UNIVERSITY OF SOUTHAMPTON

Symmetries in Toric Topology

by

Xin Fu

A thesis submitted in partial fulfillment for the
degree of Doctor of Philosophy

in the
Faculty of Social Sciences
School of Mathematical Sciences

July 2019

UNIVERSITY OF SOUTHAMPTON

ABSTRACT

FACULTY OF SOCIAL SCIENCES
SCHOOL OF MATHEMATICAL SCIENCES

Doctor of Philosophy

by Xin Fu

A polyhedral product $(\underline{X}, \underline{A})^K$ is determined by a finite simplicial complex K and m pairs of topological spaces. The functorial property of polyhedral products implies two types of symmetries of polyhedral products induced by symmetries of simplicial complexes and by group actions on the topological pairs. In this thesis, we consider these two types of symmetries.

In the case of G-polyhedral products induced by a simplicial G-complex K, we show that the homotopy decomposition $\Sigma(X, A)^K$ due to Bahri-Bendersky-Cohen-Gitler [3] is homotopy G-equivariant after suspension. It implies a homological decomposition of $H_i((X, A)^K)$ in terms of G-representations, which we rely on to study the representation stability of polyhedral product in the sense of Church-Farb [15].

The torus actions on moment-angle complexes $\mathcal{Z}_K$ is a special case of actions on polyhedral products induced by actions on the topological pairs. In this thesis, we compute the cohomology of the quotient $\mathcal{Z}_K/S^1$ under free circle actions and introduce a chain complex $(C_*(L), \delta)$ whose homology isomorphic to $H^*(\mathcal{Z}_K/S^1; R)$. For certain cases K, we determine the homotopy types of $\mathcal{Z}_K/S^1$.

Contents

Academic Thesis: Declaration of Authorship

I, Xin Fu, declare that this thesis and the work presented in it are my own and has been generated by me as the result of my own original research.

Symmetries in toric topology.

I confirm that:

1. This work was done wholly or mainly while in candidature for a research degree at this University;
2. Where any part of this thesis has been submitted for a degree or any other qualification at this University or any other institution, this has been clearly stated;
3. Where I have consulted the published works of others, this has always been clearly attributed;
4. Where I have quoted from the work of others, the source is always given. With the exception of such quotations, this thesis is entirely my own work;
5. I have acknowledged all main sources of help;
6. Where the thesis is based on work done by myself or jointly with others, I have made clear exactly what was done by others and what I have contributed myself.

Signed:

Date:

Acknowledgements

First I would like to express my deep gratitude to my supervisor Prof. Jelena Grbić. This thesis would not have been done without her guidance on math projects. Throughout my past four years, she offered a lot of helpful presentation skills and was very patient to read through and improve my research writing. Her encouragement helped me build confidence and push me to achieve a better goal. Moreover, in my final year, she gave me useful career advice and helped me to apply a postdoc.

I would also like to thank the Mathematical department at Southampton, for providing me the funding to support my studies.

Thanks to all the PhD students for organising and joining a lot of activities: seminars, reading groups, board games and random walks. In particular, thanks to those people for a lot of mathematical discussions: Ingrid and Larry for homotopy theory, Robin, Fabio and Matt B. for algebras, Jiawen for geometry. Thanks to Abi for exploring the world of toric topology and encouraging each other together. Thanks to all YRM 2018 organisers for working nicely in a team. Thanks to George, Matt S. and Guy for reading through the draft of this thesis and useful comments.

My friends in China, Shanshan and Fangting, I am always grateful for your encouragement whenever I met difficulties.

Special thanks to my parents for supporting me all the time and my sister for solving the family issues without letting me worry about them. Grandpa, I am grateful that you encouraged me to read books and articles since I was little. Missing you.

Notation

$\mathbb{R}$	The field of real numbers
$\mathbb{C}$	The filed of complex numbers
$\mathbb{Z}$	The ring of integers
$\mathbf{k}$	A field or $\mathbb{Z}$
R	The commutative ring with unit
1_R	The multiplicative identity of R
$R[m] = R[v_1, \ldots, v_m]$	The polynomial ring with m variable over R
$v_\sigma = v_{i_1} \ldots v_{i_l}$	A monomial in $R[m]$ for $\sigma = \{i_1, \ldots, i_l\} \subseteq [m]$
$A \otimes B$	The tensor product of R-algebras A and B over R
$[m] = \{1, 2, \ldots, m\}$	An m-element set
$\|A\|$	The cardinality of a finite set A
$A \sqcup B$	The disjoint union of finite sets A and B
K	A finite simplicial complex
K'	The barycentric subdivision of K
K_J	Full subcomplex of K to the vertex set J
$R[K]$	The Stanley-Reisner ring of K
I_K	The Stanley-Reisner ideal of K
P^n	A simple n-polytope
∂P^n	The boundary of a simple polytope P^n
K_P	The nerve complex of P^n
$\emptyset$	The empty set
Δ^{m-1}	The $(m-1)$-simplex
$\mathbb{I} = [0, 1]$	The unit interval
$\mathbb{I}^m$	The m-unit cube
T^m	The m-torus
Δ^k_m	The k-skeleton of Δ^{m-1}
$\mathbb{P}$	The set of minimal missing faces of K
$\Delta_{\mathbb{P}}$	The simplex on the vertex set $\mathbb{P}$
$K_{\mathbb{P}}$	The simplicial complex with minimal missing faces $\mathbb{P}$
$\Lambda^*[\mathbb{P}]$	The exterior algebra with basis elements in $\mathbb{P}$
$\mathcal{P}$	A finite poset (partially ordered set)
$X \vee Y$	The wedge sum of based spaces X and Y

$X \wedge Y$	The smash product of based spaces X and Y
$\Sigma X = S^1 \wedge X$	The suspension of a based space X
$K * L$	The join of simplicial complexes K and L
$X * Y$	The join of based spaces X and Y
$\mathrm{Cone}\, X$	The cone on a based space X
$(\underline{X}, \underline{A})^K$	A polyhedral product
$\mathcal{Z}_K = (D^2, S^1)^K$	A moment-angle complex
$\mathcal{Z}_m$	The moment-angle complex corresponding to m disjoint points
$DJ_K = (BS^1, *)^K$	A Davis-Januszkiewicz space
$(\underline{X}, \underline{A})^{\wedge K}$	A polyhedral smash product
$X \simeq Y$	Spaces X and Y are homotopy equivalent
$X \cong Y$	Spaces X and Y are homeomorphic
$M \cong N$	R-modules M and N are isomorphic
Σ_m	The symmetric group of degree m
$H \leqslant G$	H is a subgroup of G
G/H	The left coset space of groups $H \leqslant G$
$\mathcal{Z}_K/S^1_d$	The quotient space of $\mathcal{Z}_K$ under the diagonal S^1-action

Chapter 1

Introduction

Back in 1977, Hochster [26] decomposed the Tor-algebra $\mathrm{Tor}_{R[m]}(R[K], R)$ into the cohomologies of full subcomplexes of K as R-modules, where K is a finite simplicial complex. In 2000, Buchstaber and Panov [8] first introduced the space known as the moment-angle complex $\mathcal{Z}_K$ and proved that its cohomolgy is isomorphic to the Tor-algebra as R-algebras. It made possible to show that Hochster's decomposition is an isomorphism of algebras, a result due to Baskakov [4]. Since then, studies of moment-angle complexes include the studies of connections among topological properties of $\mathcal{Z}_K$, algebraic properties of $\mathrm{Tor}_{R[m]}(R[K], R)$ and combinatorial structures of K.

For example, Grbić and Theriault [22] proved that if K is shifted, then the homotopy type of $\mathcal{Z}_K$ is a wedge of spheres. This is a topological version of the algebraic statement that if K is shifted, then all multiplications in $\mathrm{Tor}_{R[m]}(R[K], R)$ vanish. Bahri-Bendersky-Cohen-Gitler [3] proved the existence of a homotopy splitting of moment-angle complexes which leads to a homological decomposition, a topological proof of Hochster's decomposition.

A polyhedral product $(\underline{X}, \underline{A})^K$ is defined by a subspace of a product space determined by a finite simplicial complex K and a sequence of topological pairs, which is a homotopy theoretical generalisation of moment-angle complexes. The studies of polyhedral products via homotopy theory extend applications to other area, such as combinatorics and geometry.

In particular, one of the fundamental properties of polyhedral products is functoriality, which indicates two types of symmetries, one from the symmetries of simplicial complexes and one from group actions on the topological pairs. Our work in this thesis consists of two projects by considering these two types of symmetries of polyhedral products.

In [20], we considered how the symmetries of simplicial complexes influence the symmetries of polyhedral products. If a finite group G acts on K simplicially, then $(X, A)^K$ is a

CW-G-complex. We showed that the homotopy decomposition [3] of $\Sigma(X,A)^K$ is then G-equivariant after suspension. In the case of Σ_m-polyhedral products, we give criteria on simplicial Σ_m-complexes which imply representation stability of Σ_m-representations $\{H_i((X,A)^{K_m})\}$ in the sense of Church-Farb [15]. This content is contained in Chapter 3.

Actions on polyhedral products can also arise from coordinatewise actions on the topological pair. In this case, we are interested in the polyhedral product $(\operatorname{Cone} G, G)^K$ which is a G^m-invariant subspace of $(\operatorname{Cone} G)^m$, where G is a compact Lie group. By taking $G = S^1$ or $\mathbb{Z}_2$, this action specialises to T^l-actions $(1 \le l \le m)$ on moment-angle complexes $\mathcal{Z}_K$ or $\mathbb{Z}_2^l$-actions on real moment-angle complexes $\mathbb{R}\mathcal{Z}_K$, respectively. Here T^l (resp. $\mathbb{Z}_2^l$) acts on $\mathcal{Z}_K$ (resp. $\mathbb{R}\mathcal{Z}_K$) as a subtorus group of T^m (resp. $\mathbb{Z}_2^m$). These quotient spaces $\mathcal{Z}_K/T^l$ and $\mathbb{R}\mathcal{Z}_K/\mathbb{Z}_2^l$, known as toric spaces, play a key role in toric topology.

Cai [11] introduced a differential graded algebra $(\mathcal{R}/I_K, d)$ with homology algebraically isomorphic to the integral cohomology of $\mathbb{R}\mathcal{Z}_K$. Choi and Park [13] considered the $\mathbb{Z}_2^l$-action on $\mathcal{R}/I_K$ and deduced an analogous Hochster's formula for the cohomology of $\mathbb{R}\mathcal{Z}_K/\mathbb{Z}_2^l$. Panov [35] identified the cohomology of the quotient space of $\mathcal{Z}_K/H$ to an appropriate Tor-algebra (4.4) if the subtorus H acts freely on $\mathcal{Z}_K$.

However, the proofs of Hochster's formula and Choi-Park's formula can not be generalised to the case of partial quotients of $\mathcal{Z}_K$, since Hochster's formula relies on a multigrading structure, which does not exist in the case of $\mathcal{Z}_K/H$ in general, and Choi-Park's formula relies on a result which is only valid for finite group actions. This makes it more difficult to find a generalised Hochster-type formula for the cohomology of $\mathcal{Z}_K/H$.

In the last chapter, we consider a special case of torus actions on $\mathcal{Z}_K$ when H is of rank $r = 1$. In this case, the circle action on $\mathcal{Z}_K$ by $(s_1, \dots, s_m)$ determines a chain complex $(C_*(L), \delta)$ (Construction 4.4.5), for which we relate the cohomology of the quotient space $\mathcal{Z}_K/S^1$ to homology groups $H_*(L, \delta)$. In the end, we finish this thesis by studying the homotopy type of $\mathcal{Z}_K/S^1$ for certain K.

I shall summarise the results on the work in representation stability of polyhedral products. This is a joint work with my supervisor Jelena Grbić (see Chapter 3).

Theorem 3.3.3. *Let K be a simplicial G-complex on $[m]$. Then there is a homotopy G-decomposition*

$$\theta\colon \Sigma^2(X,A)^K \simeq \Sigma^2 \bigvee_{J\subseteq[m]} (X,A)^{\wedge K_J}$$

where the G-action on $\Sigma^2(X,A)^K$ is induced by the G-action on X^m, and the G-action on the right hand side is a permutation of wedge summands by the G-action on $2^{[m]}$.

Passing to homology groups, we obtain the following statement.

Theorem 3.3.5. *Let K be a simplicial G-complex on $[m]$. Then there exists an isomorphism of $\mathbf{k}G$-modules*

$$\widetilde{H}_i((X,A)^K;\mathbf{k}) \cong \bigoplus_{J\subseteq[m]} \widetilde{H}_i((X,A)^{\wedge K_J};\mathbf{k}) \cong \bigoplus_{J\in[m]/G} \mathrm{Ind}_{G_J}^{G}\widetilde{H}_i((X,A)^{\wedge K_J};\mathbf{k})$$

where G acts on the middle term by permuting the summands such that $g\cdot\widetilde{H}_i((X,A)^{\wedge K_J};\mathbf{k}) = \widetilde{H}_i((X,A)^{K_{g\cdot J}};\mathbf{k})$, $[m]/G$ is a set of representatives of the orbit of $2^{[m]}\setminus\emptyset$ under G and G_J is the stabiliser of J.

If a symmetric group Σ_m of degree m acts on K, then the homology group $H_i((X,A)^K;\mathbf{k})$ is a Σ_m-representation over $\mathbf{k}$. Thus a sequence of simplicial Σ_m-complexes

$$\ldots \subseteq K_{m-1} \subseteq K_m \subseteq K_{m+1} \subseteq \ldots$$

gives rise to a sequence of Σ_m-representations

$$\ldots \longrightarrow H_i((X,A)^{K_{m-1}};\mathbf{k}) \longrightarrow H_i((X,A)^{K_m};\mathbf{k}) \longrightarrow H_i((X,A)^{K_{m+1}};\mathbf{k}) \longrightarrow \ldots \quad (1.1)$$

We also considered sufficient conditions for a sequence of simplicial Σ_m-complexes such that the induced sequence of Σ_m-representations (1.1) satisfies the property of representation stability.

Theorem 3.4.12. *Let $\{K_m, i_m\}$ be a consistent sequence of finite simplicial complexes and X be a connected, based CW-complex of finite type with a based subcomplex A. Suppose that $\{K_m, i_m\}$ is completely surjective and stabiliser consistent. Then the consistent sequence of Σ_m-representations $\{\widetilde{H}_i((X,A)^{K_m};\mathbf{k}), i_{m*}\}$ for $\mathrm{char}\,\mathbf{k}=0$ is uniformly representation stable.*

In my second project regarding to torus actions on moment-angle complexes, we first apply the Taylor resolution and Koszul resolution to the Tor-algebra (4.4), which implies two differential graded algebras whose cohomologies are isomorphic to $H^*(\mathcal{Z}_K/H)$ as algebras.

Taylor algebra. The differential graded algebra $(\Lambda^*[\mathbb{P}]\otimes R[x_1,\ldots,x_r],d)$ is defined by

$$\begin{aligned} d(\sigma_{i_1}\ldots\sigma_{i_p}) =& \sum_{1\le t\le i}(-1)^{t-1}\delta_p(x_1,\ldots,x_r)\sigma_{i_1}\ldots\hat{\sigma}_{i_t}\ldots\sigma_{i_p} \\ dx_j =& 0 \end{aligned}$$

where $\delta_p(x_1,\ldots,x_r) = \prod\limits_{i\in\mathcal{S}_\sigma\setminus\mathcal{S}_{\partial_t\sigma}} (\sum\limits_{j=1}^{r} s_{ij}x_j)$ if $\mathcal{S}_\sigma \neq \mathcal{S}_{\partial_t\sigma}$ and 1 otherwise.

The bigradings of $\sigma_{i_1}\ldots\sigma_{i_p}$ and x_j are given by

$$\mathrm{bideg}\,\sigma_{i_1}\ldots\sigma_{i_p} = (-p, 2|\mathcal{S}_\sigma|) \text{ and } \mathrm{bideg}\,x_j = (0,2).$$

Koszul algebra. The differential graded algebra $(\Lambda[u_1,\dots,u_{m-r}]\otimes R[K],d')$ is defined by

$$d'u_i=\lambda_{i1}v_1+\dots\lambda_{im}v_m \text{ and } d'v_j=0.$$

The bigradings of u_i and v_j are given by

$$\operatorname{bideg} u_i=(-1,2) \text{ and } \operatorname{bideg} v_j=(0,2).$$

Theorem (Theorem 4.3.3, Proposition 4.3.6)**.** *There exist isomorphisms of R-algebras*

$$\begin{aligned}H^*(\mathcal{Z}_K/H;R)\cong \operatorname{Tor}_{H^*(B(T^m/H);R)}(R[K],R)&\cong H(\Lambda^*[\mathbb{P}]\otimes R[x_1,\dots,x_r],d)\\ &\cong H(\Lambda[u_1,\dots,u_{m-r}]\otimes R[K],d').\end{aligned}$$

Specialising to free circle actions on $\mathcal{Z}_K$, one can construct a filtration $\{L_p\mid 1\le p\le m\}$ of simplicial complexes on the vertex set $\mathbb{P}$,

$$L_p=\{\{\sigma_{l_1},\dots,\sigma_{l_j}\}\subseteq\mathbb{P}\mid |\sigma_{l_1}\cup\dots\cup\sigma_{l_j}|\le p\}$$

where $\mathbb{P}$ denotes the set of all minimal missing faces of K.

The circle action on $\mathcal{Z}_K$ by $(s_1,\dots,s_m)$ induces a chain complex $(C_*(L_p;\delta))$ for each L_p defined in Construction 4.4.5. The result follows next.

Theorem 4.4.11. *Suppose that S^1 acts freely on $\mathcal{Z}_K$. Then there exists an isomorphism of R-algebras*

$$H^*(\mathcal{Z}_K/S^1;R)\cong\bigoplus_{2p-j\ge 0}H_{j-1}(L_p,\delta).$$

In the case of the diagonal action, the homology $H_*(L_p,\delta)$ is the reduced simplicial homology of the simplicial complex L_p.

Corollary 4.4.12. *There is an isomorphism of R-algebras*

$$H^*(\mathcal{Z}_K/S^1_d;R)\cong\bigoplus_{2p-j\ge 0}\widetilde{H}_{j-1}(L_p;R).$$

We continue to consider homotopy types of of partial quotients $\mathcal{Z}_K/S^1$ under free circle actions for certain K.

Theorem 4.5.4. *Let S^1 acts freely on $\mathcal{Z}_K$ by $(s_1,\dots,s_m)$. Assume that there exits a vertex $v\in K$ such that $s_v=\pm1$.*

(a) *There exist homotopy equivalences*

$$F_{\text{Link}}\simeq\mathcal{Z}_{\text{Link}_K(v)},\quad F_{\text{Rest}}\simeq\mathcal{Z}_{\text{Rest}_K(v)},\quad F_{\text{Star}}\simeq\mathcal{Z}_{\text{Link}_K(v)}/S^1.$$

(b) *The quotient space* $\mathcal{Z}_K/S^1$ *is the homotopy pushout of the diagram* $\mathcal{Z}_{\mathrm{Link}_K(v)}/S^1 \xleftarrow{q} \mathcal{Z}_{\mathrm{Link}_K(v)} \xrightarrow{\iota} \mathcal{Z}_{\mathrm{Rest}_K(v)}$, *where* ι *is a map induced by the simplicial inclusion* $\mathrm{Link}_K(v) \longrightarrow \mathrm{Rest}_K(v)$ *and* q *is a quotient map.*

Let $\mathcal{Z}_{\Delta^k_m}$ denote the moment-angle complex corresponding to the k-skeleton of Δ^{m-1}. We identify the homotopy type of the homotopy cofibre of the quotient map $\mathcal{Z}_{\Delta^k_m} \longrightarrow \mathcal{Z}_{\Delta^k_m}/S^1_d$, which consequently gives the homotopy type of $\mathcal{Z}_{\Delta^k_m}/S^1_d$ under the diagonal action.

Theorem 4.5.10. *Let* $C_{k,m}$ *denote the homotopy cofibre of the quotient map* $\mathcal{Z}_{\Delta^k_m} \longrightarrow \mathcal{Z}_{\Delta^k_m}/S^1_d$. *Then there exists a homotopy equivalence*

$$C_{k,m} \simeq \mathbb{C}P^{k+2} \vee (\bigvee_{i=1}^{k+1} S^{2i-1} * \mathcal{Z}_{\Delta^{k+1-i}_{m-i}}) \vee (S^{2k+3} * T^{m-k-2}).$$

Corollary 4.5.13. *The homotopy type of* $\mathcal{Z}_{\Delta^k_m}/S^1_d$ *is* $\mathcal{Z}_{\Delta^k_{m-1}} \vee C_{k-1,m-1}$.

Chapter 2

Background

The cohomology of moment-angle complexes and their quotient spaces closely relates to the Tor-algebras $\mathrm{Tor}_{R[t_1,\ldots,t_{m-r}]}(R[K], R)$ (4.4) of K, which are one of the main objects of research in commutative algebra. The study of these objects provides topological approaches to the study of algebraic properties of these algebras.

2.1 Preliminaries in commutative algebra

We start with some basic combinatorial definitions.

Definition 2.1.1. An abstract simplicial complex K on $[m] = \{1, 2, \ldots, m\}$ is a collection of subsets of $[m]$ such that

(1) if $\sigma \in K$, then any subset of σ also belongs to K;

(2) if σ and τ are in K, then the intersection $\sigma \cap \tau$ is in K.

We always assume that $\emptyset \in K$.

The finite set $[m]$ is called the vertex set of K. A ghost vertex i of K is whenever $i \in [m]$ but $i \notin K$. For example, consider $K = \{\emptyset, \{1\}\}$ on $[2]$. Then $\{2\}$ is a ghost vertex of K on $[2]$. The dimension of a simplex σ of K is $\dim \sigma = |\sigma| - 1$, where $|\sigma|$ is the cardinality of σ. The dimension of the simplicial complex K is the maximal dimension of its simplices (i.e. $\dim K = \max_{\sigma \in K} |\sigma| - 1$).

Example 2.1.2. The boundary of a simplex Δ^{m-1} is a simplicial complex of dimension $(m-2)$. The k-skeleton Δ^k_m of a simplex Δ^{m-1} consists of all subsets of $[m]$ with cardinality at most $k+1$. This is a simplicial complex of dimension k.

Construction 2.1.3. Let K_1 and K_2 be simplicial complexes on $[m_1]$ and $[m_2]$, respectively. The *join* $K_1 * K_2$ of two simplicial complexes is a simplicial complex on the vertex set $[m_1] \sqcup [m_2]$ whose faces are of the form $\sigma \sqcup \tau$, where $\sigma \in K_1$ and $\tau \in K_2$ and $\sqcup$ denotes the disjoint union.

Example 2.1.4. Let $K_1 = \underset{1}{\bullet}\ \ \underset{2}{\bullet}$ and $K_2 = \underset{3}{\bullet}\ \ \underset{4}{\bullet}$ be a disjoint union of two points. Then the join of K_1 and K_2 is the boundary of a square.

Example 2.1.5. The cone on K, denoted by Cone$K$, is the join of K and one single vertex.

Definition 2.1.6. For $J \subseteq [m]$, the *full subcomplex* K_J is the subcomplex of K whose vertex set is J, that is

$$K_J = \{\sigma \cap J \mid \sigma \in K\}.$$

Definition 2.1.7. An n-polytope P^n is called *simple* if there are exactly n codimension-one faces meeting at each vertex. These codimension-one faces are called facets of P^n.

Examples of simple polytopes include m-gons, prisms, the m-simplex Δ^m and the m-cube $\mathbb{I}^m$, where $\mathbb{I} = [0, 1]$. A typical non-example is the cone on an m-gon when $m \geq 4$.

Definition 2.1.8. Let P^n be an simple n-polytope and let $\mathcal{F} = \{F_1, \dots, F_m\}$ be the facets of P^n. The *nerve complex* ∂P^* of P^n is the boundary of its dual polytope P^* ([10, p.4]), which is a simplicial complex, denoted by K_P, on the vertex set $\mathcal{F}$, where $\{F_{i_1}, \dots, F_{i_l}\}$ forms a simplex of K_P if and only if the intersection $F_{i_1} \cap \dots \cap F_{i_l}$ is non-empty. Note that K_P is a triangulation of an $(n-1)$-sphere.

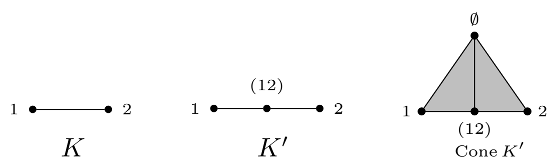
For example, if P is a simplex Δ^n or an m-gon, then the nerve complex K_P is the boundary of P since P and P^* are combinatorially isomorphic.

Definition 2.1.9. The *barycentric subdivision* K' of K is a simplicial complex on the vertex set $\{\sigma \in K \mid \sigma \neq \emptyset\}$, where $(\sigma_{i_1}, \dots, \sigma_{i_l})$ forms a face of K' if and only if there exist face inclusions $\sigma_{i_1} \subsetneq \dots \subsetneq \sigma_{i_l}$, where $\sigma_i \subsetneq \sigma_j$ means that σ_i is a proper face of σ_j. The cone on K', denoted by Cone K', can be seen as a simplicial complex on the vertex set $\{\sigma \in K\}$ (including the empty face of K) constructed in the same way by adding the empty set $\emptyset$ as the cone vertex.

Example 2.1.10. Let K be the 1-simplex. Then the abstract simplicial complex K' is

$$K' = \{\emptyset, \{1\}, \{2\}, \{1, 2\}, \{\{1\}, \{1, 2\}\}, \{\{2\}, \{1, 2\}\}\}.$$

Geometrically, we have



The shading denotes the face of ConeK'

Definition 2.1.11. Let $(\mathcal{P}, \leq)$ be a finite poset (partially ordered set). The *order complex* $\Delta(\mathcal{P})$ is a simplicial complex on the vertex set given by elements of $\mathcal{P}$, where the simplicies are the tuples $(p_{i_1}, \ldots, p_{i_l})$ such that $p_{i_1} < \ldots < p_{i_l}$ in $\mathcal{P}$.

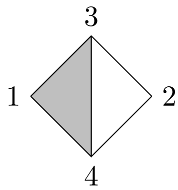
For example, a simplicial complex K has an associated poset $\bar{K}$, which has elements consisting of faces of K, ordered by the reverse inclusion, i.e., $\sigma \leq \tau$ if and only if $\tau \subseteq \sigma$. Thus, the empty set $\emptyset$ is the maximal element of $\bar{K}$. Also, the order complex $\Delta(\bar{K})$ is the simplicial complex ConeK' defined in Definition 2.1.9.

2.1.1 Stanley-Reisner ring and Tor-algebras

Let R be a commutative ring with unit, let 1_R denote the multiplicative identity of R and let $\mathbf{k}$ be a field or $\mathbb{Z}$. The tensor product $\otimes$ is taken over R, unless otherwise stated.

Definition 2.1.12. The Stanley-Reisner ring $R[K]$ of a simplicial complex K on $[m]$ is a quotient ring $R[K] = R[v_1, \cdots, v_m]/I_K$, where $I_K = (v_\sigma \colon \sigma \notin K)$ is the Stanley-Reisner ideal generated by those monomials $v_\sigma = \prod\limits_{i \in \sigma} v_i$ which correspond to non-faces σ of K.

Example 2.1.13. Let K be the following simplicial complex.



Then the Stanley-Reisner ring of K is $R[K] = R[v_1, v_2, v_3, v_4]/(v_1v_2, v_2v_3v_4)$.

The underlying R-module of $R[K]$ is free, which is infinite dimensional unless $K = \emptyset$.

Lemma 2.1.14. *[10, Proposition 3.1.9] The Stanley-Reisner ring $R[K]$ has a basis consisting of 1_R and $v_{i_1}^{a_1} \ldots v_{i_l}^{a_l}$ with $a_j > 0$ and $\{i_1, \ldots, i_l\} \in K$, as an R-module.*

The quotient homomorphism $R[v_1, \cdots, v_m] \longrightarrow R[K]$ gives $R[K]$ a module structure over $R[m] = R[v_1, \cdots, v_m]$. Moreover, R is an $R[m]$-module induced by the homomorphism $R[m] \xrightarrow{\epsilon} R$ which sends each v_i of $R[m]$ to 0. Therefore, a Tor-module $\mathrm{Tor}_{R[m]}(R[K], R)$ is established. In fact, it is an R-algebra induced by the Koszul algebra.

Construction 2.1.15 (Koszul resolution)**.** The Koszul resolution of R over $R[m]$ is a long exact sequence of free $R[m]$-modules defined by

$$\begin{aligned} &0 \longrightarrow \Lambda^m[u_1, \ldots, u_m] \otimes R[v_1, \ldots, v_m] \xrightarrow{d} \Lambda^{m-1}[u_1, \ldots, u_m] \otimes R[v_1, \ldots, v_m] \xrightarrow{d} \\ &\ldots \xrightarrow{d} \Lambda^1[u_1, \ldots, u_m] \otimes R[v_1, \ldots, v_m] \xrightarrow{d} R[v_1, \ldots, v_m] \xrightarrow{\epsilon} R \longrightarrow 0 \end{aligned} \tag{2.1}$$

where $\Lambda[u_1, \ldots, u_m]$ is an exterior algebra on m generators and $\Lambda^i[u_1, \ldots, u_m]$ denotes the R-module generated by u_J of length i, i.e., $u_J = u_{j_1} \ldots u_{j_i}$ with $J = \{j_1, \ldots, j_i\} \subseteq [m]$. Here the bigradings of u_i and v_i are given by

$$\operatorname{bideg} u_i = (-1, 2) \text{ and } \operatorname{bideg} v_i = (0, 2).$$

The differentials on the algebraic generators are defined by

$$du_i = v_i \text{ and } dv_i = 0.$$

These extend to differentials on $\Lambda[u_1, \ldots, u_m] \otimes R[v_1, \ldots, v_m]$ according to the *Leibniz identity* (i.e. $d(a \cdot b) = da \cdot b + (-1)^{\deg(a)} a \cdot db$).

Applying $- \otimes_{R[m]} R[K]$ to (2.1), we have a differential graded algebra

$$(\Lambda[u_1, \ldots, u_m] \otimes R[K], d), du_i = v_i \text{ and } dv_i = 0 \tag{2.2}$$

such that $\operatorname{Tor}_{R[m]}(R[K], R) \cong H(\Lambda[u_1, \cdots, u_m] \otimes R[K], d)$.

Recall that the tensor product $A \otimes B$ of two R-algebras A and B is an R-algebra by

$$(a_1 \otimes b_1) \cdot (a_2 \otimes b_2) = (a_1 a_2) \otimes (b_1 b_2) \tag{2.3}$$

where $a_i \in A$ and $b_i \in B$, $i = 1, 2$.

Definition 2.1.16. The differential graded algebra (2.2) is known as *Koszul algebra*, whose multiplication is the tensor product (2.3) of the exterior algebra $\Lambda[u_1, \ldots, u_m]$ and the Stanley-Reisner ring $R[K]$ over R.

The isomorphism $\operatorname{Tor}_{R[m]}(R[K], R) \cong H(\Lambda[u_1, \cdots, u_m] \otimes R[K], d)$ gives an R-algebra structure on $\operatorname{Tor}_{R[m]}(R[K], R)$.

Definition 2.1.17. We refer to $\operatorname{Tor}_{R[m]}(R[K], R)$ as the *Tor-algebra* of a simplicial complex K over R.

Example 2.1.18. Consider $K = \emptyset$ on $[m]$. Then the Stanley-Reisner ring

$$R[\emptyset] = R[v_1, \ldots, v_m]/(v_1, \ldots, v_m) = R.$$

In this case, the differentials on the Koszul algebra are trivial. Thus, $\operatorname{Tor}_{R[m]}(R, R) = \Lambda[u_1, \cdots, u_m]$.

In [34], a quotient algebra $\mathcal{R}^*(K)$ of the Koszul algebra was introduced whose underlying R-module is finite dimensional and its cohomology is isomorphic to $\operatorname{Tor}_{R[m]}(R[K], R)$.

Construction 2.1.19. Define the quotient algebra

$$\mathcal{R}^*(K) = \Lambda[u_1, \cdots, u_m] \otimes R[K]/(u_i v_i = v_i^2 = 0, 1 \leq i \leq m)$$

where the differentials d and bigrading are given by

$$du_i = v_i,\ dv_i = 0 \text{ and } \operatorname{bideg} u_i = (-1, 2),\ \operatorname{bideg} v_i = (0, 2).$$

There exists an R-basis $\{u_J v_I \mid J \cap I = \emptyset, J \subseteq [m], I \in K\}$ of the underlying R-module of $\mathcal{R}^*(K)$. Denote by $\iota\colon \mathcal{R}^*(K) \longrightarrow \Lambda[u_1, \ldots, u_m] \otimes R[K]$ an R-homomorphism which sends the basis element $u_J v_I$ identically and let $\rho\colon \Lambda[u_1, \ldots, u_m] \otimes R[K] \longrightarrow \mathcal{R}^*(K)$ be the quotient homomorphism. Thus, $\rho\iota = \mathrm{id}$. Though $\iota\rho \neq \mathrm{id}$, the next statement says that there exists a cochain homotopy h between $\iota\rho$ and id. Since the quotient homomorphism ρ is a map of algebras, ρ induces an isomorphism of algebras on cohomology.

Lemma 2.1.20. *[10, 34] There exists a cochain homotopy*

$$h_i\colon \Lambda^i[u_1, \ldots, u_m] \otimes R[K] \longrightarrow \Lambda^{i+1}[u_1, \ldots, u_m] \otimes R[K]$$

such that $dh - hd = \mathrm{id} - \iota\rho$. Hence, the quotient homomorphism ρ induces isomorphisms of R-algebras

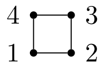
$$Tor_{R[m]}(R[K], R) \cong H(\Lambda[u_1, \cdots, u_m] \otimes R[K], d) \cong H(\mathcal{R}^*(K), d).$$

Example 2.1.21. Let K be the boundary of an $(m-1)$-simplex. Then $R[K] = R[m]/(v_1 \ldots v_m)$. The R-basis of $\mathcal{R}^*(K)$ consists of $\{u_J v_I \mid J \cap I = \emptyset, I \in K\}$. The differentials on generators $u_J v_I$ of $\mathcal{R}^*(K)$ are

$$du_J v_I = \begin{cases} 0 & \text{if } J = \{j\} \text{ and } I = [m] \setminus \{j\} \\ \sum\limits_{j \in J} \epsilon(j, J) u_{J \setminus \{j\}} v_{j \cup I} & \text{otherwise} \end{cases}$$

where $\epsilon(j, J) = (-1)^{r-1}$ if j sits at the r-th position of J. The generators of $\operatorname{Ker} d^{-1,*}$ (i.e. cocyles of bigrading $(-1, *)$) are $\{u_j v_{[m] \setminus \{j\}}, 1 \leq j \leq m\}$. Since $d(u_{j_1} u_{j_2} v_{[m] \setminus \{j_1, j_2\}}) = u_{j_2} v_{[m] \setminus j_1} - u_{j_1} v_{[m] \setminus j_1}$, $\mathrm{Tor}_{R[m]}^{-1,2m}(R[K], R) \cong R$ generated by $u_1 v_2 v_3 \ldots v_m$. Let $\alpha = \sum r_{J,I} u_J v_I \in \operatorname{Ker} d^{-2,*}$. We can write $\alpha = u_1 \alpha_1 + \alpha_2$, where $\alpha_1 \in \mathcal{R}^{-1,*}$ and $\alpha_2 \in \mathcal{R}^{-2,*}$ whose generators do not contain u_1 as a factor. Hence, $d\alpha = \alpha_1 v_1 - u_1 d\alpha_1 + d\alpha_2 = 0$ so that $\alpha_1 v_1 + d\alpha_2 = 0$ and $d\alpha_1 = 0$. It follows that $\alpha_1 = 0$ since $\alpha_1 \in \operatorname{Ker} d^{-1,*}$ and the supporting generators $u_j v_I$ of α_1 satisfying $|I| \leq m-2$. So $\alpha = \alpha_2$ whose supporting generators do not contain u_1. Repeating this, gives $\alpha = 0$. Similarly, we have $\operatorname{Ker} d^{-j,*} = 0$ for $j \geq 2$. Hence, we have

$$\mathrm{Tor}_{R[m]}^{0,0}(R[K], R) = R \text{ and } \mathrm{Tor}_{R[m]}^{-1,2m}(R[K], R) = R.$$

Example 2.1.22. Let K be the boundary of a 4-gon, 

. Then $R[K] \cong R[4]/(v_1v_3, v_2v_4)$. The non-trivial differentials on generators $u_J v_I$ ($J \cap I = \emptyset$ and $I \in K$) of $\mathcal{R}^*(K)$ are

$$d(u_j v_i) = v_j v_i \text{ if } \{j,i\} \in K$$

$$d(u_{j_1}u_{j_2}v_i) = \begin{cases} u_{j_2}v_{j_1}v_i & \text{if } j_1 \cup i \in K \text{ and } j_2 \cup i \notin K \\ -u_{j_1}v_{j_2}v_i & \text{if } j_1 \cup i \notin K \text{ and } j_2 \cup i \in K \\ u_{j_2}v_{j_1}v_i - u_{j_1}v_{j_2}v_i & \text{if } j_1 \cup i \text{ and } j_2 \cup i \in K \end{cases}$$

$$d(u_{[m]\setminus i}v_i) = \sum_{j \in [m]\setminus i} \pm u_{[m]\setminus\{j,i\}}v_iv_j.$$

The cocycles of bigrading $(-1,*)$ has a basis

$$\{u_1v_3, u_1v_2v_3, u_1v_3v_4, u_3v_1, u_3v_1v_2, u_3v_1v_4, u_2v_4, u_2v_1v_4, u_2v_3v_4, u_4v_2, u_4v_1v_2, u_4v_2v_3\}.$$

Since $du_1u_3 = u_3v_1 - u_1v_3$ and $du_2u_4 = u_4v_2 - u_2v_4$, $\mathrm{Tor}^{-1,4}_{R[m]}(R[K], R) \cong R \oplus R$ generated by u_1v_3 and u_2v_4. A direct calculation implies that $\mathrm{Tor}^{-2,8}(R[K], R)$ is generated by $u_1u_2v_3v_4$ and $\mathrm{Tor}^{0,0}_{R[m]}(R[K], R) = R$ (the ground ring R).

2.1.2 Hochster's Formula

Hochster's formula is a useful tool for computing $\mathrm{Tor}_{R[m]}(R[K], R)$ by calculating the reduced simplicial cohomology of full subcomplexes K_J of K. Recall that for $J \subseteq [m]$, the full subcomplex K_J is defined by $K_J = \{\sigma \cap J \mid \sigma \in K\}$.

Theorem 2.1.23 (Hochster [26])**.** *Let K be a simplicial complex K on $[m]$. There are isomorphisms of R-modules*

$$\mathrm{Tor}^{-i,2j}_{R[m]}(R[K], R) = \bigoplus_{J \subset [m], |J| = j} \widetilde{H}^{j-i-1}(K_J; R).$$

Note that $\widetilde{H}^{-1}(\emptyset; R) = R$.

Next, we give a sketch proof of Hochster's formula which can be adapted to to show Proposition 4.3.11. It relies on the multigrading structure of $\mathcal{R}^*(K)$ (Construction 2.1.19).

Construction 2.1.24 (multigrading)**.** The multigrading of $\mathcal{R}^*(K)$ is defined on the R-basis elements by

$$\mathrm{mdeg}\, u_{J\setminus I}v_I = (-|J \setminus I|, 2J).$$

Let $J \subseteq [m]$. Define $\mathcal{R}^{*,2J}(K)$ to be the submodule of $\mathcal{R}^*(K)$ generated by basis elements $\{u_{J\setminus I}v_I \in \mathcal{R}^*(K) \mid I \in K\}$. Since $d(u_{J\setminus I}v_I) = \sum_{j \in J\setminus I, j\cup I \in K} \pm u_{J\setminus\{j\cup I\}}v_{j\cup I}$, where $u_{J\setminus\{j\cup I\}}v_{j\cup I} \in \mathcal{R}^{*,2J}(K)$, thus $\mathcal{R}^{*,2J}(K)$ is a cochain subcomplex of $\mathcal{R}^*(K)$.

We also have a decomposition $\mathcal{R}^{-i,2j}(K) = \bigoplus_{J\subseteq[m],|J|=j} \mathcal{R}^{-i,2J}(K)$ which induces a decomposition of the Tor-module $\mathrm{Tor}_{R[m]}^{-i,2j}(R[K],R) = \bigoplus_{J\subseteq[m],|J|=j} \mathrm{Tor}_{R[m]}^{-i,2J}(R[K],R)$.

Let σ^* denote the cochain basis element of $C^*(K;R)$ corresponding to an oriented simplex $\sigma \in K$. For any $J \subseteq [m]$, there exists a cochain isomorphism

$$\begin{aligned} f\colon C^{p-1}(K_J;R) &\longrightarrow \mathcal{R}^{p-|J|,2J}(K) \\ \sigma^* &\longmapsto \epsilon(\sigma,J)u_{J\setminus\sigma}v_\sigma \end{aligned}$$

where $\epsilon(\sigma,J) = \prod_{j\in\sigma} \epsilon(j,J)$ and $\epsilon(j,J) = (-1)^{r+1}$ if j sits at the r-th position in J with J written increasingly. We refer to [10, Theorem 3.2.9] for a detailed proof that f commutes with the differentials. Hence, f induces isomorphisms on cohomology

$$\widetilde{H}^{p-1}(K_J;R) \cong \mathrm{Tor}_{R[m]}^{p-|J|,2J}(R[K],R).$$

2.1.3 Taylor resolution

The Taylor resolution [39] of the Stanley-Reisner ring $R[K]$ is constructed as follows. Let $\mathbb{P} = \{\sigma_1,\ldots,\sigma_p\}$ consist of all minimal missing faces of K and let $\Lambda^*[\mathbb{P}]$ be an exterior algebra on generators corresponding to elements in $\mathbb{P}$ over R.

Construction 2.1.25 (Taylor resolution)**.** The Taylor resolution of $R[K]$ over $R[v_1,\ldots,v_m]$ is given by

$$\begin{aligned} \ldots \longrightarrow \Lambda^i[\mathbb{P}]\otimes R[v_1,\ldots,v_m] \xrightarrow{d} \Lambda^{i-1}[\mathbb{P}]\otimes R[v_1,\ldots,v_m] \xrightarrow{d} \\ \ldots \xrightarrow{d} \Lambda^1[\mathbb{P}]\otimes R[v_1,\ldots,v_m] \xrightarrow{d} R[v_1,\ldots,v_m] \longrightarrow R[K] \longrightarrow 0 \end{aligned} \tag{2.4}$$

where the differential operation d is defined by

$$\begin{aligned} d(\sigma_{j_1}\sigma_{j_2}\cdots\sigma_{j_q}) &= \sum_{0\le t\le q} (-1)^{t-1} \frac{\mathrm{lcm}(v_{\sigma_{j_1}}, v_{\sigma_{j_2}},\cdots,v_{\sigma_{j_q}})}{\mathrm{lcm}(v_{\sigma_{j_1}},\cdots,\widehat{v}_{\sigma_{j_t}},\cdots,v_{\sigma_{j_q}})}\sigma_{j_1}\cdots\widehat{\sigma}_{j_t}\cdots\sigma_{j_q} \\ d(v_i) &= 0,\ 0\le i\le m \end{aligned}$$

and $\mathrm{lcm}(v_{\sigma_{j_1}}, v_{\sigma_{j_2}},\cdots,v_{\sigma_{j_q}})$ stands for the least common multiple, $\widehat{\sigma}_{j_t}$ and $\widehat{v}_{\sigma_{j_t}}$ mean that σ_{j_t} and $v_{\sigma_{j_t}}$ are omitted, respectively.

The multigrading of $\Lambda^*[\mathbb{P}]\otimes R[v_1,\ldots,v_m]$ is given by

$$\mathrm{mdeg}\,(\sigma_{j_1}\sigma_{j_2}\cdots\sigma_{j_q}) = (-q, \mathrm{lcm}(v_{\sigma_{j_1}}, v_{\sigma_{j_2}},\cdots,v_{\sigma_{j_q}}))$$

and its bigrading is given by

$$\mathrm{bideg}\,(\sigma_{j_1}\sigma_{j_2}\cdots\sigma_{j_q}) = (-q, 2|\sigma_{j_1}\cup\ldots\cup\sigma_{j_q}|) \text{ and } \mathrm{bideg}\, v_j = (0,2).$$

Let $\sigma_{i_1}\ldots\sigma_{i_t} \in \Lambda^t[\mathbb{P}]$ and let $\sigma_{j_1}\ldots\sigma_{j_q} \in \Lambda^q[\mathbb{P}]$. The multiplication $\times$ of the Taylor resolution (2.4) is defined by

$$\sigma_{i_1}\ldots\sigma_{i_t} \times \sigma_{j_1}\ldots\sigma_{j_q} = \begin{cases} \frac{\operatorname{lcm}(v_{\sigma_1},\ldots,v_{\sigma_{i_t}})\operatorname{lcm}(v_{\sigma_{j_1}},\ldots,v_{\sigma_{j_q}})}{\operatorname{lcm}(v_{\sigma_{i_1}},\ldots,v_{\sigma_{i_t}},v_{\sigma_{j_1}},\ldots,v_{\sigma_{j_q}})}\sigma_{i_1}\ldots\sigma_{i_t}\sigma_{j_1}\ldots\sigma_{j_q} & \text{if } \{i_1,\ldots,i_t\}\cap\{j_1,\ldots,j_q\}=\emptyset \\ 0 & \text{otherwise.} \end{cases}$$

Applying $-\otimes_{R[m]} R$ to (2.4), since R is an $R[m]$-module by sending all v_i to zero, it reduces to a differential graded algebra $(\Lambda^*[\mathbb{P}], d)$, where the differential operation d is defined by

$$d(\sigma_{j_1}\sigma_{j_2}\cdots\sigma_{j_q}) = \sum_{1\le t\le q} (-1)^{t-1}\delta_t\sigma_{j_1}\cdots\widehat{\sigma}_{j_t}\cdots\sigma_{j_q}. \tag{2.5}$$

Here $\delta_t = 1$ if $\mathcal{S}_\sigma = \mathcal{S}_{\partial_t\sigma}$ and zero otherwise, where $\mathcal{S}_\sigma = \sigma_{j_1}\cup\ldots\cup\sigma_{j_q}$ and $\mathcal{S}_{\partial_t\sigma} = \sigma_{j_1}\cup\ldots\cup\widehat{\sigma}_{j_t}\cup\ldots\cup\sigma_{j_q}$.

The differential graded algebra $(\Lambda^*[\mathbb{P}], d)$ is called the Taylor algebra whose multigrading is defined as follows.

Construction 2.1.26 (multigrading [27])**.** The multigrading of $\Lambda^*[\mathbb{P}]$ is given by

$$\operatorname{mdeg}\sigma_{j_1}\ldots\sigma_{j_q} = (-q, 2(\sigma_{j_1}\cup\ldots\cup\sigma_{j_q})).$$

For any $J\subseteq[m]$, let $\Lambda^{*,2J}$ be the submodule of $\Lambda^*[\mathbb{P}]$ generated by elements of multigrading $(*, 2J)$. By the definition of the differential (2.5), $\Lambda^{*,2J}$ is closed under the differential. Thus, $\Lambda^{*,2J}$ is a cochain subcomplex of $\Lambda^*[\mathbb{P}]$. The decomposition $\Lambda^*[\mathbb{P}] = \bigoplus_{J\subseteq[m]}\Lambda^{*,2J}[\mathbb{P}]$ induces a decomposition on cohomology. Hence, the next statement follows.

Theorem 2.1.27 ([41])**.** *Let K be a simplicial complex on $[m]$ and let $\mathbb{P}$ consist of all minimal missing faces of K. Then we have an isomorphism of rings*

$$Tor_{R[m]}(R[K], R) \cong \bigoplus_{J\subseteq[m]} H_i(\Lambda^{-i,2J}[\mathbb{P}], d).$$

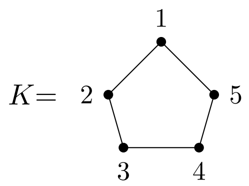
Here the product on $\bigoplus_{J\subseteq[m]} H(\Lambda^{*,2J}[\mathbb{P}], d)$ is given by

$$[c]\times[c'] = [c\cdot c']\cdot\delta_{J,J'}$$

where $[c]\in H_q(\Lambda^{*,2J}, d)$ and $[c']\in H_p(\Lambda^{*,2J'}, d)$, the cycle $c\cdot c'$ is the product of c and c' in $\Lambda^{*,*}[\mathbb{P}]$ and $\delta_{J,J'} = 1$ if $J\cap J'=\emptyset$, and $\delta_{J,J'}=0$ if $J\cap J'\neq\emptyset$.

Let us remark that the exterior algebra $\Lambda^*[\mathbb{P}]$ in the Taylor resolution $\Lambda^*[\mathbb{P}]\otimes R[v_1,\ldots,v_m]$ of $R[K]$ is taken over a finite set corresponding to the generators of the Stanley-Reisner ideal I_K. Thus, theoretically $\mathbb{P}$ in Theorem 2.1.27 can be given by any finite set $\{\sigma_1,\ldots,\sigma_l \mid \sigma\in 2^{[m]}\}$ which contains all minimal missing faces of K. However, in order that the calculation of $\mathrm{Tor}_{R[m]}(R[K], R)$ is as efficient as possible, we state the result that $\mathbb{P}$ only contains all minimal missing faces.

Example 2.1.28. Let K be the boundary of a 5-gon.



Take $\mathbb{P} = \{\tau_{13} = \{1,3\}, \tau_{14} = \{1,4\}, \tau_{24} = \{2,4\}, \tau_{25} = \{2,5\}, \tau_{35} = \{3,5\}\}$ to be the minimal missing faces of K. By the definition (2.5) of the differential operation d, the non-trivial differentials are:

$$d_5(\tau_{13}\tau_{14}\tau_{24}\tau_{25}\tau_{35}) = \tau_{14}\tau_{24}\tau_{25}\tau_{35} - \tau_{13}\tau_{24}\tau_{25}\tau_{25} + \tau_{13}\tau_{14}\tau_{25}\tau_{35} - \tau_{13}\tau_{14}\tau_{24}\tau_{35} + \tau_{13}\tau_{14}\tau_{24}\tau_{25};$$
$$d_4(\tau_{13}\tau_{14}\tau_{24}\tau_{25}) = -\tau_{13}\tau_{24}\tau_{25} + \tau_{13}\tau_{14}\tau_{25}; \quad d_4(\tau_{13}\tau_{14}\tau_{24}\tau_{35}) = -\tau_{13}\tau_{24}\tau_{35} + \tau_{13}\tau_{14}\tau_{35};$$
$$d_4(\tau_{13}\tau_{14}\tau_{25}\tau_{35}) = \tau_{14}\tau_{25}\tau_{35} - \tau_{13}\tau_{14}\tau_{25}; \quad d_4(\tau_{13}\tau_{24}\tau_{25}\tau_{35}) = \tau_{13}\tau_{24}\tau_{35} - \tau_{13}\tau_{24}\tau_{25};$$
$$d_4(\tau_{14}\tau_{24}\tau_{25}\tau_{35}) = -\tau_{14}\tau_{25}\tau_{35} + \tau_{14}\tau_{24}\tau_{35};$$
$$d_3(\tau_{13}\tau_{14}\tau_{24}) = -\tau_{13}\tau_{24}; \quad d_3(\tau_{13}\tau_{14}\tau_{35}) = \tau_{14}\tau_{35}; \quad d_3(\tau_{14}\tau_{24}\tau_{25}) = -\tau_{14}\tau_{25};$$
$$d_3(\tau_{24}\tau_{25}\tau_{35}) = -\tau_{24}\tau_{35}; \quad d_3(\tau_{13}\tau_{25}\tau_{35}) = \tau_{13}\tau_{25}.$$

As an R-module, $\mathrm{Tor}_{R[5]}(R[K], R)$ has a basis given by

$$\left\{ \begin{array}{cccccc} 1 & \tau_{13} & \tau_{14} & \tau_{24} & \tau_{25} & \tau_{35} \\ \tau_{13}\tau_{24}\tau_{25} & \tau_{13}\tau_{14} & \tau_{13}\tau_{35} & \tau_{14}\tau_{24} & \tau_{24}\tau_{25} & \tau_{25}\tau_{35} \end{array} \right\}.$$

It follows that

$$\mathrm{Tor}^{-1,4}_{R[m]}(R[K], R) \cong R^{\oplus 5}, \text{ generated by } \tau_{13}, \tau_{14}, \tau_{24}, \tau_{25}, \tau_{35};$$
$$\mathrm{Tor}^{-2,6}_{R[m]}(R[K], R) \cong R^{\oplus 5}, \text{ generated by } \tau_{13}\tau_{14}, \tau_{13}\tau_{35}, \tau_{14}\tau_{24}, \tau_{24}\tau_{25}, \tau_{25}\tau_{35};$$
$$\mathrm{Tor}^{-3,10}_{R[m]}(R[K], R) \cong R, \text{ generated by } \tau_{13}\tau_{24}\tau_{25}.$$

2.2 Moment-angle Complexes

In this section, we recall definitions and properties of moment-angle complexes.

Let K be a simplicial complex on $[m]$. The moment-angle complex $\mathcal{Z}_K$ is a union of products of discs and circles which is a T^m-invariant subspace of $(D^2)^m$, where the T^m-action on $(D^2)^m$ is induced by a coordinatewise multiplication of complex numbers, viewing $(D^2)^m$ as a subset of the complex space $\mathbb{C}^m$.

Definition 2.2.1. The moment-angle complex $\mathcal{Z}_K$ associated to a simplicial complex K on $[m]$ is defined by

$$\mathcal{Z}_K = \bigcup_{\sigma \in K} (D^2, S^1)^\sigma$$

where $(D^2, S^1)^\sigma = Y_1 \times \cdots \times Y_m$ and $Y_i = \begin{cases} D^2 & \text{if } i \in \sigma \\ S^1 & \text{if } i \notin \sigma. \end{cases}$

For example, if K is the boundary of a simplex with m vertices, then

$$\begin{aligned} \mathcal{Z}_K &= (D^2 \times D^2 \times \cdots \times S^1) \cup (D^2 \times \cdots \times S^1 \times D^2) \cup \cdots \cup (S^1 \times D^2 \times \cdots \times D^2) \\ &= \partial((D^2)^m) \cong S^{2m-1}. \end{aligned}$$

Proposition 2.2.2. *[10, Theorem 4.1.4] If K is a triangulation of an $(n-1)$-sphere with m vertices, then the corresponding moment-angle complex $\mathcal{Z}_K$ is a closed $(m+n)$-manifold.*

We refer to $\mathcal{Z}_K$ as a moment-angle manifold when K is a triangulation of a sphere. For example, let P^n be a simple polytope (Definition 2.1.7) with facets $\mathcal{F} = \{F_1, \ldots, F_m\}$. The nerve complex K_P of P^n (Definition 2.1.8) is a triangulation of $(n-1)$-sphere. Hence $\mathcal{Z}_{K_P}$ is a moment-angle manifold.

2.2.1 Cohomology of $\mathcal{Z}_K$

The cellular decomposition of $\mathcal{Z}_K$ is described as a subspace of $(D^2)^m$. The disc D^2 has three cells e^0, e^1, e^2 of dimensions $0, 1, 2$, respectively. The cells of $(D^2)^m$ are parametrised by subsets $I, J \subseteq [m]$ with $I \cap J = \emptyset$. That is to say, a cell denoted by $\kappa(J, I)$ is equal to $e_1 \times \cdots \times e_m$ in $(D^2)^m$, where e_i is the 2-dimensional cell e^2 if $i \in I$, e_i is the 1-dimensional cell e^1 if $i \in J$, and e_i is the point e^0 if $i \in [m] \setminus (I \cup J)$. Since $\mathcal{Z}_K$ is a subcomplex of $(D^2)^m$ determined by the simplicial complex K, the cells of $\mathcal{Z}_K$ are those cells $\kappa(J, I)$ where $I \in K$. Thus the cellular cochain complex $\mathcal{C}^*(\mathcal{Z}_K; R)$ of $\mathcal{Z}_K$ has a basis of cochains $\{\kappa(J, I)^* \mid I \in K, I \cap J = \emptyset\}$, where each $\kappa(J, I)^*$ corresponds to a cell $\kappa(J, I)$ of $\mathcal{Z}_K$.

Recall that the quotient differential graded algebra $\mathcal{R}^*(K)$ (Construction 2.1.19) is defined by

$$\mathcal{R}^*(K) = \Lambda[u_1, \ldots, u_m] \otimes R[K]/(u_i v_i = v_i^2 = 0, 1 \leq i \leq m)$$

where $du_i = v_i$ and $dv_i = 0$. There is a bijection between the R-basis of $\mathcal{C}^*(\mathcal{Z}_K; R)$ and $\mathcal{R}^*(K)$ which commutes with differentials.

Lemma 2.2.3 ([10, 34]). *The cellular cochain algebra $\mathcal{C}^*(\mathcal{Z}_K; R)$ is isomorphic to the algebra $\mathcal{R}^*(K)$. Therefore, there is an isomorphism of cohomology rings between $H^*(\mathcal{Z}_K; R)$ and $H(\mathcal{R}^*(K))$.*

Now combining Lemma 2.1.20, Theorem 2.1.23 and Theorem 2.1.27, we have the following result.

Theorem 2.2.4 ([10, 4]). *There are isomorphisms of R-algebras*

$$H^*(\mathcal{Z}_K; R) \cong \mathrm{Tor}_{R[m]}(R[K], R) \cong \bigoplus_{J \subseteq [m]} \widetilde{H}^*(K_J; R).$$

where the ring structure on the right hand side is given by the canonical maps

$$H^{k-|I|-1}(K_I; R) \otimes H^{l-|J|-1}(K_J; R) \to H^{k+l-|I|-|J|-1}(K_{I \cup J}; R)$$

which are induced by simplicial maps $K_{I\cup J} \to K_I * K_J$ for $I \cap J = \emptyset$ and zero otherwise.

Moreover, it has been proven in [4] that the isomorphisms of Hochster's formula

$$\widetilde{H}^{p-1}(K_J; R) \cong \mathrm{Tor}_{R[m]}^{-p+1,2J}(R[K]; R)$$

are functorial with respect to simplicial maps. An important application of these functorial properties is seen when one considers the G-actions on $\mathcal{Z}_K$ induced by a simplicial G-complex K, which will induce a $\mathbf{k}G$-module on $H^*(\mathcal{Z}_K; \mathbf{k})$ compatible with the decomposition of Hochster's formula. This lays the foundation to consider the representation stability of moment-angle complexes in Chapter 3.

We have seen that $\mathcal{Z}_K$ is homeomorphic to S^{2m-1} when K is the boundary of an $(m-1)$-simplex. Let us calculate its cohomology by applying Hochster's formula.

Example 2.2.5. Let K be the boundary of an $(m-1)$-simplex. In this case, only when $J = \emptyset$ and $J = [m]$, the reduced simplicial cohomology of K_J is nontrivial. By Theorem 2.2.4,

$$\begin{aligned} H^0(\mathcal{Z}_K; R) &\cong \widetilde{H}^{-1}(\emptyset; R) = R; \\ H^{2m-1}(\mathcal{Z}_K; R) = H^{-1,2m}(\mathcal{Z}_K; R) &\cong \widetilde{H}^{m-2}(K_{[m]}; R) \cong R. \end{aligned}$$

Moreover, examples (2.1.22 and 2.1.28) give the cohomology groups of $\mathcal{Z}_K$ when K is the boundary of a 4-gon or 5-gon. That is, let K be the boundary of a square, then

$$H^l(\mathcal{Z}_K; R) \cong \begin{cases} R & \text{if } l = 0 \\ R \oplus R & \text{if } l = 3 \\ R & \text{if } l = 6 \\ 0 & \text{otherwise.} \end{cases}$$

Let K be the boundary of a 5-gon, then

$$H^l(\mathcal{Z}_K;R) \cong \begin{cases} R & \text{if } l = 0,7 \\ R^{\oplus 5} & \text{if } l = 3,4 \\ 0 & \text{otherwise} \end{cases}$$

In fact, by [5, 31], when K is the boundary of an m-gon ($m \geq 4$), $\mathcal{Z}_K$ is homeomorphic to a connected sum of sphere products

$$\mathcal{Z}_K \cong \mathop{\#}_{k=3}^{m-1} (S^k \times S^{m+2-k})^{\#(k-2)\binom{m-2}{k-1}}.$$

2.2.2 Partial Quotients of Moment-angle Complexes

The purpose of this section is to give an alternative definition of moment-angle complexes and their quotient spaces under torus actions. The cohomological properties of partial quotients of $\mathcal{Z}_K$ will be studied in Chapter 4.

Let $\mathbb{I}^m$ denote the standard unit m-cube with $\mathbb{I} = [0,1]$. For $J \subset I \subset [m]$, any face of $\mathbb{I}^m$ can be written as

$$C_{J\subset I} = \{(y_1,\ldots,y_m) \in \mathbb{I}^m \mid y_j = 0 \text{ for } j \in J,\ y_j = 1 \text{ for } j \notin I\}$$

and vertices of $\mathbb{I}^m$ are indexed by subsets of $[m]$

$$C_{I\subset I} = \{(y_1,\ldots,y_m) \in \mathbb{I}^m \mid y_j = 0 \text{ for } j \in I,\ y_j = 1 \text{ for } j \notin I\}.$$

In particular, $C_{\emptyset\subset\emptyset} = (1,\ldots,1)$.

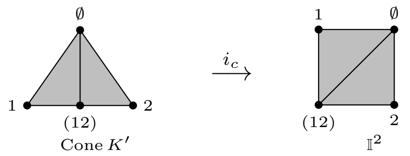
Denote by $2^{[m]}$ the power set of $[m]$.

Construction 2.2.6 (triangulation of $\mathbb{I}^m$ [10])**.** Let Δ^{m-1} be an $(m-1)$-simplex. The simplicial complex $\mathrm{Cone}(\Delta^{m-1})'$ is on the vertex set $2^{[m]}$. The identity map $\mathrm{id}\colon 2^{[m]} \longrightarrow 2^{[m]}$, viewed as a map between the vertex set of $\mathrm{Cone}(\Delta^{m-1})'$ and the vertex set of $\mathbb{I}^m$, extends linearly to a homeomorphism i_c

$$\begin{aligned} i_c\colon \mathrm{Cone}(\Delta^{m-1})' &\longrightarrow \mathbb{I}^m \\ \sigma_{i_1} \subsetneq \ldots \subsetneq \sigma_{i_l} &\longmapsto C_{\sigma_{i_1}\subset\sigma_{i_l}}. \end{aligned} \tag{2.6}$$

The image of i_c gives a triangulation of $\mathbb{I}^m$. Note that $i_c(\emptyset) = C_{\emptyset\subset\emptyset} = (1,\ldots,1)$.

Example 2.2.7. Let us illustrate the triangulation of $\mathbb{I}^2$. The vertices of $\mathbb{I}^2$ correspond to subsets of $[2]$ by $i_c(\emptyset) = (1,1)$, $i_c(\{1\}) = (0,1)$, $i_c(\{2\}) = (1,0)$ and $i_c(\{1,2\}) = (0,0)$. The resulting homeomorphism $i_c\colon \mathrm{Cone}(\Delta^1)' \longrightarrow \mathbb{I}^2$ is extended linearly by this correspondence. See the picture below.



Let K be a simplicial complex on $[m]$. Then K is a subcomplex of Δ^{m-1} and Cone K' is a subcomplex of Cone$(\Delta^{m-1})'$. Each face $F_{(\sigma_i)}$ $(\sigma_i \in K)$ of Cone K' corresponds to a chain $\sigma_0 \subsetneq \sigma_1 \subsetneq \ldots \subsetneq \sigma_l$ of faces of K and $i_c(F_{(\sigma_i)}) = C_{\sigma_0 \subseteq \sigma_l}$ by (2.6). The codimensional-one faces of $F_{(\sigma_i)}$ are these chains $\{\sigma_0 \subsetneq \ldots \subsetneq \hat{\sigma}_j \subsetneq \ldots \subsetneq \sigma_l \mid 0 \leq j \leq l\}$ where ˆ denotes omission. The interior int $F_{(\sigma_i)}$ consists of points in $F_{(\sigma_i)}$ which are not in its codimensional-one faces. Hence if $x \in$ Cone K', there is a unique face $F_{(\sigma_i)} \in$ Cone K' such that $x \in$ int $F_{(\sigma_i)}$.

The image of ConeK' under the map i_c provides a triangulation of a cubical subcomplex of $\mathbb{I}^m$. Denote by $cc(K)$ this underlying cubical subcomplex of the image of Cone K' under the map i_c. Then the moment-angle complex $\mathcal{Z}_K = (D^2, S^1)^K = \bigcup_{\sigma \in K} (D^2, S^1)^\sigma$ is a pullback of the diagram

$$\begin{array}{ccc} \mathcal{Z}_K & \longrightarrow & (D^2)^m \\ \downarrow & & \downarrow{\scriptstyle \mu} \\ cc(K) & \xhookrightarrow{\;i\;} & \mathbb{I}^m \end{array} \tag{2.7}$$

where i is an inclusion and $\mu(z_1, \ldots, z_m) = (|z_1|, \ldots, |z_m|)$.

For any $\sigma \subseteq [m]$, denote by T^σ the coordinate $|\sigma|$-subtorus

$$T^\sigma = \{(t_1, \cdots, t_m) \in T^m \mid t_j = 1 \text{ for } j \notin \sigma\} \leqslant T^m. \tag{2.8}$$

In particular, $T^\emptyset$ is the trivial subgroup $\{\mathbf{1}\}$ of T^m. Now let us introduce a homeomorphism of $(D^2)^m$ which, together with the pullback square (2.7), gives an alternative definition of $\mathcal{Z}_K$. Since D^2 is homeomorphic to $(\mathbb{I} \times S^1)/(0,t) \sim (0,t')$, there is a homeomorphism

$$(D^2)^m \cong \mathbb{I}^m \times T^m/\sim \tag{2.9}$$

where $(\mathbf{x}, \mathbf{t}_1) \sim (\mathbf{y}, \mathbf{t}_2)$ if and only if $\mathbf{x} = \mathbf{y}$ and $\mathbf{t}_1^{-1}\mathbf{t}_2 \in T^{\omega(x)}$ and $\omega(\mathbf{x}) = \{i \in [m] \mid x_i = 0\}$. This equivalence $(\mathbf{x}, \mathbf{t}_1) \sim (\mathbf{x}, \mathbf{t}_2)$ (2.9) is equivalent to saying that the coordinate $t_1^j = t_2^j$ if $x_j \neq 0$, where t_1^j and t_2^j represent the j-th coordinates of $\mathbf{t}_1$ and $\mathbf{t}_2$, respectively.

By the definition of i_c (2.6), $\omega(i_c(\mathbf{x})) = \sigma_0$ for which $\mathbf{x} \in$ int $F_{(\sigma_i)}$ and $F_{(\sigma_i)}$ corresponds to the chain $\sigma_0 \subsetneq \ldots \subsetneq \sigma_l$ of faces of K. Since $cc(K)$ is homeomorphic to Cone K' by i_c, together with the pullback diagram (2.7) and the homeomorphism (2.9), there is a homeomorphism $\mathcal{Z}_K \cong$ Cone $K' \times T^m/\sim$, where $(\mathbf{x}, \mathbf{t}_1) \sim (\mathbf{y}, \mathbf{t}_2)$ if and only if $\mathbf{x} = \mathbf{y} \in$ int $F_{(\sigma_i)}$ and $\mathbf{t}_1^{-1}\mathbf{t}_2 \in T^{\sigma_0}$. Moreover, the homeomorphism (2.9) $(D^2)^m \cong \mathbb{I}^m \times T^m/\sim$ is T^m-equivariant, where T^m acts on the second coordinate of $\mathbb{I}^m \times T^m/\sim$. Hence there is a T^m-action on $\operatorname{Cone} K' \times T^m/\sim$ by a T^m-action on the second coordinate and the homeomorphism $\mathcal{Z}_K \cong \operatorname{Cone} K' \times T^m/\sim$ is T^m-equivariant.

We obtain an alternative definition of moment-angle complexes, which was introduced in [17].

Definition 2.2.8. The moment-angle complex $\mathcal{Z}_K$ is T^m-equivariantly homeomorphic to

$$\operatorname{Cone} K' \times T^m/\sim \tag{2.10}$$

where $(\mathbf{x}, \mathbf{t}_1) \sim (\mathbf{y}, \mathbf{t}_2)$ if and only if $\mathbf{x} = \mathbf{y} \in \operatorname{int} F_{(\sigma_i)}$ and $\mathbf{t}_1^{-1}\mathbf{t}_2 \in T^{\sigma_0}$.

Example 2.2.9. Let K be the disjoint union of two points. Then the geometrical realisation $|\operatorname{Cone} K'|$ is homeomorphic to the interval $[0,1]$. Then by (2.10), $\mathcal{Z}_K \cong \mathbb{I} \times S^1_1 \times S^1_2/\sim$, where $(0,t_1,t_2) \sim (0,t_1',t_2)$ and $(1,t_1,t_2) \sim (1,t_1,t_2')$. The space $\mathbb{I} \times S^1_1 \times S^1_2/\sim$ is exactly the join of S^1 and S^1, which is S^3.

Example 2.2.9 illustrates moment-angle manifolds obtained from simple polytopes when K (the two disjoint points) is a nerve complex of a simple polytope (the 1-simplex). A polytopal moment-angle manifold refers to a moment-angle complex $\mathcal{Z}_{K_P}$ corresponding to a nerve complex K_P of a simple polytope P^n.

Let P^n be a simple polytope with m facets, let $\mathcal{F}(P)$ denote the facet set $\{F_1, \cdots, F_m\}$ of P^n and let $I_{\mathbf{x}} = \{i \in [m] \mid \mathbf{x} \in F_i\}$ for each $\mathbf{x} \in P$.

Definition 2.2.10 ([10])**.** For a simple n-polytope P with m facets, we define

$$\mathcal{Z}_P = (P^n \times T^m)/\sim$$

where $(\mathbf{x}, \mathbf{t}_1) \sim (\mathbf{y}, \mathbf{t}_2)$ if and only if $\mathbf{x} = \mathbf{y}$ and $\mathbf{t}_1^{-1}\mathbf{t}_2 \in T^{I_{\mathbf{x}}}$.

Example 2.2.11 (odd dimensional spheres)**.** Consider P^n to be the n-simplex $\Delta^n \subseteq \mathbb{R}^{n+1}$. The moment-angle manifold associated to Δ^n is S^{2n+1}. To see this, let

$$S^n_{\geqslant} = \{(x_0, x_1, \ldots, x_n) \in S^n \mid x_i \geq 0, 0 \leq i \leq n\} \subseteq \mathbb{R}^{n+1}_{\geqslant}$$

be the part of n-sphere lying in the non-negative coordinate region of $\mathbb{R}^{n+1}$. Then there exists a homeomorphism $\psi\colon \Delta^n \longrightarrow S^n_{\geqslant}$ which maps the boundary of Δ^n onto the boundary of $S^n_{\geqslant}$ homeomorphically.

Since $S^{2n+1} = \{(z_0, z_1, \ldots, z_n) \in \mathbb{C}^{n+1} \mid |z_0|^2 + |z_1|^2 + \ldots + |z_n|^2 = 1\}$, $S^{2n+1} = \{(x_0' t_0, x_1' t_1, \ldots, x_n' t_n) \mid (x_0', \ldots, x_n') \in S^n_{\geqslant}, (t_0, \ldots, t_n) \in T^{n+1}\}$ by applying the polar coordinate $z_i = x_i' t_i$. Define a map

$$\begin{aligned} f\colon \Delta^n \times T^{n+1} &\longrightarrow S^{2n+1} \subseteq \mathbb{C}^{n+1} \\ ((x_0, \ldots, x_n), (t_0, \ldots, t_n)) &\longmapsto (x_0' t_0, \ldots, x_n' t_n) \end{aligned} \tag{2.11}$$

where $(x'_0, \ldots, x'_n) = \psi(x_0, \ldots, x_n)$.

We can check that $f(\mathbf{x}, \mathbf{t}_1) = f(\mathbf{y}, \mathbf{t}_2)$ if and only if $(\mathbf{x}, \mathbf{t}_1) \sim (\mathbf{y}, \mathbf{t}_2)$, where $\sim$ is defined as in Definition 2.2.10. To see this, $f(\mathbf{x}, \mathbf{t}_1) = f(\mathbf{y}, \mathbf{t}_2)$ if and only if $x'_j t_1^j = y'_j t_2^j$, where x'_j, y'_j and t_1^j, t_2^j represent the j-th coordinate of $\mathbf{x}', \mathbf{y}'$ and $\mathbf{t}_1, \mathbf{t}_2$, respectively. Since $x'_j t_1^j = y'_j t_2^j$ as complex numbers, we have $|x'_j t_1^j| = |y'_j t_2^j|$. Thus $x'_j = y'_j$ as x'_j, y'_j are non-negative real numbers. So $\mathbf{x} = \mathbf{y}$. Then $x'_j t_1^j = x'_j t_2^j$ if and only if when $x'_j \neq 0$, $t_1^j = t_2^j$ (when $x'_j = 0$, t_1^j and t_2^j can be any element in S^1).

Finally, since f is surjective, the map f induces a homeomorphism

$$\mathcal{Z}_{\Delta^n} = \Delta^n \times T^{n+1}/\sim \overset{\bar{f}}{\cong} S^{2n+1}.$$

It has been proven in [10, Theorem 6.2.4] that $\mathcal{Z}_P = P \times T^m/\sim$ defined in Definition 2.2.10 and the moment-angle $\mathcal{Z}_{K_P}$ complex are homeomorphism.

Theorem 2.2.12 ([10]). *The moment-angle manifold $\mathcal{Z}_{K_P}$ obtained from the nerve complex of a simple polytope P^n is T^m-equivariantly homeomorphic to $\mathcal{Z}_P$ defined in Definition 2.2.10.*

Next we introduce the quotient spaces of $\mathcal{Z}_K$ under torus actions. Let $\Lambda\colon \mathbb{Z}^m \longrightarrow \mathbb{Z}^{m-r}$ be a surjective linear map. Then the short exact sequence

$$0 \longrightarrow \mathrm{Ker}\Lambda \longrightarrow \mathbb{Z}^m \overset{\Lambda}{\longrightarrow} \mathbb{Z}^{m-r} \longrightarrow 0$$

splits ([32, Corollary 23.2]). The map Λ induces a homomorphism of tori $T^m \longrightarrow T^{m-r}$, also denoted by Λ, and a short exact sequence of tori follows

$$1 \longrightarrow \mathrm{Ker}\Lambda \longrightarrow T^m \overset{\Lambda}{\longrightarrow} T^{m-r} \longrightarrow 1.$$

For every $\sigma \in K$, let Λ_σ denote the image of T^σ (2.8) in T^{m-r} due to the composite $\Lambda|_{T^\sigma}\colon T^\sigma \longrightarrow T^m \longrightarrow T^{m-r}$. We have the following definition of quotient space $\mathcal{Z}_K/H$.

Lemma 2.2.13. *The subtorus $H = \mathrm{Ker}\Lambda$ of rank r acts on $\mathcal{Z}_K$. The quotient space $\mathcal{Z}_K/H$ is homeomorphic to the following space*

$$\mathrm{Cone}\, K' \times T^{m-r}/\sim$$

where $(\mathbf{x}, \mathbf{t}_1) \sim (\mathbf{y}, \mathbf{t}_2)$ if and only if $\mathbf{x} = \mathbf{y} \in \mathrm{int}\, F_{(\sigma_i)}$ and $\mathbf{t}_1^{-1}\mathbf{t}_2 \in \Lambda_{\sigma_0}$.

Proof. Since the homeomorphism $\mathcal{Z}_K \cong \mathrm{Cone}\, K' \times T^m/\sim$ (2.10) is T^m-equivariant, it is also H-equivariant for any subtorus $H \leqslant T^m$. Hence we have the homeomorphism

$$\mathcal{Z}_K/H \cong \mathrm{Cone}\, K' \times T^m/\sim$$

where the relation $\sim$ includes the identifications due to the subtorus H-action $(\mathrm{x}, g\mathbf{t}) \sim (\mathbf{x}, \mathbf{t})$ for $g \in H$ and the identifications of $\mathcal{Z}_K$, i.e., $(\mathbf{x}, \mathbf{t}_1) \sim (\mathbf{y}, \mathbf{t}_2)$ if $\mathbf{x} = \mathbf{y} \in \operatorname{int} F_{(\sigma_i)}$ and $\mathbf{t}_1^{-1}\mathbf{t}_2 \in T^{\sigma_0}$. The identifications due to the H-action have representatives given by the left coset space T^m/H. So $(\mathbf{x}, \mathbf{t}_1 H) \sim (\mathbf{x}, \mathbf{t}_2 H)$ if and only if $(\mathbf{t}_1 H)^{-1}(\mathbf{t}_2 H) \in T^{\sigma_0}$, which is equivalent to $\mathbf{t}_1^{-1}\mathbf{t}_2 \in \Lambda_{\sigma_0}$. The statement follows since $T^m/H \cong T^{m-r}$. □

In particular, in the case of nerve complexes K_P for a simple n-polytope P^n with m facets, the integral $n \times m$-matrix Λ is called a characteristic function if KerΛ acts on $\mathcal{Z}_P$ freely. This freeness condition is given by the following $(*)$ condition ([34, Theorem 3.12])

$(*)$ for every vertex $v \in P$ as an intersection of n facets $v = F_{j_1} \cap F_{j_2} \cap \cdots \cap F_{j_n}$, the maximal minor $\Lambda_v = \Lambda_{j_1 j_2 \cdots j_n}$ formed by the columns $j_1, \cdots, j_n$ of the matrix Λ, satisfies $det\Lambda_v = \pm 1$.

With a simple polytope P^n and a characteristic function Λ, this quotient manifold $\mathcal{Z}_P/\mathrm{Ker}\Lambda$ is known as a quasi-toric manifold $M(P^n, \Lambda)$.

Definition 2.2.14 ([17])**.** Let (P^n, Λ) satisfy the $(*)$ condition. A *quasitoric manifold* is $M(P, \Lambda) = P \times T^n/\sim_\Lambda$, where $(\mathbf{x}, \mathbf{t}_1) \sim_\Lambda (\mathbf{y}, \mathbf{t}_2)$ if $\mathbf{x} = \mathbf{y}$ and $\mathbf{t}_1^{-1}\mathbf{t}_2 \in T_{\mathbf{x},\Lambda}$ and $T_{\mathbf{x},\Lambda}$ is the image of T^{I_x} in T^n under Λ.

Example 2.2.15 (projective spaces)**.** Let $\Lambda = \Big(I_n \mid -\mathbf{1}\Big)$, where $-\mathbf{1} = (-1, \dots, -1)^t$. In this case, (Δ^n, Λ) satisfies the $(*)$ condition and $\mathrm{Ker}\Lambda \cong S^1$ acts on $\mathcal{Z}_{\Delta^n}$ diagonally. By (2.11), the homeomorphism $\bar{f}\colon \Delta^n \times T^{n+1}/\sim \longrightarrow S^{2n+1}$ is T^{n+1}-equivariant. Hence, $\bar{f}$ is S^1-equivariantly homeomorphic. Consider the diagonal action on both sides, where the quotient space of S^{2n+1} under the diagonal action is $\mathbb{C}P^n$. So $\mathcal{Z}_{\Delta^n}/\mathrm{Ker}\Lambda \cong \mathbb{C}P^n$.

2.3 Polyhedral products

The homotopy-theoretical applications in polyhedral products are beautiful, providing topological approaches to finding relations among algebraic properties and combinatorial structures. In this section, we review the definitions of polyhedral products and their related properties with an emphasis from a homotopy viewpoint.

Let K be a simplicial complex on $[m]$ and let $(\underline{X}, \underline{A}) = \{(X_1, A_1), \dots, (X_m, A_m)\}$ be a sequence of m pairs of topological spaces. For any subset $\sigma = \{i_1, \dots, i_l\} \subseteq [m]$, we introduce the following spaces

$$
\begin{aligned}
(\underline{X}, \underline{A})^\sigma &= \{(x_1, \dots, x_m) \in \prod_{j=1}^m X_j \mid x_j \in A \text{ for } j \notin I\} \\
(\underline{X}, \underline{A})^{\wedge I} &= \{x_1 \wedge \dots \wedge x_m \in X_1 \wedge \dots \wedge X_m \mid x_j \in A \text{ for } j \notin I\} \\
\underline{X}^{\wedge I} &= X_{i_1} \wedge \dots \wedge X_{i_l}.
\end{aligned}
$$

If all $X_i = X$ and $A_i = A$, the above corresponding spaces are denoted by $(X, A)^I$, $(X, A)^{\wedge I}$ and $X^{\wedge I}$, respectively.

Definition 2.3.1. A *polyhedral product* $(\underline{X}, \underline{A})^K$ is a subspace of $\prod_{i=1}^{m} X_i$, defined by

$$(\underline{X}, \underline{A})^K = \bigcup_{\sigma \in K} (\underline{X}, \underline{A})^\sigma$$

If all $X_i = X$ and $A_i = A$, write $(\underline{X}, \underline{A})^K = (X, A)^K$. We will illustrate polyhedral products with various simplicial complexes and distinct topological pairs.

Example 2.3.2. (1) If $X = D^2$ and $A = S^1$, then the corresponding polyhedral product $\mathcal{Z}_K = (D^2, S^1)^K$ is known as the moment-angle complex. If $X = BS^1$ (the classifying space of S^1) and $A = *$, then the corresponding polyhedral product $DJ_K = (BS^1, *)^K$ is called the Davis-Januszkiewicz space.

(2) Let K be a simplex Δ^{m-1}. Then $(\underline{X}, \underline{A})^K = X_1 \times \ldots \times X_m$.

(3) Let each X_i be a based space and let each A_i be the base point. If K consists of m disjoint points, then $(\underline{X}, *)^K = \bigvee_{i=1}^{m} X_i$. If $K = \partial\Delta^{m-1}$, the boundary of a simplex, then $(\underline{X}, *)^{\partial\Delta^{m-1}}$ is a fat wedge of $X_1 \times \ldots \times X_m$, i.e., a subspace with at least one coordinate to be the based point.

Here are two fundamental properties (functorial properties and retractions) of polyhedral products which are crucial to the results of this thesis.

Lemma 2.3.3 ([40])**.** (a) *The polyhedral product $(\underline{X}, \underline{A})^K$ is functorial with respect to the category of simplicial complexes and simplicial inclusions and the category of a sequence of topological spaces and continuous maps, respectively.*

(b) *For any non-empty subset $J \subseteq [m]$, the polyhedral product $(\underline{X}, \underline{A})^K$ retracts off $(\underline{X}, \underline{A})^{K_J}$.*

We give a remark regarding to maps between polyhedral products induced by simplicial inclusions.

Remark 2.3.4. Since the definition of a polyhedral product relies on the vertex set of K, if L is a subcomplex of K, then there are two types of polyhedral products associated to L. The usual notation $(\underline{X}, \underline{A})^L$ denotes the polyhedral product defined on the vertex set $V(L)$ and we use $(\underline{X}, \underline{A})^{\overline{L}}$ to denote the polyhedral product obtained by taking L on the vertex set $V(K)$, i.e., allowing the ghost vertices due to L being a subcomplex of K. Both these two spaces $(\underline{X}, \underline{A})^L$ and $(\underline{X}, \underline{A})^{\overline{L}}$ are subspaces of $(\underline{X}, \underline{A})^K$. The maps induced by simplicial inclusions considered in Lemma 2.3.3 are from $(\underline{X}, \underline{A})^L$ to $(\underline{X}, \underline{A})^K$.

Next we introduce the polyhedral smash product. The smash product $X \wedge Y$ is a quotient space $X \times Y/X \vee Y$ and the smash product $X_1 \wedge \ldots \wedge X_m$ is a quotient space $(X_1 \times \ldots \times X_m)/(\underline{X}, *)^{\partial\Delta^{m-1}}$. The join $X * Y$ of two based spaces is $\Sigma X \wedge Y$.

Definition 2.3.5. The *polyhedral smash product* of a simplicial complex K and m topological pairs $(\underline{X}, \underline{A})$ is defined as

$$(\underline{X}, \underline{A})^{\wedge K} = \bigcup_{\sigma \in K} (\underline{X}, \underline{A})^{\wedge \sigma}$$

where $(\underline{X}, \underline{A})^{\wedge \sigma} = Y_1 \wedge \ldots \wedge Y_m$ and $Y_i = X_i$ if $i \in \sigma$ and $Y_i = A_i$ if $i \notin \sigma$.

2.3.1 Homotopy decompositions

A classical splitting of a product space $X_1 \times \ldots \times X_m$ is given by ([38, Proposition 7.7.6])

$$\Sigma(X_1 \times \ldots \times X_m) \xrightarrow{\simeq} \Sigma \bigvee_{I \subseteq [m]} X^{\wedge I}$$

where I goes through non-empty subsets of $[m]$.

In [3], Bahri, Bendersky, Cohen, Gitler proved that this splitting induces a splitting of polyhedral product.

Theorem 2.3.6 ([3])**.** *Let K be a simplicial complex on $[m]$. Then there exists a homotopy equivalence*

$$\Sigma(\underline{X}, \underline{A})^K \simeq \Sigma \bigvee_{I \subseteq [m]} (\underline{X}, \underline{A})^{\wedge K_I}$$

when $(\underline{X}, \underline{A})$ consists of m pairs of connected, based CW-complexes.

Interesting families of $(\underline{X}, \underline{A})^K$ include the spaces such that either each X_i or A_i is contractible, which covers the cases of moment-angle complexes and Davis-Januszkiewicz spaces.

Theorem 2.3.7 ([3])**.** *Let X_i and A_i be closed CW complexes for all i. There are homotopy equivalences in the following cases:*

(a) *if the inclusion $A_i \longrightarrow X_i$ is null homotopic, then*

$$\Sigma(\underline{X}, \underline{A})^K \simeq \Sigma(\bigvee_{I \in K} \underline{X}^{\wedge I})$$

(b) *if all X_i are contractible, then*

$$\Sigma(\underline{X}, \underline{A})^K \simeq \Sigma(\bigvee_{I \notin K} |K_I| * \underline{A}^{\wedge I})$$

where $|K_I|$ is the geometrical realisation of the full subcomplex $K_I = \{\sigma \cap I \mid \sigma \in K\}$.

Furthermore, Grbić and Theriault [23] proved that Theorem 2.3.7(b) can be desuspended for $(\underline{CX}, \underline{X})^K$ when K is shifted. A simplicial complex K is *shifted* if there exits an order $\preceq$ on its vertex set $V(K)$ such that when $\sigma = \{i_1, \ldots, i_l\} \in K$, then for every $\sigma' = \{j_1, \ldots, j_l\}$ satisfying $j_t \preceq i_t$, $(1 \leq t \leq l)$, σ' also belongs to K. A typical example of shifted complexes is the k-skeleton Δ_m^k of a simplex, which contains all the subsets of $[m]$ with cardinality less than or equal to $k+1$.

Theorem 2.3.8 ([23]). *Let K be a simplicial complex on $[m]$. If K is shifted, then there exists a homotopy equivalence*

$$(\underline{CX}, \underline{X})^K \simeq (\bigvee_{I \notin K} |K_I| * \underline{X}^{\wedge I}).$$

This generalised a result of Porter [36] for a homotopy decomposition $(\underline{CX}, \underline{X})^{\Delta_m^k}$.

Theorem 2.3.9 ([23, 36]). *The homotopy type of $(\underline{CX}, \underline{X})^{\Delta_m^k}$ is the wedge*

$$(\underline{CX}, \underline{X})^{\Delta_m^k} \simeq \bigvee_{j=k+2}^{m} (\bigvee_{1 \leq i_1 < \ldots < i_j \leq m} \binom{j-1}{k+1} \Sigma^{k+1} X_{i_1} \wedge \ldots \wedge X_{i_j}).$$

2.3.2 Diagrams of spaces

Let $\mathrm{CAT}(K)$ be the face category of K whose objects are faces of K and morphisms are inclusions, let Top be the category of topological spaces and let CW_* be the category of connected, based CW-complexes.

Definition 2.3.10. A $\mathrm{CAT}(K)$-diagram of spaces is a functor F from the face category $\mathrm{CAT}(K)$ to the category of topological spaces.

In most cases, we are working in the category of connected, based CW complexes. For two faces $\sigma \subseteq \tau$ of K, denote by $i_{\sigma,\tau} \colon \sigma \longrightarrow \tau$ the face inclusion and $F(i_{\sigma,\tau}) \colon F(\sigma) \longrightarrow F(\tau)$ the corresponding map between spaces $F(\sigma)$ and $F(\tau)$. We would like to describe the spaces of colimit and homotopy colimit of F. Since a functor preserves the identity morphisms and compositions of morphisms, the colimit of a $\mathrm{CAT}(K)$-diagram F is the following space.

Definition 2.3.11. Let F be a $\mathrm{CAT}(K)$-diagram of CW complexes. Then the *colimit* of F is the disjoint union $\coprod_{\sigma \in K} F(\sigma)$ with certain identifications

$$\operatorname{colim} F = \coprod_{\sigma \in K} F(\sigma) / \sim$$

where $x \sim F(i_{\sigma,\tau})(x)$ for which $x \in F(\sigma)$ and all possible $\sigma \subseteq \tau$ with the face inclusion $i_{\sigma,\tau} \colon \sigma \longrightarrow \tau$.

Next we describe a construction of homotopy colimit for a CAT(K)-diagram F, following a construction of the homotopy colimit in [3, 42] for a diagram $\mathcal{P} \longrightarrow CW_*$, where $\mathcal{P}$ is a poset (partially ordered set). A CAT(K)-diagram F is equivalent to a diagram from a poset $\bar{K}$ to CW_*, where $\bar{K}$ denotes the poset associated to K which has elements consisting of faces of K, ordered by the reverse inclusion. Then the construction $\operatorname{hocolim}_{\sigma\in K} F(\sigma)$ relies on the order complex $\Delta(\bar{K})$, which is ConeK', the cone on the barycentric subdivision of K (Definition 2.1.9, 2.1.11). We adapt the construction in [3, Section 4] of homotopy colimit for a diagram $\mathcal{P} \longrightarrow CW_*$ to a CAT(K)-diagram F, since objects and morphisms in CAT(K) form a poset which is exactly $\bar{K}$.

Recall that Cone K' has a vertex set $\{\sigma \in K\}$ (Definition 2.1.9) including the empty face. For $\sigma \in K$, denote by $X(\sigma)$ the full subcomplex of Cone K' on the vertex set $\{\tau \in K \mid \sigma \subseteq \tau\}$. For faces $\sigma \subseteq \tau$ of K, then $X(\tau)$ is a subcomplex of $X(\sigma)$ and denote by $j_{\tau,\sigma}\colon X(\tau) \longrightarrow X(\sigma)$ the simplicial inclusion. Note that $X(\emptyset) =$ Cone K'. With a CAT(K)-diagram F and a subface σ of τ, there are two types of related maps α and β defined by

$$\begin{aligned} \alpha = \mathrm{id} \times F(i_{\sigma,\tau})\colon \quad & X(\tau) \times F(\sigma) \longrightarrow X(\tau) \times F(\tau) \\ \beta = j_{\tau,\sigma} \times \mathrm{id}\colon \quad & X(\tau) \times F(\sigma) \longrightarrow X(\sigma) \times F(\sigma). \end{aligned}$$

Definition 2.3.12. Given a CAT(K)-diagram F of based CW complexes, the *homotopy colimit* of F is a disjoint union $\coprod_{\sigma\in K} X(\sigma) \times F(\sigma)$ after identifications

$$\operatorname{hocolim}_{\sigma\in K} F = \Big(\coprod_{\sigma\in K} X(\sigma) \times F(\sigma)\Big)/\sim \tag{2.12}$$

where $(\mathbf{x}, u) \sim (\mathbf{x}', u')$ whenever $\alpha(\mathbf{x}, u) = \beta(\mathbf{x}', u')$.

Recall that $T^\sigma = \{(t_1, \dots, t_m) \in T^m \mid t_j = 1 \text{ if } j \notin \sigma\}$ is a $|\sigma|$-torus for $\sigma \subseteq [m]$. Thus the quotient group $T^m/T^\sigma = \{(t_1, \dots, t_m) \in t^m \mid t_j = 1 \text{ if } j \in \sigma\}$ is an $(m - |\sigma|)$-torus. For $\sigma \subseteq \tau \subseteq [m]$, there exists a quotient map $T^m/T^\sigma \longrightarrow T^m/T^\tau$ projecting t_j to 1 if $j \in \tau$ but $j \notin \sigma$.

I will show that the moment-angle complex provides a candidate for the homotopy colimit of the CAT(K)-diagram $D'(\sigma) = T^m/T^\sigma$.

Example 2.3.13 (moment-angle complex)**.** Consider a CAT(K)-diagram D' defined by $D'(\sigma) = T^m/T^\sigma$ with quotient maps $T^m/T^\sigma \longrightarrow T^m/T^\tau$ for $\sigma \subseteq \tau$ of K. We describe the homotopy colimit of D' by (2.12). First, for every $\sigma \in K$, we have $X(\sigma) \times F(\sigma) \subseteq X(\emptyset) \times F(\emptyset)$. We conclude that every element $(\mathbf{x}, \mathbf{u})$ from $X(\sigma) \times F(\sigma)$ is equivalent to the same element $(\mathbf{x}, \mathbf{u})$ in $X(\emptyset) \times F(\emptyset)$ by considering the two types of maps α and β corresponding to $\emptyset \subseteq \sigma$. Thus $\operatorname{hocolim}_{\sigma\in K} D' \simeq X(\emptyset) \times F(\emptyset)/\sim$. To describe the equivalence relation on $X(\emptyset) \times F(\emptyset)$, we rely on the transitive property of an equivalence relation. That is to say, $(\mathbf{x}, \mathbf{u}) \sim (\mathbf{x}', \mathbf{u}')$ in $X(\emptyset) \times F(\emptyset)$ if and only if there exists $\sigma \in K$ and an element $(\mathbf{y}, \mathbf{v}) \in X(\sigma) \times F(\sigma)$ such that $(\mathbf{x}, \mathbf{u}) \sim (\mathbf{y}, \mathbf{v})$ and $(\mathbf{y}, \mathbf{v}) \sim (\mathbf{x}', \mathbf{u}')$. In this way, we have $\mathbf{x} = \mathbf{y} = \mathbf{x}'$ and $\mathbf{u}_j = \mathbf{u}'_j$ for $j \notin \sigma$, where u_j and u'_j are the j-th coordinate of $\mathbf{u}$ and $\mathbf{u}'$ respectively. Note that $u_j = u'_j$ for $j \notin \sigma$ if and only if $\mathbf{u}^{-1}\mathbf{u}' \in T^\sigma$. Then, we have

$$\operatorname*{hocolim}_{\sigma \in K} D' \simeq \text{Cone}K' \times T^m / \sim \tag{2.13}$$

where $(\mathbf{x}, \mathbf{u}) \sim (\mathbf{y}, \mathbf{u}')$ if and only if for some $\sigma \in K$, $\mathbf{x} = \mathbf{y} \in X(\sigma)$ and $\mathbf{u}^{-1}\mathbf{u}' \in T^\sigma$.

We see that the space $\text{Cone}K' \times T^m / \sim$ (2.13) coincides with the alternative construction of moment-angle complexes $\mathcal{Z}_K$ defined in Definition 2.2.8, which implies the following statement.

Lemma 2.3.14. *The colimit of* $D\colon \text{CAT}(K) \longrightarrow CW_*$ *by* $D(\sigma) = (D^2, S^1)^\sigma$ *is homotopy equivalent to its homotopy colimit*

$$\mathcal{Z}_K = \operatorname*{colim}_{\sigma \in K} (D^2, S^1)^\sigma \simeq \operatorname*{hocolim}_{\sigma \in K} T^m / T^\sigma.$$

Proof. We denote by the relation $\sim_1$ in (2.13) and the relation $\sim_2$ in Definition 2.2.8 and show that these relations coincide. Let $\mathbf{x} \in X(\sigma)$ and $\mathbf{t}, \mathbf{t}' \in T^m$ such that $\mathbf{t}^{-1}\mathbf{t}' \in T^\sigma$. Since $X(\sigma)$ is the full subcomplex of $\text{Cone}\, K'$ on the set $\{\tau \in K \mid \sigma \subseteq \tau\}$, there is a unique face $F_{(\tau_i)}$ $(\tau_0 \subsetneq \ldots \subsetneq \tau_l)$ such that $\mathbf{x} \in \text{int}\, F_{(\tau_i)}$ and $\sigma \subseteq \tau_0$. Thus $\mathbf{t}^{-1}\mathbf{t}' \in T^{\tau_0}$ so that $(\mathbf{x}, \mathbf{t}) \sim_2 (\mathbf{x}, \mathbf{t}')$. On the other hand, let $(\mathbf{x}, \mathbf{t}) \sim_2 (\mathbf{x}, \mathbf{t}')$, i.e., there exists a unique face $F_{(\sigma_i)} \in \text{Cone}\, K'$ such that $\mathbf{x} \in \text{int}\, F_{(\sigma_i)}$ and $\mathbf{t}^{-1}\mathbf{t}' \in T^{\sigma_0}$. Let $\sigma = \sigma_0$. So $\mathbf{x} \in X(\sigma)$ and $\mathbf{t}^{-1}\mathbf{t}' \in T^\sigma$ which implies $(\mathbf{x}, \mathbf{t}) \sim_1 (\mathbf{x}, \mathbf{t}')$. □

Let $\Lambda\colon T^m \longrightarrow T^{m-r}$ be a homomorphism of tori induced by a surjective linear map $\mathbb{Z}^m \longrightarrow \mathbb{Z}^{m-r}$. Consider the diagram $E'\colon \text{CAT}(K) \longrightarrow CW_*$ by $E'(\sigma) = T^{m-r}/\Lambda_\sigma$ and quotient maps $T^{m-r}/\Lambda_\sigma \longrightarrow T^{m-r}/\Lambda_\tau$ for $\sigma \subseteq \tau \in K$ and Λ_σ denotes the image of T^σ in T^{m-r} by Λ. Following the argument in Example 2.3.13, we recover the quotient of $\mathcal{Z}_K$ as a homotopy colimit of E'.

Example 2.3.15 (quotients of $\mathcal{Z}_K$)**.** The homotopy colimit of the $\text{CAT}(K)$-diagram E' is given by

$$\operatorname*{hocolim}_{\sigma \in K} E'(\sigma) \simeq \mathcal{Z}_K / \text{Ker}\Lambda$$

where $(\mathbf{x}, \mathbf{t}_1) \sim (\mathbf{y}, \mathbf{t}_2)$ if and only if $\mathbf{x} = \mathbf{y} \in \text{int}\, F_{(\sigma_i)}$ and $\mathbf{t}_1^{-1}\mathbf{t}_2 \in \Lambda_{\sigma_0}$.

In particular, if $H = \text{Ker}\Lambda$ satisfies $H \cap T^\sigma = \{\mathbf{1}\}$ for any $\sigma \in K$, then $E'(\sigma) = T^{m-r}/\Lambda_\sigma \cong T^m/(T^\sigma \times H)$. This is the case of particular interest to consider free subtorus actions on $\mathcal{Z}_K$ in Chapter 4.

In general, a polydedral product $(\underline{X}, \underline{A})^K$ and a polyhedral smash product $(\underline{X}, \underline{A})^{\wedge K}$ have their associated $\text{CAT}(K)$-diagrams.

Definition 2.3.16. Let K be a simplicial complex on $[m]$ and $(\underline{X}, \underline{A})$ be m pairs of spaces.

(a) Define a *polyhedral functor* $P\colon \mathrm{CAT}(K) \longrightarrow \mathrm{Top}$ by $P(\sigma) = (\underline{X}, \underline{A})^\sigma$ and the inclusion $(\underline{X}, \underline{A})^\sigma \longrightarrow (\underline{X}, \underline{A})^\tau$ for $\sigma \subseteq \tau \in K$.

(b) Analogously, define a *polyhedral smash* functor $\widehat{P}\colon \mathrm{CAT}(K) \longrightarrow \mathrm{Top}$ by $\widehat{P}(\sigma) = (\underline{X}, \underline{A})^{\wedge\sigma}$ and the inclusion $(\underline{X}, \underline{A})^{\wedge\sigma} \longrightarrow (\underline{X}, \underline{A})^{\wedge\tau}$ for $\sigma \subseteq \tau \in K$.

Thus we have $(\underline{X}, \underline{A})^K = \bigcup\limits_{\sigma\in K} (\underline{X}, \underline{A})^\sigma = \operatorname*{colim}\limits_{\sigma\in K} P(\sigma)$ and $(\underline{X}, \underline{A})^{\wedge K} = \bigcup\limits_{\sigma\in K} (\underline{X}, \underline{A})^{\wedge\sigma} = \operatorname*{colim}\limits_{\sigma\in K} \widehat{P}(\sigma)$.

Under certain conditions, the homotopy types of $\operatorname{colim} F$ and $\operatorname{hocolim} F$ coincide.

Theorem 2.3.17 ([3]). *Let K be a simplicial complex on $[m]$ and $(\underline{X}, \underline{A})$ be m pairs of connected based CW-complexes. Let P be a polyhedral functor defined by $P(\sigma) = (\underline{X}, \underline{A})^\sigma$ for $\sigma \in K$. Then there exists a homotopy equivalence*

$$\operatorname{colim} P(\sigma) \simeq \operatorname{hocolim} P(\sigma).$$

Example 2.3.18. Consider the diagram $DJ\colon \mathrm{CAT}(K) \longrightarrow CW_*$ by $DJ(\sigma) = (BS^1, *)^\sigma$ and the inclusion $(BS^1, *)^\sigma \longrightarrow (BS^1, *)^\tau$ for $\sigma \subseteq \tau \in K$. Then the Davis-Januszkiewicz space $DJ_K = (BS^1, *)^K = \operatorname*{colim}\limits_{\sigma\in K} DJ(\sigma) \simeq \operatorname*{hocolim}\limits_{\sigma\in K} DJ(\sigma)$.

To summarise, we have considered a few diagrams from $\mathrm{CAT}(K)$ to CW_*. Let H be a subtorus of T^m such that $H \cap T^\sigma = \{\mathbf{1}\}$ for every $\sigma \in K$. For faces $\sigma \subseteq \tau$ of K, these diagrams are defined as follows.

Example 2.3.19. Here are a few diagrams which will be studied in the next section to obtain homotopy fibrations.

$D(\sigma) = (D^2, S^1)^\sigma$ with inclusions $(D^2, S^1)^\sigma \longrightarrow (D^2, S^1)^\tau$,
$D'(\sigma) = T^m/T^\sigma$ with quotient maps $T^m/T^\sigma \longrightarrow T^m/T^\tau$,
$E(\sigma) = (D^2, S^1)^\sigma/H$ with induced quotient maps $(D^2, S^1)^\sigma/H \longrightarrow (D^2, S^1)^\tau/H$,
$E'(\sigma) = T^m/(T^\sigma \times H)$ with quotient maps $T^m/(T^\sigma \times H) \longrightarrow T^m/(T^\tau \times H)$,
$DJ(\sigma) = (BS^1, *)^\sigma$ with inclusions $(BS^1, *)^\sigma \longrightarrow (BS^1, *)^\tau$.

Note that each space $(D^2, S^1)^\sigma$ is an H-invariant subspace of $\mathcal{Z}_K$ (Lemma 4.5.1).

2.3.3 Fibration sequences

The purpose of this section is to get homotopy fibrations by applying Puppe's theorem [37] to $\mathrm{CAT}(K)$-diagrams. Our exposition below follows a description due to [18, p.180].

Let $\mathcal{E}$ be a $\mathrm{CAT}(K)$-diagram of spaces and let B be a fixed space. By a map $f\colon \mathcal{E} \longrightarrow B$ bewteen $\mathcal{E}$ and B, we mean that f is a natural transformation from $\mathcal{E}$ to Top with a constant evaluation $f(\sigma) = B$ for every $\sigma \in \mathcal{E}$. With a map from $\mathcal{E}$ to a fixed space B, there exists an associated diagram of fibres by taking the objectwise homotopy fibre.

Definition 2.3.20. Let $\mathcal{E}$ be a $\mathrm{CAT}(K)$-diagram of spaces, let B be a fixed space and $f\colon \mathcal{E} \longrightarrow B$ be a map bewteen $\mathcal{E}$ and B. A $\mathrm{CAT}(K)$-diagram Fib_f of fibres is defined by taking $\mathrm{Fib}_f(\sigma)$ to be the homotopy fibre of $f_\sigma\colon \mathcal{E}(\sigma) \longrightarrow B$ and morphisms $\mathrm{Fib}_f(\sigma) \longrightarrow \mathrm{Fib}_f(\tau)$ to be the corresponding maps between fibres induced by the map $\mathcal{E}(\sigma) \longrightarrow \mathcal{E}(\tau)$ for $\sigma \subseteq \tau$ in K.

Given a map f from a $\mathrm{CAT}(K)$-diagram $\mathcal{E}$ to a fixed space B, there are two topological spaces associated. One is the homotopy fibre of an induced map $\bar{f}\colon \operatorname*{hocolim}_{\sigma\in K} \mathcal{E}(\sigma) \longrightarrow B$ and another one is $\operatorname*{hocolim}_{\sigma\in K} \mathrm{Fib}_f(\sigma)$, the homotopy colimit of the $\mathrm{CAT}(K)$-diagram of fibres induced by f. Puppe's theorem states when these two spaces have the same homotopy type.

Theorem 2.3.21 ([37, 18])**.** *Let $\mathcal{E}$ be a $\mathrm{CAT}(K)$-diagram of spaces, let B be a fixed connected space and let $f\colon \mathcal{E} \longrightarrow B$ be any map bewteen $\mathcal{E}$ and B. Assume that for $\sigma \subseteq \tau$ in $\mathrm{CAT}(K)$, the following diagram is commutative*

$$\begin{array}{ccc} \mathcal{E}(\sigma) & \longrightarrow & \mathcal{E}(\tau) \\ \downarrow & & \downarrow \\ B & = & B. \end{array}$$

Then the homotopy fibre of the induced map $\bar{f}\colon \operatorname*{hocolim}_{\sigma\in K} \mathcal{E}(\sigma) \longrightarrow B$ is homotopy equivalent to the homotopy colimit of a $\mathrm{CAT}(K)$-diagram $\mathrm{Fib_f}$ of fibres.

To explain the next statement, we recall a fact [30, p.212] that if H is a closed connected subgroup of G, then $H \hookrightarrow G \xrightarrow{\pi} G/H = \{gH \mid g \in G\}$ is a fibration (in fact it is a principal H-bundle). Let $\pi\colon G \longrightarrow G/H$ denote the "type" of quotient maps. Here "type" means all quotient maps from a group G to its left coset space G/H with $H \leqslant G$. So if H_1 and H_2 are two subtori of T^m satisfying that $H_1 \cap H_2 = \{\mathbf{1}\}$, then there is a fibration

$$(H_1 \times H_2)/H_2 \longrightarrow T^m/H_2 \longrightarrow T^m/(H_1 \times H_2)$$

by taking $(H_1 \times H_2)/H_2$ as a closed subgroup of T^m/H_2 which has a quotient group isomorphic to $T^m/(H_1 \times H_2)$. Moreover, since we assume that $H_1 \cap H_2 = \{\mathbf{1}\}$, then the composite $H_1 \xrightarrow{i_1} H_1 \times H_2 \xrightarrow{\pi} (H_1 \times H_2)/H_2$ is a goup isomorphism. Thus, there is a fibration

$$H_1 \xrightarrow{\alpha} T^m/H_2 \longrightarrow T^m/(H_1 \times H_2)$$

where α is the composite $H_1 \hookrightarrow T^m \xrightarrow{\pi} T^m/H_2$.

As H_1 is a subtorus of T^m, we have a fibration $H_1 \overset{i}{\hookrightarrow} T^m \longrightarrow T^m/H_1$. Then the quotient map $T^m \longrightarrow T^m/H_2$ gives a commutative diagram of fibrations

$$\begin{array}{ccccc} H_1 & \xrightarrow{i} & T^m & \longrightarrow & T^m/H_1 \\ \| & & \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \pi} \\ H_1 & \xrightarrow{\alpha} & T^m/H_2 & \longrightarrow & T^m/(H_1 \times H_2). \end{array} \tag{2.14}$$

Moreover, there is a fibration $G/H \longrightarrow BH \longrightarrow BG$ ([16, Proposition 2.15]) if H is a subgroup of G. Then (2.14) gives rise to a homotopy commutative diagram of fibrations

$$\begin{array}{ccccc} T^m/H_1 & \longrightarrow & BH_1 & \xrightarrow{Bi} & BT^m \\ \downarrow{\scriptstyle \pi} & & \| & & \downarrow{\scriptstyle B\pi} \\ T^m/(H_1 \times H_2) & \longrightarrow & BH_1 & \xrightarrow{B\alpha} & B(T^m/H_2). \end{array}$$

Lemma 2.3.22. *Let H be a subtorus of T^m of rank r satisfying $H \cap T^\sigma = \{\mathbf{1}\}$ for every $\sigma \in K$. Then the quotient map $\mathcal{Z}_K \xrightarrow{q} \mathcal{Z}_K/H$ makes the following diagram of homotopy fibrations commutative up to homotopy*

$$\begin{array}{ccccc} \mathcal{Z}_K & \longrightarrow & DJ_K & \xrightarrow{j} & BT^m \\ \downarrow{\scriptstyle q} & & \| & & \downarrow{\scriptstyle B\pi} \\ \mathcal{Z}_K/H & \longrightarrow & DJ_K & \xrightarrow{(B\pi)\circ j} & B(T^m/H) \end{array}$$

where j is a canonical inclusion.

Proof. If $H \cap T^\sigma$ is trivial for every $\sigma \in K$, then we have a diagram of fibrations

$$\begin{array}{ccccc} T^m/T^\sigma & \longrightarrow & BT^\sigma & \longrightarrow & BT^m \\ \downarrow & & \| & & \downarrow{\scriptstyle B\pi} \\ T^m/(T^\sigma \times H) & \longrightarrow & BT^\sigma & \longrightarrow & B(T^m/H). \end{array} \tag{2.15}$$

Consider the Davis-Januszkiewicz space as $DJ_K = (BS^1, *)^K \simeq \operatorname{hocolim}_{\sigma \in K} BT^\sigma$. The inclusion $j_\sigma \colon BT^\sigma \longrightarrow BT^m$ and its composition with the quotient map $\pi j_\sigma \colon BT^\sigma \longrightarrow B(T^m/H)$, give two maps from a CAT(K)-diagram DJ (Example 2.3.19) to fixed spaces BT^m and $B(T^m/H)$, respectively. By the fibre bundles (2.15), the CAT(K)-diagrams D' and E' (Example 2.3.19) are the induced CAT(K)-diagrams of fibres for $(BS^1, *)^K \xrightarrow{j} BT^m$ and $(BS^1, *)^K \xrightarrow{(B\pi)\circ i} B(T^m/H)$, respectively. Objectwise, the quotient map $D'(\sigma) \longrightarrow E'(\sigma)$ is the induced map between fibres.

Note that these two maps j and $(B\pi) \circ j$ satisfy the condition in Puppe's theorem. A direct consequence of Puppe's theorem is that $\operatorname{hocolim}_{\sigma \in K} D'(\sigma)$ and $\operatorname{hocolim}_{\sigma \in K} E'(\sigma)$ are the homotopy fibres of maps $DJ_K \xrightarrow{j} BT^m$ and $DJ_K \xrightarrow{(B\pi)\circ j} B(T^m/H)$, respectively.

According to the construction (2.12) of the homotopy colimit, the objectwise quotient map $D'(\sigma) \longrightarrow E'(\sigma)$ will induce a quotient map between $X(\emptyset) \times D'(\sigma)/\sim$ and $X(\emptyset) \times E'(\sigma)/\sim$. By Examples 2.3.13 and 2.3.15, these candidates (2.12) of the homotopy colimit of D' and E' are homeomorphic to $\mathcal{Z}_K$ and $\mathcal{Z}_K/H$. When we replace $X(\emptyset) \times D'(\sigma)/\sim$ and $X(\emptyset) \times E'(\sigma)/\sim$ by $\mathcal{Z}_K$ and $\mathcal{Z}_K/H$ due to the homeomorphism, the quotient map between $X(\emptyset) \times D'(\sigma)/\sim$ and $X(\emptyset) \times E'(\sigma)/\sim$ induces the quotient map between $\mathcal{Z}_K$ and $\mathcal{Z}_K/H$, since $X(\emptyset) \times D'(\sigma)/\sim$ and $\mathcal{Z}_K$ are H-equivariantly homeomorphic. □

Remark 2.3.23. It can be shown that if K does not have ghost vertices, then these two fibration sequences in Lemma 2.15 splits after loop because of the existence of sections in both cases. The long exact sequence of homotopy groups associated to $\mathcal{Z}_K/H \longrightarrow DJ_K \longrightarrow B(T^m/H)$ implies that $\mathcal{Z}_K/H$ is simply-connected.

Homological consequences. In [34, 35], Panov proved that the Eilenberg-Moore spectral sequences associated to these two fibration sequences $\mathcal{Z}_K \longrightarrow DJ_K \longrightarrow BT^m$ and $\mathcal{Z}_K/H \longrightarrow DJ_K \longrightarrow B(T^m/H)$ collapse at the E_2-term if H satisfies the condition in Lemma 2.3.22. The cohomologies of $\mathcal{Z}_K$ and $\mathcal{Z}_K/H$ follow.

Theorem 2.3.24 ([34, 35])**.** *Let H be a subtorus of T^m such that $H \cap T^\sigma = \{\mathbf{1}\}$ for every $\sigma \in K$. Then there are isomorphisms of R-algebras*

$$H^*(\mathcal{Z}_K; R) \cong \mathrm{Tor}_{R[m]}(R[K], R);$$
$$H^*(\mathcal{Z}_K/H; R) \cong \mathrm{Tor}_{H^*(B(T^m/H);R)}(R[K], R).$$

We recall that the inclusion $(BS^1, *)^K \longrightarrow BT^m$ induces the quotient homomorphism on their cohomologies. More details of this Tor-algebra $\mathrm{Tor}_{H^*(B(T^m/H);R)}(R[K], R)$ will be considered in Section 4.3.

Proposition 2.3.25 ([10])**.** *The cohomology ring $H^*(DJ_K; R)$ of Davis-Januszkiewicz spaces is isomorphic to the Stanley-Reisner ring $R[K]$. The inclusion $(BS^1, *)^K \xrightarrow{j} BT^m$ induces a quotient homomorphism between their cohomologies*

$$j^* \colon R[v_1, \ldots, v_m] \longrightarrow R[K] = R[v_1, \ldots, v_m]/I_K$$

where I_K is the Stanley-Reisner ideal.

Hence, the moment-angle complex provides a topological model of the Tor-algebra $\mathrm{Tor}_{R[m]}(R[K], R)$, which makes it possible to study combinatorial algebras by topological techniques. For instance, the homotopy splitting $\Sigma\mathcal{Z}_K \simeq \Sigma^{|J|+2} \bigvee_{J \notin K} |K_J|$ (Theorem 2.3.7) implies a decomposition of R-modules

$$H^i(\mathcal{Z}_K; R) \cong \bigoplus_{J \subseteq [m]} \widetilde{H}^{i-|J|-1}(K_J; R).$$

Grbić-Theriault [22] proved that the homotopy type of $\mathcal{Z}_K$ for shifted complexes is a wedge of spheres. It is a topological version of the algebraic statement that if K is shifted, then all multiplications in $\mathrm{Tor}_{R[m]}(R[K], R)$ vanish.

It has been stated in Section 2.1 that there are standard differential graded algebras, Koszul algebra and Taylor algebra, such that their cohomolgy algebras are algebraically isomorphic to $\mathrm{Tor}_{R[m]}(R[K], R)$. In Section 4.3, we will apply the corresponding Koszul algebra and Taylor algebra to $\mathrm{Tor}_{H^*(B(T^m/H);R)}(R[K], R)$, to calculate the cohomology of $\mathcal{Z}_K/H$.

2.3.4 Homotopy pushouts

We consider the topological pushout among polyhedral products induced by a pushout of simplicial complexes. Let K_1 and K_2 be two simplicial complexes on $[m_1]$ and $[m_2]$. Then there is a pushout of simplicial complexes

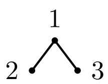
$$\begin{array}{ccc} K_1 \cap K_2 & \longrightarrow & K_2 \\ \downarrow & & \downarrow \\ K_1 & \longrightarrow & K_1 \cup K_2. \end{array} \tag{2.16}$$

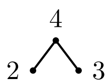
If L is a subcomplex of K, let $(\underline{X}, \underline{A})^{\overline{L}}$ denote the polyhedral product which includes ghost vertices of L in K. Say $K_1 \cup K_2$ has m vertices and take $K_1 \cap K_2$, K_1 and K_2 are subcomplexes of $K_1 \cup K_2$ on $V(K_1 \cup K_2)$. With this, we have a topological pushout.

Lemma 2.3.26 ([40])**.** *There is a pushout of topological spaces*

$$\begin{array}{ccc} (\underline{X}, \underline{A})^{\overline{K_1 \cap K_2}} & \longrightarrow & (\underline{X}, \underline{A})^{\overline{K_2}} \\ \downarrow & & \downarrow \\ (\underline{X}, \underline{A})^{\overline{K_1}} & \longrightarrow & (\underline{X}, \underline{A})^{K_1 \cup K_2} \end{array} \tag{2.17}$$

where all maps among these spaces are induced by corresponding simplicial inclusions.

Example 2.3.27. Let $K_1 =$ 

and $K_2 =$ 

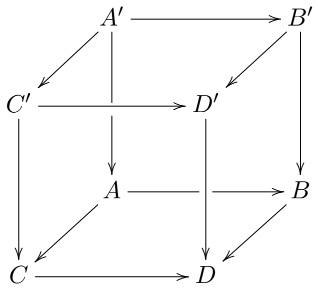
. Then $K_1 \cap K_2$ is a disjoint union of two points and $K_1 \cup K_2$ is a boundary of a square. In this case,

$$\begin{aligned} (\underline{X}, \underline{A})^{\overline{K_1 \cap K_2}} &= A_1 \times (X_2 \times A_3 \cup_{A_2 \times A_3} A_2 \times X_3) \times A_4 \\ (\underline{X}, \underline{A})^{\overline{K_1}} &= X_1 \times (X_2 \times A_3 \cup_{A_2 \times A_3} A_2 \times X_3) \times A_4 \\ (\underline{X}, \underline{A})^{\overline{K_2}} &= A_1 \times (X_2 \times A_3 \cup_{A_2 \times A_3} A_2 \times X_3) \times X_4. \end{aligned}$$

The polyhedral product $(\underline{X}, \underline{A})^{K_1 \cup K_2}$ is a union of $(\underline{X}, \underline{A})^{\overline{K_1}}$ and $(\underline{X}, \underline{A})^{\overline{K_2}}$ over the intersection $(\underline{X}, \underline{A})^{\overline{K_1 \cap K_2}}$.

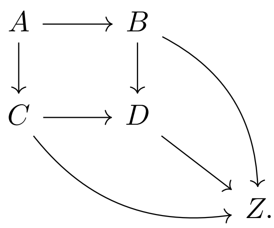
Homotopy pushouts of fibres. Here we rely on the cube lemma [29] to obtain a homotopy pushout among fibres.

Lemma 2.3.28 (cube lemma [29, 2])**.** *Consider a cube diagram whose faces are homotopy commutative.*



If the bottom square $A - B - C - D$ is a homotopy pushout and all four sided square are homotopy pullbacks, then the top square $A' - B' - C' - D'$ is also a homotopy pushout.

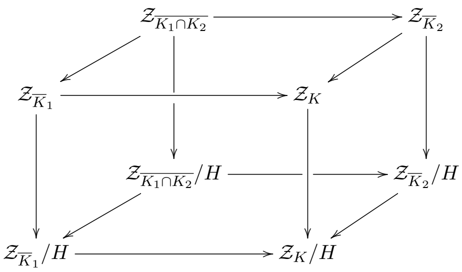
Given a map $D \longrightarrow Z$, there is a commutative diagram



A special case of cube lemma observes that the top square $A' - B' - C' - D'$ is obtained by taking the homotopy fibre, respectively, through mapping each A, B, C, D into a fixed space Z given a map $D \longrightarrow Z$. So that, if $A - B - C - D$ is a homotopy pushout, then the square of fibres on the top $A' - B' - C' - D'$ is a homotopy pushout too.

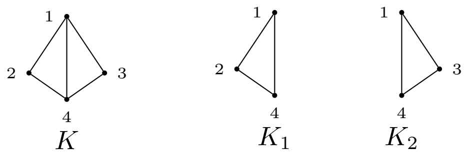
In particular, Lemma 2.3.26 provides a pushout square specialising to Davis-Januszkiewicz spaces $DJ_K = (BS^1, *)^K$. Since $(BS^1, *)$ is a pair of CW complexes, the maps between Davis-Januszkiewicz spaces induced by simplicial inclusions are cofibrations. So this pushout (2.17) in terms of Davis-Januszkiewicz spaces is also a homotopy pushout. Mapping $(BS^1, *)^K$ to BT^m and $B(T^m/H)$ as in Lemma 2.3.22, we have the homotopy fibres $\mathcal{Z}_K$ and $\mathcal{Z}_K/H$. Hence by cube lemma, there are two homotopy pushouts in terms of moment-angle complexes $\mathcal{Z}_K$ and their quotients $\mathcal{Z}_K/H$ and the maps among them are induced by simplicial inclusions in (2.16). Under the assumption of Lemma 2.3.22, the next statement follows.

Lemma 2.3.29. *Let $K = K_1 \cup K_2$ on $[m]$. Suppose that H is a subtorus of T^m such that $H \cap T^\sigma = \{\mathbf{1}\}$ for any $\sigma \in K$. There is a commutative cube diagram*



where the top and bottom are homotopy pushouts, whose maps are induced by simplicial inclusions (2.16) *and all vertical map are quotient maps.*

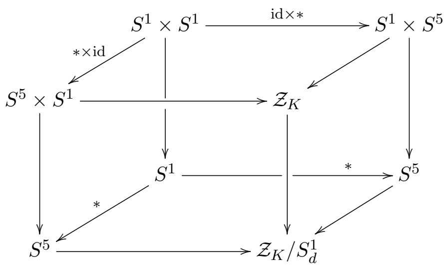
Example 2.3.30. Let K be the following simplicial complex with K_1 and K_2 pictured below. Consider the diagonal S^1-action on $\mathcal{Z}_K$.



In this case, we have the following spaces (up to homotopy)

$$\mathcal{Z}_{\overline{K_1 \cap K_2}} \simeq S^1 \times S^1,\ \mathcal{Z}_{\overline{K_1 \cap K_2}}/S^1_d \simeq S^1,\ \mathcal{Z}_{\overline{K_i}} \simeq S^1 \times S^5,\ \mathcal{Z}_{\overline{K_i}}/S^1_d \simeq S^5, i = 1, 2.$$

The diagram in Lemma 2.3.29 indicates a homotopy commutative diagram by a replacement of spaces due to homotopy equivalences



where the top and bottom square are homotopy pushout. Since the fundamental group $\pi_1(S^5)$ is trivial, the homotopy types of $\mathcal{Z}_K$ and $\mathcal{Z}_K/S^1_d$ are

$$\mathcal{Z}_K \simeq S^1 * S^1 \vee (S^1 \ltimes S^5) \vee (S^5 \rtimes S^1) \text{ and } \mathcal{Z}_K/S^1_d \simeq S^2 \vee 2S^5.$$

We will continue to consider the homotopy types of $\mathcal{Z}_{\Delta^k_m}/S^1_d$ by taking a pushout of simplicial complexes in Section 4.5.2 .

Chapter 3

Representation stability of polyhedral products

The content of this chapter is to be published as a joint article [20] with my supervisor Jelena Grbić. The idea to study representation stability was motivated from the lectures given by Benson Farb at the conference *Young Topologist Meeting* in Stockholm 2017. I started to consider the representation stability of the cohomology of moment-angle complexes corresponding to m disjoint points. My supervisor suggested to generalise it to polyhedral products which led us to work together on this project.

Moment-angle complexes $\mathcal{Z}_K = (D^2, S^1)^K$ are considered as spaces on which a torus T^l, $l \leq m$ acts. The action of the torus is induced by an S^1-action on (D^2, S^1). Extensive literature exists on the study of this action. The problem we are studying is how symmetries of a simplicial complex K influence the symmetries of the moment-angle complex $\mathcal{Z}_K$.

Church and Farb [15] introduced the theory of representation stability. The goal of representation stability is to provide a framework for generalising the classical homology stability to situations when each vector space V_m has an action of the symmetric group Σ_m (or other natural families of groups). We initiate the study of representation stability to toric topology.

If a finite group G acts simplicially on a simplicial complex K, then that action induces a G-action on polyhedral products, in particular on the moment-angle complex $\mathcal{Z}_K$. Notice that by acting simplicially on a simplicial complex K on m vertices, G is a subgroup of the symmetric group Σ_m.

In this chapter we study Σ_m-representation stability of polyhedral products. We start by analysing G-equivariant properties of the stable homotopy decomposition of moment-angle complexes $\mathcal{Z}_K$ [26, 9] and polyhedral products $(X, A)^K$ [3]. These homotopy decompositions induce $\mathbf{k}G$-module decompositions of the cohomology of moment-angle complexes and polyhedral products, respectively.

Specialising to $G = \Sigma_m$, we describe several non-trivial constructions of families of simplicial Σ_m-complexes $\mathcal{K} = \{K_m\}$ (see Constructions 3.4.7 and 3.4.8) and describe conditions on these families which together with decomposition (3.1) and Hemmer's result [25] imply uniform representation stability of Σ_m-representation of $\{\widetilde{H}_*((X,A)^{K_m};\mathbf{k})\}$ (see Theorem 3.4.12 and Corollary 3.4.14). In the case of moment-angle complexes, we construct a sequence of Σ_m-manifolds which are uniformly representation stable although not homology stable (see Proposition 3.4.15).

The uniform representation stability influences the behaviour of the Betti numbers of the i-th homology groups $\{\widetilde{H}_i((X,A)^{K_m};\mathbb{Q})\}$ and we show that in this case their growth is eventually polynomial with respect to m (see Theorem 3.5.3).

3.1 Irreducible representations of symmetric groups

We first collect definitions and properties of Σ_m-representations from [7, 21]. In this section, let G be a finite group and let $\mathbf{k}$ be a field or $\mathbb{Z}$.

Definition 3.1.1. The *group algebra* $\mathbf{k}G$ consists of elements of the form $\sum\limits_{g\in G} a_g g$, $a_g \in \mathbf{k}$, with a $\mathbf{k}$-bilinear product $\mathbf{k}G \times \mathbf{k}G \longrightarrow \mathbf{k}G$ extended uniquely by the group multiplication of G. A G-representation over $\mathbf{k}$ is a $\mathbf{k}G$-module, i.e., a module over $\mathbf{k}G$.

A G-representation over $\mathbf{k}$ can also be defined by a vector space V over $\mathbf{k}$ together with a group action $\rho\colon G \times V \longrightarrow V$ such that $\rho(g, a+b) = \rho(g,a) + \rho(g,b)$ for $g \in G$ and $a, b \in V$. We also write $g \cdot a$ to denote $\rho(g,a)$.

Let H be a subgroup of G and let V be a $\mathbf{k}H$-module. The homomorphism $\mathbf{k}H \longrightarrow \mathbf{k}G$ induced by $H \leqslant G$ gives $\mathbf{k}G$ a right $\mathbf{k}H$-module structure. Then the tensor product $\mathbf{k}G \otimes_{\mathbf{k}H} V$ is a $\mathbf{k}G$-module by $a(a' \otimes_{\mathbf{k}H} x) = (aa') \otimes_{\mathbf{k}H} x$, where $a, a' \in \mathbf{k}G$ and $x \in V$.

Definition 3.1.2. We refer to $\mathrm{Ind}_H^G V = \mathbf{k}G \otimes_{\mathbf{k}H} V$ as an *induced* $\mathbf{k}G$*-module* from a $\mathbf{k}H$-module V.

Since the H-action on G by right multiplication is free, $\mathbf{k}G$ is a free $\mathbf{k}H$-module, with a basis consisting of representatives of the left coset G/H. That is, as a $\mathbf{k}H$-module, $\mathbf{k}G \cong \bigoplus\limits_{g\in G/H} g \cdot \mathbf{k}H$, where $g \cdot \mathbf{k}H = \{\sum\limits_{h\in H} a_h(gh), a_h \in \mathbf{k}\}$. It follows that $\mathbf{k}G \otimes_{\mathbf{k}H} V$ has a G-isomorphism

$$\mathrm{Ind}_H^G V = \mathbf{k}G \otimes_{\mathbf{k}H} V \cong \bigoplus_{g\in G/H} g \cdot V$$

where $g \cdot V$ denotes $(g \cdot \mathbf{k}H) \otimes_{\mathbf{k}H} V$ and $g \cdot V \cong V$ and the G-action on $\bigoplus_{g \in G/H} g \cdot V$ is a permutation of the summands by $g \cdot V \xmapsto{g'} (g'g) \cdot V$, induced by the G-action on the left coset G/H by left multiplication, which is transitive.

On the other hand, suppose that $N = \bigoplus_{i \in I} V_i$ is a G-module and G acts transitively permuting the summands (i.e., there exists a transitive G-action on I such that $g \cdot V_i = V_{g \cdot i}$). The next statement implies $N \cong \mathrm{Ind}_H^G V$ for some H and a $\mathbf{k}H$-module V.

Proposition 3.1.3 ([7])**.** *Let $N = \bigoplus_{i \in I} V_i$ be a $\mathbf{k}G$-module as above and V be one of the summands V_i. Denote by H the isotropy group of i. Then V is an H-module and there is an isomorphism of $\mathbf{k}G$-modules $N \cong \mathrm{Ind}_H^G V$.*

A direct consequence of the above proposition follows.

Corollary 3.1.4 ([7])**.** *Let $N = \bigoplus_{i \in I} V_i$ be a $\mathbf{k}G$-module. Assume that the G-action permutes the summands according to some action of G on I. Then there exists an isomorphism of $\mathbf{k}G$-modules*

$$N \cong \bigoplus_{i \in E} \mathrm{Ind}_{G_i}^G V_i \tag{3.1}$$

where E is a set of representatives of orbits of I and G_i is the stabiliser of i in G.

Definition 3.1.5. A G-complex is a CW-complex X together with a group action G on it which permutes the cells.

A simplicial G-complex is a simplicial complex K on a vertex set $[m]$ with a G-action on $[m]$ such that the induced action on subsets of $[m]$ preserves K. Thus, the geometrical realisation of a simplicial G-complex K is a G-complex.

Example 3.1.6 (Simplicial G-complexes)**.** For a simplicial G-complex K, each chain group $C_n(K; \mathbf{k})$ is a direct sum of copies of $\mathbf{k}$, each summand corresponding to an n-simplex of K on which G acts. Denote by G_σ the stabiliser of σ, and let E_n be a set of representatives of the G-orbits of n-simplices of K. Thus, by (3.1),

$$C_n(K; \mathbf{k}) \cong \bigoplus_{\sigma \in E_n} \mathrm{Ind}_{G_\sigma}^G \mathbf{k}.$$

Definition 3.1.7. A $\mathbf{k}G$-module is *irreducible* if the only G-invariant submodules are 0 and V itself.

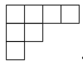
Proposition 3.1.8 ([21])**.** *Let $\mathbf{k}$ be a field of characteristic zero. For a $\mathbf{k}G$-module V, there is a unique decomposition up to isomorphism*

$$V \cong V_1^{\oplus a_1} \oplus \ldots \oplus V_k^{\oplus a_k}$$

where the V_i are non-isomorphic irreducible $\mathbf{k}G$-modules and these a_i are multiplicities of V_i in V.

A $\mathbf{k}G$-module is a G-representation over $\mathbf{k}$. Now we focus on irreducible representations of symmetric groups Σ_m, in order to study the Σ_m-representation stability. A partition $\lambda \vdash m$ of m is a sequence $\lambda = (\lambda_1, \ldots, \lambda_l)$ where $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_l \geq 1$ and $\sum_{i=1}^{l} \lambda_i = m$. The number l is called the length of λ.

Definition 3.1.9. A *Young diagram* associated to a partition $\lambda = (\lambda_1, \ldots, \lambda_l) \vdash m$ is m boxes with l rows and each i-th row contains λ_i boxes.

For example, let $\lambda = (4, 2, 1) \vdash 7$. Then its associated Young diagram is  .

The canonical tableau on a given Young diagram is numbering each box consecutively by $1, \ldots, m$ as shown

$$\begin{array}{|c|c|c|c|}\hline 1 & 2 & 3 & 4 \\ \hline 5 & 6 \\ \cline{1-2} 7 \\ \cline{1-1}\end{array} \tag{3.2}$$

Construction 3.1.10 ([21])**.** For a canonical tableau on a given Young diagram, let $P_\lambda = \{g \in \Sigma_m \mid g \text{ preserves each row}\}$ and $Q_\lambda = \{g \in \Sigma_m \mid g \text{ preserves each column}\}$. Then define a_λ and b_λ in $\mathbf{k}\Sigma_m$ by

$$a_\lambda = \sum_{g \in P_\lambda} g \text{ and } b_\lambda = \sum_{g \in Q_\lambda} \operatorname{sgn}(g) g$$

The Young symmetrizer is defined by $c_\lambda = a_\lambda b_\lambda$ in $\mathbf{k}\Sigma_m$.

For example, in (3.2), these two associated groups P_λ and Q_λ are isomorphic to $\Sigma_4 \times \Sigma_2$ and $\Sigma_3 \times \Sigma_2$, respectively.

Theorem 3.1.11 ([21])**.** *Let $\mathbf{k}$ be a field of characteristic 0 and let $V_\lambda = (\mathbf{k}\Sigma_m)c_\lambda$. Then V_λ is irreducible and each irreducible representation of Σ_m is given by V_λ for a unique partition λ.*

Definition 3.1.12. If given any partition $\lambda = (\lambda_1, \ldots, \lambda_l) \vdash k$, then for $m \geq \lambda_1 + k$, the partition $\lambda[m] = (m - k, \lambda_1, \ldots, \lambda_l)$ is called *padded partition.* Its corresponding irreducible representation is denoted by $V(\lambda)_m$.

Let $\lambda \vdash k$ and let V_λ be the corresponding irreducible Σ_k-representation. For $m \geq k$, V_λ can be seen as a $(\Sigma_k \times \Sigma_{m-k})$-representation by a trivial Σ_{m-k}-action, which is denoted by $V_\lambda \boxtimes \mathbf{k}$. Since $\Sigma_k \times \Sigma_{m-k}$ is a subgroup of Σ_m, Pieri's formula gives a decomposition of its induced Σ_m-representation.

Proposition 3.1.13 (Pieri's formula [21])**.** *Let $\mathbf{k}$ be a field of characteristic zero. Then for $m \geq k$, the induced Σ_m-representation $\operatorname{Ind}_{\Sigma_k \times \Sigma_{m-k}}^{\Sigma_m} V_\lambda \boxtimes \mathbf{k}$ has a decomposition*

$$\operatorname{Ind}_{\Sigma_k \times \Sigma_{m-k}}^{\Sigma_m} V_\lambda \boxtimes \mathbf{k} = \bigoplus_{\mu \vdash m} V_\mu$$

where μ goes over the partitions whose Young diagram is obtained by adding $m-k$ boxes to the Young diagram of λ such that no two boxes are in the same column.

Example 3.1.14. Let $\lambda = (4,2,1) \vdash 7$. Then by Pieri's formula,

$$\mathrm{Ind}_{\Sigma_7\times\Sigma_1}^{\Sigma_8} V_\lambda \boxtimes \mathbf{k} = V_{(5,2,1)} \oplus V_{(4,3,1)} \oplus V_{(4,2,2)} \oplus V_{(4,2,1,1)}$$
$$\mathrm{Ind}_{\Sigma_7\times\Sigma_2}^{\Sigma_9} V_\lambda \boxtimes \mathbf{k} = V_{(6,2,1)} \oplus V_{(5,3,1)} \oplus V_{(5,2,2)} \oplus V_{(5,2,1,1)} \oplus V_{(4,4,1)} \oplus V_{(4,3,2)} \oplus V_{(4,3,1,1)} \oplus V_{(4,2,2,1)}.$$

3.2 kG-module structures on $H^*(\mathcal{Z}_K;\mathbf{k})$

Let $\mathbf{k}$ be a field or $\mathbb{Z}$, let G be a finite group, and let K be a simplicial G-complex. We will describe G-actions on the moment-angle complex $\mathcal{Z}_K$ induced by a simplicial G-action on K.

Recall the cellular decomposition of a moment-angle complex $\mathcal{Z}_K$ in Section 2.2. A cell of $\mathcal{Z}_K$ denoted by $\kappa(L,I)$ for which $I \in K$ is equal to $e_1 \times \ldots \times e_m$ in D^{2m}, where e_i is the 2-dimensional cell e^2 if $i \in I$, e_i is the 1-dimensional cell e^1 if $i \in L$, and e_i is the point e^0 if $i \in [m] \setminus (I \cup L)$.

We start by showing that if K is a simplicial G-complex, the corresponding moment-angle complex $\mathcal{Z}_K$ is a G-complex. Let $2^{[m]}$ be the power set of $[m]$. Then the G-action on K can be extended to an action Φ on $2^{[m]}$. Specifically, $\Phi\colon G \times 2^{[m]} \longrightarrow 2^{[m]}$ is given by $\Phi(g,\{i_1,\ldots,i_l\}) = \{g\cdot i_1,\ldots,g\cdot i_l\}$, where $g \in G$ and $\{i_1,\ldots,i_l\} \subset [m]$.

The simplicial G-action on K induces a G-action on $\mathcal{Z}_K$, $\rho : G \times \mathcal{Z}_K \longrightarrow \mathcal{Z}_K$, through homeomorphisms of $\mathcal{Z}_K$ given by

$$\rho_g \cdot (z_1,\ldots,z_m) = (z_{g\cdot 1},\ldots,z_{g\cdot m}). \tag{3.3}$$

Lemma 3.2.1. *For a simplicial G-complex K, the moment-angle complex $\mathcal{Z}_K$ is a G-complex.*

Proof. A cell $\kappa(L,I)$, $I \in K$ of $\mathcal{Z}_K$ is mapped by $g \in G$ to $g\cdot\kappa(L,I) = \kappa(g\cdot L, g\cdot I)$ which is again a cell of $\mathcal{Z}_K$ as a simplicial G-action maps simplices to simplices and non-simplices to non-simplices. Thus, $\mathcal{Z}_K$ is a G-complex. ☐

The orientation on a cell $\kappa(L,I)$ is given by the orientation of I as an oriented simplex of Δ^{m-1} up to multiple $\epsilon(I, I\cup L)$. Cochains $\kappa(L,I)^*$ corresponding to the oriented cells $\kappa(L,I)$ form the basis of cochains of $\mathcal{Z}_K$. Each $g \in G$ defines a bijection on the cochain basis by $g\cdot\kappa(L,I)^* = \epsilon(g,L)\kappa(g\cdot L, g\cdot I)^*$ where $\epsilon(g,L)$ is a sign induced by the action of g on L defined to be 1 if g preserves the orientation of L and -1 otherwise. With $\sigma \subseteq L$ and both σ and L written in an increasing order, denote by $\epsilon(\sigma,L) = \prod_{j\in\sigma}\epsilon(j,L)$, where $\epsilon(j,L) = (-1)^{r-1}$ if j is the r-th element of L.

Lemma 3.2.2. *Observe that $\epsilon(g, L)$ satisfies the following identity,*

$$\epsilon(\sigma, L)\epsilon(g, L \setminus \sigma) = \epsilon(g, \sigma)\epsilon(g, L)\epsilon(g \cdot \sigma, g \cdot L). \tag{3.4}$$

In particular, if $|\sigma| = 1$, say $j \in L$ then (3.4) implies that

$$\epsilon(j, L)\epsilon(g, L \setminus \{j\}) = \epsilon(g, L)\epsilon(g \cdot j, g \cdot L). \tag{3.5}$$

Proof. Assume that $L = \{j_1, \cdots, j_l\}$ with $j_1 < \cdots < j_l$. Let $r(g, L)$ be the number of permutations such that $\{g \cdot j_1, \cdots, g \cdot j_l\}$ written in an increasing order. Then $\epsilon(g, L) = (-1)^{r(g,L)}$. Let $r(\sigma, L) = \sum_{i\in\sigma} r(i, L)$, where $r(i, L)$ is the position of i in L with L an increasing order. We have $\epsilon(\sigma, L) = (-1)^{r(\sigma,L)-|\sigma|}$. There are $r(\sigma, L) - |\sigma|$ permutations such that L is written as a disjoint union $\sigma \sqcup (L \setminus \sigma)$ with σ and $L \setminus \sigma$ in an increasing order respectively. The number of permutations such that $g \cdot \sigma$ and $g \cdot (L \setminus \sigma)$ being increasing order respectively is given by $r(g, \sigma) + r(g, L \setminus \sigma)$. We illustrate it in the following diagram, assuming that all sets appeared in the following diagram are written in an increasing order,

$$\begin{array}{ccccc} L & \xrightarrow{r(\sigma,L)-|\sigma|} & \sigma \sqcup (L \setminus \sigma) & \xrightarrow{r(g,\sigma)+r(g,L\setminus\sigma)} & g \cdot \sigma \sqcup g \cdot (L \setminus \sigma) \\ \downarrow{\scriptstyle r(g,L)} & & & & \downarrow{\scriptstyle r(g\cdot\sigma,g\cdot L)-|\sigma|} \\ g \cdot L & & \xrightarrow{=} & & g \cdot L \end{array}$$

where each number along with each arrow is the number of necessary permutations.

Therefore,

$$\begin{aligned} \epsilon(g, L) = (-1)^{r(g,L)} &= (-1)^{r(\sigma,L)-|\sigma|+r(g,\sigma)+r(g,L\setminus\sigma)+r(g\cdot\sigma,g\cdot L)-|\sigma|} \\ &= \epsilon(\sigma, L)\epsilon(g, \sigma)\epsilon(g, L \setminus \sigma)\epsilon(g \cdot \sigma, g \cdot L). \end{aligned}$$

□

Lemma 3.2.3. *Let K be a simplicial G-complex. The cellular cochain complex $C^*(\mathcal{Z}_K)$ is a cochain complex of G-modules.*

Proof. The coboundary operation on $\mathcal{C}^*(\mathcal{Z}_K)$ is given by

$$\delta\kappa(L, I)^* = \sum_{j\in L,\ j\cup I\in K} \epsilon(j, L)\kappa(L \setminus \{j\}, \{j\} \cup I)^*.$$

It is enough to show that the G-action commutes with the coboundary operator δ. As

$$\begin{aligned} \delta(g \cdot (\kappa(L, I)^*)) &= \delta(\epsilon(g, L)\kappa(g \cdot L, g \cdot I)^*) \\ &= \sum_{j\in L,\ j\cup I\in K} \epsilon(g, L)\epsilon(g \cdot j, g \cdot L)\kappa(g \cdot L \setminus \{g \cdot j\}, g \cdot j \cup g \cdot I)^* \end{aligned}$$

and

$$
\begin{aligned}
g\cdot(\delta(\kappa(L,I)^*)) &= g\cdot(\sum_{j\in L,\ j\cup I\in K}\epsilon(j,L)\kappa(L\setminus\{j\},\{j\}\cup I)^*)\\
&= \sum_{j\in L,\ j\cup I\in K}\epsilon(j,L)g\cdot(\kappa(L\setminus\{j\},\{j\}\cup I)^*)\\
&= \sum_{j\in L,\ j\cup I\in K}\epsilon(j,L)\epsilon(g,L\setminus\{j\})\kappa(g\cdot L\setminus\{g\cdot j\},g\cdot j\cup g\cdot I)^*
\end{aligned}
$$

the result follows after applying (3.5). □

Geometrizing the famous Hochster decomposition [26], Buchstaber and Panov [9, 34] together with Baskakov [4] showed that $H^*(\mathcal{Z}_K;\mathbf{k}) \cong \bigoplus_{J\subseteq[m]} \widetilde{H}^*(K_J;\mathbf{k})$ as $\mathbf{k}$-algebras, where K_J is the full subcomplex of K on J defined by $K_J = \{\sigma\cap J \mid \sigma\in K\}$. We aim to show that this is a $\mathbf{k}G$-algebra isomorphism.

Lemma 3.2.4. *Let K be a simplicial G-complex on $[m]$. Then for any subset $J\subseteq[m]$ and $g\in G$, the set $g\cdot K_J = \{g\cdot\sigma \mid \sigma\in K_J\}$ is the full subcomplex $K_{g\cdot J}$.*

Proof. Since K_J is a subcomplex of K, every subset τ of σ is in K_J if $\sigma\in K_J$. Hence for $\sigma\in K_J$, every subset τ' of $g\cdot\sigma$ is $g\cdot\tau$ for some $\tau\le\sigma$ and therefore is in $g\cdot K_J$. Thus $g\cdot K_J$ is a subcomplex of K.

To check that $g\cdot K_J$ is the full subcomplex $K_{g\cdot J}$, we observe that $g\cdot K_J = g\cdot(K\cap J) = g\cdot K\cap g\cdot J = K\cap g\cdot J = K_{g\cdot J}$. □

Denote by $\{i_0,\dots,i_p\}$ an unoriented simplex in K and by $[i_0,\dots,i_p]$ an oriented simplex in K. For an oriented p-simplex $\sigma=[i_0,\dots,i_p]$, let $\sigma^*=[i_0,\dots,i_p]^*$ denote the basis cochain in $C^p(K;\mathbf{k})$.

Next, we show that a simplicial G-action on K induces a G-action on $\bigoplus_{J\subseteq[m]} \widetilde{H}^*(K_J;\mathbf{k})$.

Lemma 3.2.5. *Let K be a simplicial G-complex. For every $g\in G$ and $J\subseteq[m]$,*

$$g\cdot\widetilde{H}^*(K_J;\mathbf{k}) = \widetilde{H}^*(K_{g\cdot J};\mathbf{k}).$$

Proof. Let $\sigma=[i_0,\dots,i_p]$ be an oriented simplex in K_J and σ^* be the corresponding base cochain in $C^p(K_J;\mathbf{k})$. Since g gives a bijection between the basis of $C^*(K_J;\mathbf{k})$ and the basis of $C^*(K_{g\cdot J};\mathbf{k})$ by $\sigma^*\mapsto g\cdot\sigma^* = \epsilon(g,\sigma)(g\cdot\sigma)^*$, the cochain complex $C^*(K_J;\mathbf{k})$ is isomorphic to $C^*(K_{g\cdot J};\mathbf{k})$ as abelian groups. As the coboundary operator d is given by

$$d\sigma^* = \sum \varepsilon_j\tau_j^*$$

where the summation of the coboundary operator extends over all $(p+1)$-simplices τ_j having σ as a face, and $\varepsilon_j = \pm 1$ is the sign with which σ appears in the expression for $\partial\tau$, we obtain the commutative diagram

$$\begin{array}{ccc} C^*(K_J;\mathbf{k}) & \xrightarrow{\cong} & C^*(K_{g\cdot J};\mathbf{k}) \\ \downarrow d & & \downarrow d \\ C^*(K_J;\mathbf{k}) & \xrightarrow{\cong} & C^*(K_{g\cdot J};\mathbf{k}). \end{array}$$

Therefore g induces an isomorphism between $\widetilde{H}^*(K_J;\mathbf{k})$ and $\widetilde{H}^*(K_{g\cdot J};\mathbf{k})$. □

We continue by showing that the G-actions on $H^*(\mathcal{Z}_K;\mathbf{k})$ and $\bigoplus_{J\subseteq[m]} \widetilde{H}^*(K_J;\mathbf{k})$ are compatible.

On $\mathcal{C}^*(\mathcal{Z}_K;\mathbf{k})$ a multigrading can be defined. Consider a subset $J\subseteq[m]$ as a vector in $\mathbb{N}^m$ whose j-th coordinate is 1 if $j\in J$, or is 0 if $j\notin J$. Define a $\mathbb{Z}\oplus\mathbb{N}^m$-grading on $\mathcal{C}^*(\mathcal{Z}_K;\mathbf{k})$ as

$$\mathcal{C}^*(\mathcal{Z}_K;\mathbf{k}) = \bigoplus_{J\subseteq[m]} \mathcal{C}^{*,2J}(\mathcal{Z}_K;\mathbf{k})$$

where $\mathcal{C}^{*,2J}(\mathcal{Z}_K;\mathbf{k})$ is the subcomplex spanned by cochains $\kappa(J\setminus I, I)^*$ with $I\subseteq J$ and $I\in K$ whose multidegree is $\operatorname{mg}\kappa(J\setminus I,I)^* = (-|J\setminus I|, J)$.

Buchstaber and Panov [10, Theorem 3.2.9] showed that there are isomorphisms between $\widetilde{H}^{p-1}(K_J;\mathbf{k})$ and $H^{p-|J|,2J}(\mathcal{Z}_K;\mathbf{k})$ which are functorial with respect to simplicial maps and are induced by the cochain isomorphisms $f_J\colon C^{p-1}(K_J;\mathbf{k}) \to \mathcal{C}^{p-|J|,2J}(\mathcal{Z}_K;\mathbf{k})$ given by

$$f_J(\sigma^*) = \epsilon(\sigma,J)\kappa(J\setminus\sigma,\sigma)^* \tag{3.6}$$

where $\sigma\in K_J$ and $\epsilon(\sigma,J) = \prod_{j\in\sigma}\epsilon(j,J)$ with $\epsilon(j,J) = (-1)^{r-1}$ if j is the r-th element of J.

The functorial property induces a commutative diagram

$$\begin{array}{ccc} C^{p-1}(K_J;\mathbf{k}) & \xrightarrow{f_J} & \mathcal{C}^{p-|J|,2J}(\mathcal{Z}_K;\mathbf{k}) \\ \downarrow g & & \downarrow g \\ C^{p-1}(K_{g\cdot J};\mathbf{k}) & \xrightarrow{f_{g\cdot J}} & \mathcal{C}^{p-|g\cdot J|,2g\cdot J}(\mathcal{Z}_K;\mathbf{k}) \end{array}$$

implying the following statement.

Lemma 3.2.6. *If K is a simplicial G-complex, then $\mathcal{C}^*(\mathcal{Z}_K;\mathbf{k})$ is multigraded isomorphic to $\bigoplus_{J\subseteq[m]} C^*(K_J;\mathbf{k})$ as* **k***G-modules.*

Passing to cohomology, we obtain the following corollary.

Corollary 3.2.7. *For a simplicial G-complex, $H^*(\mathcal{Z}_K;\mathbf{k})$ is isomorphic to $\bigoplus_{J\subseteq[m]} \widetilde{H}^*(K_J;\mathbf{k})$ as $\mathbf{k}G$-algebras.*

Proof. By [10, Theorem 4.5.8], the multiplication on $\bigoplus_{J\subseteq[m]} \widetilde{H}^*(K_J;\mathbf{k})$ is given by

$$H^i(K_I;\mathbf{k}) \otimes H^j(K_J;\mathbf{k}) \to H^{i+j}(K_{I\cup J};\mathbf{k})$$

which is induced by the simplicial inclusions $K_{I\cup J} \to K_I * K_J$ for $I \cap J = \emptyset$ and zero otherwise. Under this multiplication, the maps f_J induce a $\mathbf{k}$-algebraic isomorphism $\bigoplus_{J\subseteq[m]} \widetilde{H}^*(K_J;\mathbf{k}) \to H^*(\mathcal{Z}_K;\mathbf{k})$. Since $f_{g\cdot J} \circ g = g \circ f_J$, the maps f_J induce a $\mathbf{k}G$-algebraic isomorphism. □

Now we state the main result of this section.

Proposition 3.2.8. *Let K be a simplicial G-complex. Then there are $\mathbf{k}G$-algebra isomorphisms*

$$H^*(\mathcal{Z}_K;\mathbf{k}) \cong \bigoplus_{J\in[m]/G} \bigoplus_{g\in G/G_J} g\cdot \widetilde{H}^{*-|J|-1}(K_J;\mathbf{k})$$

where $G_J = \{g \in G \mid g\cdot J = J\}$ is the stabiliser of J and $[m]/G$ is a set of representatives of G-orbits of $2^{[m]}$.

The multiplication on $\bigoplus_{J\in[m]/G} \bigoplus_{g\in G/G_J} g\cdot \widetilde{H}^*(K_J;\mathbf{k})$ is given so that for any $I, J \in [m]/G$ and $g \in G/G_J$, $h \in G/G_I$, there is a map

$$\mu\colon g\cdot H^{k-|J|-1}(K_J;\mathbf{k}) \otimes h\cdot H^{l-|I|-1}(K_I;\mathbf{k}) = H^{k-|J|-1}(K_{g\cdot J};\mathbf{k}) \otimes H^{l-|I|-1}(K_{h\cdot I};\mathbf{k}) \to H^{k+l-|I|-|J|-1}(K_{g\cdot J\cup h\cdot I};\mathbf{k})$$

which is induced by the simplicial inclusion $K_{g\cdot J\cup h\cdot I} \longrightarrow K_{g\cdot J} * K_{h\cdot I}$ if $g\cdot J \cap h\cdot I = \emptyset$ and is a zero map otherwise.

Proof. Since by Corollary 3.2.7 $H^*(\mathcal{Z}_K;\mathbf{k}) \cong \bigoplus_{J\subseteq[m]} \widetilde{H}^*(K_J;\mathbf{k})$ as $\mathbf{k}G$-algebras, it suffices to show that the G-isomorphism

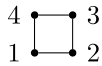
$$\bigoplus_{J\subseteq[m]} \widetilde{H}^*(K_J;\mathbf{k}) \cong \bigoplus_{J\in[m]/G} \bigoplus_{g\in G/G_J} g\cdot \widetilde{H}^*(K_J;\mathbf{k})$$

preserves the multiplications on both sides. The multiplication on $\bigoplus_{J\in[m]/G} \bigoplus_{g\in G/G_J} g\cdot \widetilde{H}^*(K_J;\mathbf{k})$ is induced by the multiplication on $\bigoplus_{J\subseteq[m]} \widetilde{H}^*(K_J;\mathbf{k})$ via the above G-isomorphism. Therefore,

$$H^*(\mathcal{Z}_K;\mathbf{k}) \cong \bigoplus_{J\in[m]/G} \bigoplus_{g\in G/G_J} g\cdot \widetilde{H}^{*-|J|-1}(K_J;\mathbf{k})$$

as $\mathbf{k}G$-algebras. □

We illustrate Proposition 3.2.8 on several examples.

Example 3.2.9. Let K be the boundary of a square, 

. It is a simplicial C_4-complex, where C_4 is the cyclic group of order 4. Write $C_4 = \{(1),(1234),(13)(24),(1432)\}$ as a subgroup of the permutation group Σ_4. A set of representatives of $2^{[4]}$ under C_4 is given by

$$E = \{\emptyset, \{1\}, \{1,2\}, \{1,3\}, \{1,2,3\}, \{1,2,3,4\}\}.$$

Taking J to be an element in E, observe that

$$\widetilde{H}^p(K_J;\mathbf{k}) = \begin{cases} \mathbf{k} & \text{when } J=\emptyset \text{ and } p=-1 \\ \mathbf{k} & \text{when } J=\{1,3\} \text{ and } p=0 \\ \mathbf{k} & \text{when } J=\{1,2,3,4\} \text{ and } p=1 \\ 0 & \text{otherwise.} \end{cases}$$

The stabilisers G_J corresponding to $J=\emptyset$, $J=\{1,3\}$ and $J=\{1,2,3,4\}$ are $G_\emptyset = C_4$, $G_{13} = \{(1),(13)(24)\}$ and $G_{1234} = C_4$, respectively. Therefore, the cohomology groups of $\mathcal{Z}_K$ are given by

$$H^i(\mathcal{Z}_K;\mathbf{k}) = \begin{cases} \mathbf{k}\oplus\mathbf{k} & \text{for } i=3 \\ \mathbf{k} & \text{for } i=0,6. \end{cases}$$

Example 3.2.10. Let $K=\Delta_m^k$ be the full k-skeleton of Δ^{m-1} which consists all subsets of $[m]$ with cardinality at most $k+1$. The permutation group Σ_m acts on K simplicially. A set of representatives of $2^{[m]}$ under the action of Σ_m can be also chosen as

$$E = \{\emptyset, \{1\}, \{1,2\}, \ldots, \{1,\ldots,m\}\}.$$

For any $J=\{1,2,\ldots,|J|\}\in E$, the stabiliser of J is the Young subgroup $\Sigma_{|J|}\times\Sigma_{m-|J|}$. If $J\in E$ with $|J|\leq k+1$, then $K_J=\Delta^{|J|-1}$. Thus $\widetilde{H}^*(K_J;\mathbf{k})=0$.

If $J\in E$ with $k+2\leq |J|\leq m$, then K_J is the full k-skeleton of $\Delta^{|J|-1}$. Recall that $\widetilde{H}^*(K_J;\mathbf{k}) = \bigoplus_c \mathbf{k}$, where $c=\binom{|J|-1}{k+1}$ if $*=k$; otherwise $\widetilde{H}^*(K_J;\mathbf{k})=0$. Therefore,

$$H^i(\mathcal{Z}_K;\mathbf{k}) = \begin{cases} \mathbf{k} & \text{where } i=0 \\ \bigoplus_c \mathbf{k} & \text{where } c=\binom{m}{|J|}\binom{|J|-1}{k+1} \text{ and } i=|J|+k+1 \\ 0 & \text{otherwise.} \end{cases}$$

Let us remark that for $k=0$, the simplicial complex K consists of m disjoint points and denote by $\mathcal{Z}_m$ the moment-angle complex corresponding to it. By Proposition 3.2.8, $H^3(\mathcal{Z}_m;\mathbf{k})$ has a basis $\{a_{ij} \mid 1\leq i<j\leq m\}$ and identifying $a_{ji}=-a_{ij}$, the symmetric group Σ_m acts on $H^3(\mathcal{Z}_m;\mathbf{k})$ by a permutation of the indices.

For $K_m = \Delta^k_m$ with k fixed and m increasing, we get a sequence of moment-angle complexes $\{\mathcal{Z}_{K_m}\}$. There exist retractions $p_m\colon \mathcal{Z}_{K_{m+1}} \longrightarrow \mathcal{Z}_{K_m}$ obtained by restricting the projection map $(D^2)^{m+1} \longrightarrow (D^2)^m$ to $\mathcal{Z}_K$. We shall consider the representation stability of the sequence $\{H^i(\mathcal{Z}_K;\mathbf{k}), p^i_m\}$ in Section 3.4.

3.3 Polyhedral products over simplicial G-complexes

Moment-angle complexes are specific examples of polyhedral products $(X,A)^K$ which are constructed from combinatorial information of a simplicial complex K and a topological pair (X,A). Our next aim is to study symmetries of polyhedral products induced by the symmetries of K. The geometric and homological properties of polyhedral products arising from simplicial Aut(K)-complexes have been studied by Ali Al-Raisi in his PhD thesis [1]. Al-Raisi proved that the map $(X,A)^K \longrightarrow \Omega\Sigma(\bigvee_{I\subseteq[m]} (X,A)^{\wedge K_I})$ is homotopy Aut(K)-equivariant.

In this section, we will give a different method for studying homotopy G-decompositions of polyhedral product $(X,A)^K$ associated with a simplicial G-complex K by studying the adjoint of the Al-Raisi map, known as the Bahri-Bendersky-Cohen-Gitler (BBCG) map (Theorem 2.3.6), after several suspensions.

If K is a simplicial G-complex, then the G-action on K induces a cellular G-action on the corresponding polyhedral product $(X,A)^K$ with respect to a pair of CW-complexes (X,A), $A \subseteq X$. Explicitly, for $\underline{x} = (x_1,\dots,x_m) \in (X,A)^K$, $g\cdot\underline{x} = (x_{g\cdot 1},\dots,x_{g\cdot m})$. Thus $(X,A)^K$ is a G-complex. If X is a G-CW-complex, then each i-th homology group $H_i(X;R)$ is an RG-module. Consider a natural G-action on ΣX by $g\cdot(\langle x,t\rangle) = \langle g\cdot x, t\rangle$ for $g \in G$. The naturality of long exact sequence for the topological pair (CX,X) implies that the isomorphism $H_{i+1}(\Sigma X;R) \cong \widetilde{H}_i(X;R)$ is an RG-isomorphism.

Consider X^m as a Σ_m-space given by $g\cdot\underline{x} = g\cdot(x_1,\dots,x_m) = (x_{g\cdot 1},\dots,x_{g\cdot m})$ for $g\in\Sigma_m$ and $x_i \in X$. There exists a Σ_m-action on the based spaces ΣX^m and $\Sigma(\bigvee_{I\subseteq[m]} X^{\wedge I})$, where I runs over the non-empty subset of $[m]$. Explicitly, for every $g\in\Sigma_m$ and $\langle\underline{x},t\rangle \in \Sigma X^m$, $g\cdot\langle\underline{x},t\rangle = \langle g\cdot\underline{x},t\rangle$. For any non-empty subset $I=\{i_1,\dots,i_l\}\subseteq[m]$, each map $g\colon \Sigma X^{\wedge I} \to \Sigma X^{\wedge g\cdot I}$ sending $\langle x_{i_1}\wedge\dots\wedge x_{i_l},t\rangle$ to $\langle x_{g\cdot i_1}\wedge\dots\wedge x_{g\cdot i_l},t\rangle$ induces a Σ_m-action on $\Sigma\bigvee_{I\subseteq[m]} X^{\wedge I}$.

Lemma 3.3.1. *There exists a homotopy equivalence*

$$\Sigma\theta_m\colon \Sigma^2 X^m \longrightarrow \Sigma^2 \bigvee_{I\subseteq[m]} X^{\wedge I}$$

that is Σ_m-equivariant.

Proof. For a non-empty set $I = \{i_1, \dots, i_l\} \subseteq [m]$, define maps $\Sigma p^{\wedge I}$ by

$$\Sigma p^{\wedge I} \colon \Sigma X^m \longrightarrow \Sigma X^{\wedge I}$$
$$\langle x_1, \dots, x_m, t\rangle \longmapsto \langle x_{i_1} \wedge \dots \wedge x_{i_l}, t\rangle$$

Let $L = 2^m - 1$. Define a comultiplication map $\delta_m \colon \Sigma X^m \longrightarrow \bigvee_{j=1}^{L} \Sigma X^m$ on ΣX^m such that
if $t \in [\frac{i}{L}, \frac{i+1}{L}]$ $(0 \leq i \leq L-1)$,

$$\delta_m(\langle x_1, \dots, x_m, t\rangle) = (*, \dots, *, \langle x_1, \dots, x_m, Lt - i\rangle, *, \dots, *)$$

where $\langle x_1, \dots, x_m, Lt-i\rangle$ is in the $(i+1)$-st wedge summand of $\bigvee_{j=1}^{L} \Sigma X^m$.

Fix an order $I_1 > I_2 > \dots > I_L$ on the finite set $\{I \subseteq [m] \mid I \neq \emptyset\}$. Let each I_j contain elements written in an increasing order. Rewrite $\Sigma(\bigvee_{I \subseteq [m]} X^{\wedge I})$ as $\Sigma X^{\wedge I_1} \vee \dots \vee \Sigma X^{\wedge I_L}$.

Consider a map $\bigvee_{I \in 2^{[m]} \setminus \emptyset} \Sigma p^{\wedge I} \colon \bigvee_{j=1}^{L} \Sigma X^m \longrightarrow \Sigma(\bigvee_{I \subseteq [m]} X^{\wedge I})$ given by

$$\bigvee_{I \in 2^{[m]} \setminus \emptyset} \Sigma p^{\wedge I} = \Sigma p^{\wedge I_1} \vee \dots \vee \Sigma p^{\wedge I_L} \colon \bigvee_{j=1}^{L} \Sigma X^m \longrightarrow \Sigma X^{\wedge I_1} \vee \dots \vee \Sigma X^{\wedge I_L}.$$

Thus the map

$$\theta_m = \bigvee_{I \in 2^{[m]} \setminus \emptyset} \Sigma p^{\wedge I} \circ \delta_m.$$

Let $g \in \Sigma_m$ and $\langle x_1, \dots, x_m, t\rangle \in \Sigma X^m$. For $t \in [\frac{i}{L}, \frac{i+1}{L}]$ $(0 \leq i \leq L-1)$, there is

$$\begin{aligned}
\theta_m \circ g(\langle x_1, \dots, x_m, t\rangle) &= \bigvee_{I \in 2^{[m]} \setminus \emptyset} \Sigma p^{\wedge I} \circ \delta_m(\langle x_{g\cdot 1}, \dots, x_{g \cdot m}, t\rangle) \\
&= \bigvee_{I \in 2^{[m]} \setminus \emptyset} \Sigma p^{\wedge I}(*, \dots, *, \underbrace{\langle x_{g\cdot 1}, \dots, x_{g\cdot m}, Lt - i\rangle}_{i+1}, *, \dots, *) \\
&= (*, \dots, *, \underbrace{\langle x_{g \cdot m_1^{(i+1)}} \wedge \dots \wedge x_{g\cdot m_s^{(i+1)}}, Lt - i\rangle}_{i+1}, *, \dots, *)
\end{aligned}$$

where $I_{i+1} = \{m_1^{(i+1)}, \dots, m_s^{(i+1)}\}$ with $m_1^{(i+1)} < \dots < m_s^{(i+1)}$.

Recall that $\bigvee_{I \in 2^{[m]} \setminus \emptyset} \Sigma p^{\wedge I} = \Sigma p^{\wedge I_1} \vee \dots \vee \Sigma p^{\wedge I_L}$ and define by $\bigvee_{I \in 2^{[m]} \setminus \emptyset} \Sigma p^{\wedge (g \cdot I)} = \Sigma p^{\wedge (g \cdot I_1)} \vee \dots \vee \Sigma p^{\wedge (g\cdot I_L)}$. Hence, $\theta_m \circ g = \bigvee_{I \in 2^{[m]} \setminus \emptyset} \Sigma p^{\wedge (g \cdot I)} \circ \delta_m$.

On the other hand, there exits a permutation T of summand $\bigvee_{j=1}^{L} \Sigma X^m$ induced by g such that $g \circ \theta_m = \bigvee_{I \in 2^{[m]} \setminus \emptyset} \Sigma p^{\wedge (g \cdot I)} \circ T \circ \delta_m$. Since g acts on a set $\{1, \dots, L\}$ by $g \cdot i$ being the unique number satisfying $I_{g \cdot i} = g \cdot I_i$ as sets, this action on $\{1, \dots, L\}$ induces a permutation T of $\bigvee_{j=1}^{L} \Sigma X^m$. Note that for $t \in [\frac{i}{L}, \frac{i+1}{L}]$ $(0 \le i \le L-1)$,

$$
\begin{aligned}
&\bigvee_{I \in 2^{[m]} \setminus \emptyset} \Sigma p^{\wedge (g \cdot I)} \circ T \circ \delta_m(\langle x_1, \dots, x_m, t \rangle) \\
=& \bigvee_{I \in 2^{[m]} \setminus \emptyset} \Sigma p^{\wedge (g \cdot I)} \circ T(*, \dots, *, \underbrace{\langle x_1, \dots, x_m, Lt - i \rangle}_{i+1}, *, \dots, *) \\
=& \bigvee_{I \in 2^{[m]} \setminus \emptyset} \Sigma p^{\wedge (g \cdot I)} (*, \dots, *, \underbrace{\langle x_1, \dots, x_m, Lt - i \rangle}_{g \cdot (i+1)}, *, \dots, *) \\
=& (*, \dots, *, \underbrace{\langle x_{g \cdot m_1^{(i+1)}} \wedge \dots \wedge x_{g \cdot m_s^{(i+1)}}, Lt - i \rangle}_{g \cdot (i+1)}, *, \dots, *)
\end{aligned}
$$

where $I_{i+1} = \{m_1^{(i+1)}, \dots, m_s^{(i+1)}\}$ with $m_1^{(i+1)} < \dots < m_s^{(i+1)}$.

Also, for $t \in [\frac{i}{L}, \frac{i+1}{L}]$ $(0 \le i \le L-1)$,

$$
\begin{aligned}
g \circ \theta_m(\langle x_1, \dots, x_m, t \rangle) &= g(*, \dots, *, \underbrace{\langle x_{m_1^{(i+1)}} \wedge \dots \wedge x_{m_s^{(i+1)}}, Lt - i \rangle}_{i+1}, *, \dots, *) \\
&= (*, \dots, *, \underbrace{\langle x_{g \cdot m_1^{(i+1)}} \wedge \dots \wedge x_{g \cdot m_s^{(i+1)}}, Lt - i \rangle}_{g \cdot (i+1)}, *, \dots, *).
\end{aligned}
$$

Thus we have $g \circ \theta_m = \bigvee_{I \in 2^{[m]} \setminus \emptyset} \Sigma p^{\wedge (g \cdot I)} \circ T \circ \delta_m$.

Since $\Sigma \delta_m$ is cocommutative, $\Sigma(g \circ \theta_m) \simeq \Sigma(\theta_m \circ g)$. □

The following statement is a consequence of Lemma 3.3.1.

Lemma 3.3.2. *a) For $g \in \Sigma_m$ and $I \subseteq [m]$, there is the homotopy commutative diagram*

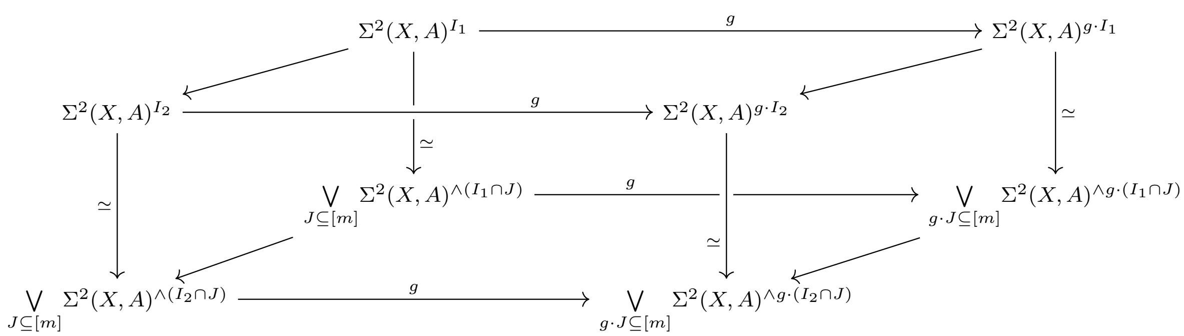
$$
\begin{array}{ccc}
\Sigma^2(X, A)^I & \xrightarrow{\simeq} & \bigvee_{J \subseteq [m]} \Sigma^2(X, A)^{\wedge (I \cap J)} \\
\Big\downarrow g & & \Big\downarrow g \\
\Sigma^2(X, A)^{g \cdot I} & \xrightarrow{\simeq} & \bigvee_{g \cdot J \subseteq [m]} \Sigma^2(X, A)^{\wedge g \cdot (I \cap J)}
\end{array}
\tag{3.7}
$$

where the vertical map g on the left is given by

$$g \cdot \langle x_1, \dots, x_m, t, s \rangle = \langle x_{g \cdot 1}, \dots, x_{g \cdot m}, t, s \rangle$$

and the vertical map g on the right maps each element in $\Sigma^2(X,A)^{\wedge(I\cap J)}$ into the corresponding one in $\Sigma^2(X,A)^{\wedge g\cdot(I\cap J)}$ via a coordinate permutation by g.

b) For an inclusion $I_1 \subseteq I_2 \subseteq [m]$, there is the diagram



(3.8)

where the four side diagrams are homotopy commutative and the top and bottom diagrams are commutative. □

Since the homotopy decomposition $\Sigma^2(X,A)^K \simeq \Sigma^2 \bigvee_{J\subseteq[m]} (X,A)^{\wedge K_J}$ is natural with respect to inclusions in K ([3, Theorem 2.10]), the next result follows immediately from the lemma above.

Theorem 3.3.3. *Let K be a simplicial G-complex with m vertices. Then there is a homotopy G-decomposition*

$$\theta\colon \Sigma^2(X,A)^K \simeq \Sigma^2 \bigvee_{J\subseteq[m]} (X,A)^{\wedge K_J} \tag{3.9}$$

where the G-action on $\Sigma^2(X,A)^K$ is induced by the G-action on X^m, and the G-action on the right hand side is induced by (3.7).

Proof. Let $\mathrm{CAT}(K)$ be the face category of K consisting of simplices of K and simplicial inclusions in K. Define two functors D and E from $\mathrm{CAT}(K)$ to CW_* by $D(\sigma) = (X,A)^\sigma$ and $E(\sigma) = \bigvee_{J\subseteq[m]} (X,A)^{\wedge(\sigma\cap J)}$ for $\sigma \in \mathrm{CAT}(K)$. For every $\sigma \in \mathrm{CAT}(K)$ and $g \in G$, diagram (3.7) implies that there exists a homotopy

$$H_g(\sigma)\colon \Sigma^2 D(\sigma) \times \mathbb{I} \longrightarrow \Sigma^2 E(g\cdot\sigma)$$

such that $H_g(\sigma)(x,0) = \theta(\sigma)$ and $H_g(\sigma)(x,1) = \theta(g\cdot\sigma)$, where $\theta(\sigma)$ is the natural homotopy equivalence between $\Sigma^2 D(\sigma)$ and $\Sigma^2 E(\sigma)$ and $\mathbb{I}$ is the interval $[0,1]$. Diagram (3.8) implies that if $\sigma,\tau \in \mathrm{CAT}(K)$, then $H_g(\sigma\cap\tau) = H_g(\sigma)|_{\Sigma^2 D(\sigma\cap\tau)\times\mathbb{I}} = H_g(\tau)|_{\Sigma^2 D(\sigma\cap\tau)\times\mathbb{I}}$. $H_g(\cdot)$ induces a natural transformation from $\Sigma^2 D(\cdot)\times\mathbb{I}$ to $\Sigma^2 E(\cdot)$.

With g fixed, $H_g(\sigma)$ will induce a continuous map $H_g\colon \mathrm{colim}\,\Sigma^2 D \times \mathbb{I} \longrightarrow \mathrm{colim}\,\Sigma^2 E$ such that $H_g(x,0) = g\theta(x)$ and $H_g(x,1) = \theta(g\cdot x)$. Therefore, θ is a homotopy G-decomposition. □

Example 3.3.4. Let K be the k-skeleton of a simplex Δ^{m-1} on which Σ_m acts by permuting vertices. By Porter [36], Grbić-Theriault [23], the homotopy type of $(CA, A)^K$ is the wedge

$$(CA, A)^K \simeq \bigvee_{j=k+2}^{m} \Big(\bigvee_{1 \le i_1 < \ldots < i_j \le m} \binom{j-1}{k+1} \Sigma^{k+1} A_{i_1} \wedge \ldots \wedge A_{i_j} \Big).$$

Although Σ_m acts on both sides this homotopy equivalence might not be a homotopy Σ_m-equivalence. However after suspending it twice, by Theorem 3.3.3 it is a homotopy equivariant map.

Considering G-equivalence (3.9) and observing the induced G-actions on the reduced homology groups, we have the following result.

Theorem 3.3.5. *Let K be a simplicial G-complex on m vertices. Then there exists a* $\mathbf{k}G$*-module isomorphism*

$$\widetilde{H}_i((X,A)^K;\mathbf{k}) \cong \bigoplus_{J \subseteq [m]} \widetilde{H}_i((X,A)^{\wedge K_J};\mathbf{k}) \cong \bigoplus_{J \in [m]/G} \mathrm{Ind}_{G_J}^{G} \widetilde{H}_i((X,A)^{\wedge K_J};\mathbf{k})$$

where G acts on the middle term by permuting the summands such that $g \cdot \widetilde{H}_i((X,A)^{\wedge K_J};\mathbf{k}) = \widetilde{H}_i((X,A)^{K_{g \cdot J}};\mathbf{k})$, $[m]/G$ is a set of representatives of the orbit of $2^{[m]} \setminus \emptyset$ under G and G_J is the stabiliser of J. □

3.4 Representation stability for polyhedral products

Let G be a finite group and $\mathbf{k}$ be a field of characteristic zero. Then a G-action on a simplicial complex K induces a G-complex structure on the corresponding polyhedral product $(X, A)^K$ and therefore its homology is a $\mathbf{k}G$-module. Since every $\mathbf{k}G$-module is a G-representation over $\mathbf{k}$, we are able to use representation theory to study the homology groups of polyhedral products associated with simplicial G-complexes. Representation stability studies a sequence of finite dimensional vector spaces such that each vector space V_m is equipped with a G_m-action and each $V_m \xrightarrow{\psi_m} V_{m+1}$ is G_m-equivariant. Here groups G_m are not arbitrary; they all belong to a fixed family of groups whose $\mathbf{k}$-linear irreducible representations are determined by some datum λ which is independent of G_m and therefore of m. One such family consists of symmetric groups Σ_m, which we will consider in this section. The idea of representation stability was firstly introduced by Church and Farb in [15, Section 2.3]. Stability in representation theory generalises a classical homological stability. A sequence $\{Y_m\}$ of groups, manifolds or topological spaces with maps $Y_m \xrightarrow{\psi_m} Y_{m+1}$ for each $i \ge 0$ is called homology stable if the map $H_i(Y_m) \xrightarrow{(\psi_m)_*} H_i(Y_{m+1})$ is an isomorphism for a sufficiently large m.

We recall the precise definition of uniformly representation stability of representations of symmetric groups according to Church and Farb [15, Definition 2.6].

Definition 3.4.1. Let $\{V_m, \psi_m\}$ be a sequence of Σ_m-representations so that the group Σ_m acts on V_{m+1} as a subgroup of Σ_{m+1}. Then it is *consistent* if each V_m decomposes as a direct sum of finite-dimensional irreducible representations.

Definition 3.4.2. Let now $\{V_m, \psi_m\}$ be a consistent sequence of Σ_m-representations over a field $\mathbf{k}$ of characteristic 0. The sequence $\{V_m, \psi_m\}$ is *uniformly representation stable* with stable range $m \geq N$ if each of the following conditions holds for all $m \geq N$.

1. Injectivity: The natural map $\psi_m \colon V_m \to V_{m+1}$ is injective.

2. Surjectivity: The Σ_{m+1}-orbit of $\psi_m(V_m)$ spans V_{m+1}.

3. Multiplicities (uniform): Decompose V_m into irreducible representations as

$$V_m = \bigoplus_{\lambda} c_{\lambda,m} V(\lambda)_m$$

with multiplicities $0 \leq c_{\lambda,m} \leq \infty$. There is some M, not depending on λ, so that for $m \geq M$ the multiplicities $c_{\lambda,m}$ are independent of m for all λ.

Let Σ_k be a subgroup Σ_m when $m \geq k$. For a Σ_k-representation V, it can be seen as a $(\Sigma_k \times \Sigma_{m-k})$-representation, where Σ_{m-1} acts trivially on V, denoted by $V \boxtimes \mathbf{k}$. Consider the induced Σ_m-representation $\mathrm{Ind}_{\Sigma_k \times \Sigma_{m-1}}^{\Sigma_m} V \boxtimes \mathbf{k}$.

Example 3.4.3 (multiplicity)**.** Let $V_{(2,1)}$ be the irreducible representation of Σ_3 corresponding to partition $(2,1)$. Then $\{V_m = \mathrm{Ind}_{\Sigma_3 \times \Sigma_{m-3}}^{\Sigma_m} V_{(2,1)} \boxtimes \mathbf{k}\}$ forms a consistent sequence of Σ_m-representations, which stabilises when $m \geq 5$. Pieri's formula (Proposition 3.1.13) implies that

$$\begin{aligned}\mathrm{Ind}_{\Sigma_3 \times \Sigma_1}^{\Sigma_4} V_{(2,1)} \boxtimes \mathbf{k} &= V_{(3,1)} \oplus V_{(2,2)} \oplus V_{(2,1,1)} \\ \mathrm{Ind}_{\Sigma_3 \times \Sigma_{m-3}}^{\Sigma_m} V_{(2,1)} \boxtimes \mathbf{k} &= V_{(m-1,1)} \oplus V_{(m-2,2)} \oplus V_{(m-2,1,1)} \oplus V_{(m-3,2,1)} \text{ for } m \geq 5 \\ &= V(1)_m \oplus V(2)_m \oplus V(1,1)_m \oplus V(2,1)_m.\end{aligned}$$

Analogously, by Pieri's formula for $m \geq 7$,

$$\mathrm{Ind}_{\Sigma_4 \times \Sigma_{m-4}}^{\Sigma_m} V_{(3,1)} \boxtimes \mathbf{k} = V_{(m-1,1)} \oplus V_{(m-2,2)} \oplus V_{(m-2,1,1)} \oplus V_{(m-3,3)} \oplus V_{(m-3,2,1)} \oplus V_{(m-4,3,1)}.$$

The sequence $\{\mathrm{Ind}_{\Sigma_4 \times \Sigma_{m-4}}^{\Sigma_m} V_{(3,1)} \boxtimes \mathbf{k}\}$ stabilises at $m \geq 7$.

In general, Hemmer [25] proved the uniform representation stability of Σ_m-representations $\{\mathrm{Ind}_{H \times \Sigma_{m-k}}^{\Sigma_m} V \boxtimes \mathbf{k}\}$ induced by an H-representation V, where $H \leq \Sigma_k$. Note that V is seen as an $(H \times \Sigma_{m-k})$-representation, where Σ_{m-k} acts on V trivially, denoted by $V \boxtimes \mathbf{k}$ for any $m \geq k$.

Theorem 3.4.4 ([25])**.** *Let H be a subgroup of Σ_k and let $m \geq k$. Then the sequence $\{\mathrm{Ind}_{H\times\Sigma_{m-k}}^{\Sigma_m} V \boxtimes \mathbf{k}\}$ is uniformly representation stable.*

In this section, we study the representation stability arising in polyhedral products over a sequence of finite simplicial Σ_m-complexes.

Definition 3.4.5. A sequence of finite simplicial complexes

$$K_0 \subseteq K_1 \subseteq K_2 \subseteq \ldots \subseteq K_m \subseteq K_{m+1} \subseteq \ldots$$

where $K_0 = \emptyset$ and each K_m is a simplicial Σ_m-complex and the simplicial inclusion $i_m \colon K_m \subseteq K_{m+1}$ is Σ_m-equivariant (Σ_m acts on K_{m+1} via $\Sigma_m \hookrightarrow \Sigma_{m+1}$) is called a *consistent sequence.*

We start by considering few families of consistent sequences of finite simplicial complexes. The main aim of the paper is to show that these consistent sequences induce the consistent sequence of Σ_m-representations of the homology of polyhedral products which are representation stable.

Example 3.4.6. (k-skeleton sequences) Fix an integer $k \geq 0$. To each m assign the k-skeleton Δ_m^k of a standard $(m-1)$-simplex,

$$\emptyset \subseteq \Delta_1^k \subseteq \ldots \subseteq \Delta_m^k \subseteq \Delta_{m+1}^k \subseteq \ldots . \tag{3.10}$$

The action of Σ_m on K_m is induced by permutations of all m vertices. Each simplicial inclusion $i_m \colon \Delta_m^k \longrightarrow \Delta_{m+1}^k$ is Σ_m-equivariant. Therefore (3.10) is consistent.

In general, if K and L are simplicial G-complexes on $V(K)$ and $V(L)$ respectively, then the G-action can be extended to the join $K * L$, as a complex on $V(K) \cup V(L)$ vertices, diagonally.

Construction 3.4.7. Fix integers $s \geq 1$ and $k_1, \ldots, k_s \geq 0$. For each $m \geq 0$, let K_m be a simplicial complex on sm vertices given by the join of $\Delta_m^{k_1}, \Delta_m^{k_2}, \ldots, \Delta_m^{k_s}$. Since each $\Delta_m^{k_i}$ is a simplicial Σ_m-complex, then K_m is also a simplicial Σ_m-complex with the Σ_m-action given by $g \cdot (\sigma_1 \sqcup \ldots \sqcup \sigma_s) = g \cdot \sigma_1 \sqcup \ldots \sqcup g \cdot \sigma_s$ for $g \in \Sigma_m$ and for each $\sigma_i \in \Delta_m^{k_i}$. Let us consider the sequence

$$\emptyset \subseteq \Delta_1^{k_1} * \ldots * \Delta_1^{k_s} \subseteq \ldots \subseteq \Delta_m^{k_1} * \ldots * \Delta_m^{k_s} \subseteq \Delta_{m+1}^{k_1} * \ldots * \Delta_{m+1}^{k_s} \subseteq \ldots .$$

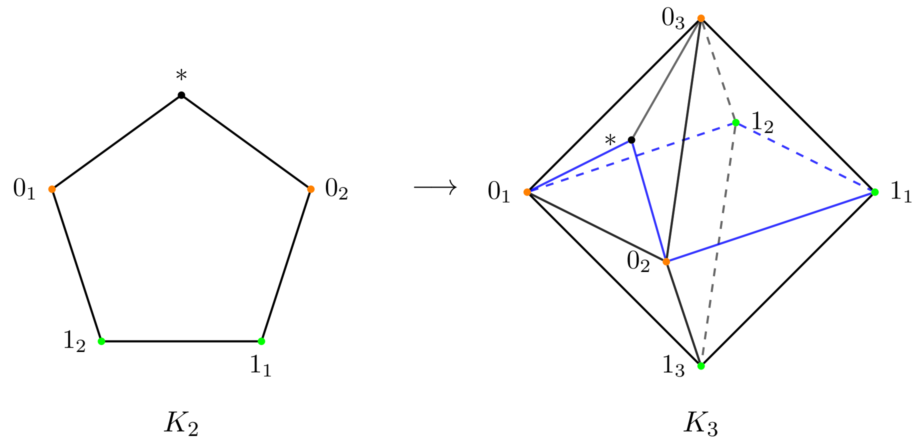
The inclusion $K_m \subseteq K_{m+1}$ is given as a join of coordinate Σ_m-equivariant inclusions $\Delta_m^{k_i} \subseteq \Delta_{m+1}^{k_i}$ and therefore it is Σ_m-equivariant.

Notice that for $s = 1$ we recover the family of k-skeleton sequences of Example 3.4.6.

Next we construct a non-tivial example of consistent sequence of finite simplicial Σ_m-complexes.

Construction 3.4.8. Let I^m be an m-cube. Consider the simplicial complex K_m obtained by taking the boundary of the dual of a simple polytope $vc(I^m)$, where $vc(I^m)$ is obtained by cutting a vertex from I^m. Note that K_1 consists of two disjoint points. K_m can also be constructed as follows. Let $S_{2m} = S_1^0 * \ldots * S_m^0$ be the join of m copies of two disjoint points, where $S_i^0 = \{0_i, 1_i\}$. Notice that S_{2m} is a triangulation of an $(m-1)$-sphere on $2m$ vertices. Then K_m is obtained from S_{2m} by deleting the interior of the $(m-1)$-face on vertices $0_1, \ldots, 0_m$ and taking the cone on it. The natural Σ_m-action on K_m is given by $g \cdot 0_i = 0_{g \cdot i}$, $g \cdot 1_i = 1_{g \cdot i}$, and g fixes the cone vertex. The inclusions $K_m \subseteq K_{m+1}$ are induced by the inclusions $S_1^0 * \ldots * S_m^0 \subseteq S_1^0 * \ldots * S_m^0 * S_{m+1}^0$ with the cone vertex mapping to itself.

For example, when $m = 2$, K_2 is simplicially isomorphic to a pentagon, and the Σ_2-action on K_2 is given by 0_1 mapping to 0_2 and 1_1 mapping to 1_2 keeping the cone vertex fixed. As shown in the picture below, the blue colour lines represent how K_2 is included into K_3.



Vertices with the same color belong to the same orbit of symmetric actions.

Definition 3.4.9. Given an integer $r \geq 0$, a consistent sequence $\mathcal{K} = \{K_m, i_m\}$ of finite simplicial Σ_m-complexes is called *r-face-stable* at degree d if for $m \geq d$ and every $\sigma \in K_m$ with dim $\sigma = r$ there exist a $g \in \Sigma_m$ and $\tau \in K_d$ such that $g \cdot i_{d,m}(\tau) = \sigma$, where $i_{d,m} = i_m \circ \ldots \circ i_d$ is a composite of the inclusions $i_d, \ldots, i_m$.

Similarly, a consistent sequence $\mathcal{K} = \{K_m, i_m\}$ of finite simplicial Σ_m-complexes is called *r-vertex-stable* at degree d if for $m \geq d$ and any collection $\{v_0, \ldots, v_r\}$ of $r+1$ vertices of K_m there exist a $g \in \Sigma_m$ and a collection $\{u_0, \ldots, u_r\}$ of $r+1$ vertices in K_d such that $g \cdot i_{d,m}(u_i) = v_i$. In particular, if $r = 0$ then $\mathcal{K}$ is called *vertex-stable*.

If a consistent sequence $\{K_m, i_m\}$ of finite simplicial Σ_m-complexes is r-vertex-stable (resp. r-face-stable) for every $r \geq 0$, we call it *completely surjective* (resp. *simplicially surjective*).

Remark 3.4.10. Note that Construction 3.4.7 and Construction 3.4.8 are completely surjective.

(i) Let $K_m = \Delta^k_m$. For $r \geq 0$, let $E_{m,r+1}$ consist of all the subsets of $[m]$ with cardinality $r+1$. Then the transitivity of Σ_m-action on $E_{m,r+1}$ implies that $\{K_m\}$ is r-vertex-stable at degree $r+1$.

(ii) Let $K_m = \Delta^{k_1}_m * \ldots * \Delta^{k_s}_m$. If $s = 2$, then $\Delta^{k_1}_m * \Delta^{k_2}_m$ is r-vertex-stable at degree $r+1$. Let J_1, J_2 be two subsets of $[m]$ with $|J_1| + |J_2| = r+1$ and $m \geq r+1$. If $J_1 \cap J_2 \neq \emptyset$, $J_1 \cap J_2$ can be seen as a subset of vertices of $\Delta^{k_1}_m$ and $\Delta^{k_2}_m$, respectively. Let $J_1^c = J_1 \setminus J_1 \cap J_2$ and $J_2^c = J_2 \setminus J_1 \cap J_2$ with cardinalities r_1 and r_2 and let $r_0 = |J_1 \cap J_2|$.

Define $g \in \Sigma_m$ by sending $\{1, \ldots, r_0\}$ to $J_1 \cap J_2$, $\{r_0+1, \ldots, r_0+r_1\}$ to J_1^c and $\{r_0 + r_1 + 1, \ldots, r_0 + r_1 + r_2\}$ to J_2^c and the complement of $\{1, \ldots, r_0 + r_1 + r_2\}$ in $[m]$ to the complement of $J_1 \cup J_2$ in $[m]$, respecting to the initial order of vertices.

Now take the subset of vertices $\{1, \ldots, r_0 + r_1\}$ of $\Delta^{k_1}_{r+1}$ and the subset of vertices $\{1, \ldots, r_0, r_0+r_1+1, \ldots, r_0+r_1+r_2\}$ of $\Delta^{k_2}_{r+1}$ satisfying $g\cdot(\{1, \ldots, r_0+r_1\} \sqcup \{1, \ldots, r_0, r_0+r_1+1, \ldots, r_0+r_1+r_2\}) = J_1 \sqcup J_2$.

If $J_1 \cap J_2 = \emptyset$, then $r_0 = 0$ and $g \in \Sigma_m$ sending $\{1, \ldots, r_1\}$ to J_1 and $\{r_1+1, \ldots, r_1+r_2\}$ to J_2 and the complement of $\{1, \ldots, r_1+r_2\}$ in $[m]$ to the complement of $J_1 \cup J_2$ in $[m]$. Inductively, K_m is completely surjective.

(iii) For any $r \geq 0$, $K_m = \partial vc(I^m)^*$ is r-vertex-stable at degree $d = r+1$. With $m \geq d$, let J be a subset of vertices of K_m and $|J| = r+1$. Write $J = J_* \sqcup J_1 \sqcup \ldots \sqcup J_m$, where J_* is either empty or the cone vertex $\{*\}$ and each $J_i \subseteq \{0_i, 1_i\}$. Since $|J| = r+1$, there are at most $r+1$ nonempty components of J, say $J_{t_1}, \ldots, J_{t_{r+1}}$. If $* \notin J$, define $g \in \Sigma_m$ by sending i to t_i if $i \leq r+1$ and to k_{i-r-1} otherwise where $\{k_1, \ldots, k_{m-r-1}\}$ is the complement of $\{t_1, \ldots, t_{r+1}\}$ in $[m]$. Now let $J' = J'_1 \sqcup \ldots \sqcup J'_{r+1}$ from the vertex set of K_{r+1} where J'_i contains 0_i or 1_i if and only if J_{t_i} contains 0_{t_i} or 1_{t_i}. If $* \in J$, consider $\widetilde{J} = J \setminus \{*\}$ and repeat the above procedure to find $g \in \Sigma_m$ and $\widetilde{J}' \in \mathrm{Ver}(K_{r+1})$ for $\widetilde{J}$. Then let $J' = J_* \sqcup \widetilde{J}'$ and $g \cdot J' = J$.

By Theorem 3.3.5, for a simplicial G-complex K on m vertices

$$\widetilde{H}_i((X,A)^K; \mathbf{k}) \cong \bigoplus_{J \in [m]/G} \mathrm{Ind}^G_{G_J} \widetilde{H}_i((X,A)^{\wedge K_J}; \mathbf{k}). \tag{3.11}$$

If a consistent sequence $\{K_m, i_m\}$ of Σ_m-complexes K_m on the vertex set $V(K_m)$ is completely surjective then the summands in (3.11) do not depend on m for sufficiently large m. We shall use Hemmer's result to study the uniformly representation stability of polyhedral products. For that the stabiliser $(\Sigma_m)_J$ needs to be of the form $H \times \Sigma_{m-k}$ for some $H \leq \Sigma_k$. Therefore we proceed by studying the stabiliser of $J \in \mathcal{P}(V(K_m))$ in Σ_m which we denote by $\mathrm{stab}(J, m)$.

Observe that for a fixed integer d, for all $m \geq d$ and for some $J \in \mathcal{P}(V(K_d))$, as J also belongs to the Σ_m-set $\mathcal{P}(V(K_m))$, there is a sequence of stabilisers

$$\begin{array}{ccccccc} \cdots & \longrightarrow & \Sigma_m & \longrightarrow & \Sigma_{m+1} & \longrightarrow & \cdots \\ & & \uparrow & & \uparrow & & \\ \cdots & \longrightarrow & \mathrm{stab}(J,m) & \longrightarrow & \mathrm{stab}(J,m+1) & \longrightarrow & \cdots. \end{array}$$

For instance, in Example 3.4.6, if $m \geq |J|$, then $J \in \mathcal{P}(V(\Delta^k_m))$ and $\mathrm{stab}(J,m) = \Sigma_{|J|} \times \Sigma_{m-|J|}$.

In Construction 3.4.7, $K_m = \Delta^{k_1}_m * \Delta^{k_2}_m * \ldots * \Delta^{k_s}_m$. Write J as a disjoint union of $J_1, \ldots, J_s$, where each J_t $(1 \leq t \leq s)$ is from the t-th component $\mathcal{P}(V(\Delta^{k_t}_m))$. Let $b(J) = \max\limits_{1 \leq t \leq s} |J_t|$. For $m \geq b(J)$, we observe the stabilisers of J in Σ_m,

$$\mathrm{stab}(J,m) = \{g \in \Sigma_m \mid g \cdot J_t = J_t, 1 \leq t \leq s\} = \bigcap_{1 \leq t \leq s} \mathrm{stab}(J_t, m)$$

where, as in Example 3.4.6, each $\mathrm{stab}(J_t, m)$ is isomorphic to $\Sigma_{|J_t|} \times \Sigma_{m-|J_t|}$. For integers $a \leq b \leq m$, we have

$$\Sigma_a \times \Sigma_{m-a} \cap \Sigma_b \times \Sigma_{m-b} = \Sigma_a \times \Sigma_{b-a} \times \Sigma_{m-b}.$$

Therefore $\mathrm{stab}(J,m) = \mathrm{stab}(J, b(J)) \times \Sigma_{m-b(J)}$ for $m \geq b(J)$, and $\Sigma_{m-b(J)}$ acts on J trivially.

We call such sequences stabiliser consistent.

Definition 3.4.11. A consistent sequence $\mathcal{K} = \{K_m, i_m\}$ of finite simplicial Σ_m-complexes is called *stabiliser consistent* if for every d and every finite set $J \in \mathcal{P}(V(K_d))$ there exists an integer $b(J)$, such that if $m \geq d$ and $m \geq b(J)$, then either Σ_m acts on J trivially or the stabiliser $\mathrm{stab}(J,m)$ is isomorphic to $\mathrm{stab}(J, b(J)) \times \Sigma_{m-b(J)}$, where $\Sigma_{m-b(J)}$ acts on J trivially.

Construction 3.4.8 also provides a stabiliser consistent sequence. If $J \in \mathcal{P}(V(K_d))$ for some d, write $J = J_* \sqcup J_1 \sqcup \ldots \sqcup J_d$, where J_* is either empty or $\{*\}$ and each $J_i \subseteq \{0_i, 1_i\}$. Since Σ_m acts on $*$ trivially, $\mathrm{stab}(J,m) = \mathrm{stab}(\tilde{J}, m)$ where $\tilde{J} = J_1 \sqcup \ldots \sqcup J_d$. Let $b(J)$ be the number of non-empty components J_t. Then for $m \geq b(J)$, $\mathrm{stab}(J,m) \cong \mathrm{stab}(J, b(J)) \times \Sigma_{m-b(J)}$ where $\Sigma_{m-b(J)}$ acts on J trivially.

As a consequence, we have the following result that states conditions on a sequence of finite simplicial complexes that will induce in homology a uniformly representation stable sequence.

Theorem 3.4.12. *Let $\{K_m, i_m\}$ be a consistent sequence of finite simplicial complexes and X be a connected, based CW-complexes of finite type with a based subcomplex A.*

Suppose that $\{K_m, i_m\}$ *is completely surjective and stabiliser consistent. Then the consistent sequence of* Σ_m*-representations* $\{\widetilde{H}_i((X,A)^{K_m};\mathbf{k}), i_{m*}\}$ *for* $\mathrm{char}\,\mathbf{k} = 0$ *is uniformly representation stable.*

Proof. By Theorem 3.3.5, we have

$$\widetilde{H}_i((X,A)^{K_m};\mathbf{k}) \cong \bigoplus_{J\in E_m} \mathrm{Ind}^{\Sigma_m}_{\mathrm{stab}(J,m)} \widetilde{H}_i((X,A)^{\wedge K_{m,J}};\mathbf{k}) \tag{3.12}$$

where E_m is a set of representatives of $\mathcal{P}(V(K_m))$ under the action Σ_m, and $\mathrm{stab}(J,m)$ is the stabiliser of J under Σ_m.

We prove that if $|J| \geq i+1$ then $\widetilde{H}_i((X,A)^{\wedge K_{m,J}};\mathbf{k})$ is trivial. By the reduced Künneth formula for path-connected spaces, it is obvious that $\widetilde{H}_i(Y_1\wedge\ldots\wedge Y_{|J|};\mathbf{k}) = 0$ if $|J| \geq i+1$, where each Y_i is either X or A. This implies that for any $\sigma \in K_{m,J}$, $\widetilde{H}_i((X,A)^{\wedge\sigma};\mathbf{k}) = 0$ if $|J| \geq i+1$. If X_1 and X_2 are connected CW-complexes with a non-empty intersection such that $\widetilde{H}_i(X_1;\mathbf{k}) = \widetilde{H}_i(X_2;\mathbf{k}) = \widetilde{H}_i(X_1\cap X_2;\mathbf{k}) = 0$ for $i \leq l$, then $\widetilde{H}_i(X_1\cup X_2;\mathbf{k}) = 0$ for $i \leq l$. As $(X,A)^{K_{m,J}}$ is a union of $(X,A)^{\sigma}$ over all $\sigma \in K_{m,J}$, inductively $\widetilde{H}_i((X,A)^{\wedge K_J};\mathbf{k})$ is trivial if $|J| \geq i+1$.

Since $\{K_m, i_m\}$ is completely surjective, if $|J| \leq i$ there exists an integer $N \geq 1$ such that if $m \geq N$, we have $E_{m+1,i} = E_{m,i} = \ldots = E_{N,i}$, where $E_{m,i} = \{J \in E_m \mid |J| \leq i\}$. Therefore the summands in (3.12) do not depend on m for $m \geq N$. On the other hand, for each $J \in E_*$ there exists an integer $b(J)$ such that for $m \geq b(J)$, either Σ_m acts on J trivially or the stabiliser $\mathrm{stab}(J,m) = \mathrm{stab}(J,b(J)) \times \Sigma_{m-b(J)}$, where $\Sigma_{m-|J|}$ acts on J trivially. In the first case, if Σ_m acts on J trivially for $m \geq b(J)$, then for any $k \leq b(J)$, Σ_k acts on J trivially because Σ_k acts on J as a subgroup of $\Sigma_{b(J)}$. As the vertex support set J of $K_{m,J}$ is fixed, the space $(X,A)^{\wedge K_{m,J}}$ will stay the same when m increases. Thus, $\widetilde{H}_i((X,A)^{\wedge K_{m,J}};\mathbf{k})$ is a fixed finite-dimensional trivial Σ_m-representation even though m varies. It follows that $\{\widetilde{H}_i((X,A)^{\wedge K_{m,J}};\mathbf{k})\}$ is uniformly representation stable.

If $\mathrm{stab}(J,m) = \mathrm{stab}(J,b(J))\times\Sigma_{m-b(J)}$, then $\widetilde{H}_i((X,A)^{\wedge K_{m,J}};\mathbf{k})$ is a $\mathrm{stab}(J,m)$-representation with a trivial $\Sigma_{m-b(J)}$-action. By [25], we have that $\mathrm{Ind}^{\Sigma_m}_{\mathrm{stab}(J,m)}\widetilde{H}_i((X,A)^{\wedge K_{m,J}};\mathbf{k})$ is uniformly representation stable.

Therefore, the sequence of Σ_m-representations $\{\widetilde{H}_i((X,A)^{K_m};\mathbf{k}), i_{m*}\}$ is uniformly representation stable as the summands do not depend on m eventually. □

Remark 3.4.13. In general, we require the simplicial maps i_m in the consistent sequence of finite simplicial complexes to be inclusions so that they induce maps of polyhedral products. However, in the case when (X,A) is a pair of topological monoids, as it is for moment-angle complexes when $(X,A) = (D^2,S^1)$, any Σ_m-simplicial map, not necessary a simplicial inclusion, can be chosen for i_m. A simplicial map $f\colon K \longrightarrow L$

induces a continuous map $(X,A)^K \longrightarrow (X,A)^L$ defined by $(x_1,\ldots,x_p) = (y_1,\ldots,y_q)$, where $y_j = \prod_{i\in f^{-1}(j)} x_i$. Here p and q are the number of vertices of K and L, respectively.

We have proved that the sequences in Constructions 3.4.7 and 3.4.8 are completely surjective and stabiliser consistent. Applying Theorem 3.4.12, we conclude the following statement.

Corollary 3.4.14. *Let $\mathcal{K}$ be one of the consistent sequences in Constructions 3.4.7 and 3.4.8 and X be a connected, based CW-complexes of finite type with a based subcomplex A. Then the consistent sequence of Σ_m-representations $\{\widetilde{H}_i((X,A)^{K_m};\mathbf{k}), i_{m_*}\}$ for* $\operatorname{char}\mathbf{k} = 0$ *is uniformly representation stable.* □

Note that since the sequence in Construction 3.4.8 provides a consistent sequence of finite simplicial complexes, given by taking the boundary of dual of simple polytopes, the corresponding moment-angle complexes are a sequence of manifolds.

Proposition 3.4.15. *Let $\mathcal{K}$ be the consistent sequence in Construction 3.4.8. Then for the moment-angle manifolds $\mathcal{Z}_{\mathcal{K}}$, the consistent sequence $\{H_*(\mathcal{Z}_{K_m};\mathbf{k}), i_{m_*}\}$ of Σ_m-representations for* $\operatorname{char}\mathbf{k} = 0$ *is uniformly representation stable.* □

Moreover, due to [5, 12], the manifold $\mathcal{Z}_{K_m}$ is diffeomorphic to

$$\mathcal{Z}_{K_m} \overset{\text{diff}}{\cong} \partial((\prod_m S^3 - D^{3m}) \times D^2) \# \mathop{\#}_{j=1}^{m} \binom{m}{j} (S^{j+2} \times S^{3m-j-1}).$$

Therefore, $H_3(\mathcal{Z}_{K_m};\mathbf{k})$ has Betti number m which means that the sequence of moment-angle manifolds $\mathcal{Z}_{K_m}$ with the maps $\mathcal{Z}_{K_m} \longrightarrow \mathcal{Z}_{K_{m+1}}$ induced by simplicial maps $K_m \longrightarrow K_{m+1}$ is not homology stable.

Let $K_m = \Delta^k_m$. Since every K_m is a full subcomplex of K_{m+1}, the moment-angle complex $\mathcal{Z}_{K_m}$ retracts off $\mathcal{Z}_{K_{m+1}}$, and the retraction map $p_m\colon \mathcal{Z}_{K_{m+1}} \longrightarrow \mathcal{Z}_{K_m}$ is Σ_m-equivariant. The uniform stability of Σ_m-representations $\{H^i(\mathcal{Z}_{K_m};\mathbf{k}), p^i_m\}$ follows immediately.

Proposition 3.4.16. *For $i \geq 2k+3$, the sequence of Σ_m-representations $\{H^i(\mathcal{Z}_{K_m};\mathbf{k}), p^i_m\}$ is uniformly representation stable.*

Proof. By Proposition 3.2.8, we have

$$H^i(\mathcal{Z}_{K_m};\mathbf{k}) \cong \bigoplus_{J\in E_m} \operatorname{Ind}^{\Sigma_m}_{\Sigma_{|J|}\times\Sigma_{m-|J|}} \widetilde{H}^{i-|J|-1}(K_{J,m};\mathbf{k})$$

where $E_m = \{\{1\},\{1,2\},\ldots,\{1,2,\ldots,m\}\}$ and $K_{J,m} = J \cap \Delta^k_m$. Thus K_J is a $(|J|-1)$-face of K, if $|J| \leq k+1$ and is the k-skeleton of $(|J|-1)$-simplex with J as its vertex set if $|J| \geq k+2$. The latter one allows an $\Sigma_{|J|}$-action. Therefore, if $|J| \leq k+1$, then $\widetilde{H}^*(K_{J,m};\mathbf{k}) = 0$. If $k+2 \leq |J| \leq m$, $\widetilde{H}^p(K_{J,m};\mathbf{k}) = \mathbf{k}$ if $p = k$, and is 0, otherwise.

The nontrivial cohomology group of $K_{J,m}$ implies that $i-|J|-1=k$ and $k+2\leq |J|\leq m$. Thus if $2k+3\leq i\leq m+k+1$, we have a Σ_m-representation isomorphism that

$$H^i(\mathcal{Z}_{K_m};\mathbf{k})\cong \mathrm{Ind}^{\Sigma_m}_{\Sigma_{|J|}\times\Sigma_{m-|J|}}\widetilde{H}^k(K_{J,m};\mathbf{k}),\ \text{with } |J|=i-k-1.$$

Hemmer [25] implies that the sequence of Σ_m-representations $\{H^i(\mathcal{Z}_{K_m};\mathbf{k}),p^i_m\}$ is uniformly representation stable. □

Example 3.4.17. When K_m consists of m disjoint points and for $i\geq 3$, as a Σ_m-representation $H^i(\mathcal{Z}_m;\mathbf{k})$ can be written explicitly as

$$H^i(\mathcal{Z}_m;\mathbf{k})=\mathrm{Ind}^{\Sigma_m}_{\Sigma_{i-1}\times\Sigma_{m-i+1}}V_{(i-2,1)}\boxtimes\mathbf{k}$$

where $V_{(i-2,1)}$ is the standard representation of Σ_{i-1}.

In particular,

$$\begin{aligned}
H^3(\mathcal{Z}_m;\mathbf{k}) =&V_{(m-1,1)}\oplus V_{(m-2,1,1)}\quad \text{for } m\geq 3;\\
H^4(\mathcal{Z}_m;\mathbf{k}) =&V_{(m-1,1)}\oplus V_{(m-2,2)}\oplus V_{(m-2,1,1)}\oplus V_{(m-3,2,1)}\quad \text{for } m\geq 5;\\
H^5(\mathcal{Z}_m;\mathbf{k}) =&V_{(m-1,1)}\oplus V_{(m-2,2)}\oplus V_{(m-2,1,1)}\oplus V_{(m-3,3)}\oplus V_{(m-3,2,1)}\oplus V_{(m-4,3,1)}\quad \text{for } m\geq 7;\\
H^6(\mathcal{Z}_m;\mathbf{k}) =&V_{(m-1,1)}\oplus V_{(m-2,2)}\oplus V_{(m-2,1,1)}\oplus V_{(m-3,3)}\oplus V_{(m-3,2,1)}\oplus V_{(m-4,4)}\\
&\oplus V_{(m-4,3,1)}\oplus V_{(m-5,4,1)}\quad \text{for } m\geq 9.
\end{aligned}$$

3.5 Applications of uniformly representation stability of polyhedral products

We finish the paper by investigating what kind of structural properties of $H_i((X,A)^K;\mathbb{Q})$ are implied by representation stability.

One of the key properties of a sequence of uniformly stable Σ_m-representations over $\mathbb{Q}$ is that their characters are eventually polynomials [14, Definition 1.4].

Denote by $\lambda\vdash m$ a partition $\lambda=(\lambda_1,\ldots,\lambda_l)$ with $\lambda_1\geq\ldots\lambda_l>0$ and $\lambda_1+\ldots+\lambda_l=m$. Let $|\lambda|$ be the sum $\lambda_1+\ldots+\lambda_l$. Given any partition λ, for any $m\geq|\lambda|+\lambda_1$, denote by $\lambda[m]=(m-|\lambda|,\lambda_1,\ldots,\lambda_l)$ (see [14, Definition 2.2.5]). Denote by $V(\lambda)_m$ the irreducible representation corresponding to partition $\lambda[m]$. The *weight* of a consistent sequence of Σ_m-representations $\{V_m,\psi_m\}$ is the maximum of $|\lambda|$ over all irreducible constituents $V(\lambda)_m$ that appears in V_m.

Example 3.5.1. For any partition $\mu\vdash n$ and $m\geq n$, applying Proposition 3.2.4 in [14], the consistent sequence $\{\mathrm{Ind}^{\Sigma_m}_{\Sigma_n\times\Sigma_{m-n}}V_\mu\boxtimes\mathbf{k}\}$ has weight n, where V_μ is the irreducible representation corresponding to μ.

Hemmer [25] (see Theorem 3.4.4) constructed a sequence of Σ_m-representations that is uniformly representation stable. Next we calculate the weight of this sequence applying the result from Example 3.5.1.

Lemma 3.5.2. *Fix an integer $n \geq 0$. Let H be a subgroup of Σ_n and V is a Σ_n-representation over a field $\mathbf{k}$ of characteristic 0. For $m \geq n$, the consistent sequence $\{\mathrm{Ind}_{H\times\Sigma_{m-n}}^{\Sigma_m} V \boxtimes \mathbf{k}\}$ has weight n.*

Proof. Observe that

$$\mathrm{Ind}_{H\times\Sigma_{m-n}}^{\Sigma_m} V \boxtimes \mathbf{k} = \mathrm{Ind}_{\Sigma_n\times\Sigma_{m-n}}^{\Sigma_m}(\mathrm{Ind}_{H\times\Sigma_{m-n}}^{\Sigma_n\times\Sigma_{m-n}}(V \boxtimes \mathbf{k})) = \mathrm{Ind}_{\Sigma_n\times\Sigma_{m-n}}^{\Sigma_m}(\mathrm{Ind}_{H}^{\Sigma_n} V) \boxtimes \mathbf{k}.$$

As Σ_n-representations, $\mathrm{Ind}_H^{\Sigma_n} V$ is decomposed as $\bigoplus_{\mu\vdash n} V_\mu^{\oplus c_\mu}$, where c_μ are multiplicities. By Example 3.5.1, $\{\mathrm{Ind}_{\Sigma_n\times\Sigma_{m-n}}^{\Sigma_m} V_\mu \boxtimes \mathbf{k}\}$ has weight n. Then $\{\mathrm{Ind}_{H\times\Sigma_{m-n}}^{\Sigma_m} V \boxtimes \mathbf{k}\}$ has weight n, as each $\mathrm{Ind}_{H\times\Sigma_{m-n}}^{\Sigma_m} V \boxtimes \mathbf{k}$ is decomposed into a finite direct sum as Σ_m-representations

$$\mathrm{Ind}_{H\times\Sigma_{m-n}}^{\Sigma_m} V \boxtimes \mathbf{k} \cong \bigoplus_{\mu\vdash n} c_\mu \mathrm{Ind}_{\Sigma_n\times\Sigma_{m-n}}^{\Sigma_m} V_\mu \boxtimes \mathbf{k}.$$

□

Given a uniformly representation stable sequence $\{V_m, \psi_m\}$, the uniform multiplicity stability implies that there exists an integer $M \geq 0$ such that V_m is decomposed into $\bigoplus_\lambda c_\lambda V(\lambda)_m$ for $m \geq M$. A classical result ([28, Example I.7.14]) in representation theory states that the character of $V(\lambda)_m$ is polynomial if $m \geq |\lambda| + \lambda_1$. Explicitly, let $a_1, a_2, \ldots$ be class functions $a_j : \Sigma_i \longrightarrow \mathbb{N}$ for any $i \geq 0$ such that $a_j(g)$ is the number of j-cycles in the cycle decomposition of g. Then, for each partition λ there exists a polynomial $P_\lambda \in \mathbb{Q}[a_1, a_2, \ldots]$, called the character polynomial corresponding to the partition λ, such that P_λ has degree $|\lambda|$ and the character $\chi_{V(\lambda)_m}(g) = P_\lambda(g)$ for all $m \geq |\lambda| + \lambda_1$ and $g \in \Sigma_m$.

We finish this chapter by looking at the growth of Betti numbers of polyhedral products.

Theorem 3.5.3. *Let $\{K, i_m\}$ and (X, A) be as in Theorem 3.4.12. Then for each $i \geq 0$, the consistent sequence $\{\widetilde{H}_i((X, A)^{K_m}; \mathbb{Q}), i_{m_*}\}$ has a finite weight. Moreover, the growth of Betti numbers of $\{\widetilde{H}_i((X, A)^{K_m}; \mathbb{Q}), i_{m_*}\}$ is eventually polynomial with respect to m.*

Proof. By Theorem 3.4.12, $\{\widetilde{H}_i((X, A)^{K_m}; \mathbb{Q}), i_{m_*}\}$ is uniformly representation stable. Thus the uniformly multiplicity stability implies that there exists an integer $N > 0$, not depending on λ, such that for all $m \geq N$, there are constant integers c_λ such that

$$\widetilde{H}_i((X, A)^{K_m}; \mathbb{Q}) \cong \bigoplus_\lambda c_\lambda V(\lambda)_m.$$

These integers c_λ are uniquely given by multiplicities defined in the irreducible components of $\widetilde{H}_i((X,A)^{K_N};\mathbb{Q})$. Therefore, the weight ω_i of sequence $\{\widetilde{H}_i((X,A)^{K_m};\mathbb{Q}), i_{m*}\}$ is the maximum $|\lambda|$ that forms a irreducible component of $\widetilde{H}_i((X,A)^{K_N};\mathbb{Q})$. Since $\widetilde{H}_i((X,A)^{K_N};\mathbb{Q})$ has finite dimension over $\mathbb{Q}$, ω_i is finite.

In particular, if $m \geq 2\omega_i$, then for all λ appearing in the above equation, $m \geq |\lambda| + \lambda_1$. Then there exists a polynomial character of $\{\widetilde{H}_i((X,A)^{K_m};\mathbb{Q})$ given by $\sum_\lambda P_\lambda$. Take g to be the identity of symmetric groups. This gives that the growth of Betti numbers of $\{\widetilde{H}_i((X,A)^{K_m};\mathbb{Q}), i_{m*}\}$ is eventually polynomial with respect to m. $\square$

Chapter 4

Torus actions on $\mathcal{Z}_K$

Let H be a torus subgroup of T^m of rank r and let $\mathcal{Z}_K$ be a moment-angle complex corresponding to a simplicial complex K on $[m]$. Then H acts on $\mathcal{Z}_K$ as a subgroup of T^m. The quotient space $\mathcal{Z}_K/H$ has underlying combinatorial data (K, Λ), where Λ is an integral matrix associated to the quotient homomorphism $T^m \longrightarrow T^m/H$. This pair (K, Λ) can be combinatorially interpreted as assigning each vertex of K a "colour" in $\mathbb{Z}^{m-r}$. Given an order on the vertex set of a coloured simplicial complex, we obtain a linear map $\Lambda\colon \mathbb{Z}^m \longrightarrow \mathbb{Z}^{m-r}$, which induces a homomorphism of tori $T^m \longrightarrow T^{m-r}$. If we require this linear map Λ to be surjective, then the induced subgroup KerΛ of T^m is a subtorus. Thus we obtain a quotient space $\mathcal{Z}_K/H$ ($H = \text{Ker}\Lambda$) under a subtorus H-action.

4.1 Preliminaries on torus groups

A homomorphism of tori $f\colon T^m \longrightarrow T^n$ induces a linear map on their Lie algebras $\mathbb{R}^m \longrightarrow \mathbb{R}^n$ and consequently induces a linear map on their fibres $\mathbb{Z}^m \longrightarrow \mathbb{Z}^n$ whose associated matrix has integral entries $\Lambda = (\lambda_{ij})_{n\times m}$ ([6, Exercise(4.15) 9, I]) after choosing a basis of $\mathbb{Z}^m$ and $\mathbb{Z}^n$. In this way, the homomorphism f is written as $f(t_1, \ldots, t_m) = (t_1^{\lambda_{11}} \ldots t_m^{\lambda_{1m}}, \ldots, t_1^{\lambda_{n1}} \ldots t_m^{\lambda_{nm}})$. In the following lemma, we will show that the induced map between cohomology rings of tori and their corresponding classifying spaces can be characterised by the associated matrix Λ.

Let R be a commutative and associative ring with unit and let 1_R denote the multiplicative identity.

Lemma 4.1.1. *Let $f\colon T^m \longrightarrow T^n$ be a group homomorphism of tori with associated integral matrix $(\lambda_{ij})_{n\times m}$.*

(a) *The induced map* f^* *on the cohomology ring of tori is given by*

$$f^*\colon H^*(T^n;R) = \Lambda_R^*[\beta_1,\ldots,\beta_n] \longrightarrow H^*(T^m;R) = \Lambda_R^*[\alpha_1,\ldots,\alpha_m]$$
$$\beta_i \longmapsto \sum_{j=1}^{m}(\lambda_{ij}\cdot 1_R)\alpha_j$$

where $\deg\beta_i = \deg\alpha_j = 1$.

(b) *The induced map* Bf^* *on the cohomology ring of classifying spaces is given by*

$$Bf^*\colon H^*(BT^n;R) = R[x_1,\ldots,x_n] \longrightarrow H^*(BT^m;R) = R[y_1,\ldots,y_m]$$
$$x_i \longmapsto \sum_{j=1}^{m}(\lambda_{ij}\cdot 1_R)y_j.$$

where $\deg x_i = \deg y_j = 2$.

Proof. (a) Since the cohomology of T^m is the exterior algebra generated by the first cohomology $H^1(T^m;R)$ ([24, Example 3.11]), it suffices to see what the map between the first cohomology groups of T^n and T^m is. Consider the first cellular chain complex $\mathcal{C}_1(T^m;R) \cong \bigoplus_m \mathcal{C}_1(S^1;R)$, which is a free R-module with a basis $\{a_i, i = 1,\ldots,m\}$, where a_i is a generator of the i-th summand of $\mathcal{C}_1(T^m;R)$. Similarly, denote by b_j, for $j = 1,\ldots,n$, the generator of the j-th summand of $\mathcal{C}_1(T^n;R)$.

Consider the restriction of f to the i-th coordinate of T^m,

$$f|_{S^1_i}\colon S^1_i \longrightarrow T^n$$
$$t \mapsto (t^{\lambda_{1i}}, t^{\lambda_{2i}},\ldots,t^{\lambda_{ni}})$$

which is of degree λ_{ji} at the j-th coordinate of the image T^n. Thus, f induces a map $f_1\colon \mathcal{C}_1(T^m;R) \longrightarrow \mathcal{C}_1(T^n;R)$ of the following form

$$f_1(a_i) = (\lambda_{1i}\cdot 1_R,\ldots,\lambda_{ni}\cdot 1_R) = (\lambda_{1i}\cdot 1_R)b_1 + \ldots + (\lambda_{ni}\cdot 1_R)b_n.$$

The first cochain $\mathcal{C}^1(T^n;R) = \mathrm{Hom}_R(\mathcal{C}_1(T^n;R),R)$ has a basis $\{b_j^*, j = 1,\ldots,n\}$, where b_j^* sends b_j to 1_R and other basis elements to zero. Similarly, $\mathcal{C}^1(T^m;R)$ has a basis $\{a_i^*, i = 1,\ldots,m\}$. Consider the induced map on the first cellular cochain

$$f^1\colon \mathcal{C}^1(T^n;R) \longrightarrow \mathcal{C}^1(T^m;R)$$
$$b_j^* \longmapsto f^1(b_j^*) = b_j^*\circ f_1.$$

Since $f^1(b_j^*)(a_i) = b_j^*\circ f_1(a_i) = b_j^*((\lambda_{1i}\cdot 1_R)b_1 + \ldots + (\lambda_{ni}\cdot 1_R)b_n) = \lambda_{ji}\cdot 1_R$, we obtain $f^1(b_j^*) = \sum_{i=1}^{m}(\lambda_{ji}\cdot 1_R)a_i^*$. Therefore, the induced map between the first cohomology groups of T^n and T^m, also denoted by f^1, has the following description

$$f^1 \colon H^1(T^n;R) \longrightarrow H^1(T^m;R)$$
$$f^1(\beta_j) \longmapsto \sum_{i=1}^{m}(\lambda_{ji}\cdot 1_R)\alpha_i^*$$

where $\{\beta_j = [b_j^*]\}$ and $\{\alpha_i = [a_i^*]\}$ form the basis of $H^1(T^n;R)$ and $H^1(T^m;R)$, respectively.

(b) By applying the Serre spectral sequence to the fibre bundle $T^n \longrightarrow ET^n \longrightarrow BT^n$, the cohomology ring $H^*(BT^n;R)$ is a polynomial ring over R whose algebraic generators x_j are in one-to-one correspondence to generators β_j of $H^*(T^n;R)$, $1 \leq j \leq n$ (see [38, Example 11.9.7]). Similarly, each algebraic generator y_i of $H^*(BT^m;R)$ corresponds to a generator α_i of $H^*(T^m;R)$, $1 \leq i \leq m$. Since $f^*(\beta_j) = \sum\limits_{i=1}^{m}(\lambda_{ji}\cdot 1_R)\alpha_i^*$ (the first statement of this lemma), therefore we have that $Bf^*(x_i) = \sum\limits_{j=1}^{m}(\lambda_{ij}\cdot 1_R)y_j$. □

Let H be a subtorus of T^m of rank r. That is, there is a homomorphism of tori

$$S \colon T^r \longrightarrow T^m \tag{4.1}$$

such that the image $S(T^r) = H$ and $S\colon T^r \longrightarrow H$ is a group isomorphism. By a slight abuse of notation, we use $S = (s_{ij})_{m\times r}$ to denote the associated integral matrix of (4.1). Then H can be written as follows

$$H = \{(t_1^{s_{11}}t_2^{s_{12}}\cdots t_r^{s_{1r}}, \ldots, t_1^{s_{m1}}t_2^{s_{m2}}\cdots t_r^{s_{mr}}) \in T^m \mid (t_1,\ldots,t_r) \in T^r\}.$$

Alternatively, since $T^m \cong \mathbb{R}^m/\mathbb{Z}^m$

$$H = \{(e^{2\pi i(s_{11}\psi_1+\ldots+s_{1r}\psi_r)}, \ldots, e^{2\pi i(s_{m1}\psi_1+\ldots+s_{mr}\psi_r)}) \mid (\psi_1,\ldots,\psi_r) \in \mathbb{R}^r\}.$$

Let us remark that H is a subtorus of T^m if and only if the column vectors of S form part of a basis of $\mathbb{Z}^m$. To see this, denote by $\mathrm{col}_{\mathbb{Z}}(S) = \mathrm{Span}_{\mathbb{Z}}\{\mathbf{s}_j \in \mathbb{Z}^m \mid 1 \leq j \leq r\}$, where $\mathbf{s}_j$ are the column vectors of S. If H is a subtorus of T^m, then $\mathrm{col}_{\mathbb{Z}}(S)$ is the spanning lattice of H. Thus $H \cong \mathrm{col}_{\mathbb{R}}(S)/\mathrm{col}_{\mathbb{Z}}(S)$, where $\mathrm{col}_{\mathbb{R}}(S) = \mathrm{Span}_{\mathbb{R}}\{\mathbf{s}_j \mid 1 \leq j \leq r\}$. Since $\mathrm{col}_{\mathbb{Z}}(S) = \mathrm{col}_{\mathbb{R}}(S) \cap \mathbb{Z}^m \cong \mathbb{Z}^r$, the column vectors of S form part of a basis of $\mathbb{Z}^m$. On the other hand, if the column vectors of S form part of a basis of $\mathbb{Z}^m$, then $\mathrm{col}_{\mathbb{Z}}(S) \cong \mathbb{Z}^r$. Thus $H \cong \mathrm{col}_{\mathbb{R}}(S)/\mathrm{col}_{\mathbb{Z}}(S)$ is a subtorus of $\mathbb{R}^m/\mathbb{Z}^m \cong T^m$. If H is a subtorus of T^m, then the quotient group T^m/H is a compact connected abelian Lie group. Thus T^m/H is isomorphic to a torus group of rank $m-r$ and the quotient homomorphism $T^m \longrightarrow T^m/H$ has an associated integral matrix $\Lambda = (\lambda_{ij})_{(m-r)\times m}$. In this way, we have $H = \mathrm{Im}\, S = \mathrm{Ker}\Lambda$. Next, we illustrate the relation between S and Λ explicitly.

There exists an integral $m\times(m-r)$-matrix S' such that the $m\times m$-matrix $M=\begin{pmatrix} S \mid S'\end{pmatrix}$ is invertible. The column vectors $\mathbf{s}_j$ $(1\le j\le m)$ of $\begin{pmatrix} S \mid S'\end{pmatrix}$ form a basis of $\mathbb{Z}^m$. Under the standard basis $\mathbf{e}_j$ of $\mathbb{Z}^m$, M is the linear isomorphism $\mathbb{Z}^m \longrightarrow \mathrm{col}_{\mathbb{Z}}(S)\oplus\mathrm{col}_{\mathbb{Z}}(S')$ by $\mathbf{s}_j=M(\mathbf{e}_j)$. Then $\Lambda=M^{-1}|_{\mathrm{col}_{\mathbb{Z}}(S')}\circ Q_2\circ M$, where Q_2 is the projection

$$\mathbb{Z}^m \overset{M}{\cong} \mathrm{col}_{\mathbb{Z}}(S)\oplus\mathrm{col}_{\mathbb{Z}}(S') \xrightarrow{Q_2} \mathrm{col}_{\mathbb{Z}}(S').$$

Similarly, let $\Lambda'=M^{-1}|_{\mathrm{col}_{\mathbb{Z}}(S)}\circ Q_1\circ M$, where Q_1 is the projection

$$\mathbb{Z}^m \overset{M}{\cong} \mathrm{col}_{\mathbb{Z}}(S)\oplus\mathrm{col}_{\mathbb{Z}}(S') \xrightarrow{Q_1} \mathrm{col}_{\mathbb{Z}}(S).$$

Then $\begin{pmatrix}\Lambda'\\ \Lambda\end{pmatrix}$ is the inverse matrix of $\begin{pmatrix} S \mid S'\end{pmatrix}$, where $\Lambda'=(\lambda'_{ij})$ is an integral $(r\times m)$-matrix and $\Lambda=(\lambda_{ij})$ is an integral $(m-r)\times m$-matrix. It follows that $\Lambda'S=I_{r\times r}$, $\Lambda S'=I_{(m-r)\times(m-r)}$ and $\Lambda S=\Lambda'S'=0$, where $I_{r\times r}$ and $I_{(m-r)\times(m-r)}$ are the identity matrices.

The following properties of integral matrices will be useful when we consider the conditions under which H acts freely on $\mathcal{Z}_K$, stated in Lemma 4.2.1. For two or more integers $a_1,\dots,a_m$, which are not all zero, denote by $\gcd(a_1,\dots,a_m)$ the largest positive integer that divides each a_i.

Lemma 4.1.2. (a) *An integral vector $(a_1,\dots,a_m)$ can be extended to an invertible integral $(m\times m)$-matrix if and only if $\gcd(a_1,\dots,a_m)=1$.*

(b) *Let S be an $m\times r$ integral matrix $(1\le r\le m)$. Then S can be extended to an invertible integral matrix if and only if the row vectors of S span $\mathbb{Z}^r$.*

Proof. (a) We proceed by induction on m. If $m=1$, a_1 is a basis of $\mathbb{Z}$ if and only if $a_1=\pm1$. Suppose the statement is true for m. Consider the case $m+1$. Let $a=\gcd(a_1,\dots,a_m)$ and $a'_i=\frac{a_i}{a}$ $(1\le i\le m)$. Since $\gcd(a_1,\dots,a_{m+1})=1$, there exist integers p and q such that $pa+qa_{m+1}=1$. By assumption, $(a'_1,\dots,a'_m)$ extends to an invertible integral matrix denoted by $\begin{pmatrix} a'_1 & \dots & a'_m\\ & A & \end{pmatrix}$. It follows that the integral matrix

$$\begin{pmatrix} a_1 & \dots & a_m & a_{m+1}\\ & A & & \mathbf{0}\\ (-1)^m q a'_1 & \dots & (-1)^m q a'_m & p\end{pmatrix}$$ is invertible.

(b) In this proof, all matrices considered are integral and by an invertible matrix, we mean that it has integral inverse (i.e., its determinant is ±1).

" $\Rightarrow$ " Since the column vectors of S form part of a basis of $\mathbb{Z}^m$, there exists S' such that $M=\begin{pmatrix} S \mid S'\end{pmatrix}$ is invertible, so $M^{-1}S=\begin{pmatrix} I_r\\ 0\end{pmatrix}$. Since the row vectors of $M^{-1}S$ are a linear combination of row vectors of S, the row vectors of S span $\mathbb{Z}^r$.

" $\Leftarrow$ " If the row vectors of S spans $\mathbb{Z}^r$, we show that S can be extended into an invertible matrix $\left(S \mid S'\right)$ by an induction on r. In the case of $r = 1$, the row vectors of S consist of m integers. These integers span $\mathbb{Z}$ if and only if their greatest common divisor $\gcd(s_1, \ldots, s_m) = 1$ by the first part of this lemma. Thus the statement is true for $r = 1$. Now assume that any integral $m \times r$-matrix can be extended to an integral invertible matrix if and only if its row vectors span $\mathbb{Z}^r$. Let us consider an $m \times (r+1)$-matrix S. Suppose its row vectors span $\mathbb{Z}^{r+1}$. It follows that there exists an $(r+1) \times m$-matrix P such that $PS = I_{r+1}$. Now let $S = \left(S' \mid \mathbf{s}_{r+1}\right)$ and $P = \begin{pmatrix} P' \\ \mathbf{p}_{r+1} \end{pmatrix}$, where $\mathbf{s}_{r+1}$ is the last column vector of S and $\mathbf{p}_{r+1}$ is the last row vector of P. Since $PS = I_{r+1}$, we have

$$P'S' = I_r, \;\; P'\mathbf{s}_{r+1} = \mathbf{0}, \;\; \mathbf{p}_{r+1}S' = \mathbf{0} \text{ and } \mathbf{p}_{r+1}\mathbf{s}_{r+1} = 1. \tag{4.2}$$

Since $P\left(S' \mid \mathbf{s}_{r+1}\right) = \left(PS' \mid P\mathbf{s}_{r+1}\right) = I_{r+1}$, the row vectors of PS' span $\mathbb{Z}^r$, which implies that the row vectors of S' span $\mathbb{Z}^r$. Thus by induction, S' can be extended to an invertible matrix $\left(S' \mid S''\right)$. Denote by $(\alpha_1, \ldots, \alpha_m)$ the column vectors of $\left(S' \mid S''\right)$. Since $\left(S' \mid S''\right)$ is invertible, there are integers $b_1, \ldots, b_m$ such that $\mathbf{s}_{r+1} = b_1\alpha_1 + \ldots + b_m\alpha_m$. Hence $\mathbf{s}_{r+1} - (b_1\alpha_1 + \ldots + b_r\alpha_r) = \sum_{j=r+1}^{m} b_j\alpha_j \in \mathrm{Col}(S'')$. Then by (4.2),

$$\mathbf{p}_{r+1}(\mathbf{s}_{r+1} - (b_1\alpha_1 + \ldots + b_r\alpha_r)) = \mathbf{p}_{r+1}\mathbf{s}_{r+1} = 1.$$

Thus $\gcd(b_{r+1}, \ldots, b_m) = 1$, which implies that $(b_{r+1}, \ldots, b_m)^t$ can be extended to an invertible $(m-r) \times (m-r)$-matrix B. Now let $Q = \left(\begin{array}{c|c} I_r & D \\ \hline 0 & B \end{array}\right)$ where $D = \left(\mathbf{b} \mid 0\right)$ and $\mathbf{b} = (b_1, \ldots, b_r)^t$. Note that Q is invertible and so is $\left(S' \mid S''\right)Q$. The minor given by the first $r+1$ column vectors of $\left(S' \mid S''\right)Q$ is exactly S. Thus, S can be extended to an invertible matrix. $\square$

4.2 Torus actions on $\mathcal{Z}_K$ and the freeness condition

Let H be a torus subgroup of T^m of rank r and let $S = (s_{ij})_{m\times r}$ and $\Lambda = (\lambda_{ij})_{(m-r)\times m}$ be the associated integral matrices of the inclusion $H \longrightarrow T^m$ and the quotient homomorphism $T^m \longrightarrow T^m/H$, respectively. The torus subgroup $H \leqslant T^m$ acts on $\mathcal{Z}_K$ and we denote by $\mathcal{Z}_K/H$ the quotient space.

Recall the subtorus $T^\sigma = \{(t_1, \ldots, t_m) \in T^m \mid t_j = 1 \text{ if } j \notin \sigma\}$ for every $\sigma \subseteq [m]$. We first characterize the condition that H acts freely on $\mathcal{Z}_K$, in terms of T^σ and the associated matrices S, Λ.

Lemma 4.2.1. *Let H be a torus subgroup of T^m of rank r acting on $\mathcal{Z}_K$. Then the following statements are equivalent:*

(a) *H acts on $\mathcal{Z}_K$ freely;*

(b) *for any $\sigma \in K$, the intersection $T^\sigma \cap H$ is trivial;*

(c) *for any $\sigma \in K$, the column vectors of the $(m - |\sigma|) \times r$-matrix $(s_{ij})_{i \notin \sigma}$ form part of basis of $\mathbb{Z}^{m-|\sigma|}$;*

(d) *for any $\sigma \in K$, the row vectors of the $(m - r) \times |\sigma|$-matrix $(\lambda_{ij})_{j \in \sigma}$ span the lattice $\mathbb{Z}^{|\sigma|}$.*

Proof. (a)$\Rightarrow$(b) For every $\sigma \in K$, there exists $\mathbf{z} = (z_1, \ldots, z_m) \in (D^2, S^1)^\sigma$ such that $z_i = 0$ if $i \in \sigma$ and $z_i = 1$ otherwise. Then the stabiliser $H_{\mathbf{z}}$ of $\mathbf{z}$ is $H \cap T^\sigma$, which is trivial if H acts on $\mathcal{Z}_K$ freely.

(a)$\Leftarrow$(b) For any point $\mathbf{z} \in \mathcal{Z}_K$, denote by $\sigma_{\mathbf{z}} = \{i \in [m] \mid |z_i| = 0\}$. Thus the stabiliser $H_{\mathbf{z}}$ of $\mathbf{z}$ is $H \cap T^\sigma$, which is trivial by the assumption. So H acts on $\mathcal{Z}_K$ freely.

(b)$\Leftrightarrow$(c) The condition of (b) is equivalent to that the product $H \times T^\sigma$ is a subtorus of T^m which has rank $r + |\sigma|$. Therefore, $r + |\sigma| \le m$ or $r \le m - |\sigma|$. The associated integral matrix $\bar{S}$ of the inclusion $H \times T^\sigma \longrightarrow T^m$ is an extended matrix of S by adding $|\sigma|$ columns $(0, \cdots, 1, \cdots, 0)^T$ with 1 as the j-th entry for $j \in \sigma$. The product $H \times T^\sigma$ is a subtorus of T^m if and only if the column vectors of $\bar{S}$ form part basis of $\mathbb{Z}^m$, which is equivalent to that the colomn vectors of $(S_{ij})_{i \notin \sigma}$ form part of basis of $\mathbb{Z}^{m-|\sigma|}$.

(b)$\Leftrightarrow$(d) The intersection $T^\sigma \cap H$ is trivial if and only if $T^\sigma \cong (T^\sigma \times H)/H$, which is equivalent to that $(T^\sigma \times H)/H$ is a subtorus of T^m/H. Thus the inclusion $(T^\sigma \times H)/H \longrightarrow T^m/H$ is induced by the composition $T^\sigma \longrightarrow T^m \longrightarrow T^m/H$, which implies that its associated matrix is given by Λ_σ, where $\Lambda_\sigma = (\lambda_{ij})_{j \in \sigma}$. By Lemma 4.1.2, the row vectors of Λ_σ span $\mathbb{Z}^{|\sigma|}$. □

Remark 4.2.2. Since T^m acts on $\mathcal{Z}_K$ coordinatewise, choosing the standard basis of T^m is equivalent to giving an order $(V(K), \preceq)$ on the vertex set of K in the following sense. The spanning lattice of T^m is given by $\mathbb{Z}^m$ and there is a one-to-one correspondence $\mathbf{e}_i \leftrightarrow v_i$, where $\{\mathbf{e}_1, \ldots, \mathbf{e}_m\}$ is the standard basis of $\mathbb{Z}^m$ and we list each vertex of K in an order $v_1 \prec \ldots \prec v_m$. In this way, the matrix Λ is called a $\mathbb{Z}^{m-r}$-color on $V(K)$, as it assigns each vertex $v_i \in V(K)$ to a vector $\lambda_i \in \mathbb{Z}^{m-r}$, where each λ_i is a column vector of Λ.

On the other hand, if give an order $v_1 \prec \ldots \prec v_m$ on $V(K)$ and assign each v_i to a vector λ_i in $\mathbb{Z}^{m-r}$, then we have an $(m-r) \times m$-matrix $\Lambda\colon \mathbb{Z}^m \longrightarrow \mathbb{Z}^{m-r}$ induced by mapping each standard basis vector $\mathbf{e}_i$ to λ_i. Thus, we have an integral matrix $\Lambda\colon \mathbb{Z}^m \longrightarrow \mathbb{Z}^{m-r}$ which gives rise to a homomorphism $T^m \xrightarrow{\Lambda} T^{m-r}$ whose kernel $H = \mathrm{Ker}\Lambda$ is a subgroup of T^m acting on $\mathcal{Z}_K$. To make sure that H is a subtorus, we require Λ to be surjective.

In this sense, one can think of the underlying combinatorial structure of the partial quotient $\mathcal{Z}_K/H$ as a pair (K, Λ), where each vertex of the simplicial complex K is assigned a vector in $\mathbb{Z}^{m-r}$. Choosing an order of vertices of K, an $(m-r) \times m$-integral matrix Λ is obtained whose column vectors are those vectors associated to vertices, listed according to the order of vertices. We say that $\mathcal{Z}_K/H$ does not depend on the order of $V(K)$ because by changing an order of the vertices, the corresponding column vectors of the matrix Λ are permutated. The resulting quotient spaces $\mathcal{Z}_K/H$ and $\mathcal{Z}_{K_1}/H_1$ are homeomorphic, where $\mathcal{Z}_{K_1}/H_1$ is obtained by changing the order of the vertices of K.

For example, the simplicial complex K in Figure 4.1 has same vectors in $\mathbb{Z}^2$ assigned to its vertices. Labelling the vertices of the simplicial complex in two different orders, the resulting simplicial complexes K_1 and K_2 are simplicial isomorphic, as seen in Figure 4.1. As abstract sets, K_1 and K_2 are different.

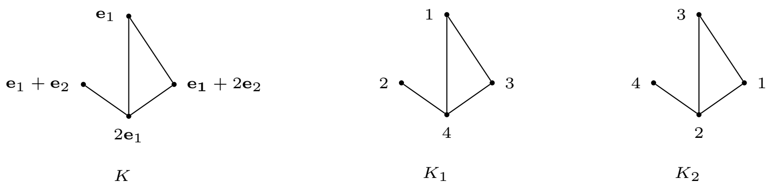


FIGURE 4.1: Simplicial complexes with different orders on the vertices.

The corresponding integral matrices are $\Lambda_1 = \begin{pmatrix} \overset{①}{1} & \overset{②}{1} & \overset{③}{1} & \overset{④}{2} \\ 0 & 1 & 2 & 0 \end{pmatrix}$ and $\Lambda_2 = \begin{pmatrix} \overset{①}{1} & \overset{②}{2} & \overset{③}{1} & \overset{④}{1} \\ 2 & 0 & 0 & 1 \end{pmatrix}$.

In this case, the quotient space $\mathcal{Z}_{K_1}/\mathrm{Ker}\Lambda_1$ is homeomorphic to $\mathcal{Z}_{K_2}/\mathrm{Ker}\Lambda_2$.

An application of Lemma 4.2.1 is that given an integral matrix Λ, one may find a simplicial complex K such that $H = \mathrm{Ker}\Lambda$ acts on $\mathcal{Z}_K$ freely.

Example 4.2.3. Let $\Lambda = (\underbrace{1, \dots, 1}_{m})$. Then by Lemma 4.2.1(d), if K is a disjoint union of m points, there is a free $\mathrm{Ker}\Lambda$-action on $\mathcal{Z}_K$. It is the only possible simplicial complex on m vertices without ghost vertices for which $\mathrm{Ker}\Lambda$ acts freely on $\mathcal{Z}_K$.

Example 4.2.4. Let K be a simplicial complex on $[m]$ without ghost vertices. Yu [43] calculated the cohomology of certain partial quotients $\mathcal{Z}_K/\mathrm{Ker}\Lambda$ (not necessarily by free actions), where Λ is obtained by permutating the column vectors of the following matrix (4.3) and each $\alpha_j \neq 0$.

$$\begin{pmatrix} \underbrace{1\dots1}_{\alpha_1} & 0 & 0 & \dots & 0 \\ 0 & \underbrace{1\dots1}_{\alpha_2} & 0 & \dots & 0 \\ 0 & 0 & \underbrace{1\dots1}_{\alpha_3} & \dots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \dots & \underbrace{1\dots1}_{\alpha_l} \end{pmatrix}_{l\times m} \tag{4.3}$$

If Λ is a matrix of this type, it induces a map f_Λ from $[m]$ to $[l]$. Then $(f_\Lambda^{-1}(1), \dots, f_\Lambda^{-1}(l))$ is a partition of the vertex set $[m]$. By abuse of notion, let $\alpha_j = f_\Lambda^{-1}(j)$.

Given a free $\mathrm{Ker}\Lambda$-action on $\mathcal{Z}_K$, the freeness condition (Lemma 4.2.1) implies that $(i_1, \dots, i_p) \in K$ if and only if each i_j belongs to a different partition α_q. Explicitly, K is a simplicial subcomplex of $[\alpha_1] * \dots * [\alpha_l]$ without ghost vertices (i.e., $\forall i \in [m]$, $i \in K$), where $[\alpha_j]$ denotes a set of α_j disjoint points.

One explicit example of Example 4.2.4 is in the following.

Example 4.2.5. In [10, Example 7.8.17], one illustrates a quasi-toric manifold over a $2k$-gon with a matrix Λ of the form

$$\begin{pmatrix} 1 & 0 & 1 & 0 & \dots & 1 & 0 \\ 0 & 1 & 0 & 1 & \dots & 0 & 1 \end{pmatrix}.$$

This Λ satisfies the condition in Example 4.2.4.

4.3 The Tor-algebras

In this section, we study the Tor-algebra $\mathrm{Tor}_{H^*(B(T^m/H);R)}(R[K], R)$ via the Koszul resolution and the Taylor resolution.

4.3.1 Panov's formula

Recall that if H is a torus subgroup of T^m acting freely on $\mathcal{Z}_K$, then the quotient space $\mathcal{Z}_K/H$ is the homotopy fibre of $DJ_K \xrightarrow{f} B(T^m/H)$ (Lemma 2.3.22, 4.2.1), where f is a composition of the inclusion $DJ_K = (BS^1, *)^K \longrightarrow BT^m$ followed by the quotient map $BT^m \xrightarrow{B\Lambda} B(T^m/H)$. Panov [35] proved that the Eilenberg-Moore spectral sequence associated to this homotopy fibration collapses at the E_2-term. Around the time of my viva, a new preprint [19] stated the cup product of the quotient space $\mathcal{Z}_K/H$ is not the standard multiplication of the Tor-algebra $\mathrm{Tor}_{H^*(B(T^m/H);R)}(R[K],R)$ as in [35]. The results in this section and Section 4.4 will still be true if 2 is a unit of R due to [19].

Theorem 2.3.24. *If H acts freely on $\mathcal{Z}_K$, then there exists an isomorphism of graded R-algebras*

$$H^*(\mathcal{Z}_K/H;R) \cong \mathrm{Tor}_{H^*(B(T^m/H);R)}(R[K],R). \tag{4.4}$$

Here the Stanley-Reisner ring $R[K]$ is an $H^*(B(T^m/H);R)$-module given by an R-homomorphism

$$H^*(B(T^m/H);R) \xrightarrow{B\Lambda^*} H^*(BT^m;R) \xrightarrow{p} R[K]$$

where p is induced by the inclusion $(BS^1,*)^K \longrightarrow BT^m$ and $B\Lambda^*$ is induced by the map $BT^m \xrightarrow{B\Lambda} B(T^m/H)$.

Precisely, since T^m/H is isomorphic to an $(m-r)$-torus group, $H^*(B(T^m/H);R)$ is a polynomial ring over R with $m-r$ variables. As convenient notations, we write $A = H^*(B(T^m/H);R) = R[t_1,\dots,t_{m-r}]$ and $H^*(BT^m;R) = R[v_1,\dots,v_m]$. Then by Lemma 4.1.1, $B\Lambda^*(t_i) = \sum\limits_{j=1}^{m}(\lambda_{ij}\cdot 1_R)v_j$ for $1 \le i \le m-r$ and by Proposition 2.3.25, the map $p\colon R[v_1,\dots,v_m] \longrightarrow R[K] = R[v_1,\dots,v_m]/I_K$ is a quotient homomorphism. The homomorphism $B\Lambda^*$ and the composition $p\circ B\Lambda^*$ define an A-module structure on $R[v_1,\dots,v_m]$ and $R[K]$, respectively.

In this way, the polynomial $R[v_1,\dots,v_m]$ is an A-algebra by sending t_i to $\sum\limits_{j=1}^{m}(\lambda_{ij}\cdot 1_R)v_j$ and the multiplication is induced by the standard multiplication of polynomial rings. The ground ring R is an A-module by mapping each t_i to zero. Since Λ can be extended into an integral invertible matrix, $R[v_1,\dots,v_m]$ is isomorphic as an A-algebra to a polynomial ring with r variables over A. We prove this in the next lemma.

Lemma 4.3.1. *Let $A[x_1,\dots,x_r]$ be a polynomial ring over A. Then there exists an isomorphism of A-algebras between $A[x_1,\dots,x_r]$ and $R[v_1,\dots,v_m]$. Consequently, the tensor product $R[v_1,\dots,v_m]\otimes_A R$ is isomorphic as an R-algebra to $R[x_1,\dots,x_r]$.*

Proof. Let $u_i = \sum\limits_{j=1}^{m}\tilde{\lambda}_{ij}v_j$, where $\tilde{\lambda}_{ij} = \lambda_{ij}$ if $i \le m-r$ and $\tilde{\lambda}_{ij} = \lambda'_{i-m+r,j}$ if $i > m-r$. Note that the matrix $\begin{pmatrix}\Lambda\\ \Lambda'\end{pmatrix}$ has an inverse matrix $M = \begin{pmatrix}S' \mid S\end{pmatrix}$. Hence $R[v_1,\dots,v_m]$ is isomorphic as an R-algebra to $R[u_1,\dots,u_m]$ by changing the basis according to M, i.e., sending v_i to $\sum\limits_{j=1}^{m}(\tilde{s}_{ij}\cdot 1_R)u_j$ where $\tilde{s}_{ij} = s'_{ij}$ if $j \le m-r$ and $\tilde{s}_{ij} = s_{i,j-m+r}$ if $j > m-r$. The composition $M\circ B\Lambda^*$ turns $R[u_1,\dots,u_m]$ into an A-module.

Since for $1 \le i \le r$

$$t_i \xmapsto{B\Lambda^*} \sum_{j=1}^{m}\lambda_{ij}v_j \xmapsto{(S'\mid S)} \sum_{j=1}^{m}\lambda_{ij}\Big(\sum_{k=1}^{m}(\tilde{s}_{jk}\cdot 1_R)u_k\Big) = u_i$$

the composition $M \circ B\Lambda^*$ sends variables $t_1, \ldots, t_r$ of A identically to variables $u_1, \ldots, u_r$. We have that $R[u_1, \ldots, u_m]$ can be seen as a polynomial ring over A with variables $u_{m-r+1}, \ldots, u_m$. Hence $R[u_1, \ldots, u_m]$ is isomorphic as an A-algebra to $A[x_1, \ldots, x_r]$ by letting x_i corresponding to u_{m-r+i}. Therefore, $R[v_1, \ldots, v_m]$ is isomorphic as an A-algebra to $A[x_1, \ldots, x_r]$. Identifying $x_i = u_{m-r+i}$ for $1 \leq i \leq r$, then $R[v_1, \ldots, v_m] \otimes_A R$ is isomorphic as an R-algebra to $R[x_1, \ldots, x_r]$ by sending each v_i to $\sum_{j=1}^{r} (s_{ij} \cdot 1_R) x_j$. □

4.3.2 Calculations of Tor-algebras

In this section, we apply the Taylor resolution and the Koszul resolution to calculate the Tor-algebra (4.4).

Taylor Resolution. Let $\mathbb{P} = \{\sigma_1, \ldots, \sigma_p\}$ be a set consisting of all the minimal missing faces of K. Denote by $\Lambda^*[\mathbb{P}]$ the exterior algebra over R with basis $\mathbb{P}$. Let $A = H^*(B(T^m/H); R)$. Then $\Lambda^i[\mathbb{P}] \otimes R[v_1, \ldots, v_m]$ is a free A-module by Lemma 4.3.1.

Construction 4.3.2 (Taylor resolution [39])**.** The Taylor resolution of $R[K]$ over A is given by

$$\begin{aligned} \ldots \longrightarrow \Lambda^i[\mathbb{P}] \otimes R[v_1, \ldots, v_m] \xrightarrow{\delta} \Lambda^{i-1}[\mathbb{P}] \otimes R[v_1, \ldots, v_m] \xrightarrow{\delta} \\ \ldots \xrightarrow{\delta} \Lambda^1[\mathbb{P}] \otimes R[v_1, \ldots, v_m] \xrightarrow{\delta} R[v_1, \ldots, v_m] \longrightarrow R[K] \longrightarrow 0 \end{aligned} \tag{4.5}$$

where for $\sigma_{j_1} \ldots \sigma_{j_q} \in \Lambda^q[\mathbb{P}]$,

$$\delta(\sigma_{j_1}\sigma_{j_2}\cdots\sigma_{j_q}) = \sum_{0 \leq p \leq q} (-1)^{p-1} \frac{\mathrm{lcm}(v_{\sigma_{j_1}}, v_{\sigma_{j_2}}, \cdots, v_{\sigma_{j_q}})}{\mathrm{lcm}(v_{\sigma_{j_1}}, \cdots, \widehat{v}_{\sigma_{j_p}}, \cdots, v_{\sigma_{j_q}})} \sigma_{j_1} \cdots \widehat{\sigma}_{j_p} \cdots \sigma_{j_q}$$

$$\delta(v_i) = 0, \; 0 \leq i \leq m$$

and $\widehat{\sigma}_{j_p}$ and $\widehat{v}_{\sigma_{j_p}}$ indicate that elements are omitted.

Let $\sigma_I = \sigma_{i_1} \ldots \sigma_{i_q} \in \Lambda^q[\mathbb{P}]$, let $\sigma_J = \sigma_{j_1} \ldots \sigma_{j_l} \in \Lambda^l[\mathbb{P}]$ and let $I = \{i_1, \ldots, i_q\}$, $J = \{j_1, \ldots, j_l\}$. The multiplication $\sigma_I \times \sigma_J$ on the Taylor resolution (4.5) is defined by

$$\sigma_I \times \sigma_J = \begin{cases} \frac{\mathrm{lcm}(v_{\sigma_{i_1}}, \ldots, v_{\sigma_{i_q}})\mathrm{lcm}(v_{\sigma_{j_1}}, \ldots, v_{\sigma_{j_q}})}{\mathrm{lcm}(v_{\sigma_{i_1}}, \ldots, v_{\sigma_{i_p}}, v_{\sigma_{j_1}}, \ldots, v_{\sigma_{j_l}})} \sigma_I \sigma_J & \text{if } I \cap J = \emptyset \\ 0 & \text{otherwise.} \end{cases} \tag{4.6}$$

The bigrading of $\Lambda^*[\mathbb{P}] \otimes R[v_1, \ldots, v_m]$ is defined by

$$\mathrm{bideg}\, \sigma_{i_1} \ldots \sigma_{i_q} = (-q, 2|\sigma_{i_1} \cup \ldots \cup \sigma_{i_q}|) \text{ and } \mathrm{bideg}\, v_j = (0, 2).$$

Due to Theorem 2.3.24, we obtain the following statement.

Theorem 4.3.3. *Let H be a torus subgroup of T^m of rank r which acts freely on $\mathcal{Z}_K$. Let $S = (s_{ij})_{m\times r}$ denote the associated integral matrix $H \leqslant T^m$. Then there exist isomorphisms of R-algebras*

$$H^*(\mathcal{Z}_K/H; R) \cong \mathrm{Tor}_A(R[K], R) \cong H(\Lambda^*[\mathbb{P}] \otimes R[x_1, \dots, x_r], d) \tag{4.7}$$

where $(\Lambda^*[\mathbb{P}] \otimes R[x_1, \dots, x_r], d)$ is defined by

$$d(\sigma_{i_1} \dots \sigma_{i_p}) = \sum_{1 \le t \le p} (-1)^{t-1} \delta_t(x_1, \dots, x_r) \sigma_{i_1} \dots \hat{\sigma}_{i_t} \dots \sigma_{i_p}$$
$$dx_j = 0 \quad 1 \le j \le r$$

and $\delta_t(x_1, \dots, x_r) = \prod_{i \in \mathcal{S}_\sigma \setminus \mathcal{S}_{\partial_t \sigma}} (\sum_{j=1}^{r} (s_{ij} \cdot 1_R) x_j)$ if $\mathcal{S}_\sigma \neq \mathcal{S}_{\partial_t \sigma}$ and 1 otherwise. Here $\mathcal{S}_\sigma = \sigma_{i_1} \cup \dots \cup \sigma_{i_p}$ and $\mathcal{S}_{\partial_t \sigma} = \sigma_{i_1} \cup \dots \cup \widehat{\sigma}_{i_t} \cup \dots \cup \sigma_{i_p}$.

Proof. Applying $- \otimes_A R$ to (4.5), we obtain a differential graded R-algebra

$$(\Lambda^*[\mathbb{P}] \otimes R[v_1, \dots, v_m] \otimes_A R, \delta \otimes_A \mathrm{id}). \tag{4.8}$$

Identifying $R[v_1, \dots, v_m] \otimes_A R \cong R[x_1, \dots, x_r]$ as R-algebras, (4.8) reduces to a differential graded R-algebra $(\Lambda^*[\mathbb{P}] \otimes R[x_1, \dots, x_r], d)$

$$\dots \longrightarrow \Lambda^p[\mathbb{P}] \otimes R[x_1, \dots, x_r] \xrightarrow{d} \dots \xrightarrow{d} \Lambda^1[\mathbb{P}] \otimes R[x_1, \dots, x_r] \xrightarrow{d} R[x_1, \dots, x_r] \longrightarrow 0$$

where the differential map d is given by

$$d(\sigma_{i_1} \dots \sigma_{i_p}) = \sum_{1 \le t \le i} (-1)^{t-1} \delta_t(x_1, \dots, x_r) \sigma_{i_1} \dots \hat{\sigma}_{i_t} \dots \sigma_{i_p}$$

and $\delta_t(x_1, \dots, x_r) = \prod_{i \in \mathcal{S}_\sigma \setminus \mathcal{S}_{\partial_t \sigma}} (\sum_{j=1}^{r} (s_{ij} \cdot 1_R) x_j)$ if $\mathcal{S}_\sigma \neq \mathcal{S}_{\partial_t \sigma}$ and 1 otherwise.

The multiplication $\sigma_I \times \sigma_J$ on $(\Lambda^*[\mathbb{P}] \otimes R[x_1, \dots, x_r], d)$ is induced from the multiplication (4.6) on the Taylor resolution, defined by

$$\sigma_I \times \sigma_J = \begin{cases} \prod_{i \in \mathcal{S}_{\sigma_I} \cap \mathcal{S}_{\sigma_J}} (\sum_{j=1}^{r} (s_{ij} \cdot 1_R) x_j) \sigma_I \sigma_J & \text{if } I \cap J = \emptyset \text{ and } \mathcal{S}_{\sigma_I} \cap \mathcal{S}_{\sigma_J} \neq \emptyset \\ \sigma_I \sigma_J & \text{if } I \cap J = \mathcal{S}_{\sigma_I} \cap \mathcal{S}_{\sigma_J} = \emptyset \\ 0 & \text{otherwise} \end{cases}$$

where σ_I and σ_J are the same notation as (4.6) and $\mathcal{S}_{\sigma_I} = \sigma_{i_1} \cup \dots \cup \sigma_{i_q}$ and $\mathcal{S}_{\sigma_J} = \sigma_{j_1} \cup \dots \cup \sigma_{j_l}$. □

Let us call the algebra $\Lambda^*[\mathbb{P}] \otimes R[x_1, \dots, x_r]$ the *Taylor algebra*.

Example 4.3.4 (complex projective spaces)**.** Let K be the boundary of a simplex Δ^{m-1}. Then $\mathbb{P} = \{\sigma\}$ where $\sigma = \{1, 2, \ldots, m\}$. The quotient space $\mathcal{Z}_K/S^1_d$ under the diagonal action is $\mathbb{C}P^{m-1}$ (Example 2.2.15). By (4.7), there is a differential graded R-algebra,

$$0 \longrightarrow \Lambda^1[\sigma] \otimes R[x] \xrightarrow{d} R[x] \longrightarrow 0,$$

where $d(\sigma) = x^{|\sigma|} = x^m$. Thus, the cohomology is given by $H^*(\mathcal{Z}_K/S^1_d; R) = R[x]/(x^m)$ and $\deg x = 2$.

More generally, we will apply Theorem 4.3.3 to the case when K is a triangulation of an $(n-1)$-sphere with m vertices. Consider a linear sequence in $\mathbb{Z}[K]$

$$t_i = \lambda_{i1}v_1 + \ldots + \lambda_{im}v_m, \; 1 \leq i \leq n \tag{4.9}$$

which gives an $n \times m$-integral matrix $\Lambda = (\lambda_{ij})$. In [10, Theorem 4.8.7], it has been proven that the linear sequence $(t_1, \ldots, t_n)$ is a linear system of parameters if and only if the kernel of $\Lambda\colon T^m \longrightarrow T^n$ is an $(m-n)$-torus acting on $\mathcal{Z}_K$ freely. Moreover, recall that an algebra is Cohen-Macaulay if it is a free finitely generated module over a polynomial subalgebra and a simplicial complex K is called Cohen-Macaulay if $\mathbf{k}[K]$ is a Cohen-Macaulay algebra for $\mathbf{k} = \mathbb{Q}$ or any finite field ([10, Definition 3.3.5, Proposition A.3.13]). Any triangulation of a sphere is Cohen-Macaulay ([10, Corollary 3.3.17]).

Therefore, if K is a triangulation of an $(n-1)$-sphere with m vertices and a sub-torus group $H \cong T^{m-n}$ acts on $\mathcal{Z}_K$ freely, then the freeness implies that the linear sequence $(t_1, \ldots, t_n)$ (4.9) is a linear system of parameters. Since $\mathbf{k}[K]$ is Cohen-Macaulay over $\mathbf{k}$, $\mathbf{k}[K]$ is a free $\mathbf{k}[t_1, \ldots, t_n]$-module ([10, Proposition A.3.3]). Hence, $\mathrm{Tor}^{-i,2j}_{H^*(B(T^m/H);\mathbf{k})}(\mathbf{k}[K], \mathbf{k})$ is trivial if $i \geq 1$ and $H^*(\mathcal{Z}_K/H; \mathbf{k}) \cong \mathrm{Tor}^{0,*}_{H^*(B(T^m/H);\mathbf{k})}(\mathbf{k}[K], \mathbf{k})$.

Corollary 4.3.5. *Let K be a triangulation of an $(n-1)$-sphere with m vertices and let H be an $(m-n)$-torus subgroup of T^m acting freely on $\mathcal{Z}_K$. Then the cohomology ring of $\mathcal{Z}_K/H$ is given by*

$$H^*(\mathcal{Z}_K/H; \mathbf{k}) = \mathbf{k}[x_1, \ldots, x_{m-n}]/(a_{\sigma,S})_{\sigma\in\mathbb{P}}$$

where $(a_{\sigma,S})_{\sigma\in\mathbb{P}}$ is the ideal generated by $a_{\sigma,S} = \prod_{i\in\sigma}(\sum_{j=1}^{m-n}(s_{ij}\cdot 1_{\mathbf{k}})x_j)$ and $\deg x_j = 2$.

Proof. According to (4.7), $\mathrm{Tor}^0_{H^*(B(T^m/H);R)}(R[K], R)$ is given by the quotient ring $R[x_1, \ldots, x_{m-n}]/\mathrm{Im}d$ where $\mathrm{Im}d$ is the ideal generated by $d(\sigma) = \prod_{i\in\sigma}(\sum_{j=1}^{m-n}(s_{ij}\cdot 1_{\mathbf{k}})x_j)$ for every $\sigma \in \mathbb{P}$. □

Koszul Resolution. The Koszul resolution of R over $R[t_1, \ldots, t_{m-r}]$ is defined by

$$(\Lambda[u_1, \ldots, u_{m-r}] \otimes R[t_1, \ldots, t_{m-r}], d), \; du_i = t_i \text{ and } dt_i = 0.$$

Tensoring it with $-\otimes_{R[t_1,\ldots,t_{m-r}]} R[K]$, we have the statement below.

Proposition 4.3.6. *Let H be a subtorus of T^m and Λ be the corresponding integral matrix of the projection $T^m \longrightarrow T^m/H$. Then there is an isomorphism of graded algebras*

$$H^*(\mathcal{Z}_K/H;R) \cong \mathrm{Tor}_{H^*(B(T^m/H);R)}(R[K],R) \cong H(\Lambda[u_1,\ldots,u_{m-r}]\otimes R[K],d) \quad (4.10)$$

where the differentials are defined by $du_i = (\lambda_{i1}\cdot 1_R)v_1 + \ldots + (\lambda_{im}\cdot 1_R)v_m$ and $dv_j = 0$.

Proof. For convenience, write $A = H^*(B(T^m/H);R) = R[t_1,\ldots,t_{m-r}]$. Recall that in Section 4.3.1, $R[K]$ is an A-algebra given by the homomorphism

$$\begin{aligned} f\colon R[t_1,\ldots,t_{m-r}] &\longrightarrow R[K] \\ t_i &\longmapsto \sum_{j=1}^{m}(\lambda_{ij}\cdot 1_R)v_j. \end{aligned}$$

There is an isomorphism of R-algebras $\beta\colon A\otimes_A R[K] \longrightarrow R[K]$, $\beta(a\otimes_A b) = f(a)b$ where $a\in A$ and $b\in R[K]$. Its algebraic inverse homomorphism is given by $\beta^{-1}(b) = 1_R\otimes_A b$. Both β and β^{-1} are R-homomorphism of algebras by properties of the tensor product. Thus there is an isomorphism of R-algebras

$$\begin{aligned} \Lambda[u_1,\ldots,u_{m-r}]\otimes A\otimes_A R[K] \cong& \Lambda[u_1,\ldots,u_{m-r}]\otimes R[K] \\ u_i\otimes a\otimes_A b \mapsto& u_i\otimes f(a)b. \end{aligned}$$

Hence the differential d on $\Lambda[u_1,\ldots,u_{m-r}]\otimes R[K]$ induced by the Koszul resolution is given by

$$du_i = f(t_i) = (\lambda_{i1}\cdot 1_R)v_1 + \ldots + (\lambda_{im}\cdot 1_R)v_m.$$

□

The algebra $\Lambda[u_1,\ldots,u_{m-r}]\otimes R[K]$ is called *Koszul algebra*, whose multiplication is the tensor product of R-algebras (2.3).

Example 4.3.7. The diagonal action on $\mathcal{Z}_K$ is always free. In this case, the associated integral matrix Λ induced by the quotient map $T^m \longrightarrow T^m/S^1_d$ is $\big(I_{m-1} \mid -\mathbf{1}\big)$, where $-\mathbf{1} = (-1,\ldots,-1)^t$. Thus the Koszul algebra $(\Lambda[u_1,\ldots,u_{m-1}]\otimes R[K],d)$ is given by

$$du_i = v_i - v_m \text{ for } 1\le i\le m-1 \text{ and } dv_j = 0 \text{ for } 1\le j\le m.$$

By Proposition 4.3.6, $H^*(\mathcal{Z}_K/S^1_d;R) \cong H_*(\Lambda[u_1,\ldots,u_{m-1}]\otimes R[K],d)$.

Bigradings. Unlike moment-angle complexes (Construction 2.1.24, 2.1.26) and quotient spaces of real moment-angle complexes ([13, Theorem 4.6]), there are no analogous multigrading structures on the Taylor algebra (4.7) and the Koszul algebra (4.10) which are closed under the differentials in general. However, the bigrading structures exist.

The bigradings of generators of the Taylor algebra $(\Lambda^*[\mathbb{P}] \otimes R[x_1, \dots, x_r], d)$ (4.7) are defined by

$$\text{bideg } \sigma_{i_1} \dots \sigma_{i_q} = (-p, 2|\sigma_{i_1} \cup \dots \cup \sigma_{i_q}|) \text{ for each } \sigma_{i_j} \in \mathbb{P} \text{ and bideg } x_j = (0, 2).$$

The bigradings of u_j and v_i in the Koszul algebra $(\Lambda^*[u_1, \dots, u_{m-r}] \otimes R[K], d)$ (4.10) are defined by

$$\text{bideg } u_j = (-1, 2) \text{ and bideg } v_i = (0, 2).$$

The Eilenberg-Moore spectral sequence indicates how the cohomology degree of $\mathcal{Z}_K/H$ corresponds to the bigrading of the Tor-algebra. Let $\Lambda^{-i,2j}[\mathbb{P}]$ and $\Lambda^{-i,2j}[u_1, \dots, u_{m-r}]$ be the corresponding R-submodule of degree $(-i, 2j)$. Similarly, let $R^{2t}[x_1, \dots, x_r]$ and $R^{2t}[K]$ be the corresponding R-submodule of degree $2t$. Then the next result follows immediately.

Theorem 4.3.8. *Let K be a simplicial complex on $[m]$ and H be a subtorus group of T^m of rank r acting on $\mathcal{Z}_K$ freely. Then there exist isomorphisms of groups*

$$\begin{aligned} H^q(\mathcal{Z}_K/H; R) &\cong \bigoplus_{-i+2j=q} \operatorname{Tor}^{-i,2j}_{H^*(B(T^m/H);R)}(R[K], R) \\ &\cong \bigoplus_{-i+2j'+2t=q} H_i(\Lambda^{-i,2j'}[\mathbb{P}] \otimes R^{2t}[x_1, \dots, x_r], d) \\ &\cong \bigoplus_{i+2t=q} H_i(\Lambda^{-i,2i}[u_1, \dots, u_{m-r}] \otimes R^{2t}[K], d). \end{aligned}$$

Proof. Let $C^*(\cdot)$ denote the singular cochain functor with coefficients in R. The algebraic isomorphism (4.4) in [35] is established by showing that the Eilenberg-Moore spectral sequence associated to the homotopy fibration $\mathcal{Z}_K/H \longrightarrow DJ_K \xrightarrow{f} B(T^m/H)$ collapses at the E_2-term and the existence of a multiplicative isomorphism

$$\Phi\colon \operatorname{Tor}_{H^*(B(T^m/H);R)}(R[K], R) \longrightarrow \operatorname{Tor}_{C^*(B(T^m/H))}(C^*(DJ_K), R).$$

Consider the pullback square

$$\begin{array}{ccc} X & \xrightarrow{\tilde{f}} & E(T^m/H) \\ \downarrow g & & \downarrow \\ DJ_K & \xrightarrow{f} & B(T^m/H) \end{array}$$

where f is a composite $DJ_K \hookrightarrow BT^m \xrightarrow{B\Lambda} B(T^m/H)$. Recall that in [30, Theorem 7.14] (the second Eilenberg-Moore theorem), there exists a filtration-preserving map

$$\theta\colon \text{total}(Q^\bullet) \otimes_{C^*(B(T^m/H);R)} C^*(E(T^m/H)) \longrightarrow C^*(X)$$

which induces an isomorphism on cohomology

$$\theta^*\colon \mathrm{Tor}_{C^*(B(T^m/H))}(C^*(DJ_K), C^*(E(T^m/H))) \longrightarrow H^*(X;R) = H^*(\mathcal{Z}_K/H;R).$$

Here $Q^\bullet \xrightarrow{\epsilon} C^*(DJ_K;R) \longrightarrow 0$ is a proper projective resolution ([30, Definition 2.18]) of $C^*(DJ_K)$ over $C^*(B(T^m/H))$. And $Q^\bullet$ forms a bicomplex. Hence there is an associated total complex $(\mathrm{total}(Q^\bullet), d)$, where $\mathrm{total}(Q^\bullet)^q = \bigoplus_{-j_1+2j_2=q} Q^{-j_1,2j_2}$. Then the map θ is defined to be a composition

$$\begin{array}{ccc} \mathrm{total}(Q^\bullet) \otimes_{C^*(B(T^m/H))} C^*(E(T^m/H)) & & \\ \downarrow{\scriptstyle \epsilon\otimes\mathrm{id}} & \searrow^{\theta} & \\ C^*(DJ_K) \otimes_{C^*(B(T^m/H))} C^*(E(T^m/H)) & \xrightarrow[\bar\alpha]{} & C^*(X) \end{array}$$

where $\bar\alpha$ is induced by the composition

$$\bar\alpha\colon C^*(DJ_K) \otimes C^*(E(T^m/H)) \xrightarrow{g^*\otimes\tilde f^*} C^*(X) \otimes C^*(X) \xrightarrow{\times} C^*(X).$$

Since $E(T^m/H)$ is contractible, there exists a chain equivalence $C^*(E(T^m/H)) \simeq R$.

Therefore, we have

$$\begin{aligned} H^q(\mathcal{Z}_K/H;R) \cong & H^q(\mathrm{total}(Q^\bullet) \otimes_{C^*(B(T^m/H))} R, d') \\ \cong & \bigoplus_{-i+2j=q} \mathrm{Tor}^{-i,2j}_{C^*(B(T^m/H))}(C^*(BT^K), R) \end{aligned}$$

where the second isomorphism is valid since the bigrading of the Tor-module is defined by the bigrading of $Q^\bullet$. As the multiplicative isomorphism Φ is degree-preserving, we have

$$\mathrm{Tor}^{-i,2j}_{C^*(B(T^m/H))}(C^*(DJ_K), R) \cong \mathrm{Tor}^{-i,2j}_{H^*(B(T^m/H);R)}(H^*(DJ_K;R), R) = \mathrm{Tor}^{-i,2j}_{H^*(B(T^m/H);R)}(R[K], R).$$

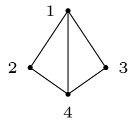
The bigradings of Tor-algebra $\mathrm{Tor}_{H^*(B(T^m/H);R)}(R[K], R)$ correspond the bigradings of the Taylor algebra (4.7) and the Koszul algebra (4.10). That is to say,

$$\begin{aligned} \mathrm{Tor}^{-i,2j}_{H^*(B(T^m/H);R)}(R[K], R) \cong & \bigoplus_{j'+t=j} H_i(\Lambda^{-i,2j'}[\mathbb{P}] \otimes R^{2t}[x_1, \ldots, x_r], d) \\ \cong & \bigoplus_{i+t=j} H_i(\Lambda^{-i,2i}[u_1, \ldots, u_{m-r}] \otimes R^{2t}[K], d). \end{aligned}$$

□

Example 4.3.9. Let K be the following simplicial complex.

K:



Then the minimal missing faces of K are $\mathbb{P} = \{\sigma_{23}, \sigma_{124}, \sigma_{134}\}$. We calculate the cohomology of the quotient $\mathcal{Z}_K/S^1_d$ under the diagonal action. The Taylor algebra (4.7) is

$$0 \longrightarrow \Lambda^3[\mathbb{P}] \otimes R[x] \xrightarrow{d_3} \Lambda^2[\mathbb{P}] \otimes R[x] \xrightarrow{d_2} \Lambda^1[\mathbb{P}] \otimes R[x] \xrightarrow{d_1} R[x] \longrightarrow 0.$$

The nontrivial differentials are

$$\begin{aligned} &d\sigma_{23} = x^2,\ d\sigma_{124} = x^3,\ d\sigma_{134} = x^3, && d\sigma_{23}\sigma_{124} = (\sigma_{124} - \sigma_{23}x)x, \\ &d\sigma_{23}\sigma_{134} = (\sigma_{134} - \sigma_{23}x)x, && d\sigma_{124}\sigma_{134} = (\sigma_{134} - \sigma_{124})x, \\ &d\sigma_{23}\sigma_{124}\sigma_{134} = \sigma_{124}\sigma_{134} - \sigma_{23}\sigma_{134} + \sigma_{23}\sigma_{124}. \end{aligned}$$

Hence, the homology groups are

$$H_i(\Lambda^*[\mathbb{P}] \otimes R[x], d) = \begin{cases} R[x]/(x^2) & \text{if } i = 0 \\ R \cdot [\sigma_{23}x - \sigma_{124}] \oplus R \cdot [\sigma_{23}x - \sigma_{134}] & \text{if } i = 1 \\ 0 & \text{otherwise} \end{cases}$$

where $R \cdot [a]$ is denoted by the R-module with the generator $[a]$.

Note that $\operatorname{bideg} x = (0, 2)$ and $\operatorname{bideg}(\sigma_{23}x - \sigma_{124}) = \operatorname{bideg}(\sigma_{23}x - \sigma_{134}) = (-1, 6)$. Therefore, we have

$$H^i(\mathcal{Z}_K/S^1_d; R) = \begin{cases} R & \text{if } i = 0, 2 \\ R \oplus R & \text{if } i = 5 \\ 0 & \text{otherwise} \end{cases}$$

in accordance with Example 2.3.30.

Example 4.3.10. Let $\mathcal{Z}_m$ be the moment-angle complex corresponding to m disjoint point. We calculate the cohomology of $\mathcal{Z}_K/\mathrm{Ker}\Lambda$, where $\Lambda = (\underbrace{1, \ldots, 1}_{m})$ (Example 4.2.3).

In this case, the Stanley-Reisner ring $R[K] = R[m]/(v_i v_j, 1 \leq i < j \leq m)$ and the Koszul algebra (4.10) is given by $(\Lambda^*[u] \otimes R[K], d)$ where $du = v_1 + \ldots + v_m$ and $d(uv_j) = v_j^2$ for $1 \leq j \leq m$. Thus, the homology groups are $H_0(\Lambda^*[u] \otimes R[K], d) \cong R[K]/(v_1 + \ldots + v_m, v_j^2, 1 \leq j \leq m)$ and zero, otherwise. This implies that

$$H^i(\mathcal{Z}_K/\mathrm{Ker}\Lambda; R) = \begin{cases} R, & \text{if } i = 0, \\ R^{\oplus(m-1)} & \text{if } i = 2, \\ 0 & \text{otherwise.} \end{cases}$$

Note that in the case $m = 2$, we have $\mathcal{Z}_K \cong S^3$ and $\mathcal{Z}_K/S^1 \simeq S^2$.

4.3.3 Yu's formula

We will apply Proposition 4.3.6 to compute $H^*(\mathcal{Z}_K/H;R)$ in Example 4.2.4 when the subtorus $H = \mathrm{Ker}\Lambda$ acts on $\mathcal{Z}_K$ freely. See [43, Theorem 1.2] for a topological approach including non-free actions. Recall that in Example 4.2.4, Λ has a particular form. Let $\mathrm{Row}\Lambda$ be a subspace of $\mathbb{Z}_2^m$ spanned by the row vectors of Λ. Then under the bijection of the vector space $\mathbb{Z}_2^m$ and the power set $2^{[m]}$ of $[m]$, the row vector space $\mathrm{Row}\Lambda$ corresponds to a subset $\mathcal{J}$ of $2^{[m]}$ consisting of $\{J \subseteq [m] \mid$ if $i \in \alpha_p$ and $i \in J$, then $\alpha_p \subseteq J\}$. By an abuse of notation, we use $\mathrm{Row}\Lambda$ to denote the set $\mathcal{J}$. Note that there is a bijection ϕ between $\mathrm{Row}\Lambda$ and the power set $2^{[l]}$ of $[l]$

$$\begin{array}{rccl} \phi\colon & \mathrm{Row}\Lambda & \longrightarrow & 2^{[l]} \\ & J & \longmapsto & \{p \mid i \in \alpha_p \text{ and } i \in J\} \end{array} \tag{4.11}$$

where l is the number of partitions of $V(K)$. Alternatively, this bijection $\phi\colon \mathrm{Row}\Lambda \longrightarrow 2^{[l]}$ is given by sending the vector $k_1\lambda_1 + \ldots + k_l\lambda_l$ to $\{i \mid k_i \neq 0\}$, where λ_i are the row vectors of Λ. The following result is known as [43, Theorem 1.2]. We give a different proof in the case of a free $\mathrm{Ker}\Lambda$-action as follows.

Proposition 4.3.11. *Let (K, Λ) satisfy the condition in Example 4.2.4. Then there exists an isomorphism of graded R-algebras*

$$H^*(\mathcal{Z}_K/\mathrm{Ker}\Lambda; R) \cong H(\Lambda[u_1, \ldots, u_l] \otimes R[K], d)$$

where the differentials are $du_i = \sum_{j\in\alpha_i} v_j, dv_p = 0, 1 \leq i \leq l, 1 \leq p \leq m$. Moreover, there are isomorphisms of homology groups

$$H^q(\mathcal{Z}_K/\mathrm{Ker}\Lambda; R) \cong \bigoplus_{J \in \mathrm{Row}\Lambda} \widetilde{H}^{q-|\phi(J)|-1}(K_J; R).$$

Next, we will prove Proposition 4.3.11 in a few lemmas, an analogue of the proof of Hochster's formula shown in [10, Section 3.2]. A sketch of this proof is contained in Section 2.1.2 of this thesis.

Construction 4.3.12. Define a quotient algebra $\mathcal{Q}^*(K)$

$$\mathcal{Q}^*(K) = \Lambda[u_1, \ldots, u_l] \otimes R[K]/(u_iv_j = v_p^2 = 0, j \in \alpha_i, 1 \leq i \leq l, 1 \leq p \leq m)$$

where $\mathcal{Q}^*(K)$ has a basis $\{u_Jv_\sigma \mid J \cap I_\alpha(\sigma) = \emptyset$ and $\sigma \in K\}$ as an R-module. The ideal $(u_iv_j = v_p^2 = 0, j \in \alpha_i, 1 \leq i \leq l, 1 \leq p \leq m)$ is closed under the differential, since $du_iv_j = v_j^2$ for $j \in \alpha_i$ and $i \in [l]$. Thus $\mathcal{Q}^*(K)$ is a differential graded algebra. Recall that $R[K]$ has a basis consisting 1_R and $v_{i_1}^{b_1} \ldots v_{i_p}^{b_p}$ with $b_j > 0$ for $\{i_1, \ldots, i_p\} \in K$ ([10, Proposition 3.1.9]). By the construction of K, if $\sigma \in K$, then each $i \in \sigma$ is from different partition α_p. For $\sigma = \{i_1, \ldots, i_p\}$, denote by $I_\alpha(\sigma) = \{j \in [l] \mid \alpha_j \cap \sigma \neq \emptyset\}$. There is a multigrading on v_σ in $\mathcal{Q}^*(K)$ defined by $\mathrm{mdeg} v_\sigma = (2a_1, \dots, 2a_l)$ where $a_i = 1$ if $i \cap I_\alpha(\sigma) \neq \emptyset$ and zero otherwise. It gives a $\mathbb{Z} \oplus \mathbb{N}^l$-grading on $\mathcal{Q}^*(K)$ by

$$\mathrm{mdeg} u_J v_\sigma = (-|J|, J \cup I_\alpha(\sigma)).$$

There exist a quotient homomorphism $\rho\colon \Lambda[u_1, \dots, u_l] \otimes R[K] \longrightarrow \mathcal{Q}^*(K)$ and an inclusion of R-modules $\iota\colon \mathcal{Q}^*(K) \longrightarrow \Lambda[u_1, \dots, u_l] \otimes R[K]$ with $\rho\iota = \mathrm{id}$.

Lemma 4.3.13. *There exists a chain homotopy* $s\colon \Lambda[u_1, \dots, u_l] \otimes R[K] \longrightarrow \Lambda[u_1, \dots, u_l] \otimes R[K]$ *such that* $ds - sd = \mathrm{id} - \iota\rho$*. Consequently, the projection* $\Lambda[u_1, \dots, u_l] \otimes R[K] \longrightarrow \mathcal{Q}^*[K]$ *induces isomorphisms on homology groups.*

Proof. The proof is similar to [10, Lemma 3.2.6]. First consider a special case when $K = [\alpha_1] * \dots * [\alpha_l]$. Let $E_l = \Lambda[u_1, \dots, u_l] \otimes R[[\alpha_1] * \dots * [\alpha_l]]$. If $l = 1$, K is a disjoint union of α_1 points. Denote by the corresponding Stanley-Reisner ring $R[\alpha_1]/I_{\alpha_1} = R[v_1, \dots, v_{\alpha_1}]/(v_i v_j, i \neq j)$. Let $x = a_0 + \sum_{i=1}^{\alpha_1} \sum_{j=1}^{q} a_{ji} v_i^j \in R[\alpha_1]/I_{\alpha_1}$. Define

$$s_1\colon E_1^{-i,*} \longrightarrow E_1^{-i-1,*}$$

by $s_1(x) = u_1[(a_{21}v_1 + \dots + a_{2\alpha_1}v_{\alpha_1}) + \dots + (a_{j1}v_1^{j-1} + \dots + a_{j\alpha_1}v_{\alpha_1}^{j-1})]$ and $s_1(ux) = 0$. The map s_1 satisfies $ds_1 - s_1 d = \mathrm{id} - \iota \circ \rho$. For a general l, since $E_l = E_{l-1} \otimes E_1$, we can inductively define $s_l = s_{l-1} \otimes \mathrm{id} + \iota_{l-1} \circ \rho_{l-1} \otimes s_1$, which is a chain homotopy between id and $\iota_l \circ \rho_l$.

Now consider any K with $V(K) = [\alpha_1] \sqcup \dots \sqcup [\alpha_l] \subseteq K \subseteq [\alpha_1] * \dots * [\alpha_l]$. Let I'_K be the ideal in E_l generated by the monomials v_σ, where $\sigma \notin K$ and each $i \in \sigma$ is from a distinct partition α_p. In this case, the Koszul algebra $\Lambda[u_1, \dots, u_l] \otimes R[K] = E_l/I'_K$. To define the chain homotopy s to $\Lambda[u_1, \dots, u_l] \otimes R[K]$, it suffices to show that

$$d(I'_K) \subseteq I'_K, \iota \circ \rho(I'_K) \subseteq I'_K, \text{ and } s(I'_K) \subseteq I'_K. \tag{4.12}$$

As an R-module, I'_K has a basis $\{u_J v_{i_1}^{b_1} \dots v_{i_p}^{b_p}\}$, where $\{i_1, \dots, i_p\} \notin K$ with each i_q sitting in a different partition α_t and $b_q > 0$. Note that $d(u_J v_{i_1}^{b_1} \dots v_{i_p}^{b_p}) = \sum_{j \in J} \pm u_{J\setminus j}(\sum_{i \in \alpha_j} v_i v_{i_1}^{b_1} \dots v_{i_p}^{b_p})$. Also notice that $v_i v_{i_1}^{b_1} \dots v_{i_p}^{b_p} = 0$ if there exists some i_q $(1 \leq q \leq p)$ such that i and i_q are from the same partition by the construction of I'_K and E_l. Thus $d(u_J v_{i_1}^{b_1} \dots v_{i_p}^{b_p}) \in I'_K$. So $d(I'_K) \subseteq I'_K$. We also have that

$$\iota\rho(u_J v_{i_1}^{b_1} \dots v_{i_p}^{b_p}) = \begin{cases} u_J v_{i_1} \dots v_{i_p} & \text{if } b_q = 1 \text{ and } J \cap I_\alpha(\sigma) = \emptyset \\ 0, & \text{otherwise.} \end{cases}$$

Hence, $\iota\rho(I'_K) \subseteq I'_K$. By the inductive definition of s_l,

$$s(u_J v_{i_1}^{b_1} \dots v_{i_p}^{b_p}) = \sum_{i_q \in \alpha_j, b_q > 1} \pm\, u_J u_j v_{i_1}^{b_1} \dots v_{i_q}^{b_q - 1} \dots v_{i_p}^{b_p}.$$

The third inclusion in (4.12) follows. □

For $L \subseteq [l]$, let $\mathcal{Q}^{*,L} = \mathrm{Span}_R\{u_{L \setminus I_\alpha(\sigma)} v_\sigma \mid \sigma \in K, I_\alpha(\sigma) \subseteq L\}$. It is a chain subcomplex since

$$du_{L \setminus I_\alpha(\sigma)} v_\sigma = \sum_{k \in L \setminus I_\alpha(\sigma)} (\pm 1) u_{L \setminus (I_\alpha(\sigma) \cup k)} (\sum_{j \in \alpha_k} v_j) v_\sigma$$

where $v_j v_\sigma$ is zero unless $j \cup \sigma \in K$. By the construction of K, if $j \cup \sigma \in K$ then $I_\alpha(j \cup \sigma) = k \cup I_\alpha(\sigma)$. Thus, $du_{L \setminus I_\alpha(\sigma)} v_\sigma \in \mathcal{Q}^{*,L}$. Moreover, the chain complex $(\mathcal{Q}^*(K), d)$ has a splitting of chain subcomplexes $\mathcal{Q}^*(K) = \bigoplus\limits_{L \subseteq [l]} \mathcal{Q}^{*,L}(K)$. In the following statement, we prove that $\mathcal{Q}^{*,L}(K)$ is isomorphic to $C^*(K_{\phi^{-1}(L)}; R)$ as cochain complexes, where ϕ is bijection (4.11) between $\mathrm{Row}\Lambda$ and subsets of $[l]$ and $K_{\phi^{-1}(L)}$ is the full subcomplex of K on $\phi^{-1}(L)$.

For $I \subseteq J \subseteq [m]$, define the sign function $\epsilon(I, J) = \prod\limits_{i \in I} \epsilon(i, I)$, where $\epsilon(i, I) = (-1)^{r+1}$ with i sitting at the r-th position of I if I is written increasingly. It satisfies that $\epsilon(j, J)\epsilon(j, j \cup I) = \epsilon(j, J \setminus I)$ for $j \in J \setminus I$ (see [10, Theorem 3.2.9]).

Lemma 4.3.14. *For $J \in \mathrm{Row}\Lambda$, there are isomorphisms of R-modules*

$$H_i(\mathcal{Q}^{-i,\phi(J)}(K), d) \cong \widetilde{H}^{|\phi(J)| - i - 1}(K_J; R).$$

Proof. For $J \in \mathrm{Row}\Lambda$ and $\phi(J) \subseteq [l]$, define

$$\begin{aligned} f\colon C^p(K_J; R) &\longrightarrow \mathcal{Q}^{|\phi(J)| - p - 1, \phi(J)}(K) \\ \chi_\sigma &\mapsto \epsilon(I_\alpha(\sigma), \phi(J)) u_{\phi(J) \setminus I_\alpha(\sigma)} v_\sigma \end{aligned}$$

where χ_σ is a base cochain corresponding to an oriented simplex σ. It suffices to prove that f is an isomorphism of chain complexes. First, by the construction of K, if $\sigma \in K$ and $|\sigma| = p + 1$, then $|I_\alpha(\sigma)| = p + 1$. Since ϕ is a bijection and $\mathcal{Q}^{|\phi(J)| - p - 1, \phi(J)}$ has a basis $\{u_{\phi(J) \setminus I_\alpha(\sigma)} v_\sigma \mid \sigma \in K, |\sigma| = p + 1, I_\alpha(\sigma) \subseteq \phi(J)\}$, f is an isomorphism. Observe that

$$\begin{aligned} f(d\chi_\sigma) &= f(\sum_{j \in J \setminus \sigma, j \cup \sigma \in K_J} \epsilon(j, j \cup \sigma) \chi_{j \cup \sigma}) \\ &= \sum_{j \in J \setminus \sigma} \epsilon(j, j \cup \sigma) \epsilon(I_\alpha(j \cup \sigma), \phi(J)) u_{\phi(J) \setminus I_\alpha(j \cup \sigma)} v_{j \cup \sigma} \\ df(\chi_\sigma) &= \sum_{k \in \phi(J) \setminus I_\alpha(\sigma)} \epsilon(I_\alpha(\sigma), \phi(J)) \epsilon(k, \phi(J) \setminus I_\alpha(\sigma)) u_{\phi(J) \setminus (k \cup I_\alpha(\sigma))} (\sum_{p \in \alpha_k} v_p) v_\sigma. \end{aligned}$$

To simplify $df(\chi_\sigma)$, note that $v_p v_\sigma = 0 \in \mathcal{Q}^{*,\phi(J)}(K)$ unless $p \cup \sigma \in K$. Thus, the summands in $df(\chi_\sigma)$ go though $k \in \phi(J) \setminus I_\alpha(\sigma)$ such that there is at least one vertex $p \in \alpha_k$ that $p \cup \sigma \in K$. There is a bijection between $\{j \in J \setminus \sigma \mid j \cup \sigma \in K_J\}$ and $\{p \in \alpha_k \mid k \in \phi(J) \setminus I_\alpha(\sigma), p \cup \sigma \in K\}$ by sending j to p. By this bijection, if $j \cup \sigma \in K_J$, then there is a unique $k \in [l]$ such that $j \in \alpha_k$ and $k \notin I_\alpha(\sigma)$. So $I_\alpha(j \cup \sigma) = k \cup I_\alpha(\sigma)$. Hence, the terms appearing in $df(\chi_\sigma)$ and $fd(\chi_\sigma)$ match. In order to show that $df(\chi_\sigma) = fd(\chi_\sigma)$, we need to show that the signs match. If $j \cup \sigma \in K_J$, then $\epsilon(j, j \cup \sigma) = \epsilon(k, k \cup I_\alpha(\sigma))$ by the construction of K. Hence, we have

$$\begin{aligned}
&\epsilon(j, j \cup \sigma)\epsilon(I_\alpha(j \cup \sigma), \phi(J)) = \epsilon(k, k \cup I_\alpha(\sigma))\epsilon(k \cup I_\alpha(\sigma), \phi(J)) \\
=&\epsilon(k, k \cup I_\alpha(\sigma))\epsilon(k, \phi(J))\epsilon(I_\alpha(\sigma), \phi(J)) = \epsilon(k, \phi(J) \setminus I_\alpha(J))\epsilon(I_\alpha(\sigma), \phi(J))
\end{aligned}$$

Therefore, f commutes with the differentials. □

We finish the algebraic proof of Proposition 4.3.11.

Proof of Proposition 4.3.11. The first statement is a direct application of the Koszul algebra. Now we prove the second statement. By Theorem 4.3.8, Lemma 4.3.13 and Lemma 4.3.14, we have

$$\begin{aligned}
H^q(\mathcal{Z}_K/\mathrm{Ker}\Lambda; R) &\cong \bigoplus_{i+2t=q} H_i(\Lambda^{-i,2i}[u_1, \ldots, u_l] \otimes R^{2t}[K], d) \\
&\cong \bigoplus_{-i+2j=q, |\phi(J)|=j} H_i(\mathcal{Q}^{-i,\phi(J)}(K), d) \\
&\cong \bigoplus_{-i+2j=q, |\phi(J)|=j} \widetilde{H}^{j-i-1}(K_J; R) \\
&= \bigoplus_{J \in \mathrm{Row}\Lambda} \widetilde{H}^{q-|\phi(J)|-1}(K_J; R)
\end{aligned}$$

which completes the proof. □

4.3.4 Comparison with the cohomology of $\mathcal{Z}_K$

Let K be a simplicial complex on $[m]$ and let $\mathbb{P}$ consist of all minimal missing faces of K. Recall that in [41] (Theorem 2.1.27), a differential graded algebra $(\Lambda^*[\mathbb{P}], \tilde{d})$ is introduced with the homology isomorphic as an R-algebra to the cohomology of $\mathcal{Z}_K$, by applying the Taylor resolution of $R[K]$ over $R[v_1, \ldots, v_m]$.

Theorem 2.1.27. *There exist isomorphisms of R-algebras*

$$H^*(\mathcal{Z}_K; R) \cong \mathrm{Tor}_{R[v_1,\ldots,v_m]}(R[K], R) \cong H_i(\Lambda^*[\mathbb{P}], \tilde{d})$$

where $\tilde{d}(\sigma_{i_1} \ldots \sigma_{i_p}) = \sum_{t=1}^{p} (-1)^{t-1} \delta_t \sigma_{i_1} \ldots \hat{\sigma}_{i_t} \ldots \sigma_{i_p}$ *and* $\delta_t = 1$ *if* $\mathcal{S}_\sigma = \mathcal{S}_{\partial_t \sigma}$ *and zero, otherwise.*

The next lemma shows that the projection map $\pi\colon \Lambda^*[\mathbb{P}]\otimes R[x_1,\ldots,x_r] \longrightarrow \Lambda^*[\mathbb{P}]$ which sends all x_i to zero is a chain map between the chain complexes $(\Lambda^*[\mathbb{P}]\otimes R[x_1,\ldots,x_r],d)$ and $(\Lambda^*[\mathbb{P}],\tilde{d})$.

Lemma 4.3.15. *The map $\pi\colon (\Lambda^*[\mathbb{P}]\otimes R[x_1,\ldots,x_r],d) \longrightarrow (\Lambda^*[\mathbb{P}],\tilde{d})$ by $x_j \mapsto 0$ is a chain map.*

Proof. Let $\sigma_{i_1}\ldots\sigma_{i_p} \in \Lambda^p[\mathbb{P}]$, let $f \in R[x_1,\ldots,x_r]$ and let $a = f(0,\ldots,0) \in R$. Then we have

$$\begin{aligned} \pi d(\sigma_{i_1}\ldots\sigma_{i_p}\otimes f) &= \pi(\sum_{1\le t\le i}(-1)^{t-1}\sigma_{i_1}\ldots\hat{\sigma}_{i_t}\ldots\sigma_{i_p}\otimes f\delta_p(x_1,\ldots,x_r)) \\ &= \sum_{1\le t\le i}(-1)^{t-1}a\delta_t(0,\ldots,0)\sigma_{i_1}\ldots\hat{\sigma}_{i_t}\ldots\sigma_{i_p} \\ \tilde{d}\pi(\sigma_{i_1}\ldots\sigma_{i_p}\otimes f) &= a\tilde{d}(\sigma_{i_1}\ldots\sigma_{i_p}) = \sum_{t=1}^{i}(-1)^{t-1}a\delta_t\sigma_{i_1}\ldots\hat{\sigma}_{i_t}\ldots\sigma_{i_p} \end{aligned}$$

Recall that by (4.7), $\delta_t(x_1,\ldots,x_r) = \prod_{i\in\mathcal{S}_\sigma\setminus S_{\partial_t\sigma}}(\sum_{j=1}^{r}s_{ij}x_j)$ if $\mathcal{S}_\sigma \neq \mathcal{S}_{\partial_t\sigma}$ and 1 otherwise. Thus $\delta_t(0,\ldots,0) = \delta_t$ and $\tilde{d}\pi = \pi d$. □

Let $R[x_1,\ldots,x_r]^+$ denote the kernel of the augmentation map $\epsilon\colon R[x_1,\ldots,x_r] \longrightarrow R$ by $\epsilon(x_j) = 0$. Since the underlying R-module of $\Lambda^*[\mathbb{P}]$ is free, we have a short exact sequence of R-modules

$$0 \longrightarrow \Lambda^*[\mathbb{P}]\otimes R[x_1,\ldots,x_r]^+ \xrightarrow{f} \Lambda^*[\mathbb{P}]\otimes R[x_1,\ldots,x_r] \xrightarrow{\pi} \Lambda^*[\mathbb{P}] \longrightarrow 0$$

where $\Lambda^*[\mathbb{P}]\otimes R[x_1,\ldots,x_r]^+$ is closed under the differential. Thus we have the following lemma.

Lemma 4.3.16. *There exists a short exact sequence of chain complexes*

$$0 \longrightarrow (\Lambda^*[\mathbb{P}]\otimes R[x_1,\ldots,x_r]^+,d) \xrightarrow{f} (\Lambda^*[\mathbb{P}]\otimes R[x_1,\ldots,x_r],d) \xrightarrow{\pi} (\Lambda^*[\mathbb{P}],\tilde{d}) \longrightarrow 0$$

where f is an inclusion of chain complexes and π is a projection by $\pi(x_j) = 0$.

It gives rise to a long exact sequence of homology groups

$$\begin{aligned} \ldots \longrightarrow H_i(\Lambda^*[\mathbb{P}]\otimes R[x_1,\ldots,x_r]^+,d) &\xrightarrow{f_i} H_i(\Lambda^*[\mathbb{P}]\otimes R[x_1,\ldots,x_r],d) \\ \xrightarrow{\pi_i} H_i(\Lambda^*[\mathbb{P}],\tilde{d}) &\xrightarrow{\phi_i} H_{i-1}(\Lambda^*[\mathbb{P}]\otimes R[x_1,\ldots,x_r]^+,d) \longrightarrow \ldots \end{aligned}$$

which induces short exact sequences for $i \ge 0$

$$0 \longrightarrow H_i(\Lambda^*[\mathbb{P}]\otimes R[x_1,\ldots,x_r]^+,d)/\mathrm{Im}\phi_{i+1} \longrightarrow H_i(\Lambda^*[\mathbb{P}]\otimes R[x_1,\ldots,x_r],d) \longrightarrow \mathrm{Ker}\phi_i \longrightarrow 0.$$

Proposition 4.3.17. *The above short exact sequences split for every $i \geq 0$, so there are isomorphisms of R-modules*

$$H_i(\Lambda^*[\mathbb{P}] \otimes R[x_1, \ldots, x_r], d) \cong H_i(\Lambda^*[\mathbb{P}] \otimes R[x_1, \ldots, x_r]^+, d)/\mathrm{Im}\phi_{i+1} \oplus \mathrm{Ker}\phi_i.$$

Proof. It suffices to define a section $s_i : \mathrm{Ker}\phi_i \longrightarrow H_i(\Lambda^*[\mathbb{P}] \otimes R[x_1, \ldots, x_r], d)$. Let $\alpha = [a] \in \mathrm{Ker}\phi_i$. Then $\phi_i(\alpha) = [d(a)] \in H_{i-1}(\Lambda^*[\mathbb{P}] \otimes R[x_1, \ldots, x_r]^+, d)$. Since $\phi_i(\alpha) = 0$, there exists an element $b \in \Lambda^i[\mathbb{P}] \otimes R[x_1, \ldots, x_r]^+$ such that $d(b) = d(a)$. Thus $a - b \in \mathrm{Ker}d$. Define $s_i(\alpha) = [a - b]$. It is easy to see that s_i is a section since $\phi_i \circ s_i(\alpha) = \phi_i([a - b]) = [a] = \alpha$. □

4.4 Circle actions on $\mathcal{Z}_K$

In this section we focus on free S^1-actions on $\mathcal{Z}_K$. Let K be a simplicial complex on $[m]$ ($K \neq \Delta^{m-1}$) and let $\gcd(s_1, \ldots, s_m) = 1$, $s_i \in \mathbb{Z}$. The S^1-action on $\mathcal{Z}_K$ by $(s_1, \ldots, s_m)$ is given by $t \cdot (z_1, \ldots, z_m) = (t^{s_1} z_1, \ldots, t^{s_m} z_m)$, where $t^{s_i} z_i$ is the product of complex numbers for any $t \in S^1$ and $(z_1, \ldots, z_m) \in \mathcal{Z}_K$.

We first adapt Lemma 4.2.1 to give conditions for S^1 acting freely on $\mathcal{Z}_K$. This will be used in Proposition 4.4.7. For two or more integers $\{a_i\}_{i \in I}$, which are not all zero, let $\gcd(a_i \colon i \in I)$ denote the largest positive integer that divides all of the a_i.

Lemma 4.4.1. *The S^1-action on $\mathcal{Z}_K$ by $(s_1, \ldots, s_m)$ is free if and only if $\gcd(s_i \colon i \in \sigma) = 1$ for every $[m] \setminus \sigma \in K$.*

Proof. In the case of circle actions, the associated matrix is $S = (s_1, \ldots, s_m)^t$. By Lemma 4.2.1, S^1 acts on $\mathcal{Z}_K$ freely if and only if for every $\tau \in K$, the vector $(s_i)_{i \notin \tau}$, obtained by deleting elements sitting at positions indexed by τ, forms part of a basis of $\mathbb{Z}^{m-|\tau|}$. By Lemma 4.1.2(a), the latter condition is equivalent to $\gcd(s_i \mid i \in [m] \setminus \tau) = 1$. The statement follows immediately by setting $\sigma = [m] \setminus \tau$ for every $\tau \in K$. □

For example, let $\mathcal{Z}_m$ be the moment-angle complex corresponding to m disjoint points. Then the S^1-action on $\mathcal{Z}_m$ by $(s_1, \ldots, s_m)$ is free if and only if for every $j \in [m]$, $\gcd(s_1, \ldots, \hat{s}_j, \ldots, s_m) = 1$, where $\hat{s}_j$ indicates omission.

Let K be a simplicial complex on $[m]$ and let $\mathbb{P} = \{\sigma_1, \ldots, \sigma_p\} \subset 2^{[m]}$ collect all minimal missing faces of K, By this property of $\mathbb{P}$, any subset of $[m]$ which contains some $\sigma \in \mathbb{P}$ is missing face of K. Then this simplicial complex K consists of the following subsets of $[m]$

$$K = 2^{[m]} \setminus \{J \subseteq [m] \mid \sigma \subseteq J \text{ for some } \sigma \in \mathbb{P}\}. \tag{4.13}$$

Lemma 4.4.2. *Let $\mathbb{P} = \{\sigma_1, \ldots, \sigma_p\}$ be the set of all minimal missing faces of K. Then the S^1-action on $\mathcal{Z}_K$ by $(s_1, \ldots, s_m)$ is free if and only if $\gcd(s_{j_1}, \ldots, s_{j_p}) = 1$ for each $j_t \in \sigma_t$ $(1 \le t \le p)$.*

Proof. Let $\{j_1, \ldots, j_p\}$ be a subset of $[m]$ with each j_t in σ_t $(1 \le t \le p)$. We show that $[m] \setminus \{j_1, \ldots, j_p\}$ are maximal faces of K (4.13).

For every $\sigma_t \in \mathbb{P}$, the intersection $\{j_1, \ldots, j_p\} \cap \sigma_t$ is non-empty, which means that every $\sigma_t \nsubseteq [m] \setminus \{j_1, \ldots, j_p\}$. Hence $[m] \setminus \{j_1, \ldots, j_p\} \in K$ by (4.13). On the other hand, if $I \in K$, then $I \notin \{J \subseteq [m] \mid \sigma \subseteq J \text{ for some } \sigma \in \mathbb{P}\}$. That is to say, for each $\sigma_t \in \mathbb{P}$, there exists $j_t \in \sigma_t$ such that $j_t \notin I$, so we have $I \subseteq [m] \setminus \{j_1, \ldots, j_p\}$ because $I \cap \{j_1, \ldots, j_p\} = \emptyset$.

By Lemma 4.4.1, we have that S^1 acts freely on $\mathcal{Z}_{K_\mathbb{P}}$ if and only if $\gcd(s_{j_1}, \ldots, s_{j_p}) = 1$ for each $j_t \in \sigma_t$. □

Remark 4.4.3. Note that $\gcd(a, bc) = 1$ if and only if $\gcd(a, b) = \gcd(a, c) = 1$. Thus, S^1 acts freely on $\mathcal{Z}_{K_\mathbb{P}}$ by $(s_1, \ldots, s_m)$ if and only if $\gcd(\prod_{j \in \sigma_1} s_j, \ldots, \prod_{j \in \sigma_p} s_j) = 1$. In particular, if $\mathbb{P} = \{\sigma\}$, then S^1 acts freely on $\mathcal{Z}_{K_\mathbb{P}}$ if and only if $s_j = \pm 1$ for every $i \in \sigma$.

Let $\mathbb{P} = \{\sigma_1, \ldots, \sigma_p\}$ be a set of minimal missing faces of K. If S^1 acts freely on $\mathcal{Z}_K$ by $(s_1, \ldots, s_m)$, then by Theorem 4.3.3, there is an isomorphism of R-algebras

$$H^*(\mathcal{Z}_K/S^1; R) \cong H(\Lambda^*[\mathbb{P}] \otimes R[x], d).$$

Here the differential d is defined by

$$\begin{aligned} d(\sigma_{i_1}\sigma_{i_2}\ldots\sigma_{i_q}) &= \sum_{1 \le t \le q} (-1)^{t-1} \Big(\prod_{j \in \mathcal{S}_\sigma \setminus \mathcal{S}_{\partial_t \sigma}} s_j \cdot 1_R \Big) \sigma_{i_1} \ldots \hat{\sigma}_t \ldots \sigma_{i_q} \otimes x^{|\mathcal{S}_\sigma \setminus \mathcal{S}_{\partial_t \sigma}|} \\ dx &= 0 \end{aligned} \tag{4.14}$$

where $\mathcal{S}_\sigma = \sigma_{i_1} \cup \ldots \cup \sigma_{i_q}$ and assume that $\prod_{j \in \emptyset} s_j \cdot 1_R = 1_R$. The bigrading of $\Lambda^*[\mathbb{P}] \otimes R[x]$ is defined by

$$\operatorname{bideg} \sigma_{i_1}\sigma_{i_2}\ldots\sigma_{i_q} = (-q, 2|\mathcal{S}_\sigma|) \text{ and } \operatorname{bideg} x = (0, 2).$$

We apply (4.14) to calculate the cohomology of the quotient space of a product of odd spheres under free circle actions in the next example. It is a different method as [33, Theorem 2].

Example 4.4.4 (free circle actions on $S^{2m_1-1} \times S^{2m_2-1}$)**.** Let $1 \le m_1 \le m_2$. Suppose that R is an integral domain. Let $K = \partial\Delta^{m_1-1} * \partial\Delta^{m_2-1}$ be the join on $m_1 + m_2$ vertices. Assume that if $m_1 = m_2 = 1$, then $\mathcal{Z}_K = S^1 \times S^1$. The minimal missing faces of K are $\mathbb{P} = \{\sigma, \tau\}$ where $\sigma = \{1, \ldots, m_1\}$ and $\tau = \{m_1 + 1, \ldots, m_1 + m_2\}$. Suppose that

S^1 acts on $\mathcal{Z}_K$ by $(a_1,\dots,a_{m_1},b_1,\dots,b_{m_2})$ and write $a=a_1\dots a_{m_1}$ and $b=b_1\dots b_{m_2}$. Then the differentials from (4.14) are

$$\begin{aligned}
d_2(\sigma\tau) &= a\cdot 1_R x^{m_1}\tau - b\cdot 1_R x^{m_2}\sigma = x^{m_1}y\\
d_1(\sigma) &= a\cdot 1_R x^{m_1}\\
d_1(\tau) &= b\cdot 1_R x^{m_2}
\end{aligned}$$

where $y = a\cdot 1_R\tau - b\cdot 1_R x^{m_2-m_1}\sigma \in \ker d_1$ and $\deg y = 2m_2-1$.

Since $d_0(x)=0$, $H_0(\Lambda[\mathbb{P}]\otimes R[x],d) = R[x]/(a\cdot 1_R x^{m_1}, b\cdot 1_R x^{m_2})$.

The circle action on $\mathcal{Z}_K$ is free if and only if each a_i is relatively prime to b_j by Lemma 4.4.2. Thus, if $a=0$ then $b=\pm 1$. If $a=0$, then $d_1(\sigma)=0$ and $x^{m_2}\sigma\in \mathrm{Im} d_2$. In this case, $H_1(\Lambda[\mathbb{P}]\otimes R[x],d) = \Lambda^1[\sigma]\otimes R[x]/(x^{m_2}\sigma)\cong \Lambda^1[\sigma]\otimes (R[x]/(x^{m_2}))$ where $\deg x = 2$ and $\deg\sigma = 2m_1-1$ and $H_0(\Lambda^*[\mathbb{P}]\otimes R[x],d)\cong R[x]/(x^{m_2})$.

If $a\neq 0$, then $H_1(\Lambda[\mathbb{P}]\otimes R[x],d) = \Lambda^1[y]\otimes (R[x]/(x^{m_1}))$.

Summarising the above, we obtain

$$H^*(\mathcal{Z}_K/S^1;R)\cong \begin{cases} R[x,y]/(y^2, x^{m_1}y, a\cdot 1_R x^{m_1}, b\cdot 1_R x^{m_2}) & \text{where } \deg y = 2m_2-1 \text{ and } \deg x = 2 \text{ if } a\neq 0\\ R[x,\sigma]/(\sigma^2, x^{m_2}) & \text{where } \deg\sigma = 2m_1-1 \text{ and } \deg x = 2 \text{ if } a = 0.\end{cases}$$

Differential (4.14) implies a chain complex as follows. We start with a simplicial complex K on $[m]$ which is not a simplex and an S^1-action on $\mathcal{Z}_K$ by $(s_1,\dots,s_m)$. Let $\mathbb{P}$ denote the set of all minimal missing faces of K. We consider simplicial complexes L whose vertex set is given by $\mathbb{P}$ and define the following chain complexes $(C_*(L),\delta)$ induced by $(s_1,\dots,s_m)$.

Construction 4.4.5. Let L be a simplicial complex on the vertex set $\mathbb{P}$. For every face F of L, define a supporting set $\mathcal{S}_F\subset [m]$ by letting $\mathcal{S}_F = \bigcup_{\sigma\in V(F)}\sigma$. Define a chain complex $(C_*(L),\delta)$ associated to L. Let $C_q(L)$ be a free R-module on basis elements of oriented q-faces F of L. Note that $C_{-1}(L)=R$ and $C_i(L)=0$ for $i\le -2$. The differential $\delta\colon C_q\longrightarrow C_{q-1}$ is defined by

$$\delta(F) = \sum_{F'\in\mathcal{F}(F)} \mathrm{sgn}(F,F')\Big(\prod_{i\in\mathcal{S}_F\setminus\mathcal{S}_{F'}} s_i\Big)F' \tag{4.15}$$

where $\mathcal{F}(F)$ consists of all facets of F and $\mathrm{sgn}(F,F')$ is 1_R if F and F' have the same orientation and -1_R otherwise. Since δ is an evaluation of differential (4.14) at $x=1$, $\delta^2=0$. Let $H_*(L,\delta)$ denote the homology groups of the chain complex $(C_*(L),\delta)$.

Let $\Delta_{\mathbb{P}}$ be a simplex on $\mathbb{P}$. Construct a filtration $\{L_p\mid 1\le p\le m\}$ of simplicial complexes on $\mathbb{P}$,

$$L_p = \{F\in\Delta_{\mathbb{P}} \mid |\mathcal{S}_F|\le p\}.$$

Note that $L_0 = \emptyset$, $L_m = \Delta_{\mathbb{P}}$ and L_p is a subcomplex of L_{p+1}. Thus each L_p is associated with a chain complex $(C_*(L_p), \delta)$ by (4.15).

Let us remark that if S^1 acts on $\mathcal{Z}_K$ diagonally, then all $s_i = 1$. In this case, the homology $H_*(L, \delta)$ is the standard reduced homology $\widetilde{H}_*(L; R)$.

Example 4.4.6. Let $K = \{v_1\}$ on $[2]$, i.e., $\{v_2\}$ is a ghost vertex of K. Thus $\mathbb{P} = \{v_2\}$ and $\mathcal{Z}_K = D^2 \times S^1$. In this case, $\Delta_{\mathbb{P}}$ is a single vertex. We consider two different S^1-actions on $\mathcal{Z}_K$ by $\omega_1 = (0, 1)$ and $\omega_2 = (1, 0)$, where the first action is free and the second is not. The associated chain complexes given by Construction (4.4.5) are

$$0 \longrightarrow C_0(\Delta_{\mathbb{P}}) \xrightarrow{\delta_0^{\omega}} R \longrightarrow 0$$

where $\delta_0^{\omega_1}(v_2) = 1_R$ and $\delta_0^{\omega_2}(v_2) = 0$. Thus $H_*(\Delta_{\mathbb{P}}, \delta^{\omega_1}) = 0$ and $H_0(\Delta_{\mathbb{P}}, \delta^{\omega_2}) = R$.

Recall that a chain complex (C_*, δ) is acyclic if its homology is trivial. As illustrated in Exmaple 4.4.6, in general, the homology of this chain complex $(C_*(\Delta_{\mathbb{P}}), \delta)$ for a simplex is not trivial. However, we show that if S^1 acts on $\mathcal{Z}_K$ freely, then $(C_*(\Delta_{\mathbb{P}}), \delta)$ is acyclic.

Proposition 4.4.7. *If S^1 acts on $\mathcal{Z}_K$ freely, then $(C_*(\Delta_{\mathbb{P}}), \delta)$ is acyclic.*

Proof. Choose $\sigma \in \mathbb{P}$ and let $\mathbb{P}'$ denote $\mathbb{P} \setminus \{\sigma\}$. If $F = [\sigma_{i_0}, \ldots, \sigma_{i_q}]$ is an oriented q-face of $\Delta_{\mathbb{P}'}$, then denote by $F\sigma$ the oriented $(q+1)$-face $[\sigma_{i_0}, \ldots, \sigma_{i_q}, \sigma]$ of $\Delta_{\mathbb{P}}$. If $c = \sum r_i F_i \in C_q(\Delta_{\mathbb{P}'})$, then $c\sigma = \sum r_i(F_i\sigma) \in C_{q+1}(\Delta_{\mathbb{P}})$. Define an R-homomorphism

$$\begin{aligned} \theta\colon C_q(\Delta_{\mathbb{P}'}) &\longrightarrow C_q(\Delta_{\mathbb{P}'}) \\ F &\longmapsto \prod_{j \in \mathcal{S}_F \cap \sigma} (s_j \cdot 1_R) F \end{aligned}$$

where we assume that $\prod\limits_{j \in \emptyset} s_j \cdot 1_R = 1_R$.

Now let $z = z_1 + z_2\sigma \in C_q(\Delta_{\mathbb{P}})$, where $z_1 \in C_q(\Delta_{\mathbb{P}'})$ and $z_2 \in C_{q-1}(\Delta_{\mathbb{P}'})$. We show that if $\delta(z) = 0$, then $(\prod\limits_{j \in \sigma} s_j) z \in \mathrm{Im}\delta$. Let $z_1 = \sum\limits_{Q \in \Delta_{\mathbb{P}'}} r_Q Q$, where the sum goes through all oriented q-faces of $\Delta_{\mathbb{P}'}$ and let $\tilde{z}_1 = \theta(z_1) = \sum\limits_{Q \in \Delta_{\mathbb{P}'}} r_Q (\prod\limits_{j \in \mathcal{S}_Q \cap \sigma} s_j) Q$. Recall that $\mathcal{F}(Q)$ denotes the set of all codimension-one faces of Q.

We do the following calculation

$$\begin{aligned} \delta(Q\sigma) &= \sum_{Q' \in \mathcal{F}(Q)} \mathrm{sgn}(Q, Q') (\prod_{j \in \mathcal{S}_{Q\sigma} \setminus \mathcal{S}_{Q'\sigma}} s_j) Q'\sigma + (-1)^{q+1} (\prod_{j \in \mathcal{S}_{Q\sigma} \setminus \mathcal{S}_Q} s_j) Q \\ \delta(\tilde{z}_1 \sigma) &= \sum_{Q \in \Delta_{\mathbb{P}'}} r_Q (\prod_{j \in \mathcal{S}_Q \cap \sigma} s_j) \delta(Q\sigma) \\ &= \sum_{Q \in \Delta_{\mathbb{P}'}} r_Q (\prod_{j \in \mathcal{S}_Q \cap \sigma} s_j) \{ \sum_{Q' \in \mathcal{F}(Q)} \mathrm{sgn}(Q, Q') (\prod_{j \in \mathcal{S}_{Q\sigma} \setminus \mathcal{S}_{Q'\sigma}} s_j) Q'\sigma + (-1)^{q+1} (\prod_{j \in \mathcal{S}_{Q\sigma} \setminus \mathcal{S}_Q} s_j) Q \}. \end{aligned}$$

Since the support $\mathcal{S}_F$ of the face $F \in \Delta_{\mathbb{P}}$ is a subset of $[m]$, we have the following

$$(\mathcal{S}_Q \cap \sigma) \cup ((\mathcal{S}_Q \cup \sigma) \setminus (\mathcal{S}_{Q'} \cup \sigma)) = (\mathcal{S}_Q \setminus \mathcal{S}_{Q'}) \cup (\mathcal{S}_{Q'} \cap \sigma)$$
$$(\mathcal{S}_Q \cap \sigma) \cup ((\mathcal{S}_Q \cup \sigma) \setminus \mathcal{S}_Q) = \sigma$$
$$(\mathcal{S}_Q \cap \sigma) \cap ((\mathcal{S}_Q \cup \sigma) \setminus (\mathcal{S}_{Q'} \cup \sigma)) = \emptyset$$
$$(\mathcal{S}_Q \cap \sigma) \cap ((\mathcal{S}_Q \cup \sigma) \setminus \mathcal{S}_Q) = \emptyset.$$

Thus,

$$\begin{aligned}
\delta(\tilde{z}_1\sigma) &= \sum_{Q\in\Delta_{\mathbb{P}'}} r_Q\{\sum_{Q'\in\mathcal{F}(Q)} \operatorname{sgn}(Q,Q')(\prod_{j\in(\mathcal{S}_Q\setminus\mathcal{S}_{Q'})\cup(\mathcal{S}_{Q'}\cap\sigma)} s_j)Q'\sigma + (-1)^{q+1}(\prod_{j\in\sigma} s_j)Q\} \\
&= \sum_{Q\in\Delta_{\mathbb{P}'}} r_Q \sum_{Q'\in\mathcal{F}(Q)} \operatorname{sgn}(Q,Q')(\prod_{(j\in\mathcal{S}_Q\setminus\mathcal{S}_{Q'})\cup(\mathcal{S}_{Q'}\cap\sigma)} s_j)Q'\sigma + (-1)^{q+1}(\prod_{j\in\sigma} s_j)z_1. \\
&= \sum_{Q\in\Delta_{\mathbb{P}'}} r_Q \sum_{Q'\in\mathcal{F}(Q)} \operatorname{sgn}(Q,Q')(\prod_{j\in(\mathcal{S}_Q\setminus\mathcal{S}_{Q'})\cup(\mathcal{S}_{Q'}\cap\sigma)} s_j)Q'\sigma + (-1)^{q+1}(\prod_{j\in\sigma} s_j)(z - z_2\sigma).
\end{aligned}$$

Next, we show that

$$\sum_{Q\in\Delta_{\mathbb{P}'}} r_Q \sum_{Q'\in\mathcal{F}(Q)} \operatorname{sgn}(Q,Q')(\prod_{j\in(\mathcal{S}_Q\setminus\mathcal{S}_{Q'})\cup(\mathcal{S}_{Q'}\cap\sigma)} s_j)Q' + (-1)^{q}(\prod_{j\in\sigma} s_j)z_2 = 0 \tag{4.16}$$

which implies that $(\prod_{j\in\sigma} s_j)z = (-1)^{q+1}\delta(\tilde{z}_1\sigma) \in \operatorname{Im}\delta$.

Let $z_2 = \sum_{F\in\Delta_{\mathbb{P}'}} r_F F$ where each $\dim F = q-1$. Since $\delta(z) = 0$, we have $-\delta(z_2\sigma) = \delta(z_1) \in C_{q-1}(\Delta_{\mathbb{P}'})$. The terms in the expression for $\delta(z_2\sigma)$ corresponding to the basis faces which contain σ as a vertex vanish. Thus,

$$\delta(z_2\sigma) = \sum_{F\in\Delta_{\mathbb{P}'}} r_F \delta(F\sigma) = \sum_{F\in\Delta_{\mathbb{P}'}} r_F(-1)^q(\prod_{j\in(\mathcal{S}_F\cup\sigma)\setminus\mathcal{S}_F} s_j)F \tag{4.17}$$

and

$$\theta(\delta(z_2\sigma)) = \sum_{F\in\Delta_{\mathbb{P}'}} r_F(-1)^q(\prod_{j\in\mathcal{S}_F\cap\sigma} s_j)(\prod_{(\mathcal{S}_F\cup\sigma)\setminus\mathcal{S}_F} s_j)F = (-1)^q(\prod_{j\in\sigma} s_j)z_2.$$

Note that $z_1 = \sum_{Q\in\Delta_{\mathbb{P}'}} r_Q Q$. Hence,

$$\delta(z_1) = \sum_{Q\in\Delta_{\mathbb{P}'}} r_Q \sum_{Q'\in\mathcal{F}(Q)} \operatorname{sgn}(Q,Q')(\prod_{j\in\mathcal{S}_Q\setminus\mathcal{S}_{Q'}} s_j)Q'. \tag{4.18}$$

As $\delta(z_1) = -\delta(z_2\sigma)$, $\theta\delta(z_1) = -\theta\delta(z_2\sigma)$. The terms appearing in (4.17) and (4.18) should match. Thus we have

$$\begin{aligned}\theta\delta(z_1) &= \sum_{Q\in\Delta_{\mathbb{P}'}} r_Q \sum_{Q'\in\mathcal{F}(Q)} \operatorname{sgn}(Q,Q')(\prod_{j\in\mathcal{S}_{Q'}\cap\sigma} s_j)(\prod_{j\in\mathcal{S}_Q\backslash\mathcal{S}_{Q'}} s_j)Q' \\ &= (-1)^{q+1}(\prod_{j\in\sigma} s_j)z_2\end{aligned}$$

which implies (4.16).

Hence, we have proven that if $\delta(z) = 0$, then $(\prod_{j\in\sigma} s_j)z$ lies in Imδ. The proof does not depend on the choice of σ. Thus, for every $\sigma \in \mathbb{P}$, $(\prod_{j\in\sigma} s_j)z$ lies in Imδ provided that $\delta(z) = 0$. The freeness condition (Remark 4.4.3) implies that $\gcd(\prod_{j\in\sigma} s_j \mid \sigma \in \mathbb{P}) = 1$. Then there are integers $\{l_\sigma\}_{\sigma\in\mathbb{P}}$ such that $\sum_{\sigma\in\mathbb{P}} l_\sigma(\prod_{j\in\sigma} s_j) = 1$. Hence, $z = \sum_{\sigma\in\mathbb{P}} l_\sigma(\prod_{j\in\sigma} s_j)z \in$ Imδ so that $(C_*(\Delta_{\mathbb{P}}), \delta)$ is acyclic. □

Our next goal is to calculate $H^*(\mathcal{Z}_K/S^1; R)$ in terms of $H_*(L_p, \delta)$, where L_p and the chain complexes $(C_*(L_p), \delta)$ are defined in Construction 4.4.5.

Define a map

$$\begin{aligned} f\colon (\Lambda^*[\mathbb{P}]\otimes R[x], d) &\longrightarrow (C_*(\Delta_{\mathbb{P}})\otimes R[x], \delta\otimes \mathrm{id}) \\ \sigma_{i_1}\ldots\sigma_{i_q} &\mapsto F_\sigma \otimes x^{|\mathcal{S}_{F_\sigma}|} \\ x^q &\mapsto x^q \end{aligned} \tag{4.19}$$

where $F_\sigma = [\sigma_{i_1},\ldots,\sigma_{i_q}]$ is an oriented simplex of $\Delta_{\mathbb{P}}$.

Since

$$(\delta\otimes\mathrm{id})\circ f(\sigma_{i_1}\ldots\sigma_{i_q}) = \delta([\sigma_{i_1},\ldots,\sigma_{i_q}]\otimes x^{|\mathcal{S}_{F_\sigma}|}) = \sum_{1\le t\le q}(-1)^{t-1}(\prod_{j\in\mathcal{S}_{F_\sigma}\backslash\mathcal{S}_{F_{\partial_t\sigma}}} s_j\cdot 1_R)\partial_t\sigma\otimes x^{|\mathcal{S}_{F_\sigma}|}$$

$$f\circ d(\sigma_{i_1}\ldots\sigma_{i_q}) = f\left(\sum_{1\le t\le q}(-1)^{t-1}(\prod_{j\in\mathcal{S}_\sigma\backslash\mathcal{S}_{\partial_t\sigma}} s_j\cdot 1_R)\partial_t\sigma\otimes x^{|\mathcal{S}_\sigma\backslash\mathcal{S}_{\partial_t\sigma}|}\right) = (\delta\otimes\mathrm{id})\circ f(\sigma_{i_1}\ldots\sigma_{i_q})$$

f is a chain map. Thus, f induces a short exact sequence of chain complexes

$$0\longrightarrow(\Lambda^*[\mathbb{P}]\otimes R[x], d)\xrightarrow{f}(C_{*-1}(\Delta_{\mathbb{P}})\otimes R[x], \delta\otimes\mathrm{id})\xrightarrow{\pi}(C_{*-1}(\Delta_{\mathbb{P}})\otimes R[x]/\mathrm{Im}f, d')\longrightarrow 0 \tag{4.20}$$

where $(C_*(\Delta_{\mathbb{P}})\otimes R[x]/\mathrm{Im}f, d')$ is a chain complex induced by $\delta\otimes\mathrm{id}$ on the quotient R-module. The quotient R-module $C_*(\Delta_{\mathbb{P}})\otimes R[x]/\mathrm{Im}f$ has a basis $\{F\otimes x^p \mid F\in\Delta_{\mathbb{P}}$ and $0\le p<|\mathcal{S}_F|\}$ and the differential d' on the basis $F\otimes x^p$ is defined by $d'(F\otimes x^p) = \sum_{F'\in\mathcal{F}(F)}\varepsilon(F,F')(\prod_{i\in\mathcal{S}_F\backslash\mathcal{S}_{F'}} s_i\cdot 1_R)F'\otimes x^p$, where $\varepsilon(F,F') = sgn(F,F')$ if $0\le p<|\mathcal{S}_{F'}|$ and is zero otherwise.

The bigrading on $C_*(\Delta_{\mathbb{P}}) \otimes R[x]/\mathrm{Im}f$ is given by setting

$$\text{bideg } F \otimes x^p = (-\dim F, 2|\mathcal{S}_F| + 2p).$$

The chain complex $(\Lambda^*[\mathbb{P}] \otimes R[x]/\mathrm{Im}f, d')$ splits in terms of relative chain complexes as follows. Consider a pair of simplicial complexes $(\Delta_{\mathbb{P}}, L_p)$ and their relative chain complex $C_*(\Delta_{\mathbb{P}}, L_p)$. Note that $C_*(\Delta_{\mathbb{P}}, L_p)$ has an R-basis $\{F \in \Delta_{\mathbb{P}} \mid |\mathcal{S}_F| > p\}$. Define a chain map

$$g\colon C_*(\Delta_{\mathbb{P}}) \otimes R[x]/\mathrm{Im}f \longrightarrow \bigoplus_{0\le p\le m-1} C_*(\Delta_{\mathbb{P}}, L_p) \tag{4.21}$$

which sends the basis element $F \otimes x^p$ to the basis element F sitting in the p-th summand $C_*(\Delta_{\mathbb{P}}, L_p)$. Note that g is a bijection between the basis elements in $C_*(\Delta_{\mathbb{P}}) \otimes R[x]/\mathrm{Im}f$ and $\bigoplus_{0\le p\le m-1} C_*(\Delta_{\mathbb{P}}, L_p)$. Thus g induces an isomorphism on homology groups for $j \ge 0$,

$$H_j(C_*(\Delta_{\mathbb{P}}) \otimes R[x]/\mathrm{Im}f, d') \cong \bigoplus_{0\le p\le m-1} H_j(\Lambda^*[\mathbb{P}] \otimes R[x], d).$$

Proposition 4.4.8. *Let S^1 act on $\mathcal{Z}_K$ freely. For $j \ge 0$, there are isomorphisms of R-modules*

$$H_j(\Lambda^*[\mathbb{P}] \otimes R[x], d) \cong H_j(C_*(\Delta_{\mathbb{P}}) \otimes R[x]/\mathrm{Im}f) \cong \bigoplus_{p=0}^{m} H_{j-1}(L_p, \delta)$$

Proof. If S^1 acts on $\mathcal{Z}_K$ freely, then $(C_*(\Delta_{\mathbb{P}}), \delta)$ is acyclic. So $(C_*(\Delta_{\mathbb{P}}) \otimes R[x], \delta \otimes \mathrm{id})$ is acyclic, as $R[x]$ is a free R-module. The first isomorphism follows from the long exact sequence induced by (4.20). The second isomorphism is true since $H_i(\Delta_{\mathbb{P}}, L_p; \delta) \cong H_{i-1}(L_p, \delta)$ for $i \ge 0$ by the long exact sequence induced by the short exact sequence of chain complexes $0 \longrightarrow C_*(L_p) \longrightarrow C_*(\Delta_{\mathbb{P}}) \longrightarrow C_*(\Delta_{\mathbb{P}}, L_p) \longrightarrow 0$. □

Together with Theorem 4.3.8, in the case of free circle actions, we have the following statement.

Proposition 4.4.9. *Let S^1 acts freely on $\mathcal{Z}_K$. There are isomorphisms of R-modules*

$$H^i(\mathcal{Z}_K/S^1; R) \cong \bigoplus_{2p-j=i} H_{j-1}(L_p, \delta). \tag{4.22}$$

Proof. By Theorem 4.3.8,

$$H^i(\mathcal{Z}_K/S^1; R) \cong \bigoplus_{-j+2p'+2q=i} H_j(\Lambda^{-j,2p'}[\mathbb{P}] \otimes R^{2q}[x], d).$$

By Proposition 4.4.8, we have $H_j(\Lambda^*[\mathbb{P}]\otimes R[x],d)\cong\bigoplus_{p=0}^{m}H_{j-1}(L_p,\delta)$. To prove (4.22), we will show that $H_{j-1}(L_p,\delta)\cong\bigoplus_{2p'+2q=p}H_j(\Lambda^{-j,2p'}[\mathbb{P}]\otimes R^{2q}[x],d)$.

Due to the isomorphism g (4.21), there is a decomposition of R-modules

$$H_j(C_*(\Delta_{\mathbb{P}})\otimes R[x]/\mathrm{Im}f,d')\cong\bigoplus_{0\le p\le m-1}H_j(\Delta_{\mathbb{P}},L_p,\delta)\otimes x^p.$$

Let $\alpha=[a]\in H_j(C_*(\Delta_{\mathbb{P}})\otimes R[x]/\mathrm{Im}f,d')$ with $d'(a)=0$. By chasing the diagrams in the zig-zag lemma induced by the short exact sequence (4.20), the isomorphism Φ_j is given by

$$\begin{aligned}\Phi_j\colon H_j(C_*(\Delta_{\mathbb{P}})\otimes R[x]/\mathrm{Im}f,d')&\longrightarrow H_j(\Lambda^*[\mathbb{P}]\otimes R[x],d)\\ [a]&\mapsto[b]\end{aligned}\tag{4.23}$$

where $b\in\Lambda^j[\mathbb{P}]\otimes R[x]$ satisfies $f(b)=\delta\otimes\mathrm{id}(a)$ (f is defined in (4.19)). Explicitly, write $[a]=[a_0]+[a_1]x+\ldots+[a_{m-1}]x^{m-1}$, where $[a_p]\in H_j(\Delta_{\mathbb{P}},L_p,\delta)$. Thus $\delta(a_p)\in C_{j-1}(L_p)$. Let $\delta(a_p)=\sum_{\dim F=j-1}r_{F,p}F$ where F are oriented $(j-1)$-faces of L_p and $r_{F,p}\in R$. Then $\Phi_j([a_p])=\sum_{\dim F=j-1}r_{F,p}[F\otimes x^{p-|\mathcal{S}_F|}]$. Note that $\mathrm{bideg}\,\Phi_j([a_p])=(-j,2p)$.

On the other hand, there are isomorphisms $H_j(\Delta_{\mathbb{P}},L_p,\delta)\cong H_{j-1}(L_p,\delta)$, which are given by sending $[a_p]$ to $[\delta(a_p)]$ for any j. Thus for every $\beta_p\in H_{j-1}(L_p,\delta)$, there exists a unique $[a_p]\in H_j(\Delta_{\mathbb{P}},L_p,\delta)$ such that $\beta_p=[\delta(a_p)]$. Since Φ_j (4.23) is an isomorphism, the following isomorphism Ψ_j is well-defined

$$\begin{aligned}\Psi_j\colon\bigoplus_p H_{j-1}(L_p,\delta)&\longrightarrow H_j(\Lambda^*[\mathbb{P}]\otimes R[x],d)\\ \beta_p&\longmapsto\Phi_j([a_p]).\end{aligned}$$

Since $\mathrm{bideg}\,\Phi_j([a_p])=(-j,2p)$, each $H_{j-1}(L_p,\delta)$ is mapped into $\bigoplus_{p'+q=p}H_j(\Lambda^{-j,2p'}[\mathbb{P}]\otimes R^{2q}[x],d)$ by Ψ_j. Thus Ψ_j induces an isomorphism

$$H_{j-1}(L_p,\delta)\xrightarrow{\cong}\bigoplus_{p'+q=p}H_j(\Lambda^{-j,2p'}[\mathbb{P}]\otimes R^{2q}[x],d).$$

□

Mulplicative structures. The differential graded algebra $(\Lambda^*[\mathbb{P}]\otimes R[x],d)$ is a special case of the Taylor algebra (4.7) when $r=1$. Thus, the homology $H_*(\Lambda^*[\mathbb{P}]\otimes R[x],d)$ has a multiplicative structure and it is possible to give a multiplication on $\bigoplus_{0\le p\le m-1}H_*(L_p,\delta)$ such that the isomorphisms in Proposition 4.4.9 are of R-algebras.

The multiplication $\times$ of $(\Lambda^*[\mathbb{P}] \otimes R[x], d)$ is defined by

$$\sigma_I \times \sigma_J = \begin{cases} (\prod_{i \in \mathcal{S}_{\sigma_I} \cap \mathcal{S}_{\sigma_J}} s_i \cdot 1_R) x^{|\mathcal{S}_{\sigma_I} \cap \mathcal{S}_{\sigma_J}|} \sigma_I \sigma_J & \text{if } I \cap J = \emptyset \text{ and } \mathcal{S}_{\sigma_I} \cap \mathcal{S}_{\sigma_J} \neq \emptyset \\ \mathcal{S}_{\sigma_I} \mathcal{S}_{\sigma_J} & \text{if } I \cap J = \mathcal{S}_{\sigma_I} \cap \mathcal{S}_{\sigma_J} = \emptyset \\ 0 & \text{otherwise} \end{cases}$$

where $\sigma_I = \sigma_{i_1} \dots \sigma_{i_q} \in \Lambda^q[\mathbb{P}]$, $\sigma_J = \sigma_{j_1} \dots \sigma_{j_l} \in \Lambda^l[\mathbb{P}]$ and $\mathcal{S}_{\sigma_I} = \sigma_{i_1} \cup \dots \cup \sigma_{i_q}$ $\mathcal{S}_{\sigma_J} = \sigma_{j_1} \cup \dots \cup \sigma_{j_l}$, and $I = \{i_1, \dots, i_q\}$, $J = \{j_1, \dots, j_l\}$.

The multiplication on $\bigoplus_{0 \leq p \leq m-1} H_*(L_p, \delta)$ is described as follows.

Construction 4.4.10. Let $\sigma_{a_1} \dots \sigma_{a_{j_1}} \in C_{j_1-1}(L_p)$ and $\sigma_{b_1} \dots \sigma_{b_{j_2}} \in C_{j_2-1}(L_q)$. Define the following linear map on $C_{j_1-1}(L_p) \oplus C_{j_2-1}(L_q)$ given by

$$C_{j_1-1}(L_p) \oplus C_{j_2-1}(L_q) \xrightarrow{\times} C_{j_1+j_2-1}(L_{p+q}) \tag{4.24}$$

$$\sigma_{a_1} \dots \sigma_{a_{j_1}} \times \sigma_{b_1} \dots \sigma_{b_{j_2}} = \begin{cases} (\prod_{i \in \mathcal{S}_{\sigma_a} \cap \mathcal{S}_{\sigma_b}} s_i \cdot 1_R) \sigma_{a_1} \dots \sigma_{a_{j_1}} \sigma_{b_1} \dots \sigma_{b_{j_2}} & \text{if } A \cap B = \emptyset \text{ and } \mathcal{S}_{\sigma_a} \cap \mathcal{S}_{\sigma_b} \neq \emptyset \\ \sigma_{a_1} \dots \sigma_{a_{j_1}} \sigma_{b_1} \dots \sigma_{b_{j_2}} & \text{if } A \cap B = \mathcal{S}_{\sigma_a} \cap \mathcal{S}_{\sigma_b} = \emptyset \\ 0 & \text{otherwise} \end{cases}$$

where $A = \{a_1, \dots, a_{j_1}\}$ and $B = \{b_1, \dots, b_{j_2}\}$.

The map (4.24) induces a multiplicative structure on $\bigoplus_{0 \leq p \leq m-1} H_*(L_p, \delta)$ such that the isomorphism (4.22) is of R-algebras. Here we conclude our main result of this section.

Theorem 4.4.11. *Suppose that S^1 acts freely on $\mathcal{Z}_K$. Then there exists an isomorphism of R-algebras*

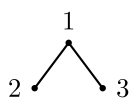
$$H^*(\mathcal{Z}_K/S^1; R) \cong \bigoplus_{2p-j \geq 0} H_{j-1}(L_p, \delta).$$

By the definition of chain complexes $(C_*(L_p), \delta)$ (Construction 4.4.5), the homology $H_*(L, \delta)$ is the standard reduced simplicial homology in the case of the diagonal action on $\mathcal{Z}_K$. A corollary of Theorem 4.4.11 follows.

Corollary 4.4.12 (diagonal action)**.** *There is an isomorphism of R-algebras*

$$H^*(\mathcal{Z}_K/S^1_d; R)) \cong \bigoplus_{2p-j \geq 0} \widetilde{H}_{j-1}(L_p; R).$$

Here are examples of free circle actions on moment-angle complexes through different simplicial complexes by Theorem 4.4.11.

Example 4.4.13. Let K be the simplicial complex 

. In this case, S^1 acts on K freely if and only if $s_2 = \pm 1$ and $s_3 = \pm 1$. Then $\mathbb{P} = \{\sigma\}$ where $\sigma = \{2, 3\}$

and $L_0 = L_1 = \emptyset$ and $L_2 = \Delta_{\mathbb{P}}$. Thus the nontrivial homology groups of $C_*(L_p, \delta)$ are $H_{-1}(L_0, \delta) = H_{-1}(L_1, \delta) = R$. By Proposition 4.4.9 we have $H^0(\mathcal{Z}_K/S^1; R) = H^2(\mathcal{Z}_K/S^1; R) = R$. The nontrivial multiplications

$$H_{-1}(L_0, \delta) \times H_{-1}(L_0, \delta) \longrightarrow H_{-1}(L_0, \delta)$$
$$H_{-1}(L_0, \delta) \times H_{-1}(L_1, \delta) \longrightarrow H_{-1}(L_1, \delta)$$

are the multiplications of the ring R. Thus $H^*(\mathcal{Z}_K/S^1; R) = R[x]/(x^2)$ where $\deg x = 2$.

We can also use Theorem 4.4.11 to calculate the cohomology of complex projective spaces.

Example 4.4.14 (projective spaces)**.** Let K be the boundary of an $(m-1)$-simplex. The quotient manifold of $\mathcal{Z}_K$ under the diagonal action is the complex projective space $\mathbb{C}P^{m-1}$. Then $\mathbb{P} = \{\sigma\}$ where $\sigma = \{1, 2, \ldots, m\}$. Thus $L_p = \emptyset$ for $p < m$ and $L_m = \Delta_{\mathbb{P}}$. The nontrivial homology groups of $C_*(L_p, \delta)$ are $H_{-1}(L_p, \delta) = R$ for $p < m$. By Proposition 4.4.9, $H^i(\mathcal{Z}_K/S^1_d; R) = R$ for $i = 2p$ and $p < m$. The multiplication on

$$\bigoplus_{0 \le p \le m-1} H_{-1}(L_p, \delta) = \bigoplus_{0 \le p \le m-1} H^{2p}(\mathcal{Z}_K/S^1_d; R) = \bigoplus_{0 \le p \le m-1} R^{2p}$$

is the multiplication of R, where $R^{2p} \cdot R^{2q} = R^{2p+2q}$ if $p + q < m$ and zero otherwise. Thus $H^*(\mathcal{Z}_K/S^1) = R[x]/(x^m)$ with $\deg x = 2$.

Example 4.4.15. Let $K, \mathbb{P}, \sigma, \tau, a, b$ be as in Example 4.4.4. Then

$$L_p = \begin{cases} \emptyset & \text{if } p < m_1 \\ \{\emptyset, \sigma\} & \text{if } m_1 \le p < m_2 \\ \{\emptyset, \sigma, \tau\} & \text{if } m_2 \le p < m_1 + m_2 \\ \Delta_{\mathbb{P}} & \text{if } p = m_1 + m_2 \end{cases} \qquad L_p = \begin{cases} \emptyset & \text{if } p < m_1 = m_2 \\ \{\emptyset, \sigma, \tau\} & \text{if } m_2 \le p < m_1 + m_2 \\ \Delta_{\mathbb{P}} & \text{if } p = m_1 + m_2 \end{cases}$$

with differential operation δ given by $\delta\sigma = a \cdot 1_R$ and $\delta\tau = b \cdot 1_R$.

If $a = 0$, then freeness condition implies that $b = \pm 1$. Thus non-trivial homology groups of $C_*(L_p, \delta)$ include

$$m_1 < m_2,\ H_j(L_p, \delta) \cong \begin{cases} R & \text{if } p < m_2 \text{ and } j = -1 \\ R \cdot [\sigma] & \text{if } m_1 \le p < m_1 + m_2 \text{ and } j = 0 \end{cases}$$
$$m_1 = m_2,\ H_j(L_p, \delta) \cong \begin{cases} R & \text{if } p < m_1 \text{ and } j = -1 \\ R \cdot [\sigma] & \text{if } m_2 \le p < m_1 + m_2 \text{ and } j = 0 \end{cases}$$

where $R \cdot [\sigma]$ is an R-module generated by $[\sigma]$.
If $a \neq 0$, non-trivial homology groups of $C_*(L_p, \delta)$ are

$$m_1 < m_2,\ H_j(L_p,\delta) \cong \begin{cases} R & \text{if } p < m_1 \text{ and } j = -1 \\ R/(a \cdot 1_R) & \text{if } m_1 \le p < m_2 \text{ and } j = -1 \\ R \cdot [b\sigma - a\tau] & \text{if } m_2 \le p < m_1 + m_2 \text{ and } j = 0 \end{cases}$$

$$m_1 = m_2,\ H_j(L_p,\delta) \cong \begin{cases} R & \text{if } p < m_1 \text{ and } j = -1 \\ R \cdot [b\sigma - a\tau] & \text{if } m_2 \le p < m_1 + m_2 \text{ and } j = 0. \end{cases}$$

By Proposition 4.4.9,

$$\text{if } a = 0 \text{ or } m_1 = m_2,\ H^i(\mathcal{Z}_K/S^1; R) = \begin{cases} R & \text{for } i = 2p, p < m_2 \\ R \cdot [\sigma] & \text{for } i = 2p-1, m_1 \le p < m_1 + m_2 \end{cases}$$

$$\text{if } a \neq 0 \text{ and } m_1 \neq m_2,\ H^i(\mathcal{Z}_K/S^1; R) = \begin{cases} R & \text{for } i = 2p, p < m_1 \\ R/(a \cdot 1_R) & \text{for } i = 2p, m_1 \le p < m_2 \\ R \cdot [b\sigma - a\tau] & \text{for } i = 2p-1, m_2 \le p < m_1 + m_2. \end{cases}$$

The multiplication $\times$ on $H^*(\mathcal{Z}_K/S^1; R)$ can be written explicitly as follows.

Suppose that $a = 0$ or $m_1 = m_2$. Consider the multiplicative identity $1_R \in H^{2p}(\mathcal{Z}_K/S^1; R)$ and $1_R \in H^{2q}(\mathcal{Z}_K/S^1; R)$. Then $1_R \times 1_R$ is equal to 1_R if $p + q < m_2$ and zero otherwise. Also, let $1_R \in H^{2p}(\mathcal{Z}_K/S^1; R)$ and $[\sigma] \in H^{2q-1}(\mathcal{Z}_K/S^1; R)$ with $p < m_2$ and $m_1 < q < m_1 + m_2$. Then $[\sigma] \times [\sigma] = 0$ and $1_R \times [\sigma] = [\sigma]$ if $p + q < m_1 + m_2$ and zero otherwise. Thus, $H^*(\mathcal{Z}_K/S^1; R) \cong R[x,\sigma]/(x^{m_2}, \sigma^2)$ where $\deg x = 2$ and $\deg \sigma = 2m_1 - 1$.

If $a \neq 0$ and $m_1 \neq m_2$, analogously, we have $H^*(\mathcal{Z}_K/S^1; R) \cong R[x,y]/(ax^{m_1}, x^{m_2}, y^2, x^{m_1}y)$ where $\deg x = 2$ and $\deg y = 2m_2 - 1$.

4.5 Homotopy types of partial quotients

In this section, we study homotopy types of $\mathcal{Z}_K/S^1$. In particular, we determine the homotopy type of the quotient space $\mathcal{Z}_{\Delta^k_m}/S^1_d$ under the diagonal action.

We first consider properties of moment-angle complexes under subtorus actions.

Lemma 4.5.1. *Let K be a simplicial complex on $[m]$ and let H be a subtorus of T^m acting on $\mathcal{Z}_K$ and $r = \operatorname{rank} H$.*

(a) *For $\sigma \in K$, $(D^2, S^1)^\sigma$ is an H-invariant subspace of $\mathcal{Z}_K$. Consequently, for any simplicial subcomplex $L \subseteq K$, $\mathcal{Z}_L$ is an H-subspace of $\mathcal{Z}_K$.*

(b) *Let* $\Phi\colon H\times \mathcal{Z}_K \longrightarrow \mathcal{Z}_K$ *be the action map. Then there exists a homeomorphism* $\mathrm{sh}\colon H\times\mathcal{Z}_K \longrightarrow H\times\mathcal{Z}_K$ *such that* $p_2\circ \mathrm{sh} = \Phi$*, where* p_2 *is a projection* $H\times\mathcal{Z}_K\longrightarrow \mathcal{Z}_K$*.*

(c) *The action map* $\Phi\colon H\times\mathcal{Z}_K\longrightarrow\mathcal{Z}_K$ *induces a map* $\bar\Phi\colon H\ltimes\mathcal{Z}_K\longrightarrow\mathcal{Z}_K$ *with a homotopy cofibre* $C_{\bar\Phi}\simeq H*\mathcal{Z}_K$*.*

Proof. (a) Since H is a subtorus of T^m, there is an isomorphism $T^r\cong H<T^m$ given by a choice of basis and an $m\times r$ integral matrix $S=(s_{ij})$ such that $g=(g_1,\dots,g_m)\in H$ has the form $g_i=t_1^{s_{i1}}\dots t_r^{s_{ir}}$ with $(t_1,\dots,t_r)\in T^r$. Let $\mathbf{z}=(z_1,\dots,z_m)\in(D^2,S^1)^\sigma$, that is, $z_i\in D^2$ if $i\in\sigma$ and $z_i\in S^1$ if $i\notin\sigma$. Recall that S^1 acts on D^2 by a rotation. Thus if $z_i\in \mathrm{Int}D^2$, then $g_i\cdot z_i\in\mathrm{Int}D^2$ and if $z_i\in\partial D^2$, then $g_i\cdot z_i\in\partial D^2$. Therefore, $g_i\cdot z_i\in D^2$ if $i\in\sigma$, otherwise $g_i\cdot z_i\in S^1$. Thus $g\cdot\mathbf{z}=(g_1\cdot z_1,\dots,g_m\cdot z_m)\in(D^2,S^1)^\sigma$.

(b) Define the shearing map $H\times\mathcal{Z}_K\xrightarrow{\mathrm{sh}}H\times\mathcal{Z}_K$ by $\mathrm{sh}(g,\mathbf{z})=(g,\Phi(g,\mathbf{z}))$ for $g\in H$ and $\mathbf{z}\in\mathcal{Z}_K$. It is a homeomorphism with inverse $\mathrm{sh}^{-1}(g,\mathbf{z})=(g,g^{-1}\mathbf{z})$. Thus $p_2\circ\mathrm{sh}=\Phi$.

(c) Let $*$ be the base point $(1,\dots,1)$ of $\mathcal{Z}_K$. Since the image $\Phi|_{H\times *}$ is in T^m and the inclusion $T^m\longrightarrow\mathcal{Z}_K$ is null homotopic, thus $\Phi|_{H\times *}$ is also null homotopic. The homotopy cofibration $H\hookrightarrow H\times\mathcal{Z}_K\longrightarrow H\ltimes\mathcal{Z}_K$ gives an induced map $\bar\Phi\colon H\ltimes\mathcal{Z}_K\longrightarrow\mathcal{Z}_K$ with $\bar\Phi\circ q\simeq\Phi$. Note that $H*\mathcal{Z}_K$ is the homotopy pushout of $H\xleftarrow{p_1}H\times\mathcal{Z}_K\xrightarrow{p_2}\mathcal{Z}_K$. By the second statement, the shearing map sh is a homeomorphism and $\Phi=p_2\circ sh$, $H*\mathcal{Z}_K$ is the homotopy pushout of $H\xleftarrow{p_1}H\times\mathcal{Z}_K\xrightarrow{\Phi}\mathcal{Z}_K$. Pinching out H, we have $C_{\bar\Phi}\simeq H*\mathcal{Z}_K$. $\square$

By Lemma 4.2.1, the subgroup H acts freely on $\mathcal{Z}_K$ if and only if $H\cap T^\sigma$ is trivial. In this case, the quotient map $q\colon\mathcal{Z}_K\longrightarrow\mathcal{Z}_K/H$ makes the following diagram commutative up to homotopy as a consequence of Lemma 2.3.22.

Lemma 4.5.2. *Let* H *be a subtorus of* T^m *acting freely on* $\mathcal{Z}_K$*. Then there is a homotopy commutative diagram of fibrations*

$$\begin{array}{ccccc}
\mathcal{Z}_K & \longrightarrow & (BS^1,*)^K & \xrightarrow{\ i\ } & BT^m\\
\downarrow{\scriptstyle q} & & \| & & \downarrow{\scriptstyle B\Lambda}\\
\mathcal{Z}_K/H & \longrightarrow & (BS^1,*)^K & \xrightarrow{(B\Lambda)\circ i} & B(T^m/H).
\end{array}$$

4.5.1 Free circle actions

Now we focus on circle actions on $\mathcal{Z}_K$. Suppose that $H=\{(t^{s_1},\dots,t^{s_m})\in T^m\mid t\in S^1\}$ is a circle subgroup T^m, where $s_i\in\mathbb{Z}$. Let Λ be the associated integral matrix of the projection $T^m\longrightarrow T^m/H$. As stated in Section 4.1, the relation between S and Λ is as follows. Since H is a circle subgroup of T^m, there exists an integral $m\times(m-1)$-matrix S' such that the $m\times m$-matrix $\begin{pmatrix}S\mid S'\end{pmatrix}$ is invertible, where $S=(s_1,\dots,s_m)$.

Then $\begin{pmatrix}\Lambda'\\ \Lambda\end{pmatrix}$ is the inverse matrix of $\begin{pmatrix}S \mid S'\end{pmatrix}$ where $\Lambda' = (\lambda'_{ij})$ is an integral $(1 \times m)$-vector and $\Lambda = (\lambda_{ij})$ is the integral $(m-1) \times m$-matrix representing the quotient map $T^m \longrightarrow T^m/H$. Following this, if $s_1 = \pm 1$, then the matrix $\begin{pmatrix} s_1 & \mathbf{0} \\ \mathbf{s} & I_{m-1}\end{pmatrix}$ has an invertible matrix $\begin{pmatrix} s_1 & \mathbf{0} \\ -s_1\mathbf{s} & I_{m-1}\end{pmatrix}$, where $\mathbf{s} = (s_2, \ldots, s_m)$. Thus $\Lambda = \begin{pmatrix}-s_1\mathbf{s} & I_{m-1}\end{pmatrix}$ such that $\mathrm{Ker}\Lambda = H$.

The next statement applies to the special case of quotient spaces $\mathcal{Z}_K/S^1$ under free circle actions when K has ghost vertices.

Lemma 4.5.3. *Suppose that $\{v\}$ is a ghost vertex of K. Let S^1 acts on $\mathcal{Z}_K$ by $(s_1, \ldots, s_m)$. If $s_v = \pm 1$, then S^1 acts on $\mathcal{Z}_K$ freely and $\mathcal{Z}_K/S^1 \simeq \mathcal{Z}_L$, where $L = K_{\bar{V}}$ is the full subcomplex of K on $\bar{V} = V(K) \setminus \{v\}$.*

Proof. Without loss of generality, we can assume $\{1\}$ is a ghost vertex of K. Then $\mathcal{Z}_K = S^1 \times \mathcal{Z}_L$, where $\mathcal{Z}_L$ is an S^1-space by $(s_2, \ldots, s_m)$. If $s_1 = \pm 1$, then S^1-action on $\mathcal{Z}_K$ is an S^1-diagonal action on the product space $S^1 \times \mathcal{Z}_L$. Let Φ, Φ^{-1} be maps $S^1 \times \mathcal{Z}_L \longrightarrow \mathcal{Z}_L$ where Φ is the group action and $\Phi^{-1}(g, \mathbf{z}) = \Phi(g^{-1}, \mathbf{z})$. Then if $s_1 = 1$, Φ^{-1} will induce an S^1-equivariant homeomorphism $\mathcal{Z}_K/S^1 = S^1 \times_{S^1} \mathcal{Z}_L \cong \mathcal{Z}_L$, whose inverse is given by sending $\mathbf{z} \in \mathcal{Z}_L$ to $[(1, \mathbf{z})] \in \mathcal{Z}_K/S^1$. Similarly, if $s_1 = -1$, then the action map Φ will induce an S^1-equivariant homeomorphism. $\square$

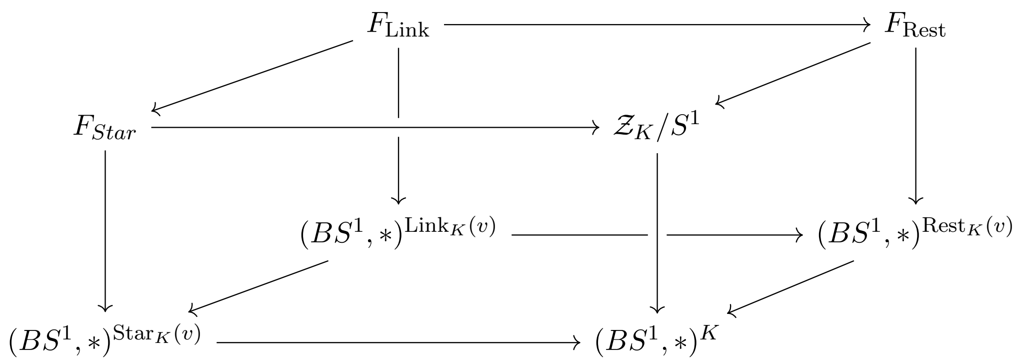
For a simplicial complex K and $v \in V(K)$, let

$$\begin{aligned}
\mathrm{Link}_K(v) &= \{\sigma \in K \mid (v) * \sigma \in K, v \notin \sigma\}\\
\mathrm{Star}_K(v) &= \{\sigma \in K \mid (v) * \sigma \in K\} = (v) * \mathrm{Link}_K(v)\\
\mathrm{Rest}_K(v) &= \{\sigma \in K \mid V(K) \setminus \{v\}\}.
\end{aligned}$$

There exists a pushout of simplicial complexes

$$\begin{array}{ccc}
\mathrm{Link}_K(v) & \longrightarrow & \mathrm{Rest}_K(v)\\
\downarrow & & \downarrow\\
\mathrm{Star}_K(v) & \longrightarrow & K
\end{array}$$

which induces a topological pushout of corresponding Davis-Januszkiewicz spaces by Lemma 2.3.26. Mapping these spaces to $B(T^m/S^1)$, denote by $F_{\mathrm{Link}}, F_{\mathrm{Star}}$ and F_{Rest} the correspond homotopy fibres, respectively. Then there is a diagram of homotopy pushouts as follows.



If a circle action on $\mathcal{Z}_K$ satisfies the condition in Lemma 4.5.3, it is possible to identify the homotopy types of these fibres for special cases.

Theorem 4.5.4. *Let S^1 acts on $\mathcal{Z}_K$ freely. Assume that there exits a vertex $v \in K$ such that $s_v = \pm 1$.*

(a) *There exist homotopy equivalences*

$$F_{\text{Link}} \simeq \mathcal{Z}_{\text{Link}_K(v)}, \ F_{\text{Rest}} \simeq \mathcal{Z}_{\text{Rest}_K(v)}, \ F_{\text{Star}} \simeq \mathcal{Z}_{\text{Link}_K(v)}/S^1.$$

(b) *The quotient space $\mathcal{Z}_K/S^1$ is the homotopy pushout of the diagram $\mathcal{Z}_{\text{Link}_K(v)}/S^1 \xleftarrow{q} \mathcal{Z}_{\text{Link}_K(v)} \xrightarrow{\iota} \mathcal{Z}_{\text{Rest}_K(v)}$, where ι is the map induced by the simplicial inclusion and q is the quotient map.*

Proof. (a) Without loss of generality, assume that $v = \{1\}$. Since $\text{Link}_K(1)$ and $\text{Rest}_K(1)$ are on the vertex set $\{2, \ldots, m\}$, $F_{\text{Link}} \simeq \mathcal{Z}_{\text{Link}_K(1)}$, $F_{\text{Rest}} \simeq \mathcal{Z}_{\text{Rest}_K(v)}$ by Lemma 4.5.3. Since S^1 acts on $\mathcal{Z}_{\text{Star}_K(1)}$ freely, its quotient $\mathcal{Z}_{\text{Star}_K(1)}/S^1$ is homotopy equivalent to the Borel construction $ES^1 \times_{S^1} \mathcal{Z}_{\text{Star}_K(1)}$, where $\mathcal{Z}_{\text{Star}_K(1)} = D^2 \times \mathcal{Z}_{\text{Link}_K(1)}$. Since S^1 acts on $\mathcal{Z}_{\text{Link}_K(1)}$ freely, $F_{\text{Star}} = \mathcal{Z}_{\text{Star}_K(1)}/S^1 \simeq ES^1 \times_{S^1} \mathcal{Z}_{\text{Star}_K(1)} \simeq ES^1 \times_{S^1} (D^2 \times \mathcal{Z}_{\text{Link}_K(1)}) \simeq \mathcal{Z}_{\text{Link}_K(1)}/S^1$.

(b) It suffices to identify the maps between these fibres. Since $s_1 = \pm 1$, the matrix Λ representing the projection $T^m \longrightarrow T^m/S^1$ is given by $\begin{pmatrix} -s_1\mathbf{s} & I_{m-1} \end{pmatrix}$ with $\mathbf{s} = (s_2, \ldots, s_m)^t$. Therefore, the composite $BT^{m-1} \xrightarrow{Bj} BT^m \longrightarrow B(T^m/S^1)$ is the identity map, where j is an inclusion of T^{m-1} to the last $m-1$ coordinates of T^m. Thus for L being $\text{Link}_K(1)$ or $\text{Rest}_K(1)$, the composite $(BS^1,*)^L \longrightarrow BT^m \xrightarrow{B\Lambda} B(T^m/S^1)$ is the standard inclusion $(BS^1,*)^L \longrightarrow BT^{m-1}$ if T^{m-1} is identified with T^m/S^1. Therefore, the map between the fibres $F_{\text{Link}} \longrightarrow F_{\text{Rest}}$ is the inclusion between the corresponding moment-angle complexes $\mathcal{Z}_{\text{Link}} \xrightarrow{\iota} \mathcal{Z}_{\text{Rest}}$.

There exists an induced free circle action on $\mathcal{Z}_{\text{Link}_K(1)}$ given by $g \cdot (z_2, \ldots, z_m) = (g^{s_2} \cdot z_2, \ldots, g^{s_m} \cdot z_m)$. We first note that $\operatorname{Im} \mathbf{s} = \{(t_2^{s_2}, \ldots, t_m^{s_m}) \mid (t_2, \ldots, t_m) \in T^{m-1}\}$ is a circle subgroup of T^{m-1}. Because we assume that $\{1\} \in K$, by Lemma 4.2.1, the freeness condition of a circle action on $\mathcal{Z}_K$ implies that $\gcd(s_2, \ldots, s_m) = 1$. To see that this induced action is free, send $(z_2, \ldots, z_m) \in \mathcal{Z}_{\mathrm{Link}_K(1)}$ to $(0, z_2, \ldots, z_m) \in \mathcal{Z}_{\mathrm{Star}_K(1)}$. The isotropy group of $(0, z_2, \ldots, z_m)$ under the original S^1-action by $(s_1, s_2, \ldots, s_m)$ is equal to the isotropy group of $(z_2, \ldots, z_m)$ under the induced S^1-action by $(s_2, \ldots, s_m)$. Since the original S^1-action acts on $\mathcal{Z}_{\mathrm{Star}_K(1)}$ freely, the isotropy group of $(0, z_2, \ldots, z_m)$ is trivial, which means that the S^1-action on $\mathcal{Z}_{\mathrm{Link}_K(1)}$ by $(s_2, \ldots, s_m)$ is free.

This circle subgroup of T^{m-1} has an associated integral matrix π representing the quotient map $T^{m-1} \longrightarrow T^{m-1}/S^1$. There is a homotopy commutative diagram of fibrations

$$\begin{array}{ccccc}
\mathcal{Z}_{\mathrm{Link}_K(1)} & \longrightarrow & (BS^1,*)^{\mathrm{Link}_K(1)} & \xrightarrow{(B\Lambda)\circ i} & B(T^m/S^1) \\
\| & & \| & & \downarrow \simeq \\
\mathcal{Z}_{\mathrm{Link}_K(1)} & \longrightarrow & (BS^1,*)^{\mathrm{Link}_K(1)} & \xrightarrow{\eta} & BT^{m-1} \\
\downarrow q & & \| & & \downarrow B\pi \\
\mathcal{Z}_{\mathrm{Link}_K(1)}/S^1 & \longrightarrow & (BS^1,*)^{\mathrm{Link}_K(1)} & \xrightarrow{\gamma=(B\pi)\circ\eta} & B(T^{m-1}/S^1) \\
\| & & \downarrow j_2 & & \downarrow j_2 \\
\mathcal{Z}_{\mathrm{Link}_K(1)}/S^1 & \longrightarrow & BS^1 \times (BS^1,*)^{\mathrm{Link}_K(1)} & \xrightarrow{\mathrm{id}\times\gamma} & BS^1 \times B(T^{m-1}/S^1)
\end{array} \tag{4.25}$$

where the top square is obtained by T^m/S^1 being identified with T^{m-1}, the second square is due to Lemma 4.5.2, q is a quotient map and j_2 is an inclusion into the second coordinate.

In fact, the homotopy fibration at the bottom row in (4.25) is equivalent to the homotopy fibration obtained by mapping $(BS^1,*)^{\mathrm{Star}_K(1)}$ to $B(T^m/S^1)$

$$F_{\mathrm{Star}} \longrightarrow (BS^1,*)^{\mathrm{Star}_K(1)} \xrightarrow{(B\Lambda)\circ i} B(T^m/S^1).$$

The relation between $(s_2, \ldots, s_m)$ and π implies that T^m/S^1 is isomorphic to $\mathrm{Im}\,\mathbf{s} \times \mathrm{Im}\,\pi$, where $\mathrm{Im}\,\mathbf{s}$ and $\mathrm{Im}\,\pi$ are torus groups with rank 1 and $m-2$, respectively. Thus there are isomorphisms $T^m/S^1 \xrightarrow{M_1} \mathrm{Im}\,\mathbf{s} \times \mathrm{Im}\,\pi \xrightarrow{M_2} S^1 \times T^{m-2}$, which are represented by an $(m-1) \times (m-1)$-integral invertible matrices M_1 and M_2. Let $M = M_2 M_1$. Composing BM with $(B\Lambda) \circ i$, we have a diagram of homotopy fibrations

$$\begin{array}{ccccc}
F_{\mathrm{Star}} & \longrightarrow & (BS^1,*)^{\mathrm{Star}_K(1)} & \xrightarrow{(B\Lambda)\circ i} & B(T^m/S^1) \\
\downarrow & & \| & & \downarrow BM \\
F & \longrightarrow & (BS^1,*)^{\mathrm{Star}_K(1)} & \xrightarrow{(BM)\circ(B\Lambda)\circ i} & BS^1 \times BT^{m-2}
\end{array} \tag{4.26}$$

where the left square is homotopy commutative and the right one is commutative and all vertical maps are homotopy equivalences.

Since $(BS^1,*)^{\mathrm{Star}_K(1)} = BS^1 \times (BS^1,*)^{\mathrm{Link}_K(1)}$, the composite $(BM)\circ B\Lambda \circ i = \mathrm{id}\times\gamma$.

Combining these homotopy commutative diagrams (4.25) and (4.26), the simplicial inclusion $\mathrm{Link}_K(1) \longrightarrow \mathrm{Star}_K(1)$ induces a quotient map of the fibres $\mathcal{Z}_{\mathrm{Link}_K(1)} \xrightarrow{q} \mathcal{Z}_{\mathrm{Link}_K(1)}/S^1$. □

In some special case, the map $\mathcal{Z}_{\mathrm{Link}_K(v)} \longrightarrow \mathcal{Z}_{\mathrm{Rest}_K(v)}$ is null homotopic. For example, if for some $v \in K$ such that $\mathrm{Link}_K(v) = \emptyset$, then $\mathcal{Z}_{\mathrm{Link}_K(v)} \longrightarrow \mathcal{Z}_{\mathrm{Rest}_K(v)}$ is null homotopic ([23, Lemma 3.3]). If so, there is a homotopy splitting of the quotient $\mathcal{Z}_K/S^1$.

Corollary 4.5.5. *Let K and S^1 satisfy the assumption in Theorem 4.5.4. Suppose that for the same vertex v, the map $\mathcal{Z}_{\mathrm{Link}_K(v)} \longrightarrow \mathcal{Z}_{\mathrm{Rest}_K(v)}$ is null homotopic. Then there exists a homotopy splitting $\mathcal{Z}_K/S^1 \simeq \mathcal{Z}_{\mathrm{Rest}_K(v)} \vee C_q$, where C_q is the homotopy cofibre of the quotient map $\mathcal{Z}_{\mathrm{Link}_K(v)} \xrightarrow{q} \mathcal{Z}_{\mathrm{Link}_K(v)}/S^1$.*

In particular, if $\mathrm{Link}_K(v) = \emptyset$, then $\mathcal{Z}_K/S^1 \simeq \mathcal{Z}_{\mathrm{Rest}_K(v)} \vee S^2 \vee (S^1 * T^{m-2})$.

Proof. If the map $\mathcal{Z}_{\mathrm{Link}_K(v)} \longrightarrow \mathcal{Z}_{\mathrm{Rest}_K(v)}$ is null homotopic, there is an iterated homotopy pushout

$$\begin{array}{ccccc}
\mathcal{Z}_{\mathrm{Link}_K(v)} & \longrightarrow & * & \longrightarrow & \mathcal{Z}_{\mathrm{Rest}_K(v)} \\
\downarrow q & & \downarrow & & \downarrow \\
\mathcal{Z}_{\mathrm{Link}_K(v)}/S^1 & \longrightarrow & C_q & \longrightarrow & \mathcal{Z}_K/S^1.
\end{array}$$

Thus the first statement follows.

If $\mathrm{Link}_K(v) = \emptyset$, then $\mathcal{Z}_{\mathrm{Link}_K(v)} \simeq T^{m-1}$ and $\mathrm{Star}_K(v) = \{v\}$. Consider the following diagram of fibration sequences

$$\begin{array}{ccccc}
\mathcal{Z}_{\emptyset} & \longrightarrow & * & \longrightarrow & B(T^m/S^1) \\
\downarrow q & & \downarrow & & \| \\
\mathcal{Z}_{\emptyset}/S^1 & \longrightarrow & BS^1_v & \longrightarrow & B(T^m/S^1) \\
p\downarrow\simeq & & \| & & \downarrow\simeq \\
\Omega BT^{m-2} & \longrightarrow & BS^1_v & \xrightarrow{\mathrm{id}\times *} & BS^1 \times BT^{m-2}.
\end{array}$$

Here the top diagram between homotopy fibrations is induced by $\emptyset \longrightarrow \{v\}$ and the bottom diagram is an equivalence of fibration sequences, proved as a special case of diagram (4.26) in Theorem 4.5.4, due to the isomorphism $T^m/S^1 \cong S^1 \times T^{m-2}$. Since p is a homotopy equivalence, we have $C_q \simeq C_{pq}$. Note that the composition pq is induced by projecting $T^m/S^1 \longrightarrow T^{m-2}$. Precisely, it is the map $T^m/S^1 \xrightarrow{\cong} S^1 \times T^{m-2} \xrightarrow{p_2} T^{m-2}$. Therefore, it remains to identify the homotopy cofibre of p_2.

Let $\pi_2\colon X\times Y \longrightarrow Y$ be a projection, where X and Y are two connected CW-complexes. Consider the following homotopy commutative diagram

$$\begin{array}{ccccc} X\times Y & \xrightarrow{\pi_1} & X & \longrightarrow & * \\ \downarrow{\scriptstyle \pi_2} & & \downarrow & & \downarrow \\ Y & \longrightarrow & X * Y & \longrightarrow & C_{\pi_2} \end{array}$$

where the left and right diagrams are homotopy pushouts. Since $X \longrightarrow X * Y$ is null homotopic, $C_{\pi_2} \simeq \Sigma X \vee X * Y$. Thus $C_q \simeq \Sigma S^1 \vee (S^1 * T^{m-2})$. □

Example 4.5.6. Denote by $\mathcal{Z}_m$ the moment-angle complex corresponding to m disjoint points. If S^1 acts freely on $\mathcal{Z}_m$ by $(s_1,\ldots,s_m)$ with some $s_j = \pm 1$, then $\mathcal{Z}_m/S^1 \simeq \mathcal{Z}_{m-1} \vee S^2 \vee (S^1 * T^{m-2})$.

4.5.2 Homotopy types of cofibres

In this section, we determine homotopy cofibres $C_{k,m}$ of quotient maps $q_{k,m}\colon \mathcal{Z}_{\Delta^k_m} \longrightarrow \mathcal{Z}_{\Delta^k_m}/S^1_d$ under the diagonal action. Note that if $K = \Delta^k_m$ is on the vertex set $\{1,\ldots,m\}$, then $\mathrm{Link}_K\{1\}$ has vertex set $\{2,\ldots,m\}$ and is simplicially isomorphic to Δ^{k-1}_{m-1}. Thus we have a pushout of simplicial complexes

$$\begin{array}{ccc} \Delta^{k-1}_{m-1} & \longrightarrow & \Delta^k_{m-1} \\ \downarrow & & \downarrow \\ (1) * \Delta^{k-1}_{m-1} & \longrightarrow & \Delta^k_m. \end{array}$$

This pushout implies homotopy pushouts of the corresponding moment-angle complexes and of their quotient spaces under the diagonal action (Lemma 2.3.29)

$$\begin{array}{ccc} S^1\times \mathcal{Z}_{\Delta^{k-1}_{m-1}} & \xrightarrow{\mathrm{id}\times *} & S^1\times \mathcal{Z}_{\Delta^k_{m-1}} \\ \downarrow{\scriptstyle *\times\mathrm{id}} & & \downarrow{\scriptstyle f_{k,m}} \\ \mathcal{Z}_{\Delta^{k-1}_{m-1}} & \longrightarrow & \mathcal{Z}_{\Delta^k_m} \end{array} \qquad \begin{array}{ccc} \mathcal{Z}_{\Delta^{k-1}_{m-1}} & \xrightarrow{\simeq *} & \mathcal{Z}_{\Delta^k_{m-1}} \\ \downarrow{\scriptstyle q_{k-1,m-1}} & & \downarrow{\scriptstyle g_{k,m}} \\ \mathcal{Z}_{\Delta^{k-1}_{m-1}}/S^1_d & \longrightarrow & \mathcal{Z}_{\Delta^k_m}/S^1_d \end{array} \tag{4.27}$$

where $f_{k,m}$ is a map induced by the simplicial inclusion $\Delta^k_{m-1} \longrightarrow \Delta^k_m$ and the map $g_{k,m}$ is induced by $f_{k,m}$ between the quotient spaces.

The left diagram in (4.27) implies an iterated homotopy pushout

$$\begin{array}{ccccc} S^1\times \mathcal{Z}_{\Delta^{k-1}_{m-1}} & \longrightarrow & S^1 & \longrightarrow & S^1\times \mathcal{Z}_{\Delta^k_{m-1}} \\ \downarrow{\scriptstyle *\times\mathrm{id}} & & \downarrow & & \downarrow{\scriptstyle f_{k,m}} \\ \mathcal{Z}_{\Delta^{k-1}_{m-1}} & \longrightarrow & S^1 * \mathcal{Z}_{\Delta^{k-1}_{m-1}} & \longrightarrow & \mathcal{Z}_{\Delta^k_m} \end{array}$$

which induces the following iterated homotopy pushout after pinching out S^1

$$\begin{array}{ccccc} S^1 \ltimes \mathcal{Z}_{\Delta^{k-1}_{m-1}} & \longrightarrow & * & \longrightarrow & S^1 \ltimes \mathcal{Z}_{\Delta^{k}_{m-1}} \\ \big\downarrow {\scriptstyle *\ltimes \mathrm{id}} & & \big\downarrow & & \big\downarrow {\scriptstyle \bar{f}_{k,m}} \\ \mathcal{Z}_{\Delta^{k-1}_{m-1}} & \longrightarrow & S^1 * \mathcal{Z}_{\Delta^{k-1}_{m-1}} & \xrightarrow{h_{k,m}} & \mathcal{Z}_{\Delta^{k}_{m}}. \end{array} \tag{4.28}$$

Thus the right square of (4.28) implies a splitting homotopy cofibration $S^1 \ltimes \mathcal{Z}_{\Delta^k_{m-1}} \xrightarrow{\bar{f}_{k,m}} \mathcal{Z}_{\Delta^k_m} \longrightarrow C_{\bar{f}_{k,m}}$, where the homotopy cofibre $C_{\bar{f}_{k,m}}$ is homotopic to $S^1 * \mathcal{Z}_{\Delta^{k-1}_{m-1}}$.

Since the map $\mathcal{Z}_{\Delta^{k-1}_{m-1}} \longrightarrow \mathcal{Z}_{\Delta^{k}_{m-1}}$ is null homotopic, the right homotopy pushout in (4.27) also implies an iterated homotopy pushout

$$\begin{array}{ccccc} \mathcal{Z}_{\Delta^{k-1}_{m-1}} & \longrightarrow & * & \longrightarrow & \mathcal{Z}_{\Delta^{k}_{m-1}} \\ \big\downarrow {\scriptstyle q_{k-1,m-1}} & & \big\downarrow & & \big\downarrow {\scriptstyle g_{k,m}} \\ \mathcal{Z}_{\Delta^{k-1}_{m-1}}/S^1_d & \longrightarrow & C_{k-1,m-1} & \xrightarrow{h'_{k,m}} & \mathcal{Z}_{\Delta^{k}_{m}}/S^1_d. \end{array} \tag{4.29}$$

The right square of (4.29) implies a splitting homotopy cofibration $\mathcal{Z}_{\Delta^k_{m-1}} \xrightarrow{g_{k,m}} \mathcal{Z}_{\Delta^k_m}/S^1_d \longrightarrow C_{g_{k,m}}$, where the homotopy cofibre $C_{g_{k,m}}$ is homotopic to $C_{k-1,m-1}$.

Lemma 4.5.7. *There exists a homotopy equivalence $\mathcal{Z}_{\Delta^k_m}/S^1_d \simeq \mathcal{Z}_{\Delta^k_{m-1}} \vee C_{k-1,m-1}$, where $C_{k-1,m-1}$ is the homotopy cofibre of the quotient map $\mathcal{Z}_{\Delta^{k-1}_{m-1}} \longrightarrow \mathcal{Z}_{\Delta^{k-1}_{m-1}}/S^1_d$.*

Hence, to determine the homotopy type of $\mathcal{Z}_{\Delta^k_m}/S^1_d$, it suffices to determine the homotopy type of $C_{k,m}$.

Lemma 4.5.8. *There exists a homotopy commutative diagram*

$$\begin{array}{ccc} S^1 \ltimes \mathcal{Z}_{\Delta^k_{m-1}} & \xrightarrow{\bar{\Phi}^{-1}} & \mathcal{Z}_{\Delta^k_{m-1}} \\ \big\downarrow {\scriptstyle \bar{f}_{k,m}} & & \big\downarrow {\scriptstyle g_{k,m}} \\ \mathcal{Z}_{\Delta^k_m} & \xrightarrow{q_{k,m}} & \mathcal{Z}_{\Delta^k_m}/S^1_d \end{array} \tag{4.30}$$

where $\bar{\Phi}^{-1}$ is induced by the map $S^1 \times \mathcal{Z}_{\Delta^k_{m-1}} \xrightarrow{\Phi^{-1}} \mathcal{Z}_{\Delta^k_{m-1}}$ given by $\Phi^{-1}(t,\mathbf{z}) = t^{-1}\cdot \mathbf{z}$.

Proof. The simplicial inclusion $\Delta^k_{m-1} \longrightarrow \Delta^k_m$ gives rise to a commutative diagram

$$\begin{array}{ccc} S^1 \times \mathcal{Z}_{\Delta^k_{m-1}} & \xrightarrow{\alpha} & S^1 \times_{S^1_d} \mathcal{Z}_{\Delta^k_{m-1}} \\ \big\downarrow {\scriptstyle f_{k,m}} & & \big\downarrow {\scriptstyle \beta} \\ \mathcal{Z}_{\Delta^k_m} & \xrightarrow{q_{k,m}} & \mathcal{Z}_{\Delta^k_m}/S^1_d \end{array}$$

where the horizontal maps α and $q_{k,m}$ are quotient maps and β is a map between quotient spaces induced by $f_{k,m}$. By Lemma 4.5.3, there is a homotopy equivalence

$$S^1 \times_{S^1_d} \mathcal{Z}_{\Delta^k_{m-1}} \overset{\eta}{\simeq} \mathcal{Z}_{\Delta^k_{m-1}}$$

where η sends $[(t, \mathbf{z})]$ to $\Phi^{-1}(t, \mathbf{z})$. It follows easily that $\eta \circ \alpha(t, \mathbf{z}) = t^{-1} \cdot \mathbf{z} = \Phi^{-1}(t, \mathbf{z})$. Thus, replacing $S^1 \times_{S^1} \mathcal{Z}_{\Delta^k_{m-1}}$ by its homotopy equivalent space $\mathcal{Z}_{\Delta^k_{m-1}}$ due to η, there is a homotopy commutative diagram,

$$\begin{array}{ccc}
S^1 \times \mathcal{Z}_{\Delta^k_{m-1}} & \xrightarrow{\Phi^{-1}} & \mathcal{Z}_{\Delta^k_{m-1}} \\
\downarrow{\scriptstyle f_{k,m}} & & \downarrow{\scriptstyle \beta\circ\eta} \\
\mathcal{Z}_{\Delta^k_m} & \xrightarrow{q_{k,m}} & \mathcal{Z}_{\Delta^k_m}/S^1_d
\end{array}$$

where $\beta \circ \eta$ coincides the map $g_{k,m}$ in the diagram (4.27), since they are the maps induced by $f_{k,m}$ after we have chosen an certain homotopy type of quotient spaces.

Since the restriction of Φ^{-1} to the first coordinate S^1 is null homotopic, we obtain the homotopy commutative diagram in the statement. □

The homotopy commutative diagram (4.30) gives rise to the following homotopy commutative diagram (see [38, Theorem 7.6.3])

$$\begin{array}{ccccc}
S^1 \ltimes \mathcal{Z}_{\Delta^k_{m-1}} & \xrightarrow{\bar{\Phi}^{-1}} & \mathcal{Z}_{\Delta^k_{m-1}} & \longrightarrow & S^1 * \mathcal{Z}_{\Delta^k_{m-1}} \\
\downarrow{\scriptstyle \bar{f}_{k,m}} & & \downarrow{\scriptstyle g_{k,m}} & & \downarrow \\
\mathcal{Z}_{\Delta^k_m} & \xrightarrow{q_{k,m}} & \mathcal{Z}_{\Delta^k_m}/S^1_d & \longrightarrow & C_{k,m} \\
\downarrow{\scriptstyle r_{k,m}} & & \downarrow{\scriptstyle r'_{k,m}} & & \downarrow \\
C_{\bar{f}_{k,m}} & \xrightarrow{\phi_{k,m}} & C_{g_{k,m}} & \longrightarrow & Q_{k,m}
\end{array} \tag{4.31}$$

where each row and column is a homotopy cofibration. The homotopy pushouts (4.28) and (4.29) imply that $C_{\bar{f}_{k,m}} \simeq S^1 * \mathcal{Z}_{\Delta^{k-1}_{m-1}}$ and $C_{g_{k,m}} \simeq C_{k-1,m-1}$ and the first and second columns of (4.31) are splitting homotopy cofibrations.

We will determine the homotopy type of $C_{k,m}$. The idea is to find simplicial complexes $L^k_{j,m}$ such that their quotient spaces under diagonal actions give the homotopy type of the cofibre of the quotient map. We firstly identify the homotopy type of maps $\phi_{k,m}$.

Lemma 4.5.9. *Let $K = \Delta^k_m$ and $L^k_{1,m} = K \cup \Delta_{\{1,2,\ldots,m-1\}}$. Then $C_{\bar{f}_{k,m}} \simeq \mathcal{Z}_{L^k_{1,m}}$ and $\mathcal{Z}_{L^k_{1,m}}/S^1_d \simeq C_{g_{k,m}}$. Under these homotopy equivalences, the maps $\phi_{k,m}$ are equivalent to the quotient maps $\mathcal{Z}_{L^k_{1,m}} \longrightarrow \mathcal{Z}_{L^k_{1,m}}/S^1_d$.*

Proof. Since $K \cap \Delta_{\{1,2,\dots,m-1\}} = \Delta^k_{m-1}$, we have a pushout of simplicial complexes

$$\begin{array}{ccc} \Delta^k_{m-1} & \longrightarrow & \Delta_{\{1,2,\dots,m-1\}} \\ \big\downarrow & & \big\downarrow \\ \Delta^k_m & \longrightarrow & L^k_{1,m}. \end{array}$$

There are two homotopy pushouts of topological spaces, one of moment-angle complexes and one of quotient spaces of moment-angle complexes

$$\begin{array}{ccc} \mathcal{Z}_{\Delta^k_{m-1}} \times S^1 & \xrightarrow{*\times \mathrm{id}} & S^1 \\ {\scriptstyle f_{k,m}}\big\downarrow & & \big\downarrow \\ \mathcal{Z}_{\Delta^k_m} & \longrightarrow & \mathcal{Z}_{L^k_{1,m}} \end{array} \quad \text{and} \quad \begin{array}{ccc} \mathcal{Z}_{\Delta^k_{m-1}} & \longrightarrow & * \\ \big\downarrow{\scriptstyle g_{k,m}} & & \big\downarrow \\ \mathcal{Z}_{\Delta^k_m}/S^1_d & \longrightarrow & \mathcal{Z}_{L^k_{1,m}}/S^1_d. \end{array}$$

Pinching out S^1 in the left pushout above, we have a homotopy cofibration

$$\mathcal{Z}_{\Delta^k_{m-1}} \rtimes S^1 \xrightarrow{\bar{f}_{k,m}} \mathcal{Z}_{\Delta^k_m} \longrightarrow \mathcal{Z}_{L^k_{1,m}}. \tag{4.32}$$

Taking the corresponding quotient spaces of (4.32) and the homotopy commutative diagram (4.31), there exists a homotopy commutative diagram of homotopy fibrations

$$\begin{array}{ccccc} \mathcal{Z}_{\Delta^k_{m-1}} \rtimes S^1 & \xrightarrow{\bar{f}_{k,m}} & \mathcal{Z}_{\Delta^k_m} & \longrightarrow & \mathcal{Z}_{L^k_{1,m}} \\ \big\downarrow & & \big\downarrow{\scriptstyle q_{k,m}} & & \big\downarrow \\ \mathcal{Z}_{\Delta^k_{m-1}} & \xrightarrow{g_{k,m}} & \mathcal{Z}_{\Delta^k_m}/S^1_d & \longrightarrow & \mathcal{Z}_{L^k_{1,m}}/S^1_d. \end{array}$$

Thus the maps $\phi_{k,m}$ in (4.31) are quotient maps up to homotopy and $C_{\bar{f}_{k,m}} \simeq \mathcal{Z}_{L^k_{1,m}}$ and $\mathcal{Z}_{L^k_{1,m}}/S^1_d \simeq C_{g_{k,m}} \simeq C_{k-1,m-1}$. □

We have identified the homotopy type of $C_{k-1,m-1}$ as $\mathcal{Z}_{L^k_{1,m}}/S^1_d$. We will continue to show that the homotopy cofibre $C_{k,m}$ has the following form.

Theorem 4.5.10. *There exists homotopy equivalences*

$$C_{k,m} \simeq \mathbb{C}P^{k+2} \vee \Big(\bigvee_{i=1}^{k+1} S^{2i-1} * \mathcal{Z}_{\Delta^{k+1-i}_{m-i}}\Big) \vee (S^{2k+3} * T^{m-k-2}).$$

The main idea of the proof of Theorem 4.5.10 is to construct a sequence of simplicial complexes $L^k_{j,m}$ and iterate to determine the homotopy types of their quotient spaces under the diagonal action. We give an explicit construction of these simplicial complexes $L^k_{j,m}$ from the k-skeleton Δ^k_m.

Denote by $\Delta_{\{i_1,\ldots,i_p\}}$ a simplex on vertices $\{i_1,\ldots,i_p\}$. Let $L^k_{0,m} = \Delta^k_m$. Define $L^k_{1,m} = \Delta^k_m \cup \Delta_{\{1,2,\ldots,m-1\}}$ and $L^k_{j,m} = L^k_{j-1,m} \cup \Delta_{\{1,\ldots,\widehat{m-j+1},\ldots,m\}}$, where $\widehat{m-j+1}$ means that this vertex is omitted.

We first prove that the simplicial inclusion $L^{k-1}_{j,m} \longrightarrow L^k_{j,m}$ induces a null homotopic map on corresponding moment-angle complexes.

Lemma 4.5.11. *For $1 \leq j \leq k+1$, the inclusion $J\colon \mathcal{Z}_{L^{k-1}_{j,m}} \longrightarrow \mathcal{Z}_{L^k_{j,m}}$ is null homotopic.*

Proof. Let $K = \bigcup\limits_{m-j+1\leq q\leq m} \Delta_{\{1,\ldots,\hat{q},\ldots,m-1\}}$. Thus $\mathcal{Z}_K = (\prod\limits_{m-j} D^2) \times \mathcal{Z}_{\partial\Delta^{j-1}}$, where $\partial\Delta^{j-1}$ is the boundary of a simplex on vertices $\{m-j+1,\ldots,m\}$. Note that $L^k_{j,m} = \Delta^k_m \cup K$ and $\mathcal{Z}_{L^k_{j,m}} = \mathcal{Z}_{\Delta^k_m} \cup \mathcal{Z}_K$.

First, there is a filtration of simplicial complexes $\Delta^{k-1}_m \subseteq (1) * \Delta^{k-1}_{m-1} \subseteq \Delta^k_m$, where Δ^{k-1}_{m-1} in the middle is on vertices $\{2,\ldots,m\}$, which implies a filtration of simplicial complexes $L^{k-1}_{j,m} \subseteq ((1) * \Delta^{k-1}_{m-1}) \cup K \subseteq L^k_{j,m}$. In particular, $((1) * \Delta^{k-1}_{m-1}) \cup K = (1) * (\Delta^{k-1}_{m-1} \cup K_1)$, where K_1 is the full subcomplex of K on vertices $\{2,\ldots,m\}$. Thus, the inclusion J factors through the corresponding moment-angle complexes

$$\mathcal{Z}_{L^{k-1}_{j,m}} \xrightarrow{i_1} D^2 \times (\mathcal{Z}_{\Delta^{k-1}_{m-1}} \cup \mathcal{Z}_{K_1}) \xrightarrow{i'_1} \mathcal{Z}_{L^k_{j,m}}.$$

By the construction of $L^k_{j,m}$, $\Delta^{k-1}_{m-1} \cup K_1 = L^{k-1}_{j,m-1}$ which is a full subcomplex of $L^{k-1}_{j,m}$ on vertices $\{2,\ldots,m\}$. Denote by r_1 the retraction $\mathcal{Z}_{L^{k-1}_{j,m}} \longrightarrow \mathcal{Z}_{L^{k-1}_{j,m-1}}$. Then the map i_1 factors through r_1 and a coordinate inclusion $\iota_1\colon \mathcal{Z}_{L^{k-1}_{j,m-1}} \longrightarrow D^2 \times \mathcal{Z}_{L^{k-1}_{j,m-1}}$ up to homotopy. Namely, there exists a diagram

$$\begin{array}{ccccc} & & \mathcal{Z}_{L^{k-1}_{j,m}} & & \\ & \swarrow^{r_1} & \downarrow{\scriptstyle i_1} & \searrow^{J} & \\ \mathcal{Z}_{L^{k-1}_{j,m-1}} & \xrightarrow[\iota_1]{} & D^2 \times \mathcal{Z}_{L^{k-1}_{j,m-1}} & \xrightarrow[i'_1]{} & \mathcal{Z}_{L^k_{j,m}} \end{array}$$

where the left triangle is homotopy commutative and the right one is commutative. In particular, the composition $i'_1\iota_1$ coincides with the map induced by the simplicial inclusion $L^{k-1}_{j,m-1} \longrightarrow L^k_{j,m}$ which has a filtration $L^{k-1}_{j,m-1} \xrightarrow{j_2} L^k_{j,m-1} \xrightarrow{j'_2} L^k_{j,m}$.

The same strategy applies for $L^{k-1}_{j,m-1} \xrightarrow{j_2} L^k_{j,m-1}$. Repeating the above procedure, there are diagrams for $1 \leq q \leq m-k-1$

$$\begin{array}{ccccc} & & \mathcal{Z}_{L^{k-1}_{j,m-q+1}} & & \\ & \swarrow^{r_q} & \downarrow{\scriptstyle i_q} & \searrow^{j_q} & \\ \mathcal{Z}_{L^{k-1}_{j,m-q}} & \xrightarrow{\iota_q} & D^2 \times \mathcal{Z}_{L^{k-1}_{j,m-q}} & \xrightarrow{i'_q} & \mathcal{Z}_{L^k_{j,m-q+1}} \\ & \searrow_{j_{q+1}} & & \nearrow_{j'_{q+1}} & \\ & & \mathcal{Z}_{L^k_{j,m-q}} & & \end{array} \tag{4.33}$$

where each $L^{k-1}_{j,m-q}$ is a full subcomplex of $L^{k-1}_{j,m-q+1}$ on vertices $\{q+1, \ldots, m\}$, the top left triangle is homotopy commutative and the other two are commutative.

If $q = m-k-1$, observe the composition $\mathcal{Z}_{L^{k-1}_{j,k+1}} \xrightarrow{j_{m-k}} \mathcal{Z}_{L^k_{j,k+1}} \xrightarrow{j'_{m-k}} \mathcal{Z}_{L^k_{j,k+2}}$. Since $L^k_{j,k+1}$ is a full subcomplex of $L^k_{j,k+2}$ on vertices $\{m-k, \ldots, m\}$, it contains all subsets of $\{m-k, \ldots, m\}$ with cardinality at most $k+1$. Thus $L^k_{j,k+1}$ is a simplex, which means that j_{m-k} is null homotopic. Chasing the homotopy commutative diagram (4.33), j_{m-k} is a factor of J up to homotopy. Hence, J is null homotopic. □

Proposition 4.5.12. *There exist homotopy equivalences $\mathcal{Z}_{L^k_{j,m}} \simeq S^1 * \mathcal{Z}_{L^{k-1}_{j-1,m-1}}$ and $\mathcal{Z}_{L^k_{j,m}}/S^1_d \simeq C_{q^{k-1}_{j-1,m-1}}$, where $C_{q^{k-1}_{j-1,m-1}}$ is the homotopy cofibre of the quotient map $\mathcal{Z}_{L^k_{j,m}} \xrightarrow{q^k_{j,m}} \mathcal{Z}_{L^k_{j,m}}/S^1_d$. Consequently, we have the homotopy types of the following spaces*

$$\mathcal{Z}_{L^k_{j,m}} \simeq \begin{cases} S^{2j-1} * \mathcal{Z}_{\Delta^{k-j}_{m-j}} & \text{if } 1 \leq j \leq k+1 \\ S^{2k+3} & \text{if } j = k+2 \end{cases}$$

and

$$\mathcal{Z}_{L^k_{j,m}}/S^1_d \simeq \begin{cases} \mathbb{C}P^{k+1} \vee (\bigvee_{i=j}^{k} S^{2i-1} * \mathcal{Z}_{\Delta^{k-i}_{m-i-1}}) \vee (S^{2k+1} * T^{m-k-2}) & \text{if } 1 \leq j \leq k+1 \\ \mathbb{C}P^{k+1} & \text{if } j = k+2. \end{cases}$$

Proof. If $1 \leq j \leq k+1$, observe that $\mathrm{Link}_{L^k_{j,m}}(m) = L^{k-1}_{j-1,m-1}$ and $\mathrm{Rest}_{L^k_{j,m}}(m) = \Delta_{\{1,\ldots,m-1\}}$. We have two homotopy pushouts of corresponding moment-angle complexes and their quotient spaces under the diagonal action

$$\begin{array}{ccc} \mathcal{Z}_{L^{k-1}_{j-1,m-1}} \times S^1 & \xrightarrow{*\times \mathrm{id}} & S^1 \\ {\scriptstyle \mathrm{id}\times *}\downarrow & & \downarrow \\ \mathcal{Z}_{L^{k-1}_{j-1,m-1}} & \longrightarrow & \mathcal{Z}_{L^k_{j,m}} \end{array} \quad \text{and} \quad \begin{array}{ccc} \mathcal{Z}_{L^{k-1}_{j-1,m-1}} & \longrightarrow & * \\ \downarrow{\scriptstyle q^{k-1}_{j-1,m-1}} & & \downarrow \\ \mathcal{Z}_{L^{k-1}_{j-1,m-1}}/S^1_d & \longrightarrow & \mathcal{Z}_{L^k_{j,m}}/S^1_d. \end{array}$$

Thus $\mathcal{Z}_{L^k_{j,m}} \simeq S^1 * \mathcal{Z}_{L^{k-1}_{j-1,m-1}}$ and $\mathcal{Z}_{L^k_{j,m}}/S^1_d \simeq C_{q^{k-1}_{j-1,m-1}}$. Iterating $\mathcal{Z}_{L^k_{j,m}} \simeq S^1 * \mathcal{Z}_{L^{k-1}_{j-1,m-1}}$, we obtain the homotopy equivalences $\mathcal{Z}_{L^k_{j,m}} \simeq S^{2j-1} * \mathcal{Z}_{\Delta^{k-j}_{m-j}}$ for $1 \le j \le k+1$.

Next consider that $\mathrm{Link}_{L^k_{j,m}}(1) = L^{k-1}_{j,m-1}$ and $\mathrm{Rest}_{L^k_{j,m}}(1) = L^k_{j,m-1}$. Consider the homotopy pushouts of corresponding moment-angle complexes and their quotient spaces under the diagonal action

$$\begin{array}{ccc} S^1 \times \mathcal{Z}_{L^{k-1}_{j,m-1}} & \xrightarrow{\mathrm{id}\times *} & S^1 \times \mathcal{Z}_{L^k_{j,m-1}} \\ {\scriptstyle *\times\mathrm{id}}\big\downarrow & & \big\downarrow {\scriptstyle f^k_{j,m}} \\ \mathcal{Z}_{L^{k-1}_{j,m-1}} & \longrightarrow & \mathcal{Z}_{L^k_{j,m}} \end{array} \quad \text{and} \quad \begin{array}{ccc} \mathcal{Z}_{L^{k-1}_{j,m-1}} & \xrightarrow{\simeq *} & \mathcal{Z}_{L^k_{j,m-1}} \\ \big\downarrow {\scriptstyle q^{k-1}_{j,m-1}} & & \big\downarrow {\scriptstyle g^k_{j,m}} \\ \mathcal{Z}_{L^{k-1}_{j,m-1}}/S^1_d & \longrightarrow & \mathcal{Z}_{L^k_{j,m}}/S^1_d. \end{array}$$

By Lemma 4.5.11, the simplicial inclusion $\mathrm{Link}_{L^k_{j,m}}(1) \longrightarrow \mathrm{Rest}_{L^k_{j,m}}(1)$ induces a null homotopic map on corresponding moment-angle complexes. Thus, there are two splitting homotopy cofibrations

$$S^1 \ltimes \mathcal{Z}_{L^k_{j,m-1}} \xrightarrow{\bar{f}^k_{j,m}} \mathcal{Z}_{L^k_{j,m}} \longrightarrow S^1 * \mathcal{Z}_{L^{k-1}_{j,m-1}}$$
$$\mathcal{Z}_{L^k_{j,m-1}} \xrightarrow{g^k_{j,m}} \mathcal{Z}_{L^k_{j,m}}/S^1_d \longrightarrow C_{q^{k-1}_{j,m-1}}.$$

Thus, there are homotopy equivalences

$$\mathcal{Z}_{L^k_{j,m}} \simeq S^1 * \mathcal{Z}_{L^{k-1}_{j,m-1}} \vee S^1 \ltimes \mathcal{Z}_{L^k_{j,m-1}} \quad \text{and} \quad C_{\bar{f}^k_{j,m}} \simeq S^1 * \mathcal{Z}_{L^{k-1}_{j,m-1}} \simeq \mathcal{Z}_{L^k_{j+1,m}}$$
$$\mathcal{Z}_{L^k_{j,m}}/S^1_d \simeq \mathcal{Z}_{L^k_{j,m-1}} \vee C_{q^{k-1}_{j,m-1}} \quad \text{and} \quad C_{g^k_{j,m}} \simeq C_{q^{k-1}_{j,m-1}} \simeq \mathcal{Z}_{L^k_{j+1,m}}/S^1_d.$$

Iterating the homotopy equivalence $\mathcal{Z}_{L^k_{j,m}}/S^1_d \simeq \mathcal{Z}_{L^k_{j,m-1}} \vee C_{q^{k-1}_{j,m-1}} \simeq \mathcal{Z}_{L^k_{j,m-1}} \vee (\mathcal{Z}_{L^k_{j+1,m}}/S^1_d)$, we have

$$\mathcal{Z}_{L^k_{j,m}}/S^1_d \simeq \mathcal{Z}_{L^k_{j,m-1}} \vee \mathcal{Z}_{L^k_{j+1,m-1}} \vee \ldots \vee \mathcal{Z}_{L^k_{k+1,m-1}} \vee (\mathcal{Z}_{L^k_{k+2,m}}/S^1_d). \tag{4.34}$$

In the end, we identify the homotopy type of $\mathcal{Z}_{L^k_{k+2,m}}/S^1_d$.

If $k=0$, then $L^0_{2,m} = \Delta_{\{1,\ldots,m-1\}} \cup \Delta_{\{1,\ldots,m-2,m\}}$, where two $(m-2)$-simplices are glued together along one common facet $\Delta_{\{1,\ldots,m-2\}}$. In this case, we have

$$\mathcal{Z}_{L^0_{2,m}} = (\prod_{m-2} D^2) \times (D^2, S^1)^{\partial\Delta^1} \simeq S^1 * S^1.$$

Since the diagonal action on $\mathcal{Z}_K$ is free, the genuine quotient space has the same homotopy type as its homotopy quotient. Hence, there is a homotopy equivalence

$$\mathcal{Z}_{L^0_{2,m}}/S^1_d \simeq ES^1 \times_{S^1_d} \mathcal{Z}_{L^0_{2,m}} = ES^1 \times_{S^1_d} ((\prod_{m-2} D^2) \times (D^2, S^1)^{\partial\Delta^1}) \simeq ES^1 \times_{S^1_d} (D^2, S^1)^{\partial\Delta^1} \simeq \mathbb{C}P^1.$$

In general, the simplicial complex $L^k_{k+2,m} = \bigcup\limits_{j=m-k-1}^{m} \Delta_{\{1,\ldots,\hat{j},\ldots,m\}}$, where $k+2$ simplices of dimension $m-2$ (the "first" $k+2$ facets of Δ^{m-1}) are glued along the common face $\Delta_{\{1,\ldots,m-k-2\}}$. Thus, $\mathcal{Z}_{L^k_{k+2,m}} = (\prod\limits_{m-k-2} D^2) \times (D^2, S^1)^{\partial\Delta^{k+1}}$. The diagonal action on $\mathcal{Z}_{L^k_{k+2,m}}$ implies that the genuine quotient space has the same homotopy type with its homotopy quotient. Hence, we have

$$\mathcal{Z}_{L^k_{k+2,m}}/S^1_d \simeq ES^1 \times_{S^1_d} \mathcal{Z}_{L^k_{k+2,m}} = ES^1 \times_{S^1_d} ((\prod_{m-k-2} D^2) \times (D^2, S^1)^{\partial\Delta^{k+1}}) \simeq (D^2, S^1)^{\partial\Delta^{k+1}}/S^1_d \simeq \mathbb{C}P^{k+1}.$$

By (4.34), there is a homotopy equivalence

$$\begin{aligned}
\mathcal{Z}_{L^k_{j,m}}/S^1_d &\simeq \mathbb{C}P^{k+1} \vee \mathcal{Z}_{L^k_{j,m-1}} \vee \mathcal{Z}_{L^k_{j+1,m-1}} \vee \ldots \vee \mathcal{Z}_{L^k_{k+1,m-1}} \\
&\simeq \mathbb{C}P^{k+1} \vee (S^{2j-1} * \mathcal{Z}_{\Delta^{k-j}_{m-j-1}}) \vee (S^{2j+1} * \mathcal{Z}_{\Delta^{k-j-1}_{m-j-2}}) \vee \ldots \vee (S^{2k+1} * T^{m-k-2}) \\
&\simeq \mathbb{C}P^{k+1} \vee (\bigvee_{i=j}^{k} S^{2i-1} * \mathcal{Z}_{\Delta^{k-i}_{m-i-1}}) \vee (S^{2k+1} * T^{m-k-2}).
\end{aligned}$$

□

Now we prove Theorem 4.5.10.

Proof of Theorem 4.5.10. The homotopy commutative diagram (4.31) shows that $C_{k,m} \simeq \mathcal{Z}_{L^{k+1}_{1,m+1}}/S^1_d$. By Proposition 4.5.12,

$$C_{k,m} \simeq \mathbb{C}P^{k+2} \vee (\bigvee_{i=1}^{k+1} S^{2i-1} * \mathcal{Z}_{\Delta^{k+1-i}_{m-i}}) \vee (S^{2k+3} * T^{m-k-2}).$$

□

Corollary 4.5.13. *The homotopy type of $\mathcal{Z}_{\Delta^k_m}/S^1_d$ is $\mathcal{Z}_{\Delta^k_{m-1}} \vee C_{k-1,m-1}$.*

Proof. The proof follows from Corollary 4.5.5. □

Bibliography

[1] Ali Al-Raisi, *Equivariance, Module Structure, Branched Covers, Strickland Maps and Cohomology Related to the Polyhedral Product Functor*, ProQuest LLC, Ann Arbor, MI, 2014, Thesis (Ph.D.)–University of Rochester. MR 3301741

[2] Martin Arkowitz, *Introduction to homotopy theory*, Springer New York, 2011.

[3] A. Bahri, M. Bendersky, F. R. Cohen, and S. Gitler, *The polyhedral product functor: a method of decomposition for moment-angle complexes, arrangements and related spaces*, Adv. Math. **225** (2010), no. 3, 1634–1668. MR 2673742

[4] I. V. Baskakov, *Cohomology of K-powers of spaces and the combinatorics of simplicial divisions*, Uspekhi Mat. Nauk **57** (2002), no. 5(347), 147–148. MR 1992088

[5] Frédéric Bosio and Laurent Meersseman, *Real quadrics in $\mathbf{C}^n$, complex manifolds and convex polytopes*, Acta Math. **197** (2006), no. 1, 53–127. MR 2285318

[6] Theodor Bröcker and Tammo tom Dieck, *Representations of compact Lie groups*, Graduate Texts in Mathematics, vol. 98, Springer-Verlag, New York, 1995, Translated from the German manuscript, Corrected reprint of the 1985 translation. MR 1410059

[7] Kenneth S. Brown, *Cohomology of groups*, Graduate Texts in Mathematics, vol. 87, Springer-Verlag, New York, 1994, Corrected reprint of the 1982 original. MR 1324339

[8] Victor M Buchstaber and Taras E Panov, *Torus actions, combinatorial topology, and homological algebra*, Russian Mathematical Surveys **55** (2000), no. 5, 825.

[9] Victor M. Buchstaber and Taras E. Panov, *Torus actions and their applications in topology and combinatorics*, University Lecture Series, vol. 24, American Mathematical Society, Providence, RI, 2002. MR 1897064

[10] ———, *Toric topology*, Mathematical Surveys and Monographs, vol. 204, American Mathematical Society, Providence, RI, 2015. MR 3363157

[11] Li Cai, *On the cohomology of polyhedral products with space pairs (D^1, S^0)*, arXiv:1708.06722v5 (2013).

[12] Liman Chen, Feifei Fan, and Xiangjun Wang, *The topology of the moment-angle manifolds–on a conjecture of S.Gitler ans S.López*, arXiv:1406.6756v2 (2016).

[13] Suyoung Choi and Hanchul Park, *On the cohomology and their torsion of real toric objects*, Forum Math. **29** (2017), no. 3, 543–553. MR 3641664

[14] Thomas Church, Jordan S. Ellenberg, and Benson Farb, *FI-modules and stability for representations of symmetric groups*, Duke Math. J. **164** (2015), no. 9, 1833–1910. MR 3357185

[15] Thomas Church and Benson Farb, *Representation theory and homological stability*, Adv. Math. **245** (2013), 250–314. MR 3084430

[16] R. L. Cohen, *The topology of fiber bundles*, Lecture notes, Lecture notes, Stanford University, 1998.

[17] Michael W. Davis and Tadeusz Januszkiewicz, *Convex polytopes, Coxeter orbifolds and torus actions*, Duke Math. J. **62** (1991), no. 2, 417–451. MR 1104531

[18] Emmanuel Dror Farjoun, *Cellular spaces, null spaces and homotopy localization*, Lecture Notes in Mathematics, vol. 1622, Springer-Verlag, Berlin, 1996. MR 1392221

[19] Matthias Franz, *The cohomology rings of smooth toric varieties and quotients of moment-angle complexes*, arXiv:1907.04791 (2019).

[20] Xin Fu and Jelena Grbić, *Simplicial G-complexes and representation stability of polyhedral products*, to appear in Algebraic & Geometric Topology, arXiv:1803.11047 (2018).

[21] William Fulton and Joe Harris, *Representation theory*, Graduate Texts in Mathematics, vol. 129, Springer-Verlag, New York, 1991, A first course, Readings in Mathematics. MR 1153249

[22] Jelena Grbić and Stephen Theriault, *The homotopy type of the complement of a coordinate subspace arrangement*, Topology **46** (2007), no. 4, 357–396. MR 2321037

[23] Jelena Grbić and Stephen Theriault, *The homotopy type of the polyhedral product for shifted complexes*, Adv. Math. **245** (2013), 690–715. MR 3084441

[24] Allen Hatcher, *Algebraic topology*, Cambridge University Press, Cambridge, 2002. MR 1867354

[25] David J. Hemmer, *Stable decompositions for some symmetric group characters arising in braid group cohomology*, J. Combin. Theory Ser. A **118** (2011), no. 3, 1136–1139. MR 2763061

[26] Melvin Hochster, *Cohen-Macaulay rings, combinatorics, and simplicial complexes*, (1977), 171–223. Lecture Notes in Pure and Appl. Math., Vol. 26. MR 0441987

[27] Jun Ma, Feifei Fan, and Xiangjun Wang, *Homology groups of simplicial complements*, Chin. Ann. Math. Ser. B **37** (2016), no. 5, 683–690. MR 3536000

[28] I. G. Macdonald, *Symmetric functions and Hall polynomials*, second ed., Oxford Classic Texts in the Physical Sciences, The Clarendon Press, Oxford University Press, New York, 2015, With contribution by A. V. Zelevinsky and a foreword by Richard Stanley, Reprint of the 2008 paperback edition [MR1354144]. MR 3443860

[29] Michael Mather, *Pull-backs in homotopy theory*, Canad. J. Math. **28** (1976), no. 2, 225–263. MR 0402694

[30] John McCleary, *A user's guide to spectral sequences*, second ed., Cambridge Studies in Advanced Mathematics, vol. 58, Cambridge University Press, Cambridge, 2001. MR 1793722

[31] Dennis McGavran, *Adjacent connected sums and torus actions*, Trans. Amer. Math. Soc. **251** (1979), 235–254. MR 531977

[32] James R. Munkres, *Elements of algebraic topology*, Addison-Wesley Publishing Company, Menlo Park, CA, 1984. MR 755006

[33] Hideki Ozeki and Fuichi Uchida, *Principal circle actions on a product of spheres*, Osaka J. Math. **9** (1972), 379–390. MR 0353348

[34] Taras E. Panov, *Cohomology of face rings, and torus actions*, Surveys in contemporary mathematics, London Math. Soc. Lecture Note Ser., vol. 347, Cambridge Univ. Press, Cambridge, 2008, pp. 165–201. MR 2388493

[35] ———, *On the cohomology of quotient spaces of moment-angle complexes*, Uspekhi Mat. Nauk **70** (2015), no. 4(424), 209–210. MR 3400574

[36] Gerald J. Porter, *Higher order Whitehead products*, Topology **3** (1965), no. 2, 123 – 135.

[37] Volker Puppe, *A remark on "homotopy fibrations"*, Manuscripta Math. **12** (1974), 113–120. MR 0365556

[38] Paul Selick, *Introduction to homotopy theory*, Fields Institute Monographs, vol. 9, American Mathematical Society, Providence, RI, 1997. MR 1450595

[39] Diana Kahn Taylor, *Ideals generated by monomials in an R-sequence*, ProQuest LLC, Ann Arbor, MI, 1966, Thesis (Ph.D.)–The University of Chicago. MR 2611561

[40] Stephen Theriault, *Toric homotopy theory*, Combinatorial and toric homotopy, Lect. Notes Ser. Inst. Math. Sci. Natl. Univ. Singap., vol. 35, World Sci. Publ., Hackensack, NJ, 2018, pp. 1–66. MR 3792460

[41] Xiangjun Wang and Qibing Zheng, *The homology of simplicial complements and the cohomology of polyhedral products*, Forum Math. **27** (2015), no. 4, 2267–2299. MR 3365798

[42] Volkmar Welker, Günter M. Ziegler, and Rade T. Živaljević, *Homotopy colimits—comparison lemmas for combinatorial applications*, J. Reine Angew. Math. **509** (1999), 117–149. MR 1679169

[43] Li Yu, *On Hochster's formula for a class of quotient spaces of moment-angle complexes*, Osaka J. Math. **56** (2019), no. 1, 33–50. MR 3908775