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THE USE OF CONFORMAL MAPPING IN SEAKEEPING CALCULATIONS

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The determination of the motion of a ship in the seaway is a difficult problem. Solutions that are approximate to some degree or another abound in the literature. One common method of solution is to use the so called 'strip theory'. This is in essence a cross-flow hypothesis introduced by Lewis and applied to ships in the first instance by Korvin-Kroukousky and Jacobs. The idea of this method is to subdivide the ship into longitudinal sections, twenty being a typical number, so that the flow across that section is found. The underlying principle behind the calculation of forces across such a cross section is that the wave profile is unchanged across all sections of the ship. This principle is referred to as the Froude-Krilov hypothesis.

The results from more advanced and fuller theories of Salvesen, Tuck and Faltinsen have been used to correlate against model test results. The analysis is particularly good for heave and pitch motions. For the three coupled lateral motions roll, sway and yaw, the degree of correlation depends entirely upon the detail that the theory uses to predict the hydrodynamic damping of auxilliary control surfaces.

The strip theory is used often uncritically for bodies that are not ship shaped.

In the strip theory a method is required to calculate the forces. In the many theories that are in the literature, two basic methods are used. The majority base their solutions around the classic papers of Ursell. In the original paper Ursell studied the oscillations of a circular cylinder heaving in regular sinusoidal waves. Ursell produced a stream function that satisfied the linear boundary conditions on the wave surface and the oscillating body for all wave frequencies.

A set of such stream functions is

$$a^{2m} \left\{ \frac{\sin 2m\theta}{r^{2m}} + \frac{k}{(2m-1)} \frac{\sin(2m-1)\theta}{r^{2m-1}} \right\} \begin{matrix}\cos \\ \sin\end{matrix} (\omega t)$$

where $m = 1, 2, 3 \ldots.$

a is the radius of the circular cylinder

$k = \omega^2/g$ the wave number

r is the radial distance from the centre of the cylinder

θ is the polar angle to a general point

To this set of stream functions must be added another stream function that represents diverging wave trans, for example,

$$\frac{gA}{\pi\omega} \{ \psi_c \cos\omega t + \psi_s \sin\omega t \}$$

where A is the amplitude at infinity of the diverging waves.

$$\psi_c = \pi e^{-kr\cos\theta} \sin(kr\cos\theta)$$

$$\psi_c = -\pi e^{-kr\cos\theta} \cos(kr\sin\theta) + \int_0^\infty \frac{e^{-kr\sin\theta}}{k^2+K^2} \left[K\sin(Kr\cos\theta) + k\cos(Kr\sin\theta)\right] dK$$

The second method of solution is based upon the source singularity distribution. The geometrical shape of the section is mathematically represented by a series of straight line segments. The velocity potential is then calculated for a distribution of source singularities, each situated at the centre of the segment. The strength of each source singularity is constant. Each source singularity satisfies the Laplaces equation, the free surface condition and the condition that there are diverging waves a long distance from the cylinder. The velocity potential boundary condition is matched to the cylinder vertical velocity to allow the strength to be calculated.

In either case, Ursell or the source singularity method, the velocity potential and stream function are found by a suitable analytic technique. The actual forces on the ship cross section are then calculated by use of a linear form of Bernoulli Theorem. The dynamic pressure due to the unsteady velocity potential component in Bernoulli Theorem has received the greatest attention in seakeeping. Since the velocity potential will have terms, in general, that have both sine and cosine variations with respect to the frequency of oscillation, then there will be forces associated with both these terms. The force that is associated with the component having the same phase as the forced oscillation of the body is called the added mass. The force that is out of phase with the forced oscillation is called the damping.

The method of solution using singularities is not suitable immediately for conformal mapping techniques so will not be discussed further herein. The method due to Ursell will now be extrapolated to be useful for ship shaped sections.

Since the method of Ursell is based upon circular sections some method is required to transform the ship section shape into circular sections without removing or changing the boundary conditions that the section has to satisfy. There are two types of transformation that allow this process to take place, they are the Joukowsky type and the Schwartz-Christoffel type. Both are conformal transformations.

Schwartz-Christoffel Transformation

This transformation has been used only in a very limited number of cases. It should prove to be superior to the Joukowski transformation when there are abrupt changes in section curvature. The form of the transformation is

$$z = a_0 \int_{\zeta} \Pi \, (\zeta_n - \zeta)^{-\theta_n/\pi} \, d\zeta + a_1$$

a_0 is a scale factor

a_1 is a constant to locate the body in the correct position in the z plane

$z = x+iy$ the points in the transformed plane, i.e. the circle plane

$\zeta = u+iv$ is the section shape

ζ_n is the point in the ship body

θ_n is the exterior angle between sides joining

This transformation will thus allow for shapes with bilge keels, chines, and rise of floor. The basic transformation maps the ship section into a straight line. Thus to use Ursell's method requires a further transformation. There are limited papers giving results from such methods, but Wendel, and Lewis are two good examples. In both cases the shape of the sections are simple and have rectangular cylinders.

Joukowski Mapping

The generalized Joukowski mapping is given by

$$Z = \sum_{n=1}^{\infty} a_n \zeta^{2-n}$$

This will map an arbitrary shape in the z-plane to a circle in the plane.

It is worthwhile exploring this equation a little further, by using the symmetry of the ship sections about the vertical centreline. Consider the following three cases:

1. Symmetry about the x-axis (vertical symmetry)

Figure 1 gives details of the section.
In this case the symmetry implies $z = \bar{z}$ the complex conjugate

$$\therefore \quad \Sigma\, a_n \zeta^{2-n} = \sum_{n=1}^{\infty} \overline{a_n}\, \bar{\zeta}^{2-n}$$

The point

$$Z = re^{i\varepsilon} = \zeta$$

$$\bar{Z} = re^{-i\varepsilon} = \bar{\zeta}$$

$$Z(\bar{\zeta}) = \Sigma\, a_n\, \bar{\zeta}^{2-n}$$

but because of symmetry

$$Z(\zeta) = Z(\bar{\zeta}) = \overline{Z(\zeta)}$$

$$\bar{\Sigma}\, a_n\, \bar{\zeta}^{(2-n)} = \Sigma\, \overline{a_n}\, \bar{\zeta}^{(2-n)}$$

because $\zeta = re^{i\theta}$ is an orthogonal function

$\therefore \quad a_n = \overline{a_n} \quad$ i.e. a_n is real.

Thus for the ship sections to have vertical symmetry the coefficients a_n must be real

2. Symmetry about the y-axis (lateral symmetry)

The details are in Figure 2.

This symmetry implies $z = -\bar{z}$

The point P corresponds to $\zeta = re^{i\theta}$
point S corresponds to $\zeta = re^{i(\pi-\theta)} = -re^{i\theta} = -\bar{\zeta}$

$$\therefore \quad \bar{Z} = -\bar{\zeta}$$

$\therefore$ using the symmetry condition

$$\Sigma\, a_n\, \zeta^{2-n} = -\overline{\Sigma\, a_n\, \zeta^{2-n}} = \Sigma\, \overline{a_n}\, \bar{\zeta}^{(2-n)}$$

but

$$Z(\zeta) = Z(-\bar{\zeta}) = \Sigma\, a_n(-\bar{\zeta})^{2-n}$$

Thus using the orthogonality condition

$$-\overline{a_n} = (-1)^{2-n} \, a_n$$

or

$$a_n = (-1)^{1-n} \, \overline{a_n}$$

This means that the series splits into two series

a_{2n+1} are real $\qquad$ $n = 0, 1, 2, \ldots.$

a_{2n} are imaginary $\qquad$ $n = 1, 2, 3, \ldots$

3. Symmetry about both the x and y axes (vertical and lateral symmetry)

Here both 1 and 2 of above apply equally,

This requires a_n real for $n = 1, 2, \ldots$

and $\quad a_{2n+1}$ real $\quad$ for $\quad n = 0, 1, 2, \ldots$

$\quad\quad a_{2n}$ imaginary $\quad$ for $\quad n = 1, 2, 3, \ldots$

This means that only the odd terms of the general Joukowski transformation are non-zero i.e.

The series of terms that are used extensively by Naval Architects and Hydrodynamicists is the last series i.e.

a_{2n+1} are real

a_{2n} are all zero

The general problem of matching a section shape to the circle is not often attempted. Instead the series is truncated to a finite number of terms. The number of terms being as small as three in the so called Lewis transformation.

A) Lewis Form

This derivative of the general Joukowski transformation is

$$Z = a_0 \left(\zeta + a_1 \zeta^{-1} + a_3 \zeta^{-3}\right)$$

In this case only three terms are present, so in principal any three section properties will allow the calculation of the terms. Practice is that the section half beam, section draught and section area are provided to solve for a_0, a_1 and a_3.

Figure 3 gives details of the z and ζ planes.

The radius of the circle is taken to be 1 unit.

The points $x = B/2 \quad y = 0$ corresponds to $\theta = 0$

$x = 0 \quad y = T$ corresponds to $\theta = \pi/2$

$$B/2 = a_0(1 + a_1 + a_3)$$

and

$$T = a_0(1 - a_1 + a_3)$$

The sectional area A is found from

$$A = 2\int_{x=0}^{B/2} y dx$$

where

$$x = a_0 \left[(1+a_1)\cos\theta + a_3 \cos 3\theta\right]$$

$$y = a_0 \left[(1-a_1)\sin\theta - a_3 \sin 3\theta\right]$$

On evaluation

$$A = \frac{\pi}{2} a_0^2 \left[1 - a_1^2 - 3a_3^2\right]$$

The area coefficient σ is defined with respect to the circumscribing rectangle.

$$\sigma = A/BT$$

$$\therefore \quad \sigma = \frac{\pi}{4}\left[\frac{1 - a_1^2 - 3a_3^2}{(1+a_3)^2 - a_1^2}\right]$$

if the ratio $H = B/2T$ is used together with the variable ν, where

$$\nu = 1 + a_3 \quad \text{and} \quad r = a_1/\nu$$

$$\therefore \quad r = \frac{H-1}{H+1}$$

which is a quadratic in ν when re-arranged gives

$$\left[3 + r^2 + (1-r^2)\frac{4\sigma}{\pi}\right]\nu^2 - 6\nu + 2 = 0$$

This allows ν to be found analytically as

$$\nu = \frac{3 + \sqrt{9-2c_1}}{c_1}$$

where

$$c_1 = 3 + r^2 + (1-r^2)\, 4\sigma/\pi$$

So the coefficients of the Lewis form can be found to be

$$a_0 = B/2\,(1 + a_1 + a_3)$$

$$a_1 = r\,(a_3 + 1)$$

$$a_3 = \frac{3 + \sqrt{9 - 2c_1} - c_1}{c_1}$$

It has been shown, Wilson and Loader, that there are a range of area coefficients , and ratios H for which the Lewis form produces symmetrical and non re-entrant forms. These ranges are:

$$\frac{3\pi}{32}\left(3 + \frac{H}{4}\right) > \sigma > \frac{3\pi}{32}\left(2 - \frac{1}{H}\right) \qquad H > 1$$

and $\frac{3\pi}{32}(3 + \frac{1}{4H}) > \sigma > \frac{3\pi}{32}(2-H)$ for $H < 1$

If the ship is also limited to be of conventional shape i.e. no bulbous bows or tunnel forms then there is a maximum value of the area coefficient.

$$\sigma = \frac{\pi}{32}(H + \frac{1}{H} + 10)$$

Figure 4 shows the shape that is obtained by using the Lewis form on a rectangular section. It can be seen that the matching of the transformed section to that of original shape is poor.

To try to redress this problem many researchers have included other terms. The inclusion of the a_5 term was first used by Landweber and Macagno and this seems to have been forgotten by Loukakis, Perakis and Papoulias in their work on seakeeping.

The ultimate method is obviously where the transformed section matches extremely well that of the original section. Thus with the previous definition of the transformation viz

$$Z = a_0 \left[\zeta + \sum_{n=1}^{N} a_{2n-1} \zeta^{-(2n-1)}\right]$$

and

$$\zeta = re^{i\theta}$$

$$x = a_0 \left[\cos\theta + \sum_{n=1}^{N} a_{2n-1} \cos(2n-1)\theta\right]$$

$$y = a_0 \left[\sin\theta - \sum_{n=1}^{N} a_{2n-1} \sin(2n-1)\theta\right]$$

The problem usually requires that N the number of coefficients is minimized to achieve a good fit to the data points. The data points are usually supplied at m values. There will be thus m-1 equations in x and a different m-1 equation in y.

Thus there are 2m-2 equations. The problem is usually posed to make the data points which made an angle α_i (to some datum line) correspond to a series of angles θ_i in the transformed plane. These transformed angles will not be generally known.

Thus the normal method of solution is to make the number of equations 2m-2 equal to the number of unknowns m+N

$$\therefore \quad 2m - 2 = m+N$$

$$\text{i.e.} \quad m = N+2$$

If this approach is used a Newton-Raphson technique can be used. Alternatively a set of 2m-2 equations in N+m unknown can be solved by an optimizing technique that minimizes the least squares distance from the transformed section from the original section. In this case the effect of increasing N can be seen quite easily.

In either case a starting solution is needed and the usual method is to calculate the Lewis form coefficients a_0, a_1 and a_3. The effect of increasing the size of N can be seen in the figures 5-7 for the rectangular and ship shaped sections.

In conclusion the use that conformal mapping techniques have had on the study of seakeeping have been profound. Without this method it is doubtful that the predictive techniques now enjoyed by naval architects would exist.

References

1. Korvin-Kroukovsky, B.V. and Jacobs, W. Pitching and Heaving motions of a ship in regular waves. Tr. SNAME, 65, 500-632, 1957.

2. Ladweder, L. and Macagno, M. Added mass of a three parameter family of two dimensional forces oscillation in a free surface. J.S.R., March 1959.

3. Lewis, F.M. The inertia of the water surrounding a vibrating ship. Tr. SNAME, 37, 1-20, 1929.

4. Loukakis, T.A., Perakis, A.N. and Papoulias, F.A. The effect of some hull form parameters on the seakeeping behaviour of surface ships. NTUA Report No. 9-1983.

5. Salvesen, N., Tuck, E.O. and Faltinsen, O. Ship motions and Sea loads. Tr. SNAME, 78, 1970.

6. Ursell, F. On the heaving motion of a circular cylinder on the surface of a fluid. Q.J.M. + A.M., Vol. 11, Pt 2, 1949.

7. Wendel, K. Hydrodynamische Massen und Hydrodymanische Massentragheits Momente. Sahrbuck der Schiff bautechnische Gesellschaft, 44, 1950.

8. Wilson, P.A. and Loader, P. General Theory of UCLVERT and UCLAT programs. ARE Tech. Mem., July 1985.

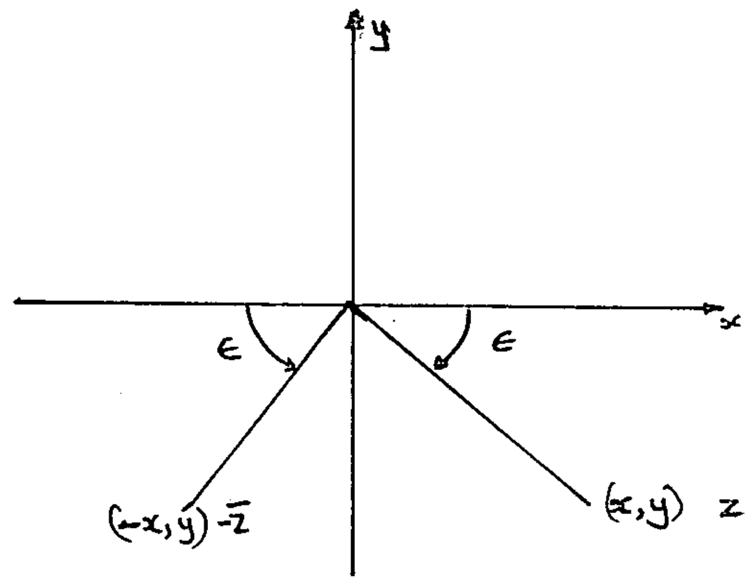


FIGURE 2

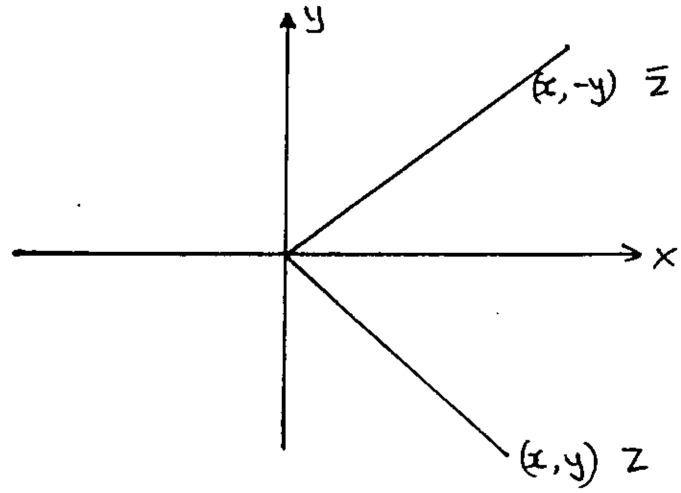


FIGURE 1

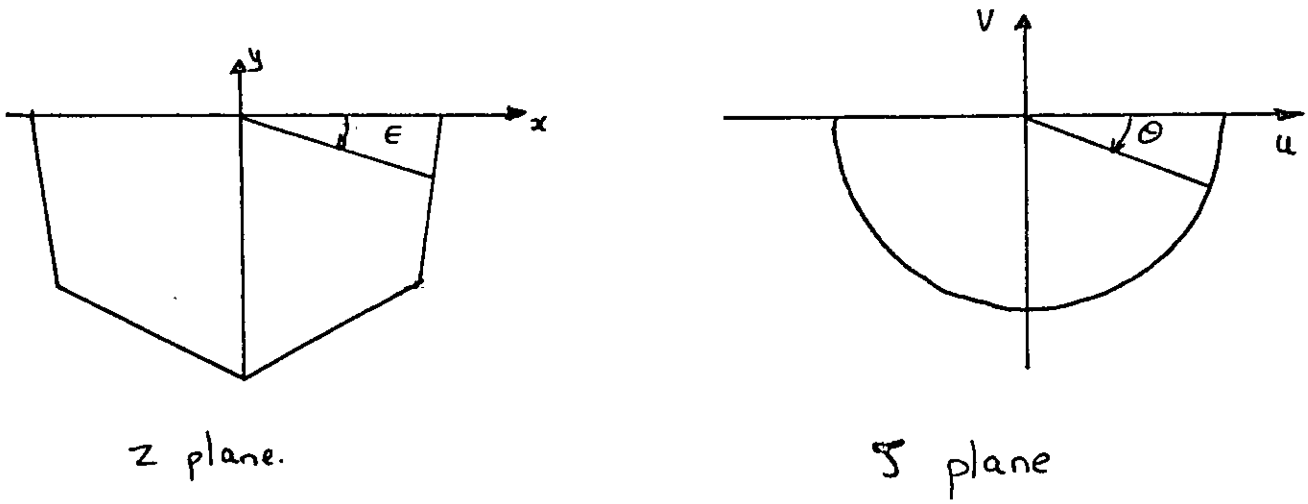
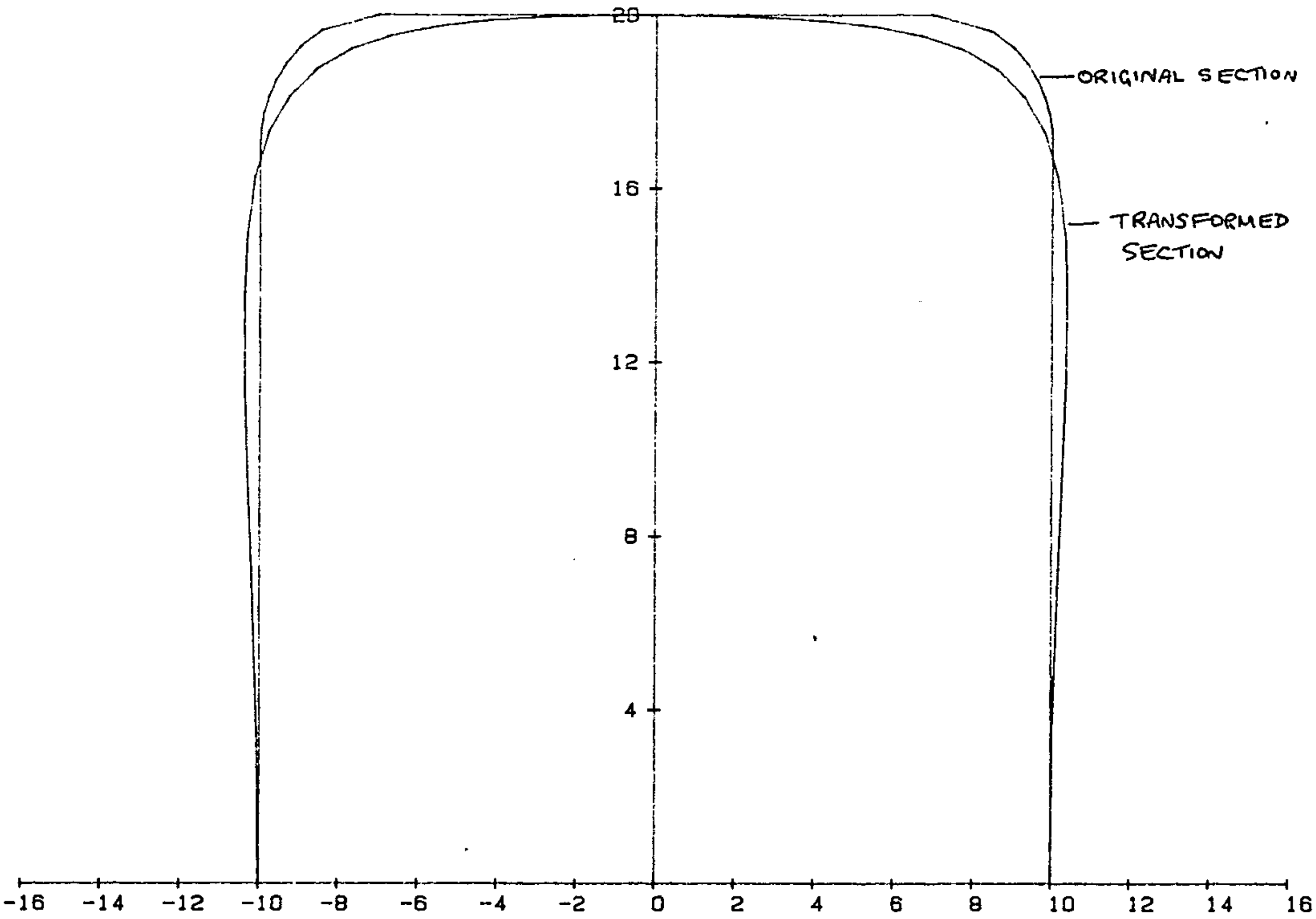


FIGURE 3

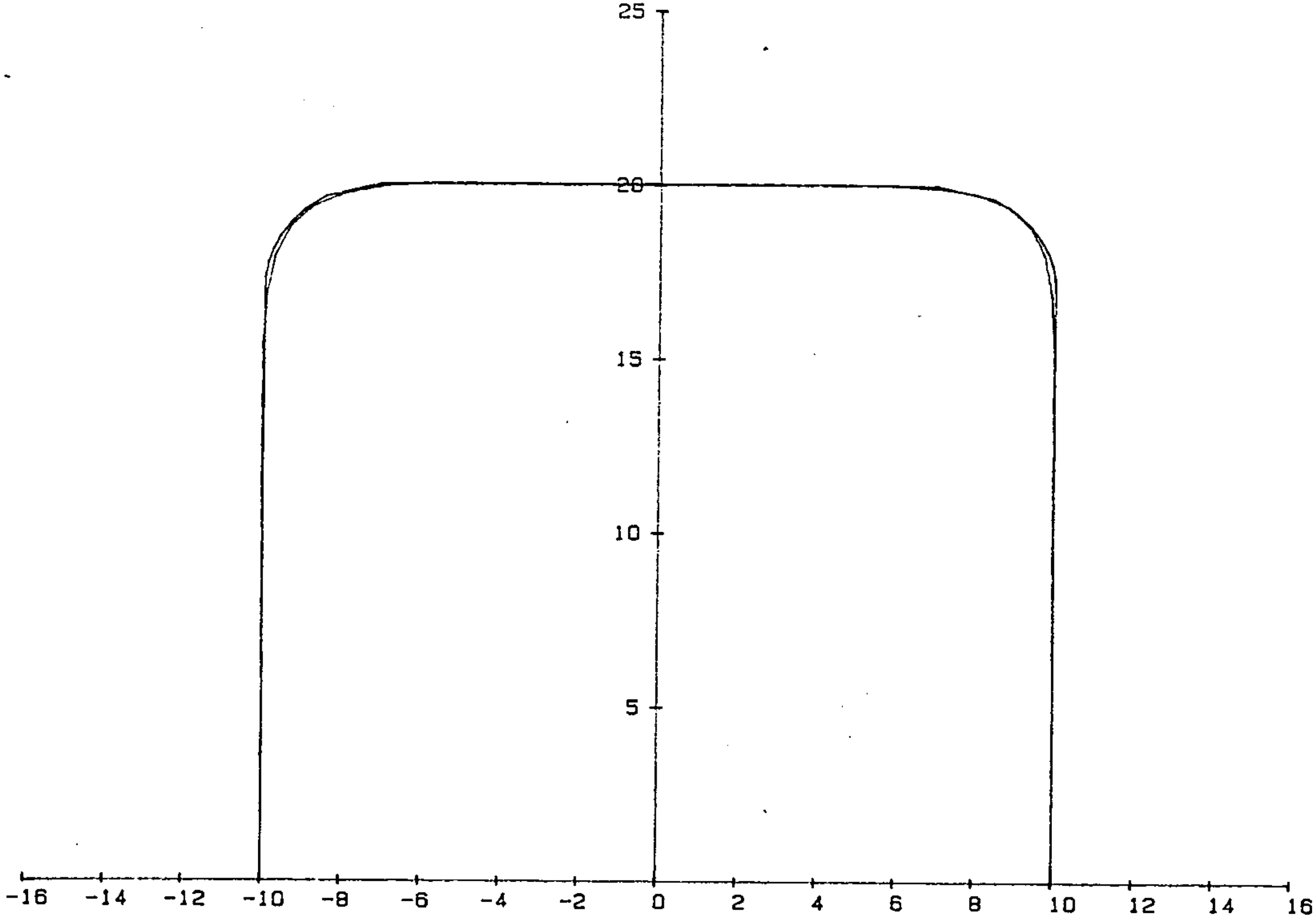
Number of terms 3

Figure No. 4



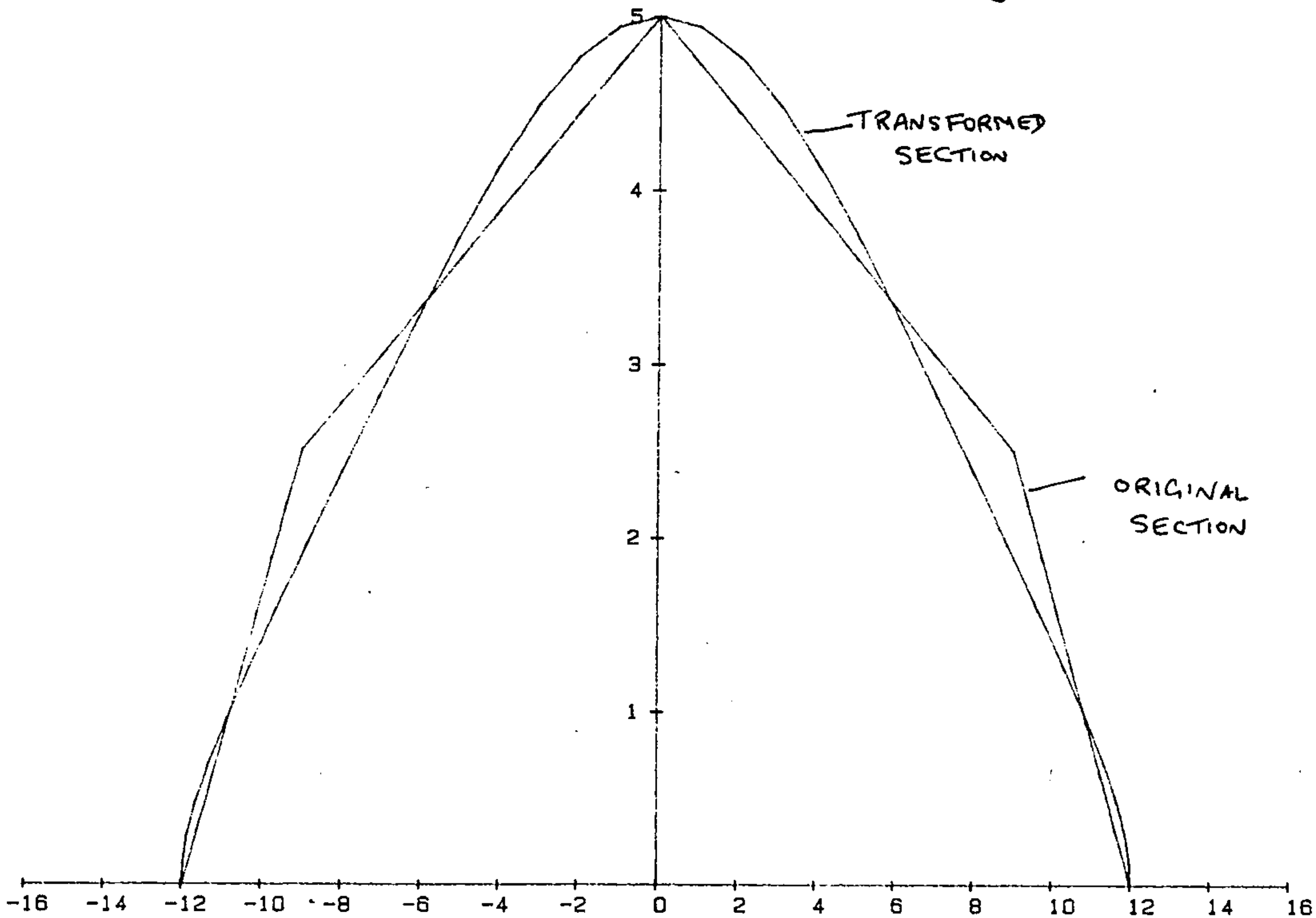
Number of terms 5

Figure No. 5



Number of terms 3

Figure No. 6



Number of terms 8

Figure No. 7