appendix we prove the following extension of Lemma 5.5, using a similar argument to the one that Leibman uses to reduce [15, Proposition 3.21] to [15, Proposition 2.15].

**Proposition A.1.** *Let $G$ be a group, let $N$ be a finitely generated torsion-free $s$-step nilpotent group, and let $\varphi : G \to N$ be a polynomial mapping of degree $d$. Then there is a torsion-free $ds$-step nilpotent quotient $G'$ of $G$ and a polynomial mapping $\hat{\varphi} : G' \to N$ of degree $d$ such that, writing $\pi : G \to G'$ for the quotient homomorphism, we have $\varphi = \hat{\varphi} \circ \pi$.*

Given a group $G$ we write

$$G = \gamma_1(G) \rhd \gamma_2(G) \rhd \dots$$

for the lower central series of $G$. Following [17], we define the *generalised commutator subgroups* $\overline{\gamma_i(G)}$ of $G$ via

$$\overline{\gamma_i(G)} = \{x \in G : \exists n \in \mathbb{N} \text{ such that } x^n \in \gamma_i(G)\},$$

noting that $G/\overline{\gamma_i(G)}$ is torsion-free $(i-1)$-step nilpotent.

**Lemma A.2.** *Let $G$ be a group and let $x \in \gamma_i(G)$. Then there exists a finitely generated subgroup $\Gamma = \Gamma(x,i) < G$ such that $x \in \gamma_i(\Gamma)$. If instead $x \in \overline{\gamma_i(G)}$ then there exists a finitely generated subgroup $\Lambda = \Lambda(x,i) < G$ such that $x \in \overline{\gamma_i(\Lambda)}$*

*Proof.* To start with we assume that $x \in \gamma_i(G)$. In the case $i = 1$ the lemma is satisfied by taking $\Gamma(x,1) = \langle x \rangle$, so we may assume that $i \geq 2$. If $x \in \gamma_i(G)$ this implies that there exist elements $y_1, \dots, y_k \in \gamma_{i-1}(G)$ and $z_1, \dots, z_k \in G$ such that $x = \prod_{j=1}^k [y_j, z_j]$, and so by induction on $i$ we may take

$$\Gamma(x,i) = \langle \Gamma(y_1, i-1), \dots, \Gamma(y_k, i-1), z_1, \dots, z_k \rangle.$$

If instead $x \in \overline{\gamma_i(G)}$ then by definition there exists $n \in \mathbb{N}$ such that $x^n \in \gamma_i(G)$, and so we may take $\Lambda(x,i) = \langle \Gamma(x^n, i), x \rangle$. □

*Proof of Proposition A.1.* It is sufficient to show that for every $x \in G$ and $c \in \overline{\gamma_{ds+1}(G)}$ we have $\varphi(xc) = \varphi(x)$. Following Leibman's proof of [15, Proposition 3.21], we may assume by Lemma A.2 that $G$ is finitely generated. It then follows from [15, Corollary 1.18] that $\varphi(G)$ lies in a finitely generated subgroup of $N$, and so we may also assume that $N$ is finitely generated. The proposition then follows from Lemma 5.5 and [15, Proposition 3.15]. □

## B Hyperbolic groups

The following argument was communicated by Yago Antolín, and shows that generic subgroups of hyperbolic groups are free, with respect to the uniform probability measure on the balls given by a finite generating set. In particular, the degree of nilpotence with respect to such a measure is zero for any non-elementary hyperbolic group.

These techniques and the result are well known to experts, and we include it here for completeness.

As previously, let $F_r$ denote the free group of rank $r$. For a group, $G$, generated by a (finite) set $X$, we let $\mathbb{B}_X(n)$ denote the ball of radius $n$, and for an element $g \in G$, we denote by $|g|_X$ the word length of $g$. Let $\mu_n$ be the uniform probability measure on the ball of radius $n$ in $G$ with respect to $X$.

**Theorem B.1.** *Let $G$ be a non-elementary hyperbolic group with a finite generating set $X$. For every $r \in \mathbb{N}$*

$$\lim_{n\to\infty} \frac{|\{(g_1, \dots, g_r) \in \mathbb{B}_X(n)^r \mid \langle g_1, \dots, g_r \rangle \cong F_r\}|}{|\mathbb{B}_X(n)|^r} = 1,$$

*and the limit converges exponentially fast.*

We note that the analogous theorem with respect to sequences of measures $(\mu^{*n})_{n=1}^{\infty}$ corresponding to the steps of the random walk on $G$ was proved in [12].

The following Corollary is immediate:

**Corollary B.2.** *Let $G$ be a non-elementary hyperbolic group with a finite generating set, $X$, and write $M = (\mu_n)_{n=1}^{\infty}$ for the sequence of uniform measures on the balls $\mathbb{B}_X(n)$. Then $\mathrm{dc}_M^k(G) = 0$.*

Throughout, $G$ is a non-elementary hyperbolic group (i.e. a hyperbolic group that is not virtually cyclic) and $X$ a finite generating set of $G$. We assume that $\Gamma(G, X)$ is $\delta$-hyperbolic. There are many equivalent definitions of Gromov hyperbolicity, (see for example [4, Chapter III.H.1.17]), for convenience we will use the one that says that geodesic triangles are $\delta$-thin. In particular, if $x, y, z \in G$, and $\alpha$ is a geodesic with endpoints in $x$ and $y$, $\beta$ a geodesic with endpoints in $x, z$ and $\gamma$ a geodesics with endpoints $y, z$ then we have that for points $v \in \alpha$ and $u \in \beta$ with $\mathrm{d}(x, u) = \mathrm{d}(x, v) \leq (y \cdot z)_x \coloneqq \frac{1}{2}(\mathrm{d}(x, y) + \mathrm{d}(x, z) - \mathrm{d}(y, z))$ one has that $\mathrm{d}(u, v) \leq \delta$.

Since $G$ has exponential growth, $\lim \sqrt[n]{|\mathbb{B}_X(n)|} = \lambda > 1$. A result of Coornaert [6] states that there are positive constants $A$, $B$ and $n_0$ such that

$$A\lambda^n \leq |\mathbb{B}_X(n)| \leq B\lambda^n \tag{B.1}$$

for all $n \geq n_0$.

*Remark* B.3. From the submultiplicativity of the function $|\mathbb{B}_X(n)|$ it follows that the limit $\lim \sqrt[n]{|\mathbb{B}_X(n)|}$ exists and hence for every $\varepsilon > 0$ there exists $n_\varepsilon$, $A$ and $B$ such that for all $n > n_\varepsilon$,

$$A(\lambda - \varepsilon)^n \leq |\mathbb{B}_X(n)| \leq B(\lambda + \varepsilon)^n.$$

One can prove Theorem B.1 using this weaker fact. However, for simplicity of exposition, we have preferred to use (B.1).

**Lemma B.4** (Delzant [9, Lemma 1.1.])**.** *Let $(x_n)$ be a sequence of points on a $\delta$-hyperbolic metric space such that $\mathrm{d}(x_{n+2}, x_n) \geq \max(\mathrm{d}(x_{n+2}, x_{n+1}), \mathrm{d}(x_{n+1}, x_n)) + 2\delta + a$. Then $\mathrm{d}(x_n, x_m) \geq a|m - n|$.*

**Lemma B.5.** *There exists a constant $D_0 = D_0(\delta) \geq 0$ such that the following holds.*

*Let $g_1, g_2, \ldots, g_r \in G$ satisfying that for all $a, b \in \{g_1, \ldots, g_r\}^{\pm 1}$ with $a \neq b^{-1}$ the inequality*

$$|ab|_X \geq \max\{|a|_X, |b|_X\} + D_0 \tag{B.2}$$

*holds. Then $\langle g_1, \ldots, g_n\rangle$ is a free subgroup with basis $\{g_1, \ldots, g_n\}$.*

*Proof.* Take $D_0 \geq 2\delta + 1$. Let $w$ be any reduced word on $Z = \{g_1, \ldots, g_r\}^{\pm 1}$ and denote by $w_i$ the prefix of length $i$ (as a word in $Z$). Then $\mathrm{d}(w_i, w_{i+2}) = \mathrm{d}(1, w_i^{-1} w_{i+1}) = |ab|_X$ for some $a, b \in Z$ with $a \neq b^{-1}$ (since $w$ is reduced). Thus, it follows from Lemma B.4 that $|w|_X \geq \ell_Z(w)$, where $\ell_Z(w)$ denotes the length of $w$ as a word in $Z$. $\square$

We will find bounds on the number of elements in $\mathbb{B}_X(n)$ not satisfying (B.2). There are two different cases to be considered: $a = b$ and $a \neq b$.

**Lemma B.6.** *There is $D_1 = D_1(\delta, D_0) \geq 0$ such that the cardinality of the set*

$$AA(n) = \{g \in \mathbb{B}_X(n) \mid |g^2|_X < |g|_X + D_0\}$$

*is bounded above by $|\mathbb{B}_X(\frac{n}{2} + D_1)|$.*

*Proof.* Let $g \in \mathbb{B}_X(n)$ with $|g^2|_X < |g|_X + D_0$. Then $(1 \cdot g^2)_g > |g|_X/2 - D_0/2$. Let $w$ be a geodesic word over $X$ representing $g$. Suppose that $w = w_\iota w' w_\tau$, where $w_\iota$ and $w_\tau$ are the prefix and suffix of $w$ of length $(1, g^2)_g$, respectively. Then, there exists $t \in \mathbb{B}_X(\delta)$ such that $w_\tau w_\iota =_G t$. Thus $w_\iota^{-1} g w_\iota = w' t$, and therefore $g$ is conjugated to an element of length at most $D_0 + \delta$ by an element of length at most $n/2 - D_0/2$. Hence, the cardinality of $AA(n)$ is bounded above by $|\mathbb{B}_X(n/2 - D_0/2)||\mathbb{B}_X(D_0 + \delta)| \leq |\mathbb{B}_X(\frac{n}{2} + D_1)|$ for some $D_1$. $\square$

*Remark* B.7. Note that by Lemma B.5, if $g \notin AA(n)$ then $g$ has infinite order. Thus, in particular, the above Lemma implies that the number of finite order elements in the ball of radius $n$ is at most $|\mathbb{B}_X(\frac{n}{2} + D_1)|$. This appears in [7].

**Lemma B.8.** *Let $\varepsilon \in (3/4, 1)$, $n \in \mathbb{N}$ and $g \in G$. Suppose that $|g|_X > \varepsilon n$. Then there exists $D_2 = D_2(\delta, D_0, \varepsilon) \geq 0$ such that the cardinality of the set*

$$AB(g, n) = \{h \in \mathbb{B}_X(n) \mid |h|_X > \varepsilon n, |gh|_X < n + D_0\}$$

*is bounded above by $|\mathbb{B}_X(\frac{3n}{4} + D_2)|$.*

*Proof.* Let $h \in \mathbb{B}_X(n)$ with $|h|_X > \varepsilon n$ and $|gh|_X < n + D_0$. Then

$$(1 \cdot gh)_h > \varepsilon n - n/2 - D_0/2 > n/4 - D_0/2.$$

Let $u$ and $v$ be geodesic words over $X$ representing $g$ and $h$ respectively. Suppose that $u = u_1 u_2$ and $v = v_1 v_2$, where $u_2$ and $v_1$ have length $(1, gh)_h$. Note that $|v_2|_X \leq 3n/4 + D_0/2$. Then there exists $t \in \mathbb{B}_X(\delta)$ such that $u_2 v_1 =_G t$. Thus, $u_2 h = t v_2$, and so $AB(g, n)$ is contained in $u_2^{-1} \mathbb{B}_X(\delta) \mathbb{B}_X(3/4n + D_0/2)$. $\square$

*Proof of Theorem B.1.* Fix $\varepsilon \in (3/4, 1)$.

Let
$$P_0(n) = \frac{|\{(g_1,\ldots,g_r) \in (\mathbb{B}_X(n) - \mathbb{B}_X(\varepsilon n))^r \mid g_i \notin AA(n)\}|}{|\mathbb{B}_X(n)|^r}.$$

For $n \gg 0$, we have from Lemma B.6 and (B.1)

$$\begin{aligned}
P_0(n) &\geq \frac{(|\mathbb{B}_X(n)| - A\lambda^{\varepsilon n} - A\lambda^{n/2+D_1})^r}{|\mathbb{B}_X(n)|^r} \\
&\geq \frac{(|\mathbb{B}_X(n)| - 2A\lambda^{\varepsilon n})^r}{|\mathbb{B}_X(n)|^r} \\
&\geq 1 - \frac{\sum_{k=1}^r \binom{r}{k} |\mathbb{B}_X(n)|^{r-k}(2A\lambda^{\varepsilon n})^k}{|\mathbb{B}_X(n)|^r} \\
&\geq 1 - \sum_{k=1}^r \binom{r}{k} \left(\frac{2A\lambda^{\varepsilon n}}{B\lambda^n}\right)^k \\
&\geq 1 - \frac{C_1}{\lambda^{(1-\varepsilon)n}}
\end{aligned}$$

where $C_1$ is some constant depending on $A$, $B$ and $r$.

For $j = 1,\ldots,r$, let

$$P_j(n) = \frac{|\{(g_1,\ldots,g_r) \in (\mathbb{B}_X(n) - \mathbb{B}_X(\varepsilon n))^r \mid g_j \notin AB(g_i^{\pm 1}, n) \text{ for } i \neq j\}|}{|\mathbb{B}_X(n)|^r}.$$

For $n \gg 0$, we have from Lemma B.8 and (B.1)

$$\begin{aligned}
P_j(n) &\geq \frac{(|\mathbb{B}_X(n)| - |\mathbb{B}_X(\varepsilon n)|)^r - (2r-2)|\mathbb{B}_X(3n/4 + D_2)|(|\mathbb{B}_X(n)| - |\mathbb{B}_X(\varepsilon n)|)^{r-1}}{|\mathbb{B}_X(n)|^r} \\
&\geq \frac{(|\mathbb{B}_X(n)| - |\mathbb{B}_X(\varepsilon n)|)^r}{|\mathbb{B}_X(n)|^r} - \frac{(2r-2)|\mathbb{B}_X(3n/4 + D_2)|(|\mathbb{B}_X(n)|)^{r-1}}{|\mathbb{B}_X(n)|^r} \\
&\geq 1 - \frac{C_1}{\lambda^{(1-\varepsilon)n}} - \frac{(2r-2)A\lambda^{3n/4+D_2}}{B\lambda^n} \\
&\geq 1 - \frac{C_1}{\lambda^{(1-\varepsilon)n}} - \frac{C_2}{\lambda^{n/4}}
\end{aligned}$$

where $C_2$ is some constant depending on $A$,$B$ and $r$.

Thus, for $i = 0,1,\ldots,r$ $\lim_{n\to\infty} P_i(n) = 1$ converges exponentially fast. By Lemma B.5 we have that for $n \gg 0$

$$1 \geq \frac{|\{(g_1,\ldots,g_r) \in \mathbb{B}_X(n)^r \mid \langle g_1,\ldots,g_r\rangle \cong F_r\}|}{|\mathbb{B}_X(n)|^r} \geq 1 - \sum_{i=0}^r (1 - P_i(n))$$

and taking limits, we see that the probability that an $r$-tuple freely generates a free group converges to 1 exponentially fast. $\square$

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# Paper 3

## ACYLINDRICAL HYPERBOLICITY OF GROUPS ACTING ON QUASI-MEDIAN GRAPHS AND EQUATIONS IN GRAPH PRODUCTS

MOTIEJUS VALIUNAS

Abstract. In this paper we study group actions on quasi-median graphs, or 'CAT(0) prism complexes', generalising the notion of CAT(0) cube complexes. We consider hyperplanes in a quasi-median graph $X$ and define the *contact graph* $\mathcal{C}X$ for these hyperplanes. We show that $\mathcal{C}X$ is always quasi-isometric to a tree, generalising a result of Hagen [16], and that under certain conditions a group action $G \curvearrowright X$ induces an *acylindrical* action $G \curvearrowright \mathcal{C}X$, giving a quasi-median analogue of a result of Behrstock, Hagen and Sisto [5].

As an application, we exhibit an acylindrical action of a graph product on a quasi-tree, generalising results of Kim and Koberda for right-angled Artin groups [18], [19]. We show that for many graph products $G$, the action we exhibit is the 'largest' acylindrical action of $G$ on a hyperbolic metric space. We use this to show that the graph products of equationally Noetherian groups over finite graphs of girth $\geq 5$ are equationally Noetherian, generalising a result of Sela [25].

## 1 Introduction

Group actions on CAT(0) cube complexes occupy a central role in geometric group theory. Such actions have been used to study many interesting classes of groups, such as right-angled Artin and Coxeter groups, many small cancellation and 3-manifold groups, and even finitely presented infinite simple groups, constructed by Burger and Mozes in [7]. Study of CAT(0) cube complexes is aided by their rich combinatorial structure, introduced by Sageev in [24].

In the present paper we study *quasi-median graphs*, which can be viewed as a generalisation of CAT(0) cube complexes; see Definition 2.1. In particular, one may think of quasi-median graphs as 'CAT(0) prism complexes', consisting of *prisms* – cartesian products of (possibly infinite dimensional) simplices – glued together in a non-positively curved way. In his PhD thesis [10], Genevois introduced cubical-like combinatorial structure and geometry to study a wide class of groups acting on quasi-median graphs, including graph products, certain wreath products, and diagram products.

In particular, given a quasi-median graph $X$, we study *hyperplanes* in $X$: that is, the equivalence classes of edges of $X$, under the equivalence relation generated by letting two edges be equivalent if they induce a square or a triangle. Two hyperplanes are said to *intersect* if two edges defining those hyperplanes are adjacent in a square, and *osculate* if two edges defining those hyperplanes are adjacent but do not belong to a square; see

Definition 2.2. This allows us to define two other graphs related to $X$, which turn out to be useful in the study of groups acting on $X$.

**Definition 1.1.** Let $X$ be a quasi-median graph. We define the *contact graph* $\mathcal{C}X$ and the *crossing graph* $\Delta X$ as follows. For the vertices, let $V(\mathcal{C}X) = V(\Delta X)$ be the set of hyperplanes of $X$. Two hyperplanes $H, H'$ are then adjacent in $\Delta X$ if and only if $H$ and $H'$ intersect; hyperplanes $H, H'$ are adjacent in $\mathcal{C}X$ if and only if $H$ and $H'$ either intersect or osculate.

For a CAT(0) cube complex $X$, Hagen has shown that the contact graph $\mathcal{C}X$ is a quasi-tree – that is, it is quasi-isometric to a tree [16, Theorem 4.1]. Here we generalise this result to quasi-median graphs.

**Theorem A.** *Let $X$ be a quasi-median graph. Then the contact graph $\mathcal{C}X$ is a quasi-tree.*

We prove Theorem A in Section 3.2.

In this paper we study acylindrical hyperbolicity of groups acting on quasi-median graphs.

**Definition 1.2.** Suppose a group $G$ acts on a metric space $(X, d)$ by isometries. Such an action is said to be *acylindrical* if for every $\varepsilon > 0$, there exist constants $D_\varepsilon, N_\varepsilon > 0$ such that for all $x, y \in X$ with $d(x, y) \geq D_\varepsilon$, the number of elements $g \in G$ satisfying

$$d(x, x^g) \leq \varepsilon \qquad \text{and} \qquad d(y, y^g) \leq \varepsilon$$

is bounded above by $N_\varepsilon$. Moreover, an action $G \curvearrowright X$ by isometries on a hyperbolic metric space $X$ is said to be *non-elementary* if orbits under this action is unbounded and $G$ is not virtually cyclic.

A group $G$ is then said to be *acylindrically hyperbolic* if it possesses a non-elementary acylindrical action on a hyperbolic metric space.

Acylindrically hyperbolic groups form a large family, including hyperbolic and relatively hyperbolic groups, mapping class groups of most surfaces, and $\mathrm{Out}(F_n)$ for $n \geq 3$ [23]. This family also includes 'most' hierarchically hyperbolic groups [5, Corollary 14.4], and in particular 'most' groups $G$ that act properly and cocompactly on a CAT(0) cube complex with a 'factor system': see [5]. The following result shows that, more generally, many groups acting on quasi-median graphs are acylindrically hyperbolic.

In the following theorem, we say a group action $G \curvearrowright X$ is *special* if there are no two hyperplanes $H, H'$ of $X$ such that $H$ and $H'$ intersect but $H^g$ and $H'$ osculate for some $g \in G$, and there is no hyperplane $H$ that intersects or osculates with $H^g \neq H$ for some $g \in G$. We say a collection $\mathcal{S}$ of sets is *uniformly finite* if there exists a constant $D \in \mathbb{N}$ such that each $S \in \mathcal{S}$ has cardinality $\leq D$.

**Theorem B.** *Let $G$ be a group acting specially on a quasi-median graph $X$, and suppose vertices in $\Delta X/G$ have uniformly finitely many neighbours.*

(i) *If $\Delta X$ is connected and $\Delta X/G$ has finitely many vertices, then the inclusion $\Delta X \hookrightarrow \mathcal{C}X$ is a quasi-isometry.*

(ii) *If stabilisers of vertices under $G \curvearrowright X$ are uniformly finite, then the induced action $G \curvearrowright \mathcal{C}X$ is acylindrical. In particular, if the orbits under $G \curvearrowright \mathcal{C}X$ are unbounded, then $G$ is either virtually cyclic or acylindrically hyperbolic.*

We prove part (i) of Theorem B in Section 3.1, and part (ii) in Section 4.

Note that a large class of examples of group actions on CAT(0) cube complexes with a factor system comes from special actions [5, Corollaries 8.8 and 14.5]. Theorem B (ii) generalises this result to quasi-median graphs. We also show that several other hierarchically hyperbolic space-like results on CAT(0) cube complexes generalise to quasi-median graphs: for instance, existence of 'hierarchy paths', see [5, Theorem A (2)] and Proposition 3.1.

The main application of Theorems A and B we give is to study graph products of groups. In particular, let $\Gamma$ be a simplicial graph and let $\mathcal{G} = \{G_v \mid v \in V(\Gamma)\}$ be a collection of non-trivial groups. The *graph product* $\Gamma\mathcal{G}$ of the groups $G_v$ over $\Gamma$ is defined as the group

$$\Gamma\mathcal{G} = \left( \mathop{\ast}_{v \in V(\Gamma)} G_v \right) \Big/ \left\langle\!\left\langle g_v^{-1} g_w^{-1} g_v g_w \,\middle|\, g_v \in G_v, g_w \in G_w, (v,w) \in E(\Gamma) \right\rangle\!\right\rangle .$$

For example, for a complete graph $\Gamma$ we have $\Gamma\mathcal{G} \cong \prod_{v \in V(\Gamma)} G_v$, while for discrete $\Gamma$ we have $\Gamma\mathcal{G} \cong \ast_{v\in V(\Gamma)} G_v$. The applicability of the results above to graph products follows from the following result of Genevois.

**Theorem 1.3** (Genevois [10, Propositions 8.2 and 8.11])**.** *Let $\Gamma$ be a simplicial graph, let $\mathcal{G} = \{G_v \mid v \in V(\Gamma)\}$ be a collection of non-trivial groups, and let $S = \bigcup_{v \in V(\Gamma)} G_v \setminus \{1\} \subseteq \Gamma\mathcal{G}$. Then the Cayley graph $X$ of $\Gamma\mathcal{G}$ with respect to $S$ is quasi-median. Moreover, the action of $\Gamma\mathcal{G}$ on $X$ is free on vertices, special, and the quotient $\Delta X/\Gamma\mathcal{G}$ is isomorphic to $\Gamma$.*

An important subclass of graph products are right-angled Artin groups (RAAGs): indeed, if $G_v \cong \mathbb{Z}$ then $\Gamma\mathcal{G}$ is the RAAG associated to $\Gamma$. In this case, a vertex $v \in V(\Gamma)$ is usually identified with a generator of $G_v$. In [18] Kim and Koberda constructed the *extension graph* $\Gamma^e$ of a RAAG $G = \Gamma\mathcal{G}$ as a graph with vertex set $V(\Gamma^e) = \{v^g \in G \mid g \in G, v \in V(\Gamma)\}$, where $g^v$ and $h^w$ are adjacent in $\Gamma^e$ if and only if they commute as elements of $G$. This graph turns out to be the same as the crossing graph $\Delta X$ of the Cayley graph $X$ defined in Theorem 1.3.

In fact, Kim and Koberda showed that, given that $|V(\Gamma)| \geq 2$ and both $\Gamma$ and its complement $\Gamma^C$ are connected, $\Gamma^e$ is quasi-isometric to a tree [18] and the action of $G$ on $\Gamma^e$ by conjugation is non-elementary acylindrical [19]. In this paper we generalise these results to arbitrary graph products; this follows as a special case of Theorems A and B. As a special case, we recover hyperbolicity of the extension graph $\Gamma^e$ and acylindricity of the action $\Gamma\mathcal{G} \curvearrowright \Gamma^e$, providing an alternative (shorter and more geometric) argument to the ones presented in [18] and [19]. In the following corollary, a graph $\Gamma$ is said to have *bounded degree* if there exists a constant $D \in \mathbb{N}$ such that each vertex of $\Gamma$ has degree $\leq D$.

**Corollary C.** *Let $\Gamma$ be a simplicial graph, let $\mathcal{G} = \{G_v \mid v \in V(\Gamma)\}$ be a collection of non-trivial groups, and let $X$ be the quasi-median graph defined in Theorem 1.3. Then $\mathcal{C}X$ is a quasi-tree, and if $\Gamma$ has bounded degree then the induced action $\Gamma\mathcal{G} \curvearrowright \mathcal{C}X$ is acylindrical. Moreover, if $|V(\Gamma)| \geq 2$ and the complement $\Gamma^C$ of $\Gamma$ is connected, then either $\Gamma\mathcal{G} \cong C_2 * C_2$ is the infinite dihedral group, or this action is non-elementary.*

The hyperbolicity of $\mathcal{C}X$ and the acylindricity of the action follow immediately from Theorems A, B and 1.3, while non-elementarity is shown in Section 5.1.

It is worth noting that Minasyan and Osin have already shown in [22] that if $|V(\Gamma)| \geq 2$ and the complement of $\Gamma$ is connected, then $\Gamma\mathcal{G}$ is either infinite dihedral or acylindrically hyperbolic. However, their proof is not direct and does not provide an explicit acylindrical action on a hyperbolic space. The aim of Corollary C is to describe such an action.

We also show that in many cases the action of $\Gamma\mathcal{G}$ on $\mathcal{C}X$ is, in the sense of Abbott, Balasubramanya and Osin [1], the 'largest' acylindrical action of $\Gamma\mathcal{G}$ on a hyperbolic metric space: see Section 5.2. In particular, we show that many graph products are strongly $\mathcal{AH}$-accessible. This generalises the analogous result for right-angled Artin groups [1, Theorem 2.19 (c)].

**Corollary D.** *Let $\Gamma$ be a finite simplicial graph and let $\mathcal{G} = \{G_v \mid v \in V(\Gamma)\}$ be a collection of infinite groups. Suppose that for each isolated vertex $v \in V(\Gamma)$, the group $G_v$ is strongly $\mathcal{AH}$-accessible. Then $\Gamma\mathcal{G}$ is strongly $\mathcal{AH}$-accessible. Furthermore, if $\Gamma$ has no isolated vertices, then the action $\Gamma\mathcal{G} \curvearrowright \mathcal{C}X$, where $X$ is as in Theorem 1.3, is the largest acylindrical action of $\Gamma\mathcal{G}$ on a hyperbolic metric space.*

We prove Corollary D in Section 5.2.

*Remark* 1.4. After the first version of this preprint was made available, it has been brought to the author's attention that most of the results stated in Corollary C follow from the results in [9], [11], [13]. Moreover, a special case of Corollary D (when the vertex groups $G_v$ are hierarchically hyperbolic) follows from the results in [2], [6]. See Remarks 5.4 and 5.5 for details.

As an application, we use Corollary C to study the class of equationally Noetherian groups, defined as follows.

**Definition 1.5.** Given $n \in \mathbb{N}$, let $F_n$ denote the free group of rank $n$ with a free basis $X_1, \ldots, X_n$. Given a group $G$, an element $s \in F_n$ and a tuple $(g_1, \ldots, g_n) \in G^n$, we write $s(g_1, \ldots, g_n) \in G$ for the element obtained by replacing every occurence of $X_i$ in $s$ with $g_i$, and evaluating the resulting word in $G$. Given a subset $S \subseteq F_n$, the *solution set* of $S$ in $G$ is

$$V_G(S) = \{(g_1, \ldots, g_n) \in G^n \mid s(g_1, \ldots, g_n) = 1 \text{ for all } s \in S\}.$$

A group $G$ is said to be *equationally Noetherian* if for any $n \in \mathbb{N}$ and any subset $S \subseteq F_n$, there exists a *finite* subset $S_0 \subseteq S$ such that $V_G(S_0) = V_G(S)$.

Many classes of groups are known to be equationally Noetherian. For example, groups that are linear over a field – in particular, right-angled Artin groups – are equationally Noetherian [4, Theorem B1]. It is easy to see that the class of equationally Noetherian groups is preserved under taking subgroups and direct products; a deep and non-trivial argument shows that the same is true for free products:

**Theorem 1.6** (Sela [25, Theorem 9.1])**.** *Let $G$ and $H$ be equationally Noetherian groups. Then $G * H$ is equationally Noetherian.*

Using methods of Groves and Hull developed for acylindrically hyperbolic groups [15], we generalise Theorem 1.6 to a wider class of graph products.

**Theorem E.** *Let $\Gamma$ be a finite simplicial triangle-free and square-free graph, and let $\mathcal{G} = \{G_v \mid v \in V(\Gamma)\}$ be a collection of equationally Noetherian groups. Then the graph product $\Gamma\mathcal{G}$ is equationally Noetherian.*

We prove Theorem E in Section 6.

The paper is structured as follows. In Section 2, we define quasi-median graphs and give several results that are used in later sections. In Section 3, we analyse the geometry of the contact graph and its relation to crossing graph, and prove Theoren A and Theorem B (i). In Section 4, we consider the action of a group $G$ on a quasi-median graph $X$, and prove Theorem B (ii). In Section 5, we consider the particular case when $G = \Gamma\mathcal{G}$ is a graph product and $X$ is the quasi-median graph associated to it, and deduce Corollaries C and D. In Section 6, we apply these results to prove Theorem E.

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## 2 Preliminaries

Throughout the paper, we use the following conventions and notation. By a graph $X$, we mean an undirected simple (simplicial) graph, and we write $V(X)$ and $E(X)$ for the vertex and edge sets of $X$, respectively. Moreover, we write $d_X(-,-)$ for the combinatorial metric on $X$ – thus, we view $X$ as a geodesic metric space. We consider the set $\mathbb{N}$ of natural numbers to include 0.

Given a group $G$, all actions of $G$ on a set $X$ are considered to be right actions, $\theta : X \times G \to X$, and are written as $\theta(x, g) = x^g$ or $\theta(x, g) = xg$. Note that this results in perhaps unusual terminology when we consider a Cayley graph $\mathrm{Cay}(G, S)$: in our case it has edges of the form $(g, sg)$ for $g \in G$ and $s \in S$.

### 2.1 Quasi-median graphs

In this section we introduce quasi-median graphs and basic results that we use throughout the paper. Most of the definitions and results in this section were introduced by Genevois in his thesis [10]. We therefore refer the interested reader to [10] for further discussion and results on applications of quasi-median graphs to geometric group theory.

**Definition 2.1.** Let $X$ be a graph, let $x_1, x_2, x_3 \in V(X)$ be three vertices, and let $k \in \mathbb{N}$. We say a triple $(y_1, y_2, y_3) \in V(X)^3$ is a *$k$-quasi-median* of $(x_1, x_2, x_3)$ if (see Figure 3.1(a)):

(i) $y_i$ and $y_j$ lie on a geodesic between $x_i$ and $x_j$ for any $i \neq j$;

(ii) $k = d_X(y_1, y_2) = d_X(y_1, y_3) = d_X(y_2, y_3)$; and

(iii) $k$ is as small as possible subject to (i) and (ii).

We say $(y_1, y_2, y_3) \in V(X)^3$ is a *quasi-median* of $(x_1, x_2, x_3) \in V(X)^3$ if it is a $k$-quasi-median for some $k$. A 0-quasi-median is called a *median*.

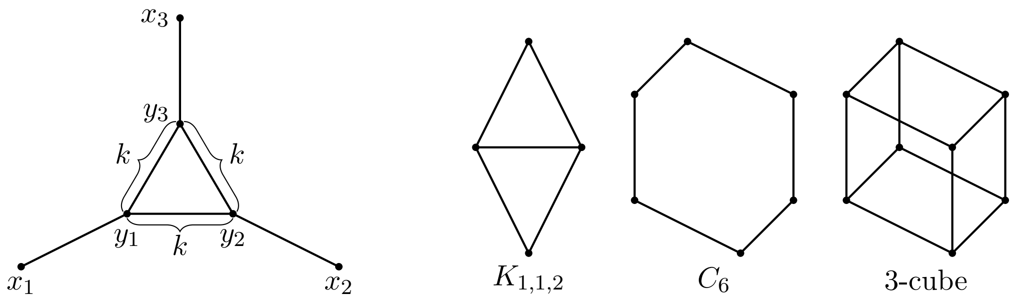
We say a graph $X$ is a *quasi-median graph* if (see Figure 3.1(b)):

(i) every triple of vertices has a unique quasi-median;

(ii) $K_{1,1,2}$ is not isomorphic to an induced subgraph of $X$; and

(iii) if $Y \cong C_6$ is a subgraph of $X$ such that the embedding $Y \hookrightarrow X$ is isometric, then the convex hull of $Y$ in $X$ is isomorphic to the 3-cube.

There are many equivalent characterisations of quasi-median graphs: see [3, Theorem 1]. In this paper we think of quasi-median graphs as generalisations of median graphs.




(a) A $k$-quasi-median $(y_1, y_2, y_3)$ of $(x_1, x_2, x_3)$. (b) The graphs $K_{1,1,2}$, $C_6$ and the 3-cube.

FIGURE 3.1: Graphs appearing in Definition 2.1.

Recall that a graph $X$ is called a *median graph* if every triple of vertices of $X$ has a unique median. In particular, every median graph is quasi-median; more precisely, it is known that a graph is median if and only if it is quasi-median and triangle-free: see [10, Corollary 2.92], for instance.

In what follows, a *clique* is a maximal complete subgraph, a *triangle* is a complete graph on 3 vertices, and a *square* is a complete bipartite graph on two sets of 2 vertices each.

**Definition 2.2.** Let $X$ be a quasi-median graph. Let $\sim$ be the equivalence relation on $E(X)$ generated by the equivalences $e \sim f$ when $e$ and $f$ either are two sides of a triangle or opposite sides of a square. A *hyperplane* $H$ is an equivalence class $[e]$ for some $e \in E(X)$; in this case, we say $H$ is the hyperplane *dual* to $e$ (or, alternatively, $H$ is the hyperplane dual to any clique containing $e$). Given a hyperplane $H$ dual to $e \in E(X)$, the *carrier* of $H$, denoted by $\mathcal{N}(H)$, is the full subgraph of $X$ induced by $[e] \subseteq E(X)$; a *fibre* of $H$ is a connected component of $\mathcal{N}(H) \setminus J$, where $J$ is the union of the interiors of all the edges in $[e]$.

Given two edges $e, e' \in E(X)$ with a common endpoint ($p$, say) that do not belong to the same clique, let $H$ and $H'$ be the hyperplanes dual to $e$ and $e'$, respectively. We then say $H$ and $H'$ *intersect* (or *intersect at* $p$) if $e$ and $e'$ are adjacent edges in a square, and we say $H$ and $H'$ *osculate* (or *osculate at* $p$) otherwise.

Finally, given two vertices $p, q \in V(X)$ and a hyperplane $H$, we say $H$ *separates* $p$ from $q$ if every path between $p$ and $q$ contains an edge dual to $H$. More generally, we say $H$ *separates* two subgraphs $P, Q \subseteq X$ if $H$ does not separate any two vertices of $P$ or any two vertices of $Q$, but it separates a vertex of $P$ from a vertex of $Q$. Given a path $\gamma$ in $X$, we also say $H$ *crosses* $\gamma$ if $\gamma$ contains an edge dual to $H$.

Another important concept in the study of quasi-median graphs are gated subgraphs. Such subgraphs coincide with *convex subgraphs* for median graphs, but in general form a larger class in quasi-median graphs.

**Definition 2.3.** Let $X$ be a quasi-median graph, let $Y \subseteq X$ be a full subgraph, and let $v \in V(X)$. We say $p \in V(Y)$ is a *gate* for $v$ in $Y$ if, for any $q \in V(Y)$, there exists a geodesic in $X$ between $v$ and $q$ passing through $p$. We say a full subgraph $Y \subseteq X$ is a *gated subgraph* if every vertex of $X$ has a gate in $Y$.

The following result says that the subgraphs of interest to us are gated. Here, by convention, given two graphs $Y$ and $Z$ we denote by $Y \times Z$ the 1-skeleton of the square complex obtained as a cartesian product of $Y$ and $Z$.

**Proposition 2.4** (Genevois [10, Proposition 2.15])**.** *Let $X$ be a quasi-median graph, $H$ a hyperplane dual to a clique $C$, and $F$ a fibre of $H$. Then $\mathcal{N}(H)$, $C$ and $F$ are gated subgraphs of $X$. Moreover, there exists a graph isomorphism $\Psi : \mathcal{N}(H) \to F \times C$, and the cliques dual to $H$ (respectively the fibres of $H$) are precisely the subgraphs $\Psi^{-1}(\{p\} \times C)$ for vertices $p \in V(F)$ (respectively $\Psi^{-1}(F \times \{p\})$ for vertices $p \in V(C)$).*

## 2.2 Special actions

In this section we describe the hypotheses that we impose on group actions on quasi-median graphs. We first define what it means for an action on a quasi-median graph to be special.

**Definition 2.5.** Let $X$ be a quasi-median graph, and let $G$ be a group acting on it by graph isomorphisms. We say the action $G \curvearrowright X$ is *special* if

(i) no two hyperplanes in the same orbit under $G \curvearrowright X$ intersect or osculate; and

(ii) given two hyperplanes $H$ and $H'$ that intersect, $H^g$ and $H'$ do not osculate for any $g \in G$.

Special actions on CAT(0) cube complexes were introduced by Haglund and Wise in [17]. Notably, there it is shown that, in our terminology, if a group $G$ acts specially, cocompactly and without 'orientation-inversions' of hyperplanes on a CAT(0) cube complex $X$, then the fundamental group of the quotient $X/G$ embeds in a right-angled Artin group.

It is clear from Proposition 2.4 that no hyperplane in a quasi-median graph can self-intersect or self-osculate. The next lemma says that, moreover, the action of the trivial group on a quasi-median graph is special. Recall that two hyperplanes are said to *interosculate* if they both intersect and osculate.

**Lemma 2.6.** *In a quasi-median graph $X$, no two hyperplanes can interosculate.*

*Proof.* Suppose for contradiction that hyperplanes $H$ and $H'$ intersect at $p$ and osculate at $q$ for some $p, q \in V(X)$, and assume without loss of generality that $p$ and $q$ are chosen in such a way that $d_X(p, q)$ is as small as possible. It is clear that $p \neq q$: see, for instance, [10, Lemma 2.13]. On the other hand, since $\mathcal{N}(H)$ and $\mathcal{N}(H')$ are gated (and therefore convex) by Proposition 2.4, and as $p, q \in \mathcal{N}(H) \cap \mathcal{N}(H')$, it follows that a geodesic between $p$ and $q$ lies in $\mathcal{N}(H) \cap \mathcal{N}(H')$. In particular, if $r$ is a vertex on this geodesic, then $H$ and $H'$ either intersect at $r$ or osculate at $r$; by minimality of $d_X(p,q)$, it then follows that $d_X(p,q) = 1$.

Let $e$ be the edge joining $p$ and $q$, and let $K$ be the hyperplane dual to $e$. It follows from Proposition 2.4 that $K \neq H$ and $K \neq H'$: indeed, if we had $K = H$ (say), then $K = H$ and $H'$ would intersect at $q$, contradicting the choice of $q$. Thus $K$ is distinct from $H$ and $H'$, and so $e$ belongs to a fibre of $H$ and a fibre of $H'$. It then follows from Proposition 2.4 that $K$ intersects both $H$ and $H'$ at $q$, and that the graph $Y$ shown in Figure 3.2 is a subgraph of $X$.

We now claim that the embedding $Y \hookrightarrow X$ is isometric. Indeed, as $H$, $H'$ and $K$ are distinct hyperplanes, no two vertices $p', q' \in V(Y)$ with $d_Y(p', q') = 2$ can be joined by an edge in $X$, as that would create a triangle in $X$ with edges dual to different hyperplanes. It is thus enough to show that if $p', q' \in V(Y)$ and $d_Y(p', q') = 3$, then $d_X(p', q') = 3$. Up to relabelling $H$, $H'$ and $K$, we may assume without loss of generality that $p' = s$ and $q' = q$. Now it is clear that $d_X(s, q) \neq 1$: otherwise, $p_1 s$ and $q_1 q$ are opposite sides in a square in $X$, contradicting the fact that $H \neq H'$. Thus, suppose for contradiction that $d_X(s,q) = 2$. But then the triple $(p_1, s, t)$ is a quasi-median of $(p_1, s, q)$ for some vertex $t \in V(X)$, and the edges $p_1 s$, $p_1 t$, $q_1 q$ are dual to the same hyperplane, again contradicting the fact that $H \neq H'$. Thus the embedding $Y \hookrightarrow X$ is isometric, as claimed.

But now the embedding of the $C_6 \subseteq Y$ formed by vertices $s$, $p_1$, $q_1$, $q$, $q_2$ and $p_2$ into $X$ is also isometric, and so the convex hull of this $C_6$ in $X$ is a 3-cube. Thus there exists a vertex $u \in V(X)$ joined by edges to $s$, $p_2$ and $q_2$. This implies that $H$ and $H'$ intersect at $q$, contradicting the choice of $q$. Thus $H$ and $H'$ cannot interosculate. $\square$

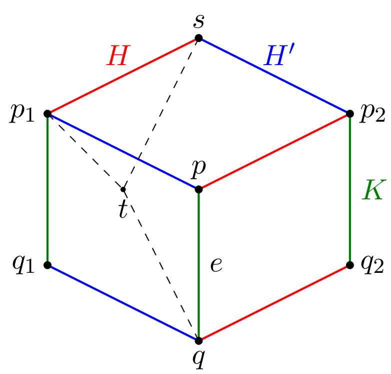



FIGURE 3.2: Proof of Lemma 2.6: the graph $Y$ (solid edges) and the vertex $t \in V(X)$.

*Remark* 2.7. We use Lemma 2.6 in the following setting. Let $\gamma$ be a geodesic in a quasi-median graph $X$, let $e$ and $e'$ be two consecutive edges of $\gamma$, and let $H$ and $H'$ be the hyperplanes dual to $e$ and $e'$, respectively. Suppose that $H$ and $H'$ intersect. It then follows from Lemma 2.6 that $H$ and $H'$ cannot osculate at the common endpoint $p$ of $e$ and $e'$, and therefore $H$ and $H'$ must intersect at $p$. In particular, $X$ contains a square with edges $e$, $e'$, $f$ and $f'$, in which $f$ and $f'$ are the edges opposite to $e$ and $e'$, respectively. We may then obtain another geodesic $\gamma'$ in $X$ (with the same endpoints as $\gamma$) by replacing the subpath $ee'$ of $\gamma$ with $f'f$. We refer to the operation of replacing $\gamma$ by $\gamma'$ as *swapping $e$ and $e'$ on $\gamma$*.

## 2.3 Geodesics in quasi-median graphs

Here we record two results on geodesics in a quasi-median graph. The first one of these is due to Genevois.

**Proposition 2.8** (Genevois [10, Proposition 2.30])**.** *A path in a quasi-median graph $X$ is a geodesic if and only if it intersects any hyperplane at most once. In particular, the distance between two vertices of $X$ is equal to the number of hyperplanes separating them.* □

**Lemma 2.9.** *Let $p, q, r \in V(X)$ be vertices of a quasi-median graph $X$ such that some hyperplane separates $q$ from $p$ and $r$. Then there exists a hyperplane $C$ separating $q$ from $p$ and $r$ and geodesics $\gamma_p$ (respectively $\gamma_r$) between $q$ and $p$ (respectively $q$ and $r$) such that $q$ is an endpoint of the edges of $\gamma_p$ and $\gamma_r$ dual to $C$.*

*Proof.* Let $C$ be a hyperplane separating $q$ from $p$ and $r$, let $\gamma_p$ (respectively $\gamma_r$) be a geodesic between $q$ and $p$ (respectively $q$ and $r$), and let $c_p$ and $c_r$ be the edges of $\gamma_p$ and $\gamma_r$ (respectively) dual to $C$. Let $q_p$ and $q_p'$, $q_r$ and $q_r'$ be the endpoints of $c_p$, $c_r$ (respectively), labelled so that $C$ does not separate $q$, $q_p$ and $q_r$. Suppose, without loss of generality, that $\gamma_p$ and $C$ are chosen in such a way that $d_X(q, q_p)$ is as small as possible, and that $\gamma_r$ is chosen so that $d_X(q, q_r)$ is as small as possible (subject to the choice of $\gamma_p$ and $C$). See Figure 3.3.

We first claim that $q = q_p$. Indeed, suppose not, and let $c_p' \neq c_p$ be the other edge of $\gamma_p$ with endpoint $q_p$. Let $C_p'$ be the hyperplane dual to $c_p'$. Then $C_p'$ does not separate $q_p$ and $p$ (as $\gamma_p$ is a geodesic), nor $q$ and $r$ (by minimality of $d_X(q, q_p)$), but it separates $q_p$ (and so $p$) from $q$ (and so $r$). On the other hand, $C$ separates $q_p$ from $p$ (as $\gamma_p$ is a geodesic) and $q$ from $r$ (as $\gamma_r$ is a geodesic). Therefore, $C$ and $C_p'$ must intersect. But then we may swap $c_p$ and $c_p'$ on $\gamma_p$ (see Remark 2.7), contradicting minimality of $d_X(q, q_p)$. Thus we must have $q = q_p$.

We now claim that $q = q_r$. Indeed, suppose not, and let $c_r' \neq c_r$ be the other edge of $\gamma_r$ with endpoint $q_r$. Let $C_r'$ be the hyperplane dual to $c_r'$. Then $C_r'$ does not separate $q$ and $q_p'$ (as $C$ is the only hyperplane separating $q = q_p$ and $q_p'$), nor $q_r$ and $r$ (as $\gamma_r$ is a geodesic), but it separates $q$ (and so $q_p'$) from $q_r$ (and so $r$). On the other hand, $C$ separates $q_r$ from $r$ (as $\gamma_r$ is a geodesic) and $q$ from $q_p'$. Therefore, $C$ and $C_r'$ must intersect. But then we may swap $c_r$ and $c'_r$ on $\gamma_r$, contradicting minimality of $d_X(q, q_r)$. Thus we must have $q = q_r$. $\square$

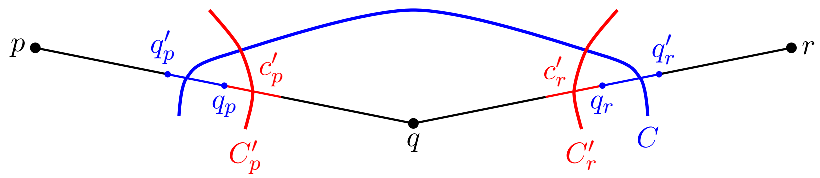



FIGURE 3.3: Proof of Lemma 2.9.

# 3 Geometry of the contact graph

Here we analyse the geometry of the contact graph $\mathcal{C}X$ of a quasi-median graph $X$. In Section 3.1 we show that, under certain conditions, $\mathcal{C}X$ is quasi-isometric to $\Delta X$, and prove Theorem B (i). In Section 3.2 we prove that $\mathcal{C}X$ is a quasi-tree (Theorem A).

## 3.1 Contact and crossing graphs

The following proposition allows us to lift geodesics in $\mathcal{C}(X)$ back to $X$. This generalises the existence of 'hierarchy paths' in CAT(0) cube complexes [5, Theorem A(2)] to arbitrary quasi-median graphs. Moreover, the same result applies when $\mathcal{C}X$ is replaced by $\Delta X$, as long as $\Delta X$ is connected.

**Proposition 3.1.** *Let $\Gamma = \mathcal{C}X$ or $\Gamma = \Delta X$, and let $A, B \in V(\Gamma)$ be hyperplanes in the same connected component of $\Gamma$. Let $p \in V(X)$ (respectively $q \in V(X)$) be a vertex in $\mathcal{N}(A)$ (respectively $\mathcal{N}(B)$). Then there exists a geodesic $A = A_0, \dots, A_m = B$ in $\Gamma$ and vertices $p = p_0, \dots, p_{m+1} = q \in V(X)$ such that $p_i \in \mathcal{N}(A_{i-1}) \cap \mathcal{N}(A_i)$ for $1 \leq i \leq m$ and $d_X(p, q) = \sum_{i=0}^{m} d_X(p_i, p_{i+1})$.*

*Proof.* By assumption, there exists a geodesic $A = A_0, A_1, \dots, A_m = B$ in $\Gamma$. For $1 \leq i \leq m$, let $p_i \in V(X)$ be a vertex in the carriers of both $A_{i-1}$ and $A_i$, and let $p_0 = p$, $p_{m+1} = q$. Suppose that the $A_i$ and the $p_i$ are chosen in such a way that $D = \sum_{i=0}^{m} d_X(p_i, p_{i+1})$ is as small as possible. We claim that $D = d_X(p, q)$.

Let $\gamma_i$ be a geodesic between $p_i$ and $p_{i+1}$ for $0 \leq i \leq m$. Suppose for contradiction that $D > d_X(p, q)$: this means that $\gamma_0\gamma_1 \cdots \gamma_m$ is not a geodesic. Therefore, there exists a hyperplane $C$ separating $p_i$ and $p_{i+1}$ as well as $p_j$ and $p_{j+1}$ for some $i < j$. Let $c_i$ (respectively $c_j$) be the edge of $\gamma_i$ (respectively $\gamma_j$) dual to $C$.

As hyperplane carriers are gated (and therefore convex), any hyperplane separating $p_i$ and $p_{i+1}$ either is or intersects $A_i$ for $0 \leq i \leq m$. Now note that $j - i \leq 2$: indeed, we have $d_\Gamma(A_i, C) \leq 1$ and $d_\Gamma(A_j, C) \leq 1$, so $j - i = d_\Gamma(A_i, A_j) \leq 1 + 1 = 2$. In particular, $j - i \in \{1, 2\}$.

We now claim that $j = i + 1$. Indeed, suppose for contradiction that $j = i + 2$. Let $p'_{i+1}$ (respectively $p'_{i+2}$) be the endpoint of $c_i$ (respectively $c_{i+2}$) closer to $p_i$ (respectively $p_{i+3}$). Then we have

$$\begin{aligned} &d_X(p_i, p_{i+1}) + d_X(p_{i+1}, p_{i+2}) + d_X(p_{i+2}, p_{i+3}) \\ &\quad = d_X(p_i, p'_{i+1}) + d_X(p'_{i+1}, p_{i+1}) + d_X(p_{i+1}, p_{i+2}) + d_X(p_{i+2}, p'_{i+2}) + d_X(p'_{i+2}, p_{i+3}) \\ &\quad \geq d_X(p_i, p'_{i+1}) + d_X(p'_{i+1}, p'_{i+2}) + d_X(p'_{i+2}, p_{i+3}), \end{aligned} \tag{3.1}$$

with equality if and only if $\gamma'_i \gamma_{i+1} \gamma'_{i+2}$ is a geodesic, where $\gamma'_i$ (respectively $\gamma'_{i+2}$) is the portion of $\gamma_i$ (respectively $\gamma_{i+2}$) between $p'_{i+1}$ and $p_{i+1}$ (respectively $p_{i+2}$ and $p'_{i+2}$). But $\gamma'_i \gamma_{i+1} \gamma'_{i+2}$ cannot be a geodesic as it passes through two edges dual to $C$, and so strict inequality in (3.1) holds. We may then replace $A_{i+1}$, $p_{i+1}$ and $p_{i+2}$ with $C$, $p'_{i+1}$ and $p'_{i+2}$, respectively, contradicting minimality of $D$. Thus $j = i + 1$, as claimed.

Therefore, $C$ separates $p_{i+1}$ from $p_i$ and $p_{i+2}$. By Lemma 2.9, we may assume (after modifying $C$, $\gamma_i$ and $\gamma_{i+1}$ if necessary) that $p_{i+1}$ is an endpoint of both $c_i$ and $c_{i+1}$. As $c_i$ and $c_{i+1}$ are dual to the same hyperplane, it follows that they belong to the same clique. In particular (as carriers of hyperplanes are gated and so contain their triangles) this whole clique belongs to $\mathcal{N}(A_i) \cap \mathcal{N}(A_{i+1})$. If $r_{i+1} \neq p_{i+1}$ is the other endpoint of $c_i$, then $d_X(p_i, r_{i+1}) < d_X(p_i, p_{i+1})$ and $d_X(r_{i+1}, p_{i+2}) \leq d_X(p_{i+1}, p_{i+2})$. We may therefore replace $p_{i+1}$ by $r_{i+1}$, contradicting minimality of $D$. Thus $D = d_X(p, q)$, as claimed. □

Taking $\Gamma = \Delta X$ and $p = q$ in Proposition 3.1 immediately gives the following.

**Corollary 3.2.** *Let $A, B \in V(\Delta X)$ be hyperplanes in the same connected component of $\Delta X$ osculating at a point $p \in V(X)$. Then there exists a geodesic $A = A_0, \ldots, A_m = B$ in $\Delta X$ such that $A_{i-1}$ and $A_i$ intersect at $p$ for $1 \leq i \leq m$.* □

**Lemma 3.3.** *Suppose a group $G$ acts on $X$ specially with $N$ orbits of hyperplanes. Let $A$ and $B$ be hyperplanes that osculate and belong to the same connected component of $\Delta X$. Then $d_{\Delta X}(A, B) \leq \max\{2, N - 1\}$.*

*Proof.* Let $p \in V(X)$ be such that $A$ and $B$ osculate at $p$. By Corollary 3.2, there exists a geodesic $A = A_0, A_1, \ldots, A_m = B$ in $\Delta X$ such that $A_{i-1}$ and $A_i$ intersect at $p$ for each $i$. Let $i_1, \ldots, i_k \in \mathbb{N}$, satisfying $0 = i_1 < i_2 < \cdots < i_k = m + 1$, be such that $A_{i_j}^G = A_{i_{j+1}-1}^G$ for $1 \leq j \leq k - 1$ (for instance, we may take $i_j = j - 1$). Suppose this is done so that $k$ is as small as possible. Clearly, this implies $A_{i_j}^G \neq A_{i_{j'}}^G$ whenever $1 \leq j < j' \leq k - 1$: otherwise, we may replace $i_1, \ldots, i_k$ by $i_1, \ldots, i_j, i_{j'+1}, \ldots, i_k$, contradicting minimality of $k$. In particular, $k \leq N + 1$; as $m \geq 2$, note also that $k \geq 2$. We will consider the cases $k = 2$ and $k \geq 3$ separately.

Suppose first that $k \geq 3$. We claim that $i_{j+1} - i_j \leq 1$ whenever $1 \leq j \leq k-1$. Indeed, note that whenever $1 \leq j \leq k-2$, $p \in \mathcal{N}(A_{i_j}) \cap \mathcal{N}(A_{i_{j+1}})$, and so $A_{i_j}$ and $A_{i_{j+1}}$ must either intersect or osculate. But $A_{i_{j+1}}$ intersects $A_{i_{j+1}-1}$, and $A_{i_j}^G = A_{i_{j+1}-1}^G$: therefore, as the action $G \curvearrowright X$ is special, it follows that $A_{i_j}$ and $A_{i_{j+1}}$ must intersect. In particular, $i_{j+1} - i_j = d_{\Delta X}(A_{i_j}, A_{i_{j+1}}) \leq 1$ for $1 \leq j \leq k-2$. For $j = k-1$, we may similarly note that $\mathcal{N}(A_{i_j-1}) \cap \mathcal{N}(A_{i_{j+1}-1}) \neq \varnothing$ and so $A_{i_j-1}$ and $A_m = A_{i_{j+1}-1}$ must intersect: thus $i_{j+1} - i_j = d_{\Delta X}(A_{i_j-1}, A_{i_{j+1}-1}) \leq 1$ in this case as well. In particular, we get

$$d_{\Delta X}(A, B) = m = i_k - i_1 - 1 = \left( \sum_{j=1}^{k-1} (i_{j+1} - i_j) \right) - 1 \leq k - 2 \leq N - 1,$$

as required.

Suppose now that $k = 2$. Similarly to the case $k \geq 3$, we may note that $p \in \mathcal{N}(A_1) \cap \mathcal{N}(A_m)$, and so, as $A_0$ and $A_1$ intersect and as $A_0^G = A_m^G$, it follows that $A_1$ and $A_m$ intersect. Thus $m - 1 = d_{\Delta X}(A_1, A_m) \leq 1$ and so $d_{\Delta X}(A, B) = m \leq 2$, as required. □

*Proof of Theorem B (i).* It is clear that $d_{\mathcal{C}X}(A, B) \leq d_{\Delta X}(A, B)$ for any hyperplanes $A$ and $B$, as $\Delta X$ is a subgraph of $\mathcal{C}X$. Conversely, Lemma 3.3 implies that $d_{\Delta X}(A, B) \leq \overline{N} d_{\mathcal{C}X}(A, B)$ for any hyperplanes $A$ and $B$, where $\overline{N} = \max\{2, N-1\}$. □

*Remark* 3.4. We note that all the assumptions for Theorem B (i) are necessary. Indeed, it is clear that $\Delta X$ needs to be connected. To show necessity of the other two conditions, consider the following. Let $G_0 = \langle S \mid R \rangle$ be the group with generators $S = \{a_{i,j} \mid (i,j) \in \mathbb{Z}^2\}$ and relators $R = \bigcup_{(i,j) \in \mathbb{Z}^2} \{a_{i,j}^2, [a_{i,j}, a_{i,j+1}], [a_{i,j}, a_{i+1,j}]\}$; this is the (infinitely generated) right-angled Coxeter group associated to a 'grid' in $\mathbb{R}^2$: a graph $\Gamma$ with $V(\Gamma) = \mathbb{Z}^2$, where $(i,j)$ and $(i',j')$ are adjacent if and only if $|i-i'| + |j-j'| = 1$. Let $X$ be the Cayley graph of $G_0$ with respect to $S$.

Then $X$ is a quasi-median (and, indeed, median) graph by [10, Proposition 8.2]. Furthermore, by the results in [10, Chapter 8], $\Delta X$ is connected, and if $H_{i,j}$ is the hyperplane dual to the edge $(1, a_{i,j})$ of $X$ (for $(i,j) \in \mathbb{Z}^2$) then $d_{\mathcal{C}X}(H_{0,0}, H_{i,j}) \leq 1$ but $d_{\Delta X}(H_{0,0}, H_{i,j}) = |i| + |j|$ for all $(i,j)$. Thus the inclusion $\Delta X \hookrightarrow \mathcal{C}X$ cannot be a quasi-isometry. Moreover, by Theorem A, we know that $\mathcal{C}X$ is a quasi-tree, whereas the inclusion into $\Delta X$ of the subgraph spanned by $\{H_{i,j} \mid (i,j) \in \mathbb{Z}^2\}$ (which is isomorphic to the 'grid' $\Gamma$) is isometric, and so $\Delta X$ cannot be hyperbolic – therefore, $\Delta X$ and $\mathcal{C}X$ are not quasi-isometric in this case.

It follows from [10, Proposition 8.11] that the usual action of $G_0$ on $X$ is special – however, there are infinitely many orbits of hyperplanes under this action. On the other hand, let $G = G_0 \rtimes \mathbb{Z}^2$, where the action of $\mathbb{Z}^2 = \langle x, y \mid xy = yx \rangle$ on $G_0$ is given by $a_{i,j}^{x^n y^m} = a_{i+n,j+m}$; this can be thought of as an example of a *graph-wreath product*, see

[20] for details. Then it is easy to see that the action of $G$ on $G_0$ extends to an action of $G$ on $X$. This action is transitive on hyperplanes, and therefore not special.

## 3.2 Hyperbolicity

We show here that $\mathcal{C}X$ is a quasi-tree, proving Theorem A.

**Proposition 3.5.** *Let $A, B \in V(\mathcal{C}X)$ be two hyperplanes such that $d_{\mathcal{C}X}(A,B) \geq 2$. Then there exists a midpoint $M$ of a geodesic between $A$ and $B$ in $\mathcal{C}X$ and a hyperplane $C$ separating $\mathcal{N}(A)$ and $\mathcal{N}(B)$ such that $d_{\mathcal{C}X}(M,C) \leq 3/2$.*

*Proof.* By Proposition 2.4, we know that $\mathcal{N}(A)$ and $\mathcal{N}(B)$ are gated. It then follows from [10, Lemma 2.36] that there exist vertices $p \in V(\mathcal{N}(A))$ and $q \in V(\mathcal{N}(B))$ such that any hyperplane separating $p$ from $q$ also separates $\mathcal{N}(A)$ from $\mathcal{N}(B)$. Let $A = A_0, \dots, A_m = B \in V(\Delta X)$ and $p = p_0, \dots, p_{m+1} = q \in V(X)$ be as given by Proposition 3.1 in the case $\Gamma = \mathcal{C}X$, and let $M$ be the midpoint of the former geodesic. It is clear that $\mathcal{N}(A_i) \cap \mathcal{N}(A_j) = \varnothing$ whenever $|i-j| \geq 2$; in particular, $p_i \neq p_{i+1}$ whenever $1 \leq i \leq m-1$.

Now let $i = \lfloor m/2 \rfloor \in \{1, \dots, m-1\}$, and let $C$ be any hyperplane separating $p_i$ and $p_{i+1}$. By the choice of the $p_j$, there exists a geodesic between $p$ and $q$ passing through $p_i$ and $p_{i+1}$: therefore, $C$ separates $p$ and $q$. Hence, by the choice of $p$ and $q$, $C$ also separates $\mathcal{N}(A)$ from $\mathcal{N}(B)$. Finally, note that as $C$ separates $p_i, p_{i+1} \in \mathcal{N}(A_i)$, we have $d_{\mathcal{C}X}(A_i, C) \leq 1$. Therefore,

$$d_{\mathcal{C}X}(M,C) \leq d_{\mathcal{C}X}(M,A_i) + d_{\mathcal{C}X}(A_i,C) = \left|\frac{m}{2} - i\right| + d_{\mathcal{C}X}(A_i,C) \leq \frac{1}{2} + 1 = \frac{3}{2},$$

as required. □

**Definition 3.6.** For a connected graph $\Gamma$ and two vertices $v, w \in V(\Gamma)$ we say a point $m \in \Gamma$ is a *midpoint* between $v$ and $w$ if $d_\Gamma(m,v) = d_\Gamma(m,w) = \frac{1}{2} d_\Gamma(v,w)$.

Let $D \in \mathbb{N}$. A connected graph $\Gamma$ is said to *satisfy the $D$-bottleneck criterion* if for any vertices $v, w \in V(\Gamma)$, there exists a midpoint $m$ between $v$ and $w$ such that any path between $v$ and $w$ passes through a point $p$ such that $d_\Gamma(p,m) \leq D$.

**Theorem 3.7** (Manning [21, Theorem 4.6])**.** *A connected graph $\Gamma$ is a quasi-tree if and only if there exists a constant $D$ such that $\Gamma$ satisfies the $D$-bottleneck criterion.* □

*Remark* 3.8. In [21], the statement of this theorem is given for a general geodesic metric space (not necessarily a graph), and the definition of bottleneck criterion given there is stronger: instead of taking $v, w$ to be vertices of $\Gamma$ in Definition 3.6, Manning allows $v, w$ to be any points of $\Gamma$. However, as any point in a graph is within distance $\frac{1}{2}$ of a vertex, it is easy to see that in our setting the definition given here is equivalent to the one given in [21] (up to possibly modifying the constant $D$).

*Proof of Theorem A.* We claim that $\mathcal{C}X$ satisfies the 5/2-bottleneck criterion.

Let $A, B \in V(\mathcal{C}X)$ be two hyperplanes. If $d_{\mathcal{C}X}(A,B) < 2$, then any path between $A$ and $B$ passes through $A$, and $d_{\mathcal{C}X}(A,M) = d_{\mathcal{C}X}(A,B)/2 < 1 < 5/2$ for any midpoint $M$ between $A$ and $B$, so the 5/2-bottleneck criterion is satisfied.

On the other hand, if $d_{\mathcal{C}X}(A,B) \geq 2$, then let $M$ and $C$ be as given by Proposition 3.5. Let $A = B_0, B_1, \ldots, B_n = B$ be any path in $\mathcal{C}X$ between $A$ and $B$, and choose vertices $q_1, \ldots, q_n \in V(X)$ such that $q_i \in \mathcal{N}(B_{i-1}) \cap \mathcal{N}(B_i)$ for all $i$. As $q_1 \in \mathcal{N}(A)$, $q_n \in \mathcal{N}(B)$, and as $C$ separates $A$ and $B$, it follows that $C$ separates $q_1$ and $q_n$, and so it separates $q_i$ and $q_{i+1}$ for some $i$. But as $q_i, q_{i+1} \in \mathcal{N}(B_i)$, it follows that $d_{\mathcal{C}X}(C, B_i) \leq 1$. In particular,

$$d_{\mathcal{C}X}(M, B_i) \leq d_{\mathcal{C}X}(M,C) + d_{\mathcal{C}X}(C,B_i) \leq \frac{3}{2} + 1 = \frac{5}{2},$$

and so again the 5/2-bottleneck criterion is satisfied.

In particular, Theorem 3.7 implies that $\mathcal{C}X$ is a quasi-tree. $\square$

# 4 Acylindricity

In this section we prove Theorem B (ii). To do this, in Section 4.1 we introduce the notion of contact sequences (see Definition 4.2) and show the main technical result we need to prove Theorem B (ii): namely, Proposition 4.3. In Section 4.2 we use this to deduce Theorem B (ii).

Throughout this section, let $X$ be a quasi-median graph.

## 4.1 Contact sequences

**Lemma 4.1.** *Let $Y \leq X$ be a gated subgraph and let $\mathcal{H}$ be a collection of hyperplanes in $X$. Let $Y'_{\mathcal{H}} \subseteq V(X)$ be the set of vertices $v \in V(X)$ for which there exists a vertex $p_v \in V(Y)$ such that all hyperplanes separating $v$ from $p_v$ are in $\mathcal{H}$. Then the full subgraph $Y_{\mathcal{H}}$ of $X$ spanned by $Y'_{\mathcal{H}}$ is gated.*

*Proof.* Suppose for contradiction that $Y_{\mathcal{H}}$ is not gated, and let $v \in V(X)$ be a vertex that does not have a gate in $Y_{\mathcal{H}}$. Let $p$ be the gate for $v$ in $Y$. Let $\hat{p}$ be a vertex of $Y_{\mathcal{H}}$ on a geodesic between $v$ and $p$ with $d_X(v,\hat{p})$ minimal. By our assumption, $\hat{p}$ is not a gate for $v$ in $Y_{\mathcal{H}}$, and so there exists a vertex $\hat{q} \in V(Y_{\mathcal{H}})$ such that no geodesic between $v$ and $\hat{q}$ passes through $\hat{p}$. Let $q$ be the gate of $\hat{q}$ in $Y$. Let $\gamma_p$, $\gamma_q$, $\delta$, $\hat{\delta}$, $\eta$ be geodesics between $\hat{p}$ and $p$, $\hat{q}$ and $q$, $p$ and $q$, $\hat{p}$ and $\hat{q}$, $v$ and $\hat{p}$ (respectively), as shown in Figure 3.4.

Since both $\eta$ and $\hat{\delta}$ are geodesics, and since $\eta\hat{\delta}$ is not (by the choice of $\hat{q}$), it follows from Lemma 2.9 that we may assume, without loss of generality, that there exists a hyperplane $C$ and edges $c_1$, $c_2$ of $\eta$, $\hat{\delta}$ (respectively), both of which are dual to $C$ and have $\hat{p}$ as an endpoint. But as $p$ is the gate for $v$ in $Y$, as $\eta\gamma_p$ is a geodesic by the choice of $\hat{p}$, and as $q \in Y$, it follows that $\eta\gamma_p\delta$ is a geodesic. Therefore, by Proposition 2.8 $H$ cannot cross $\gamma_p\delta$, and so $H$ does not separate $\hat{p}$ and $q$. As $H$ separates $\hat{p}$ and $\hat{q}$, it follows that $H$ separates $\hat{q}$ and $q$ and so crosses $\gamma_q$. In particular, since $\hat{q} \in V(Y_{\mathcal{H}})$ and since $q \in V(Y)$ is a gate for $\hat{q}$ in $Y$, it follows that all hyperplanes crossing $\gamma_q$ are in $\mathcal{H}$, and therefore $H \in \mathcal{H}$. But then the endpoint $p' \neq \hat{p}$ of $c_1$ is separated from $p \in V(Y)$ only by hyperplanes in $\mathcal{H}$; this contradicts the choice of $\hat{p}$. Thus $Y_{\mathcal{H}}$ is gated, as claimed. $\square$

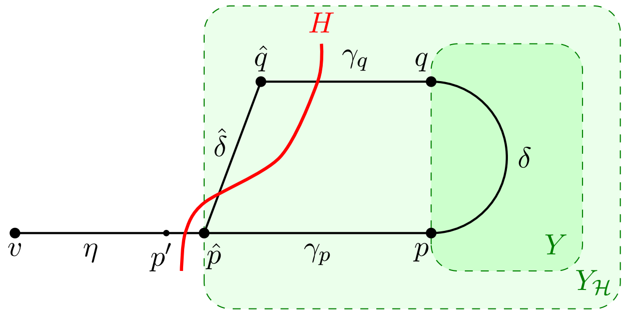



FIGURE 3.4: Proof of Lemma 4.1.

Now let a group $G$ act on a quasi-median graph $X$. This induces an action of $G$ on the crossing graph $\Delta X$. Let $\mathcal{H}$ be the set of orbits of vertices under $G \curvearrowright \Delta X$ – alternatively, the set of orbits of hyperplanes under $G \curvearrowright X$. We may regard each element of $\mathcal{H}$ as a collection of hyperplanes – thus, for instance, given $\mathcal{H}_0 \subseteq \mathcal{H}$ we may write $\bigcup \mathcal{H}_0$ for the set of all hyperplanes whose orbits are elements of $\mathcal{H}_0$.

Let $n \in \mathbb{N}$, and let $\mathcal{H}_1, \dots, \mathcal{H}_n$ be subsets of $\mathcal{H}$. Pick a vertex (a 'basepoint') $o \in V(X)$, and define the subgraphs $Y_0, \dots, Y_n \subseteq X$ inductively: set $Y_0 = \{o\}$ and $Y_i = (Y_{i-1})_{\bigcup \mathcal{H}_i}$ for $1 \leq i \leq n$. By Lemma 4.1, $Y_n$ is a gated subgraph. We denote $Y_n$ as above by $Y(o, \mathcal{H}_1, \dots, \mathcal{H}_n)$, and we denote the gate for $v \in V(X)$ in $Y_n$ by $\mathfrak{g}(v; o, \mathcal{H}_1, \dots, \mathcal{H}_n)$.

**Definition 4.2.** Let $H, H' \in V(\mathcal{C}X)$, and let $p, p' \in V(X)$ be such that $p \in \mathcal{N}(H)$ and $p' \in \mathcal{N}(H')$. Let $n = d_{\mathcal{C}X}(H, H')$. Given any geodesic $H = H_0, \dots, H_n = H'$ in $\mathcal{C}X$ and vertices $p = p_0, p_1, \dots, p_{n+1} = p' \in V(X)$ such that $p_i, p_{i+1} \in \mathcal{N}(H_i)$ for $0 \leq i \leq n$, we call $\mathfrak{S} = (H_0, \dots, H_n, p_0, \dots, p_{n+1})$ a *contact sequence* for $(H, H', p, p')$.

Given a contact sequence $\mathfrak{S} = (H_0, \dots, H_n, p_0, \dots, p_{n+1})$ for $(H, H', p, p')$ and a vertex $v \in V(X)$, we say $(g_0, \dots, g_n) \in V(X)^{n+1}$ is the *$v$-gate* for $\mathfrak{S}$ if $g_i$ is the gate for $v$ in $\mathcal{N}(H_i)$ for $0 \leq i \leq n$. We furthermore denote $(d_X(p_n, g_n), \dots, d_X(p_0, g_0))$ and $(d_X(p_1, g_0), \dots, d_X(p_{n+1}, g_n))$ by $\mathfrak{C}_{\diagup}(\mathfrak{S}, v)$ and $\mathfrak{C}_{\diagdown}(\mathfrak{S}, v)$, respectively. We say a contact sequence $\mathfrak{S}$ for $(H, H', p, p')$ is *$v$-minimal* if for any other contact sequence $\mathfrak{S}'$ for $(H, H', p, p')$ we have either $\mathfrak{C}_{\diagup}(\mathfrak{S}, v) \leq \mathfrak{C}_{\diagup}(\mathfrak{S}', v)$ or $\mathfrak{C}_{\diagdown}(\mathfrak{S}, v) \leq \mathfrak{C}_{\diagdown}(\mathfrak{S}', v)$ in the lexicographical order.

Finally, suppose a group $G$ acts on $X$. Given a vertex $v \in V(X)$ and a contact sequence $\mathfrak{S} = (H_0, \ldots, H_n, p_0, \ldots, p_{n+1})$ for $(H, H', p, p')$ with a $v$-gate $(g_0, \ldots, g_n)$, we say the tuple $(\mathcal{H}_0, \ldots, \mathcal{H}_n, \mathcal{H}'_0, \ldots, \mathcal{H}'_n)$, where $\mathcal{H}_i, \mathcal{H}'_i \subseteq V(\mathcal{C}X/G)$, is the $(v, G)$-*orbit sequence* for $\mathfrak{S}$ if

$$\mathcal{H}_i = \{H^G \mid H \in V(\mathcal{C}X) \text{ separates } p_i \text{ from } g_i\}$$

and

$$\mathcal{H}'_i = \{H^G \mid H \in V(\mathcal{C}X) \text{ separates } p_{i+1} \text{ from } g_i\}$$

for $0 \leq i \leq n$.

It is clear that given any $H$, $H'$, $p$ and $p'$ as in Definition 4.2, there exists a contact sequence for $(H, H', p, p')$. As the lexicographical order is a well-ordering of $\mathbb{N}^n$, it follows that a $v$-minimal contact sequence exists as well.

**Proposition 4.3.** *Suppose a group $G$ acts specially on a quasi-median graph $X$. Let $H, H' \in V(\mathcal{C}X)$, let $p, p' \in V(X)$ be such that $p \in \mathcal{N}(H)$ and $p' \in \mathcal{N}(H')$, and let $v \in V(X)$. Let $\mathfrak{S} = (H_0, \ldots, H_n, p_0, \ldots, p_{n+1})$ be a $v$-minimal contact sequence for $(H, H', p, p')$ with $v$-gate $(g_0, \ldots, g_n)$ and $(v, G)$-orbit sequence $(\mathcal{H}_0, \ldots, \mathcal{H}_n, \mathcal{H}'_0, \ldots, \mathcal{H}'_n)$. Write $\mathfrak{g}_i := \mathfrak{g}(v; p, \mathcal{H}_0, \ldots, \mathcal{H}_i)$ and $\mathfrak{g}'_i := \mathfrak{g}(v; p', \mathcal{H}'_n, \ldots, \mathcal{H}'_i)$ for $0 \leq i \leq n$. Then,*

(i) $\mathfrak{g}_n = \mathfrak{g}'_0$;

(ii) *no two hyperplanes in $\bigcup \mathcal{H}_i$ and $\bigcup \mathcal{H}'_j$ (respectively) osculate whenever $i > j$; and*

(iii) *for $1 \leq i \leq n$, the hyperplanes separating $\mathfrak{g}_{i-1}$ from $\mathfrak{g}_i$ (respectively $\mathfrak{g}'_i$ from $\mathfrak{g}'_{i-1}$) are precisely the hyperplanes separating $p_i$ from $g_i$ (respectively $p_i$ from $g_{i-1}$).*

*Proof.* Induction on $n$.

For $n = 0$, we claim that $\mathfrak{g}_0 = g_0$. Indeed, by definition of $\mathcal{H}_0$ we have $g_0 \in Y(p_0, \mathcal{H}_0)$, and so there exists a geodesic $\eta$ between $p$ and $v$ passing through $g_0$ and $\mathfrak{g}_0$. Suppose for contradiction $\mathfrak{g}_0 \neq g_0$, let $a \subseteq \eta$ be the edge with endpoint $g_0$ such that the other endpoint $q_a \neq g_0$ of $a$ satisfies $d_X(v, g_0) > d_X(v, q_a)$, and let $A$ be the hyperplane dual to $a$; see Figure 3.5(a). Then $g_0 \in \mathcal{N}(H_0) \cap \mathcal{N}(A)$, and so $H_0$ and $A$ either coincide, or intersect, or osculate. As $A$ separates $p$ and $\mathfrak{g}_0$, we know that $A^g$ separates $p$ and $g_0$ and so $A^g$ and $H_0$ either coincide or intersect for some $g \in G$. Thus, as the action $G \curvearrowright X$ is special, it follows that $A$ and $H_0$ cannot osculate, and therefore they either coincide or intersect. But then we also have $q_a \in \mathcal{N}(H_0)$, contradicting the choice of $g_0$. Therefore, $\mathfrak{g}_0 = g_0$, as claimed. A symmetric argument shows that $\mathfrak{g}'_0 = g_0$, and so the conclusion of the proposition is clear.

Suppose now that $n \geq 1$, and let $\hat{\mathfrak{g}}'_i = \mathfrak{g}(v; p_n, \mathcal{H}'_{n-1}, \ldots \mathcal{H}'_i)$ for $0 \leq i \leq n$ (so that $\hat{\mathfrak{g}}'_n = p_n$). Notice that $(H_0, \ldots, H_{n-1}, p_0, \ldots, p_n)$ is a $v$-minimal contact sequence for $(H, H_{n-1}, p, p_n)$. Thus, by the inductive hypothesis we have (i') $\mathfrak{g}_{n-1} = \hat{\mathfrak{g}}'_0$;

(ii') no two hyperplanes in $\bigcup \mathcal{H}_i$ and $\bigcup \mathcal{H}'_j$ (respectively) osculate whenever $n-1 \geq i > j$;

(iii') for $1 \leq i \leq n-1$, the hyperplanes separating $\mathfrak{g}_{i-1}$ from $\mathfrak{g}_i$ (respectively $\hat{\mathfrak{g}}'_i$ from $\hat{\mathfrak{g}}'_{i-1}$) are precisely the hyperplanes separating $p_i$ from $g_i$ (respectively $p_i$ from $g_{i-1}$).

Moreover, let $\hat{\mathfrak{g}}_i = \mathfrak{g}(v; p_1, \mathcal{H}_1, \ldots \mathcal{H}_i)$ for $0 \leq i \leq n$ (so that $\hat{\mathfrak{g}}_0 = p_1$), and notice that $(H_1, \ldots, H_n, p_1, \ldots, p_{n+1})$ is a $v$-minimal contact sequence for $(H_1, H', p_1, p')$. Thus, by the inductive hypothesis we have

(i") $\hat{\mathfrak{g}}_n = \mathfrak{g}'_1$;

(ii") no two hyperplanes in $\bigcup \mathcal{H}_i$ and $\bigcup \mathcal{H}'_j$ (respectively) osculate whenever $i > j \geq 1$;

(iii") for $2 \leq i \leq n$, the hyperplanes separating $\hat{\mathfrak{g}}_{i-1}$ from $\hat{\mathfrak{g}}_i$ (respectively $\mathfrak{g}'_i$ from $\mathfrak{g}'_{i-1}$) are precisely the hyperplanes separating $p_i$ from $g_i$ (respectively $p_i$ from $g_{i-1}$).

Finally, the proof of the $n = 0$ case above shows that $\mathfrak{g}(v; p_i, \mathcal{H}_i) = g_i = \mathfrak{g}(v; p_{i+1}, \mathcal{H}'_i)$ for $0 \leq i \leq n$.

Now let $q = \hat{\mathfrak{g}}_{n-1}$ and note that we also have $q = \hat{\mathfrak{g}}'_1$: this is clear if $n = 1$ and follows from the inductive hypothesis if $n \geq 2$. Let $\mathcal{A}, \mathcal{B}, \mathcal{A}', \mathcal{B}' \subseteq V(\Delta X)$ be the sets of hyperplanes separating $q$ from $\mathfrak{g}_{n-1}$, $q$ from $\mathfrak{g}'_1$, $\mathfrak{g}_{n-1}$ from $\mathfrak{g}_n$, $\mathfrak{g}'_1$ from $\mathfrak{g}_n$, respectively; see Figure 3.5(b). We claim that $\mathcal{A} = \mathcal{A}'$ and $\mathcal{B} = \mathcal{B}'$. We will show this in steps, proving part (ii) of the Proposition along the way.

**$\mathcal{A} \cap \mathcal{B} = \varnothing$:** Suppose for contradiction that there exists some hyperplane $A \in \mathcal{A} \cap \mathcal{B}$. As $A \in \mathcal{A}$, we know that $A$ separates $\hat{\mathfrak{g}}'_1$ from $\hat{\mathfrak{g}}'_0$, and so by (iii') above it also separates $p_1$ from $g_0$: thus $d_{\mathcal{C}X}(H_0, A) \leq 1$. Similarly, as $A \in \mathcal{B}$, by (iii") above we know that $A$ separates $p_n$ from $g_n$ and therefore $d_{\mathcal{C}X}(H_n, A) \leq 1$. Hence, $n = d_{\Delta X}(H_0, H_n) \leq 2$, and so either $n = 1$ or $n = 2$.

Let $\alpha$, $\beta$ be geodesics between $p_1$ and $g_0$, $p_n$ and $g_n$, respectively, and let $a \subseteq \alpha$ and $b \subseteq \beta$ be the edges dual to $A$. As $a$ and $b$ lie on geodesics with endpoint $v$, we may pick endpoints $q_a$ and $q_b$ of $a$ and $b$, respectively, such that $A$ does not separate $q_a$, $q_b$ and $v$.

Suppose first that $n = 2$: see Figure 3.5(c). Note that in this case $H_0, A, H_2$ is a geodesic in $\mathcal{C}X$ and that $d_X(p_2, g_2) > d_X(q_b, g_2)$ and $d_X(p_1, g_0) > d_X(q_a, g_0)$. Moreover, since $q_a$ lies on a geodesic between $p_1$ and $g_0$, we have $q_a \in \mathcal{N}(H_0)$; similarly, $q_b \in \mathcal{N}(H_2)$. Furthermore, by the construction we know that $q_a, q_b \in \mathcal{N}(A)$.

We may therefore replace $p_1$, $p_2$ and $H_1$ by $q_a$, $q_b$ and $A$, respectively, contradicting $v$-minimality of $\mathfrak{S}$. Thus $n \neq 2$.

Suppose now that $n = 1$. Then $A$ separates $p_1$ from both $g_0$ and $g_1$. By Lemma 2.9, we may then without loss of generality assume that $p_1$ is an endpoint (distinct from $q_a$ and $q_b$) of both $a$ and $b$. Now note that both $a$ and $b$ are edges on a geodesic between $p_1$ and $v$, so we must have $a = b$, and in particular $q_a = q_b$; see Figure 3.5(d). Since $A$ separates $p_1$ from both $g_0$ and $g_1$, it intersects or coincides with both $H_0$ and $H_1$, and so $q_a \in \mathcal{N}(H_0) \cap \mathcal{N}(H_1)$. We may therefore replace $p_n$ by $q_a$; but we have $d_X(p_1, g_1) > d_X(q_a, g_1)$ and $d_X(p_1, g_0) > d_X(q_a, g_0)$, contradicting $v$-minimality of $\mathfrak{S}$. Thus no such hyperplane $A \in \mathcal{A} \cap \mathcal{B}$ can exist and so $\mathcal{A} \cap \mathcal{B} = \varnothing$, as claimed.

**$\mathcal{A} \cap \mathcal{B}' = \varnothing$:** This is clear, as $\mathfrak{g}_{n-1} = \hat{\mathfrak{g}}_0'$ lies on a geodesic between $q = \hat{\mathfrak{g}}_1'$ and $\mathfrak{g}_n$, and so no hyperplane can separate $\mathfrak{g}_{n-1}$ from both $q$ and $\mathfrak{g}_n$.

**$\mathcal{A}' \cap \mathcal{B} = \varnothing$:** Let $\mathcal{C}$ be the set of hyperplanes separating $\mathfrak{g}_n$ and $v$. We first claim that $\mathcal{A}' \cap \mathcal{B} = \mathcal{A}' \cap \mathcal{C}$. Indeed, let $B \in \mathcal{A}' \cap \mathcal{B}$. Since $B \in \mathcal{B}$, it separates $q$ and $\mathfrak{g}_1'$; as $\mathfrak{g}_1' = \hat{\mathfrak{g}}_n$ lies on a geodesic between $q$ and $v$, $B$ cannot separate $\mathfrak{g}_1'$ and $v$. But as $B \in \mathcal{A}'$, it separates $\mathfrak{g}_1'$ and $\mathfrak{g}_n$, and so $B$ must separate $\mathfrak{g}_n$ and $v$. Therefore, $B \in \mathcal{A}' \cap \mathcal{C}$. Conversely, let $C \in \mathcal{A}' \cap \mathcal{C}$. Since $C \in \mathcal{C}$, it separates $\mathfrak{g}_n$ and $v$; as $\mathfrak{g}_n$ lies on a geodesic between $q$ and $v$, $C$ cannot separate $q$ and $\mathfrak{g}_n$. But as $C \in \mathcal{A}'$, it separates $\mathfrak{g}_1'$ and $\mathfrak{g}_n$, and so $C$ must separate $q$ and $\mathfrak{g}_1'$. Therefore, $C \in \mathcal{A}' \cap \mathcal{B}$, and so $\mathcal{A}' \cap \mathcal{B} = \mathcal{A}' \cap \mathcal{C}$, as claimed.

Now suppose for contradiction that there exists a hyperplane $A \in \mathcal{A}' \cap \mathcal{B} = \mathcal{A}' \cap \mathcal{B} \cap \mathcal{C}$. Let $\gamma$ be a geodesic between $\mathfrak{g}_n$ and $v$, and let $c \subseteq \gamma$ be the edge dual to $A$. By Lemma 2.9, we may without loss of generality assume that $\mathfrak{g}_n$ is an endpoint of $c$: see Figure 3.5(e).

Now let $q_c \neq \mathfrak{g}_n$ be the other endpoint of $c$. Note that since $A \in \mathcal{B}$, we have $A^G \in \mathcal{H}_n$. Therefore, it follows that $q_c$ is separated from $\mathfrak{g}_{n-1}$ only by hyperplanes in $\bigcup \mathcal{H}_n$; as $d_X(v, \mathfrak{g}_n) > d_X(v, q_c)$, this contradicts the definition of $\mathfrak{g}_n$. Thus $\mathcal{A} \cap \mathcal{B}' = \varnothing$, as claimed.

**$\mathcal{A} \subseteq \mathcal{A}'$ and $\mathcal{B} \subseteq \mathcal{B}'$:** As $\mathcal{A} \cap \mathcal{B} = \varnothing = \mathcal{A} \cap \mathcal{B}'$, every hyperplane separating $q$ and $\mathfrak{g}_{n-1}$ does not separate $q$ and $\mathfrak{g}_1'$, nor $\mathfrak{g}_{n-1}$ and $\mathfrak{g}_n$, thus it separates $\mathfrak{g}_1'$ and $\mathfrak{g}_n$. It follows that $\mathcal{A} \subseteq \mathcal{A}'$. Similarly, as $\mathcal{A} \cap \mathcal{B} = \varnothing = \mathcal{A}' \cap \mathcal{B}$, we get $\mathcal{B} \subseteq \mathcal{B}'$.

**Part (ii):** By (ii') and (ii") above, it is enough to show that no two hyperplanes in $\bigcup \mathcal{H}_n$ and $\bigcup \mathcal{H}_0'$ (respectively) osculate. Thus, let $A$ (respectively $B$) be a hyperplane separating $p_1$ and $g_0$ (respectively $p_n$ and $g_n$), so that $A^G \in \mathcal{H}_0'$ and $B^G \in \mathcal{H}_n$. It is now enough to show that $A^g$ and $B^h$ do not osculate for any $g, h \in G$.

But as $A$ separates $p_1$ from $g_0$, we know from (iii') that it also separates $\hat{\mathfrak{g}}_1' = q$ from $\hat{\mathfrak{g}}_0' = \mathfrak{g}_{n-1}$, that is, $A \in \mathcal{A}$. Similarly, as $B$ separates $p_n$ and $g_n$, we know from (iii”) that $B \in \mathcal{B}$. But as $\mathcal{A} \cap \mathcal{B} = \varnothing = \mathcal{A} \cap \mathcal{B}'$ and as $\mathcal{B} \subseteq \mathcal{B}'$, it follows that $A$ separates $q$ and $\mathfrak{g}_1'$ from $\mathfrak{g}_{n-1}$ and $\mathfrak{g}_n$, while $B$ separates $q$ from $\mathfrak{g}_1'$ and $\mathfrak{g}_{n-1}$ from $\mathfrak{g}_n$. Therefore, $A$ and $B$ must intersect. But as the action $G \curvearrowright X$ is special, it follows that $A^g$ and $B^h$ do not osculate for any $g, h \in G$. Thus no two hyperplanes in $\bigcup \mathcal{H}_n$ and $\bigcup \mathcal{H}_0'$ (respectively) osculate, and so part (ii) holds, as required.

**$\mathcal{A}' \cap \mathcal{B}' = \varnothing$:** Suppose for contradiction that $A \in \mathcal{A}' \cap \mathcal{B}'$ is a hyperplane. Let $\alpha'$ be a geodesic between $\mathfrak{g}_1'$ and $\mathfrak{g}_n$, let $a \subseteq \alpha'$ be the edge dual to $A$, and let $q_a, q_a'$ be the endpoints of $a$ so that $A$ does not separate $\mathfrak{g}_1'$ and $q_a$. Suppose, without loss of generality, that $\alpha'$ and $A$ are chosen in such a way that $d_X(\mathfrak{g}_1', q_a)$ is as small as possible.

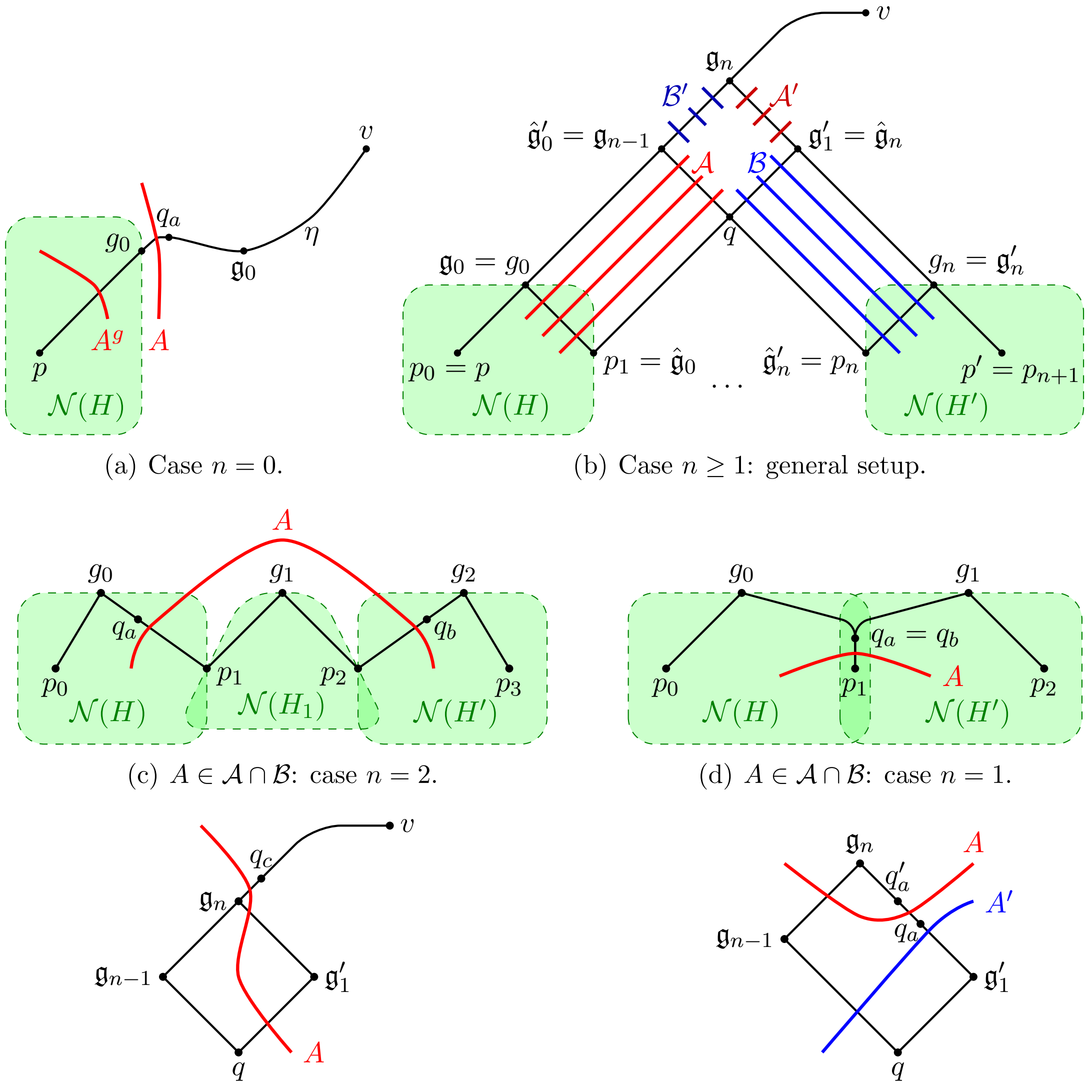
We now claim that $\mathfrak{g}_1' = q_a$. Indeed, suppose not, and let $a' \neq a$ be the other edge on $\alpha'$ with endpoint $q_a$. Let $A' \in \mathcal{A}'$ be the hyperplane dual to $a'$; see Figure 3.5(f). Then $A'$ does not separate $q$ and $\mathfrak{g}_1'$ (as $\mathcal{A}' \cap \mathcal{B} = \varnothing$), nor $\mathfrak{g}_{n-1}$ and $\mathfrak{g}_n$ (by minimality of $d_X(\mathfrak{g}_1', q_a)$), but it separates $\mathfrak{g}_1'$ (and so $q$) from $\mathfrak{g}_n$ (and so $\mathfrak{g}_{n-1}$). In particular, $A' \in \mathcal{A}$, and so $A' \in \bigcup \mathcal{H}_0'$. On the other hand, $A \in \mathcal{B}' \subseteq \bigcup \mathcal{H}_n$, and so $A$ and $A'$ cannot oscullate by part (ii). It follows that $A$ and $A'$ must intersect, and therefore we may swap $a$ and $a'$ on $\alpha'$, contradicting minimality of $d_X(\mathfrak{g}_1', q_a)$. Thus $\mathfrak{g}_1' = q_a$, as claimed.

But now $q_a'$ is separated from $q$ just by hyperplanes in $\bigcup \mathcal{H}_n$. Furthermore, $A$ cannot separate $\mathfrak{g}_n$ and $v$ (as $\mathfrak{g}_n$ lies on a geodesic between $\mathfrak{g}_{n-1}$ and $v$, and as $A$ separates $\mathfrak{g}_{n-1}$ and $\mathfrak{g}_n$), nor $\mathfrak{g}_n$ and $q_a'$ (as $\alpha'$ is a geodesic), but $A$ separates $q_a'$ (and so $\mathfrak{g}_n$ and $v$) from $\mathfrak{g}_1'$. In particular, $d_X(v, \mathfrak{g}_1') > d_X(v, q_a')$, contradicting the fact that $\mathfrak{g}_1' = \hat{\mathfrak{g}}_n$. Thus $\mathcal{A}' \cap \mathcal{B}' = \varnothing$, as required.

**$\mathcal{A} = \mathcal{A}'$ and $\mathcal{B} = \mathcal{B}'$:** We have already shown $\mathcal{A} \subseteq \mathcal{A}'$ and $\mathcal{B} \subseteq \mathcal{B}'$. Conversely, as $\mathcal{A}' \cap \mathcal{B} = \varnothing = \mathcal{A}' \cap \mathcal{B}'$, every hyperplane separating $\mathfrak{g}_1'$ and $\mathfrak{g}_n$ does not separate $q$ and $\mathfrak{g}_1'$, nor $\mathfrak{g}_{n-1}$ and $\mathfrak{g}_n$, thus it separates $q$ and $\mathfrak{g}_{n-1}$. It follows that $\mathcal{A}' \subseteq \mathcal{A}$ and so $\mathcal{A} = \mathcal{A}'$. Similarly, as $\mathcal{A} \cap \mathcal{B}' = \varnothing = \mathcal{A}' \cap \mathcal{B}'$, we get $\mathcal{B} = \mathcal{B}'$.

Now part (iii) of the Proposition follows immediately. Indeed, given (iii’) and (iii”), it is enough to show that the hyperplanes separating $\mathfrak{g}_{n-1}$ from $\mathfrak{g}_n$ (respectively $\mathfrak{g}_1'$ from $\mathfrak{g}_0'$) are precisely the hyperplanes separating $\hat{\mathfrak{g}}_{n-1}$ from $\hat{\mathfrak{g}}_n$ (respectively $\hat{\mathfrak{g}}_1'$ from $\hat{\mathfrak{g}}_0'$). But this, and so (iii), follows from the fact that $\mathcal{A} = \mathcal{A}'$ and $\mathcal{B} = \mathcal{B}'$.

Finally, we are left to show part (i). We know that $\mathcal{A}' = \mathcal{A} \subseteq \bigcup \mathcal{H}_0'$, and so $\mathfrak{g}_n \in Y(\mathfrak{g}_1', \mathcal{H}_0') \subseteq Y(p', \mathcal{H}_n', \ldots, \mathcal{H}_0')$. In particular, there exists a geodesic between $v$ and $\mathfrak{g}_n$ passing through $\mathfrak{g}(v; p', \mathcal{H}_n', \ldots, \mathcal{H}_0') = \mathfrak{g}_0'$. But a symmetric argument can show that there exists a geodesic between $v$ and $\mathfrak{g}_0'$ passing through $\mathfrak{g}_n$. Thus $\mathfrak{g}_n = \mathfrak{g}_0'$, proving (i). $\square$




(a) Case $n = 0$.

(b) Case $n \geq 1$: general setup.

(c) $A \in \mathcal{A} \cap \mathcal{B}$: case $n = 2$.

(d) $A \in \mathcal{A} \cap \mathcal{B}$: case $n = 1$.

(e) $A \in \mathcal{A}' \cap \mathcal{B}$.

(f) $A \in \mathcal{A}' \cap \mathcal{B}'$.

FIGURE 3.5: Proof of Proposition 4.3.

## 4.2 Consequences of Proposition 4.3

**Corollary 4.4.** *Suppose a group $G$ acts specially on a quasi-median graph $X$. Let $H, H', K, K' \in V(\mathcal{C}X)$, and let $p, p', v, v' \in V(X)$ be such that $p \in \mathcal{N}(H)$, $p' \in \mathcal{N}(H')$, $v \in \mathcal{N}(K)$ and $v' \in \mathcal{N}(K')$. Suppose that $d_{\mathcal{C}X}(H, K) \geq d_{\mathcal{C}X}(H, H') + d_{\mathcal{C}X}(K, K') + 3$.*

*If $\mathfrak{S}$ is a $v$-minimal contact sequence for $(H, H', p, p')$, then $\mathfrak{S}$ is also $v'$-minimal. Furthermore, if $(\mathcal{H}_0, \ldots, \mathcal{H}_n, \mathcal{H}'_0, \ldots, \mathcal{H}'_n)$ is the $(v, G)$-orbit sequence for $\mathfrak{S}$, then we have $\mathfrak{g}(v; p, \mathcal{H}_0, \ldots, \mathcal{H}_n) = \mathfrak{g}(v'; p, \mathcal{H}_0, \ldots, \mathcal{H}_n)$.*

*Proof.* Let $m = d_{\mathcal{C}X}(K, K')$, and let $K = K_0, \ldots, K_m = K'$ be a geodesic in $\Delta X$. For $1 \leq i \leq m$, choose a vertex $v_i \in \mathcal{N}(K_{i-1}) \cap \mathcal{N}(K_i)$; let also $v_0 = v$ and $v_{m+1} = v'$. Let $n = d_{\mathcal{C}X}(H, H')$.

Given a contact sequence $\mathfrak{S} = (H_0, \ldots, H_n, p_0, \ldots, p_{n+1})$ for $(H, H', p, p')$ and any $v \in V(X)$, the tuples $\mathfrak{C}_{\diagup}(\mathfrak{S}, v)$ and $\mathfrak{C}_{\diagdown}(\mathfrak{S}, v)$ only depend on the gates for $v$ in the $\mathcal{N}(H_i)$, $0 \leq i \leq n$. In particular, if for all hyperplanes $A \in V(\mathcal{C}X)$ with $d_{\Delta X}(H, A) \leq n$ the gates for $v$ and $v'$ in $\mathcal{N}(A)$ coincide, then the set of $v$-minimal contact sequences for $(H, H', p, p')$ coincides with the set of $v'$-minimal ones.

Thus, let $A \in V(\mathcal{C}X)$ be a hyperplane with $d_{\Delta X}(H, A) \leq n$, and suppose for contradiction that $g \neq g'$, where $g$ and $g'$ are the gates for $v$ and $v'$ (respectively) in $\mathcal{N}(A)$. Let $B$ be a hyperplane separating $g$ and $g'$. Since $B$ separates two points in $\mathcal{N}(A)$, we must have $d_{\mathcal{C}X}(A, B) \leq 1$, and so $d_{\mathcal{C}X}(H, B) \leq n + 1$. On the other hand, as $B$ separates the gates of $v$ and $v'$ in a gated subgraph, $B$ must also separate $v = v_0$ and $v' = v_{m+1}$. Thus $B$ must separate $v_i$ and $v_{i+1}$ for some $i \in \{0, \ldots, m\}$. As $v_i, v_{i+1} \in \mathcal{N}(K_i)$, it follows that $d_{\mathcal{C}X}(B, K_i) \leq 1$. In particular, $d_{\mathcal{C}X}(B, K) \leq d_{\mathcal{C}X}(B, K_i) + d_{\mathcal{C}X}(K_i, K) \leq i + 1 \leq m + 1$. But then $d_{\mathcal{C}X}(H, K) \leq d_{\mathcal{C}X}(H, B) + d_{\mathcal{C}X}(B, K) \leq n + m + 2$, contradicting our assumption. Thus we must have $g = g'$, and so the set of $v$-minimal contact sequences for $(H, H', p, p')$ coincides with the set of $v'$-minimal ones. In particular, $\mathfrak{S}$ is a $v'$-minimal structural sequence for $(H, H', p, p')$, and so the conclusion of Proposition 4.3 holds if $v$ is replaced by $v'$ as well.

Now suppose for contradiction that the vertices $\mathfrak{g}_n(v) = \mathfrak{g}(v; p, \mathcal{H}_0, \ldots, \mathcal{H}_n)$ and $\mathfrak{g}_n(v') = \mathfrak{g}(v'; p, \mathcal{H}_0, \ldots, \mathcal{H}_n)$ do not coincide. Let $B$ be a hyperplane separating $\mathfrak{g}_n(v)$ from $\mathfrak{g}_n(v')$. Then $B$ separates gates for $v$ and $v'$ in a gated subgraph, and so as above we get $d_{\mathcal{C}X}(B, K) \leq m + 1$. On the other hand, since $B$ separates $\mathfrak{g}_n(v)$ from $\mathfrak{g}_n(v')$, it follows that $B$ separates $p$ from either $\mathfrak{g}_n(v)$ or $\mathfrak{g}_n(v')$: without loss of generality, suppose the former. Then $B$ must separate $\mathfrak{g}(v; p, \mathcal{H}_0, \ldots, \mathcal{H}_{j-1})$ and $\mathfrak{g}(v; p, \mathcal{H}_0, \ldots, \mathcal{H}_j)$ for some $j \in \{0, \ldots, n\}$. By Proposition 4.3 (iii), it then follows that $B$ separates $p_j$ from $g_j$, and so $d_{\mathcal{C}X}(B, H_j) \leq 1$; in particular, $d_{\mathcal{C}X}(H, B) \leq d_{\mathcal{C}X}(H, H_j) + d_{\mathcal{C}X}(H_j, B) \leq j + 1 \leq n + 1$. Therefore, $d_{\mathcal{C}X}(H, K) \leq d_{\mathcal{C}X}(H, B) + d_{\mathcal{C}X}(B, K) \leq n + m + 2$, again contradicting our assumption. Thus we must have $\mathfrak{g}_n(v) = \mathfrak{g}_n(v')$, as required. □

**Lemma 4.5.** *Suppose $G$ acts specially on $X$. Let $D \in \mathbb{N}$, and suppose every vertex of $\Delta X/G$ has at most $D$ neighbours. If $v, w \in V(X)$, then there exist at most $(D + 1)^2$ hyperplanes $H \in V(\mathcal{C}X)$ such that $w \in \mathcal{N}(H)$ and $w$ is not the gate for $v$ in $\mathcal{N}(H)$.*

*Proof.* Let $\mathcal{U} \subseteq V(X)$ be the set of vertices $u \in V(X)$ such that $d_X(u, w) = 1$ and $d_X(v, w) = d_X(v, u) + 1$. We claim that $|\mathcal{U}| \leq D + 1$. Indeed, suppose there exist $k$ distinct vertices $u_1, \ldots, u_k \in \mathcal{U}$, and let $H_i$ be the hyperplane separating $w$ and $u_i$ for $1 \leq i \leq k$. It is clear that $H_i \neq H_j$ whenever $i \neq j$: indeed, if $H_i = H_j = H$ then by Proposition 2.8 $H$ cannot separate $v$ from either $u_i$ or $u_j$, and therefore $u_i = u_j$, hence $i = j$. Since $w \in \mathcal{N}(H_i) \cap \mathcal{N}(H_j)$ for every $i, j$ and since the action $G \curvearrowright X$ is special, it also follows that $H_i^G \neq H_j^G$ whenever $i \neq j$.

We now claim that $H_i$ and $H_j$ intersect for every $i \neq j$. Indeed, $H_i$ cannot separate

$u_i$ from $v$ (by Proposition 2.8), nor $w$ from $w_j$ (as $H_i \neq H_j$), but it does separate $w$ (and so $u_j$) from $u_i$ (and so $v$). On the other hand, a symmetric argument shows that $H_j$ separates $w$ and $u_i$ from $u_j$ and $v$. Thus $H_i$ and $H_j$ must intersect, as claimed. Therefore, $d_{\Delta X}(H_i, H_j) = 1$ and so, as $H_i^G \neq H_j^G$, we have $d_{\Delta X/G}(H_i^G, H_j^G) = 1$. In particular, $\{H_1^G, \ldots, H_k^G\}$ are $k$ vertices of a clique in $\Delta X/G$, and so by our assumption it follows that $k \leq D+1$. Thus $|\mathcal{U}| \leq k$, as claimed.

Now let $u \in \mathcal{U}$, and let $\mathcal{H} \subseteq V(\mathcal{C}X)$ be the set of hyperplanes $H \in V(\mathcal{C}X)$ such that $u, w \in \mathcal{N}(H)$. It is then enough to show that $|\mathcal{H}| \leq D+1$. Thus, let $H_1, H_2, \ldots, H_k \in \mathcal{H}$ be $k$ distinct hyperplanes, where $H_1$ is the hyperplane separating $u$ and $w$. As $w \in \mathcal{N}(H_i) \cap \mathcal{N}(H_j)$ for every $i,j$ and as $G \curvearrowright X$ is special, it is clear that $H_i^G \neq H_j^G$ for any $i \neq j$. Furthermore, it is clear (see, for instance, Proposition 2.4) that $H_1$ and $H_j$ intersect for every $j \neq 1$. In particular, $d_{\Delta X}(H_1, H_j) = 1$, and so $d_{\Delta X/G}(H_1^G, H_j^G) = 1$. As by assumption $H_1^G$ has at most $D$ neighbours in $\Delta X/G$, it follows that $k \leq D+1$, and so $|\mathcal{H}| \leq D+1$, as required. □

**Theorem 4.6.** *Suppose a group $G$ acts specially on a quasi-median graph $X$, and suppose there exists some $D \in \mathbb{N}$ such that $|\operatorname{Stab}_G(w)| \leq D$ for any $w \in V(X)$ and any vertex of $\Delta X/G$ has at most $D$ neighbours. Then the induced action $G \curvearrowright \mathcal{C}X$ is acylindrical, and the acylindricity constants $D_\varepsilon$ and $N_\varepsilon$ can be expressed as functions of $\varepsilon$ and $D$ only.*

*Proof.* Let $\varepsilon \in \mathbb{N}$. We claim that the acylindricity condition in Definition 1.2 is satisfied for $D_\varepsilon = 2\varepsilon + 6$ and $N_\varepsilon = N^{2(\varepsilon+3)} D/(N-1)^2$, where $N = (D+1)^2 2^{D+1}$.

Indeed, let $h, k \in \Delta X$ be such that $d_{\mathcal{C}X}(h,k) \geq D_\varepsilon$. Let $H, K \in V(\mathcal{C}X)$ be hyperplanes such that $d_{\mathcal{C}X}(H,h) \leq 1/2$ and $d_{\mathcal{C}X}(K,k) \leq 1/2$, and note that we have $d_{\mathcal{C}X}(H,K) \geq D_\varepsilon - 1 = 2\varepsilon + 5$. Let $\mathcal{G}_\varepsilon(h,k) = \{g \in G \mid d_{\mathcal{C}X}(h,h^g) \leq \varepsilon, d_{\mathcal{C}X}(k,k^g) \leq \varepsilon\}$, and note that we have $\mathcal{G}_\varepsilon(h,k) \subseteq \mathcal{G}_{\varepsilon+1}(H,K)$. We thus aim to show that $|\mathcal{G}_{\varepsilon+1}(H,K)| \leq N_\varepsilon$.

Pick vertices $v \in \mathcal{N}(K)$ and $p \in \mathcal{N}(H)$, and an element $g \in \mathcal{G}_{\varepsilon+1}(H,K)$. Suppose that $\mathfrak{S} = (H_0, \ldots, H_n, p_0, \ldots, p_{n+1})$ is a $v$-minimal contact sequence for $(H, H^g, p, p^g)$ with $v$-gate $(g_0, \ldots, g_n)$ and $(v,G)$-orbit sequence $(\mathcal{H}_0, \ldots, \mathcal{H}_n, \mathcal{H}_0', \ldots, \mathcal{H}_n')$; as $g \in \mathcal{G}_{\varepsilon+1}(H,K)$, we have $n \leq \varepsilon + 1$. For $0 \leq i \leq n$, set $\mathfrak{g}_i = \mathfrak{g}(v; p, \mathcal{H}_0, \ldots, \mathcal{H}_i)$ and $\mathfrak{g}_i' = \mathfrak{g}(v; p^g, \mathcal{H}_n', \ldots, \mathcal{H}_i')$; let also $\mathfrak{g}_{-1} = p$ and $\mathfrak{g}_{n+1}' = p^g$.

We first claim that there exist hyperplanes $A_0, \ldots, A_n \in V(\mathcal{C}X)$ such that $\mathfrak{g}_{i-1}, \mathfrak{g}_i \in \mathcal{N}(A_i)$ for each $i$. Indeed, this is clear if $g_i = p_{i+1}$ for each $i$, as in that case we may simply take $A_i = H_i$ for each $i$. Otherwise, let $k \in \{0, \ldots, n\}$ be minimal such that $g_k \neq p_{k+1}$, and let $A$ be a hyperplane separating $g_k$ and $p_{k+1}$ such that $g_k \in \mathcal{N}(A)$. For $0 \leq i \leq k-1$ we may take $A_i = H_i$, while for $k \leq i \leq n$ we can show (by induction on $i$, say) that $\mathfrak{g}_i \in \mathcal{N}(A)$. Indeed, the base case ($i = k$) is clear by construction; and if $\mathfrak{g}_{i-1} \in \mathcal{N}(A)$ for some $i > k$ and $\mathfrak{g}_{i-1} = q_0, \ldots, q_m = \mathfrak{g}_i$ is a geodesic in $X$, then $A$ cannot osculate with the hyperplane separating $q_{j-1}$ and $q_j$ by Proposition 4.3 (ii) and (iii), and so $q_j \in \mathcal{N}(A)$ by induction on $j$. Thus we may take $A_i = A$ for $k \leq i \leq n$, so that $\mathfrak{g}_{i-1}, \mathfrak{g}_i \in \mathcal{N}(A_i)$ for each $i$, as claimed. A symmetric argument shows that there exist hyperplanes $B_n, \ldots, B_0 \in V(\mathcal{C}X)$ such that $\mathfrak{g}'_{i+1}, \mathfrak{g}'_i \in \mathcal{N}(B_i)$ for each $i$.

Now, we may pass the sequence $(\mathfrak{g}_{-1}, \ldots, \mathfrak{g}_n)$ to a subsequence $(\mathfrak{g}_{k_0}, \ldots, \mathfrak{g}_{k_a})$ by removing those $\mathfrak{g}_i$ for which $\mathfrak{g}_{i-1} = \mathfrak{g}_i$. It then follows that $\mathfrak{g}_{k_{i-1}} \neq \mathfrak{g}_{k_i}$ and that $\mathfrak{g}_{k_{i-1}}, \mathfrak{g}_{k_i} \in \mathcal{N}(A_{k_i})$ for $1 \leq i \leq a$, where $a \leq n+1 \leq \varepsilon+2$. Similarly, we may pass the sequence $(\mathfrak{g}'_{n+1}, \ldots, \mathfrak{g}'_0)$ to a subsequence $(\mathfrak{g}_{k'_0}, \ldots, \mathfrak{g}_{k'_b})$ such that $\mathfrak{g}_{k'_{i-1}} \neq \mathfrak{g}_{k'_i}$ and that $\mathfrak{g}_{k'_{i-1}}, \mathfrak{g}'_{k_i} \in \mathcal{N}(B_{k_i})$ for $1 \leq i \leq b$, where $b \leq n+1 \leq \varepsilon+2$.

Now as $d_{\mathcal{C}X}(H, H^g) + d_{\mathcal{C}X}(K, K^g) + 3 \leq 2(\varepsilon+1)+3 = 2\varepsilon+5 \leq d_{\mathcal{C}X}(H, K)$, it follows from Corollary 4.4 that $\mathfrak{S}$ is also a $v^g$-minimal contact sequence and that $\mathfrak{g}(v; p, \mathcal{H}_0, \ldots, \mathcal{H}_n) = \mathfrak{g}(v^g; p, \mathcal{H}_0, \ldots, \mathcal{H}_n)$. Therefore, by Proposition 4.3 (i) and the discussion above,

$$\begin{aligned} \mathfrak{g}(v; p, \mathcal{H}_{k_1}, \ldots, \mathcal{H}_{k_a}) &= \mathfrak{g}(v; p, \mathcal{H}_0, \ldots, \mathcal{H}_n) = \mathfrak{g}(v^g; p, \mathcal{H}_0, \ldots, \mathcal{H}_n) \\ &= \mathfrak{g}(v^g; p^g, \mathcal{H}'_n, \ldots, \mathcal{H}'_0) = \mathfrak{g}(v; p, \mathcal{H}'_n, \ldots, \mathcal{H}'_0)^g \\ &= \mathfrak{g}(v; p, \mathcal{H}'_{k'_1}, \ldots, \mathcal{H}'_{k'_b})^g. \end{aligned} \tag{4.1}$$

As the stabiliser of $\mathfrak{g}(v; p, \mathcal{H}_{k_1}, \ldots, \mathcal{H}_{k_a})$ has cardinality $\leq D$, it follows that, given any subsets $\mathcal{H}_{k_1}, \ldots, \mathcal{H}_{k_a}, \mathcal{H}'_{k'_1}, \ldots, \mathcal{H}'_{k'_b} \subseteq V(\mathcal{C}X/G)$, there are at most $D$ elements $g \in G$ satisfying (4.1). But as $\mathfrak{g}_{k_{i-1}} \neq \mathfrak{g}_{k_i}$, as $\mathfrak{g}_{k_i}$ lies on a geodesic between $\mathfrak{g}_{k_{i-1}}$ and $v$, and as $\mathfrak{g}_{k_{i-1}}, \mathfrak{g}_{k_i} \in \mathcal{N}(A_i)$, it follows from Lemma 4.5 that there are at most $(D+1)^2$ possible choices for $A_{k_i}$ (for $1 \leq i \leq a$). Moreover, given a choice of $A_{k_i}$, as $\mathcal{H}_{k_i} \subseteq \operatorname{star}_{\Delta X/G}(A^G_{k_i})$ and by assumption $|\operatorname{star}_{\Delta X/G}(A^G_{k_i})| \leq D+1$, there exist at most $2^{D+1}$ choices for $\mathcal{H}_{k_i}$. It follows that there exist at most $N^a$ choices for the subsets $\mathcal{H}_{k_1}, \ldots, \mathcal{H}_{k_a} \subseteq V(\mathcal{C}X/G)$, where $N = (D+1)^2 2^{D+1}$; similarly, there exist at most $N^b$ choices for the subsets $\mathcal{H}'_{k'_1}, \ldots, \mathcal{H}'_{k'_b} \subseteq V(\mathcal{C}X/G)$. In particular,

$$|\mathcal{G}_{\varepsilon+1}(H, K)| \leq D \left( \sum_{a=0}^{\varepsilon+2} N^a \right) \left( \sum_{b=0}^{\varepsilon+2} N^b \right) < D \left( \frac{N^{\varepsilon+3}}{N-1} \right)^2 = N_\varepsilon,$$

as required. □

## 5 Application to graph products

We use this section to deduce results about graph products from Theorems A and B: namely, we show Corollary C in Section 5.1 and Corollary D in Section 5.2. Throughout this section, let $\Gamma$ be a simplicial graph, let $\mathcal{G} = \{G_v \mid v \in V(\Gamma)\}$ be a collection of non-trivial groups, and let $X$ be the quasi-median graph associated to $\Gamma\mathcal{G}$, as given by Theorem 1.3. We will use the following result.

**Theorem 5.1** (Genevois [10, Section 8.1]; Genevois–Martin [13, Theorem 2.13]). *For $v \in V(\Gamma)$, let $H_v$ be the hyperplane dual to the clique $G_v \subseteq X$. Then any hyperplane $H$*

*in $X$ is of the form $H_v^g$ for some $v \in V(\Gamma)$ and $g \in \Gamma\mathcal{G}$. Moreover, the vertices in $\mathcal{N}(H)$ are precisely $\Gamma_{\mathrm{star}(v)}\mathcal{G}_{\mathrm{star}(v)}g \subseteq V(X)$.*

*Remark* 5.2. Due to our convention to consider only right actions, the Cayley graph $X = \mathrm{Cay}(\Gamma\mathcal{G}, S)$ defined in Theorem 1.3 is the *left* Cayley graph: for $s \in S$ and $g \in \Gamma\mathcal{G}$, an edge labelled $s$ joins $g \in V(X)$ to $sg \in V(X)$. Therefore, contrary to the convention in [10] and [13], the vertices in the carrier of a hyperplane will form a *right* coset of $\Gamma_{\mathrm{star}(v)}\mathcal{G}_{\mathrm{star}(v)}$ for some $v \in V(\Gamma)$.

### 5.1 Acylindrical hyperbolicity

Here we prove Corollary C. It is clear from Theorem 1.3 that we may apply Theorems A and B to the quasi-median graph $X$ associated to a graph product $\Gamma\mathcal{G}$. In particular, it follows that the contact graph $\mathcal{C}X$ is a quasi-tree and $\Gamma\mathcal{G}$ acts on it acylindrically. We thus only need to show that, given that $|V(\Gamma)| \geq 2$ and the complement $\Gamma^C$ of $\Gamma$ is connected, the action $\Gamma\mathcal{G} \curvearrowright \mathcal{C}X$ is non-elementary.

**Lemma 5.3.** *Let $H$ be a hyperplane in $X$. Then the following are equivalent:*

(i) *$\mathcal{C}X$ is unbounded;*

(ii) *$\Gamma^C$ is connected and $|V(\Gamma)| \geq 2$.*

*Proof.* We first show (i) $\Rightarrow$ (ii). Indeed, if $\Gamma$ is a single vertex $v$, then $X$ is a single clique and so $\mathcal{C}X$ is a single vertex. On the other hand, if $\Gamma^C$ is disconnected, then we have a partition $V(\Gamma) = A \sqcup B$ where $A$ and $B$ are adjacent and non-empty. In particular, $\Gamma\mathcal{G} = \Gamma_A\mathcal{G}_A \times \Gamma_B\mathcal{G}_B$, and so any vertex $g \in \Gamma\mathcal{G}$ of $X$ can be expressed as $g = g_A g_B$ for some $g_A \in \Gamma_A\mathcal{G}_A$ and $g_B \in \Gamma_B\mathcal{G}_B$. Thus, if $H \in V(\mathcal{C}X)$ then by Theorem 5.1, $\mathcal{N}(H) = \Gamma_{\mathrm{star}(v)}\mathcal{G}_{\mathrm{star}(v)}g_A g_B$ for some $g_A \in \Gamma_A\mathcal{G}_A$, $g_B \in \Gamma_B\mathcal{G}_B$ and $v \in V(\Gamma)$: without loss of generality, suppose $v \in A$. Then $g_B \in \Gamma_B\mathcal{G}_B \leq \Gamma_{\mathrm{star}(v)}\mathcal{G}_{\mathrm{star}(v)}$ and $g_A \in \Gamma_A\mathcal{G}_A \leq \Gamma_{\mathrm{star}(u)}\mathcal{G}_{\mathrm{star}(u)}$ for any $u \in B$, and so $g_A \in \mathcal{N}(H) \cap \mathcal{N}(H_u)$; therefore, $d_{\mathcal{C}X}(H, H_u) \leq 1$. Since $1 \in \mathcal{N}(H_u) \cap \mathcal{N}(H_v)$ and so $d_{\mathcal{C}X}(H_u, H_v) \leq 1$ for any $u, v \in V(\Gamma)$, it follows that $d_{\mathcal{C}X}(H, H') \leq 3$ for any $H, H' \in \mathcal{C}X$ and so $\mathcal{C}X$ is bounded, as required.

To show (ii) $\Rightarrow$ (i), suppose that $\Gamma$ is a graph with at least 2 vertices and connected complement. Thus, there exists a closed walk $(v_0, v_1, \ldots, v_\ell)$ on the complement of $\Gamma$ that visits every vertex – in particular, we have $v_i \in V(\Gamma)$ with $v_\ell = v_0$ and $v_{i-1} \neq v_i$, $(v_{i-1}, v_i) \notin E(\Gamma)$ for $1 \leq i \leq \ell$. Pick arbitrary non-identity elements $g_i \in G_{v_i}$ for $i = 1, \ldots, \ell$, and consider the element $g = g_1 \cdots g_\ell \in \Gamma\mathcal{G}$.

Now let $n \in \mathbb{N}$, and let $A, B \in V(\mathcal{C}X)$ be such that $1 \in \mathcal{N}(A)$ and $g^n \in \mathcal{N}(B)$. Let $A = A_0, \ldots, A_m = B$ be the geodesic in $\mathcal{C}X$ and let $1 = p_0, \ldots, p_{m+1} = g^n$ be the vertices in $X$ given by Proposition 3.1. It follows from the normal form theorem for graph products [14, Theorem 3.9] that $\overbrace{(g_1 \cdots g_\ell) \cdots (g_1 \cdots g_\ell)}^{n}$ is the *unique* normal form for the element $g^n$. In particular, as geodesics in $X$ are precisely the words spelling out normal forms of elements of $\Gamma\mathcal{G}$, we have $p_i = g_{\ell n - c_i + 1} g_{\ell n - c_i + 2} \cdots g_{\ell n}$, where $0 = c_0 \leq c_1 \leq \cdots \leq c_{m+1} = \ell n$ and indices are taken modulo $\ell$.

We now claim that $c_{i+1} - c_i < \ell$ for each $i$. Indeed, suppose $c_{i+1} - c_i \geq \ell$ for some $i$. Note that, as $p_i, p_{i+1} \in \mathcal{N}(A_i)$, it follows from Theorem 5.1 that $\Gamma_{\mathrm{star}(v)}\mathcal{G}_{\mathrm{star}(v)} p_i = V(\mathcal{N}(A_i)) = \Gamma_{\mathrm{star}(v)}\mathcal{G}_{\mathrm{star}(v)} p_{i+1}$ for some $v \in V(\Gamma)$, and hence $p_{i+1}p_i^{-1} \in \Gamma_{\mathrm{star}(v)}\mathcal{G}_{\mathrm{star}(v)}$. But as $g_{\ell n - c_{i+1}} g_{\ell n - c_{i+1} + 1} \cdots g_{\ell n - c_i - 1}$ is a normal form for $p_{i+1}p_i^{-1}$ (where indices are taken modulo $\ell$), it follows that $v_j \in \mathrm{star}(v)$ for $\ell n - c_{i+1} \leq j < \ell n - c_i$ (with indices again modulo $\ell$). But as by assumption $c_{i+1} - c_i \geq \ell$ and as $\{v_1, \ldots, v_\ell\} = V(\Gamma)$, this implies that $\mathrm{star}(v) = V(\Gamma)$, and so $v$ is an isolated vertex of $\Gamma^C$. This contradicts the fact that $\Gamma^C$ is connected; thus $c_{i+1} - c_i < \ell$ for each $i$, as claimed.

In particular, we get $\ell n = \sum_{i=0}^{m}(c_{i+1} - c_i) < (m+1)\ell$, and so $m + 1 > n$. Thus $d_{\mathcal{C}X}(A, B) = m \geq n$ and so $\mathcal{C}X$ is unbounded, as required. □

It is now easy to deduce when the action of $\Gamma\mathcal{G}$ on $\mathcal{C}X$ is non-elementary acylindrical.

*Proof of Corollary C.* By the argument above, we only need to show the last part. Thus, suppose that $\Gamma$ is a graph with at least 3 vertices and connected complement. Then, by Lemma 5.3, the graph $\mathcal{C}X$ is unbounded. In particular, given any $H \in V(\mathcal{C}X)$ and $n \in \mathbb{N}$, we may pick $H' \in V(\mathcal{C}X)$ such that $d_{\mathcal{C}X}(H, H') \geq n+1$. Since the action $\Gamma\mathcal{G} \curvearrowright X$ is transitive on vertices, it follows that given any vertex $p \in \mathcal{N}(H)$ there exists $g \in \Gamma\mathcal{G}$ such that $p^g \in \mathcal{N}(H')$, and in particular $d_{\mathcal{C}X}(H^g, H') \leq 1$. Thus $d_{\mathcal{C}X}(H, H^g) \geq n$, and so the action $\Gamma\mathcal{G} \curvearrowright \mathcal{C}X$ has unbounded orbits.

We now claim that $\Gamma\mathcal{G}$ is not virtually cyclic. Indeed, since $|V(\Gamma)| \geq 3$ and $\Gamma^C$ is connected, $\Gamma^C$ contains a path of length 2, and so there exist vertices $v_1, v_2, w \in \Gamma$ such that $v_1 \nsim w \nsim v_2$. Let $A = \{v_1, v_2, w\}$ and $H = \Gamma_{\{v_1,v_2\}}\mathcal{G}_{\{v_1,v_2\}}$ (so either $H \cong G_{v_1} \times G_{v_2}$ or $H \cong G_{v_1} * G_{v_2}$). Since the groups $G_v$ are non-trivial for each $v \in V(\Gamma)$, we have $|H| \geq 4 > 2$ and so $\Gamma_A\mathcal{G}_A \cong G_w * H$ has infinitely many ends. In particular, since the subgroup $\Gamma_A\mathcal{G}_A \leq \Gamma\mathcal{G}$ is not virtually cyclic, neither is $\Gamma\mathcal{G}$, as required. □

*Remark* 5.4. After appearance of the first version of this preprint, it has been brought to the author's attention that most of the results stated in Corollary C have already been proved by Genevois. In [11, Theorem 2.38], Genevois shows that $\Delta X$ is quasi-isometric to a tree whenever it is connected and $\Gamma$ is finite, so in particular, by Theorem B (i), $\mathcal{C}X$ is a quasi-tree as well. Moreover, methods used by Genevois to prove [9, Theorem 22] can be adapted to show that the action of $\Gamma\mathcal{G}$ on $\mathcal{C}X$ is non-uniformly acylindrical; here, the *non-uniform acylindricity* of an action $G \curvearrowright X$ is a weaker version of acylindricity, defined by replacing the phrase 'is bounded above by $N_\varepsilon$' by 'is finite' in Definition 1.2. Corollary C strengthens this statement.

### 5.2 $\mathcal{AH}$-accessibility

Here we study $\mathcal{AH}$-accessibility, introduced in [1] by Abbott, Balasubramanya and Osin, of graph products. In particular, we show that if $\Gamma$ is connected, non-trivial, and the groups in $\mathcal{G}$ are infinite, then the action of $\Gamma\mathcal{G}$ on $\mathcal{C}X$ is the 'largest' acylindrical action of $\Gamma\mathcal{G}$ on a hyperbolic metric space. Hence we prove Corollary D.

We briefly recall the terminology of [1]. Given two isometric actions $G \curvearrowright X$ and $G \curvearrowright Y$ of a group $G$, we say $G \curvearrowright X$ *dominates* $G \curvearrowright Y$, denoted $G \curvearrowright Y \preceq G \curvearrowright X$, if there exist $x \in X$, $y \in Y$ and a constant $C$ such that

$$d_Y(y, y^g) \leq C d_X(x, x^g) + C$$

for all $g \in G$. The actions $G \curvearrowright X$ and $G \curvearrowright Y$ are said to be *weakly equivalent* if $G \curvearrowright X \preceq G \curvearrowright Y$ and $G \curvearrowright Y \preceq G \curvearrowright X$. This partitions all such actions into equivalence classes.

It is easy to see that $\preceq$ defines a preorder on the set of all isometric actions of $G$ on metric spaces. Therefore, $\preceq$ defines a partial order on the set of equivalence classes of all such actions. We may restrict this to a partial order on the set $\mathcal{AH}(G)$ of equivalence classes of *acylindrical* actions of $G$ on a *hyperbolic* space. We then say the group $G$ is *$\mathcal{AH}$-accessible* if the partial order $\mathcal{AH}(G)$ has a largest element (which, if exists, must necessarily be unique), and we say $G$ is *strongly $\mathcal{AH}$-accessible* if a representative of this largest element is a Cayley graph of $G$.

Recall that for an action $G \curvearrowright X$ by isometries with $X$ hyperbolic, an element $g \in G$ is said to be *loxodromic* if, for some (or any) $x \in X$, the map $\mathbb{Z} \to X$ given by $n \mapsto x^{g^n}$ is a quasi-isometric embedding. It is clear from the definitions that the 'largest' action $G \curvearrowright X$ will also be *universal*, in the sense that every element of $G$, that is loxodromic with respect to *some* acylindrical action of $G$ on a hyperbolic space, will be loxodromic with respect to $G \curvearrowright X$.

In [1, Theorem 2.19 (c)], it is shown that the all right-angled Artin groups are $\mathcal{AH}$-accessible (and more generally, so are all hierarchically hyperbolic groups – in particular, groups acting properly and cocompactly on a CAT(0) cube complex possessing a factor system [2, Theorem A]). Here we generalise this result to 'most' graph products of infinite groups. The proof is very similar to that of [1, Lemma 7.16].

*Proof of Corollary D.* It is easy to show – for instance, by Theorem 5.1 – that $\mathcal{C}X$ is ($G$-equivariantly) quasi-isometric to the Cayley graph of $\Gamma\mathcal{G}$ with respect to the generating set $\bigcup_{v \in V(\Gamma)} \Gamma_{\mathrm{star}(v)}\mathcal{G}_{\mathrm{star}(v)}$.

We prove the statement by induction on $|V(\Gamma)|$. If $|V(\Gamma)| = 1$ ($V(\Gamma)| = \{v\}$, say), then $v$ is an isolated vertex of $\Gamma$ and so, by the assumption, $\Gamma\mathcal{G} \cong G_v$ is strongly $\mathcal{AH}$-accessible.

Suppose now that $|V(\Gamma)| \geq 2$. If $\Gamma$ has an isolated vertex ($\Gamma = \Gamma_A \sqcup \{v\}$ for some partition $V(\Gamma) = A \sqcup \{v\}$, say), then $\Gamma\mathcal{G} \cong \Gamma_A\mathcal{G}_A * G_v$ is hyperbolic relative to $\{\Gamma_A\mathcal{G}_A, G_v\}$. By the induction hypothesis, both $\Gamma_A\mathcal{G}_A$ and $G_v$ are strongly $\mathcal{AH}$-accessible, and hence, by [1, Theorem 7.9], so is $\Gamma\mathcal{G}$. If, on the other hand, the complement $\Gamma^C$ of $\Gamma$ is disconnected ($\Gamma^C = \Gamma_A^C \sqcup \Gamma_B^C$ for some partition $V(\Gamma) = A \sqcup B$, say), then $\Gamma\mathcal{G} \cong \Gamma_A\mathcal{G}_A \times \Gamma_B\mathcal{G}_B$ is not acylindrically hyperbolic by [23, Corollary 7.2], as both $\Gamma_A\mathcal{G}_A$ and $\Gamma_B\mathcal{G}_B$ are infinite. It then follows from [1, Example 7.8] that $\Gamma\mathcal{G}$ is strongly $\mathcal{AH}$-accessible; it also follows that *any* acylindrical action of $\Gamma\mathcal{G}$ on a hyperbolic metric space ($\Gamma\mathcal{G} \curvearrowright \mathcal{C}X$, say) represents the largest element of $\mathcal{AH}(\Gamma\mathcal{G})$.

Hence, we may without loss of generality assume that $\Gamma$ is a graph with no isolated vertices and connected complement. It then follows that $|V(\Gamma)| \geq 4$, and so by Corollary C, $\mathcal{C}X$ is a hyperbolic metric space and $\Gamma\mathcal{G}$ acts on it non-elementarily acylindrically. It is easy to see from Theorem 5.1 that, given two hyperplanes $H, H' \in V(\mathcal{C}X)$, they are adjacent in $\mathcal{C}X$ if and only if there exist distinct $u, v \in V(\Gamma)$ and $g \in \Gamma\mathcal{G}$ such that $H = H_u^g$ and $H' = H_v^g$. It follows that the quotient space $\mathcal{C}X/\Gamma\mathcal{G}$ is the complete graph on $|V(\Gamma)|$ vertices, and in particular, the action $\Gamma\mathcal{G} \curvearrowright \mathcal{C}X$ is cocompact.

Moreover, it follows from Theorem 5.1 that the stabiliser of an arbitrary vertex $H_v^g$ of $\mathcal{C}X$ is precisely $G \cong (\Gamma_{\mathrm{star}(v)}\mathcal{G}_{\mathrm{star}(v)})^g \cong G_v^g \times (\Gamma_{\mathrm{link}(v)}\mathcal{G}_{\mathrm{link}(v)})^g$. Since $\Gamma$ has no isolated vertices, $\mathrm{link}(v) \neq \varnothing$, and so, as all groups in $\mathcal{G}$ are infinite, both $G_v^g$ and $(\Gamma_{\mathrm{link}(v)}\mathcal{G}_{\mathrm{link}(v)})^g$ are infinite groups. Thus, $G$ is a direct product of two infinite groups, and so – by [23, Corollary 7.2], say – $G$ does not possess a non-elementary acylindrical action on a hyperbolic space. Since $G$ is not virtually cyclic, for every acylindrical action of $\Gamma\mathcal{G}$ on a hyperbolic space $Y$, the induced action of $G$ on $Y$ has bounded orbits. It then follows from [1, Proposition 4.13] that $\Gamma\mathcal{G}$ is strongly $\mathcal{AH}$-accessible – and in particular, $\Gamma\mathcal{G} \curvearrowright \mathcal{C}X$ represents the largest element of $\mathcal{AH}(\Gamma\mathcal{G})$. □

*Remark* 5.5. Corollary D gives some explicit descriptions for the class of hierarchically hyperbolic groups, introduced by Behrstock, Hagen and Sisto in [5]. In particular, a result by Berlai and Robbio [6, Theorem C] says that if all vertex groups $G_v$ are hierarchically hyperbolic with the intersection property and clean containers, then the same can be said about $\Gamma\mathcal{G}$. Moreover, Abbott, Behrstock and Durham show in [2, Theorem A] that all hierarchically hyperbolic groups are $\mathcal{AH}$-accessible, which implies Corollary D in the case when the vertex groups $G_v$ are hierarchically hyperbolic with the intersection property and clean containers.

More precisely, every hierarchically hyperbolic group $G$ comes with an action on a space $\mathcal{X}$, such that there exist projections $\pi_Y : \mathcal{X} \to 2^{\hat{\mathcal{C}}Y}$ to some collection of $\delta$-hyperbolic spaces $\{\hat{\mathcal{C}}Y \mid Y \in \mathfrak{S}\}$, where $\mathfrak{S}$ is a partial order that contains a (unique) largest element, $S \in \mathfrak{S}$, say. Moreover, the action of $G$ on $\mathcal{X}$ induces an action of $G$ on (a space quasi-isometric to) $\mathcal{U} = \bigcup_{x \in \mathcal{X}} \pi_S(x) \subseteq \hat{\mathcal{C}}S$, and in [5, Theorem 14.3] it is shown that this action is acylindrical. In [2], this construction is modified so that the action

$G \curvearrowright \mathcal{U}$ represents the largest element of $\mathcal{AH}(G)$. If $\Gamma$ is connected, non-trivial, and the groups $G_v$ are infinite and hierarchically hyperbolic (with the intersection property and clean containers), then the proof of Corollary D gives this action $\Gamma\mathcal{G} \curvearrowright \mathcal{U}$ explicitly. This is potentially useful for studying hierarchical hyperbolicity of graph products.

*Remark* 5.6. Note that the condition on the $G_v$ being infinite is necessary for the proof to work. Indeed, suppose $\Gamma = \overset{a}{\bullet}\!\!-\!\!\overset{b}{\bullet}\!\!-\!\!\overset{c}{\bullet}\!\!-\!\!\overset{d}{\bullet}$ is a path of length 3, and $G_v = \langle g_v \rangle \cong C_2$ for each $v \in V(\Gamma)$, so that $\Gamma\mathcal{G}$ is the right-angled Coxeter group over $\Gamma$. Notice that $\Gamma\mathcal{G} \cong A *_C B$, where $A = G_b \times (G_a * G_c)$, $B = G_c \times G_d$ and $C = G_c$. In particular, since $C$ is finite, $\Gamma\mathcal{G}$ is hyperbolic relative to $\{A, B\}$. Hence the Cayley graph $\mathrm{Cay}(\Gamma\mathcal{G}, A \cup B)$ is hyperbolic and the usual action of $\Gamma\mathcal{G}$ on it is acylindrical.

It is easy to verify from the normal form theorem for amalgamated free products that the element $g_b g_d$ will be loxodromic with respect to $\Gamma\mathcal{G} \curvearrowright \mathrm{Cay}(\Gamma\mathcal{G}, A \cup B)$. However, as $g_b g_d \in \Gamma_{\mathrm{star}(c)}\mathcal{G}_{\mathrm{star}(c)}$, we know that $g_b g_d$ stabilises the hyperplane dual to $G_c \subseteq V(X)$ under the action of $\Gamma\mathcal{G}$ on $\mathcal{C}X$, and so $g_b g_d$ is not loxodromic with respect to $\Gamma\mathcal{G} \curvearrowright \mathcal{C}X$. In particular, the equivalence class of $\Gamma\mathcal{G} \curvearrowright \mathcal{C}X$ cannot be the largest element of $\mathcal{AH}(\Gamma\mathcal{G})$. It is straightforward to generalise this argument to show that if $c \in V(\Gamma)$ is a separating vertex of a connected finite simplicial graph $\Gamma$, then for *any* graph product $\Gamma\mathcal{G}$ with $G_c$ finite, the action $\Gamma\mathcal{G} \curvearrowright \mathcal{C}X$ will not be the 'largest' one.

On the other hand, note that this particular group $\Gamma\mathcal{G}$ (and indeed any right-angled Coxeter group) will be $\mathcal{AH}$-accessible: see [2, Theorem A (4)].

## 6 Equational Noetherianity of graph products

In this section we prove Theorem E. To do this, we use the methods that Groves and Hull exhibited in [15]. Here we briefly recall their terminology.

The approach to equationally Noetherian groups used in [15] is through sequences of homomorphisms. In particular, let $G$ be any group, let $F$ be a finitely generated group and let $\varphi_i : F \to G$ be a sequence of homomorphisms ($i \in \mathbb{N}$). Let $\omega : \mathcal{P}(\mathbb{N}) \to \{0, 1\}$ be a non-principal ultrafilter. We say a sequence of properties $(P_i)_{i\in\mathbb{N}}$ holds *$\omega$-almost surely* if $\omega(\{i \in \mathbb{N} \mid P_i \text{ holds}\}) = 1$. We define the *$\omega$-kernel* of $F$ with respect to $(\varphi_i)$ to be

$$F_{\omega,(\varphi_i)} = \{f \in F \mid \varphi_i(f) = 1 \ \omega\text{-almost surely}\};$$

we write $F_\omega$ for $F_{\omega,(\varphi_i)}$ if the sequence $(\varphi_i)$ is clear. It is easy to check that $F_\omega$ is a normal subgroup of $F$. We say *$\varphi_i$ factors through $F_\omega$ $\omega$-almost surely* if $F_\omega \subseteq \ker(\varphi_i)$ $\omega$-almost surely.

The idea behind all these definitions is the following result.

**Theorem 6.1** (Groves and Hull [15, Theorem 3.5]). *Let $\omega$ be a non-principal ultrafilter. Then the following are equivalent for any group $G$:*

(i) *$G$ is equationally Noetherian;*

(ii) *for any finitely generated group $F$ and any sequence of homomorphisms $(\varphi_i)$ from $F$ to $G$, $\varphi_i$ factors through $F_\omega$ $\omega$-almost surely.*

*Remark* 6.2. Note that Definition 1.5 differs from the usual definition of equationally Noetherian groups, as we do not allow 'coefficients' in our equations: that is, we restrict to subsets $S \subseteq F_n$ instead of $S \subseteq G * F_n$. However, the two concepts coincide when $G$ is finitely generated – see [4, §2.2, Proposition 3]. We use this (weaker) definition of equationally Noetherian groups as it is more suitable for our methods. In particular, we use an equivalent characterisation of equationally Noetherian groups given by Theorem 6.1.

In this section we prove Theorem E. In Section 6.1, we introduce 'admissible' graphs and show that being equationally Noetherian is preserved under taking graph products over admissible graphs. In Section 6.3, we show that indeed all graphs of girth $\geq 5$ are admissible.

## 6.1 Reduction to sequences of linking homomorphisms

Suppose now that the group $G$ acts by isometries on a metric space $(X, d)$. As before, let $F$ be a finitely generated group, $\omega$ a non-principal ultrafilter, and $(\varphi_i : F \to G)_{i=1}^{\infty}$ a sequence of homomorphisms. Pick a finite generating set $S$ for $F$. We say that the sequence of homomorphisms $(\varphi_i)$ is *non-divergent* if

$$\lim_{\omega} \inf_{x \in X} \max_{s \in S} d(x, x^{\varphi_i(s)}) < \infty.$$

We say that $(\varphi_i)$ is *divergent* otherwise. It is easy to see that this does not depend on the choice of a generating set for $F$.

The main technical result of [15] states that in case $X$ is hyperbolic and the action of $G$ on $X$ is non-elementary acylindrical, it is enough to consider non-divergent sequences of homomorphisms (cf Theorem 6.1).

**Theorem 6.3** (Groves and Hull [15, Theorem B]). *Let $X$ be a hyperbolic metric space and $G$ a group acting non-elementarily acylindrically on $X$. Suppose that for any finitely generated group $F$ and any non-divergent sequence of homomorphisms $(\varphi_i : F \to G)$, $\varphi_i$ factors through $F_\omega$ $\omega$-almost surely. Then $G$ is equationally Noetherian.*

We now consider the particular case when $G$ is a graph product and $X$ is the extension graph. Thus, as before, let $\Gamma$ be a finite simplicial graph and let $\mathcal{G} = \{G_v \mid v \in V(\Gamma)\}$

be a collection of non-trivial groups. It turns out that in this case we may reduce any non-divergent sequence of homomorphisms to one of the following form: see the proof of Theorem 6.6.

**Definition 6.4.** Let $\mathcal{F} = \{F_v \mid v \in V(\Gamma)\}$ be a collection of finitely generated groups, and let $\varphi : \Gamma\mathcal{F} \to \Gamma\mathcal{G}$ be a homomorphism. We say $\varphi$ is *linking* if $\varphi(F_v) \subseteq \Gamma_{\mathrm{link}(v)}\mathcal{G}_{\mathrm{link}(v)}$ for each $v \in V(\Gamma)$. We say the graph $\Gamma$ is *admissible* if for every collection of non-trivial equationally Noetherian groups $\mathcal{G} = \{G_v \mid v \in V(\Gamma)\}$ and every sequence of linking homomorphisms $(\varphi_i : \Gamma\mathcal{F} \to \Gamma\mathcal{G})_{i=1}^{\infty}$, $\varphi_i$ factors through $(\Gamma\mathcal{F})_\omega$ $\omega$-almost surely.

The proof of Theorem 6.6 uses the following result.

**Lemma 6.5.** *Full subgraphs of admissible graphs are admissible.*

*Proof.* Let $\Gamma$ be a admissible graph, let $\mathcal{G} = \{G_v \mid v \in V(\Gamma)\}$ be a collection of non-trivial equationally Noetherian groups, and let $\mathcal{F} = \{F_v \mid v \in V(\Gamma)\}$ be a collection of finitely generated groups. Let $A \subseteq V(\Gamma)$, so that $\Gamma_A$ is a full subgraph of $\Gamma$, and let $(\varphi_i^A : \Gamma_A\mathcal{F}_A \to \Gamma_A\mathcal{G}_A)_{i=1}^{\infty}$ be a sequence of linking homomorphisms. Let $\omega$ be a non-principal ultrafilter. We aim to show that $\varphi_i^A$ factors through $(\Gamma_A\mathcal{F}_A)_\omega$ $\omega$-almost surely.

Note that we have a canonical retraction $\rho_A : \Gamma\mathcal{F} \to \Gamma_A\mathcal{F}_A$, defined on vertex groups by $\rho_A(f) = f$ if $f \in F_v$ for $v \in A$, and $\rho_A(f) = 1$ if $f \in F_v$ for $v \notin A$. We also have a canonical inclusion of subgroup $\iota_A : \Gamma_A\mathcal{G}_A \to \Gamma\mathcal{G}$. For each $i$, let $\varphi_i = \iota_A \circ \varphi_i^A \circ \rho_A : \Gamma\mathcal{F} \to \Gamma\mathcal{G}$. It is easy to see that the $\varphi_i$ are linking homomorphisms. In particular, since $\Gamma$ is admissible, we have $(\Gamma\mathcal{F})_\omega \subseteq \ker\varphi_i$ $\omega$-almost surely. Moreover, since $\iota_A$ is injective, we obtain

$$\ker\varphi_i = \rho_A^{-1}(\ker\varphi_i^A) \text{ for each } i \qquad \text{and} \qquad (\Gamma\mathcal{F})_\omega = \rho_A^{-1}((\Gamma_A\mathcal{F}_A)_\omega).$$

As $\rho_A$ is surjective, it follows that $(\Gamma_A\mathcal{F}_A)_\omega \subseteq \ker\varphi_i^A$ $\omega$-almost surely, and so $\Gamma_A$ is admissible, as required. □

**Theorem 6.6.** *For any admissible graph $\Gamma$ and any collection $\mathcal{G} = \{G_v \mid v \in V(\Gamma)\}$ of equationally Noetherian groups, the graph product $\Gamma\mathcal{G}$ is equationally Noetherian.*

*Proof.* We proceed by induction on $|V(\Gamma)|$. If $|V(\Gamma)| = 1$ ($V(\Gamma) = \{v\}$, say) then $\Gamma\mathcal{G} \cong G_v$, and so the result is clear. Thus, assume that $|V(\Gamma)| \geq 2$.

If $\Gamma$ is disconnected, then we have a partition $V(\Gamma) = A \sqcup B$ into non-empty subsets such that $\Gamma = \Gamma_A \sqcup \Gamma_B$, and so $\Gamma\mathcal{G} \cong \Gamma_A\mathcal{G}_A * \Gamma_B\mathcal{G}_B$. By Lemma 6.5, both $\Gamma_A$ and $\Gamma_B$ are admissible, and so by the induction hypothesis, both $\Gamma_A\mathcal{G}_A$ and $\Gamma_B\mathcal{G}_B$ are equationally Noetherian. By Theorem 1.6, $\Gamma\mathcal{G}$ is equationally Noetherian as well, as required. Thus, without loss of generality, we may assume that $\Gamma$ is connected.

Similarly, if the complement of $\Gamma$ is disconnected, then we have a partition $V(\Gamma) = A \sqcup B$ such that $\Gamma\mathcal{G} \cong \Gamma_A\mathcal{G}_A \times \Gamma_B\mathcal{G}_B$. As before, $\Gamma_A\mathcal{G}_A$ and $\Gamma_B\mathcal{G}_B$ are equationally Noetherian by the induction hypothesis. It is clear from the definition that a direct product $G \times H$ of equationally Noetherian groups $G$ and $H$ is equationally Noetherian: indeed, this follows from the cartesian product decomposition $V_{G\times H}(S) = V_G(S) \times V_H(S)$, for any $S \subseteq F_n$. Thus $\Gamma\mathcal{G}$ is equationally Noetherian in this case as well.

Therefore, we may without of loss of generality assume that $\Gamma$ is a connected graph with a connected complement and $|V(\Gamma)| \geq 2$ (and, therefore, $|V(\Gamma)| \geq 4$). In this case, Corollary C shows that $\mathcal{C}X$ is a hyperbolic metric space and the action of $\Gamma\mathcal{G}$ on it is non-elementary acylindrical. We thus may use Theorem 6.3 to show that $\Gamma\mathcal{G}$ is equationally Noetherian.

In particular, let $F$ be a finitely generated group and let $(\varphi_i : F \to \Gamma\mathcal{G})_{i\in\mathbb{N}}$ be a non-divergent sequence of homomorphisms. By Theorem 6.3, it is enough to show that $\varphi_i$ factors through $F_\omega$ $\omega$-almost surely.

We proceed as in the proof of [15, Theorem D]. Let $S$ be a finite generating set for $F$. Note that, by Theorem 5.1, we may conjugate each $\varphi_i$ (if necessary) to assume that the minimum (over all hyperplanes $H$ in $X$) of $\max_{s\in S} d_{\mathcal{C}X}(H, H^{\varphi_i(s)})$ is attained for $H = H_u$ for some $u \in V(\Gamma)$. Moreover, it is easy to see from Theorem 5.1 that $|\|g\|_* - d_{\mathcal{C}X}(H_u, H_u^g)| \leq 1$ for any $g \in \Gamma\mathcal{G}$ and $u \in V(\Gamma)$, where we write $\|g\|_*$ for the minimal integer $\ell \in \mathbb{N}$ such that $g = g_1 \cdots g_\ell$ and $g_i \in \Gamma_{\mathrm{star}(v_i)}\mathcal{G}_{\mathrm{star}(v_i)}$ for some $v_i \in V(\Gamma)$. In particular, since the sequence $(\varphi_i)$ is non-divergent, it follows that

$$\lim_\omega \max_{s\in S} \|\varphi_i(s)\|_* < \infty.$$

It follows that for each $s \in S$, there exists $\hat{n}_s \in \mathbb{N}$ such that $\|\varphi_i(s)\|_* = \hat{n}_s$ $\omega$-almost surely. Moreover, for each $s \in S$, there exist $\hat{v}_{s,1}, \ldots, \hat{v}_{s,\hat{n}_s} \in V(\Gamma)$ such that we have

$$\varphi_i(s) = \hat{g}_{i,s,1} \cdots \hat{g}_{i,s,\hat{n}_s}$$

with $\hat{g}_{i,s,j} \in \Gamma_{\mathrm{star}(\hat{v}_{s,j})}\mathcal{G}_{\mathrm{star}(\hat{v}_{s,j})}$ $\omega$-almost surely. But since we have $\Gamma_{\mathrm{star}(v)}\mathcal{G}_{\mathrm{star}(v)} = G_v \times \Gamma_{\mathrm{link}(v)}\mathcal{G}_{\mathrm{link}(v)}$ for each $v \in V(\Gamma)$, we can write $\hat{g}_{i,s,j} = g_{i,s,2j-1}g_{i,s,2j}$, where $g_{i,s,2j-1} \in G_{\hat{v}_{s,j}} \leq \Gamma_{\mathrm{link}(v_{s,2j-1})}\mathcal{G}_{\mathrm{link}(v_{s,2j-1})}$ with any choice of vertex $v_{s,2j-1} \in \mathrm{link}(\hat{v}_{s,j})$ (which exists since $\Gamma$ is connected and $|V(\Gamma)| \geq 2$), and $g_{i,s,2j} \in \Gamma_{\mathrm{link}(v_{s,2j})}\mathcal{G}_{\mathrm{link}(v_{s,2j})}$ with $v_{s,2j} = \hat{v}_{s,j}$. It follows that, after setting $n_s = 2\hat{n}_s$, we may write

$$\varphi_i(s) = g_{i,s,1} \cdots g_{i,s,n_s}$$

with $g_{i,s,j} \in \Gamma_{\mathrm{link}(v_{s,j})}\mathcal{G}_{\mathrm{link}(v_{s,j})}$ $\omega$-almost surely.

Now for each $s \in S$, define abstract letters $h_{s,1}, \dots, h_{s,n_s}$. For each $v \in V(\Gamma)$, let

$$H_v = \{h_{s,j} \mid v_{s,j} = v\},$$

and let $F_v = F(H_v)$, the free group on $H_v$. Let $\mathcal{F} = \{F_v \mid v \in V(\Gamma)\}$, and consider the graph product $\Gamma\mathcal{F}$. We can define a map from $S$ to $\Gamma\mathcal{F}$ by sending $s \in S$ to $h_{s,1} \cdots h_{s,n_s}$. Let $N$ be the normal subgroup of $\Gamma\mathcal{F}$ generated by images of all the relators of $F$ under this map. This gives a group homomorphism $\rho : F \to \Gamma\mathcal{F}/N$.

The map $\hat{\varphi}_i : \Gamma\mathcal{F}/N \to \Gamma\mathcal{G}$, obtained by sending $h_{s,j}N$ to $g_{i,s,j}$, is $\omega$-almost surely a well-defined homomorphism. Indeed, all the relators in $\Gamma\mathcal{F}/N$ are either of the form $[h_{s_1,j_1}, h_{s_2,j_2}] = 1$ if $v_{s_1,j_1} \sim v_{s_2,j_2}$ in $\Gamma$, or of the form $\phi(\{h_{s,1} \cdots h_{s,n_s} \mid s \in S\})$, where $\phi(S)$ is a relator in $F$. Both of these $\omega$-almost surely map to the identity under $\hat{\varphi}_i$: the former because $[g_{i,s_1,j_1}, g_{i,s_2,j_2}] = 1$ in $G$ if $v_{s_1,j_1} \sim v_{s_2,j_2}$ in $\Gamma$, and the latter because $\varphi_i$ is a well-defined homomorphism. It is also clear that $\varphi_i = \hat{\varphi}_i \circ \rho$ $\omega$-almost surely.

Now let $\pi : \Gamma\mathcal{F} \to \Gamma\mathcal{F}/N$ be the quotient map. Then, by construction, the homomorphisms $\varphi_i' = \hat{\varphi}_i \circ \pi : \Gamma\mathcal{F} \to \Gamma\mathcal{G}$ are linking (when they are well-defined). Since $\Gamma$ is admissible and the groups $G_v$ are equationally Noetherian, it follows that $\varphi_i'$ factors through $(\Gamma\mathcal{F})_\omega$ $\omega$-almost surely. Since $\pi$ is surjective, this implies that $(\Gamma\mathcal{F}/N)_\omega \subseteq \ker \hat{\varphi}_i$ $\omega$-almost surely. Thus $\varphi_i = \hat{\varphi}_i \circ \rho$ factors through $F_\omega = \rho^{-1}((\Gamma\mathcal{F}/N)_\omega)$ $\omega$-almost surely, as required. □

We expect that the class of equationally Noetherian groups is closed under taking arbitrary graph products. Although we are not able to show this in full generality, in the next subsection we show that any triangle-free and square-free graph $\Gamma$ is admissible, and therefore, by Theorem 6.6, the class of equationally Noetherian groups is closed under taking graph products over such graphs $\Gamma$.

### 6.2 Digression: dual van Kampen diagrams

Before embarking on a proof of Theorem E, let us define the following notion. Following methods of [8] and [19], we consider *dual van Kampen diagrams* for words representing the identity in $\Gamma\mathcal{G}$; recently, dual van Kampen diagrams for graph products have been independently introduced by Genevois in [12]. Here we explain their construction and properties.

We consider van Kampen diagrams in the quasi-median graph $X$ given by Theorem 1.3, viewed as a Cayley graph. In particular, note that we have a presentation

$$\Gamma\mathcal{G} = \langle S \mid R_\triangle \sqcup R_\square \rangle \tag{6.1}$$

with generators

$$S = \bigsqcup_{v \in V\Gamma} (G_v \setminus \{1\})$$

and relators of two types: the 'triangular' relators

$$R_\triangle = \bigsqcup_{v \in V(\Gamma)} \{ghk^{-1} \mid g, h, k \in G_v \setminus \{1\}, gh = k \text{ in } G_v\}$$

and the 'rectangular' relators

$$R_\square = \bigsqcup_{(v,w) \in E(\Gamma)} \{[g_v, g_w] \mid g_v \in G_v \setminus \{1\}, g_w \in G_w \setminus \{1\}\}.$$

We now dualise the notion of van Kampen diagrams with respect to the presentation (6.1). Let $D \subseteq \mathbb{R}^2$ be a van Kampen diagram with boundary label $w$, for some word $w \in S^*$ representing the identity in $\Gamma\mathcal{G}$, with respect to the presentation (6.1). It is convenient to pick a colouring $V(\Gamma) \to \mathbb{N}$ and to colour edges of $D$ according to their labels. Suppose that $w = g_1 \cdots g_n$ for some syllables $g_i$, and let $e_1, \ldots, e_n$ be the corresponding edges on the boundary of $D$. We add a 'vertex at infinity' $\infty$ somewhere on $\mathbb{R}^2 \setminus D$, and for each $i = 1, \ldots, n$, we attach to $D$ a triangular 'boundary' face whose vertices are the endpoints of $e_i$ and $\infty$. We get the dual van Kampen diagram $\Delta$ corresponding to $D$ by taking the dual of $D$ as a polyhedral complex and removing the face corresponding to $\infty$: thus, $\Delta$ is a tesselation of a disk. See Figure 3.6.

We lift the colouring of edges in $D$ to a colouring of edges of $\Delta$: this gives a corresponding vertex $v \in V(\Gamma)$ for each internal edge of $\Delta$. We say a 1-subcomplex (a subgraph) of $\Delta$ is a *$v$-component* (or just a *component*) for some $v \in V(\Gamma)$ if it is a maximal connected subgraph each of whose edges correspond to the vertex $v$. We call a vertex of $\Delta$ an *intersection point* (respectively *branch point*, *boundary point*) if it comes from a triangular (respectively rectangular, boundary) face in $D$. It is easy to see that boundary, intersection and branch points lying on a component $C$ will be precisely the vertices of $C$ of degree 1, 2 and 3, respectively.

The following Lemma says that, without loss of generality, we may always assume that components of dual van Kampen diagrams do not contain cycles. It is a special case of [12, Proposition 1.1].

**Lemma 6.7.** *Let $w \in S^*$ be a word representing the identity element in $\Gamma\mathcal{G}$. Then there exists a dual van Kampen diagram $\Delta$ for $w$ such that each component of $\Delta$ is a tree.*

*Proof.* Let $D$ be a van Kampen diagram for $w$ with the corresponding dual van Kampen diagram $\Delta$. Suppose a $v$-component $C$ of $\Delta$ (for some $v \in V(\Gamma)$) contains a cycle $C_0 \subseteq C$. Then $C_0$ corresponds to a 'corridor' $K_0 \subseteq D$: that is, a subcomplex $K_0$ homeomorphic to an annulus or, in 'degenerate' cases, a disk. The *interior* $\mathrm{int}(K_0)$ of $K_0$ will consist

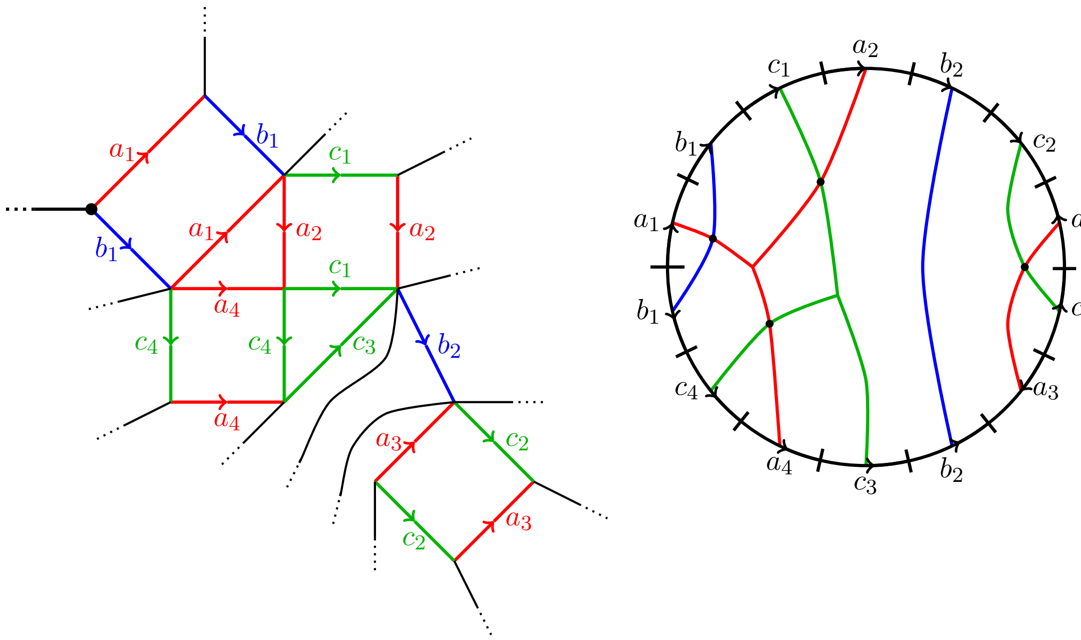



FIGURE 3.6: Van Kampen diagram ($D$, left) and its dual ($\Delta$, right) with the word $a_1b_1c_1a_2b_2c_2a_3^{-1}c_2^{-1}a_3b_2^{-1}c_3^{-1}a_4^{-1}c_4^{-1}b_1^{-1}$ as its boundary label, where $a_i \in G_a$ with $a_1a_2 = a_4$, $b_i \in G_b$, $c_i \in G_c$ with $c_4c_3 = c_1$, and $b \sim a \sim c$ in $\Gamma$. The black edges on $D$ represent the boundary faces attached: the non-visible endpoint of each black edge is the point $\infty$. The dual van Kampen diagram $\Delta$ contains 6 components in total: 2 components corresponding to each of the vertices $a$, $b$ and $c$.

of faces and edges that correspond to vertices and edges of $C_0$. Note that his will *not* have the usual meaning if $K_0$ is homeomorphic to a disk, as vertices contained in the 'usual' interior of $K_0$ and edges joining them will not belong to $\mathrm{int}(K_0)$. Thus $\mathrm{int}(K_0)$ separates $D$ into two connected components: the *inside* and the *outside* of $K_0$.

Fix $e$ a directed edge $e$ in $\mathrm{int}(K_0)$ with initial vertex in the inside of $K_0$, and let $g \in G_v$ be the label of $e$. We then construct a new van Kampen diagram $D'$ from $D$ as follows. Given any directed edge $e'$ in $\mathrm{int}(K_0)$ with initial vertex in the inside of $K_0$ and label $g' \in G_v$, we replace the label of $e'$ with $g^{-1}g'$. By construction, the resulting diagram will have one or more edges labelled by the trivial element. Each face containing such an edge (we call it a *bad* face) will either be a triangular face with other two edges having the same (non-identity) labels, or a rectangular one with two opposite edges labelled by the trivial element. In either case we can remove such a face by gluing the two edges labelled by non-identity elements. We remove all the bad faces in such a way, and call the resulting diagram $D'$. The corresponding dual van Kampen diagram $\Delta'$ will be identical to $\Delta$ apart from some of the edges of $C_0$ removed (along with vertices that would otherwise have degree 2 in $\Delta'$). Thus $\Delta'$ has strictly fewer cycles contained in a single component than $\Delta$, and so we may repeat this procedure to obtain a dual van Kampen diagram in which each component is a tree. $\square$

### 6.3 Graphs of large girth

Here we aim to show that all (finite simplicial) graphs of girth $\geq 5$ – that is, graphs that are both triangle-free and square-free – are admissible. Thus, let $\Gamma$ be a finite simplicial graph, and let $\mathcal{F} = \{F_v \mid v \in V(\Gamma)\}$ and $\mathcal{G} = \{G_v \mid v \in V(\Gamma)\}$ be two collections of groups, with all $F_v$ finitely generated and all $G_v$ equationally Noetherian. Let $\omega$ be a non-principal ultrafilter. For each $i \in \mathbb{N}$, let $\varphi_i : \Gamma\mathcal{F} \to \Gamma\mathcal{G}$ be a linking homomorphism (in the sense of Definition 6.4).

Notice that, given a homomorphism $\varphi : \Gamma\mathcal{F} \to \Gamma\mathcal{G}$, there are only finitely many choices for the subsets $\operatorname{supp}(\varphi(F_v)) \subseteq \operatorname{link}(v)$ for $v \in V(\Gamma)$. Therefore, there exist subsets $A_v \subseteq \operatorname{link}(v)$ such that $A_v = \operatorname{supp}(\varphi_i(F_v))$ for all $v \in V(\Gamma)$ $\omega$-almost surely. We will fix these subsets $A_v$ throughout this subsection. The next result characterises combinatorial restrictions that must be imposed on the $A_v$.

**Lemma 6.8.** *If $\Gamma$ has girth $\geq 4$, then for any $v \sim w$ we have $a_v \sim a_w$ for all $a_v \in A_v$ and $a_w \in A_w$. In particular, if $\Gamma$ has girth $\geq 5$, then either $A_v \subseteq \{w\}$ or $A_w \subseteq \{v\}$ whenever $v \sim w$.*

*Proof.* First, we prove the first statement. Let $i \in \mathbb{N}$ be such that $A_u = \operatorname{supp}(\varphi_i(F_v))$ for $u \in \{v, w\}$, and let $g_u \in \varphi_i(F_u)$ be an element such that $a_u \in \operatorname{supp}(g_u)$ for $u \in \{v, w\}$. Since $\varphi_i$ is a homomorphism, $[g_v, g_w] = 1$. Let $\Delta$ be a dual van Kampen diagram corresponding to the word $p_v^{-1}p_w^{-1}p_vp_w$ for some reduced words $p_v$, $p_w$ representing $g_v$, $g_w$, respectively, and let $\partial_v$ and $\partial_v'$ (respectively $\partial_w$ and $\partial_w'$) be the intervals on the boundary of $\Delta$ that spell out $p_v$ (respectively $p_w$).

Let $P_v$ (respectively $P_w$) be a $a_v$-component (respectively $a_w$-component) of $\Delta$ that has a boundary point on $\partial_v$ (respectively $\partial_w$). Notice that no other boundary point of $P_v$ lies on $\partial_v$ since $p_v$ is reduced. Notice also that as $A_v \subseteq \operatorname{link}(v)$ and $A_w \subseteq \operatorname{link}(w)$, and as by assumption $\Gamma$ is triangle-free, we have $A_v \cap A_w = \varnothing$ – in particular, $a_v \notin A_w$. Thus $P_v$ cannot have boundary points on either $\partial_w$ or $\partial_w'$.

As $P_v$ must have at least two boundary points, this implies that $P_v$ must have a boundary point on $\partial_v'$. Similarly, $P_w$ must have a boundary point on $\partial_w'$. But then $P_v$ and $P_w$ intersect, implying that $a_v \sim a_w$, as required. This proves the first statement.

The second statement of the Lemma now follows from the first one under the additional assumption that $\Gamma$ is square-free. □

As an immediate consequence, we obtain the following result.

**Corollary 6.9.** *If $\Gamma$ has girth $\geq 5$ and $v \in V(\Gamma)$ has $|A_v| \geq 2$, then $|A_w| \leq 1$ for all $w \sim v$.* □

This implies the existence of 'non-rigid' vertices if $\Gamma$ has girth $\geq 5$, in the following sense. The idea behind this is that there are transformations that allow us to move boundary points of components corresponding to non-rigid vertices in certain dual van Kampen diagrams: see Lemma 6.11.

**Definition 6.10.** We call a vertex $v \in V(\Gamma)$ $(\varphi_i)$*-rigid* (or simply *rigid*) if there exists $w \in V(\Gamma)$ such that $v \in A_w$ and $|A_w| \geq 2$. Otherwise, $v$ is called *non-rigid.*

Given a subset $A \subseteq V(\Gamma)$, we write $\iota_A : \Gamma_A\mathcal{F}_A \to \Gamma\mathcal{F}$ for the canonical inclusion, and $\rho_A : \Gamma\mathcal{G} \to \Gamma_A\mathcal{G}_A$ for the canonical retraction. We then may define further homomorphisms

$$\varphi_i^{(v,1)} = \rho_{V(\Gamma)\setminus\mathrm{link}(v)} \circ \varphi_i : \Gamma\mathcal{F} \to \Gamma_{V(\Gamma)\setminus\mathrm{link}(v)}\mathcal{G}_{V(\Gamma)\setminus\mathrm{link}(v)}$$

and

$$\varphi_i^{(v,2)} = \rho_{V(\Gamma)\setminus\{v\}} \circ \varphi_i : \Gamma\mathcal{F} \to \Gamma_{V(\Gamma)\setminus\{v\}}\mathcal{G}_{V(\Gamma)\setminus\{v\}}.$$

In addition, given any $v \in V(\Gamma)$, we define

$$B(v) = \{w \in V(\Gamma) \mid A_w = \{v\}\}.$$

If $v$ is non-rigid, then we may 'decompose' the homomorphisms $\varphi_i$ into ones with a 'smaller' domain. In particular, $\varphi_i$ $\omega$-almost surely restricts to homomorphisms

$$\varphi_i^{(v,3)} = \varphi_i \circ \iota_{B(v)} : \Gamma_{B(v)}\mathcal{F}_{B(v)} \to G_v$$

and

$$\varphi_i^{(v,4)} = \varphi_i \circ \iota_{V(\Gamma)\setminus B(v)} : \Gamma_{V(\Gamma)\setminus B(v)}\mathcal{F}_{V(\Gamma)\setminus B(v)} \to \Gamma_{V(\Gamma)\setminus\{v\}}\mathcal{G}_{V(\Gamma)\setminus\{v\}}.$$

For $j \in \{1,2,3,4\}$, let $(\Gamma\mathcal{F})_\omega^{(v,j)}$ be the $\omega$-kernel for the sequence of homomorphisms $\left(\varphi_i^{(v,j)}\right)_{i=1}^\infty$.

**Lemma 6.11.** *Suppose $v \in V(\Gamma)$ is non-rigid. Then $\omega$-almost surely we have*

$$\ker(\varphi_i) = \left\langle\!\!\left\langle \iota_{B(v)}\left(\ker\varphi_i^{(v,3)}\right) \cup \iota_{V(\Gamma)\setminus B(v)}\left(\ker\varphi_i^{(v,4)}\right) \cup \left[\ker\varphi_i^{(v,1)}, \ker\varphi_i^{(v,2)}\right] \right\rangle\!\!\right\rangle. \quad (6.2)$$

*Moreover, the $\omega$-kernel for the sequence $(\varphi_i)_{i=1}^\infty$ is*

$$(\Gamma\mathcal{F})_\omega = \left\langle\!\!\left\langle \iota_{B(v)}\left((\Gamma\mathcal{F})_\omega^{(v,3)}\right) \cup \iota_{V(\Gamma)\setminus B(v)}\left((\Gamma\mathcal{F})_\omega^{(v,4)}\right) \cup \left[(\Gamma\mathcal{F})_\omega^{(v,1)}, (\Gamma\mathcal{F})_\omega^{(v,2)}\right] \right\rangle\!\!\right\rangle.$$

*Proof.* We first prove that (6.2) holds $\omega$-almost surely. The inclusion $(\supseteq)$ is clear, and so we only need to prove the inclusion $(\subseteq)$.

Let $i \in \mathbb{N}$ be such that $\mathrm{supp}(\varphi_i(F_w)) = A_w$ for all $w \in V(\Gamma)$: this happens $\omega$-almost surely. Let $g \in \ker(\varphi_i)$ be a cyclically reduced element. Consider an expression $g = g_1 \cdots g_n$, with $g_j \in F_{v_j}$ for some $v_1, \ldots, v_n \in V(\Gamma)$. We will look at $g_1 \cdots g_n$ as a *cyclic word* throughout, that is, we will not distinguish between $g_1 \cdots g_n$ and its cyclic permutations.

We will perform two types of transformations of the cyclic word $g_1 \cdots g_n$, which will not change whether or not the resulting element is contained in either side of (6.2):

(A) *Transpositions*: if, for some $k \leq \ell \leq m$, we have $\varphi_i(g_{k+1} \cdots g_\ell) \in \Gamma_{\mathrm{link}(v)}\mathcal{G}_{\mathrm{link}(v)}$ and $\varphi_i(g_{\ell+1} \cdots g_m) \in G_v$, then we may transpose the corresponding subwords of the cyclic word $g_1 \cdots g_n$: replace the (cyclic) word $g_{k+1} \cdots g_k$ with the word $g_{\ell+1} \cdots g_m g_{k+1} \cdots g_\ell g_{m+1} \cdots g_k$. By construction, we have

$$g_{k+1} \cdots g_\ell \in \ker \varphi_i^{(v,1)}$$

and

$$g_{\ell+1} \cdots g_m \in \ker \varphi_i^{(v,2)},$$

so this transformation multiplies $g$ by a conjugate of the element

$$[g_{\ell+1} \cdots g_m, g_{k+1} \cdots g_\ell] \in \left[\ker \varphi_i^{(v,1)}, \ker \varphi_i^{(v,2)}\right].$$

(B) *Removals*: if, for some $k, \ell$, we have $\varphi_i(g_{k+1} \cdots g_\ell) = 1$ and $v_j \in B(v)$ for $j = k+1, \ldots, \ell$, then we may remove the corresponding subword of $g_1 \cdots g_n$: that is, replace the (cyclic) word $g_{k+1} \cdots g_k$ with the word $g_{\ell+1} \cdots g_k$. By construction, this transformation multiplies $g$ by a conjugate of the element

$$(g_{k+1} \cdots g_\ell)^{-1} \in \iota_{B(v)}\left(\ker \varphi_i^{(v,3)}\right).$$

Let $\Delta$ be a dual van Kampen diagram for the word $\varphi_i(g_1) \cdots \varphi_i(g_n)$, where the elements $\varphi_i(g_j)$ are represented by reduced words. We will prove that $g$ is contained in the right-hand side of (6.2) by induction on $n$. The base case, $n = 0$, is clear. Without loss of generality, we may assume that $\varphi_i(g_j) \neq 1$ for each $j$. Indeed, if $\varphi_i(g_j) = 1$ for some $j$ then we may replace the cyclic word $g_{j+1} \cdots g_j$ with $g_{j+1} \cdots g_{j-1}$ by multiplying $g$ by a conjugate of $g_j^{-1} \in \iota_{B(v)}\left(\ker \varphi_i^{(v,3)}\right)$ or $g_j^{-1} \in \iota_{V(\Gamma)\setminus B(v)}\left(\ker \varphi_i^{(v,4)}\right)$, depending on whether or not $v_j \in B(v)$. This reduces the length of the word representing $g$, and so we are done by the induction hypothesis.

If $\Delta$ does not contain any $v$-components, then we are done: indeed, this means that $v_j \notin B(v)$ for all $j$ and so $g \in \iota_{V(\Gamma)\setminus B(v)}\left(\ker \varphi_i^{(v,4)}\right)$. Otherwise, let $P$ be a $v$-component of $\Delta$.

Since $v$ is non-rigid, it follows that we may write $g_1 \cdots g_n$ (or some its cyclic permutation) as $h_1 k_1 \cdots h_m k_m$, where any boundary point on the interval on the boundary of $\Delta$ corresponding to $\varphi_i(h_j)$ (respectively $\varphi_i(k_j)$) is (respectively is not) a boundary point of $P$. Notice that the $h_j$ consist only of syllables from $G_w$ for $w \in B(v)$, and that

$\varphi_i(h_1 \cdots h_m) = 1$. If $m = 1$, then we are done: indeed, in that case $h_1 = 1$, and so we may remove the subword $h_1$ from $g_1 \cdots g_n$, as explained in (B) above. This reduces the length of a word representing $g$, so we are done by the induction hypothesis.

Suppose now that $m \geq 2$. If $Q$ is any component of $\Delta$ having a boundary point on the interval $\partial_j$ corresponding to $\varphi_i(k_j)$, then either $Q$ intersects $P$, or all other boundary points of $Q$ are on $\partial_j$. It follows that $\varphi_i(k_j) \in \Gamma_{\mathrm{link}(v)}\mathcal{G}_{\mathrm{link}(v)}$; as $P$ is a $v$-component, it is also clear that $\varphi_i(h_j) \in G_v$. Thus we may transpose subwords $h_j$ and $k_j$ of $g_1 \cdots g_n$ for any $j$, as explained in (A) above. This also can be done with minimal changes to $\Delta$: see Figure 3.7. In particular, this rearranges boundary points in $\Delta$ without changing whether or not a specific boundary point belongs to $P$. This reduces the value of $m$ for the corresponding word, and so after $m - 1$ such transpositions we return to the case $m = 1$. We are then done by the previous paragraph. This proves (6.2).

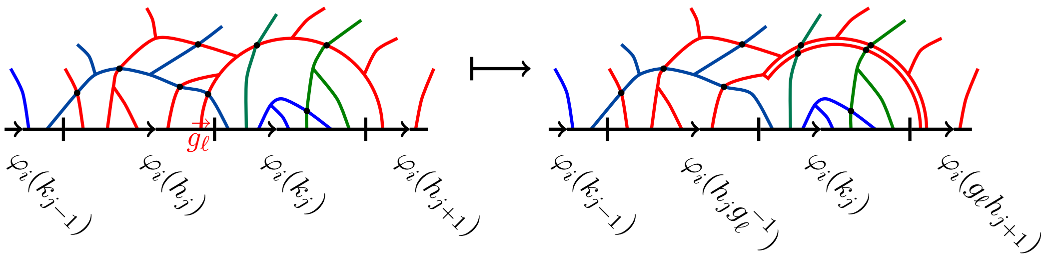



FIGURE 3.7: Proof of Lemma 6.11: transposing $k_j$ and the last syllable $g_\ell$ of $h_j$. We transpose $h_j$ and $k_j$ by performing finitely many operations like these. $P$ is shown in red, other components in other colours.

Finally, for the second statement, notice that in the proof above, the only operations we do to the cyclic word $g_1 \cdots g_n$ are transpositions (A) or removals (B) of its subwords, and there are finitely many operations of this form. The number of these operations is also bounded as a function of $n$: for instance, we may assume that no permutation of syllables of $g_1 \cdots g_n$ is obtained more than once while performing the procedure, and so there are at most $n!$ transpositions of subwords performed until we remove a subword. Thus some particular sequence of transpositions and removals of subwords happens $\omega$-almost surely, which implies the second statement. □

By combining Corollary 6.9 with Lemma 6.11, we obtain the following.

**Theorem 6.12.** *Any finite graph $\Gamma$ of girth $\geq 5$ is admissible.*

*Proof.* We will induct on $|V(\Gamma)|$; the base case, $|V(\Gamma)| = 1$, is clear. Now assume that $\Gamma$ is a graph of girth $\geq 5$ with $|V(\Gamma)| \geq 2$ and that every graph $\hat{\Gamma}$ of girth $\geq 5$ with $|V(\hat{\Gamma})| < |V(\Gamma)|$ is admissible.

Note that $\Gamma$ has at least one non-rigid vertex. Indeed, it is clear that any vertex $v$ such that $|A_w| \leq 1$ for all $w \sim v$ is non-rigid. Thus, if $\Gamma$ contains a vertex $v$ with $|A_v| \geq 2$

then, by Corollary 6.9, $v$ is non-rigid. On the other hand, if $\Gamma$ contains no vertices $v$ with $|A_v| \geq 2$, then no vertices of $\Gamma$ are rigid.

Without loss of generality, we can assume that $\Gamma$ is connected – indeed, if it is not then $\Gamma\mathcal{G} \cong \Gamma_A\mathcal{G}_A * \Gamma_B\mathcal{G}_B$ for some partition $V(\Gamma) = A \sqcup B$. By the inductive hypothesis, $\Gamma_A$ and $\Gamma_B$ are admissible, and therefore, by Theorem 6.6, $\Gamma_A\mathcal{G}_A$ and $\Gamma_B\mathcal{G}_B$ are equationally Noetherian. It then follows from Theorem 1.6 that $\Gamma\mathcal{G}$ is equationally Noetherian as well, and so (by Theorem 6.1) $\Gamma$ is admissible, as required. We will therefore assume here that $\Gamma$ is connected.

Now let $v$ be a non-rigid vertex of $\Gamma$. As $\Gamma$ is connected, $\mathrm{link}(v) \neq \varnothing$. Therefore, by inductive hypothesis, the graphs $\Gamma_{V(\Gamma)\setminus\mathrm{link}(v)}$ and $\Gamma_{V(\Gamma)\setminus\{v\}}$ are admissible, and consequently, by Theorem 6.6, the groups $\Gamma_{V(\Gamma)\setminus\mathrm{link}(v)}\mathcal{G}_{V(\Gamma)\setminus\mathrm{link}(v)}$ and $\Gamma_{V(\Gamma)\setminus\{v\}}\mathcal{G}_{V(\Gamma)\setminus\{v\}}$ are equationally Noetherian. As $G_v$ is also equationally Noetherian, it follows that for every $j \in \{1,2,3,4\}$ we have

$$(\Gamma\mathcal{F})_\omega^{(v,j)} \subseteq \ker\varphi_i^{(v,j)}$$

$\omega$-almost surely. The result now follows from Lemma 6.11. □

*Proof of Theorem E.* This is immediate from Theorems 6.6 and 6.12. □

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