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Confidence intervals of Gini coefficient under unequal probability sampling

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Abstract

We propose an estimator for the Gini coefficient, based on a ratio of means. We show how bootstrap and empirical likelihood can be combined to construct confidence intervals. Our simulation study shows the estimator proposed is usually less biased than customary estimators. The observed coverages of the empirical likelihood confidence interval proposed are also closer to the nominal value.
Keywords: Bootstrap, empirical likelihood, inclusion probability, survey weight, sampling design

1. Introduction

Gini's (1914) coefficient is widely used indicator for measuring income inequality in a wide range of area of economics and finance (e.g. Koshevoy and Mosler, 1997; Ogwang, 2000; Gajdos and Weymark, 2005). The Gini coefficient is defined as the ratio of the area that lies between the 45° line and the Lorenz's (1905) curve given by

$$\mathcal{L}(x) := \frac{1}{E(Y)} \int_0^x y \, dF_Y(y), \tag{1}$$

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where $F_Y(\cdot)$ is the cumulative distribution function of a positive random variable Y and $E(Y)$ is the expectation of Y. An excellent review of various formulations of the Gini coefficient can be found in Giorgi and Gigliarano (2017).

Surveys are usually used to estimated the Gini coefficient. However, sampled unit are rarely selected independently with equal probability, because of sample selection, which involves stratification and unequal probabilities. Two customary estimators for unequal probability sampling can be found in the literature (e.g. Langel and Tillé, 2013, for a review). They are defined by (13) and (14) in §5. The proposed estimator is different and based on a ratio, which allows to express it as an empirical likelihood estimator. Single stage designs are considered in this paper. The approach proposed can be extended for multi-stage by using Berger's (2018*a*) approach.

Variance estimation of the Gini coefficient has been widely studied in the literature (e.g. Nair, 1936; Hoeffding, 1948; Glasser, 1962; Sendler, 1979; Beach and Davidson, 1983; Gastwirth and Gail, 1985; Schezhtman and Yitzhaki, 1987; Sandström et al., 1985, 1988; Nygård and Sandström, 1989; Yitzhaki, 1991; Shao, 1994; Binder and Kovaćević, 1995; Bishop et al., 1997; Karagiannis and Kovaćević, 2000; Ogwang, 2000; Giles, 2004; Modarres and Gastwirth, 2006; Davidson, 2009). Yitzhaki (1991) and Qin et al. (2010) proposed a variance estimator under stratified random samples. Asymptotic variance under stratified and clustered survey data can be found in Bhattacharya (2007). Berger (2008) proposed a jackknife variance estimator under unequal probability sampling. Langel and Tillé (2013) provided a comprehensive literature review on variance estimation for the Gini coefficient.

Sandström et al. (1988) has developed a confidence interval for the Gini coefficient based on normal approximation. Mills and Zandvakili (1997) consider the use of bootstrap methods to compute interval estimates for the Gini coefficient. Qin et al. (2010) proposed pseudoempirical likelihood confidence intervals for the Gini coefficient under simple random samples, using bootstrap and empirical likelihood methods. Qin et al.'s (2010) approach requires estimating the distribution function, and is not designed for unequal probability sampling. Other empirical likelihood intervals with independent and identically observations can be found in Peng (2011). Empirical likelihood confidence intervals are range preserving; that is, the lower bound and the upper bound cannot be outside the parameter space $[0, 1]$ of the Gini coefficient. The bounds are driven by the distribution observed from the data, rather than an asymptotic distribution. Empirical likelihood also offers the possibility of using some auxiliary information which may improve the estimation of the Gini coefficient (Berger and Torres, 2016). A review of empirical likelihood under unequal probability sampling can be found in Berger (2018*b*). Note that the confidence intervals proposed do require an effective sample size or a design-effect, unlike the pseudoempirical likelihood approach (Wu and Rao, 2006) for unequal probability sampling.

In §2, we define the Gini coefficient. The estimator proposed is defined in §3. In §4, we show how bootstrap and empirical likelihood can be combined to construct confidence intervals. The empirical likelihood confidence intervals have the advantage of having bounds within the range of the Gini coefficient. Linearisation will not be required for empirical likelihood confidence intervals. Our simulation study in §5 shows that the estimator proposed can be more efficient than the customary estimator (e.g. Berger, 2008; Langel and Tillé, 2013). The coverages of the proposed empirical likelihood confidence interval are usually not significantly different from the nominal value.

2. The Gini Coefficient

Let $Y \geqslant 0$ denote a positive random variable with a distribution function $F_Y(y)$. The Gini coefficient is defined by

$$G_0 := \frac{2}{E(Y)} \int_0^{\infty} yF_Y(y)dF_Y(y) - 1 = 1 - \frac{1}{E(Y)} \int_0^{\infty} \big\{1 - F_Y(y)\big\}^2 dy\cdot \qquad (2)$$

Yitzhaki (1998) proposed an alternative expression of G_0 based on the minimum

$$Z := \min\big\{Y_a, Y_b\big\}$$

of two independent copies Y_a and Y_b of Y. Since $Z \geqslant 0$, we always have that $E(Z) = \int_0^{\infty}\{1 - F_Z(z)\}dz$, where $F_Z(z)$ denotes the cumulative distribution of Z. Furthermore, since Z is the minimum of two random variables with the same distribution, we have that $F_Z(z) = 1-\{1-F_Y(z)\}^2$. This implies $E(Z) = \int_0^{\infty}\{1-F_Y(z)\}^2 dz$. Thus, (2) gives Yitzhaki's (1998) alternative expression (see also Peng, 2011),

$$G_0 = 1 - \frac{E(Z)}{E(Y)}\cdot \qquad (3)$$

Let U be a finite population of N units, where N is a fixed quantity which is not necessarily known. Consider that we have N independent copies $\{Y_i : i \in U\}$ of Y. Let $\{y_i : i \in U\}$ be the realisation of these copies.

The empirical equivalent of $E(Z)$ is therefore

$$\bar{y}_U^* \quad := \quad \frac{1}{N(N-1)} \sum_{i \in U} \sum_{\substack{j \in U \\ j \neq i}} \min\{y_i, y_j\} = \frac{1}{N} \sum_{i \in U} y_i^*\cdot$$

where

$$y_i^* \quad := \quad \frac{1}{N-1}\sum_{\substack{j\in U\\ j\neq i}} \min\{y_i, y_j\}.$$

Thus, the empirical version of (3) is the finite population parameter

$$G_U := 1 - \frac{\bar{y}_U^*}{\bar{y}_U}, \tag{4}$$

where

$$\bar{y}_U \quad := \quad \frac{1}{N}\sum_{i\in U} y_i. \tag{5}$$

3. Estimation of the Gini coefficient

Suppose that a sample S is randomly selected from U. We observe the values y_i for the sampled units $i \in S$. We shall use Neyman's (1938) design-based approach; that is, the sampling distribution is conditional on $\{y_i : i \in U\}$ and driven by the random selection of S. Thus, the values $\{y_i : i \in U\}$ and the parameter G_U will be treated as constants.

We consider that the population U is broken up into disjoint strata $U_1, \dots,$ $U_h, \dots,$ U_H and $\cup_{h=1}^{H} U_h = U$. Within each stratum U_h, a sample of n_h units is selected with-replacement with unequal selection probabilities P_i, where $\sum_{i\in U_h} P_i = 1$. Let $\pi_i = n_h P_i$, when $i \in U_h$. Let S_h be the set of n_h labels for stratum U_h. where $S = \cup_{h=1}^{H} S_h$ and $n = \sum_{h=1}^{H} n_h$. We assume that we have a with-replacement or without-replacement sampling design with negligible sampling fractions, in order to justify the bootstrap approach. Fortunately, in practice, the Gini coefficient is estimated from social surveys which are often based on negligible sampling fractions. The negligible sampling fraction is only needed for variance estimation and confidence intervals.

The estimator proposed for (4) is

$$\widehat{G}_\pi := 1 - \frac{\bar{y}^*_\pi}{\bar{y}_\pi}, \tag{6}$$

where $\bar{y}^*_\pi$ and $\bar{y}_\pi$ denote Hájek's (1971) estimators given by

$$\begin{aligned}
\bar{y}_\pi &:= \frac{1}{\widehat{N}} \sum_{i\in S} \frac{y_i}{\pi_i}, \\
\bar{y}^*_\pi &:= \frac{1}{\widehat{N}} \sum_{i\in S} \frac{\widehat{y}^*_i}{\pi_i}, \\
\widehat{y}^*_i &:= \frac{1}{\widehat{N} - \pi_i^{-1}} \sum_{\substack{j\in S \\ j\neq i}} \frac{1}{\pi_j} \min\{y_i, y_j\}, \\
\widehat{N} &:= \sum_{i\in S} \pi_i^{-1}.
\end{aligned}$$

The advantage of (6) is the fact that it does not involve the estimation $F_Y(y)$. Note that (6) reduces to Yitzhaki's (1998) under simple random sampling with a single stratum (see also Peng, 2011; Giorgi and Gigliarano, 2017).

Rescaled bootstrap (Rao et al., 1992; Rust and Rao, 1996) can be used for variance estimation. This method is based on bootstrap weights (Rust and Rao, 1996) given by

$$w_i^{(b)} := \frac{r_i n}{\pi_i(n-1)} \tag{7}$$

where r_i is the number of times i-th unit is selected, by bootstrap. The variance between the bootstrap replicates can be used as a variance estimate. A bootstrap confidence interval based on the bootstrap quantiles can be derived (the so-called "*histogram approach*").

The theory of bootstrap is well established, and little needs to be added. However, empirical likelihood is a new emerging topic, and little has been done on empirical likelihood confidence intervals for Gini, under unequal probability sampling. Peng's (2011) approach assumed an independent and identically distributed setting. Qin et al.'s (2010) method is based on simple random sampling. In §4, we show how an empirical likelihood confidence interval can be constructed with unequal probability sampling, in conjunction with bootstrap.

4. Empirical likelihood confidence intervals

In this §, we show how Berger and Torres's (2016) approach can be combined with bootstrap. Empirical likelihood is based on estimating equations. It can be shown that (6) is the solution to

$$\sum_{i \in S} \frac{1}{\pi_i} e(y_i, \widehat{y}_i^*, G) = 0 \cdot \tag{8}$$

where

$$e(y_i, \widehat{y}_i^*, G) \quad := \quad y_i(G-1) + \widehat{y}_i^* \cdot \tag{9}$$

By substituting (9) within (8), we obtain $\sum_{i \in S} \pi_i^{-1} y_i(G-1) + \sum_{i \in S} \pi_i^{-1} \widehat{y}_i^* = (G-1)\widehat{N}\bar{y}_\pi + \widehat{N}\bar{y}_\pi^* = 0$. The solution to the last equation is indeed (6).

Berger and Torres's (2012; 2014; 2016) "*empirical log-likelihood function*" is defined by

$$\ell_{\max}(G) := \max_{p_i : i \in S} \left\{ \sum_{i \in S} \log(p_i) : p_i > 0, \sum_{i \in S} \frac{p_i}{\pi_i} e(y_i, \widehat{y}_i^*, G) = 0, \sum_{i \in S} p_i \boldsymbol{\delta}_i = \frac{\overrightarrow{\boldsymbol{n}}}{n} \right\}, \tag{10}$$

where G denotes a value within the parameter space, $\boldsymbol{\delta}_i$ is the vector of stratification variables defined by

$$\boldsymbol{\delta}_i := \left(\delta_{i1}, \ldots, \delta_{ih}, \ldots, \delta_{iH}\right)^\top$$

and $\overrightarrow{\boldsymbol{n}}$ is the strata allocation given by

$$\overrightarrow{\boldsymbol{n}} := \sum_{i \in S} \widetilde{\boldsymbol{\delta}}_i = \big(n_1, \ldots, n_h, \ldots, n_H\big)^\top.$$

Within (10), we have two types of constraints. The constraint involving G is a moment condition which contains the standard sampling weights π_i^{-1}. We also have a stratification constraint $\sum_{i \in S} p_i \boldsymbol{\delta}_i = \overrightarrow{\boldsymbol{n}} n^{-1}$, which is not motivated by moment conditions. The function (10) reduces to Owen's (1988) empirical log-likelihood function when we have a single stratum and $\pi_i = n/N$, $\forall i \in U$. The advantage of (10) is that it can be used as a standard likelihood function for design-based inference. Note that (10) differs from Peng's (2011) approach, even with a single stratum and $\pi_i = n/N$, because Peng's (2011) approach is based on splitting the sample randomly into two sub-samples of same size.

The "*maximum empirical likelihood estimator*" $\widehat{G}_{EL}$ is defined as the quantity which maximises $\ell_{\max}(G)$. Berger and Torres (2016) show that this implies that $\widehat{G}_{EL}$ is the solution to (8). Thus, $\widehat{G}_{EL} = \widehat{G}_\pi$.

The empirical likelihood approach can be also used for confidence intervals based upon (6). Consider the "*empirical log-likelihood ratio statistic*"

$$\widehat{r}(G) := 2\Big\{\ell_{\max}(\widehat{G}) - \ell_{\max}(G)\Big\}. \tag{11}$$

Berger and Torres (2016) showed that the empirical log-likelihood ratio statistic converges to a ancillary quadratic form, when $G = G_0$. Unfortunately, this quadratic form will not necessarily converge to a χ^2-distribution, because the $\widehat{y}_i^*$ are estimated. In other word, this quadratic form is an ancillary statistics with an unknown distribution. We shall approximate this distribution using bootstrap.

In order to compute a α-level confidence interval, we would need to know the upper α-quantile of the distribution of $\widehat{r}(G_0)$. This distribution upper can be approximated by the bootstrap distribution. Consider the rescaled bootstrap sampling weights given by (7). Let $\widehat{r}(G)^b$ the b-th bootstrap value of (11) based on bootstrap sampling weights given by (7), with $G = \widehat{G}_\pi$. The α-level bootstrap confidence interval is

$$\Big[\min\big\{G : \widehat{r}(G) \leqslant r_\alpha\big\} \ ; \ \max\big\{G : \widehat{r}(G) \leqslant r_\alpha\big\} \Big], \tag{12}$$

where r_α is the α-quantile of $\{\widehat{r}(\widehat{G}_\pi)^1, \ldots, \widehat{r}(\widehat{G}_\pi)^b, \ldots, \widehat{r}(\widehat{G}_\pi)^B\}$. Note that $\widehat{r}(G)$ is a convex non-symmetric function with a minimum at $G = \widehat{G}_\pi$. This interval can be found by using any root search method, such that the Brent (1973) and Dekker's (1969) method, since the bounds are the two roots of $\widehat{r}(G) - r_\alpha = 0$. This can be achieved numerically by calculating $\widehat{r}(G)$ for several values of G.

The empirical likelihood confidence intervals cannot be disjoint because $\widehat{r}(G)$ is always convex, because of the strict concavity of the function $\sum_{i \in S} \log(p_i)$ within (10).

5. Simulation studies

Two customary estimators can be found in the literature (e.g. Berger, 2008; Langel and Tillé, 2013). They are given by

$$\widehat{G}_\pi^{(1)} := \frac{2}{\widehat{N}\bar{y}_\pi} \sum_{i \in S} \frac{y_i}{\pi_i} \widehat{F}_\pi(y_i) - 1, \tag{13}$$

$$\widehat{G}_\pi^{(2)} := \frac{1}{2\widehat{N}^2 \bar{y}_\pi} \sum_{i \in S} \sum_{j \in S} \frac{1}{\pi_i \pi_j} |y_i - y_j|, \tag{14}$$

where

$$\widehat{F}_\pi(y_i) := \frac{1}{\widehat{N}} \sum_{i \in S} \frac{1}{\pi_i} I\{y_i < y\}\cdot$$

In this §, we compare via simulation the estimator proposed $\widehat{G}_\pi$ in (6) with (13) and (14). We also compare their variance estimators and coverages of their 95% confidence intervals. Our simulation study will show the estimator proposed (6) can be less biased than (13) and (14). The observed coverages of the empirical likelihood confidence interval are also closer to the nominal value.

We generated $N = 10\,000$ population values y_i from different distributions as in Davidson (2009), Qin et al. (2010) and Peng (2011), namely the χ^2, exponential, lognormal, Pareto and Weibull distributions. The different values of G_0 defined by (2) are given in Table 1. We selected 2000 randomized systematic samples of size $n = 200$ and 500. The inclusion probabilities π_i are generated from a linear model with y_i as covariate, in order to obtain a correlation of 0.7 between π_i and y_i. We chosen this correlation to highlight the effect of the design. We use $B = 1000$ replicates for the bootstrap procedures.

In Table 1, we have the observed relative bias (RB) and mean squared error (MSE) given by

$$\begin{aligned} RB(\widehat{G}) &:= \frac{\widehat{E}(\widehat{G}) - G_0}{G_0} \times 100\%, \\ MSE(\widehat{G}) &:= \widehat{E}\{(\widehat{G} - G_0)^2\} \end{aligned}$$

for $\widehat{G} = \widehat{G}_\pi$, $\widehat{G}_\pi^{(1)}$ and $\widehat{G}_\pi^{(2)}$. Here, $\widehat{E}(\cdot)$ denotes the means over the 2000 observed values. The RB of $\widehat{G}_\pi$ is slightly smaller than with $\widehat{G}_\pi^{(2)}$. The RB of $\widehat{G}_\pi^{(1)}$ tends to be the smallest for large value of G_0. However, $\widehat{G}_\pi^{(1)}$ has the largest RB with small G_0. The MSE of $\widehat{G}_\pi$ and $\widehat{G}_\pi^{(2)}$ are similar. The MSE of $\widehat{G}_\pi^{(1)}$ is slightly larger when $n = 200$. With $n = 500$, all the MSE are similar. From Table 1, we conclude that $\widehat{G}_\pi$ tends to have the smallest bias with a MSE comparable to one observed for $\widehat{G}_\pi^{(2)}$.

In Table 2, we have the observed coverages of the 95% confidence intervals. For $\widehat{G}_\pi$, we consider two confidence intervals: The "*bootstrap confidence inter-*

val" based on the 2.5% and 97.5% quantiles of the bootstrap (column "*Boot*"), and the empirical likelihood confidence intervals (12) (column "*EL*"). The usual confidence intervals based on linearised variance estimates are used for $\widehat{G}_\pi^{(1)}$ and $\widehat{G}_\pi^{(2)}$. The quantity G_0 is the target parameter on which the confidence intervals are based upon. The relative bias of the variance estimator

$$RB\{\widehat{V}(\widehat{G})\} := \frac{\widehat{E}\{\widehat{\mathrm{V}}(\widehat{G})\} - \mathrm{V}(\widehat{G})}{\mathrm{V}(\widehat{G})} \times 100\%$$

are given in the last three columns, where $\mathrm{V}(\widehat{G})$ denotes the observed variance. The bootstrap variance is used for $\widehat{G}_\pi$. For $\widehat{G}_\pi^{(1)}$ and $\widehat{G}_\pi^{(2)}$, we use the linearisation variance estimates (e.g. Berger, 2008; Langel and Tillé, 2013) based on Hartley and Rao's (1962) variance estimator.

The observed coverages of the empirical likelihood approach are usually not significantly different from 95%, when the other coverages are different from 95%. The low coverages of $\widehat{G}_\pi^{(1)}$ and $\widehat{G}_\pi^{(2)}$ can be explained by lack of normality. With small values of G_0, the lower bounds of $\widehat{G}_\pi^{(1)}$ and $\widehat{G}_\pi^{(2)}$ can be negative. This could also explain the low coverage of $\widehat{G}_\pi^{(1)}$. When the coverage of the empirical likelihood approach is significantly different from 95%, the other coverages are also significantly different (distributions $\Gamma(\alpha = 5, \beta = 1)$, $Exp(\lambda = 1)$ and χ_1^2). The distribution $\Gamma(\alpha = 10, \beta = 1)$ is an exception, because $\widehat{G}_\pi^{(1)}$ has the best coverage, but with a biased variance estimator. We have observed one sample of size $n = 200$ with a negative lower bound for the confidence interval of (13). This occurs with the data generated from a χ^2-distribution.

The RB of the variance of $\widehat{G}_\pi$ can be large with $n = 200$, because they are based on bootstrap. However, with $n = 500$, all the RB are similar, and $\widehat{G}_\pi$ may have the smallest RB. When $n = 200$, we have larger RB for large value of G_0 (distributions $Exp(\lambda = 1)$ and χ_1^2 and $\Gamma(\alpha = 0.2, \beta = 1)$).

In Table 3, we have the observed average length of the 95% confidence intervals as well as the observed "*coefficient of variation*" (CV) of the lengths. The average length is very similar and in line with the coverages observed in Table 2, because confidence intervals with large coverage tend to be larger. The confidence interval tends to produce wider confidence intervals on average, because it has the largest observed coverage.

A small CV implies more stable confidence intervals, but this does not imply observed coverages closer to 95%. The CV of the bootstrap confidence intervals tends to be the smallest, but with observed coverage significantly different from 95%. For the Pareto and Weibull distribution, the CV of (11) is slightly larger than the other confidence intervals, which have coverages usually different from 95%. This effect is more pronounced with $n = 200$. With the Gamma and χ^2-distributions, we have a small CV with bootstrap and $\widehat{G}_{\pi}^{(2)}$, but with very low coverages.

6. Discussion

Our simulation study shows the estimator proposed is usually less biased than the customary estimators. The observed coverages of the empirical likelihood confidence interval proposed are also closer to the nominal value. We considered a single stage design. However, the approach proposed can be extended for multi-stage design with unit non-response, using Berger's (2018*a*) approach combined with bootstrap. Auxiliary information has not been considered for simplicity. Calibration weights can be used within (6). The empirical likelihood approach proposed can also take into account of some auxiliary information, by adding additional constraints within (10) (see Berger and Torres, 2016; Berger, 2018*a*,*b*, for more details). These additional constraints imply that $\widehat{G}_{EL}$ will be different but usually close to $\widehat{G}_{\pi}$, because $\widehat{G}_{EL}$ is based on calibrated weights.

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Table 1: Relative bias (%) and mean squared error (MSE) of $\widehat{G}_\pi$, $\widehat{G}_\pi^{(1)}$ and $\widehat{G}_\pi^{(2)}$ for several distributions. G_0 is given by (3). The rows are sorted according to G_0.

			Relative bias (%)			MSE $\times 10,000$		
n	Distributions	G_0	$\widehat{G}_\pi$	$\widehat{G}_\pi^{(1)}$	$\widehat{G}_\pi^{(2)}$	$\widehat{G}_\pi$	$\widehat{G}_\pi^{(1)}$	$\widehat{G}_\pi^{(2)}$
200	Pareto($\alpha = 10, \beta = 1$)	0.05	-0.4	8.5	-0.9	0.2	0.4	0.2
	Weibull($\alpha = 10, \beta = 1$)	0.07	-0.7	6.3	-1.2	0.2	0.3	0.2
	Pareto($\alpha = 5, \beta = 1$)	0.11	1.0	5.1	0.6	0.8	1.1	0.8
	Weibull($\alpha = 5, \beta = 1$)	0.13	-0.6	2.8	-1.0	0.6	0.7	0.6
	$\Gamma(\alpha = 10, \beta = 1)$	0.18	-0.9	2.0	-1.3	1.2	1.5	1.2
	$\Gamma(\alpha = 5, \beta = 1)$	0.25	-2.0	0.2	-2.4	2.4	2.6	2.5
	LogN($\mu = 0, \sigma = 0.5$)	0.28	-0.6	0.9	-1.0	1.8	1.8	1.8
	Exp($\lambda = 1$)	0.50	-0.9	-0.3	-1.4	6.2	6.0	6.4
	χ_1^2	0.64	-1.9	-1.4	-2.3	8.8	8.1	9.7
	$\Gamma(\alpha = 0.2, \beta = 1)$	0.80	0.0	0.1	-0.3	3.5	3.5	3.6
500	Pareto($\alpha = 10, \beta = 1$)	0.05	-0.2	3.4	-0.3	0.1	0.1	0.1
	Weibull($\alpha = 10, \beta = 1$)	0.07	-0.7	2.1	-0.9	0.1	0.1	0.1
	Pareto($\alpha = 5, \beta = 1$)	0.11	0.9	2.6	0.8	0.3	0.4	0.3
	Weibull($\alpha = 5, \beta = 1$)	0.13	-0.5	0.9	-0.7	0.2	0.2	0.2
	$\Gamma(\alpha = 10, \beta = 1)$	0.18	-0.5	0.7	-0.7	0.5	0.5	0.5
	$\Gamma(\alpha = 5, \beta = 1)$	0.25	-1.7	-0.9	-1.9	1.1	1.0	1.1
	LogN($\mu = 0, \sigma = 0.5$)	0.28	-0.4	0.2	-0.6	0.7	0.7	0.7
	Exp($\lambda = 1$)	0.50	-0.9	-0.7	-1.1	2.5	2.4	2.6
	χ_1^2	0.64	-1.7	-1.5	-1.9	4.6	4.3	4.9
	$\Gamma(\alpha = 0.2, \beta = 1)$	0.80	-0.1	0.0	-0.2	1.4	1.4	1.4

Table 2: Observed coverages (%) of 95% confidence intervals of $\widehat{G}_\pi$ (bootstrap and empirical likelihood), $\widehat{G}_\pi^{(1)}$ and $\widehat{G}_\pi^{(2)}$. Relative bias RB$\{\widehat{V}(\widehat{G})\}$ (%), of the bootstrap variance estimator of $\widehat{G}_\pi$, and the linearised variance of $\widehat{G}_\pi^{(1)}$ and $\widehat{G}_\pi^{(2)}$. Several distributions are considered. The rows are sorted according to G_0 (see Table 1 for the values of G_0).

		Coverages (%)				RB$\{\widehat{V}(\widehat{G})\}$ (%)		
		$\widehat{G}_\pi$						
n	Distributions	Boot	(11)	$\widehat{G}_\pi^{(1)}$	$\widehat{G}_\pi^{(2)}$	$\widehat{G}_\pi$	$\widehat{G}_\pi^{(1)}$	$\widehat{G}_\pi^{(2)}$
200	Pareto$(\alpha = 10, \beta = 1)$	94.2	94.6	83.2†	94.0†	2.6	1.6	1.6
	Weibull$(\alpha = 10, \beta = 1)$	93.5†	94.6	85.6†	93.4†	4.8	3.7	4.0
	Pareto$(\alpha = 5, \beta = 1)$	94.4	95.1	92.6†	94.3	4.4	2.1	2.2
	Weibull$(\alpha = 5, \beta = 1)$	93.5†	94.5	93.3†	93.3†	3.9	2.7	3.2
	$\Gamma(\alpha = 10, \beta = 1)$	91.1†	92.9†	94.6	90.6†	-7.6	-17.5	-5.5
	$\Gamma(\alpha = 5, \beta = 1)$	89.4†	92.7†	93.4†	89.1†	-10.9	-2.6	-9.9
	LogN$(\mu = 0, \sigma = 0.5)$	93.7†	95.4	94.8	93.7†	-1.0	-1.7	-2.0
	Exp$(\lambda = 1)$	90.7†	92.7†	92.7†	90.4†	-13.7	-14.9	-14.4
	χ^2_1	89.1†	90.4†	90.8†	89.3†	-19.7	-20.8	-18.8
	$\Gamma(\alpha = 0.2, \beta = 1)$	94.7	95.4	94.4	94.6	-5.7	0.0	-2.8
500	Pareto$(\alpha = 10, \beta = 1)$	95.1	95.2	90.1†	94.3	4.9	-0.9	-0.9
	Weibull$(\alpha = 10, \beta = 1)$	93.5†	94.6	85.6†	93.4†	4.8	3.7	4.0
	Pareto$(\alpha = 5, \beta = 1)$	95.1	94.8	93.1†	94.3	4.2	-3.6	-3.9
	Weibull$(\alpha = 5, \beta = 1)$	93.2†	95.1	94.4	93.0†	0.6	-3.7	-3.5
	$\Gamma(\alpha = 10, \beta = 1)$	93.4†	94.7	94.4	93.2†	1.5	-2.7	-0.5
	$\Gamma(\alpha = 5, \beta = 1)$	88.6†	92.9†	92.3†	88.3†	-1.2	2.2	-1.9
	LogN$(\mu = 0, \sigma = 0.5)$	94.4	95.4	94.6	94.2	-1.0	-1.7	-2.0
	Exp$(\lambda = 1)$	93.1†	93.8†	93.7†	92.8†	-6.0	-7.7	-8.2
	χ^2_1	87.7†	87.5†	89.0†	87.6†	3.0	-1.6	0.6
	$\Gamma(\alpha = 0.2, \beta = 1)$	95.5	96.0	94.6	95.2	8.6	0.6	1.6

† Coverage rates significantly different from 95%: p-value $\leqslant 0.05$.

Table 3: Observed Average Length of 95% confidence intervals and observed coefficient of the lengths variation (CV) in percent. The rows are sorted according to G_0 (see Table 1 for the values of G_0).

		Average Lengths				CV(Lengths) %			
		$\widehat{G}_\pi$				$\widehat{G}_\pi$			
n	Distributions	Boot	(11)	$\widehat{G}_\pi^{(1)}$	$\widehat{G}_\pi^{(2)}$	Boot	(11)	$\widehat{G}_\pi^{(1)}$	$\widehat{G}_\pi^{(2)}$
200	Pareto($\alpha = 10, \beta = 1$)	0.017	0.018	0.017	0.017	16.4	18.1	16.2	16.2
	Weibull($\alpha = 10, \beta = 1$)	0.016	0.016	0.016	0.016	13.5	13.3	13.3	13.3
	Pareto($\alpha = 5, \beta = 1$)	0.036	0.037	0.035	0.035	14.6	15.1	14.2	14.2
	Weibull($\alpha = 5, \beta = 1$)	0.030	0.030	0.029	0.030	11.8	11.7	11.5	11.5
	$\Gamma(\alpha = 10, \beta = 1)$	0.039	0.040	0.041	0.039	23.1	38.4	66.9	28.5
	$\Gamma(\alpha = 5, \beta = 1)$	0.054	0.056	0.057	0.054	23.7	37.8	43.8	28.8
	LogN($\mu = 0, \sigma = 0.5$)	0.052	0.053	0.052	0.052	11.9	12.7	15.9	13.3
	Exp($\lambda = 1$)	0.089	0.092	0.089	0.090	19.3	30.9	26.2	23.7
	χ_1^2	0.089	0.095	0.091	0.091	27.4	38.5	40.1	33.8
	$\Gamma(\alpha = 0.2, \beta = 1)$	0.076	0.077	0.074	0.076	9.0	9.5	8.1	9.0
500	Pareto($\alpha = 10, \beta = 1$)	0.011	0.011	0.011	0.011	11.0	11.1	10.4	10.4
	Weibull($\alpha = 10, \beta = 1$)	0.010	0.010	0.010	0.010	8.6	8.6	8.3	8.3
	Pareto($\alpha = 5, \beta = 1$)	0.023	0.023	0.022	0.022	9.3	9.4	8.6	8.6
	Weibull($\alpha = 5, \beta = 1$)	0.019	0.019	0.019	0.019	8.2	8.1	7.6	7.6
	$\Gamma(\alpha = 10, \beta = 1)$	0.025	0.026	0.026	0.025	20.9	47.5	79.4	23.9
	$\Gamma(\alpha = 5, \beta = 1)$	0.036	0.037	0.036	0.035	22.0	37.6	33.9	26.8
	LogN($\mu = 0, \sigma = 0.5$)	0.033	0.033	0.032	0.032	9.9	9.9	12.0	10.9
	Exp($\lambda = 1$)	0.058	0.060	0.058	0.058	20.1	34.9	27.5	23.9
	χ_1^2	0.060	0.064	0.060	0.061	32.3	52.4	49.5	39.1
	$\Gamma(\alpha = 0.2, \beta = 1)$	0.048	0.048	0.046	0.046	5.7	6.0	5.1	5.1