Cluster Groups: Groups with Presentations Arising from Cluster Algebras

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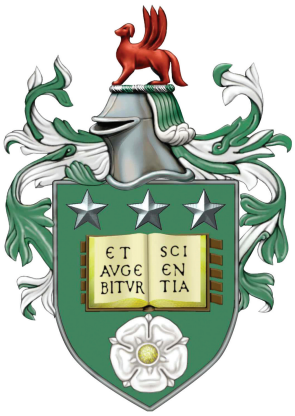

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Abstract

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Each quiver appearing in a seed of a cluster algebra determines a corresponding group, which we call a cluster group, defined via a presentation. Grant and Marsh showed that, for quivers appearing in seeds of cluster algebras of finite type, the associated cluster groups are isomorphic to finite reflection groups, thus are finite Coxeter groups. There are many well-established results for Coxeter presentations and we are interested in whether cluster group presentations possess comparable properties.

As for finite Coxeter groups, we can consider parabolic subgroups of cluster groups. We prove that, in the type A_n case, an isomorphism exists between the lattice of subsets of the set of defining generators of the cluster group and the lattice of its parabolic subgroups. Moreover, we show each parabolic subgroup has a presentation given by restricting the presentation of the whole group. In addition, we provide a method for obtaining a positive companion basis of the quiver.

For more general quivers, we prove an alternative exchange lemma for the associated cluster group by showing that each element has a factorisation with respect to a given parabolic subgroup. In the type A_n case, we also consider elements whose reduced expressions all begin with a certain fixed generator and examine the form of these reduced expressions.

Finally, we provide an alternative proof to the known fact that the length function on a parabolic subgroup of a Coxeter group of type A_n agrees with the length function on the whole group and discuss an analogous conjecture for cluster groups.

Contents

Introduction

Cluster groups arise from the theory of cluster algebras and are closely connected to finite Coxeter groups.

A finite Coxeter group is an abstract group arising from a Coxeter presentation, a group presentation whose relations are of a certain form. An outline of the rich history of research of these groups is given in [7, Historical Note], where it is explained how Coxeter groups evolved from the theory of finite reflection groups when [10] proved that each finite reflection group can be defined by a Coxeter presentation. Conversely, [11] later showed that every finite Coxeter group has a realisation as a reflection group, thus classifying the finite Coxeter groups.

The finite reflection groups can be classified via the irreducible root systems and their corresponding Dynkin diagrams [33, Chapter 2].

The applications of Coxeter groups are widespread throughout algebra [7], analysis [27], applied mathematics [9] and geometry [20]. However, the many combinatorial properties of Coxeter groups make them an interesting topic of research in their own right (see [4]).

Similarly, the article [37] discusses the tremendous growth of the relatively new theory of cluster algebras which has resulted from the field's connections with a wide range of subjects. Relating group presentations to cluster algebras is one such recent development.

Cluster algebras were introduced by Fomin and Zelevinsky in order to construct a model for dual canonical bases in semisimple groups [23]. However, their applications have spread into numerous areas of mathematics such as combinatorics [24], quiver representation theory [45] and Poisson geometry [26].

A cluster algebra is a subalgebra of the field of rational functions, $\mathbb{F} = \mathbb{Q}(u_1, u_2, ..., u_n)$, on a finite number of indeterminates. It is determined by an initial input known as a seed, consisting of a free generating set of $\mathbb{F}$ over $\mathbb{Q}$ and an integral skew-symmetrizable matrix, called the exchange matrix. When the exchange matrix is skew-symmetric, it is represented by a quiver and the cluster algbera is called skew-symmetric. A combinatorial process called mutation is repeatedly applied to the initial seed, yielding more seeds, and from this process a generating set for the cluster algebra is produced. The elements of this generating set are known as the cluster variables of the cluster algebra. A cluster algebra is

of finite type if it has finitely many cluster variables. The classification of cluster algebras of finite type [24, Theorem 1.9] shows that a skew-symmetric cluster algebra is of finite type if and only if there is a seed containing a quiver which is an oriented simply-laced Dynkin diagram. Moreover, it is shown that the cluster variables of the cluster algebra are in bijection with the almost positive roots of the root system of the same Dynkin type. Thus the classification of the cluster algebras of finite type demonstrates a connection between Coxeter groups and cluster algebras.

This connection was further developed in [3], which associated to each quiver appearing in a seed of a cluster algebra of finite type a group presentation and showed that the corresponding group is invariant under mutation. For oriented Dynkin diagrams, this presentation is precisely a Coxeter presentation. Consequently, the group is isomorphic to a finite reflection group of the same Dynkin type as the cluster algebra from which the quiver arises. In addition, similar presentations have been defined for affine Coxeter groups [19] and Artin braid groups, in the simply-laced case [28] and the finite type case [29]. Considering the group presentation associated to each mutation-Dynkin quiver in [28], together with the additional set of relations that specify that the square of each generator is equal to the identity, [28, Lemma 2.5] shows the resulting group is isomorphic to the group with presentation defined by [3]. Thus, we obtain another presentation associated to *any* quiver appearing in a skew-symmetric cluster algebra of finite type that gives a group that is isomorphic to a finite reflection group of the same Dynkin type. It is this group presentation, based on the work done in [28], that is considered here. These presentations make sense for any cluster quiver and so we are interested in considering them more generally. Due to the context given above, we call the corresponding group a 'cluster group'.

There is significant existing research into the fields of both cluster algebras and finite Coxeter groups. The theory of cluster groups provides an interesting connection between these influential topics. However, cluster group presentations have yet to be thoroughly studied as an independent concept. This thesis aims to build a body of results for cluster group presentations. In particular, it contains three main results which demonstrate that certain properties of the Coxeter presentations of a finite reflection group are transferable to the cluster group presentation.

As for Coxeter groups, we can define a parabolic subgroup of a cluster group to be a subgroup generated by a subset of the set of defining generators in the cluster group presentation. The first main result, given in Chapter 6, shows that there exists a lattice isomorphism between the collection of subsets of the generators of a cluster group presentation associated to quivers of mutation-Dynkin type A_n and the set of parabolic subgroups of the cluster group. Chapter 6 also contains the second main result, that each parabolic subgroup has a cluster group presentation given by restricting the presentation

of the whole group. The final result appears in Chapter 7 and shows any cluster group associated to a cluster quiver satisfies a property similar to the 'Exchange Property' for Coxeter groups (see [4, Theorem 1.5.1]).

An outline of the thesis is as follows.

In Chapter 1 we will present the necessary background material for root systems, finite reflection groups, and finite Coxeter groups. This includes details on the classification of the finite reflection groups via root systems, Cartan matrices and the Dynkin diagrams as well as a brief summary of the properties of finite Coxeter groups that will motivate some of our lines of research into cluster groups.

Chapter 2 will set out the definition and classification of the cluster algebras of finite type, including the interpretation of this classification for skew-symmetric cluster algebras using the simply-laced Dynkin diagrams. An additional focus will be given to the properties of quivers appearing in skew-symmetric cluster algebras of finite type A_n. In particular, we will show how a graph, known as the braid graph, can be associated to a quiver of mutation-Dynkin type A_n.

Given $n \geq 0$, the finite Coxeter group of type A_n is isomorphic to the symmetric group, Σ_{n+1} [7, Chapter VI, Section 4.7]. This property will be used to prove our first main result: a lattice isomorphism theorem for cluster groups of mutation-Dynkin type A_n. In preparation, Chapter 3 recalls key definitions and results for set partitions, lattices and lattice homomorphisms, as well as the Young subgroups of Σ_{n+1}.

The focus of Chapter 4 is to formally introduce a cluster group associated to a cluster quiver. Elementary results will be given for cluster groups, such as using results from [28] to check invariance of the associated cluster group under mutation of the quiver. Furthermore, we will give definitions of the length function and parabolic subgroups, analogous to those in the theory of Coxeter groups.

Chapter 5 and Chapter 6 contain a study of the cluster groups associated to quivers of mutation-Dynkin type A_n. Chapter 5 shows how the braid graph can be used to construct a group isomorphism between the associated cluster group and Σ_{n+1} and to obtain a positive companion basis of the quiver. The first two main results of the thesis are established in Chapter 6. Namely, we use results from Chapter 5 to prove that there exists a lattice isomorphism between the collection of subsets of the generators of the cluster group presentation associated to quivers of mutation-Dynkin type A_n and the set of parabolic subgroups of the cluster group. This result is achieved by associating a set partition to each subset of the set of generators and constructing and composing lattice isomorphisms between a lattice of set partitions, a lattice of Young subgroups and the lattice of parabolic subgroups of the cluster group. In Chapter 6, we give cluster group presentations of the parabolic subgroups of cluster groups of this type. We note that the results appearing in Chapter 6, along with several results from Section 4.2 and Section 5.2, have been pub-

lished in [52]

Finally, in Chapter 7 we consider cluster groups more generally and prove an 'alternative' exchange lemma for cluster group presentations. The proof relies on the existence of a factorisation of a reduced expression of each element with respect to a given parabolic subgroup. In this chapter we also further explore the format of reduced expressions of elements in the cluster groups of mutation-Dynkin type A_n. It is shown that there exists a reduced expression conforming to a particular arrangement for elements whose reduced expressions all begin in a fixed generator. The chapter concludes by giving a new approach to proving the known fact that the length function on a parabolic subgroup of a finite Coxeter group of type A_n agrees with the length function on the whole group. We conjecture that the parabolic subgroups of a cluster groups possess the same property and discuss a possible approach to proving this.

Chapter 1

Finite Reflection Groups

1.1 Introduction

The inter-play between root systems, Cartan matrices, finite reflection groups and finite Coxeter groups is crucial in the study of cluster groups and so we begin by establishing the definitions of these mathematical objects and summarising their classifications.

It is well known that the equivalence classes of the Cartan matrices of finite type classify the irreducible crystallographic root systems of real inner product spaces, up to isomorphism, via the Dynkin diagrams. Root systems, the concept of which originates from Lie theory [17], are subsets of real inner product spaces whose elements satisfy certain criteria.

Furthermore, each root system, Φ, of a real inner product space, V, gives rise to a finite reflection group, W, acting on V. This is done by taking W to be the group generated by the set of reflections in the hyperplanes orthogonal to the elements of Φ [17, Section 11.3.1]. In fact, every finite reflection group can be generated in this way [33, Theorem 1.5]. Under this correspondence, the crystallographic root systems correspond to the crystallographic finite reflection groups, which are known as the Weyl groups [33, Section 2.9].

The theory of finite reflection groups is well-established and it has been shown that each finite reflection group has a Coxeter presentation [33, Theorem 1.9], a group presentation with certain restrictions on the defining relations. Any group with a Coxeter presentation is called a *Coxeter group*. Every finite Coxeter group can be viewed as a (not necessarily crystallographic) finite reflection group on a real inner product space [33, Section 5.3].

Any Coxeter group can be represented by its *Coxeter graph*, which defines the Coxeter presentation of the group, and these graphs classify the Coxeter groups [7, Chapter VI, Section 4, Theorem 1].

In this chapter, we will give the key definitions of root systems, finite reflection groups and Coxeter groups and look in further detail at the correspondence between Cartan matrices of finite type, crystallographic root systems, finite reflection groups and finite Coxeter groups. Where relevant, we will demonstrate how the Dynkin diagrams can be used to represent these structures.

1.2 Root Systems

In this section we will define an irreducible crystallographic root system in a real inner product space and a Cartan matrix of finite type. We will outline how the isomorphism classes of the irreducible crystallographic root systems are in bijection with the equivalence classes of the Cartan matrices of finite type, which are classified by the Dynkin diagrams.

Let V be a real inner product space where the inner product of any $\alpha, \beta \in V$ is denoted by (α, β).

Definition 1.2.1. *[7, Chapter V, Section 2] A **reflection** on V is a linear operator on V which sends some $\alpha \in V$ to $-\alpha$ and fixes the hyperplane, H_α, orthogonal to α. Such a reflection is denoted by s_α and is given by the formula:*

$$s_\alpha(\beta) = \beta - 2\frac{(\alpha,\beta)}{(\alpha,\alpha)}\alpha, \qquad \beta \in V.$$

For any $\alpha \in V$, the reflection s_α is an orthogonal linear transformation [40, Lemma 4.1.3]. That is, reflections preserve the inner product.

Definition 1.2.2. *[33, Section 1.2] A **root system** in V is a finite subset Φ of $V \setminus \{0\}$ such that, for all $\alpha \in \Phi$, the following hold.*

(1) *The reflection s_α restricted to Φ gives a bijection from Φ onto itself (i.e. $s_\alpha(\Phi) = \Phi$).*

(2) $\Phi \cap \mathbb{R}\alpha = \{\pm\alpha\}$.

*A root system, Φ, of V is **crystallographic** if, for all $\alpha, \beta \in \Phi$, the following condition is also satisfied.*

(3) $2\frac{(\alpha,\beta)}{(\alpha,\alpha)} \in \mathbb{Z}$.

Remark 1.2.3. Some authors also require that Φ spans V [17] and, in the affine case, the group generated by the corresponding reflections, given the discrete topology, acts properly on V [38, Section 1.2]. However, excluding these additional assumptions doesn't affect the development of the results cited in this chapter.

Let Φ be a root system in V.

Definition 1.2.4. *[17, Definition 11.7] If Φ cannot be written as a disjoint union of two non-empty subsets Φ_1 and Φ_2 such that $(\alpha,\beta)=0$ for all $\alpha\in\Phi_1$, $\beta\in\Phi_2$, then Φ is said to be **irreducible**.*

Definition 1.2.5. *[17, Definition 11.19] The root system Φ' of the real inner product space V' is **isomorphic** to Φ if there exists a vector space isomorphism $\varphi: V\longrightarrow V'$ such that $\varphi(\Phi)=\Phi'$ and $2\frac{(\alpha,\beta)}{(\alpha,\alpha)}=2\frac{(\varphi(\alpha),\varphi(\beta))}{(\varphi(\alpha),\varphi(\alpha))}$, for all $\alpha,\beta\in\Phi$.*

Definition 1.2.6. *[33, Section 1.3] A subset Π of Φ is a **simple system** for Φ if the following conditions hold.*

(1) *Π is a basis for $\mathbb{R}\Phi$ (as a subspace of V).*

(2) *For all $\alpha\in\Phi$, α can be written as a linear combination of elements in Π such that the coefficients are either all non-negative or all non-positive.*

Example 1.2.7. For $n\geq 1$, take $V=\mathbb{R}^{n+1}$ together with the dot product on V. Consider the basis $\{e_1,...,e_{n+1}\}$ of V, where e_i is the vector with 0 in every entry except the i^{th} entry, which is equal to 1. We define:

$$\Phi=\{\pm(e_i-e_j):1\leq i<j\leq n+1\}\subseteq V$$

and

$$\Pi=\{\alpha_i=e_i-e_{i+1}:1\leq i\leq n\}\subseteq\Phi.$$

Then Φ is a root system with simple system Π. To see this, we first show that, for each $\alpha\in\Phi$, $s_\alpha|_\Phi:\Phi\longrightarrow\Phi$ is a bijection. For any $1\leq,i,j,k,l\leq n+1$, where $i\neq j$ and $k\neq l$, we have

$$s_{(e_i-e_j)}(e_k-e_l)=(e_k-e_l)-[(e_k,e_i)-(e_l,e_i)-(e_k,e_j)+(e_l,e_j)](e_i-e_j).$$

If i,j,k,l are all distinct, then $[(e_k,e_i)-(e_l,e_i)-(e_k,e_j)+(e_l,e_j)]=0$, so $s_{(e_i-e_j)}(e_k-e_l)=(e_k-e_l)\in\Phi$.

If $i=k$, $j\neq l$ then $s_{(e_i-e_j)}(e_k-e_l)=(e_k-e_l)-(e_k-e_j)=(e_j-e_l)\in\Phi$.

If $i\neq k$, $j=l$ then $s_{(e_i-e_j)}(e_k-e_l)=(e_k-e_l)-(e_i-e_l)=(e_k-e_i)\in\Phi$.

If $i=l$, $j\neq k$ then $s_{(e_i-e_j)}(e_k-e_l)=(e_k-e_l)+(e_l-e_j)=(e_k-e_j)\in\Phi$.

If $i\neq l$, $j=k$ then $s_{(e_i-e_j)}(e_k-e_l)=(e_k-e_l)+(e_i-e_k)=(e_i-e_l)\in\Phi$.

If $i=k$ and $j=l$ or $i=l$ and $j=k$ then $s_{(e_i-e_j)}(e_k-e_l)=(e_k-e_l)\pm 2(e_k-e_l)=(e_l-e_k)\in\Phi$.

So $s_\alpha(\Phi)\subseteq\Phi$. As $s_\alpha : V\longrightarrow V$ is a bijection, $s|_\Phi$ is injective. So $s_\alpha(\Phi)=\Phi$, as required.

Clearly, $\{\alpha,-\alpha\}\subseteq\Phi\cap\mathbb{R}\alpha$ for all $\alpha\in\Phi$. Conversely, if $\beta\in\Phi\cap\mathbb{R}\alpha$ then $\beta=r\alpha$, for some $r\in\mathbb{R}$, and $\beta=\pm(e_i-e_j)$, for some $1\leq i<j\leq n$. Equating these, as α is of the form $\pm(e_p-e_q)$ for some $1\leq p<q\leq n$, it follows that $r=\pm 1$ and $\alpha=(e_i-e_j)$. So $\beta\in\{\alpha,-\alpha\}$. That is, $\{\alpha,-\alpha\}=\Phi\cap\mathbb{R}\alpha$.

We now show that Π is a simple system. For each $1\leq i<j\leq n+1$, we have

$$(e_i-e_j)=(e_i-e_{i+1})+(e_{i+1}-e_{i+2})+\ldots+(e_{j-1}-e_j).$$

So every element of Φ can be written as a linear combination of elements in Π (so Π spans $\mathbb{R}\Phi$) with either all non-negative or all non-positive coefficients.

Moreover, if $\Sigma_{i=1}^n r_i(e_i-e_{i+1})=0$, for some $r_i\in\mathbb{R}$, then $0=\Sigma_{i=1}^n r_i(e_i-e_{i+1})=r_1e_1+\Sigma_{i=2}^n(r_i-r_{i-1})e_i-r_ne_{n+1}$. Thus $r_1=0$ and $r_n=0$ and it follows that $r_i=0$ for all $1\leq i\leq n$. Thus the elements of Π are linearly independent and so, as we have already shown that Π spans $\mathbb{R}\Phi$, Π is a basis for $\mathbb{R}\Phi$. So Π satisfies Definition 1.2.6.

Theorem 1.2.8. *[33, Theorem 1.3(b)] Every root system has a simple system.*

Definition 1.2.9. *[17, Section 11.31] Let Φ be a root system in a real inner product space, V. We define a group associated to Φ generated by reflections on V by:*

$$W_\Phi=\langle s_\alpha : \alpha\in\Phi\rangle.$$

*When Φ is crystallographic, we call W_Φ the **Weyl group** of Φ.*

Lemma 1.2.10. *[17, Lemma 11.12] (or [33, Section 1.2]) Let Φ be a root system. Then W_Φ is finite.*

Remark 1.2.11. The proof of [17, Lemma 11.12] requires that Φ spans the inner product space. However, this requirement can be accommodated by considering Φ as a root system of $\mathbb{R}\Phi$. Moreover, only crystallographic root systems are being considered in [17, Lemma 11.12] but the proof continues to hold when it is generalized to all root systems.

Theorem 1.2.12. *[17, Theorem 11.16] Suppose that Π and Π' are both simple systems of Φ. Then there exists an element $w \in W_\Phi$ such that $\Pi' = \{w(\alpha) : \alpha \in \Pi\}$.*

In order to classify the irreducible crystallographic root systems, we need to define a Cartan matrix of finite type.

Definition 1.2.13. *[36, Section 1.1] A **generalised Cartan matrix** is an integral $n \times n$ matrix, $A = (a_{ij})$, such that*

(a) For all $1 \leq i \leq n$, $a_{ii} = 2$.

(b) For all $i \neq j$, $a_{ij} \leq 0$.

(c) For all $i \neq j$, if $a_{ij} = 0$ then $a_{ji} = 0$.

If A also satisfies the additional condition:

(d) [36, Section 4] The principal minors of A are positive.

*then A is a **Cartan matrix of finite type**.*

Definition 1.2.14. *Two $n \times n$ Cartan matrices of finite type, $A = (a_{ij})$ and $A' = (a'_{ij})$, are **equivalent** if there exists a permutation σ of $\{1, 2, ..., n\}$ such that $A' = (a_{\sigma(i),\sigma(j)})$.*

It is easy to see that Definition 1.2.14 defines an equivalence relation on the set of Cartan matrices of finite type. These equivalence classes are classified by the Dynkin diagrams.

Definition 1.2.15. *[36, Chapter 4] Given an $n \times n$ Cartan matrix, A, of finite type, the **Dynkin diagram corresponding to** A, denoted by $S(A)$, is the graph on n vertices where, if $|a_{ij}| \neq 0$, the vertices i and j are joined by $|a_{ij}|$ edges. Moreover, if $|a_{ij}| \geq |a_{ji}|$ and $|a_{ij}| > 1$ then we add an arrow to the edges, pointing towards i.*

Note that, up to relabelling of the vertices, the Dynkin diagram, $S(A)$, determines the Cartan matrix of finite type, A, up to equivalence. That is, $S(A) = S(A')$ if and only if A and A' are equivalent.

Definition 1.2.16. *[36, Section 0.3] An $n \times n$ Cartan matrix, $A = (a_{ij})$, is **indecomposable** if there is no non-trivial subsets $I, J \subset \{1, ..., n\}$ such that $I \cap J = \emptyset$, $I \cup J = \{1, ..., n\}$ and $a_{ij} = 0$ whenever $i \in I, j \in J$.*

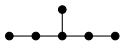
Theorem 1.2.17. *[36, Theorem 4.8(a)] If A is an indecomposable Cartan matrix of finite type then $S(A)$ is one of the following connected diagrams:*

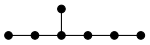
A_n, $n \geq 1$

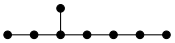
B_n, $n \geq 2$

C_n, $n \geq 3$

D_n, $n \geq 4$ 

E_6 

E_7 

E_8 

F_4

G_2

We refer to the Dynkin diagrams A_n, D_n, E_6, E_7 and E_8 as the **simply-laced Dynkin diagrams** and their corresponding equivalence classes of Cartan matrices of finite type as the **simply-laced Cartan matrices of finite type**.

Definition 1.2.18. *[7, Chapter VI, Section 1.5, Definition 3] Let Φ be a crystallographic root system in a real inner product space V with simple system $\Pi = \{\alpha_1, ..., \alpha_n\}$. The **Cartan matrix** of Φ is the matrix given by $A = (a_{ij})$, where*

$$a_{ij} = 2\frac{(\alpha_i, \alpha_j)}{(\alpha_i, \alpha_i)}.$$

By Theorem 1.2.12, together with the fact that reflections preserve the inner product of V, the Cartan matrix of a root system does not depend on the choice of simple system (up to equivalence).

Example 1.2.19. Consider the root system Φ with simple system Π as given in Example 1.2.7. In the case $n = 3$, the corresponding Cartan matrix:

$$A = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix}$$

From Definition 1.2.15, the corresponding Dynkin diagram, $S(A)$, will be of type A_3:

$$S(A) = A_3, \quad \bullet\!\!-\!\!\bullet\!\!-\!\!\bullet$$

In general, for $n \geq 1$, the corresponding Cartan matrix for the root system Φ will be the $n \times n$ matrix:

$$A = \begin{pmatrix} 2 & -1 & 0 & 0 & \dots & 0 \\ -1 & 2 & -1 & 0 & \dots & 0 \\ 0 & -1 & 2 & -1 & \dots & 0 \\ \vdots & & & \ddots & & \vdots \\ 0 & 0 & 0 & \dots & 2 & -1 \\ 0 & 0 & 0 & \dots & -1 & 2 \end{pmatrix}$$

It follows that $S(A)$ will be the Dynkin diagram of type A_n:

$$S(A) = A_n, \quad \bullet\!\!-\!\!\bullet\!\!-\!\!\cdots\!\!-\!\!\bullet\!\!-\!\!\bullet$$

Theorem 1.2.20. *[34, Section 11.1] Two crystallographic root systems are isomorphic if and only if their corresponding Cartan matrices are equivalent.*

Theorem 1.2.21. *Any Cartan matrix corresponding to an irreducible, crystallographic root system is indecomposable and of finite type.*

Therefore Definition 1.2.18 defines an injective mapping from the isomorphism classes of irreducible, crystallographic root systems and the equivalence classes of indecomposable Cartan matrices of finite type. In fact, the following theorem shows that this mapping is a bijection.

Theorem 1.2.22. *[17, Section 13.2] Every Cartan matrix of finite type gives rise to a crystallographic root system.*

1.3 Finite Reflection Groups and Coxeter Presentations

In this section we will see how every finite reflection group arises from a root system and vice versa.

Definition 1.3.1. *A finite reflection group is a finite group generated by a set of reflections on a real inner product space.*

Note that while a finite reflection group is necessarily generated by a finite set of reflections, a reflection group which is generated by a finite set of reflections is not necessarily finite [33, Chapter 4].

By Lemma 1.2.10, if Φ is a root system in a real inner product space, V, then W_Φ is a finite reflection group. Conversely, given a finite reflection group, W, acting on a real inner product space V, we can obtain a root system in V in the following way. For each reflection $s_\alpha \in W$, consider the line, $\mathbb{R}\alpha$, orthogonal to H_α. Clearly, for all $\beta \in V$, $\beta \in \mathbb{R}\alpha$ if and only if $w(\beta) \in \mathbb{R}w(\alpha)$ for all $w \in W$. Furthermore, we have the following proposition.

Proposition 1.3.2. *[33, Proposition 1.2] For any $s_\alpha \in W$, $s_{w(\alpha)} \in W$ for all $w \in W$.*

That is, each element of W permutes the set of lines $L = \{\mathbb{R}\alpha : s_\alpha \in W\}$. This, together with the fact that reflections preserve the inner product on V, means that the collection of all vectors of the same length lying on the lines in L, satisfies the definition of a root system in V. Let Φ represent this root system. By Theorem 1.2.8, we can fix some simple system Π of Φ.

Theorem 1.3.3. *[33, Theorem 1.5] For any finite reflection group, W, acting on a real inner product space V, there exists a root system Φ in V such that $W = W_\Phi$.*

By Lemma 1.2.10, from each root system Φ, we can define a finite reflection group and, by Theorem 1.3.3, each finite reflection group arises from a root system. The finite reflection groups arising from crystallographic root systems (i.e. the Weyl groups) are also referred to as the **crystallographic reflection groups** (see [33, Section 2.8]).

Remark 1.3.4. Non-isomorphic root systems can give rise to isomorphic reflection groups. For example, the root systems of type B_n and C_n give rise to isomorphic reflection groups [33, Section 2.9].

In order to classify the finite reflection groups completely, we first need to define a Coxeter presentation and its Coxeter graph.

Definition 1.3.5. *[44, Section 1.1] Let X be a set. A **word** on X is an ordered tuple of elements in $X \sqcup X^{-1}$, where $X^{-1} = \{x^{-1} : x \in X\}$ is a copy of X. We denote the tuple $w = (w_1, w_2, ..., w_k)$ by $w = w_1 w_2 ... w_k$.*

Definition 1.3.6. *[44, Section 1.1] Consider the two words on a set X given by $u = u_1 u_2 ... u_j$ and $w = w_1 w_2 ... w_k$. The **concatenation** of u with w is the word $uw = u_1 u_2 ... u_j w_1 w_2 ... w_k$.*

Definition 1.3.7. *[44, Section 1.4] A **reduced word** on a set X is a word on X in which x and x^{-1} are never adjacent for any $x \in X$. The **empty word**, denoted by e, is the unique 0-tuple of elements in $X \sqcup X^{-1}$ and is a reduced word.*

Definition 1.3.8. *[44, Section 1.4] Given a set X, the **free group generated by** X, denoted by $F(X)$, is the set of all reduced words on X under multiplication given by concatenation of words, followed by deleting all pairs $x^{-1}x$ and xx^{-1}. Note that this is well-defined as two different reductions on a word will result in the same reduced word [44, Theorem 1.2].*

Definition 1.3.9. *[44, Section 1.2] Given a subset, R, of a free group, $F(X)$, the **normal closure of** R is the normal subgroup of $F(X)$:*

$$N^R = \{w_1^{-1} r_1^{\varepsilon_1} w_1 w_2^{-1} r_2^{\varepsilon_2} w_2 ... w_n^{-1} r_n^{\varepsilon_n} w_n : w_i \in F(X), r_i \in R, \varepsilon_i \in \{\pm 1\}, n \geq 0\}$$

Taking an index set I, let R be a set of relations of the form $x_i = y_i$, where $x_i, y_i \in F(X)$, for $i \in I$. After replacing R with the subset $\{x_i y_i^{-1} : i \in I\}$ of $F(X)$, we construct N^R as given in the definition above.

Definition 1.3.10. *[44, Section 2.1] Given a set, X, together with a set, R, of relations on the elements of X, the **group presentation** $\langle X|R\rangle$ denotes the quotient group $F(X)/N^R$. We say that the group $F(X)/N^R$ (or any group isomorphic to $F(X)/N^R$) has group presentation $\langle X|R\rangle$.*

Definition 1.3.11. *[33, Section 5.1] A group, W, is a **Coxeter group** if there exists a group presentation of W of the form*

$$\langle S|R\rangle$$

*where $S = \{s_1, .., s_n\}$ is a non-empty set and R is the set of relations of the form $(s_i s_j)^{m(i,j)}$, for all $s_i, s_j \in S$, where $m(i,j) \in \mathbb{N} \cup \{\infty\}$ with $m(i,j) = 1$ if $i = j$, and $m(i,j) \geq 2$ otherwise. If $m(i,j) = \infty$ then no relation occurs on s_i and s_j. The pair (W,S) is known as a **Coxeter system**.*

Definition 1.3.12. *[7, Chapter IV, Section 1.9] Given a Coxeter system, (W,S), the **Coxeter graph** of W is the graph on n vertices, labelled by the set $\{1,...,n\}$, where i and j are joined by an edge when $m(i,j) \geq 3$. If $m(i,j) > 3$, the edge is labelled by $m(i,j)$.*

Definition 1.3.13. *[7, Chapter IV, Section 1.9] A Coxeter system, (W,S), is **irreducible** if the underlying graph of its Coxeter graph is connected.*

Example 1.3.14. Consider the group, G, arising from the Coxeter presentation

$$\langle s_1, s_2 | (s_i s_j)^{m(i,j)} \rangle$$

where

$$m(i,j) = \begin{cases} 1 & \text{if } i = j \\ 3 & \text{if } i \neq j \end{cases}$$

So

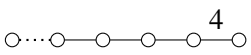
$$G = \{e, s_1, s_2, s_1 s_2, s_2 s_1, s_1 s_2 s_1\}$$

and has Coxeter graph:

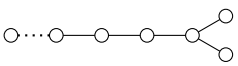
Theorem 1.3.15. *[7, Chapter VI, Section 4.1, Theorem 1] Let (W,S) and (W',S) be irreducible finite Coxeter systems. Then W and W' are isomorphic if and only if W and W' have isomorphic Coxeter graphs. Moreover, if (W,S) is an irreducible finite Coxeter system, the Coxeter graph of W is one of the following graphs:*

A_n, $n \geq 1$,

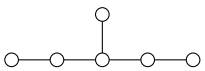
B_n, $n \geq 2$,



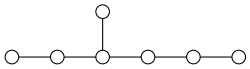
D_n, $n \geq 4$,



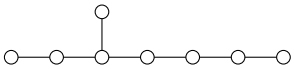
E_6



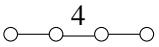
E_7

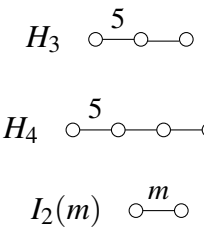


E_8



F_4





Therefore, the irreducible finite Coxeter groups are classified by their Coxeter graphs. In general, the Coxeter graph of a finite Coxeter group must be a disjoint union of the graphs in Theorem 1.3.15.

Theorem 1.3.16. *[33, Theorem 1.9] Let Φ be a root system in a real inner product space V. Then the finite reflection group, W_Φ, has a group presentation:*

$$\langle \{s_\alpha : \alpha \in \Pi\} | (s_\alpha s_\beta)^{m(\alpha,\beta)} = e \rangle$$

where $m(\alpha,\beta)$ is the order of the element $(s_\alpha s_\beta) \in W_\Phi$, for each $\alpha, \beta \in \Pi$.

That is, $W_\Phi = \langle s_\alpha : \alpha \in \Pi \rangle$ is subject only to the relations given above. As $m(\alpha,\alpha) = 1$ and, for all $\alpha, \beta \in \Pi$, $m(\alpha,\beta) \geq 2$ when $\alpha \neq \beta$, the finite reflection group associated to a root system is a finite Coxeter group. Thus the isomorphism classes of the finite reflection groups are also determined by the Coxeter graphs in Theorem 1.3.15. In fact, as we shall see in Section 1.4, every finite Coxeter group can be viewed as a finite reflection group, so the finite Coxeter groups and the finite reflection groups coincide.

In particular, we note that the Weyl groups have Coxeter presentations. We recall that a finite reflection group, W, is a Weyl group if $W = W_\Phi$ for some crystallographic root system Φ. Consider the Dynkin diagram of Φ. By replacing any double edge with a single edge labelled by 4 and any triple edge with a single edge labelled by 6, we obtain the Coxeter graph of W. Consequently, we are able to read the Coxeter presentation of W_Φ from the Dynkin diagram of Φ [7, Chapter VI, Section 4.2].

Remark 1.3.17. It follows that the isomorphism classes of the Weyl groups are determined by the underlying graphs of the Dynkin diagrams. Moreover, by considering these underlying graphs, we can conclude that the only non-isomorphic irreducible crystallographic root systems giving rise to isomorphic finite reflection groups are those of type B_n and C_n. For all other Dynkin diagrams, the corresponding finite reflection groups are pairwise non-isomorphic.

Example 1.3.18. Consider the root system Φ of type A_n, described in Example 1.2.7, whose Cartan matrix is give in Example 1.2.19. From the underlying graph of the Dynkin diagram of type A_n, we see that the finite reflection group W_Φ has Coxeter presentation:

$$\langle s_{\alpha_1}, s_{\alpha_2}, \ldots, s_{\alpha_n} | (s_{\alpha_i} s_{\alpha_j})^{m(i,j)} \rangle$$

where

$$m(i,j) = \begin{cases} 1 & \text{if } |i-j| = 0 \\ 2 & \text{if } |i-j| > 1 \\ 3 & \text{if } |i-j| = 1 \end{cases}$$

1.4 Properties of Coxeter Groups

Coxeter presentations give an insightful perspective into the Coxeter groups as they possess strong properties. Following [33, Section 5.3], we will provide a geometric interpretation of a given finite Coxeter group, showing it can be viewed as a finite reflection group. We will also outline some of the defining characteristics of finite Coxeter groups.

To begin, the following proposition shows that a finite Coxeter group has order at least 2.

Proposition 1.4.1. *[33, Proposition 5.1] Let (W,S) be a finite Coxeter system. There is a unique surjective homomorphism*

$$\varepsilon : W \longrightarrow \{\pm 1\},$$
$$\varepsilon : s_i \longmapsto -1.$$

It follows that each generator $s_i \in S$ has order 2 in W.

Given a finite Coxeter system, (W,S), we can view W as a finite reflection group on the vector space V over $\mathbb{R}$ with basis $\{\alpha_i : 1 \leq i \leq n\}$ and bilinear form

$$B(\alpha_i, \alpha_j) = -2\cos\left(\frac{\pi}{m(i,j)}\right).$$

For each $s_i \in S$, we define a linear transformation, σ_i, on V by

$$\sigma_i(\beta) = \beta - B(\alpha_i, \beta)\alpha_i, \qquad \beta \in V.$$

By calculation, it is clear that σ_i sends α_i to $-\alpha_i$, fixes H_{α_i} pointwise, preserves the bilinear form and has order 2 in $GL(V)$. We note that V is now a real vector space with bilinear form and each σ_i can be considered as a more general kind of reflection in the hyperplane orthogonal to α_i.

Proposition 1.4.2. *[33, Proposition 5.3, Corollary 5.4] There exists a unique, injective homomorphism*

$$\sigma : W \longrightarrow GL(V)$$

$$\sigma : s_i \longmapsto \sigma_i$$

such that any element in the image of σ preserves the bilinear form on V. Moreover, the order of $s_i s_j$ in W is precisely $m(i,j)$, for all $s_i, s_j \in S$.

Using this geometric representation of W, we can view W as a finite reflection group on V.

Theorem 1.4.3. *[33, Theorem 6.4] If W is a finite Coxeter group then W is a finite reflection group.*

As every finite reflection group has a Coxeter presentation, we have that the finite reflection groups are precisely the finite Coxeter groups.

For ease, we will denote $\sigma(w)(\alpha_i)$ simply by $w(\alpha_i)$, for each $w \in W$. We define a root system in V by

$$\Phi = \{w(\alpha_i) : w \in W, 1 \leq i \leq n\} \subseteq V$$

We call the elements of Φ the **roots** of W. Each root, $w(\alpha_i) \in \Phi$, can be written as some linear combination of the basis elements, α_j, with real coefficients:

$$w(\alpha_i) = \sum_{s_j \in S} c_j \alpha_j$$

If the coefficients, c_j, are either all positive or all negative, we say $w(\alpha_i)$ is **positive** and **negative**, respectively. The roots are a useful tool for proving results about Coxeter groups. Before outlining some of these results, we require some key definitions.

From the definition of group presentations, any element w of W can be written as

$$w = s_{i_1}^{a_1} s_{i_2}^{a_2} \ldots s_{i_r}^{a_r},$$

where $1 \leq i_j \leq n$ and $a_j = \pm 1$, for all $1 \leq j \leq r$. As each generator is of order 2, it follows that w can be written in the form

$$w = s_{i_1} s_{i_2} \ldots s_{i_r}.$$

The **length** of w, $l(w)$, is the smallest r such that $w = s_{i_1} s_{i_2} \ldots s_{i_r}$, for some $s_{i_j} \in S$, and a **reduced expression** of w is any expression of w as a product of $l(w)$ elements of S [33, Section 5.2].

Theorem 1.4.4. *[33, Theorem 5.4] For any $w \in W$ and $s_i \in S$, if $l(ws_i) < l(w)$ then $w(\alpha_i)$ is a negative root and if $l(ws_i) > l(w)$ then $w(\alpha_i)$ is a positive root.*

This gives the following corollary.

Corollary 1.4.5. *[33, Section 5.4] Any root $\alpha \in \Phi$ is either positive or negative.*

Thus $\{\alpha_i : 1 \leq i \leq n\}$ is a simple system of Φ. We call the elements of this set the **simple roots** of W.

For any subset $I \subseteq S$, W_I denotes the subgroup of W generated by I. Any subgroup of W which can be obtained in this way is called a **parabolic subgroup** of W [33, Section 5.4]. Given any $w \in W_I$, we can also consider the length of w with respect to I. That is, let $l_I(w)$ be the smallest r such that $w = s_{i_1}s_{i_2}...s_{i_r}$ for some $s_{i_j} \in I$. We also define the set

$$W^I = \{w \in W : l(ws_i) > l(w) \quad \forall s_i \in I\}.$$

Lemma 1.4.6. *[4, Lemma 2.4.3] For any finite Coxeter system (W,S) and subset $I \subseteq S$, $w \in W^I$ if and only if no reduced expression of W ends in an element of I.*

Proposition 1.4.7. *[4, Proposition 2.4.4] For any finite Coxeter system (W,S) and subset $I \subseteq S$, every $w \in W$ has a unique factorisation*

$$w = ab, \qquad \text{for some } a \in W^I, b \in W_I$$

such that $l(w) = l(a) + l(b)$.

Theorem 1.4.8. *[33, Theorem 5.5] For any finite Coxeter system (W,S) and $I \subseteq S$ the following hold.*

(a) With the given values $m(i,j)$, (W_I, I) is a finite Coxeter system.

(b) $l = l_I$.

(c) The collection of parabolic subgroups of W forms a lattice under inclusion and the assignment $I \longmapsto W_I$ defines a lattice isomorphism between the lattice of subsets of S and the lattice of parabolic subgroups of W.

(d) S is a minimal generating set for W.

A group, W, with generating set S, each of whose elements is of order 2, is said to satisfy the **Exchange Property** if the following holds.

If $w = s_1...s_r$ is an expression of $w \in W$ and $s \in S$ is such that $l(sw) < l(w)$ then $sw = s_1...\hat{s_j}...s_r$ for some $1 \leq j \leq r$.

The Exchange Property is a characterizing result for Coxeter groups.

Theorem 1.4.9. ***(The Exchange Lemma)*** *[4, Theorem 1.5.1] A group, W, with generating set S such that each element of S is of order 2 satisfies the Exchange Property if and only if (W,S) is a Coxeter system.*

It can be useful to reinterpret the Exchange Property as the **Deletion Property**:

If $w = s_1...s_r$ is an expression of $w \in W$ and $l(w) < r$, then $w = s_1...\hat{s_i}...\hat{s_j}...s_r$ for some $1 \leq i < j \leq r$.

A group, W, with generating set S, such that each element of S is of order 2, satisfies the Exchange Property if and only if it satisfies the Deletion Property [4, Theorem 1.5.1], meaning either one can be used as an identifying property of a Coxeter groups.

Consider an expression $w = s_{i_1}...s_{i_r}$ (not necessarily reduced) of an element $w \in W$. For each $1 \leq j \leq r$, we define the root $\beta_j = s_{i_r}s_{i_{r-1}}...s_{i_{j+1}}(\alpha_{i_j})$, where β_{i_r} is interpreted as α_{i_r}. These roots provide insight into the given expression of $w \in W$.

Lemma 1.4.10. *[33, Section 5.6] Given an expression $w = s_{i_1}...s_{i_r}$ of $w \in W$, the following hold.*

(a) The expression $s_{i_1}...s_{i_r}$ is reduced if and only if $\beta_j \neq \beta_k$ for all $1 \leq j,k \leq r$, $j \neq k$.

(b) The set of roots $\{\beta_j : 1 \leq j \leq r\} \subseteq \Phi$ is precisely the set of positive roots sent to negative roots by w.

(c) The length of w is equal to the number of positive roots turned negative by w. That is, $l(w) = |\{\beta_j : 1 \leq j \leq r\}|$, if $w = s_{i_1}...s_{i_r}$ is reduced.

By definition, each root $\alpha \in \Phi$ can be written as $\alpha = w(\alpha_i)$, for some $w \in W$ and $1 \leq i \leq n$. By considering the action on V, it can be seen that $s_\alpha = ws_{\alpha_i}w^{-1}$ [33, Section 5.7]. Thus the following equality holds.

$$\{s_\alpha : \alpha \in \Phi\} = \bigcup_{w \in W} wSw^{-1}.$$

Finally, we have the following useful results for the reflections in W.

Lemma 1.4.11. *[33, Section 5.7] The assignment $\alpha \longmapsto s_\alpha$ defines a bijection between the set of positive roots and the set of reflections in W.*

Lemma 1.4.12. *[33, Lemma 5.7 and 5.8] If $\alpha, \beta \in \Phi$ and $\beta = w(\alpha)$ for some $w \in W$, then $s_\beta = ws_\alpha w^{-1}$.*

Chapter 2

Cluster Algebras

2.1 Introduction

The topic of cluster groups arises from the theory of cluster algebras, which were introduced by Fomin and Zelevinsky in order to construct a model for dual canonical bases in semisimple groups [23].

A (skew-symmetric) cluster algebra [23, Section 2] is a subalgebra of the field of rational functions, $\mathbb{F} = \mathbb{Q}(u_1, u_2, ..., u_n)$, on a finite number of indeterminates. It is determined by an initial input, known as a *seed*, consisting of a free generating set of $\mathbb{F}$ over $\mathbb{Q}$ and a cluster quiver i.e. a quiver with no loops or two-cycles. A combinatorial process called *mutation* is repeatedly applied to produce more seeds. A generating set for the cluster algebra is then given by the union of the seeds. The elements of this generating set are known as the *cluster variables* of the cluster algebra. This notion of mutating, which involves a local change in the quiver relating to a choice of vertex, is at the heart of the definition of a cluster algebra.

A cluster algebra is of *finite type* if it has finitely many cluster variables. The classification [25] of the cluster algebras of finite type corresponds to the classification of the finite crystallographic root systems via Cartan matrices and their corresponding Dynkin diagrams. In this classification, it is shown that a skew-symmetric cluster algebra is of finite type if and only if it has a seed containing a cluster quiver which is an orientation of a (possibly disconnected) simply-laced Dynkin diagram. Thus, each skew-symmetric cluster algebra of finite type contains a seed whose quiver defines a Coxeter presentation giving rise to the reflection group of the same Dynkin type.

This correspondence suggests a connection between root systems and cluster algebras of finite type. In fact, [24, Theorem 1.9] shows that the cluster variables of a cluster algebra of finite type are in bijection with the *almost positive roots* of the corresponding root system.

In this chapter, we will recall the definitions of a quiver, quiver mutation and a cluster algebra as well as how cluster quivers can be used to record the strong isomorphism classes of the skew-symmetric cluster algebras. We will state the classification theorem for cluster algebras of finite type via the Cartan matrices of finite type and examine the role this classification gives to the skew-symmetric cluster algebras of finite type in the theory of finite reflection groups.

In the final sections, we will outline some existing background theory relating to quivers appearing in the skew-symmetric cluster algebras of finite type that will be useful, in later chapters, for proving results about cluster groups associated to these quivers. Namely, we will recall how the quivers arising from cluster algebras of finite type A_n, for $n \geq 1$, arise from triangulations of polygons and how to obtain a graph, known as the braid graph of the quiver, from these triangulations.

Finally, we will see the definition of a companion basis of a quiver appearing in the skew-symmetric cluster algebras of finite type and recall some useful results about them.

2.2 Quivers and Quiver Mutation

In this section we define a vertex-labelled cluster quiver and the mutation of a cluster quiver at a fixed vertex.

Definition 2.2.1. *A **quiver** is a quadruple $Q = (Q_0, Q_1, s, t)$ where Q_0 and Q_1 are sets and $s : Q_1 \longrightarrow Q_0$ and $t : Q_1 \longrightarrow Q_0$ are functions. We interpret Q as a directed graph where Q_0 is the set of vertices and Q_1 is the set of arrows and the functions s and t are such that $s(\alpha) \in Q_0$ determines the source of the arrow $\alpha \in Q_1$ and $t(\alpha) \in Q_0$ determines the target.*

Example 2.2.2. The quiver given by $Q = (\{1,2,3\}, \{\alpha, \beta\}, s, t)$ where s and t are defined by $s(\alpha) = 1, s(\beta) = 3$ and $t(\alpha) = t(\beta) = 2$ is interpreted as the directed graph:

$$1 \xrightarrow{\alpha} 2 \xleftarrow{\beta} 3$$

Definition 2.2.3. *A **cluster quiver** is a quiver with no loops or 2-cycles.*

Definition 2.2.4. *An **isomorphism** between two quivers $Q = (Q_0, Q_1, s, t)$ and $P = (P_0, P_1, s', t')$ is a pair of bijections $f = (f_0, f_1)$ where*

$$f_0 : Q_0 \longrightarrow P_0 \text{ and } f_1 : Q_1 \longrightarrow P_1$$

satisfy

$$f_0(t(\alpha)) = t'(f_1(\alpha)) \text{ and } f_0(s(\alpha)) = s'(f_1(\alpha)) \text{ for all } \alpha \in Q_1$$

*Two quivers are **isomorphic** if there exists an isomorphism between them.*

Definition 2.2.5. *A **vertex-fixing isomorphism** between two quivers is a quiver isomorphism $f = (f_0, f_1)$ such that f_0 is the identity map. Two quivers are **vertex-fixing isomorphic** if there exists an vertex-fixing isomorphism between them.*

We can partition the collection of all quivers into isomorphism classes where two quivers lie in the same isomorphism class if and only if they are isomorphic. An isomorphism class of quivers can be viewed as a quiver in which the vertices and arrows are unlabelled.

Similarly, we can partition the collection of all quivers into vertex-fixing isomorphism classes: two quivers lie in the same vertex-fixing isomorphism class if and only if they are vertex-fixing isomorphic. A vertex-fixing isomorphism class of quivers can be viewed as a quiver in which the vertices are labelled but the arrows are not. We may refer to such quivers as **vertex-labelled quivers**.

Note that if two quivers lie in the same vertex-fixing isomorphism class, they will lie in the same isomorphism class. Thus the vertex-fixing isomorphism classes partition the isomorphism classes.

Going forward, we shall consider cluster quivers up to vertex-fixing isomorphism. Thus our quivers will have labelled vertices but unlabelled arrows. For instance, Auslander-Reiten quivers are of this type [50, Chapter 3]. In this way, we follow the cluster algebra literature (for example, [40]).

Definition 2.2.6. *Let Q be a cluster quiver with vertex set V. The **mutation** of Q at the vertex $v \in V$ is the quiver $\mu_v(Q)$, which is obtained by applying the following steps:*

(a) For each path in Q of length two passing through the vertex v, $u \longrightarrow v \longrightarrow w$, add an arrow $u \longrightarrow w$.

Note that this takes into account multiplicity of arrows, i.e. if there are x arrows $u \longrightarrow v$ and y arrows $v \longrightarrow w$ there would be xy arrows added from u to w.

(b) Reverse each arrow adjacent to v.

(c) Repeatedly delete pairs of arrows forming 2-cycles until there are no 2-cycles.

Note that this operation is well-defined as, no matter which pairs of arrows we choose to delete, the same number of arrows will be deleted in each direction.

Definition 2.2.7. *Two cluster quivers are **mutation-equivalent** if there exists a finite sequence of mutations transforming one into the other.*

2.3 Cluster Algebras

The purpose of this section is to define a cluster algebra using both matrix and quiver notation. We will discuss the classification of the cluster algebras of finite type and interpret this classification for both types of notation. We remark that the formal definition of cluster algebras includes *frozen variables* [40, Definition 2.1.1]. However, we consider only the cluster algebras with no frozen variables (i.e. coefficient-free cluster algebras), so these will not be defined.

Definition 2.3.1. *An $n \times n$ matrix, $B = (b_{ij})$, is **skew-symmetrizable** if there exists an $n \times n$ diagonal matrix, $D = (d_{ij})$, with $d_{ii} > 0$, such that DB is skew-symmetric. That is, $d_{ii}b_{ij} = -d_{jj}b_{ji}$.*

Definition 2.3.2. *For any $1 \leq k \leq n$, the **matrix mutation** of an $n \times n$ skew-symmetrizable integer matrix, $B = (b_{ij})$, at $1 \leq k \leq n$ is the $n \times n$ matrix $\mu_k(B) = (b'_{ij})$, where*

$$b'_{ij} = \begin{cases} -b_{ij} & \text{if } i = k \text{ or } j = k \\ b_{ij} + \frac{|b_{ik}|b_{kj} + |b_{kj}|b_{ik}}{2} & \text{otherwise} \end{cases}$$

Example 2.3.3.

$$B = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}, \qquad \mu_2(B) = \begin{pmatrix} 0 & -1 & 1 \\ 1 & 0 & -1 \\ -1 & 1 & 0 \end{pmatrix}$$

We note in the above example that the mutation of the skew-symmetrizable matrix B at 2 is also skew-symmetrizable. In fact, this will always be the case.

Lemma 2.3.4. *[25, Section 4] For any $1 \leq k \leq n$, the matrix mutation, $\mu_k(B)$, of an $n \times n$ skew-symmetrizable integer matrix B is skew-symmetrizable.*

Fix a field $\mathbb{F} = \mathbb{Q}(u_1, ..., u_n)$ of rational functions in a finite number of indeterminates.

Definition 2.3.5. *A **seed** in $\mathbb{F}$ is a pair, $(\boldsymbol{x}, B)$, where $\boldsymbol{x} = \{x_1, ..., x_n\}$ is a free generating set of $\mathbb{F}$ over $\mathbb{Q}$ and $B = (b_{ij})$ is an $n \times n$ skew-symmetrizable integer matrix. The matrix B is the **exchange matrix** of the seed while the set $\boldsymbol{x}$ is the **cluster** of the seed.*

Definition 2.3.6. *Two seeds, $(\boldsymbol{x}, B)$ and $(\boldsymbol{x}', B')$, where $\boldsymbol{x} = \{x_1, .., x_n\}$ and $\boldsymbol{x}' = \{x'_1, .., x'_n\}$, are **equivalent** if there exists a permutation σ of $\{1, ..., n\}$ such that, for all $1 \leq k \leq n$, $x_{\sigma(k)} = x'_k$, and $B' = (b_{\sigma(i)\sigma(j)})$.*

Example 2.3.7. Let $\mathbb{F} = \mathbb{Q}(u_1, u_2)$. Then the seeds $(\{x_1, x_2\}, B)$ and $(\{x_1', x_2'\}, B')$ in $\mathbb{F}$, where

$$B = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \qquad B' = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix},$$

$x_1 = u_1, x_2 = u_2$ and $x_1' = u_2, x_2' = u_1$ are equivalent via the permutation $\sigma = (12)$

We extend the definition of the mutation of a skew-symmetrizable matrix B at $1 \leq k \leq n$, to the mutation of a seed $(\boldsymbol{x}, B)$ at k.

Definition 2.3.8. *For any $1 \leq k \leq n$, the **mutation of a seed**, $(\boldsymbol{x}, B)$, at k is the seed $\mu_k(\boldsymbol{x}, B) = (\boldsymbol{x}', B')$ where $B' = \mu_k(B)$ and $\boldsymbol{x}' = (\boldsymbol{x} \setminus \{x_k\}) \cup \{x_k'\}$, where x_k' is the element of $\mathbb{F}$ satisfying the following **exchange relation**:*

$$x_k x_k' = \prod_{\substack{j=1 \\ b_{kj}>0}}^{n} x_j^{b_{kj}} + \prod_{\substack{j=1 \\ b_{kj}<0}}^{n} x_j^{-b_{kj}}. \tag{2.1}$$

Example 2.3.9. Let $\mathbb{F} = \mathbb{Q}(u_1, u_2, u_3)$ and consider the seed $(\{x_1, x_2, x_3\}, B)$ where

$$B = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}$$

Then $\mu_2(\{x_1, x_2, x_3\}, B) = (\{x_1, \frac{x_1 + x_3}{x_2}, x_3\}, B')$ where

$$B' = \begin{pmatrix} 0 & -1 & 1 \\ 1 & 0 & -1 \\ -1 & 1 & 0 \end{pmatrix}$$

Definition 2.3.10. *Two seeds, $(\boldsymbol{x}, B)$ and $(\boldsymbol{x}', B')$, are **mutation-equivalent** if there is a finite sequence of mutations taking one to the other.*

Lemma 2.3.11. *[40, Lemma 2.1.5] For every seed, $(\boldsymbol{x}, B)$, in $\mathbb{F}$ and $1 \leq k \leq n$, $\mu_k^2(\boldsymbol{x}, B) = (\boldsymbol{x}, B)$.*

Definition 2.3.12. *Given an initial seed $(\boldsymbol{x}, B)$ in $\mathbb{F}$, let $\mathscr{S}$ be the set of all seeds mutation-equivalent to $(\boldsymbol{x}, B)$. The **cluster algebra**, $\mathscr{A}(\boldsymbol{x}, B)$, over $\mathbb{Q}$ is the $\mathbb{Q}$-subalgebra of $\mathbb{F}$ generated by the union of the clusters appearing in the seeds of $\mathscr{S}$.*

We will refer to the elements of $\mathscr{S}$ as **the seeds of the cluster algebra** $\mathscr{A}(\boldsymbol{x}, B)$ and the elements of any cluster appearing in a seed in $\mathscr{S}$ as **cluster variables**.

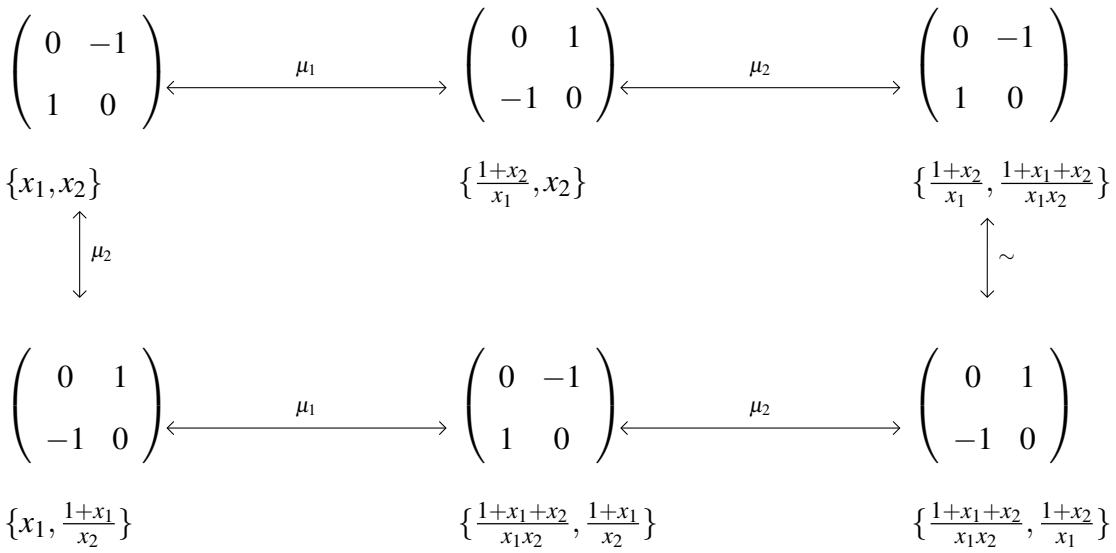


Figure 2.1: $\sim$ denotes that seeds are equivalent.

Definition 2.3.13. *[23, Definition 7.4] Given a cluster algebra, $\mathscr{A}(\boldsymbol{x}, B)$, the **exchange graph** of $\mathscr{A}(\boldsymbol{x}, B)$ is the graph whose vertex set is equal to $\mathscr{S}$ with an edge between two seeds if and only if there is a mutation transforming the seeds into one another.*

Example 2.3.14. Let $\mathbb{F} = \mathbb{Q}(u_1, u_2)$. Then Figure 2.1 represents the exchange graph of the the cluster algebra $\mathscr{A}(\boldsymbol{x}, B)$, where $\boldsymbol{x} = \{x_1, x_2\}$ and

$$B = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

It can be seen from the exchange graph that, up to equivalence, $\mathscr{A}(\boldsymbol{x}, B)$ has a finite number of seeds.

In the cases when the exchange matrix of a seed $(\boldsymbol{x}, B)$ in $\mathbb{F}$ is a skew-symmetric matrix, we can record B as a cluster quiver, $Q(B)$, with vertex set $Q(B)_0 = \{1, ..., n\}$, such that there are b_{ij} arrows from i to j when $b_{ij} \geq 0$ and b_{ji} arrows from j to i when $b_{ij} < 0$.

Conversely, let Q be a cluster quiver on n vertices and define q_{ij} to be the number of arrows from vertices i to j in Q. Then the matrix $B(Q) = (b_{ij})$ where

$$b_{ij} = q_{ij} - q_{ji}$$

will be an $n \times n$ skew-symmetric matrix. Moreover, $Q(B(Q)) = Q$.

Example 2.3.15.

$$B = \begin{pmatrix} 0 & -1 & 1 \\ 1 & 0 & -1 \\ -1 & 1 & 0 \end{pmatrix}$$

$$Q(B):\quad 1 \overset{\frown}{\longleftarrow} 2 \longleftarrow 3$$

Lemma 2.3.16. *[40, Lemma 2.3.1] The map $Q \longmapsto Q(B)$ defines a bijection between the set of skew-symmetric integer matrices and the set of cluster quivers.*

We note that the quiver mutation of a cluster quiver, Q, given by Definition 2.2.6 corresponds with the matrix mutation of $B(Q)$, as given in Definition 2.3.2. Thus any seed, $(\boldsymbol{x},B)$, in $\mathbb{F}$ consisting of a skew-symmetric matrix can be equivalently recorded by $(\boldsymbol{x},Q)$, where $Q = Q(B)$ is a cluster quiver. In this quiver notation, the exchange relation, (2.1), becomes

$$x_k x_k' = \prod_{j=1}^{n} x_j^{q_{kj}} + \prod_{j=1}^{n} x_j^{q_{jk}}.$$

We refer to the cluster algebras whose seeds contain skew-symmetric matrices as the **skew-symmetric** cluster algebras.

Definition 2.3.17. *A cluster algebra is of **finite type** if it has a finite number of seeds.*

Example 2.3.18. Let $\mathbb{F} = \mathbb{Q}(u_1,u_2)$ and consider the seed $(\{x_1,x_2\},Q)$ where Q is the quiver:

$$1 \Longrightarrow 2$$

This seed is mutation-equivalent to an infinite number of distinct seeds [40, Section 2.5].

Definition 2.3.19. *Take $\mathbb{F} = \mathbb{Q}(u_1,\ldots,u_m)$ and $\mathbb{F}' = \mathbb{Q}(u_1',\ldots,u_m')$. Two cluster algebras $\mathscr{A}(\boldsymbol{x},B) \subseteq \mathbb{F}$ and $\mathscr{A}(\boldsymbol{x}',B') \subseteq \mathbb{F}'$ are **strongly isomorphic** if there exists a field isomorphism, $\sigma : \mathbb{F} \longrightarrow \mathbb{F}'$, such that $(\sigma(\boldsymbol{x}),B)$ is a seed of $\mathscr{A}(\boldsymbol{x}',B')$.*

If a skew-symmetrizable matrix B appears in some seed, $(\boldsymbol{x}'',B)$, of the cluster algebra $\mathscr{A}(\boldsymbol{x}',B')$, then we can construct a field isomorphism

$$\sigma : \mathbb{F} \longrightarrow \mathbb{F}',$$
$$\sigma : x_v \longmapsto x_v'',$$

giving $\sigma(\boldsymbol{x}) = \boldsymbol{x}'$. Thus, $\mathscr{A}(\boldsymbol{x},B)$ and $\mathscr{A}(\boldsymbol{x}',B')$ are strongly isomorphic if and only if B appears in some seed of $\mathscr{A}(\boldsymbol{x}',B')$. That is, if and only if B and B' are mutation-equivalent. So we can denote the strong isomorphism class of the cluster algebra $\mathscr{A}(\boldsymbol{x},B)$ by $\mathscr{A}(B)$, where $\mathscr{A}(B) = \mathscr{A}(B')$ if and only if B and B' are mutation-equivalent. For the skew-symmetric cluster algebras, using quiver notation we denote the strong isomorphism class of the cluster algebra $\mathscr{A}(\boldsymbol{x},Q)$ by $\mathscr{A}(Q)$ where $\mathscr{A}(Q) = \mathscr{A}(Q')$ if and only if Q and Q' are mutation-equivalent.

The strong isomorphism classes of the cluster algebras of finite type were classified by Fomin and Zelevinksy [24], who showed that the classification corresponds to the classification of Cartan matrices of finite type.

Definition 2.3.20. *[40, Section 5.1] Given an $n \times n$ integer matrix $B = (b_{ij})$, the **Cartan counterpart** of B is the $n \times n$ generalised Cartan matrix $A(B) = (a_{ij})$ where $a_{ij} = 2$ if $i = j$ and $a_{ij} = -|b_{ij}|$ if $i \neq j$.*

Theorem 2.3.21. *[24, Theorem 1.4] Every cluster algebra lying in the same strong isomorphism class is simultaneously of finite or infinite type. There is a canonical bijection between the Cartan matrices of finite type and the strong isomorphism classes of cluster algebras of finite type, where the Cartan matrix, A, of finite type is mapped to the strong isomorphism class $\mathscr{A}(B)$ such that $A(B) = A$.*

Definition 2.3.22. *Let Δ be a simply-laced Dynkin diagram. Any vertex-labelled orientation of Δ is called a **cluster quiver of Dynkin type** Δ. Following [22], we call a connected Dynkin diagram **indecomposable** and a disconnected Dynkin diagram **decomposable**, where the type is determined by specifying each type of the connected (thus indecomposable) components. The decomposable Dynkin diagram that is the disjoint union of the Dynkin diagrams Δ_1 and Δ_2 is denoted by $\Delta_1 \sqcup \Delta_2$.*

Definition 2.3.23. *A cluster quiver is of **mutation-Dynkin type** Δ if it is mutation-equivalent to a cluster quiver of Dynkin type Δ.*

We remark that any mutation-Dynkin quiver has a unique type [25, Theorem 1.7].

Proposition 2.3.24. *(a) [22, Proposition 4.14] Any full subquiver of an indecomposable quiver of mutation-Dynkin type is a disjoint union of quivers of mutation-Dynkin type.*

(b) [32, Theorem 1.1] Any full subquiver of a mutation-Dynkin type A_n quiver is of mutation-Dynkin type $A_{n_1} \sqcup \ldots \sqcup A_{n_r}$ for some $r \geq 1$.

We consider a skew-symmetric cluster algebra of finite type, $\mathscr{A}(Q)$, where Q is a cluster quiver. From the definitions, under the canonical bijection, $A = A(B(Q))$ will be the corresponding Cartan matrix of finite type.

Suppose that Q is mutation-equivalent to an orientation, $\overrightarrow{\Delta}$, of the simply-laced Dynkin diagram, $\Delta = S(A)$, where A is a simply-laced Cartan matrix of finite type. Then $\mathscr{A}(Q) =$

$\mathscr{A}(\overrightarrow{\Delta})$. Clearly, $A(B(\overrightarrow{\Delta})) = A$. So, by Theorem 2.3.21, $\mathscr{A}(Q)$ is a cluster algebra of finite type.

Conversely, suppose $\mathscr{A}(Q)$ is of finite type. By Theorem 2.3.21, $\mathscr{A}(Q) = \mathscr{A}(Q')$ where $A(B(Q'))$ is a Cartan matrix of finite type. So $S(A(B(Q')))$ is a simply-laced Dynkin diagram of type Δ and Q' is some orientation, $\overrightarrow{\Delta}$, of Δ. That is, $\mathscr{A}(Q) = \mathscr{A}(\overrightarrow{\Delta})$ and so Q is mutation-equivalent to $\overrightarrow{\Delta}$. Thus, for skew-symmetric cluster algebras, Theorem 2.3.21 is equivalent to the following.

Theorem 2.3.25. *Let Q be a cluster quiver. The skew-symmetric cluster algebra, $\mathscr{A}(Q)$, is of finite type if and only if Q is mutation-equivalent to an orientation of a simply-laced Dynkin diagram.*

Thus, by Theorem 2.3.21, up to strong isomorphism, the skew-symmetric cluster algebras of finite type are classified by the simply-laced Dynkin diagrams. We say that $\mathscr{A}(Q)$ is of finite type Δ if Q is mutation-equivalent to an orientation of the simply-laced Dynkin diagram Δ.

In Chapter 2, we saw that the Weyl groups are the finite reflection groups arising from crystallographic root systems. For a Dynkin diagram, Δ, we say that the Weyl group, W, is of type Δ when it corresponds to the crystallographic root system of type Δ. As the Cartan matrices classify the cluster algebras of finite type, to each strong isomorphism class, $\mathscr{A}(B)$, we can associate the Weyl group of the same Dynkin type.

Recall that the Dynkin diagrams define Coxeter presentations of the Weyl groups. Thus, when Δ is simply-laced, Q is a cluster quiver and $\mathscr{A}(Q)$ is a skew-symmetric cluster algebra of finite type Δ, there exists a seed of $\mathscr{A}(Q)$ whose quiver defines a Coxeter presentation of the Weyl group of type Δ.

2.4 Quivers of Mutation-Dynkin Type A_n

In this section, we will consider only the quivers of mutation-Dynkin type A_n, for $n \geq 1$. These particular quivers arise from triangulations of polygons [22]. We will give the formal definition of a triangulation of a polygon and outline how to obtain a quiver of mutation-Dynkin type A_n from a given triangulation. We will also define the braid graph of a given triangulation which, in later chapters, will be used to contruct an isomorphism between a cluster group of mutation-Dynkin type A_n and the symmetric group, Σ_{n+1}.

In a convex polygon, P, a **diagonal** is a line in the interior of P which connects two non-adjacent vertices and only touches the boundary of the polygon at its endpoints. A **triangulation** of P is a decomposition of the polygon into triangles by a maximal set of

non-crossing diagonals. We remark that every triangulation of a convex n-gon has exactly $n-2$ triangles and $n-3$ diagonals.

Every triangulation, $\mathscr{T}$, of a convex $(n+3)$-gon gives rise to an indecomposable quiver of mutation-Dynkin type A_n, denoted by $Q_{\mathscr{T}}$ [8].

Definition 2.4.1. *[21, Section 4] For a triangulation $\mathscr{T}$ of a convex $(n+3)$-gon, $Q_{\mathscr{T}}$ is the quiver whose vertices are in bijection with the diagonals of $\mathscr{T}$. Moreover, there exists an arrow from the vertex i to the vertex j in $Q_{\mathscr{T}}$ if and only if the corresponding diagonals d_i and d_j bound a common triangle, where d_j immediately precedes d_i in the anticlockwise orientation of the triangle.*

Theorem 2.4.2. *[8, Lemma 2.1] (and also from [21, Example 6.6]) For a triangulation $\mathscr{T}$ of a convex $(n+3)$-gon, $Q_{\mathscr{T}}$ is of mutation-Dynkin type A_n. Conversely, every quiver Q of mutation-Dynkin type A_n is of the form $Q_{\mathscr{T}}$ for some triangulation, $\mathscr{T}$, of an $(n+3)$-gon.*

A **flip** along the diagonal d_i is defined in the following way. Suppose $d_i = XY$ where X and Y are distinct vertices of P. Then d_i is adjacent to two triangles, XYA and XYB where A and B are vertices of P, distinct from X and Y. A flip along the diagonal d_i consists of deleting d_i and adding the new diagonal $d_i' = AB$ to form a new triangulation [21, Definition 3.5].

As mentioned in [21, Proposition 3.8], the articles [30], [31] and [46] give the following proposition.

Proposition 2.4.3. *[30], [31], [46] Any two triangulations of a polygon are connected by a sequence of flips along diagonals.*

Proposition 2.4.4. *[8, Lemma 2.1] Let $\mathscr{T}$ be a triangulation of a convex polygon and let $\mathscr{T}'$ be the triangulation obtained by flipping $\mathscr{T}$ along the diagonal d_i. Then $Q_{\mathscr{T}'} = \mu_i(Q_{\mathscr{T}})$.*

The following gives a neat summary of the above which can be found in [49]. Let $\mathbb{T}$ be the set of all triangulations of the $(n+3)$-gon P and $\mathscr{M}_n$ be the set of quivers of mutation-Dynkin type A_n. Define a function

$$\gamma : \mathbb{T} \longrightarrow \mathscr{M}_n,$$

$$\gamma : \mathscr{T} \longmapsto Q_{\mathscr{T}}.$$

This map is surjective as every indecomposable quiver Q of mutation-Dynkin type A_n arises from a triangulation of an $(n+3)$-gon. We define the following equivalence relation on $\mathbb{T}$:

$\mathscr{T} \sim \mathscr{T}'$ if and only if $\mathscr{T}'$ can be obtained from $\mathscr{T}$ by a clockwise rotation of P.

The map γ induces a surjective map $\tilde{\gamma}: \mathbb{T}/\sim \longrightarrow \mathscr{M}_n$.

Theorem 2.4.5. *[49, Theorem 3.5] For $n \geq 2$, the map $\tilde{\gamma}: \mathbb{T}/\sim \longrightarrow \mathscr{M}_n$ is bijective.*

Suppose that Q is a quiver of mutation-Dynkin type with connected components $Q_1, ..., Q_r$ of types $A_{n_1}, ..., A_{n_r}$, respectively, where $n_1, ..., n_r \in \mathbb{Z}^+$ and $n = \sum_{i=1}^{r} n_i$, with $r \geq 1$. From the above, it follows that Q arises from a triangulation of the disjoint union of $P_1, ..., P_r$ where P_i is a convex (n_i+3)-gon for each $1 \leq i \leq r$. Moreover, any triangulation of this disjoint union of polygons admits a quiver of mutation-Dynkin type $A_{n_1} \sqcup ... \sqcup A_{n_r}$.

This correspondence between triangulations of polygons and quivers of mutation-Dynkin type A_n allows us to consider the possible local configurations of a given vertex in a quiver of mutation-Dynkin type A_n.

Definition 2.4.6. *Let Q be a quiver of mutation-Dynkin type. A **chordless cycle** in Q is a cycle in the underlying graph of Q such that there is no edge in the full subgraph on the vertices of the cycle that does not lie in the cycle.*

Proposition 2.4.7. *[32], [23, Proposition 9.7] In a quiver of mutation-Dynkin type, all chordless cycles are oriented.*

Lemma 2.4.8. *[32, Theorem 1.1] (and also [42, Lemma 5.5]) In a quiver of mutation-Dynkin type A_n, the only cycles are 3-cycles. Moreover, any two 3-cycles share at most one common vertex.*

By the **local configuration** of a given vertex i in a quiver, we mean the vertices incident to i. By [32, Theorem 1.1] (and also [42, Section 5]), any vertex i in a mutation-Dynkin quiver of type A_n has valency between 1 and 4 and must have one of the following local configurations:

If i has valency 1:

$$i \longrightarrow j \qquad \text{or} \qquad i \longleftarrow j$$

If i has valency 2:

$$k \longleftarrow i \longrightarrow j \qquad \text{or} \qquad k \longleftarrow i \longleftarrow j \qquad \text{or} \quad k \longrightarrow i \longleftarrow j \qquad \text{or} \quad i \overset{\frown}{\longleftrightarrow} j \longrightarrow k$$

If i has valency 3:

$$l \longleftarrow i \overset{\frown}{\longleftrightarrow} j \longrightarrow k \qquad \text{or} \quad l \longrightarrow i \overset{\frown}{\longleftrightarrow} j \longrightarrow k$$

If i has valency 4:

$$k \overleftrightarrow{\longrightarrow} j \longrightarrow i \overleftarrow{\longleftarrow} p \longleftarrow q$$

Remark 2.4.9. Theorem 2.4.2 gives an alternative proof of Theorem 2.3.24(b) since taking a full subquiver of a mutation-Dynkin type A_n quiver is equivalent to cutting along a set of diagonals of the corresponding triangulation of an $(n+3)-$gon to obtain a disjoint union of triangulations of smaller polygons.

We now define the braid graph of a given triangulation. These graphs will, in later chapters, be a useful tool for studying cluster groups of mutation-Dynkin type A_n.

Definition 2.4.10. *[28, Definition 3.1] Let $\mathscr{T}$ be a triangulation of a convex $(n+3)$-gon, P. The **braid graph of** $\mathscr{T}$ is the graph $\Gamma_{\mathscr{T}} = (V_{\mathscr{T}}, E_{\mathscr{T}})$ where $V_{\mathscr{T}}$ are the vertices of V and $E_{\mathscr{T}}$ are the edges, defined in the following way. The vertices $V_{\mathscr{T}}$ are in bijection with the triangles of $\mathscr{T}$ in P and there exists an edge between two vertices if and only if the corresponding triangles share a common diagonal in $\mathscr{T}$.*

Remark 2.4.11. The braid graph described in Definition 2.4.10 is more commonly known as the **dual graph** of a triangulation [14].

For each $\mathscr{T}, \mathscr{T}' \in \mathbb{T}$ with $\mathscr{T} \sim \mathscr{T}'$, it is clear that $\Gamma_{\mathscr{T}}$ is isomorphic to $\Gamma_{\mathscr{T}'}$.

Let Q be a quiver of mutation-Dynkin type $A_{n_1}, ..., A_{n_r}$, where $n_1, ..., n_r \in \mathbb{Z}^+$ and $n = \sum_{i=1}^{r} n_i$, with $r \geq 1$, and take a triangulation, $\mathscr{T}$, giving rise to Q. As $\Gamma_{\mathscr{T}}$ is independent of the choice of the triangulation giving rise to Q, it makes sense to refer to $\Gamma_{\mathscr{T}}$ as the braid graph of Q. Thus we will denote it by Γ_Q.

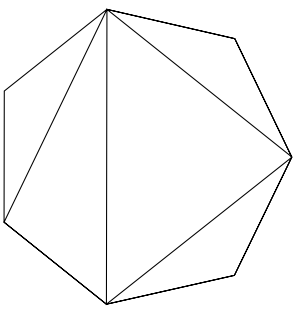
Moreover, Γ_Q is a connected tree on $n+r$ vertices and, as each triangle can be bounded by between 1 and 3 diagonals, the valency of each vertex is equal to 1, 2 or 3.

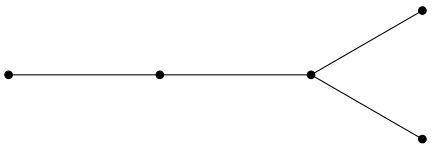
Example 2.4.12. Let Q be the quiver

$$1 \longleftarrow 2 \longleftarrow 3 \longleftarrow 4$$

Then $\mathscr{T}_Q$ and Γ_Q are:

$\mathscr{T}_Q$:



Γ_Q: 

2.5 Companion Bases

In Section 1.3, we saw how results can be proved about a Weyl group by examining the action of its elements on the corresponding root system. Fixing a simple system, $\{\alpha_1, ..., \alpha_n\}$, of the corresponding root system, we also established that the Dynkin diagram of a given simply-laced Weyl group allows us to read off the inner product values of pairs of simple roots. As

$$m(\alpha_i, \alpha_j) = \begin{cases} 2, & \text{when there is no edge between } i \text{ and } j \text{ in } \Delta \\ 3, & \text{when there is an edge between } i \text{ and } j \text{ in } \Delta \end{cases}$$

we have

$$(\alpha_i, \alpha_j) = -2\cos\left(\frac{\pi}{m(\alpha_i, \alpha_j)}\right) = \begin{cases} 0, & \text{when there is no edge between } i \text{ and } j \text{ in } \Delta \\ -1, & \text{when there is an edge between } i \text{ and } j \text{ in } \Delta \end{cases}$$

That is, $\{\alpha_i : 1 \leq i \leq n\}$ is a *companion basis* of the Dynkin diagram. Companion bases are $\mathbb{Z}$-bases of the integral root lattice of a root system of a simply-laced Dynkin diagram and were implicitly constructed in [2].

Definition 2.5.1. *[42, Definition 4.1] Let Q be a mutation-Dynkin quiver and Φ be the root system of the same Dynkin type. A **companion basis** for Q is a subset $C = \{\gamma_i : 1 \leq i \leq n\}$ of Φ such that C is a $\mathbb{Z}$-basis for $\mathbb{Z}\Phi$ and for all $1 \leq i \neq j \leq n$, $|(\gamma_i, \gamma_j)|$ equals the number of arrows between i and j in Q.*

Let Q be a mutation-Dynkin quiver on n vertices with companion basis $C = \{\gamma_i : 1 \leq i \leq n\} \subseteq \Phi$.

Theorem 2.5.2. *[42, Theorem 6.1] The set $C' = \{\gamma'_i : 1 \leq i \leq n\} \subseteq \Phi$ given by*

$$\gamma'_i = \begin{cases} s_{\gamma_k}(\gamma_i), & \text{when there is an arrow from } i \text{ to } k \text{ in } Q; \\ \gamma_i, & \text{otherwise} \end{cases}$$

is a companion basis for $\mu_k(Q)$, for any $1 \leq k \leq n$.

It follows from Theorem 2.5.2 that a companion basis exists for all quivers of mutation-Dynkin type. The following lemma provides another way of obtaining a companion basis of a mutation-Dynkin quiver from an existing one, which will be useful in Section 5.3.

Lemma 2.5.3. *[42, Lemma 4.2] Take any subset I of $\{1,...,n\}$. The subset $C' = \{\gamma_i' : 1 \leq i \leq n\} \subseteq \Phi$ where*

$$\gamma_i' = \begin{cases} \gamma_i, & \text{if } i \notin I \\ -\gamma_i, & \text{if } i \in I \end{cases}$$

is also a companion basis of Q.

For the following results, we take Q to be any quiver of mutation-Dynkin type A_n. Let $C = \{\gamma_i : 1 \leq i \leq n\}$ be a companion basis of Q. By definition, C is a $\mathbb{Z}$-basis for $\mathbb{Z}\Phi$, meaning each $\alpha \in \Phi$ can be written as a unique linear combination of elements of C with coefficients in $\mathbb{Z}$:

$$\alpha = \sum_{i=1}^{n} c_i^{\alpha} \gamma_i, \qquad c_i^{\alpha} \in \mathbb{Z}.$$

A **path** in a graph is a sequence of pairwise distinct vertices with the property that each vertex in the sequence is joined by an edge to the subsequent vertex. For a cluster quiver, Q, an **unoriented path** is a path in the underlying graph of Q.

Definition 2.5.4. *For each $\alpha \in \Phi$, we define the **support of α with respect to** C to be the set $Supp_C(\alpha) = \{i : c_i^{\alpha} \neq 0\} \subseteq \{1,2,...,n\}$. If the elements of $Supp_C(\alpha)$ are precisely the vertices of an oriented path, p, in Q then we say α has support p.*

The unoriented paths in Q give information on the coefficients c_i^{α} for each $\alpha \in \Phi^+$. The next result follows from the structure of the quiver, as described in [42, Lemma 5.7].

Lemma 2.5.5. *[32, Theorem 1.1] For any two vertices i and j of Q, there exists a unique shortest unoriented path in Q between i and j.*

We say that a path p in Q is a **shortest unoriented path** if it is the unique shortest unoriented path in Q between i and j for some vertices i and j. Note that no shortest unoriented path will follow two sides of a 3-cycle.

Given a shortest unoriented path, p, in Q, the following propositions give a way of finding a positive root with support p.

Proposition 2.5.6. *[42, Proposition 5.10] Let p be an unoriented path in Q along the vertices $i_0, i_1, ..., i_m$. For any $1 \leq j \leq m$, we have*

$$s_{i_j} s_{i_{j-1}} ... s_{i_1}(\gamma_{i_0}) = \gamma_{i_0} + \sum_{k=1}^{m} (-1)^k \left(\prod_{l=1}^{k} (\gamma_{i_l}, \gamma_{i_{l-1}}) \right) \gamma_{i_k}.$$

Lemma 2.5.7. *[42, Lemma 5.8] The number of shortest unoriented paths in Q equals the number of positive roots in Φ.*

It follows that every positive root has support p for some shortest unoriented path p in Q.

Proposition 2.5.8. *[42, Proposition 5.11] There is a bijective correspondence between the unoriented paths in Q and the set of positive roots in Φ, where each unoriented path p in Q corresponds to the unique positive root that has support p.*

Corollary 2.5.9. *[42, Theorem 5.3] Let Q be a quiver of mutation-Dynkin type A_n and suppose $C = \{\gamma_i : 1 \leq i \leq n\}$ is a companion basis of Q. Each $\alpha \in \Phi$ can be written as a unique linear combination of elements of C with coefficients in $\{0, \pm 1\}$:*

$$\alpha = \sum_{i=1}^{n} c_i^{\alpha} \gamma_i \qquad c_i^{\alpha} \in \{0, \pm 1\}$$

Chapter 3

The Symmetric Group and Young Subgroups

3.1 Introduction

In this chapter we will consider the finite reflection groups associated to root systems of type A_n. We will recall some definitions and results that will be useful for studying the cluster groups of mutation-Dynkin type A_n and see that a finite reflection group of type A_n is isomorphic to the symmetric group on $n+1$ elements, Σ_{n+1} [47, Proposition 3.4]. We will provide a group presentation for Σ_{n+1}, defined by a graph on n edges, and define the Young subgroups of Σ_{n+1}. We will also give the definitions of a lattice and a set partition and demonstrate how the set of Young subgroups of Σ_{n+1} and the collection of set partitions of $\{1,2,...,n+1\}$ are isomorphic as lattices [5].

3.2 The Symmetric Group

Take $V = \mathbb{R}^{n+1}$, with basis $\{e_1,...,e_{n+1}\}$, for some $n \geq 1$, and consider the root system $\Phi = \{e_i - e_j : 1 \leq i \neq j \leq n+1\}$ in V. By Example 1.2.7, Φ is a root system of type A_n and we have a simple system $\Pi = \{\alpha_i = e_i - e_{i+1} : 1 \leq i \leq n\} \subseteq \Phi$. Recall that the finite reflection group W_Φ has Coxeter presentation:

$$\langle s_{\alpha_1}, s_{\alpha_2}, ..., s_{\alpha_n} | (s_{\alpha_i} s_{\alpha_j})^{m(i,j)} \rangle$$

where

$$m(i,j) = \begin{cases} 1 & \text{if } |i-j| = 0 \\ 2 & \text{if } |i-j| > 1 \\ 3 & \text{if } |i-j| = 1 \end{cases}$$

Lemma 3.2.1. *[7, Chapter VI, Section 4.7] Let Φ be a root system of type A_n, for some $n \geq 1$. Then there exists a group isomorphism*

$$\sigma : W_\Phi \longrightarrow \Sigma_{n+1},$$
$$\sigma : s_{\alpha_i} \longmapsto (i, i+1).$$

Remark 3.2.2. The proof of Lemma 3.2.1 observes that each generator, s_{α_i}, of W permutes the set of basis elements $B = \{e_1, ..., e_{n+1}\}$. Specifically,

$$s_{\alpha_i}(e_j) = \begin{cases} e_i & \text{if } j = i+1 \\ e_{i+1} & \text{if } j = i \\ e_j & \text{otherwise.} \end{cases}$$

It is then possible to construct a group homomorphism

$$\overline{\sigma} : W_\Phi \longrightarrow \Sigma_B,$$
$$\overline{\sigma} : s_{\alpha_i} \longmapsto (e_i, e_{i+1}).$$

By composing $\overline{\sigma}$ with the group isomorphism between Σ_B and Σ_{n+1} which maps e_i to i, we obtain the isomorphism $\sigma : W_\Phi \longrightarrow \Sigma_{n+1}$. It follows that for any $w \in W$ and $\alpha_i \in \Pi$, we have

$$w(\alpha_i) = w(e_i - e_{i+1}) = w(e_i) - w(e_{i+1}) = e_{\sigma(w)(i)} - e_{\sigma(w)(i+1)}.$$

Definition 3.2.3. *[41, Section 1.2] A graph is a **tree** when it is connected and contains no cycles.*

Remark 3.2.4. There exists a unique path between any two vertices of a tree.

Definition 3.2.5. *[41, Section 1.3] A graph is **planar** if it can be drawn in the plane so that no two edges intersect except at a common endpoint.*

We note that every tree is planar. Let Γ be a tree. We denote by Σ_Γ the symmetric group on the vertices of Γ. That is, the group of permutations of the set of vertices of Γ.

Proposition 3.2.6. *[47, Proposition 3.4] For a tree Γ, the group Σ_Γ is generated by the set $X_\Gamma = \{\sigma : \sigma \text{ is an edge of } \Gamma\}$ subject to the relations:*

(1) $\sigma^2 = e$ *for all* $\sigma \in X_\Gamma$.

(2) If $\sigma_1, \sigma_2 \in X_\Gamma$ *are disjoint then*

$$\sigma_1\sigma_2 = \sigma_2\sigma_1.$$

(3) If $\sigma_1, \sigma_2 \in X_\Gamma$ *have one common vertex then*

$$\sigma_1\sigma_2\sigma_1 = \sigma_2\sigma_1\sigma_2.$$

(4) If $\sigma_1, \sigma_2, \sigma_3 \in X_\Gamma$ *have a single vertex in common and lie in clockwise order, then*

$$\sigma_1\sigma_2\sigma_3\sigma_1 = \sigma_2\sigma_3\sigma_1\sigma_2.$$

Remark 3.2.7. By [3, Lemma 4.1], (4) is equivalent to the following relation.

(4^*) If $\sigma_1, \sigma_2, \sigma_3 \in X_\Gamma$ have a single vertex in common and lie in clockwise order, then

$$\sigma_1\sigma_2\sigma_3\sigma_1 = \sigma_2\sigma_3\sigma_1\sigma_2 = \sigma_3\sigma_1\sigma_2\sigma_3.$$

An additional relation is given in [47] for when Γ is not a tree. However, we will only require the four relations given in Proposition 3.2.6.

Remark 3.2.8. Given the relations $(1) - (3)$, [28, Lemma 2.5] shows the relations given in (4) are equivalent to the cycle relation given in the group presentations defined in [3].

In [47], a version of Proposition 3.2.6 is given for the braid group, in which an edge σ is interpreted as a braid which twists the strands corresponding to the endpoints of the edge. In the context of the symmetric group, we view σ as the transposition which interchanges the endpoints of the corresponding edge. As noted in [47, Remark 3.4], the proof that the statement is true for the symmetric group is similar to the proof given for [47, Proposition 3.4] for the braid group.

3.3 Lattices, Set Partitions and Young Subgroups

In this section, we recall some key definitions and examples of lattices and set partitions.

Definition 3.3.1. *[12, Definition 2.4] A **lattice** is a partially ordered set in which every two-element subset has both a least upper bound (the 'join') and a greatest lower bound (the 'meet'). For any two elements X and Y of a lattice, we denote the **join** of X and Y by $X \vee Y$ and the **meet** of X and Y by $X \wedge Y$.*

We give three examples of lattices which will be useful in later chapters.

Example 3.3.2. (1.) Let X be a set. The power set of X forms a partially ordered set under inclusion which is a lattice.

(2.) Recall that a **set partition** ρ of a non-empty set X is a collection of non-empty subsets of X such that

(a) $X = \bigcup \rho$

(b) $\alpha_1 \cap \alpha_2 = \emptyset$ for all distinct pairs $\alpha_1, \alpha_2 \in \rho$.

We call the elements of a partition ρ the **parts** of ρ. When $\rho = \{\alpha_1, ..., \alpha_k\}$ is a partition of a set X we employ an abuse of notation by writing $\rho = \bigsqcup_{j=1}^{k} \alpha_j$.

The collection of set partitions of a set X forms a partially ordered set under the refinement ordering. That is, for each pair of partitions ρ, ρ' of X,

$$\rho \leq \rho' \Leftrightarrow \text{ every part of } \rho \text{ is a subset of some part of } \rho'.$$

Moreover, [43, Theorem 5.15.1] outlines how the meet and join of any two partitions of the same set are obtained, which we describe below.

The partitions of X are in bijection with the equivalence relations on X. So for each partition $\rho = \bigsqcup_{j=1}^{k} \alpha_j$ of X we have a corresponding equivalence relation, R_ρ, on X where for each $x, y \in X$,

$$xR_\rho y \Leftrightarrow x, y \in \alpha_j \text{ for some } 1 \leq j \leq k.$$

From any two equivalence relations R_1 and R_2 on a set X, we construct new equivalence relations, denoted by $R_1 \cap R_2$ and $t(R_1 \cup R_2)$, in the following way. For any $x, y \in X$,

(a) $x(R_1 \cap R_2)y$ if and only if xR_1y and xR_2y

(b) $x(t(R_1 \cup R_2))y$ if and only if there exist $z_0, ..., z_m \in X$ such that $x = z_0, y = z_m$ and either $z_i R_1 z_{i+1}$ or $z_i R_2 z_{i+1}$ for all $0 \leq i \leq m-1$.

Note that $t(R_1 \cup R_2)$ is the transitive closure of the binary relation $R_1 \cup R_2$ on X.

Given two partitions ρ_1, ρ_2 of the set X, $\rho_1 \wedge \rho_2$ is the partition corresponding to $R_1 \cap R_2$ and $\rho_1 \vee \rho_2$ is the partition corresponding to $t(R_1 \cup R_2)$ [43, Theorem 5.15.1].

(3.) Let G be a group. Then the set of subgroups of G forms a partially ordered set under inclusion. This forms a lattice and, for any subgroups G_1, G_2 of G, $G_1 \vee G_2 = \langle G_1 \cup G_2 \rangle$ and $G_1 \wedge G_2 = G_1 \cap G_2$.

Definition 3.3.3. *[12, Definition 2.13] Let L be a lattice. A non-empty subset K of L is a* ***sublattice*** *of L if, for every $x, y \in K$, $x \vee y, x \wedge y \in K$.*

Definition 3.3.4. *[12, Definition 2.16] Let L and K be lattices. A map $\phi : L \longrightarrow K$ is a* ***lattice homomorphism*** *if for all $x, y \in L$, $\phi(x \vee y) = \phi(x) \vee \phi(y)$ and $\phi(x \wedge y) = \phi(x) \wedge \phi(y)$. Moreover, ϕ is a* ***lattice isomorphism*** *if it is a bijective lattice homomorphism.*

Proposition 3.3.5. *[12, Proposition 2.4(ii)] Let L and K be lattices. For any map $\phi : L \longrightarrow K$, ϕ is a lattice isomorphism if and only if ϕ is an order-isomorphism.*

Definition 3.3.6. *[35] For some $n, k \in \mathbb{N}$, let $\rho = \{\alpha_j : 1 \leq j \leq k\}$ be a set partition of $\{1, ..., n\}$. The subgroup of Σ_n given by*

$$\Sigma_{\alpha_1} \times \Sigma_{\alpha_2} \times ... \times \Sigma_{\alpha_k}$$

where $\Sigma_{\alpha_j} = \{\sigma \in \Sigma_n : \sigma(m) = m, \forall m \notin \alpha_j\}$ is called the ***Young subgroup corresponding to*** *ρ and is denoted by $Y(\rho)$.*

Given a set $X = \{1, ..., n\}$, let $\mathscr{P}$ be the lattice of set partitions of X and let $\mathscr{Y} = \{Y(\rho) : \rho \in \mathscr{P}\}$. The following result is well-known (see e.g. [5]), but we include a proof for the convenience of the reader.

Proposition 3.3.7. *The set $\mathscr{Y}$ is a lattice under inclusion and there exists a lattice isomorphism*

$$\psi : \mathscr{P} \longrightarrow \mathscr{Y},$$
$$\psi : \rho \longmapsto Y(\rho).$$

Proof. Let $\mathscr{S}$ be the set of subgroups of Σ_n. Thus $\mathscr{S}$ is a lattice under inclusion. We begin by showing there exists an injective lattice homomorphism

$$\psi : \mathscr{P} \longrightarrow \mathscr{S},$$
$$\psi : \rho \longmapsto Y(\rho).$$

As $\mathscr{Y} = im(\psi)$, we will then be able to conclude that $\mathscr{Y}$ is a sublattice of $\mathscr{S}$ and that ψ induces a lattice isomorphism between $\mathscr{P}$ and $\mathscr{Y}$.

In order to show that ψ is a lattice homomorphism, we must show that for any $\rho, \rho' \in \mathscr{P}$, $\psi(\rho \vee \rho') = \psi(\rho) \vee \psi(\rho')$ and $\psi(\rho \wedge \rho') = \psi(\rho) \wedge \psi(\rho')$. That is,

$$Y(\rho \vee \rho') = \langle Y(\rho) \cup Y(\rho') \rangle,$$

and

$$Y(\rho \wedge \rho') = Y(\rho) \cap Y(\rho').$$

To do this, it will be useful to show that ψ is order-preserving. That is, for any $\rho, \rho' \in \mathscr{P}$, if $\rho \leq \rho'$ then $Y(\rho) \subseteq Y(\rho')$. If $\rho \leq \rho'$, where $\rho = \bigsqcup_{j=1}^{a} \alpha_j$ and $\rho' = \bigsqcup_{j=1}^{b} \alpha'_j$, then by definition, every part α_j of ρ is a subset of some part α'_l of ρ'. Thus for some $1 \leq l \leq b, \Sigma_{\alpha_j} \subseteq \Sigma_{\alpha'_l}$ for every $1 \leq j \leq a$, giving $Y(\rho) \subseteq Y(\rho')$. By [12, Proposition 2.19], it follows from the fact that ψ is order-preserving that for all $\rho, \rho' \in \mathscr{P}$, $\psi(\rho \vee \rho') \supseteq \psi(\rho) \vee \psi(\rho')$ and $\psi(\rho \wedge \rho') \subseteq \psi(\rho) \wedge \psi(\rho')$. That is,

$$Y(\rho \vee \rho') \supseteq \langle Y(\rho) \cup Y(\rho') \rangle,$$

and

$$Y(\rho \wedge \rho') \subseteq Y(\rho) \cap Y(\rho').$$

So it remains to show that

$$Y(\rho \vee \rho') \subseteq \langle Y(\rho) \cup Y(\rho') \rangle,$$

and

$$Y(\rho \wedge \rho') \supseteq Y(\rho) \cap Y(\rho').$$

Recall that $\rho \wedge \rho'$ is the partition corresponding to the equivalence relation $R_\rho \cap R_{\rho'}$. If $\sigma \in Y(\rho) \cap Y(\rho')$ then $\sigma \in (\prod_{i=1}^{a} \Sigma_{\alpha_i}) \cap (\prod_{j=1}^{b} \Sigma_{\alpha'_j})$, where $\rho = \bigsqcup_{i=1}^{a} \alpha_i$ and $\rho' = \bigsqcup_{j=1}^{b} \alpha'_j$. That is,

$$\sigma = \sigma_1 \sigma_2 .... \sigma_a = \sigma'_1 \sigma'_2 .... \sigma'_b$$

where $\sigma_i \in \Sigma_{\alpha_i}$ and $\sigma'_j \in \Sigma_{\alpha'_j}$, for each $1 \leq i \leq a$ and $1 \leq j \leq b$.

Let $\rho \wedge \rho' = \bigsqcup_{l=1}^{c} \beta_l$. For a vertex x of Γ_Q, we have

$$x \in \alpha_i$$

and

$$x \in \alpha'_j,$$

for some $1 \leq i \leq a$ and $1 \leq j \leq b$. In particular, $x \in \alpha_i \cap \alpha'_j$. Thus

$$\sigma(x) \in \alpha_i$$

and

$$\sigma(x) \in \alpha'_j$$

giving $\sigma(x) \in \alpha_i \cap \alpha'_j$. So $\sigma \in \Sigma_{\alpha_i \cap \alpha'_j}$.

Clearly, for all $u, v \in \alpha_i \cap \alpha'_j$ we have $uR_\rho v$ and $vR_{\rho'}v$. So, by definition, $u(R_\rho \cap R_{\rho'})v$, meaning there exists some $1 \leq l \leq c$ such that $\alpha_i \cap \alpha'_j \subseteq \beta_l$. This gives

$$\sigma \in \Sigma_{\alpha_i \cap \alpha'_j} \subseteq \Sigma_{\beta_l} \subseteq Y(\rho \wedge \rho').$$

So $Y(\rho \wedge \rho') = Y(\rho_I) \cap Y(\rho_J)$.

Next, we show that $Y(\rho \vee \rho') \subseteq \langle Y(\rho) \cup Y(\rho') \rangle$. Let $\rho = \bigsqcup_{i=1}^{a} \alpha_i$, $\rho' = \bigsqcup_{j=1}^{b} \alpha'_j$ and $\rho \vee \rho' = \bigsqcup_{l=1}^{c} \beta_l$. We consider any $1 \leq l \leq c$ and show $\Sigma_{\beta_l} \subseteq \langle Y(\rho) \cup Y(\rho') \rangle$.

Recall that $\rho \vee \rho'$ is the partition corresponding to the equivalence relation $t(R_\rho \cup R_{\rho'})$. That is, for any $x, y \in \{1, ..., n\}$, x and y lie in the same part of $\rho \vee \rho'$ if and only if there exists $z_0, ..., z_m \in \{1, ..., n\}$ such that $x = z_0, y = z_m$ and, for all $0 \leq i \leq m-1$, either $z_i R_\rho z_{i+1}$ or $z_i R_{\rho'} z_{i+1}$. Equivalently, z_i, z_{i+1} lie in the same part of ρ or z_i, z_{i+1} lie in the same part of ρ'. Choosing any $1 \leq l \leq c$, we can write $\beta_l = \{p_1, ..., p_q\} \subseteq \{1, ..., n\}$. So $\Sigma_{\beta_l} \cong \Sigma_q$ by an isomorphism $v \longmapsto p_v$. Since Σ_q is generated by the set $\{(v, v+1) : 1 \leq v \leq q-1\}$, Σ_{β_l} is generated by the set $\{(p_v, p_{v+1}) : 1 \leq v \leq q-1\}$.

We show that $(p_v, p_{v+1}) \in \langle Y(\rho) \cup Y(\rho') \rangle$, for any $1 \leq v \leq q-1$ and $1 \leq l \leq c$.

As p_v and p_{v+1} lie in the same part of $\rho \vee \rho'$, there exists $p_v = z_0, z_1, ..., z_m = p_{v+1} \in \{1, ..., n\}$ such that, for all $0 \leq i \leq m-1$, z_i, z_{i+1} lie in the same part of ρ or z_i, z_{i+1} lie in the same part of ρ'.

In the case when z_i, z_{i+1} lie in the same part of ρ, $(z_i, z_{i+1}) \in Y(\rho)$. In the case when z_i, z_{i+1} lie in the same part of ρ', $(z_i, z_{i+1}) \in Y(\rho')$. So $(z_i, z_{i+1}) \in \langle Y(\rho) \cup Y(\rho') \rangle$ for all $0 \leq i \leq m-1$. Thus

$$\begin{aligned}(p_v, p_{v+1}) &= (p_v, z_1)(z_1, z_2)...(z_{m-2}, z_{m-1})(p_{v+1}, z_{m-1})(z_{m-1}, z_{m-2})...(z_2, z_1)(z_1, p_v) \\ &\in \langle Y(\rho) \cup Y(\rho') \rangle.\end{aligned}$$

As the generators of Σ_{β_l} all lie in $\langle Y(\rho) \cup Y(\rho') \rangle$, it follows that $\Sigma_{\beta_l} \subseteq \langle Y(\rho) \cup Y(\rho') \rangle$, for all $1 \leq l \leq c$, and so $Y(\rho \vee \rho') \subseteq \langle Y(\rho) \cup Y(\rho') \rangle$. Thus $Y(\rho \vee \rho') = \langle Y(\rho) \cup Y(\rho') \rangle$. The map ψ is therefore a lattice homomorphism.

Finally, we show that ψ is injective. Suppose $Y(\rho) = Y(\rho')$ for some $\rho = \bigsqcup_{j=1}^{a} \alpha_j$ and $\rho' = \bigsqcup_{j=1}^{b} \alpha'_j$. So $Y(\rho) \subseteq Y(\rho')$. Thus for any $1 \leq j \leq a$, $\Sigma_{\alpha_j} \subseteq Y(\rho) \subseteq Y(\rho')$. Every $\sigma \in Y(\rho')$ can be written in the form $\sigma = \sigma_1 ... \sigma_{a'}$ where $\sigma_j \in \Sigma_{\alpha'_j}$. By definition of $\Sigma_{\alpha'_j}$, $\sigma_j(m) = m$ for any $m \notin \alpha'_j$. Thus $\sigma(m) \in \alpha'_j$ for all $m \in \alpha'_j$. Suppose there exists $1 \leq p \neq q \leq b$ such that $\alpha_j \cap \alpha'_p \neq \emptyset$ and $\alpha_j \cap \alpha'_q \neq \emptyset$. So we can choose $x \in \alpha_j \cap \alpha'_p$ and $y \in \alpha_j \cap \alpha'_q$ and take $\sigma = (x, y) \in \Sigma_{\alpha_j}$. Thus $\sigma = (x, y) \in Y(\rho')$ but $x \in \alpha'_p$ and $\sigma(x) = y \in \alpha'_q$, contradicting that, for all $1 \leq j \leq b$ and $\sigma \in Y(\rho)$ $\sigma(m) \in \alpha'_j$ for all $m \in \alpha'_j$. Thus $\alpha_j \subseteq \alpha'_l$ for some $1 \leq l \leq b$ and so by definition $\rho \leq \rho'$. As $Y(\rho) = Y(\rho')$, we also have that $Y(\rho') \subseteq Y(\rho)$ so, by the same argument, $\rho' \leq \rho$, meaning $\rho = \rho'$.

As $\mathscr{Y} = im(\psi)$, we conclude that $\mathscr{Y}$ is a sublattice of $\mathscr{S}$ and

$$\tilde{\psi} : \mathscr{P} \longrightarrow \mathscr{Y},$$

$$\tilde{\psi} : \rho \longmapsto Y(\rho)$$

is a lattice isomorphism. □

Chapter 4

Cluster Groups

4.1 Introduction

Let Q be a cluster quiver. In previous chapters we saw that the cluster algebra $\mathscr{A}(Q)$ is of finite type if and only if Q is mutation-equivalent to a orientation of a simply-laced Dynkin diagram. Moreover, as the Cartan matrices of finite type classify the cluster algebras of finite type, Q cannot be mutation-equivalent to two oriented simply-laced Dynkin diagrams of different type. We say that $\mathscr{A}(Q)$ is of finite type Δ if Q is of mutation-Dynkin type Δ, for a simply-laced Dynkin diagram Δ. That is, when there is a seed of $\mathscr{A}(Q)$ whose quiver is an orientation of Δ.

We also saw that the Dynkin diagrams define Coxeter presentations giving rise to finite reflection groups. In the simply-laced cases, these finite reflection groups will be pairwise non-isomorphic. Thus, when $\mathscr{A}(Q)$ is of finite type Δ, there is a seed of $\mathscr{A}(Q)$ whose quiver defines a Coxeter presentation of the finite reflection group of the same Dynkin type.

The article [3] associates to each mutation-Dynkin quiver a group presentation and shows that the corresponding group is isomorphic to a finite reflection group. Moreover, this reflection group is of the same Dynkin type as the cluster algebra from which the quiver arises.

Several subsequent publications have built on this work. The article [19] provides similar presentations for affine Coxeter groups while [28] and [29], independently of one another, extended the results of [3] to provide presentations for Artin braid groups in the simply-laced case and the finite type case, respectively. Moreover, it is shown that the groups corresponding to both of these presentations are invariant under mutation of the quiver.

Considering the group presentation associated to each mutation-Dynkin quiver in [28], together with the additional set of relations that specify that the square of each generator

is equal to the identity, [28, Lemma 2.5] shows the resulting group is isomorphic to the group with presentation defined by [3]. Thus, we obtain another presentation associated to a mutation-Dynkin quiver that gives a group that is isomorphic to a finite reflection group. It is this group presentation based on the work done in [28] that is considered here. The consequence of this result is that we are able to define a group presentation associated to the quiver appearing in *any* seed of a skew-symmetric cluster algebra of finite type Δ, that gives rise to the finite reflection group of type Δ. These presentations make sense for any cluster quiver and so we are interested in considering them more generally. Due to the context given above, we call the corresponding group a 'cluster group' with the aim of exploring its properties in general.

As outlined in Section 1.4, there are many well-established results for Coxeter presentations and we are interested in which of these properties hold for cluster group presentations.

In this chapter, we will define the cluster group associated to a cluster quiver and verify that, for quivers appearing in seeds of skew-symmetric cluster algebras of finite type, the cluster group is invariant under mutation of the quiver and thus isomorphic to the finite reflection group of the same Dynkin type. Finally, we will give definitions of the length function and parabolic subgroups for cluster group presentations, which are analogous to those appearing in the theory of Coxeter groups.

4.2 The Cluster Group Associated to a Cluster Quiver

We begin by defining the cluster group associated to a cluster quiver, which is based on [28, Definition 2.2]. We will use [28, Proposition 2.9] to establish that, for quivers of mutation-Dynkin type Δ, the corresponding group is isomorphic to the finite reflection group of type Δ.

Definition 4.2.1. *Let Q be a cluster quiver. Following [19], if i and j are vertices of Q which are joined by a single arrow, then we call this arrow* ***simple****.*

Definition 4.2.2. *Let Q be a cluster quiver with vertex set Q_0. The* ***cluster group associated to*** *Q, denoted by G_Q, is defined to be the group with group presentation $\langle T|R\rangle$ where*

$$T = \{t_i : i \in Q_0\}$$

and R is the following set of relations:

(a) *For all $i \in Q_0$, $t_i^2 = e$.*

(b) ***The braid relations:*** *For all $i, j \in Q_0$,*

(i) $t_i t_j = t_j t_i$ *if there is no arrow between i and j in Q.*

(ii) $t_i t_j t_i = t_j t_i t_j$ *if i and j are joined by a simple arrow in Q.*

(c) ***The cycle relations:***

$$t_{i_1} t_{i_2} t_{i_3} ... t_{i_r} t_{i_1} ... t_{i_{r-2}} = t_{i_2} t_{i_3} ... t_{i_r} t_{i_1} t_{i_2} t_{i_3} ... t_{i_{r-1}}$$

for every chordless, oriented cycle

$$i_1 \longrightarrow i_2 \longrightarrow i_3 \longrightarrow ... \longrightarrow i_r \longrightarrow i_1$$

in Q, in which all arrows are simple.

Remark 4.2.3. By [28, Lemma 2.4], the relations given in (c) are written in minimal form but could be replaced by the following, more symmetric, relation.

(c^*)

$$\begin{aligned} t_{i_1} t_{i_2} t_{i_3} ... t_{i_r} t_{i_1} ... t_{i_{r-2}} &= t_{i_2} t_{i_3} ... t_{i_r} t_{i_1} t_{i_2} t_{i_3} ... t_{i_{r-1}} \\ &= t_{i_3} ... t_{i_r} t_{i_1} t_{i_2} t_{i_3} t_{i_4} ... t_{i_r} \\ &= ... \\ &= t_{i_r} t_{i_1} t_{i_2} t_{i_3} ... t_{i_r} t_{i_1} ... t_{i_{r-3}} \end{aligned}$$

for every chordless, oriented cycle

$$i_1 \longrightarrow i_2 \longrightarrow i_3 \longrightarrow ... \longrightarrow i_r \longrightarrow i_1$$

in Q, in which all arrows are simple.

We also note that the relations $(b)(ii)$ and (c) are defined in order to be compatible with the group presentations associated to diagrams of finite type given in [28] and the group presentations associated to diagrams of affine type given in [19].

Moreover, given the relations (a) and (b), [28, Lemma 2.5] shows the relations given in (c) are equivalent to the cycle relation given in the group presentations defined in [3].

We note that for general cluster quivers the corresponding cluster group is not necessarily invariant under mutation of the quiver [19, Section 7].

Suppose that Q is a decomposable cluster quiver with connected components $Q_1, ..., Q_r$. Consider the cluster group presentation associated to each connected component of Q.

Theorem 4.2.4. *[6, Theorem 1] Let G_1 and G_2 be groups with group presentations $\langle X_1|R_1\rangle$ and $\langle X_2|R_2\rangle$, respectively. Then $G_1 \times G_2$ has presentation $\langle X_1 \cup X_2|R_1 \cup R_2 \cup B\rangle$ where B is the set of relations $\{x_1x_2 = x_2x_1 : x_1 \in X_1, x_2 \in X_2\}$.*

Lemma 4.2.5. *If Q is a decomposable cluster quiver with indecomposable components $Q_1, .., Q_r$ then*

$$G_Q = G_{Q_1} \times G_{Q_2} \times ... \times G_{Q_r}.$$

Proof. Let Q be a decomposable cluster quiver with indecomposable components $Q_1, .., Q_r$. Consider the cluster group, G_Q, arising from the cluster group presentation $\langle T|R\rangle$ defined by Q.

For each $1 \leq i \neq j \leq r$, let T_i be the set of generators of G_{Q_i} and R_i the set of relations. We define the set of relations

$$B_{ij} = \{t_it_j = t_jt_i : t_i \in T_i, t_j \in T_j\}.$$

By Theorem 4.2.4,

$$G_{Q_1} \times G_{Q_2} \times ... \times G_{Q_r} = \langle T_1 \cup ... \cup T_r|R_1 \cup ... \cup R_r \cup B\rangle$$

where $B = \bigcup_{i=1}^{r} (\bigcup_{\substack{1\leq j\leq r, \\ j\neq i}} B_{ij})$.

As the Q_i are the connected components of Q, no edge exists between any two vertices lying in different components. Noting that $T = T_1 \sqcup ... \sqcup T_r$, it follows that $R = R_1 \cup ... \cup R_r \cup B$, meaning $\langle T_1 \cup ... \cup T_r|R_1 \cup ... \cup R_r \cup B\rangle$ is precisely the cluster group presentation of Q, $\langle T|R\rangle$. Thus $G_Q = G_{Q_1} \times G_{Q_2} \times ... \times G_{Q_r}$.

□

It is easy enough to prove the following lemma.

Lemma 4.2.6. *If Q and P are isomorphic cluster quivers, then $G_Q \cong G_P$.*

In the light of Lemma 4.2.6, we note that G_Q only relies on Q up to isomorphism and so, without loss of generality, we can suppose the vertex set of Q is the set $\{1, ..., n\}$. The cluster groups associated to more general cluster quivers will be isomorphic to the groups associated with these quivers.

As discussed previously, [28, Lemma 2.5] shows the cluster group associated to a mutation-Dynkin quiver is isomorphic to the group with the group presentation defined in [3]. In fact, [28, Lemma 2.5] shows that the cycle relations appearing in the cluster

group presentations for mutation-Dynkin quivers are equivalent to those appearing in the group presentations defined in [3]. Moreover, [3, Theorem 5.4] shows that the groups arising from these group presentations are invariant under mutation of the quiver. It follows that the cluster groups associated to mutation-Dynkin quivers are invariant under mutation. Alternatively, as we have done below, by a simple extension of the proof of [28, Proposition 2.9] it can be shown directly that when Q is a mutation-Dynkin quiver, the corresponding cluster group will be invariant under mutation of Q.

Lemma 4.2.7. *Let Q be a cluster quiver of mutation-Dynkin type. If Q and Q' are mutation-equivalent then $G_Q \cong G_{Q'}$.*

Proof. We prove the result first for indecomposable quivers of mutation-Dynkin type. Let Q and Q' be mutation-equivalent indecomposable cluster quivers, on $n \geq 1$ vertices, of mutation-Dynkin type.

Suppose $Q' = \mu_k(Q)$ for some $1 \leq k \leq n$. Let the defining set of generators for G_Q and $G_{Q'}$ be denoted by $T = \{t_i : 1 \leq i \leq n\}$ and $T' = \{t'_i : 1 \leq i \leq n\}$, respectively, and denote the identity elements of by e and e', respectively. Furthermore, let F_Q and $F_{Q'}$ denote the free group on T and T', respectively.

We can define a group homomorphism

$$\varphi_k : F_Q \longrightarrow G_{Q'},$$

$$\varphi_k(t_i) = \begin{cases} (t'_k t'_i t'_k), & \text{when there is some arrow in } Q \text{ from } i \text{ to } k \\ t'_i, & \text{otherwise.} \end{cases}.$$

We claim that φ_k induces a group homomorphism

$$\tilde{\varphi}_k : G_Q \longrightarrow G_{Q'}.$$

To show that $\tilde{\varphi}_k$ is well-defined, we must show that the $\varphi_k(t_i)$ satisfy the defining relations of $G_{Q'}$. For all $1 \leq i \leq n$, we have:

$$\begin{aligned} \varphi_k(t_i)^2 &= \begin{cases} (t'_k t'_i t'_k)^2, & \text{when there is some arrow in } Q \text{ from } i \text{ to } k \\ (t'_i)^2, & \text{otherwise} \end{cases} \\ &= \begin{cases} (t'_k)^2, & \text{when there is some arrow in } Q \text{ from } i \text{ to } k \\ (t'_i)^2, & \text{otherwise} \end{cases} \\ &= e'. \end{aligned}$$

By [28, Proposition 2.9] (using [28, Lemma 2.4]), the $\varphi_k(t_i)$ satisfy the remaining defining relations of $G_{Q'}$. Thus $\tilde{\varphi}_k$ is well-defined.

It remains to show that $\tilde{\varphi}_k$ has an inverse. We can define a group homomorphism

$$\psi_k : F_{Q'} \longrightarrow G_Q,$$

$$\psi_k(t_i') = \begin{cases} (t_k t_i t_k), & \text{when there is some arrow in } Q' \text{ from } k \text{ to } i \\ t_i, & \text{otherwise.} \end{cases}$$

Note that, by Definition 2.2.6(a), there exists an arrow from i to k in Q if and only if there exists an arrow from k to i in $\mu_k(Q) = Q'$.

In an analogous way, it can be shown that ψ_k induces a well-defined group homomorphism

$$\tilde{\psi}_k : G_{Q'} \longrightarrow G_Q.$$

We show that $\tilde{\psi}_k$ is the inverse of $\tilde{\varphi}_k$:

For each $1 \leq i \leq n$, we have

$$\begin{aligned} \tilde{\varphi}_k \circ \tilde{\psi}_k(t_i') &= \begin{cases} \tilde{\varphi}_k((t_k t_i t_k)), & \text{when there is some arrow in } Q' \text{ from } k \text{ to } i \\ \tilde{\varphi}_k(t_i), & \text{otherwise} \end{cases} \\ &= \begin{cases} (t_k' t_k' t_i' t_k' t_k'), & \text{when there is some arrow in } Q' \text{ from } k \text{ to } i \\ t_i', & \text{otherwise} \end{cases} \\ &= t_i'. \end{aligned}$$

$$\begin{aligned} \text{Similarly, } \tilde{\varphi}_k^{-1} \circ \tilde{\varphi}_k(t_i) &= \begin{cases} \tilde{\varphi}_k^{-1}((t_k' t_i' t_k')), & \text{when there is some arrow in } Q \text{ from } i \text{ to } k \\ \tilde{\varphi}_k^{-1}(t_i'), & \text{otherwise} \end{cases} \\ &= \begin{cases} (t_k(t_k t_i t_k)t_k), & \text{when there is some arrow in } Q \text{ from } i \text{ to } k \\ t_i, & \text{otherwise} \end{cases} \\ &= t_i. \end{aligned}$$

Thus $\tilde{\varphi}_k$ is an isomorphism and so $G_Q \cong G_{\mu_k(Q)}$. By the transitivity of isomorphisms, if Q' is any mutation-equivalent cluster quiver to Q then $G_Q \cong G_{Q'}$.

Suppose that Q is a decomposable cluster quiver with connected components $Q_1, ..., Q_r$. By Lemma 4.2.5, $G_Q = G_{Q_1} \times ... \times G_{Q_r}$. Fixing any $1 \leq k \leq n$, the vertex k lies in an indecomposable component Q_j, for some $1 \leq j \leq r$, and $\mu_k(Q) = Q_1 \sqcup ... \sqcup \mu_k(Q_j) \sqcup ... \sqcup Q_r$.

By Lemma 4.2.5, $G_{\mu_k(Q)} = G_{Q_1} \times ... \times G_{\mu_k(Q_j)} \times ... \times G_{Q_r}$ and by the above, $G_{\mu_k(Q_j)} \cong G_{Q_j}$. Thus

$$\begin{aligned} G_{\mu_k(Q)} &= G_{Q_1} \times ... \times G_{\mu_k(Q_j)} \times ... \times G_{Q_r} \\ &\cong G_{Q_1} \times ... \times G_{Q_j} \times ... \times G_{Q_r} = G_Q. \end{aligned}$$

By the transitivity of isomorphisms, if Q' is any mutation-equivalent cluster quiver to Q then $G_Q \cong G_{Q'}$. □

As discussed in Chapter 1, the crystallographic root systems are classified by the Dynkin diagrams and each of these root systems gives rise to a finite reflection group, which is a finite Coxeter group. Let Δ be a simply-laced Dynkin diagram. In this case, the Coxeter presentation of the corresponding finite reflection group, W_Δ, is obtained from Δ in the following way.

Take a vertex labelling of Δ by the set $\{1,..,n\}$. Let W_Δ be the group with group presentation $\langle S|R\rangle$ where $S = \{s_1, ..., s_n\}$ and R is the set of relations $\{(s_i s_j)^{m(i,j)} : 1 \leq , i, j \leq n\}$, where

$$m(i,j) = \begin{cases} 2, & \text{when there is no arrow between } i \text{ and } j \text{ in } \Delta; \\ 3, & \text{when there is an arrow between } i \text{ and } j \text{ in } \Delta. \end{cases}$$

Fix some orientation, $\overrightarrow{\Delta}$, of Δ. As Δ contains no cycles, the cluster group presentation of $G_{\overrightarrow{\Delta}}$ is given by:

$$G_{\overrightarrow{\Delta}} = \langle t_1, ..., t_n | t_i^2 = (t_i t_j)^{p(i,j)} = e \rangle$$

where

$$\begin{aligned} p(i,j) &= \begin{cases} 2, & \text{when there is no arrow between } i \text{ and } j; \\ 3, & \text{when there is an arrow between } i \text{ and } j. \end{cases} \\ &= m(i,j). \end{aligned}$$

In this case, $G_{\overrightarrow{\Delta}}$ does not depend on the orientation of Δ, so we denote it simply by G_Δ. Comparing these group presentations, it is easy to see that

$$\begin{aligned} \varphi : G_\Delta &\longrightarrow W_\Delta, \\ \varphi : t_i &\longmapsto s_i \end{aligned}$$

defines an isomorphism between G_Δ and W_Δ. If Q is of mutation-Dynkin type Δ, then Q is mutation-equivalent to some orientation of Δ. By Lemma 4.2.7, $G_Q \cong G_\Delta \cong W_\Delta$. Thus, when Q is a quiver of mutation-Dynkin type, the associated cluster group is isomorphic to the finite reflection group of the same Dynkin type.

Theorem 4.2.8. *[3, Theorem 5.4], [28, Lemma 2.5] When Q is a quiver of mutation-Dynkin type Δ, then G_Q is isomorphic to the finite reflection group arising from the root system of type Δ.*

Remark 4.2.9. By the definition of the cluster group associated to a quiver Q, there is a surjective map from the set of vertices, V, of Q to the set of defining generators, T, of G_Q. In general, it is not clear that this map is injective. That is, it may be that $t_i = t_j$ in G_Q for distinct $i, j \in V$. However, from the proof of Lemma 4.2.7, if Q is a quiver of mutation-Dynkin type Δ then the set of defining generators, T, of G_Q is a subset of the set of reflections in the Coxeter system (W_Δ, S). Moreover, by Theorem 4.2.8, T generates W_Δ. It is well known that W_Δ cannot be generated by fewer than $|S|$ reflections (for a proof of this, see [18, Lemma 2.1]). As $|S| = |T|$, the map between V and T must be injective. This gives the following result.

Lemma 4.2.10. *Let Q be a quiver of mutation-Dynkin type and consider the associated cluster group, G_Q. Then $t_i \neq t_j$ in G_Q for any two distinct vertices i and j of Q.*

The following result is analogous to Proposition 1.4.1 for Coxeter groups.

Proposition 4.2.11. *For a cluster quiver Q, there is a surjective homomorphism*

$$\varepsilon : G_Q \longrightarrow \{\pm 1\}$$
$$\varepsilon : t_i \longmapsto -1.$$

It follows that the order of each generator $t_i \in T$ of G_Q is 2.

Proof. As in the proof of Lemma 4.2.6, we define a map from F_Q and show this induces a group homomorphism on G_Q. We begin by defining a group homomorphism:

$$\tilde{\varepsilon} : F_Q \longrightarrow \{\pm 1\}$$
$$\tilde{\varepsilon} : t_i \longmapsto -1.$$

where $\{\pm 1\}$ is considered as a multiplicative group. To show that $\tilde{\varepsilon}$ induces ε, we must show that each relation in the cluster presentation of G_Q lies in the kernel of $\tilde{\varepsilon}$.

(a) For all $1 \leq i \leq n$, $\tilde{\varepsilon}(t_i^2) = (-1)^2 = 1$.

(b) For all $1 \leq i, j \leq n$, $i \neq j$, such that i and j are not connected by an edge in Q, $\tilde{\varepsilon}((t_i t_j)^2) = (-1)^4 = 1$. For all $1 \leq i, j \leq n$, $i \neq j$, such that i and j are connected by an edge in Q, $\tilde{\varepsilon}((t_i t_j)^3) = (-1)^6 = 1$.

(c) For every chordless oriented cycle of length r:

$$i_1 \longrightarrow i_2 \longrightarrow i_3 \longrightarrow ... \longrightarrow i_r \longrightarrow i_1$$

in Q,

$$\tilde{\varepsilon}((t_{i_1} t_{i_2} t_{i_3} ... t_{i_r} t_{i_1} ... t_{i_{r-2}})(t_{i_{r-1}} ... t_{i_3} t_{i_2} t_{i_1} t_{i_r} ... t_{i_3} t_{i_2})) = (-1)^{2(2r-2)} = 1.$$

Thus ε is a surjective group homomorphism. As -1 has order 2 in $\{\pm 1\}$, each generator, t_i, of G_Q is of order 2. □

Remark 4.2.12. It can be seen from the proof of Proposition 4.2.11 that such a surjective homomorphism exists for any group arising from a group presentation whose relations are all of even length (and so the generators of this group will have even order at least 2).

4.3 The Length Function

In [33, Section 5.2], the length function for a Coxeter group is defined along with five basic properties. Below, we adapt the definition of the length function for cluster groups and show that these properties still hold.

Let Q be a cluster quiver. By definition of the presentation of a group, any element w of G_Q can be written as

$$w = t_{i_1}^{a_1} t_{i_2}^{a_2} ... t_{i_r}^{a_r},$$

where $1 \leq i_j \leq n$ and $a_j = \pm 1$, for all $1 \leq j \leq r$. As $t_i^2 = e$, for all $1 \leq i \leq n$, it follows that w can be written in the form

$$w = t_{i_1} t_{i_2} ... t_{i_r}.$$

Analogous to the Coxeter group case, we make the following definitions. The **length** of $w \in G_Q$, $l(w)$, is the smallest r such that $w = t_{i_1} t_{i_2} ... t_{i_r}$ for $t_{i_j} \in T$ and a **reduced expression** of w is any expression of w as a product of $l(w)$ elements of T.

Lemma 4.3.1. *For all $w_1, w_2 \in G_Q$ and $t_i \in T$ the following properties hold.*

(1) $l(w_1) = l(w_1^{-1})$.

(2) $l(w_1) = 1 \Leftrightarrow w_1 \in T$.

(3) $l(w_1 w_2) \leq l(w_1) + l(w_2)$.

(4) $l(w_1 w_2) \geq l(w_1) - l(w_2)$.

(5) $l(w_1) - 1 \leq l(w_1 t_i) \leq l(w_1) + 1$.

(6) $l(w_1 t_i) \neq l(w_1)$ *and* $l(t_i w_1) \neq l(w_1)$

Proof. Using arguments similar to those in the Coxeter case [33, Section 5.2] we have the following.

(1) If $l(w_1) = r$ then there exists an expression $w_1 = t_{i_1} t_{i_2} ... t_{i_r}$ with $t_{i_j} \in T$ for each $1 \leq j \leq r$. Thus $w_1^{-1} = t_{i_r} t_{i_2} ... t_{i_1}$ and so $l(w_1^{-1}) \leq l(w_1)$. Similarly, $l(w_1^{-1}) \geq l(w_1)$ giving $l(w_1) = l(w_1^{-1})$.

(2) If $l(w_1) = 1$, then there exists an expression $w_1 = t_i$ for some $t_i \in T$, meaning $w_1 \in T$. By Proposition 4.2.11, each generator t_i has order at least 2 (in fact, the order is equal to 2). Thus $t_i \neq e$ for all $1 \leq i \leq n$. So the converse, that $l(t_i) = 1$ for all $1 \leq i \leq n$, also holds.

(3) If $l(w_1) = r_1$ and $l(w_2) = r_2$ then there exist expressions $w_1 = t_{i_1} t_{i_2} ... t_{i_{r_1}}$ and $w_2 = t_{j_1} t_{j_2} ... t_{j_{r_2}}$ and so there is an expression $w_1 w_2 = t_{i_1} t_{i_2} ... t_{i_{r_1}} t_{j_1} t_{j_2} ... t_{j_{r_2}}$, meaning $l(w_1 w_2) \leq r_1 + r_2$.

(4) Note that $w_1 = w_1 w_2 w_2^{-1}$, so $l(w_1) = l(w_1 w_2 w_2^{-1})$. By (3) and (1), $l(w_1 w_2 w_2^{-1}) \leq l(w_1 w_2) + l(w_2^{-1}) = l(w_1 w_2) + l(w_2)$. Rearranging gives $l(w_1 w_2) \geq l(w_1) - l(w_2)$.

(5) Applying (3) and (2), we have $l(w_1 t_i) \leq l(w_1) + l(t_i) = l(w_1) + 1$. Applying (4) and (2), we have $l(w_1 t_i) \geq l(w_1) - l(t_i) = l(w_1) - 1$.

(6) Using the group homomorphism $\varepsilon : G_Q \longrightarrow \{\pm 1\}$ defined in Proposition 4.2.11, we have $\varepsilon(w) = (-1)^{l(w)}$ for all $w \in G_Q$. If $l(w_1 t_i) = l(w_1)$ for some $w_1 \in G_Q$ and $t_i \in T$, then $\varepsilon(w_1) = (-1)^{l(w_1)} = (-1)^{l(w_1 t_i)} = \varepsilon(w_1 t_i) = \varepsilon(w_1)\varepsilon(t_i) = -\varepsilon(w_1)$. As $\varepsilon(w_1) \in \{\pm 1\}$, this is not possible. By similar reasoning, $l(t_i w_1) \neq l(w_1)$.

□

Remark 4.3.2. In general, the length of an element w in a group with group presentation $\langle X|R\rangle$ is given by the length of the shortest expression $w = x_1...x_r$ where $x_j \in X \cup X^{-1}$ [44, Section 1.1]. Taking this definition of length, the properties (1), (3) and (4) in Lemma 4.3.1 hold for a group with arbitrary presentation.

It is easy to see that (2) and (6) fail for the trivial group given by the presentation $\langle t|t = e\rangle$. However, the property

(2^*) $w_1 \in X \cup X^{-1} \Rightarrow l(w_1) \leq 1$

holds for any group with group presentation $\langle X|R\rangle$. We can then apply (2^*), (3) and (4) to prove that (5) holds for a group with arbitrary presentation.

The proofs of (2) and (6) rely on the fact that the order of each generator is at least 2 and Proposition 4.2.11, respectively. Thus, by Remark 4.2.12, we have that Lemma 4.3.1 will hold for any group arising from a group presentation whose relations are all of even length and whose generators are of order 2 (with (1), (3), (4) and (5) holding for every group).

Definition 4.3.3. *Let Q be a cluster quiver. For an expression $w = t_{i_1}...t_{i_k}$ of $w \in G_Q$ (not necessarily reduced), we call the set $\{t_{i_1},...,t_{i_k}\}$ the **support** of the expression.*

4.4 Parabolic Subgroups

For a Coxeter group (W,S), a parabolic subgroup is a subgroup generated by the elements of some subset I of S and such a group is denoted by W_I. We consider such subgroups for cluster groups, defined in an analogous way.

Definition 4.4.1. *Let G_I denote the subgroup of G_Q generated by the elements of the subset I of T. A (standard) **parabolic subgroup** of G_Q is a subgroup of the form G_I for some $I \subseteq T$.*

Lemma 4.4.2. *Let Q be a quiver of mutation-Dynkin type and consider the associated cluster group G_Q. For any $I \in \mathscr{I}$, $G_I \cap T = I$.*

Proof. Clearly, $I \subseteq G_I \cap T$. Conversely, if $t \in T \setminus I$ and $t \in G_I$, then $T \setminus \{t\}$ is a generating set for G_Q, contradicting Remark 4.2.9. Thus $G_I \cap \{t_1,..,t_n\} = I$.

□

We have seen, in Section 1.4, that the finite Coxeter groups have some useful properties. For example, Proposition 1.4.7 gives the existence of a unique coset representative for a parabolic subgroup, Theorem 1.4.8 shows that each parabolic subgroup is a finite Coxeter group in its own right and defines a lattice isomorphism between the set of subsets of the defining generators and the collection of parabolic subgroups and Theorem 1.4.9 establishes the Exchange Lemma. In the next chapter, we begin to explore if cluster group presentations possess properties comparable to those for finite Coxeter groups.

Chapter 5

Positive Companion Bases

5.1 Introduction

In this chapter, we will consider only the cluster groups associated to quivers of mutation-Dynkin type with connected components $Q_1,...,Q_r$ of types $A_{n_1},...,A_{n_r}$, respectively, where $n_1,...,n_r \in \mathbb{Z}^+$ and $n = \Sigma_{i=1}^{r} n_i$, with $r \geq 1$. First, let Q be a quiver of this type with $r = 1$. In previous chapters, we saw that G_Q is isomorphic to the finite Coxeter group of type A_n and so is isomorphic to the symmetric group, Σ_{n+1}. We have also seen how a triangulation, $\mathscr{T}_Q$, of an $(n+3)$-gon arises from Q and how the braid graph of Q, Γ_Q, is obtained from $\mathscr{T}_Q$. Furthermore, these results can be extended to the case when $r \geq 1$. These properties provide useful tools with which to study the associated cluster groups.

We recall from Section 1.4, that each finite Coxeter system, (W,S), can be viewed as a finite reflection group of the real vector space V, with basis $\{\alpha_1,...,\alpha_n\}$, where the set $\Phi = \{w(\alpha_i) : w \in W, 1 \leq i \leq n\}$ forms a root system of V. Moreover, each root, $\alpha \in \Phi$, can be written as a linear combination of the α_i whose coefficients are either all positive or all negative. This property is a useful resource for proving results about the finite Coxeter groups. For example, Theorem 1.4.4 means that for each $w \in W$ and $1 \leq i \leq n$, the positivity of $w(\alpha_i)$ provides information concerning the length of ws_i. Lemma 1.4.10 demonstrates how the set of positive roots turned negative by $w \in W$ can be established from a reduced expression of w and is directly linked to the length of w. These roots can also be used to determine whether a given expression of w is reduced or not.

In this chapter, we aim to build an analogous theory of roots for cluster groups of mutation-Dynkin type $A_{n_1},...,A_{n_r}$. To begin, we will show that the braid graph, defined in Section 2.4, defines an isomorphism between G_Q and a Young subgroup of Σ_{n+r}.

We will then prove that there exists a companion basis, $C = \{\gamma_i : 1 \leq i \leq n\}$, of Q such

that, for all $1 \leq i \neq j \leq n$, we have:

$$(\gamma_i, \gamma_j) = \begin{cases} 1, & \text{if } i \text{ and } j \text{ are joined by an arrow in } Q \\ 0, & \text{if } i \text{ and } j \text{ are not joined by an arrow in } Q. \end{cases}$$

We call such a companion basis a *positive companion basis* of Q. We will show how a positive companion basis, C, can be constructed explicitly from the braid graph of Q and that every root, $\alpha \in \Phi$, can be written as a linear combination of the elements of C, where the non-zero coefficients have absolute value 1 and alternate in sign.

Finally, motivated by Lemma 1.4.10, we will define the associated roots of a given expression of $w \in G_Q$ and explore what information these associated roots yield about the set $\{w(\gamma_i) : w \in G_Q, 1 \leq i \leq n\}$ and the structure of G_Q.

5.2 The Braid Graph Isomorphism

Let Q be a quiver of mutation-Dynkin type with connected components $Q_1, ..., Q_r$ of types $A_{n_1}, ..., A_{n_r}$, respectively, where $n_1, ..., n_r \in \mathbb{Z}^+$ and $n = \Sigma_{i=1}^r n_i$, with $r \geq 1$. In Section 2.4, we saw that the diagonals of the triangulation $\mathscr{T}_Q$ are in bijection with the vertices of Q. The braid graph, Γ_Q, is obtained from $\mathscr{T}_Q$ by taking a vertex for every triangle of $\mathscr{T}_Q$ where an edge exists between two vertices if and only if the corresponding triangles share a common diagonal in $\mathscr{T}_Q$. It follows that the edges of the braid graph of Q are in bijection with the diagonals of $\mathscr{T}_Q$. We fix a labelling of Γ_Q by the set $\{1, 2, ..., n+r\}$ and denote the endpoints of the edge, E_i, of Γ_Q corresponding to the vertex i of Q by x_i and y_i. I.e. $E_i = \{x_i, y_i\}$.

By Theorem 4.2.8, we know that G_Q is isomorphic to the finite reflection group of the same Dynkin type and so, by transitivity, G_Q is isomorphic to Σ_{n+r}. We will use the above labelling of Γ_Q to define a direct isomorphism between G_Q and Σ_{n+r}.

First, we will show that when $r = 1$, the braid graph defines an isomorphism between G_Q and Σ_{n+1}. We will then extend this to the case when $r \geq 1$.

Given a tree, Γ, Proposition 3.2.6 gave a group presentation for Σ_Γ, the permutations of the vertices of Γ. As Γ_Q is a tree with vertex set $\{1, 2, ..., n+1\}$, Proposition 3.2.6 yields a presentation of Σ_{n+1} which we will use to prove the following lemma.

Lemma 5.2.1. *Let Q be a quiver of mutation-Dynkin type A_n and consider the braid graph, Γ_Q. For each vertex i of Q, let $x_i, y_i \in \{1, 2, ..., n+1\}$ denote the endpoints of the edge E_i in Γ_Q (so $x_i \neq y_i$). There exists an isomorphism*

$$\pi_Q : G_Q \longrightarrow \Sigma_{n+1},$$

$$\pi_Q : t_i \longmapsto (x_i, y_i).$$

Proof. By Proposition 3.2.6, there exists a group presentation of Σ_{n+1} with generating set $\{(x_i,y_i) : \{x_i,y_i\} \text{ is an edge in } \Gamma_Q\}$ subject to the relations $(1)-(4)$. As these correspond to the relations in the cluster presentation of G_Q (see Remark 3.2.8 and Remark 4.2.3) via the bijection between the vertices of Q (and so the set of defining generators of G_Q) and the edges of the braid graph, it is clear that π_Q is an isomorphism. □

Next, we consider the case when $r \geq 1$. Let P be the disjoint union of $P_1,...,P_r$, where each P_i is a convex (n_i+3)-gon for each $1 \leq i \leq r$. As previously discussed, there exists a triangulation $\mathscr{T}_i$ of P_i giving rise to Q_i for each $1 \leq i \leq r$. Let $\mathscr{T}$ be the collection of these triangulations. Thus $\mathscr{T}$ will be a triangulation of P giving rise to Q. It follows that Γ_Q will be the graph that is the disjoint union of $\Gamma_{Q_1},....,\Gamma_{Q_r}$. Note that Γ_Q will contain n edges and $n+r$ vertices, each with valency 1, 2 or 3 (as the only cycles occurring in quivers of mutation-Dynkin type A_n are 3-cycles).

For each $1 \leq i \leq r$, choose any labelling of the vertices of Γ_{Q_i} by the set $N_i = \{(\Sigma_{j=1}^{i-1} n_j)+i,...,(\Sigma_{j=1}^{i} n_j)+i\}$, taking $N_1 = \{1,2,...,n_1+1\}$.

Consequently, the vertex sets of the connected components of Γ_Q define the parts of a set partition, $\rho_Q = \sqcup_{j=1}^{r} N_j$, of the set $\{1,...,n+r\}$. We show that there exists an isomorphism between G_Q and the Young subgroup corresponding to ρ_Q.

Lemma 5.2.2. *Let Q be a quiver of mutation-Dynkin type $A_{n_1} \sqcup ... \sqcup A_{n_r}$ and consider the braid graph of Q, $\Gamma_Q = \Gamma_{Q_1} \sqcup ... \sqcup \Gamma_{Q_r}$. For each $1 \leq i \leq r$, choose any labelling of the vertices of Γ_{Q_i} by the set $N_i = \{(\Sigma_{j=1}^{i-1} n_j)+i,...,(\Sigma_{j=1}^{i} n_j)+i\}$, taking $N_1 = \{1,2,...,n_1+1\}$. Take the set partition $\rho_Q = \sqcup_{j=1}^{r} N_j$ of $\{1,...,n+r\}$ and, for each vertex i of Q, let $x_i,y_i \in \{1,2,...,n+r\}$ denote the endpoints of the edge E_i in Γ_Q (so $x_i \neq y_i$). There exists an isomorphism*

$$\pi_Q : G_Q \longrightarrow Y(\rho_Q),$$
$$\pi_Q : t_i \longmapsto (x_i,y_i).$$

Proof. By Lemma 4.2.5, we have that

$$G_Q = G_{Q_1} \times ... \times G_{Q_r}.$$

By Lemma 5.2.1, for each $1 \leq j \leq r$ and any labelling of Γ_{Q_j} by $\{1,...,n_j+1\}$, there exists an isomorphism

$$\pi_{Q_j} : G_{Q_j} \longrightarrow \Sigma_{n_j+1},$$
$$\pi_{Q_j} : t_i \longmapsto (x_i',y_i'),$$

where x_i' and y_i' are the endpoints of E_i, for some distinct $x_i', y_i' \in \{1, ..., n_i + 1\}$. Let $p_j : \Sigma_{n_j+1} \longrightarrow \Sigma_{N_j}$ be a relabelling of Γ_{Q_j} to the induced labelling by Γ_Q. So the following map is an isomorphism:

$$p_j \circ \pi_{Q_j} : G_{Q_j} \longrightarrow \Sigma_{N_j},$$

$$p_j \circ \pi_{Q_j} : t_i \longmapsto (x_i, y_i),$$

where $x_i, y_i \in N_j \subseteq \{1, 2, ..., n + r\}$ are the endpoints of E_i in the chosen labelling of Γ_Q. So we can define an isomorphism

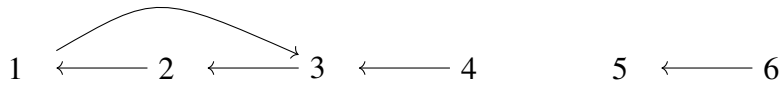
$$\pi_Q : G_{Q_1} \times ... \times G_{Q_r} \longrightarrow \Sigma_{N_1} \times ... \times \Sigma_{N_r}$$

by $\pi|_{G_{Q_j}} = p_j \circ \pi_{Q_j}$, for each $1 \leq j \leq r$.

Noting that $G_Q = G_{Q_1} \times ... \times G_{Q_r}$ and $Y(\rho_Q) = \Sigma_{N_1} \times ... \times \Sigma_{N_r}$, we have obtained the desired isomorphism.

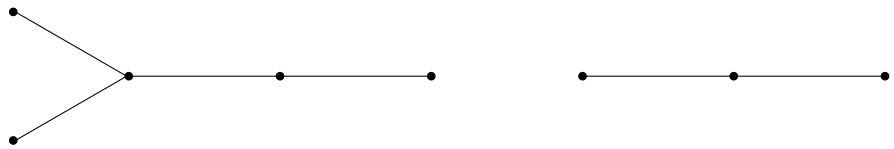
□

Example 5.2.3. Let Q be the quiver



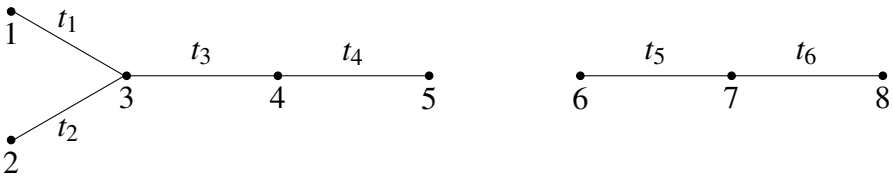
So Q is a quiver of mutation- Dynkin type $A_4 \sqcup A_2$ with braid graph:

Γ_Q:



Let us take the following labelling of Γ_Q by the set $\{1, 2, 3, 4, 5, 6, 7, 8\}$.

Γ_Q:



With respect to this labelling, $\rho_Q = \{\{1, 2, 3, 4, 5\}, \{6, 7, 8\}\}$ and π_Q is the isomorphism:

$$\pi_Q : G_Q \longrightarrow Y(\rho_Q),$$
$$\pi_Q : t_1 \longmapsto (1,3),$$
$$\pi_Q : t_2 \longmapsto (2,3),$$
$$\pi_Q : t_3 \longmapsto (3,4),$$
$$\pi_Q : t_4 \longmapsto (4,5),$$
$$\pi_Q : t_5 \longmapsto (6,7),$$
$$\pi_Q : t_6 \longmapsto (7,8).$$

5.3 Positive Companion Bases

As described in Section 1.4, a finite Coxeter system, (W,S), can be viewed as a finite reflection group on the vector space, V, over $\mathbb{R}$ with basis $\Pi = \{\alpha_i : 1 \leq i \leq n\}$ together with a bilinear form. In particular, the set $\Phi = \{w(\alpha_i) : w \in W, 1 \leq i \leq n\}$ forms a root system in V in which every root can be written as a linear combination of the elements in Π, where all the coefficients are either all positive or all negative. This property of Φ makes it a useful tool for studying the finite Coxeter groups.

Suppose that Q is mutation-equivalent to an orientation, $\overrightarrow{\Delta}$, of the simply-laced Dynkin diagram, Δ. By Lemma 4.2.7, there exists an isomorphism

$$\varphi : G_Q \longrightarrow W,$$

where W is the finite Coxeter group of type Δ. Under this isomorphism, the defining generators of G_Q are mapped to elements of the set of reflections, $\{ws_iw^{-1} : w \in W, s_i \in S\}$, of the Coxeter system (W,S). This means we can study how the elements of G_Q act on V. However, for each $1 \leq i \leq n$, the reflection $\varphi(t_i)$ is defined according to the finite sequence of mutations transforming Q into $\overrightarrow{\Delta}$ and, in general, we do not know the exact sequence of mutations. This motivates us to make a change of basis for V that will allow us to more easily examine how G_Q acts on V, rather than via the isomorphism φ. In particular, we would like to find a companion basis for Q.

In Section 2.5 we recalled the definition of a companion basis for Q and that, by Theorem 2.5.2, a companion basis exists for any quiver of mutation-Dynkin type. Moreover, Theorem 2.5.2 provides a method of obtaining a companion basis for Q from Π, via the sequence of mutations taking $\overrightarrow{\Delta}$ to Q.

Given a companion basis $C = \{\gamma_i : 1 \leq i \leq n\}$ of Q, we define the set:

$$\Phi_C = \{w(\gamma_i) : w \in G_Q, 1 \leq i \leq n\}.$$

By definition, $C \subseteq \Phi$. So for each $1 \leq i \leq n$, there exist $w_i \in W$ and $1 \leq j \leq n$ such that $\gamma_i = w_i(\alpha_j)$. Note that $w_i(\alpha_j) = \varphi^{-1}(w_i)(\alpha_j)$ for all $w_i \in W$ and $1 \leq j \leq n$. Thus, for any $w \in G_Q$, $w(\gamma_i) = w(\varphi^{-1}(w_i)(\alpha_j)) = \varphi(w)w_i(\alpha_j) \in \Phi$, meaning $\Phi_C \subseteq \Phi$ for any companion basis C of Q. Therefore, we will still refer to the elements of Φ_C as **roots** and we will call the elements of C the **simple roots** of the cluster group G_Q.

Recall from Section 2.5 that each root $\gamma \in \Phi_C$ can be written as a unique linear combination of the form $\gamma = \sum_{i=1}^{n} c_i\gamma_i$, for some $c_i \in \mathbb{Z}$. In order to restrict the possible values of a given c_i, we look to find a companion basis for Q which always yields a non-negative value when taking the inner product of any two of its elements.

Definition 5.3.1. *Let Q be a quiver of mutation-Dynkin type with companion basis $C = \{\gamma_i : 1 \leq i \leq n\}$. If C is such that, for all $1 \leq i \neq j \leq n$, we have $(\gamma_i, \gamma_j) \geq 0$ then we call C a **positive companion basis** of Q.*

In this section, we will prove the existence of a positive companion basis of Q, in the case when Q is a quiver of mutation-Dynkin type $A_{n_1} \sqcup ... \sqcup A_{n_r}$, for some $n_i \in \mathbb{Z}^+$ where $n = \Sigma_{i=1}^{r} n_i$. We will give a method for obtaining such a basis from the braid graph of Q and examine the linear combinations of the roots in Φ_C.

Let Q be a quiver of mutation-Dynkin type $A_{n_1} \sqcup ... \sqcup A_{n_r}$, for some $n_i \in \mathbb{Z}^+$ where $n = \Sigma_{i=1}^{r} n_i$. As quivers of mutation-Dynkin type $A_{n_1} \sqcup ... \sqcup A_{n_r}$ are simply laced, given a companion basis $C = \{\gamma_i : 1 \leq i \leq n\}$ of Q we remark that, for all $1 \leq i \neq j \leq n$, we have

$$|(\gamma_i, \gamma_j)| = \begin{cases} 1, & \text{if } i \text{ and } j \text{ are joined by an arrow in } Q \\ 0, & \text{if } i \text{ and } j \text{ are not joined by an arrow in } Q. \end{cases}$$

By Corollary 2.5.9, any root $\gamma \in \Phi_C$ can be written as a unique linear combination of the form $\gamma = \sum_{i=1}^{n} c_i\gamma_i$, for some $c_i \in \{-1, 0, 1\}$.

Lemma 5.3.2. *[2, Proposition 1.4] Let Q be a mutation-Dynkin quiver with companion basis $C = \{\gamma_i : 1 \leq i \leq n\}$. For every chordless, oriented cycle*

$$i_1 \longrightarrow i_2 \longrightarrow i_3 \longrightarrow ... \longrightarrow i_m \longrightarrow i_1$$

in Q, the following holds:

$$\prod_{j=1}^{m} (\gamma_{i_j}, \gamma_{i_{j+1 (mod m)}}) < 0.$$

Remark 5.3.3. As Q is of mutation-Dynkin type $A_{n_1} \sqcup \ldots \sqcup A_{n_r}$, by Lemma 2.4.8, every oriented cycle in Q is a 3-cycle. By Lemma 5.3.2, for any companion basis $C = \{\gamma_i : 1 \leq i \leq n\}$ of Q and every 3-cycle

$$i_1 \longrightarrow i_2 \longrightarrow i_3 \longrightarrow i_1$$

in Q, either $(\gamma_{i_j}, \gamma_{i_k}) = 1$ for all pairs (j,k), with $1 \leq j,k \leq 3$ and $j \neq k$ or $(\gamma_{i_j}, \gamma_{i_k}) = 1$ holds for exactly one pair, with the remaining two inner products equal to -1.

Proposition 5.3.4. *If Q is a quiver of mutation-Dynkin type $A_{n_1} \sqcup \ldots \sqcup A_{n_r}$, where $n = \Sigma_{i=1}^{r} n_i$, then there exists a positive companion basis of Q.*

Proof. By Theorem 2.5.2, there exists some companion basis $C = \{\gamma_i : 1 \leq i \leq n\} \subseteq \Phi$ of Q. Moreover, as Q is of mutation-Dynkin type $A_{n_1} \sqcup \ldots \sqcup A_{n_r}$, C is such that, for all $i \neq j$, we have

$$(\gamma_i, \gamma_j) = \begin{cases} \pm 1 & \text{if } i \text{ and } j \text{ are joined by an arrow in } Q, \\ 0 & \text{if } i \text{ and } j \text{ are not joined by an arrow in } Q. \end{cases}$$

We will use C to construct a new companion basis of Q, $\overline{C} = \{\overline{\gamma}_i : 1 \leq i \leq n\}$, such that, for all $i \neq j$, we have

$$(\overline{\gamma}_i, \overline{\gamma}_j) = \begin{cases} 1 & \text{if } i \text{ and } j \text{ are joined by an arrow in } Q, \\ 0 & \text{if } i \text{ and } j \text{ are not joined by an arrow in } Q. \end{cases}$$

To begin, we label each edge in the underlying graph of Q, with respect to the companion basis C, as follows:

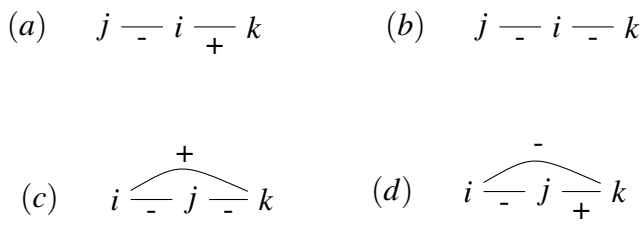
> For an edge with endpoints i and j, label the edge with "$+$" if $(\gamma_i, \gamma_j) = 1$ and "$-$" if $(\gamma_i, \gamma_j) = -1$.

If all edges are labelled "$+$", then C is a positive companion basis. So we assume that there exists an edge with endpoints i and j labelled by "$-$". It follows i and j lie in the same connected component, Q_l, of Q and that $(\gamma_i, \gamma_j) = -1$.

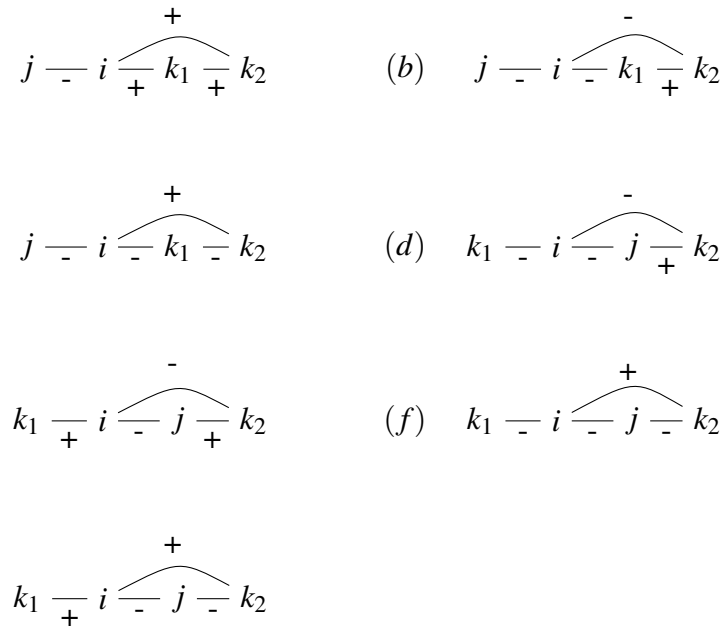
Using the possible local configurations identified in Section 2.4, and taking into account Remark 5.3.3, we consider the local configuration of i in the underlying graph of Q_l with this labelling.

(1) If i has valency 1 then the local configuration at i must be:

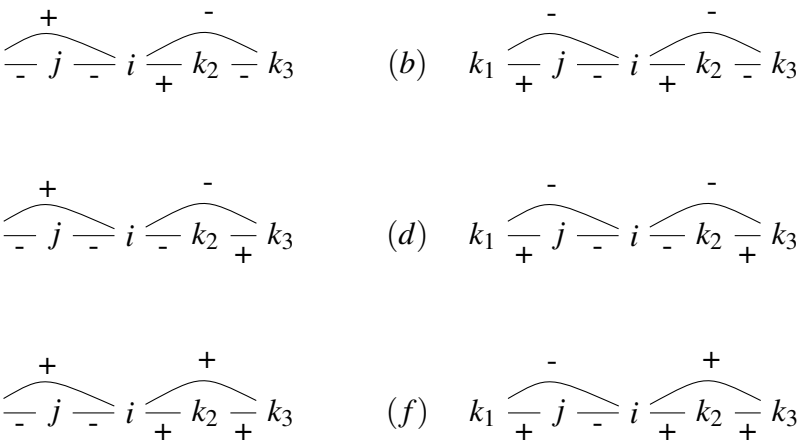
(2) If i has valency 2 then the local configuration at i must be one of the following:



(3) If i has valency 3 then the local configuration at i must be one of the following:



(4) If i has valency 4 then the local configuration at i must be one of the following:



By Lemma 2.5.3, we can replace any $\gamma_i \in C$ with $-\gamma_i$ and obtain a companion basis of Q.

In the cases of $(1), (2b), (2d), (3b), (3d)$ or $(4d)$, replace γ_i with $\gamma_i' = -\gamma_i$ to obtain a new companion basis C'.

In the cases of $(2a), (2c), (3a), (3g)$ or $(4e)$, replace γ_j with $\gamma_j' = -\gamma_j$ to obtain a new companion basis C'.

In the cases of $(3c), (3f)$ or $(4f)$, replace γ_j and γ_{k_1} with $\gamma_j' = -\gamma_j$ and $\gamma_{k_1}' = -\gamma_{k_1}$ to obtain a new companion basis C'.

In the cases of $(4a)$, replace γ_j and γ_{k_3} with $\gamma_j' = -\gamma_j$ and $\gamma_{k_3}' = -\gamma_{k_3}$ to obtain a new companion basis C'.

In the case of $(3e)$ or $(4c)$, replace γ_i and γ_{k_1} with $\gamma_i' = -\gamma_i$ and $\gamma_{k_1}' = -\gamma_{k_1}$ to obtain a new companion basis C'.

In the case of $(4b)$, replace γ_i and γ_{k_2} with $\gamma_i' = -\gamma_i$ and $\gamma_{k_2}' = -\gamma_{k_2}$ to obtain a new companion basis C'.

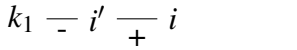
For all other elements of C', take $\gamma_q' = \gamma_q$, defining a new companion basis $C' = \{\gamma_q' : 1 \leq q \leq n\}$ such that $(\gamma_i', \gamma_q) \geq 0$ for all $q \neq i$.

Now, assuming that we are able to, pick a vertex i' of Q_l such that:

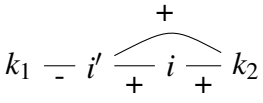
$(*)$ $\quad i'$ is incident to i and there exists some $q \neq i'$ such that $(\gamma_{i'}', \gamma_q) < 0$.

Note that q cannot lie in the local configuration of i as otherwise $(\gamma_{i'}', \gamma_q) \geq 0$ by our previous step. We consider the local configuration of i'. It cannot be that i' has valency 1 as it is incident to both i and q, for $i \neq q$.

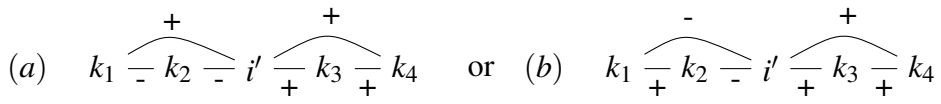
(1) If i' has valency 2:



(2) If i' has valency 3:



(3) If i' has valency 4:



In the cases of $(1),(2)$, replace γ'_{k_1} with $\gamma''_{k_1} = -\gamma'_{k_1}$ to obtain a new companion basis C''.

In the case of $(3a)$, replace γ'_{k_2} with $\gamma''_{k_2} = -\gamma'_{k_2}$ to obtain a new companion basis C''.

In the cases of $(3b)$, replace γ'_{k_1} with $\gamma''_{k_1} = -\gamma'_{k_1}$ and γ'_{k_2} with $\gamma''_{k_2} = -\gamma'_{k_2}$ to obtain a new companion basis C''.

For all other elements of C'', take $\gamma''_q = \gamma'_q$, defining a new companion basis $C'' = \{\gamma''_q : 1 \leq q \leq n\}$ such that, in the labelling of the underlying graph of Q with respect to C'', every edge with an endpoint equal to either i or i' is labelled by "$+$".

Repeat this process for every vertex incident to i that satisfies the condition $(*)$. This will yield a companion basis such that, in the labelling of the underlying graph of Q with respect to this companion basis, every edge with an endpoint equal to either i or any i' incident to i is labelled by "$+$". Once this is complete, or if there is no vertex incident to i satisfying the condition $(*)$, repeat the process above for every vertex lying in the local configuration of each i' incident to i, and so on.

As n_l is finite, eventually we will obtain a companion basis $\overline{C}$ such that in the labelling of the underlying graph of Q with respect to $\overline{C}$, every edge in the connected component Q_l is labelled by "$+$". After repeating this process for each connected component of Q, we obtain the required companion basis. □

Remark 5.3.5. Proposition 5.3.4 can also be proved using existing results of [2], [42] and [48] as outlined below.

By Theorem 2.5.2, there exists a companion basis, $C = \{\gamma_1, ..., \gamma_n\}$, of Q. Let $A = (a_{ij})$ be the matrix given by $a_{ij} = (\gamma_i, \gamma_j)$ for each $1 \leq i, j \leq n$. Then A is a positive quasi-Cartan companion of $B(Q)$ [42, Definition 3.2]. Moreover, (according to the the definition of an admissible quasi-Cartan companion given in [48, Section 1]) A is admissible, by [2, Proposition 1.4].

Next, let $A' = (a'_{ij})$ be the matrix given by

$$a'_{ij} = \begin{cases} 2 & \text{if } i = j, \\ 1 & \text{if } i \text{ is connected to } j \text{ in } Q, \\ 0 & \text{if } i \text{ is not connected to } j \text{ in } Q. \end{cases}$$

As Q is simply-laced, A' will be a quasi-Cartan companion of $B(Q)$. Moreover, as the only oriented cycles in quivers of mutation-Dynkin type A_n are 3-cycles (Lemma 2.4.8), for every oriented cycle in Q there will be an odd number of edges $\{i, j\}$ such that $a'_{ij} > 0$. I.e. A' is admissible.

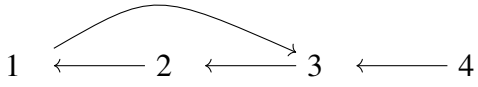
By [48, Theorem 1.4], A' can be obtained from A by a sequence of simultaneous sign changes in rows and columns. This defines a corresponding sequence of sign changes to elements in C (where a sign change in the i^{th} row and column corresponds to replacing γ_i by $-\gamma_i$), giving a set $C' = \{\gamma'_1, ..., \gamma'_n\}$. By Lemma 2.5.3, C' will be a companion basis of Q. Moreover, $a'_{ij} = (\gamma'_i, \gamma'_j)$ for each $1 \leq i, j \leq n$. Thus, C' is a positive companion basis of Q.

The proof of Proposition 5.3.4 describes the sequence of sign changes in C that give C', according to the configuration of the quiver.

From here onwards, take C to be a positive companion basis of Q.

Remark 5.3.6. We cannot define 'positive' and 'negative' roots with respect to C, as we can for the simple system Π. This is because the linear expansion of a root $\gamma \in \Phi_C$ does not necessarily have all positive or all negative coefficients, as shown in the following counter-example.

Example 5.3.7. Take Q to be the following quiver of mutation-Dynkin type A_4:



Taking $C = \{\gamma_i : 1 \leq i \leq 4\}$ to be a positive companion basis of Q, we have $t_1(\gamma_2) = \gamma_2 - \gamma_1$.

Note that as $\Phi_C \subseteq \Phi$, each $\gamma \in \Phi_C$ can be written as a linear combination of the elements of the simple system Π, so will still be positive or negative in the conventional sense.

Lemma 5.3.8. *Let Q be a quiver of mutation-Dynkin type $A_{n_1} \sqcup ... \sqcup A_{n_r}$, where $n = \Sigma_{i=1}^{r} n_i$, with positive companion basis $C = \{\gamma_i : 1 \leq i \leq n\}$. Then each $\gamma \in \Phi_C$ can be written in the form*

$$\gamma = \gamma_{i_0} - \gamma_{i_1} + \gamma_{i_2} - ... \pm \gamma_{i_m}$$

where $i_0, i_1, ..., i_m$ is the unique shortest unoriented path in Q between i_0 and i_m.

Proof. By Proposition 2.5.6 and Proposition 2.5.8, each $\gamma \in \Phi_C$ can be written in the following way:

$$\gamma = \gamma_{i_0} + \sum_{k=1}^{m} (-1)^k \left(\prod_{l=1}^{k} (\gamma_{i_l}, \gamma_{i_{l-1}}) \right) \gamma_{i_k},$$

where $i_0, i_1, ..., i_m$ is the unique shortest unoriented path in Q between i_0 and i_m. As C is a positive companion basis of Q, and Q is a quiver of mutation-Dynkin type $A_{n_1} \sqcup ... \sqcup A_{n_r}$, it follows that, for each $1 \leq k \leq m$, $(\gamma_{i_l}, \gamma_{i_{l-1}}) = 1$ for all $1 \leq l \leq k$. So,

$$\gamma = \gamma_{i_0} - \gamma_{i_1} + \gamma_{i_2} - ... \pm \gamma_{i_m}.$$

□

Remark 5.3.9. Note that, for $w \in G_Q$ and $1 \leq i \leq n$, we still have the property that for any $\gamma \in \Phi_C$, the reflection $s_{w(\gamma_i)}$ is equal to the element $wt_iw^{-1} \in G_Q$. This can be calculated directly, as for the Coxeter case in [33, Section 5.7], by considering $wt_iw^{-1}(\lambda)$ for some $\lambda \in V$.

Next, we show that when Q is a quiver of mutation-Dynkin type $A_{n_1} \sqcup ... \sqcup A_{n_r}$, we can explicitly obtain a positive companion basis for Q directly from the braid graph of Q.

Take $V = \mathbb{R}^{n+r}$ with basis $\{e_1, ..., e_{n+r}\}$. We have seen in earlier chapters that the set $\Phi = \Phi_1 \sqcup \ldots \sqcup \Phi_r$, where

$$\Phi_k = \{e_i - e_j : \Sigma_{l=1}^{k-1} n_k + k \leq i \neq j \leq \Sigma_{l=1}^{k} n_k + k\}$$

(taking $\Phi_1 = \{e_i - e_j : 1 \leq i \neq j \leq n_1 + 1\}$) is a root system in V of type $A_{n_1} \sqcup ... \sqcup A_{n_r}$ and has simple system $\Pi = \Pi_1 \sqcup \ldots \sqcup \Pi_r$ where

$$\Pi_k = \{\alpha_i = e_i - e_{i+1} : \Sigma_{l=1}^{k-1} n_k + k \leq i \leq \Sigma_{l=1}^{k} n_k + (k-1)\}$$

(taking $\Pi_1 = \{e_i - e_{i+1} : 1 \leq i \leq n_1\}$). To explicitly construct a positive companion basis $C \subseteq \Phi$ of Q from the braid graph of Q, we need to establish the following.

Definition 5.3.10. *[1, Section 1.2] Let $\overrightarrow{\Gamma}$ be an oriented graph. For a vertex, v, of $\overrightarrow{\Gamma}$, the **in-degree** of v, denoted by $deg^-(v)$, is the number of arrows in $\overrightarrow{\Gamma}$ whose head is v.*

*Similarly, the **out-degree** of v, denoted by $deg^+(v)$, is the number of arrows in $\overrightarrow{\Gamma}$ whose tail is v. If $deg^-(v) = 0$, v is called a **source** and if $deg^+(v) = 0$, v is called a **sink**.*

Lemma 5.3.11. *For any $n \in \mathbb{N}$, $n \geq 2$, every tree, Γ, on n vertices has an orientation, $\overrightarrow{\Gamma}$, such that every vertex is either a source or a sink.*

Proof. We proceed by induction on n. If Γ is a connected tree on two vertices, v and v', joined by an edge E, choose an orientation of Γ such that E has tail v' and head v. So v' is a source and v is a sink in this orientation.

Suppose Γ is a tree on $k \in \mathbb{N}$ vertices and that every tree on fewer vertices has an orientation in which every vertex is either a source or a sink.

By [51, Corollary 4.3], as Γ is a tree it contains a vertex, v, of degree one.

Consider the subgraph, Γ', of Γ obtained by deleting the vertex v and the adjacent edge. By induction, this subgraph has an orientation, $\overrightarrow{\Gamma'}$, in which every vertex is either a source or a sink.

As v is of degree one, there is a unique vertex v' of Γ such that v and v' are joined by an edge, E. We take $\overrightarrow{\Gamma}$ to be the orientation induced by $\overrightarrow{\Gamma'}$ together with the following orientation of E. If v' is a source in $\overrightarrow{\Gamma'}$, orientate E with tail v' and head v. So v' is a source and v is a sink in $\overrightarrow{\Gamma}$. If v' is a sink in $\overrightarrow{\Gamma'}$, orientate E with head v' and tail v. So v' is a sink and v is a source in $\overrightarrow{\Gamma}$. Thus $\overrightarrow{\Gamma}$ is such that every vertex is either a source or a sink.

□

Let Γ_Q be the braid graph of Q with some fixed labelling of the vertices by $\{1,...,n+r\}$. As Γ_Q is a disjoint union of connected trees, by Lemma 5.3.11, we can choose an orientation, $\overrightarrow{\Gamma}_Q$, of Γ_Q such that every vertex is either a sink or a source. By construction, the arrows of $\overrightarrow{\Gamma}_Q$ are in bijection with the vertices of Q. Let $E_i = (x_i, y_i)$ be the arrow of $\overrightarrow{\Gamma}_Q$ corresponding to the vertex i of Q, where $x_i \in \{1,...,n+r\}$ is the tail of E_i and $y_i \in \{1,...,n+r\}$ is the head of E_i. The following lemma describes how to obtain a positive companion basis, $C = \{\gamma_i : 1 \leq i \leq n\}$, for Q, where each γ_i is defined by the endpoints of the arrow E_i in $\overrightarrow{\Gamma}_Q$.

Lemma 5.3.12. *Let Q be a quiver of mutation-Dynkin type $A_{n_1} \sqcup ... \sqcup A_{n_r}$, where $n = \Sigma_{i=1}^{r} n_i$, and let $\overrightarrow{\Gamma}_Q$ be an orientation of the braid graph of Q with a fixed labelling of the vertices by the set $\{1,...,n+r\}$ such that every vertex is either a sink or a source. Let $E_i = (x_i, y_i)$ be the arrow in $\overrightarrow{\Gamma}_Q$ corresponding to the vertex i of Q, where x_i is the tail of*

E_i and y_i is the head of E_i. Then

$$C = \{\gamma_i = e_{x_i} - e_{y_i} : 1 \leq i \leq n\}$$

is a positive companion basis of Q.

Proof. For $C = \{\gamma_i = e_{x_i} - e_{y_i} : 1 \leq i \leq n\}$ to be a positive companion basis of Q, we need to show that C satisfies three conditions:

(1) $C \subseteq \Phi$.

(2) For all $1 \leq i \neq j \leq n$,

$$(\gamma_i, \gamma_j) = \begin{cases} 1 & \text{if } i \text{ and } j \text{ are joined by an arrow in } Q, \\ 0 & \text{if } i \text{ and } j \text{ are not joined by an arrow in } Q. \end{cases}$$

(3) C is a $\mathbb{Z}$-basis for $\mathbb{Z}\Phi$.

Take $1 \leq i \leq n$ and consider the arrow $E_i = (x_i, y_i)$ in $\overrightarrow{\Gamma}_Q$. As x_i and y_i are joined by an arrow, they lie in the same connected component, $\overrightarrow{\Gamma}_Q^k$, of $\overrightarrow{\Gamma}_Q$. Thus $x_i, y_i \in \{\sum_{l=1}^{k-1} n_l + k, ..., \sum_{l=1}^{k} n_l + k\}$. So $\pm(e_{x_i} - e_{y_i}) \in \Phi_k \subseteq \Phi$, meaning $\gamma_i \in \Phi$ for all $1 \leq i \leq n$. So condition (1) is met.

Suppose $1 \leq i \neq j \leq n$ are connected by an arrow in Q. By construction, the arrows in Q are in bijection with the vertices of $\overrightarrow{\Gamma}_Q$. Let v be the vertex in Γ_Q corresponding to the arrow joining i and j in Q. Then v is either a sink or a source in $\overrightarrow{\Gamma}_Q$. Supposing v is a sink, then $E_i = (v_i, v)$ and $E_j = (v_j, v)$ where v, v_i, v_j are pairwise distinct vertices of $\overrightarrow{\Gamma}_Q$. Thus $\gamma_i = e_{v_i} - e_v$ and $\gamma_j = e_{v_j} - e_v$ and so $(\gamma_i, \gamma_j) = 1$.

Supposing v is a source, then $E_i = (v, v_i)$ and $E_j = (v, v_j)$. Thus $\gamma_i = e_v - e_{v_i}$ and $\gamma_j = e_v - e_{v_j}$ and so $(\gamma_i, \gamma_j) = 1$.

Suppose $1 \leq i \neq j \leq n$ are not connected by an arrow in Q. Then the edges E_i and E_j share no common vertex and so $\gamma_i = e_{v_i} - e_{u_i}$ and $\gamma_j = e_{v_j} - e_{u_j}$ for pairwise distinct vertices v_i, u_i, v_j, u_j. So $(\gamma_i, \gamma_j) = 0$. So condition (2) is met.

As $|C| = n$, to show that C is a $\mathbb{Z}$-basis for $\mathbb{Z}\Phi$ we only need to show that C spans $\mathbb{Z}\Phi$. That is, every element of Φ can be written as a linear combination of elements in C with integer coefficients.

For ease of notation, suppose that $r = 1$. For any $w \in W$ and $1 \leq i \leq n$, $w(\alpha_i) = e_{\sigma(w)(i)} - e_{\sigma(w)(i+1)}$, by Remark 3.2.2.

So $\sigma(w)(i) = x_0$ and $\sigma(w)(i+1) = x_m$ for some distinct $x_0, x_m \in \{1, ..., n+1\}$. So x_0 and x_m are vertices in the braid graph of Q and there exists a unique shortest path between them. Suppose this path is given by the sequence of vertices $x_0, x_1, x_2, ...x_{m-1}, x_m$. For each $0 \leq l \leq m-1$, denote the edge $\{x_l, x_{l+1}\}$ in Γ_Q by E_{j_l}.

Assuming the vertex x_0 is a source in the oriented graph $\overrightarrow{\Gamma}_Q$, we have

$$\begin{aligned}
\gamma_{j_0} &= e_{x_0} - e_{x_1} \\
\gamma_{j_1} &= e_{x_2} - e_{x_1} \\
\gamma_{j_2} &= e_{x_2} - e_{x_3} \\
&\vdots \\
\gamma_{j_{m-1}} &= \begin{cases} e_{x_{m-1}} - e_{x_m}, & \text{if } m \text{ is odd} \\ e_{x_m} - e_{x_{m-1}}, & \text{if } m \text{ is even} \end{cases}
\end{aligned}$$

Then

$$\begin{aligned}
w(\alpha_i) = e_{x_0} - e_{x_m} &= (e_{x_0} - e_{x_1}) - (e_{x_1} - e_{x_2}) + ... \pm (e_{x_{m-1}} - e_{x_m}) \\
&= \gamma_{j_0} - \gamma_{j_1} + \gamma_{j_2} - ... \pm \gamma_{j_{m-1}}.
\end{aligned}$$

If the vertex x_0 is a sink in the oriented graph $\overrightarrow{\Gamma}_Q$, then $w(\alpha_i) = -\gamma_{j_0} + \gamma_{j_1} - \gamma_{j_2} + ... \pm \gamma_{j_{m-1}}$. Thus C spans $\mathbb{Z}\Phi$. □

Remark 5.3.13. As C is a companion basis we have that $C \subseteq \Phi$ and so $\Phi_C \subseteq \Phi$. Moreover, as $t_{j_{m-1}}...t_{j_1}(\gamma_{j_0}) = \gamma_{j_0} - \gamma_{j_1} + \gamma_{j_2} - ... \pm \gamma_{j_{m-1}}$ (and $-t_{j_{m-1}}...t_{j_1}(\gamma_{j_0}) = -\gamma_{j_0} + \gamma_{j_1} - \gamma_{j_2} + ... \pm \gamma_{j_{m-1}}$), the proof of Lemma 5.3.12 demonstrates that $\Phi_C = \Phi$.

5.4 Associated Roots

Let w be an element of the finite Coxeter system (W, S). In Section 1.4 we defined a set of roots $\{\beta_j : 1 \leq j \leq r\}$ associated to a given expression $w = s_{i_1}...s_{i_r}$, where $\beta_{i_r} = \alpha_{i_r}$ and $\beta_j = s_{i_r} s_{i_{r-1}} ... s_{i_{j+1}}(\alpha_{i_j})$ for each $1 \leq j \leq r-1$. Lemma 1.4.10 showed how these roots provide an insight into the given expression of $w \in W$. For instance, the expression is reduced if and only if $\beta_j \neq \beta_k$ for all $1 \leq j, k \leq r$. Moreover, $l(w) = |\{\beta_j : 1 \leq j \leq r\}|$. In particular, it can be shown that this set of roots is precisely the set of positive roots, α, such that $w(\alpha)$ is negative. It follows that the set $\{\beta_j : 1 \leq j \leq r\}$ remains the same no matter which expression we take for $w \in W$.

In this section we will use the positive companion basis established in the previous section to define a similar set of roots associated to an expression of an element, w, of a cluster group of mutation-Dynkin type $A_{n_1} \sqcup ... \sqcup A_{n_r}$. We will show that the set of these roots can vary depending on the expression of w and that if a given expression is reduced, then the associated roots will be distinct. However, we will provide a counterexample to demonstrate that the converse of this statement is not true.

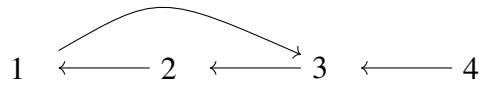
Definition 5.4.1. *Let Q be a quiver of mutation-Dynkin type $A_{n_1} \sqcup ... \sqcup A_{n_r}$ with the positive companion basis, $C = \{\gamma_i = e_{x_i} - e_{y_i} : 1 \leq i \leq n\}$, shown to exist in Lemma* 5.3.12. *For $w \in G_Q$, the **roots associated to the expression** $w = t_{i_1}...t_{i_k}$ are the roots*

$$\beta_j = t_{i_k} t_{i_{k-1}} ... t_{i_{j+1}}(\gamma_{i_j}),$$

for each $1 \leq j \leq k$, where $\beta_{i_k} = \gamma_{i_k}$. We denote the set $\{\beta_{i_j} : 1 \leq j \leq k\} \subseteq \Phi_C$ by $\Phi(t_{i_1}...t_{i_k})$.

In general, the associated roots will vary over the expressions for w, as shown in the following example.

Example 5.4.2. Take Q to be the following quiver of mutation-Dynkin type A_4:



As $t_2^2 = e$ and $t_2 t_1 t_3 t_2 = t_1 t_3 t_2 t_1$, by the cycle relation, we have that:

$$t_1 t_2 t_1 t_3 = t_1 t_2 t_1 t_3 t_2 t_2 = t_1 t_1 t_3 t_2 t_1 t_2 = t_3 t_2 t_1 t_2.$$

However, taking $t_1 t_2 t_1 t_3$ we have

$$\begin{aligned}
\beta_1 &= t_3 t_1 t_2(\gamma_1) = \gamma_3 - \gamma_2 \\
\beta_2 &= t_3 t_1(\gamma_2) = \gamma_2 - \gamma_1 \\
\beta_3 &= t_3(\gamma_1) = \gamma_3 - \gamma_1 \\
\beta_4 &= \gamma_3
\end{aligned}$$

So $\Phi(t_1 t_2 t_1 t_3) = \{\gamma_3 - \gamma_2, \gamma_2 - \gamma_1, \gamma_3 - \gamma_1, \gamma_3\}$, while $\Phi(t_3 t_2 t_1 t_2) = \{\gamma_2, \gamma_1 - \gamma_2, -\gamma_1, \gamma_3\}$.

Proposition 5.4.3. *Let Q be a quiver of mutation-Dynkin type $A_{n_1} \sqcup ... \sqcup A_{n_r}$ and take $w \in G_Q$. If $w = t_{i_1}...t_{i_k}$ is a reduced expression, then $|\Phi(t_{i_1}...t_{i_k})| = k$.*

Proof. Suppose $w = t_{i_1}...t_{i_k}$ is a reduced expression of $w \in G_Q$. If $|\Phi(t_{i_1}...t_{i_k})| \neq k$ then $\beta_{i_j} = \beta_{i_l}$ for some $j \neq l$. Without loss of generality, we can assume that $j < l$. By definition, this means $t_{i_k}t_{i_{k-1}}...t_{i_{j+1}}(\gamma_{i_j}) = t_{i_k}t_{i_{k-1}}...t_{i_{l+1}}(\gamma_{i_j})$ which implies $\gamma_{i_l} = t_{i_l}...t_{i_{j+1}}(\gamma_{i_j})$.

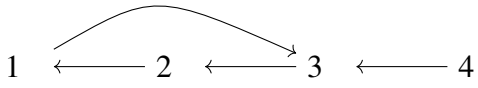
By Remark 5.3.9, we have $t_{i_l} = (t_{i_l}...t_{i_{j+1}})t_{i_j}(t_{i_l}...t_{i_{j+1}})^{-1}$, giving $t_{i_j}t_{i_{j+1}}...t_{i_{l-1}} = t_{i_{j+1}}...t_{i_l}$. Thus

$$w = t_{i_1}...t_{i_j}...t_{i_{l-1}}t_{i_l}...t_{i_k} = t_{i_1}...t_{i_{j-1}}(t_{i_{j+1}}...t_{i_l})t_{i_l}...t_{i_k} = t_{i_1}...\hat{t}_{i_j}...\hat{t}_{i_l}...t_{i_k},$$

contradicting that $t_{i_1}...t_{i_k}$ was a reduced expression. □

To show that the converse of Proposition 5.4.3 is not true, we provide the following counterexample.

Example 5.4.4. Take Q to be the following quiver of mutation-Dynkin type A_4:



Consider the roots associated to the following reduced expression $w = t_1t_2t_1t_3t_1$. This is not a reduced expression, as we saw in Example 5.4.2 that $t_1t_2t_1t_3 = t_3t_2t_1t_2$ and so (using that $t_2t_1t_2 = t_1t_2t_1$ and $t_1^2 = e$) we have $w = t_1t_2t_1t_3t_1 = t_3t_2t_1t_2t_1 = t_3t_1t_2t_1t_1 = t_3t_1t_2$, so $l(w) \leq 3$. However, $\Phi(t_1t_2t_1t_3t_1) = \{\gamma_1, \gamma_3 - \gamma_1, -\gamma_3, \gamma_2, \gamma_3 - \gamma_2\}$, so $|\Phi(t_1t_2t_1t_3t_1)| = 5 > 3 \geq l(w)$.

Let $w \in G_Q$ and take any expression $w = t_{i_1}...t_{i_k}$. We define the transposition

$$\tau_j = (\pi_Q(t_{i_k}...t_{i_{j-1}})(x_{i_j}), \pi_Q(t_{i_k}...t_{i_{j-1}})(y_{i_j})).$$

By Remark 3.2.2, we have that $\beta_j = e_{\tau_j(x_{i_j})} - e_{\tau_j(y_{i_j})}$. As shown in the following proposition, the product of these transpositions over $1 \leq j \leq k$ yields the permutation $\pi_Q(t_{i_1}...t_{i_k})$. This result is analogous to [4, Section 1.3].

Proposition 5.4.5. $\pi_Q(t_{i_1}...t_{i_k}) = \tau_k...\tau_1$.

Proof. We proceed by induction on k. For ease, we will denote π_Q by π. If $k = 1$ then $\pi(t_{i_1}) = (x_{i_1}, y_{i_1})$, so $\pi(t_{i_1}) = (x_{i_1}, y_{i_1}) = \tau_1$.

Now suppose $k > 1$ and that the statement is true for all expressions of shorter length. Consider $w' = wt_{i_k} = t_{i_1}...t_{i_{k-1}} = t_{i'_1}...t_{i'_{k'}}$, where $k' = k-1$ and $i'_j = i_j$ for each $1 \leq j \leq k'$. To distinguish between the expressions, for $1 \leq j \leq k-1$ let

$$\tau'_j = (\pi(t_{i'_{k'}}...t_{i'_{j-1}})(x_{i_j}), \pi(t_{i'_{k'}}...t_{i'_{j-1}})(y_{i_j})) = (\pi(t_{i_{k-1}}...t_{i_{j-1}})(x_{i_j}), \pi(t_{i_{k-1}}...t_{i_{j-1}})(y_{i_j}))$$

and, for each $1 \leq j \leq k$,

$$\tau_j = (\pi(t_{i_k}...t_{i_{j-1}})(x_{i_j}), \pi(t_{i_k}...t_{i_{j-1}})(y_{i_j})).$$

Comparing τ'_j and τ_j, we have

$$\begin{aligned}
\tau_j &= (\pi(t_{i_k}...t_{i_{j-1}})(x_{i_j}), \pi(t_{i_k}...t_{i_{j-1}})(y_{i_j})) \\
&= (\pi(t_{i_k})\pi(t_{i_{k-1}}...t_{i_{j-1}})(x_{i_j}), \pi(t_{i_k})\pi(t_{i_{k-1}}...t_{i_{j-1}})(y_{i_j})) \\
&= ((x_{i_k}, y_{i_k})\pi(t_{i_{k-1}}...t_{i_{j-1}})(x_{i_j}), (x_{i_k}, y_{i_k})\pi(t_{i_{k-1}}...t_{i_{j-1}})(y_{i_j})) \\
&= (x_{i_k}, y_{i_k})(\pi(t_{i_{k-1}}...t_{i_{j-1}})(x_{i_j}), \pi(t_{i_{k-1}}...t_{i_{j-1}})(y_{i_j}))(x_{i_k}, y_{i_k}) \\
&= \pi(t_{i_k})\tau'_j\pi(t_{i_k})
\end{aligned}$$

By induction,

$$\pi(t_{i_1}...t_{i_{k-1}}) = \tau'_{k-1}...\tau'_1.$$

Thus, noting that $\tau_k = (x_{i_k}, y_{i_k}) = \pi(t_{i_k})$, we have

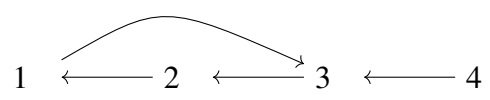
$$\begin{aligned}
\tau_k\tau_{k-1}...\tau_1 &= \tau_k\pi(t_{i_k})\tau'_{k-1}...\tau'_1\pi(t_{i_k}) = \tau_k\pi(t_{i_k})\pi(t_{i_1}...t_{i_{k-1}})\pi(t_{i_k}) \\
&= \pi(t_{i_k})\pi(t_{i_k})\pi(t_{i_1}...t_{i_{k-1}})\pi(t_{i_k}) = \pi(t_{i_k}^2)\pi(t_{i_1}...t_{i_{k-1}}t_{i_k}) \\
&= \pi(t_{i_1}...t_{i_{k-1}}t_{i_k}).
\end{aligned}$$

So by the principle of mathematical induction, the statement holds for expressions of any length.

□

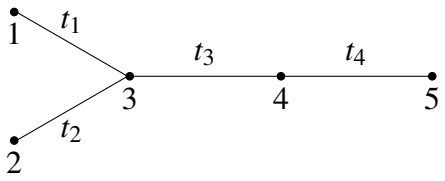
Remark 5.4.6. An alternative proof for Proposition 5.4.5, using an argument analogous to [4, Section 1.3] in the Coxeter case, can be given as follows. Given an expression $t_{i_1}...t_{i_k}$, define $t'_j = w_j t_{i_j} w_j^{-1}$, where $w_j = t_{i_1}...t_{i_{j-1}}$ for each $1 \leq j \leq k$. Thus $t_{i_1}...t_{i_k} = t'_{i_k}t'_{i_{k-1}}...t'_{i_1}$ and so $\pi_Q(t_{i_1}...t_{i_k}) = \pi_Q(t'_{i_k}t'_{i_{k-1}}...t'_{i_1}) = \pi_Q(t'_{i_k})\pi_Q(t'_{i_{k-1}})...\pi_Q(t'_{i_1})$. For any $1 \leq j \leq k$, we have that $\pi_Q(t'_{i_j}) = \pi_Q(w_j t_{i_j} w_j^{-1}) = \pi_Q(w_j)(x_{i_j}, y_{i_j})\pi_Q(w_j^{-1}) = (\pi_Q(w_j)(x_{i_j}), \pi_Q(w_j)(y_{i_j})) = \tau_j$. Thus Proposition 5.4.5 follows.

Example 5.4.7. Let Q be the quiver



So Q is a quiver of mutation-Dynkin type A_4. Let us take the following labelling of Γ_Q by the set $\{1,2,3,4,5\}$.

Γ_Q:



So there exists a group isomorphism

$$\pi : G_Q \longrightarrow Y(\rho_Q),$$
$$\pi : t_1 \longmapsto (1,3),$$
$$\pi : t_2 \longmapsto (2,3),$$
$$\pi : t_3 \longmapsto (3,4),$$
$$\pi : t_4 \longmapsto (4,5),$$

Take $w = t_1t_2t_1t_3 \in G_Q$. Then $\pi(w) = (1,3)(2,3)(1,3)(3,4) = (1,2)(3,4)$.

Moreover,

$$\tau_1 = (\pi(t_3t_1t_2)(1), \pi(t_3t_1t_2)(3)) = (4,2)$$
$$\tau_2 = (\pi(t_3t_1)(2), \pi(t_3t_1)(3)) = (1,2)$$
$$\tau_3 = (\pi(t_3)(1), \pi(t_3)(3)) = (1,4)$$
$$\tau_4 = (3,4)$$

So $\tau_1\tau_2\tau_3\tau_4 = (4,2)(1,2)(1,4)(3,4) = (1,2)(3,4) = \pi(w)$.

Chapter 6

A Lattice Isomorphism Theorem

6.1 Introduction

Suppose Q is a quiver of mutation-Dynkin type with connected components $Q_1, ..., Q_r$ of types $A_{n_1}, ..., A_{n_r}$, respectively, where $n_1, ..., n_r \in \mathbb{Z}^+$ and $n = \Sigma_{i=1}^{r} n_i$.

Let $\mathscr{I}$ denote the set of subsets of the generating set, T, of G_Q. Given a cluster quiver Q and $I \in \mathscr{I}$, we use G_I to denote the subgroup of the cluster group G_Q generated by I and we call any subgroup of G_Q obtained in this way a (standard) parabolic subgroup. In this chapter, we will prove two main results about parabolic subgroups of cluster groups of mutation-Dynkin type $A_{n_1} \sqcup ... \sqcup A_{n_r}$. In the first section, we will prove the following.

Theorem. *(Theorem* 6.2.9*) Suppose Q is a quiver of mutation-Dynkin type with connected components of mutation-Dynkin type A and consider the associated cluster group $G_Q = \langle T|R\rangle$. There exists a lattice isomorphism*

$$\Phi : \mathscr{I} \longrightarrow \tilde{\mathscr{G}},$$

$$\Phi : I \longmapsto G_I,$$

where $\mathscr{I}$ is the collection of subsets of the set of defining generators T and $\tilde{\mathscr{G}}$ denotes the collection of all the parabolic subgroups of G_Q (i.e. subgroups generated by a subset $I \subseteq T$).

In the case when the quiver is an oriented simply-laced Dynkin diagram, Δ, the associated cluster group presentation is precisely the Coxeter presentation of type Δ. By [3] and [28, Lemma 2.5], the cluster group presentations associated to mutation-Dynkin quivers give groups which are isomorphic to finite reflection groups and thus are finite Coxeter groups. Alternatively, as shown in Lemma 4.2.7, this can also be shown directly by a

simple extension of [28, Proposition 2.9]. Under this isomorphism, the defining generators of G_Q are mapped to elements of the set of reflections, $\{ws_iw^{-1} : w \in W_\Delta, s_i \in S\}$, of the Coxeter system (W_Δ, S). Thus for any $I \subseteq T$, the parabolic subgroup G_I is isomorphic to a reflection subgroup of W_Δ. Reflection subgroups of Coxeter groups are themselves Coxeter groups [15].

Each finite Coxeter group has an associated root system and conversely each root system defines a Coxeter group (see [33]). Suppose Δ is a Dynkin diagram of type A_n and W_Δ is the corresponding Coxeter group with root system Φ. For any reflection subgroup W' of W_Δ, the subset

$$\Psi = \{\alpha : s_\alpha \in W'\} \subseteq \Phi$$

is a subsystem of Φ and W' is the Coxeter group defined by Ψ [16]. By [16, Corollary 1], the Dynkin diagram of Ψ will be of type $A_{n'}$ for some $n' \leq n$. Thus W' will be of type $A_{n'}$. It follows that, when Q is of mutation-Dynkin type A_n, each parabolic subgroup of G_Q is isomorphic to a cluster group associated to a quiver of type $A_{n'}$, for some $n' \leq n$. In fact, our second main result shows that each parabolic subgroup has a cluster group presentation given by restricting the presentation of the whole group.

Given any $I \subseteq T$, let Q_I denote the full subquiver of Q on the vertices corresponding to the elements of I. To distinguish between the defining generators of G_Q and G_{Q_I}, we label the generators of the cluster presentation of G_{Q_I} by t_i', for each $t_i \in I$.

Theorem. *(Theorem* 6.3.1*) Suppose Q is a quiver of mutation-Dynkin type with connected components of mutation-Dynkin type A and consider the associated cluster group $G_Q = \langle T|R\rangle$. For any $I \subseteq T$, there exists an isomorphism*

$$G_{Q_I} \longrightarrow G_I,$$
$$t_i' \longmapsto t_i,$$

where G_I denotes the subgroup of G_Q generated by I and G_{Q_I} is the cluster group associated to Q_I, the full subquiver of Q on the vertices corresponding to the elements of I, where the generators of the cluster presentation of G_{Q_I} are labelled by t_i', for each $t_i \in I$.

This theorem is our second main result of this chapter, and will be proved in the second section.

There are many well-established results for Coxeter presentations and, as previously discussed, we are interested in whether cluster groups possess properties comparable to those of Coxeter groups. Theorem 6.2.9 is analogous to the following theorem for Coxeter groups.

Theorem 6.1.1. *[33, Theorem 5.5(c)] Let (W,S) be a Coxeter system. The assignment $I \mapsto W_I$ defines a lattice isomorphism between the collection of subsets of S and the collection of parabolic subgroups W_I of W.*

Similarly, analogously to our second main result, any parabolic subgroup of a Coxeter group has a Coxeter presentation [33, Theorem 5.5(a)].

6.2 A Lattice Isomorphism Theorem

In [33], the proof of Theorem 6.1.1 examines how an element of the finite Coxeter group acts on the corresponding root system. We approach the problem of proving an analogous lattice isomorphism theorem for cluster groups of mutation-Dynkin type $A_{n_1} \sqcup ... \sqcup A_{n_r}$ in a different way. Our method will require the definitions and results of set partitions and Young subgroups provided in Chapter 3.

Suppose Q is a quiver of mutation-Dynkin type with connected components $Q_1, ..., Q_r$ of types $A_{n_1}, ..., A_{n_r}$, respectively, where $n_1, ..., n_r \in \mathbb{Z}^+$ and $n = \Sigma_{i=1}^{r} n_i$. Take $\mathscr{T}$ to be a triangulation of the disjoint union of (n_i+3)-gons giving rise to Q and consider the braid graph, Γ_Q, of Q. Fix a labelling of Γ_Q by the set $\{1, ..., n+r\}$. By construction, there is a bijection between the set of generators of G_Q and the set of edges of Γ_Q and recall that E_i represents the edge in Γ_Q corresponding to the vertex i of Q for each $1 \leq i \leq n$. Moreover, from the chosen labelling of the braid graph, we obtain a set partition ρ of $\{1, ..., n+r\}$ by taking the parts of ρ to be the vertex sets of the connected components of Γ_Q. Suppose E_i has endpoints $x_i, y_i \in \{1, 2, ..., n+r\}$ (with $x_i \neq y_i$). Then, by Lemma 5.2.2, there exists an isomorphism

$$\pi_Q : G_Q \longrightarrow Y(\rho),$$
$$\pi_Q : t_i \longmapsto (x_i, y_i).$$

Thus, as shown in Lemma 4.2.10, the edges of the braid graph are in bijection with the defining generators of G_Q. Moreover, $t_i \neq t_j$ for any distinct vertices i and j of Q. That is, there is a bijection between the vertices of Q and the defining generators of G_Q.

Let $\mathscr{I}$ be the power set of $T = \{t_1, ..., t_n\}$. This is a lattice under inclusion. Moreover, let $\mathscr{G}$ be the set of subgroups of G_Q (so $\mathscr{G}$ is a lattice under inclusion) and let $\tilde{\mathscr{G}} = \{G_I : I \in \mathscr{I}\}$, i.e. $\tilde{\mathscr{G}}$ is the collection of parabolic subgroups of G_Q. Let $\mathscr{P}$ be the collection of set partitions of $\{1, ..., n+r\}$ and $\mathscr{Y}$ the set of Young subgroups of Σ_{n+r}. By Example 3.3.2, $\mathscr{P}$ is a lattice under the refinement ordering. By Proposition 3.3.7, the set $\mathscr{Y}$ is a

lattice under inclusion and there exists a lattice isomorphism

$$\psi : \mathscr{P} \longrightarrow \mathscr{Y},$$
$$\psi : \rho \longmapsto Y(\rho).$$

Definition 6.2.1. *Given $I \in \mathscr{I}$ we obtain a set partition $\rho_I \in \mathscr{P}$ in the following way. Let Γ_I be the graph obtained from Γ_Q by deleting all edges E_i such that $t_i \notin I$. As Γ_Q is a connected tree, Γ_I will consist of some connected components, $\Gamma_I^1, \Gamma_I^2, ..., \Gamma_I^k$, each of which is a full subgraph of Γ_Q. We define $\rho_I = \sqcup_{j=1}^k \alpha_j$ where α_j is the vertex set of the connected component Γ_I^j.*

Let $\tilde{\mathscr{P}} = \{\rho_I : I \in \mathscr{I}\}$ and $\tilde{\mathscr{Y}} = \{Y(\rho) : \rho \in \tilde{\mathscr{P}}\}$. In this section, we will show that there exists a lattice isomorphism:

$$\phi : \mathscr{I} \overset{\text{Lem 6.2.7}}{\longrightarrow} \tilde{\mathscr{Y}},$$
$$\phi : I \longmapsto Y(\rho_I).$$

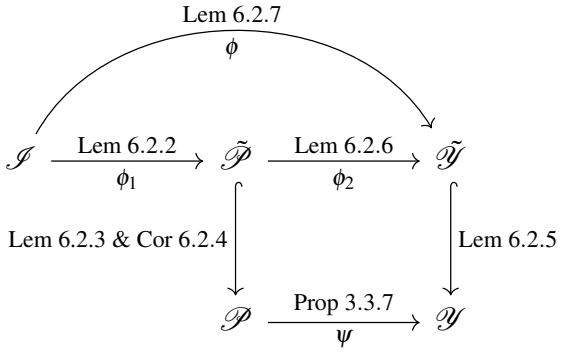
To do this, we will show there exists an order isomorphism:

$$\phi_1 : \mathscr{I} \overset{\text{Lem 6.2.2}}{\longrightarrow} \tilde{\mathscr{P}},$$
$$\phi_1 : I \longmapsto \rho_I$$

and prove that $\tilde{\mathscr{P}}$ is a sublattice of $\mathscr{P}$ under the refinement ordering. Thus, by Proposition 3.3.5, ϕ_1 will be a lattice isomorphism. We will further show that $\tilde{\mathscr{Y}}$ is a sublattice of $\mathscr{Y}$ under inclusion and use Proposition 3.3.7 to show the following map is a lattice isomorphism.

$$\phi_2 : \tilde{\mathscr{P}} \overset{\text{Lem 6.2.6}}{\longrightarrow} \tilde{\mathscr{Y}},$$
$$\phi_2 : \rho_I \longmapsto Y(\rho_I).$$

By composing ϕ_1 and ϕ_2, we obtain the desired lattice isomorphism between $\mathscr{I}$ and $\tilde{\mathscr{Y}}$. This approach is summarised in the following diagram.



Finally, we will show that, for each $I \in \mathscr{I}$, the parabolic subgroup G_I is isomorphic to the Young subgroup $Y(\rho_I)$ and use this to prove that a lattice isomorphism exists between $\mathscr{I}$ and $\tilde{\mathscr{G}}$.

Lemma 6.2.2. *There exists an order-isomorphism.*

$$\phi_1 : \mathscr{I} \longrightarrow \tilde{\mathscr{P}},$$
$$\phi_1 : I \longmapsto \rho_I.$$

Proof. To show that ϕ_1 is an order isomorphism, we must show that for any $I, J \in \mathscr{I}$, $I \subseteq J$ if and only if $\rho_I \leq \rho_J$.

Suppose $I \subseteq J$. Then Γ_I is a subgraph of Γ_J, meaning the vertex set of each connected component of Γ_I is a subset of the vertex set of some component of Γ_J and so $\rho_I \leq \rho_J$.

If $\rho_I \leq \rho_J$ then each part of ρ_I is a subset of a part of ρ_J, meaning the vertex set of each connected component of Γ_I is a subset of the vertex set of some component of Γ_J. Recall Γ_I is obtained by deleting all edges of Γ_Q corresponding to all $t_i \notin I$. If there existed some $t_i \in I \setminus J$, then E_i would lie in a connected component of Γ_I, so the vertex set of this component would contain the endpoints of E_i. However, as $t_i \notin J$, E_i would not lie in any connected component of Γ_J. Thus the endpoints of E_i would lie in separate connected components of Γ_J, contradicting the fact that the vertex set of each connected component of Γ_I is a subset of the vertex set of some component of Γ_J. Thus $I \subseteq J$.

□

We go on to show that $\tilde{\mathscr{P}}$ is a lattice and so ϕ_1 is a lattice isomorphism.

Lemma 6.2.3. *The set $\tilde{\mathscr{P}}$ is a lattice under the refinement ordering.*

Proof. We prove that $\tilde{\mathscr{P}}$ is a lattice under refinement by showing that it is a sublattice of $\mathscr{P}$.

By definition, to show that $\tilde{\mathscr{P}}$ is a sublattice, we must show that for any $I, J \in \mathscr{I}$, $\rho_I \vee \rho_J, \rho_I \wedge \rho_J \in \tilde{\mathscr{P}}$. We show that

$$\rho_{I \cap J} = \rho_I \wedge \rho_J, \tag{6.1}$$
$$\rho_{I \cup J} = \rho_I \vee \rho_J. \tag{6.2}$$

For ease of notation we will write R_K for the equivalence relation corresponding to the partition $\rho_K \in \tilde{\mathscr{P}}$. Noting Example 3.3.2 (2.), we need to show that $\rho_{I \cap J}$ is precisely

the partition corresponding to the equivalence relation $R_I \cap R_J$ and $\rho_{I\cup J}$ is precisely the partition corresponding to the equivalence relation $t(R_I \cup R_J)$.

Let $\rho_I = \sqcup_{i=1}^{a} \alpha_i$, $\rho_J = \sqcup_{j=1}^{b} \alpha'_j$ and $\rho_{I\cap J} = \sqcup_{l=1}^{c} \beta_l$. First, we show that for all $x, y \in \{1, ..., n+r\}$,

$$xR_{I\cap J}y \Leftrightarrow x(R_I \cap R_J)y.$$

Recall that for any $K \in \mathscr{I}$, ρ_K is obtained by deleting the edges in Γ_Q corresponding to the t_i not in K then taking the parts of ρ_K to be the vertex sets of the connected components of this resulting graph, denoted by Γ_K.

As $I \cap J \subseteq I, J$ and as Γ_Q is a tree, for any two distinct vertices x and y lying in the same connected component of $\Gamma_{I\cap J}$, the unique path from x to y in Γ_Q must consist only of edges corresponding to some $t_i \in I \cap J$. Therefore there exists a path from x to y in both Γ_I and Γ_J. Thus a connected component in $\Gamma_{I\cap J}$ is a subgraph of some connected component in both Γ_I and in Γ_J. So $xR_{I\cap J}y$ implies $x(R_I \cap R_J)y$.

Conversely, suppose two distinct vertices x and y lie in the same component in Γ_I and the same component in Γ_J. Then there exists a path in Γ_Q between x and y consisting only of edges corresponding to some $t_i \in I$ and a path between x and y consisting only of edges corresponding to some $t_i \in J$. However, Γ_Q is a tree, meaning any existing path between x and y is unique. Thus the edges in the path between x and y in Γ_Q consist only of edges corresponding to some $t_i \in I \cap J$, giving that x and y lie in the same connected component of $\Gamma_{I\cap J}$. So $x(R_I \cap R_J)y$ implies $xR_{I\cap J}y$.

As the equivalence relation corresponding to $\rho_{I\cap J}$ is precisely $R_I \cap R_J$, it must be that $\rho_{I\cap J} = \rho_I \wedge \rho_J$, so (6.1) is shown.

Finally, we show that for all $x, y \in \{1, ..., n+r\}$,

$$xR_{I\cup J}y \Leftrightarrow x(t(R_I \cup R_J))y.$$

For any $x, y \in \{1, ..., n+r\}$, $xR_{I\cup J}y$ if and only if x and y lie in the same connected component of $\Gamma_{I\cup J}$. This occurs if and only if there exists a unique path in Γ_Q between x and y consisting only of edges corresponding to elements of $I \cup J$. That is, if and only if there exists distinct $z_0, ..., z_m \in \{1, ..., n+r\}$ such that $x = z_0, y = z_m$ and for all $1 \leq i \leq m$ either $t_i \in I$ or $t_i \in J$, where t_i corresponds to the edge in Γ_Q with endpoints (z_{i-1}, z_i). In other words, there exist distinct $z_0, ..., z_m \in \{1, ..., n+r\}$ such that $x = z_0, y = z_m$ and for all $1 \leq i \leq m-1$ either $z_i R_1 z_{i+1}$ or $z_i R_2 z_{i+1}$. So $xR_{I\cup J}y \Leftrightarrow x(t(R_I \cup R_J))y$.

Hence both (6.1) and (6.2) are shown, so $\rho_I \vee \rho_J, \rho_I \wedge \rho_J \in \tilde{\mathscr{P}}$ for all $I, J \in \mathscr{I}$ and so $\tilde{\mathscr{P}}$ is a sublattice of $\mathscr{P}$. □

From Lemma 6.2.2 together with Lemma 6.2.3 and Proposition 3.3.5 we conclude the following.

Corollary 6.2.4. *The set $\tilde{\mathscr{P}}$ is a sublattice of $\mathscr{P}$ isomorphic to $\mathscr{I}$.*

We now show that $\tilde{\mathscr{Y}}$ is a sublattice of $\mathscr{Y}$.

Lemma 6.2.5. *The set $\tilde{\mathscr{Y}}$ is a sublattice of $\mathscr{Y}$.*

Proof. For all $I, J \in \mathscr{I}$,

$$\begin{aligned} Y(\rho_I) \vee Y(\rho_J) &= \psi(\rho_I) \vee \psi(\rho_J) \\ &= \psi(\rho_I \vee \rho_J) \text{ by Proposition 3.3.7} \\ &= \psi(\rho_{I \cup J}) \text{ by Lemma 6.2.3} \\ &= Y(\rho_{I \cup J}) \in \tilde{\mathscr{Y}} \end{aligned}$$

and

$$\begin{aligned} Y(\rho_I) \wedge Y(\rho_J) &= \psi(\rho_I) \wedge \psi(\rho_J) \\ &= \psi(\rho_I \wedge \rho_J) \text{ by Proposition 3.3.7} \\ &= \psi(\rho_{I \cap J}) \text{ by Lemma 6.2.3} \\ &= Y(\rho_{I \cap J}) \in \tilde{\mathscr{Y}}. \end{aligned}$$

Thus $\tilde{\mathscr{Y}}$ is a sublattice of $\mathscr{Y}$. □

Lemma 6.2.6. *There exists a lattice isomorphism*

$$\begin{aligned} &\phi_2 : \tilde{\mathscr{P}} \longrightarrow \tilde{\mathscr{Y}}, \\ &\phi_2 : \rho_I \longmapsto Y(\rho_I). \end{aligned}$$

Proof. It follows from Proposition 3.3.7 that

$$\begin{aligned} &\psi|_{\tilde{\mathscr{P}}} : \tilde{\mathscr{P}} \longrightarrow \mathscr{Y}, \\ &\psi|_{\tilde{\mathscr{P}}} : \rho_I \longmapsto Y(\rho_I) \end{aligned}$$

is an injective lattice homomorphism. As $\tilde{\mathscr{Y}} = im(\psi|_{\tilde{\mathscr{P}}})$, we conclude that $\psi|_{\tilde{\mathscr{P}}}$ induces the lattice isomorphism:

$$\phi_2 : \tilde{\mathscr{P}} \longrightarrow \tilde{\mathscr{Y}},$$
$$\phi_2 : \rho_I \longmapsto Y(\rho_I).$$

□

Thus $\tilde{\mathscr{Y}}$ is a sublattice of $\mathscr{Y}$ isomorphic to $\tilde{\mathscr{P}}$ and so isomorphic to $\mathscr{I}$.

Lemma 6.2.7. *There exists a lattice isomorphism*

$$\phi : \mathscr{I} \longrightarrow \tilde{\mathscr{Y}},$$
$$\phi : I \longmapsto Y(\rho_I).$$

Proof. Let

$$\phi : \mathscr{I} \longrightarrow \tilde{\mathscr{Y}}$$

be the composition $\phi = \phi_2 \circ \phi_1$. Then

$$\phi(I) = Y(\rho_I)$$

for all $I \in \mathscr{I}$. By Lemma 6.2.3 and Lemma 6.2.6, ϕ is a lattice isomorphism. □

Recall that $\mathscr{G}$ is the set of subgroups of G_Q (so $\mathscr{G}$ is a lattice under inclusion) and $\tilde{\mathscr{G}} = \{G_I : I \in \mathscr{I}\}$ (the set of parabolic subgroups of G_Q). Our next aim is to show that $\tilde{\mathscr{G}}$ is a sublattice of $\mathscr{G}$. For this we need the following lemmas.

Lemma 6.2.8. *For $I \in \mathscr{I}$, the map*

$$\pi_Q|_{G_I} : G_I \longrightarrow Y(\rho_I),$$
$$\pi_Q|_{G_I} : t_i \longmapsto (x_i, y_i)$$

is a group isomorphism. That is, $G_I \cong Y(\rho_I)$.

Proof. For ease of notation, we write $\pi|_I$ in place of $\pi_Q|_{G_I}$.

Clearly, $im(\pi|_I) \subseteq Y(\rho_I)$ as for each $t_i \in I$, $E_i = \{x_i, y_i\}$ lies in some connected component Γ_I^j of Γ_I. It follows that $x_i, y_i \in \alpha_j$ and so $\pi|_I(t_i) = (x_i, y_i) \in \Sigma_{\alpha_j} \subseteq Y(\rho_I)$.

Since π_Q is an isomorphism, $\pi|_I$: $G_I \longmapsto Y(\rho_I)$ is injective. Hence it is enough to show that $im(\pi|_I) = Y(\rho_I)$ in order to show that $\pi|_I$: $G_I \longrightarrow Y(\rho_I)$ is an isomorphism.

Let $\rho_I = \sqcup_{j=1}^{k} \alpha_j$. For any $1 \leq j \leq k$, $\alpha_j = \{m_1, ..., m_u\} \subseteq \{1, ..., n+r\}$, we have $\Sigma_{\alpha_j} \cong \Sigma_u$ induced by the map

$$\tilde{\pi} : l \longmapsto m_l$$

As the set of elementary transpositions $\{(l, l+1) : 1 \leq j \leq u-1\}$ generates Σ_u, the set $\{(m_l, m_{l+1}) : 1 \leq j \leq u-1\}$ generates Σ_{α_j}. For any $1 \leq l \leq u-1$, there exists a unique path in Γ_I^j between the vertices m_l and m_{l+1} (as $m_l, m_{l+1} \in \alpha_j$):

$$m_l = x_1 \overset{E_{i_1}}{\text{——}} x_2 \overset{E_{i_2}}{\text{——}} x_3 \overset{E_{i_3}}{\text{——}} ... \overset{E_{i_{p-1}}}{\text{——}} x_p = m_{l+1}$$

and

$$\begin{aligned}(m_l, m_{l+1}) &= (x_1, x_2)(x_2, x_3)...(x_{p-2}, x_{p-1})(x_{p-1}, x_p)(x_{p-2}, x_{p-1}) \\ &...(x_2, x_3)(x_1, x_2) \\ &= \pi \mid_I (t_{i_1} t_{i_2} ... t_{i_{p-1}} t_{i_p} t_{i_{p-1}} ... t_{i_2} t_{i_1})\end{aligned}$$

where $t_{i_q} \in I$ for each $1 \leq q \leq p$, as each edge E_{i_q} lay in Γ_I^j. Thus $\pi \mid_I$ is surjective and so $G_I \cong Y(\rho_I)$. □

Theorem 6.2.9. *Suppose Q is a quiver of mutation-Dynkin type with connected components of mutation-Dynkin type A and consider the associated cluster group $G_Q = \langle T | R \rangle$. Then the collection of parabolic subgroups of G_Q, denoted by $\tilde{\mathscr{G}}$, is a sublattice of the set of subgroups of G_Q under inclusion and there exists a lattice isomorphism*

$$\Phi : \mathscr{I} \longrightarrow \tilde{\mathscr{G}},$$

$$\Phi : I \longmapsto G_I,$$

where $\mathscr{I}$ is the collection of subsets, I, of the set of defining generators, T, and G_I is the subgroup generated by I.

Proof. Suppose Q is a quiver of mutation-Dynkin type with connected components $Q_1, ..., Q_r$ of types $A_{n_1}, ..., A_{n_r}$, respectively, where $n_1, ..., n_r \in \mathbb{Z}^+$ and $n = \Sigma_{i=1}^{r} n_i$. Let $Sub(\Sigma_{n+r})$ denote the set of subgroups of Σ_{n+r}. The group isomorphism

$$(\pi_Q)^{-1} : \Sigma_{n+r} \longrightarrow G_Q$$

induces a lattice isomorphism

$$\begin{aligned}Sub(\Sigma_{n+r}) &\longrightarrow \mathscr{G}, \\ H &\longmapsto (\pi_Q)^{-1}(H).\end{aligned}$$

Note that, by Lemma 6.2.8, $(\pi_Q)^{-1}(Y(\rho_I)) = G_I$ for each $Y(\rho_I) \in \tilde{\mathscr{Y}}$. Thus $\tilde{\mathscr{G}}$ is the image of the sublattice $\tilde{\mathscr{Y}}$ under this induced lattice isomorphism. It follows that $\tilde{\mathscr{G}}$ is a sublattice of $\mathscr{G}$ and that taking the restriction of the induced lattice isomorphism to the sublattice $\tilde{\mathscr{Y}}$ of $Sub(\Sigma_{n+r})$ yields the following lattice isomorphism

$$\begin{aligned} \tilde{\mathscr{Y}} &\longrightarrow \tilde{\mathscr{G}}, \\ Y(\rho_I) &\longmapsto G_I. \end{aligned}$$

Composing this map with ϕ (Lemma 6.2.7) will give the desired lattice isomorphism Φ. □

6.3 Cluster Group Presentations of Parabolic Subgroups

Suppose Q is a quiver of mutation-Dynkin type with connected components $Q_1, ..., Q_r$ of type $A_{n_1}, ..., A_{n_r}$, respectively, where $n_1, ..., n_r \in \mathbb{Z}^+$ and $n = \Sigma_{i=1}^r n_i$. In this section, we prove that the parabolic subgroups of the cluster group G_Q have presentations given by restricting the cluster group presentation of the whole group.

Consider a subset $I \subseteq T$. Let Q_I denote the full subquiver, Q_I, of Q on vertices corresponding to the elements of I. Consider the group with presentation given by taking the defining generators to be the set I and the set of relations to be all the corresponding defining relations of the cluster presentation of G_Q that consist only of elements of I. The group corresponding to this presentation is a cluster group of mutation-Dynkin type as it is precisely G_{Q_I}. By Proposition 2.3.24, Q_I will be a disjoint union of quivers $Q'_1, ..., Q'_{r'}$ of mutation-Dynkin type $A_{n'_1}, ..., A_{n'_{r'}}$, respectively, for some $1 \leq n'_i \leq n$ and $r' \geq 1$.

Consider the parabolic subgroup, G_I, generated by I. From Lemma 6.2.8, G_I is isomorphic to $Y(\rho_I)$. We will show that there exists an isomorphism between G_{Q_I} and $Y(\rho_I)$ and so, by transitivity, the parabolic subgroup G_I has a cluster group presentation associated to Q_I.

In particular, we will show that this isomorphism gives the following commutative diagram.

$$\begin{array}{ccc} G_I & \longrightarrow & G_Q \\ \downarrow & \circlearrowleft & \uparrow \\ Y(\rho_I) & \longrightarrow & G_{Q_I} \end{array}$$

To distinguish between the defining generators of G_Q and G_{Q_I}, we label the generators of the cluster presentation of G_{Q_I} by t'_i, for each $t_i \in I$.

Theorem 6.3.1. *Suppose Q is a quiver of mutation-Dynkin type with connected components of mutation-Dynkin type A and consider the associated cluster group $G_Q = \langle T|R \rangle$.*

For any $I \subseteq T$, there exists an isomorphism

$$G_{Q_I} \longrightarrow G_I,$$
$$t_i' \longmapsto t_i,$$

where G_I denotes the subgroup of G_Q generated by I and G_{Q_I} is the cluster group associated to Q_I, the full subquiver of Q on the vertices corresponding to the elements of I, where the generators of the cluster presentation of G_{Q_I} are labelled by t_i', for each $t_i \in I$.

Proof. Suppose Q is a quiver of mutation-Dynkin type with connected components $Q_1, ..., Q_r$ of type $A_{n_1}, ..., A_{n_r}$ for some $n_1, ..., n_r \in \mathbb{Z}^+$. We show that there exists an isomorphism

$$G_{Q_I} \longrightarrow Y(\rho_I),$$
$$t_i' \longmapsto \pi_I(t_i).$$

We begin by considering the braid graphs Γ_Q and Γ_{Q_I} and show that $\Gamma_I = \Gamma_{Q_I}$.

Let $\mathscr{T}$ be a triangulation giving rise to Q. As discussed in Remark 2.4.9, a triangulation $\mathscr{T}'$ giving rise to Q_I can be obtained from $\mathscr{T}$ by cutting along each of the diagonals lying in the set $\{d_i : t_i \notin I\}$.

We consider the braid graph, Γ_{Q_I} of Q_I. We remark that in obtaining $\mathscr{T}'$, no triangle of $\mathscr{T}$ will have been deleted. Thus the number of vertices of Γ_{Q_I} equals the number of vertices in Γ_Q, which equals the number of vertices in Γ_I. Moreover, all triangles in $\mathscr{T}'$ will have the same orientation as in $\mathscr{T}$ except those which were bounded by a diagonal that was 'cut'. As one diagonal of $\mathscr{T}$ bounds exactly two triangles, the set of edges in Γ_{Q_I} will equal the set of edges in Γ_Q minus the set of edges which correspond to one of the 'cut' diagonals. These are precisely the edges corresponding to all elements of $T \setminus I$. Thus Γ_{Q_I} is equal to the graph obtained from Γ_Q by deleting the edges corresponding to the $t_i \notin I$, which is precisely Γ_I.

The fixed labelling of Γ_Q induces a labelling on the vertices Γ_{Q_I} from which we obtain a set partition ρ' of $\{1, ..., n+r\}$. As $\Gamma_{Q_I} = \Gamma_I$, this set partition is precisely ρ_I meaning $Y(\rho') = Y(\rho_I)$.

By Lemma 5.2.2, there exists an isomorphism

$$\pi_{Q_I} : G_{Q_I} \longrightarrow Y(\rho'),$$
$$\pi_{Q_I} : t_i' \longmapsto (x_i, y_i)$$

where x_i and y_i are the endpoints of the edge in Γ_{Q_I} corresponding to t_i'. Moreover, as $\Gamma_{Q_I} = \Gamma_I$ and applying by Lemma 6.2.8, the following map is an isomorphism.

$$\pi_Q \mid_I : G_I \longrightarrow Y(\rho_I),$$
$$\pi_Q \mid_I : t_i \longmapsto (x_i, y_i).$$

As $Y(\rho') = Y(\rho_I)$, we can define $\pi := (\pi_Q \mid_I)^{-1} \circ \pi_{Q_I}$, which will be the desired isomorphism,

$$\pi_Q \mid_I^{-1} \circ \pi_{Q_I} : G_{Q'} \longrightarrow G_I,$$
$$\pi_Q \mid_I^{-1} \circ \pi_{Q_I} : t_i' \longrightarrow t_i.$$

□

Chapter 7

An Alternative Exchange Lemma for Cluster Groups

7.1 Introduction

In this chapter we will examine the properties of cluster groups associated to cluster quivers more generally.

Recall that for a finite Coxeter system, (W,S), by Proposition 1.4.7, for any $I \subseteq S$ there exists a unique factorisation of each element $w \in W$ of the form $w = ab$ such that $l(a) + l(b) = l(w)$, where $a \in W^I$ and $b \in W_I$. This factorisation is unique as, by an application of the Deletion Property, it can be shown that a is the unique element of the set $wW_I \cap W^I$ [4, Proposition 2.4.4]. Further, a is the unique minimal length element of the coset wW_I [4, Corollary 2.4.5]. Additionally, by Lemma 1.4.6, W^I consists of the elements of W whose reduced expressions do not end in elements of I and Theorem 1.4.4 stated that the length function on W_I equals the length function on W. Finally, by Theorem 1.4.9, known as the Exchange Lemma, each finite Coxeter group satisfies the Exchange Property and the Exchange Lemma states that this is, in fact, a characterizing property for finite Coxeter groups.

In this chapter, we aim to prove results for cluster groups that are analogous to these properties. In particular, we will prove a result for cluster groups associated to any cluster quiver which is analogous to Proposition 1.4.7. However, in this case the factorisation will not necessarily be unique. We will then use the existence of this factorisation to prove an alternative exchange lemma for cluster groups. We will then restrict once more to cluster groups associated to quivers of mutation-Dynkin type A_n and, motivated by Lemma 1.4.6, we consider the possible forms of reduced expressions of similarly defined subsets of cluster groups. In the final section, we study the length function restricted to a

parabolic subgroup.

7.2 Cluster Group Exchange Lemma

Let Q be a cluster quiver and consider the associated cluster group, G_Q. Analogous to the Coxeter case, for each $I \subseteq T$ we define the following sets.

$$G^I = \{w \in G_Q : l(wt_i) > l(w) \ \ \forall \ \ t_i \in I\}$$

$$^IG = \{w \in G_Q : l(t_i w) > l(w) \ \ \forall \ \ t_i \in I\}$$

Proposition 7.2.1. *Let Q be a cluster quiver and consider the associated cluster group, G_Q. For any $I \subseteq T$, every $w \in G_Q$ has a factorisation*

$$w = ab, \qquad \text{for some } a \in G^I, b \in G_I$$

such that $l(w) = l(a) + l(b)$.

Proof. We proceed by induction on $l(w)$. If $l(w) = 1$ then $w = t_i$, for some $1 \leq i \leq n$. If $t_i \in I$, then we choose $a = e, b = t_i$. If $t_i \notin I$, then we claim that $l(t_i t_k) > l(t_i)$ for all $t_k \in I$ and so we choose $a = t_i, b = e$. Taking any $t_k \in I$, if $l(t_i t_k) < l(t_i)$ then $l(t_i t_k) = 0$, by Lemma 4.3.1 (5), as $l(t_i) = 1$. Thus $t_i t_k = e$. It follows that $t_i = t_k \in I$, contradicting that $t_i \notin I$. As $l(t_i t_k) \neq l(t_i)$ by Lemma 4.3.1 (6), it must be that $l(t_i t_k) > l(t_i)$.

Suppose $l(w) = r \geq 1$ and that the statement holds for every element of G_Q of shorter length. If $w \in G^I$ then we choose $b = e$ and $a = w$. Similarly, if $w \in G_I$ then we choose $a = e$ and $b = w$. So we need only consider the case when $w \notin G^I$ and $w \notin G_I$.

As $w \notin G^I$, there exists $t_k \in I$ such that $l(wt_k) < l(w)$. By Lemma 4.3.1 (5), it follows that $l(wt_k) = l(w) - 1 < r$. By induction, there exists $a' \in G^I$ and $b' \in G_I$ such that $wt_k = a'b'$ and

$$l(wt_k) = l(w) - 1 = l(a') + l(b').$$

Let $a = a'$ and $b = b't_k$. Then $ab = a'b't_k = (wt_k)t_k = wt_k^2 = w$. It remains to show that $l(b't_k) = l(b') + 1$, giving $l(a) + l(b) = l(a') + l(b't_k) = l(a') + l(b') + 1 = l(wt_k) + 1 = l(w)$.

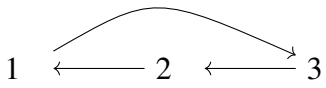
We assume, for a contradiction, that $l(b't_k) < l(b')$. That is, by Lemma 4.3.1 (5), $l(b't_k) = l(b') - 1$. We have already established that $wt_k = a'b'$, so $w = a'b't_k$, and that

$l(wt_k) = l(a') + l(b')$. Thus

$$\begin{aligned} l(w) = l(a'b't_k) &\leq l(a') + l(b't_k) \\ &= l(a') + (l(b') - 1) \\ &= (l(a') + l(b')) - 1 \\ &= l(wt_k) - 1 \\ &< l(wt_k), \end{aligned}$$

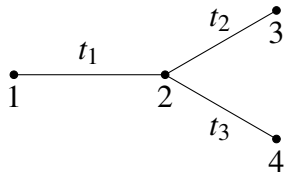
contradicting that $l(wt_k) < l(w)$. Since $l(b't_k) \neq l(b')$ by Lemma 4.3.1 (6), we have $l(b't_k) > l(b')$ and so $l(b't_k) = l(b') + 1$ by Lemma 4.3.1 (5). Therefore, $l(w) = l(a) + l(b)$. Finally, we note that $a = a' \in G^I$ and, as $t_k, b' \in G_I$, we have that $b \in G_I$. Thus we have obtained the required factorisation of w. □

Example 7.2.2. Let Q be the quiver



We consider the braid graph of Q with the following labelling.

Γ_Q:



With respect to this labelling, π_Q is the isomorphism:

$$\begin{aligned} \pi_Q &: G_Q \longrightarrow \Sigma_4, \\ \pi_Q &: t_1 \longmapsto (1,2), \\ \pi_Q &: t_2 \longmapsto (2,3), \\ \pi_Q &: t_3 \longmapsto (2,4). \end{aligned}$$

Take $I = \{t_1, t_2\} \subseteq T$ and consider $w = t_2t_3t_1t_2 \in G_Q$. By Proposition 4.2.11, to show that this expression of w is reduced, we only need to show that there exists no expression of w of length 0 or 2. To do this, we can show that $\pi_Q(w) \neq \pi_Q(w')$ for any w' such that $l(w') \in \{0,2\}$. The computation and comparisons of the permutations $\pi_Q(w)$ and

$\pi_Q(w')$ were performed using Maple^{TM} ([39]). The exact code for this example is given in Appendix A.1.

Take $a = t_2t_3$ and $b = t_1t_2$. Clearly, $b \in G_I$. It remains to show that $a \in G^I$ and so we have a factorisation of w as given in Lemma 7.2.1. To show $a \in G^I$, we show that $t_2t_3t_1$ and $t_2t_3t_2$ are reduced expressions in G_Q. Again, by Proposition 4.2.11, we only need to show that there exist no expressions of these elements of length 1. As $\pi_Q(t_2t_3t_1) = (2,3)(2,4)(1,2) = (1,4,3,2)$, it is clear that $\pi_Q(t_2t_3t_1) \neq \pi_Q(t_j)$, meaning $t_2t_3t_1 \neq t_j$, for all $1 \leq j \leq 3$. Similarly, as $\pi_Q(t_2t_3t_2) = (2,3)(2,4)(2,3) = (3,4)$, $t_2t_3t_2 \neq t_j$ for all $1 \leq j \leq 3$. Thus $a \in G^I$.

Remark 7.2.3. The proof of Proposition 7.2.1 relies only on Lemma 4.3.1 holding for the cluster group. Thus, by Remark 4.3.2, we have that any group arising from a group presentation whose relations are all of even length will have the factorisation property given in Proposition 7.2.1.

As stated in Proposition 1.4.7, in the Coxeter case this factorisation exists but is furthermore unique. The element in the factorisation lying in the set W^I is shown to be the unique element of wW_I of minimal length [33, Proposition 1.10]. The uniqueness of these minimal length elements distinguish them as coset representatives and they are referred to as the *minimal coset representatives* [33, Section 1.10]. Thus the set $wW_I \cap W^I$ contains only one element, namely the minimal coset representative of wW_I. The uniqueness of the factorisation for elements of Coxeter groups is a consequence of the Deletion Property and is not a property that is transferable to the factorisations of elements of cluster groups shown to exist in Proposition 7.2.1.

Proposition 7.2.4. *There exists a cluster quiver Q and $I \subseteq T$ such that an element $w \in G_Q$ has distinct factorisations of the form*

$$w = ab,$$

where $a \in G^I, b \in G_I$ and $l(a) + l(b) = l(w)$.

Proof. Let Q be the same quiver considered in Example 7.2.2, along with the same labelling of Γ_Q. Once more, take $I = \{t_1, t_2\} \subseteq T$ and $w = t_2t_3t_1t_2 \in G_Q$. We saw in Example 7.2.2 that this is a reduced expression of w and that by taking $a = t_2t_3$ and $b = t_1t_2$ we obtain a factorisation of w as described in Lemma 7.2.1.

Note that we can apply a cycle relation to w to obtain the expression $w = t_1t_2t_3t_1$. As this is of length 4, it is also reduced.

Let $a' = t_1t_2t_3$ and $b' = t_1$. As this expression for a' is a subexpression of a reduced expression of w, it must be reduced. Moreover, we claim that $a' \in G^I$. As $w = a't_1$ and $l(w) = l(a') + 1$, it must be that $l(a't_1) > l(a')$, otherwise $w = t_1t_2t_3t_1$ would not be a reduced expression.

So it remains to show that $l(a't_2) > l(a')$. By Proposition 4.2.11, we only need to show that there exists no expression of $a't_2$ of length 0 or 2. As $\pi_Q(a't_2) = (1,2)(2,3)(2,4)(2,3) = (1,2)(3,4)$, we can conclude that $a't_2 \neq e$. The computations required to verify that $l(a't_2) \neq 2$ were performed using Maple^{TM} ([39]) and the exact code for this is given in Appendix A.2.

Finally, we verify the factorisations are distinct by observing that $\pi_Q(b) = (1,2)(2,3) = (1,2,3) \neq (1,2) = \pi_Q(b')$, so it must be that $b \neq b'$. Furthermore, if $a = a'$ then it would follow that $b = b'$, so this cannot be the case either.

Thus, we obtain the factorisations $w = ab = a'b'$ where $l(w) = l(a) + l(b) = l(a') + l(b')$ for distinct $a, a' \in G^I$ and distinct $b, b' \in G_I$. □

The counterexample given in the proof of Proposition 7.2.4 demonstrates that for a cluster group G_Q, it is possible for the set $wG_I \cap G^I$ to have more than one element for some $I \subseteq T$ and $w \in G_Q$.

To strengthen Proposition 7.2.1, it could be further investigated whether the set $wG_I \cap G^I$ contains any distinguishing elements, such as a unique element of minimal length. However, Proposition 7.2.1 is still useful without uniqueness and can be used to prove a result for cluster groups which is comparable to the Exchange Property.

Lemma 7.2.5. *Let Q be a cluster quiver. For any $w \in G_Q$ and $t_i \in T$, if $l(t_iw) < l(w)$ then the following hold.*

(a) There exists a reduced expression of w beginning with t_i.

(b) There exists a reduced expression $w = t_{i_1}...t_{i_q}$ such that $t_it_{i_1}...t_{i_{j-1}} = t_{i_1}...t_{i_{j-1}}t_{i_j}$ for some $1 \leq j \leq q$.

Proof. Let Q be a cluster quiver and suppose $l(t_iw) < l(w)$ for some $w \in G_Q$ and $t_i \in T$.

(a) If $l(t_iw) < l(w)$ then $l(w^{-1}t_i) < l(w^{-1})$. Let $I = \{t_i\}$ and consider the parabolic subgroup $G_I = \{e, t_i\}$. By Proposition 7.2.1, there exists a factorisation $w^{-1} =$

ab where $a \in G^I, b \in G_I$ and $l(a)+l(b) = l(w)$. As $l(w^{-1}t_i) < l(w^{-1})$, we must have that $w^{-1} \notin G^I$, meaning $b \neq e$, thus $b = t_i$. That is, $w^{-1} = at_i$ where $a \in G^I$ and $l(w^{-1}) = l(b)+1$. Taking a reduced expression $a = t'_{i_1}...t'_{i_k}$, we obtain a reduced expression $w^{-1} = t'_{i_1}...t'_{i_k}t_i$, meaning $w = t_i t'_{i_k}...t'_{i_1}$ is a reduced expression of w beginning with t_i.

(b) We can easily see that such an expression by taking reduced expression $w = t_{i_1}...t_{i_k}$ such that $t_{i_1} = t_i$, which exists by (a), and choosing $j = 1$.

□

Remark 7.2.6. (a) By Remark 7.2.3, this 'alternative exchange lemma' will hold for any group arising from a group presentation whose relations are all of even length. This is because the proof of Lemma 7.2.5 relies only on the fact that Proposition 7.2.1 holds for cluster groups.

(b) An analogous proof for part (a) of Lemma 7.2.5 can be used to show that if $l(t_i w) < l(w)$ then there exists a reduced expression of w ending with t_i.

It only takes a small step to extend this result to one resembling the Exchange Lemma for Coxeter groups.

Lemma 7.2.7. *Let Q be a cluster quiver and take any $w \in G_Q$. If $t_i \in T$ is such that $l(t_i w) < l(w)$ then there exists a reduced expression $w = t_{i_1}...t_{i_k}$ such that $t_i w = t_{i_1}...\hat{t}_{i_j}...t_{i_k}$ for some $1 \leq j \leq k$.*

Proof. Take any $w \in G_Q$ and suppose $t_i \in T$ is such that $l(t_i w) < l(w)$. By Lemma 7.2.5 (b), there exists a reduced expression $w = t_{i_1}...t_{i_k}$ such that $t_i t_{i_1}...t_{i_{j-1}} = t_{i_1}...t_{i_{j-1}}t_{i_j}$ for some $1 \leq j \leq k$. That is, $t_i = t_{i_1}...t_{i_{j-1}}t_{i_j}t_{i_{j-1}}...t_{i_1}$. Thus

$$\begin{aligned} t_i w &= (t_{i_1}...t_{i_{j-1}}t_{i_j}t_{i_{j-1}}...t_{i_1})t_{i_1}...t_{i_k} \\ &= t_{i_1}...t_{i_{j-1}}t_{i_{j+1}}...t_{i_k} \\ &= t_{i_1}...\hat{t}_{i_j}...t_{i_k} \end{aligned}$$

□

The following result is a consequence of Lemma 7.2.5.

Lemma 7.2.8. *Let Q be a cluster quiver on n vertices and fix any $1 \leq i \leq n$. Take $I = T \setminus \{t_i\}$ and $w \in G_Q$ such that $w \neq e$. Then $w \in {}^{I}G$ if any only if all reduced expressions of w begin in t_i.*

Proof. If $I = T \setminus \{t_i\}$ then

$$ {}^{I}G = \{w \in G_Q : l(t_j w) > l(w) \ \ \forall \ \ j \neq i\}.$$

If $w \in G_Q$ is such that every reduced expression of w begins in t_i, then $w \in {}^{I}G$. This is because, if there existed $j \neq i$ such that $l(t_j w) < l(w)$, then, by Lemma 7.2.5, there would exist a reduced expression of w beginning in t_j where $j \neq i$, a contradiction. Conversely, suppose $w \in {}^{I}G$ and $w \neq e$. If w has a reduced expression beginning in t_j for some $j \neq i$ then $l(t_j w) < l(w)$, contradicting that $w \in {}^{I}G$. Thus every reduced expression of w must begin in t_i. We conclude that any element of G_Q not equal to the identity element lies in ${}^{I}G$ if and only if all of its reduced expressions begin in t_i.

□

Whilst Lemma 7.2.7 is comparable to the Exchange Property, the Exchange Property holds for any given reduced expression of an element of a Coxeter group. Thus Lemma 7.2.7 is not as strong as a condition. However, it would be further strengthened if the following conjecture could be proved.

Conjecture 7.2.9. Let Q be a cluster quiver and take any $w \in G_Q$. If $t_i \in T$ is such that $l(t_i w) < l(w)$, then given any reduced expression $w = t_{i_1} ... t_{i_k}$ there exists a reduced expression of $t_i w$ whose support is equal to the set $\{t_{i_1}, ..., \hat{t}_{i_j}, ..., t_{i_q}\}$, for some $1 \leq j \leq k$.

Remark 7.2.10. By Remark 7.2.6(a), we can see that Lemma 7.2.7 will hold for any group arising from a group presentation such that the square of each generator gives the identity and whose relations are all of even length.

7.3 Reduced Expressions of Elements in ${}^{I}G$

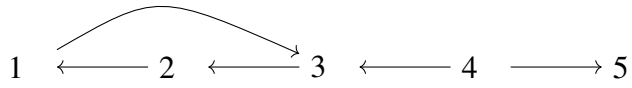
In this section we will only consider quivers of mutation-Dynkin type $A_{n_1} \sqcup ... \sqcup A_{n_r}$, where $n_1, ..., n_r \in \mathbb{Z}^+$ and $n = \sum_{i=1}^{r} n_i$, with $r \geq 1$. Let Q be a quiver of this type and recall in Section 7.2, for $I \subseteq T$, we defined the set

$$ {}^{I}G = \{w \in G_Q : l(t_i w) > l(w) \ \ \forall \ \ t_i \in I\}.$$

For $I = T \setminus \{t_i\}$, by Lemma 7.2.8, we have that any element of G_Q not equal to the identity element lies in ${}^I G$ if and only if all of its reduced expressions begin in t_i. In this final section of the chapter, we consider the forms of the reduced expressions of elements lying in ${}^I G$, for a subset $I = T \setminus \{t_i\}$, for some $1 \leq i \leq n$.

To begin, we establish some notation. For $1 \leq i \neq j \leq n$ such that i and j are connected by an edge in Q, let Γ^i_j denote the set of $t_m \in T$ such that $m \neq j$ (so $t_m \neq t_j$ by Lemma 4.2.10) and there exists a path between j and m in the underlying graph of Q that does not pass through i or any $k \neq j$ such that i and k are connected by an edge in Q.

Example 7.3.1. Let Q be the quiver



Then $\Gamma^2_3 = \{t_4, t_5\}$ but $\Gamma^2_1 = \emptyset$.

Remark 7.3.2. Note that $t_j \notin \Gamma^i_j$ for all $1 \leq j \leq n$. Moreover, as Q is of mutation-Dynkin type $A_{n_1} \sqcup \ldots \sqcup A_{n_r}$, the underlying graph of Q has no cycles of length greater than 3, by Lemma 2.4.8. It follows that each element of Γ^i_j will commute with every element of $T \setminus (\Gamma^i_j \cup \{t_j\})$.

Let $\overline{w}(\Gamma^i_j)$ represent a reduced expression of a general element in G_Q whose support is contained in Γ^i_j. By the above,

$$t_k \overline{w}(\Gamma^i_j) = \overline{w}(\Gamma^i_j) t_k \tag{7.1}$$

for all $t_k \in T \setminus (\Gamma^i_j \cup \{t_j\})$.

Lemma 7.3.3. *Let Q be a quiver of mutation-Dynkin type on n vertices with connected components of type A and consider the associated cluster group, G_Q. For any $1 \leq i \leq n$ and $w \in G_Q$, if there exists a reduced expression $w = t_{i_1} t_{i_2} \ldots t_{i_k}$ where $t_{i_j} \neq t_i$ for all $1 \leq j \leq k$, then w has a reduced expression of the form:*

$$\left(\prod_{u=1}^{N} \overline{w}(\Gamma^i_{j_u}) \right) \left(\prod_{v=1}^{N'} t_{j'_v} \overline{w}(\Gamma^i_{j'_v}) \right), \tag{7.2}$$

where the vertices j_u are distinct for all $1 \leq u \leq N$ and j_u and j'_v are connected to i in Q, for all $1 \leq u \leq N$ and $1 \leq v \leq N'$.

Moreover, this reduced expression can be obtained from the expression $w = t_{i_1} t_{i_2} \ldots t_{i_k}$ by applying a sequence of commutations.

Proof. Let Q be a quiver of mutation-Dynkin type $A_{n_1} \sqcup ... \sqcup A_{n_r}$, where $n_1, ..., n_r \in \mathbb{Z}^+$ and $n = \sum_{i=1}^{r} n_i$, with $r \geq 1$. Note that by Lemma 4.2.10 we can assume that, for all vertices i and j of Q, $t_i \neq t_j$ if and only if $i \neq j$. We proceed by induction on $l(w)$. Suppose that $l(w) = 1$ and $w = t_{i_1}$ for some $t_{i_1} \in T$ where $i_1 \neq i$. If i_1 is connected to i by an edge in Q then this reduced expression is of the form (7.2) by taking $N = 0$ and $N' = 1$. If i_1 is not connected to i by an edge in Q, then $t_{i_1} \in \Gamma^i_{j_u}$ for some j_u connected to i and so this reduced expression is of the form (7.2) by taking $N = 1$, $N' = 0$ and $t_{i_1} = \overline{w}(\Gamma^i_{j_1})$.

Suppose $l(w) = k > 1$ and any element with a reduced expression of shorter length containing no t_i terms has a reduced expression of the form (7.2). If there exists a reduced expression $w = t_{i_1} t_{i_2} ... t_{i_k}$ such that $t_{i_j} \neq t_i$ for all $1 \leq j \leq k$, then we consider $w' = t_{i_1} w = t_{i_2} ... t_{i_k}$. Clearly, this is an element of shorter length with a reduced expression containing no t_i terms. By induction there exists a reduced expression of the form:

$$t_{i_1} w = \left(\prod_{u=1}^{N} \overline{w}(\Gamma^i_{j_u}) \right) \left(\prod_{v=1}^{N'} t_{j'_v} \overline{w}(\Gamma^i_{j'_v}) \right),$$

containing $k-1$ terms, where j_u and j'_v are connected to i in Q for all $1 \leq u \leq N$, $1 \leq v \leq N'$ and the j_u are distinct over $1 \leq u \leq N$. Moreover, this reduced expression can be obtained from $t_{i_2} ... t_{i_k}$ by applying a sequence of commutations. Thus

$$w = t_{i_1} \left(\prod_{u=1}^{N} \overline{w}(\Gamma^i_{j_u}) \right) \left(\prod_{v=1}^{N'} t_{j'_v} \overline{w}(\Gamma^i_{j'_v}) \right),$$

which consists of k terms and so is a reduced expression. If i_1 is not connected to i by an edge in Q, then $t_{i_1} \in \Gamma^i_p$ for some p connected to i and so we take $j_0 = p$. Thus $\overline{w}(\Gamma^i_{j_0}) = t_{i_1}$ and

$$w = \left(\prod_{u=0}^{N} \overline{w}(\Gamma^i_{j_u}) \right) \left(\prod_{v=1}^{N'} t_{j_v} \overline{w}(\Gamma^i_{j_v}) \right)$$

which is of the form (7.2) (after relabelling).

It remains to consider when i_1 is connected to i by an edge in Q. In this case, if $i_1 = j_u$ for some $1 \leq u \leq N$ then we can commute t_{i_1} through and extend the first appearing $\overline{w}(\Gamma^i_{i_1})$. By Remark 7.3.2, $\overline{w}(\Gamma^i_{j_{u_1}}) \overline{w}(\Gamma^i_{j_{u_2}}) = \overline{w}(\Gamma^i_{j_{u_2}}) \overline{w}(\Gamma^i_{j_{u_1}})$ for all $j_{u_1} \neq j_{u_2}$ meaning we can assume, without loss of generality, that $i_1 = j_N$. Moreover, as the j_u are distinct, $j_u \neq i_1$ for all $1 \leq u \leq N-1$. Consequently, $t_{i_1} \overline{w}(\Gamma^i_{j_u}) = \overline{w}(\Gamma^i_{j_u}) t_{i_1}$ for all $1 \leq u \leq N-1$. Thus

$$w = \left(\prod_{u=1}^{N-1} \overline{w}(\Gamma^i_{j_u}) \right) \left(t_{i_1} \overline{w}(\Gamma^i_{i_1}) \right) \left(\prod_{v=1}^{N'} t_{j'_v} \overline{w}(\Gamma^i_{j'_v}) \right).$$

By absorbing t_{i_1} into the second product, we obtain

$$w = \left(\prod_{u=1}^{N-1} \overline{w}(\Gamma^i_{j_u})\right)\left(\prod_{v=0}^{N'} t_{j'_v}\overline{w}(\Gamma^i_{j'_v})\right),$$

where $j'_0 = i_1$ and $\overline{w}(\Gamma^i_{j'_0}) = e$. After relabelling, this gives an expression of the required form. As the only relations we have applied were legal commutations, this expression contains k terms, thus is reduced.

If $i_1 \neq j_u$, for any $1 \leq u \leq N$, then $t_{i_1}\overline{w}(\Gamma^i_{j_u}) = \overline{w}(\Gamma^i_{j_u})t_{i_1}$, for all $1 \leq u \leq N$. Thus

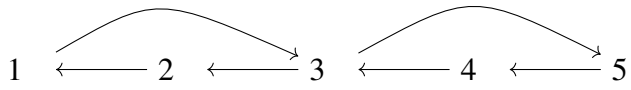
$$w = \left(\prod_{u=1}^{N} \overline{w}(\Gamma^i_{j_u})\right) t_{i_1} \left(\prod_{v=1}^{N'} t_{j'_v}\overline{w}(\Gamma^i_{j'_v})\right).$$

By absorbing t_{i_1} into the second product, we obtain the expression

$$w = \left(\prod_{u=1}^{N} \overline{w}(\Gamma^i_{j_u})\right)\left(\prod_{v=0}^{N'} t_{j'_v}\overline{w}(\Gamma^i_{j'_v})\right),$$

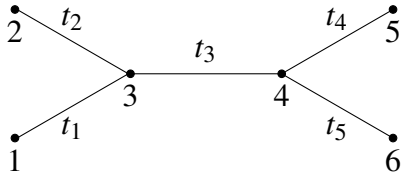
where $j'_0 = i_1$. After relabelling, we obtain an expression of the form (7.2). Again, as the only relations we have applied were legal commutations, this expression contains k terms and so is reduced. □

Example 7.3.4. Let Q be the quiver



Take the following labelling of Γ_Q by the set $\{1,2,3,4,5,6\}$.

Γ_Q:



With respect to this labelling, π_Q is the isomorphism:

$$\pi_Q : G_Q \longrightarrow \Sigma_6,$$

$$\pi_Q : t_1 \longmapsto (1,3),$$

$$\pi_Q : t_2 \longmapsto (2,3),$$

$$\pi_Q : t_3 \longmapsto (3,4),$$

$$\pi_Q : t_4 \longmapsto (4,5),$$

$$\pi_Q : t_5 \longmapsto (4,6).$$

Take $i = 1$. So $\Gamma_2^1 = \emptyset$ and $\Gamma_3^1 = \{t_4, t_5\}$. Consider the element $w = t_3t_2t_5t_3t_4 \in G_Q$. To show this expression is reduced, by Proposition 4.2.11, we only need to show that there exists no expression of w of length 1 or 3. The computations required to verify this were performed using Maple^{TM} ([39]) and the exact code for this example is given in Appendix A.3. So we may assume the expression is reduced.

By applying the commutation $t_2t_5 = t_5t_2$, we obtain the expression

$$w = t_3t_5t_2t_3t_4,$$

which is a reduced expression of the form (7.2) where $N = 0$ (so the j_u are trivially distinct) and $N' = 3$ with $j'_1 = t_3, j'_2 = t_2$ and $j'_3 = t_3$ and $\overline{w}(\Gamma^1_{j'_1}) = t_5$, $\overline{w}(\Gamma^1_{j'_2}) = e$ and $\overline{w}(\Gamma^1_{j'_3}) = t_4$.

Proposition 7.3.5. *Let Q be a quiver of mutation-Dynkin type on n vertices with connected components of type A, and consider the associated cluster group, G_Q. For any $1 \leq i \leq n$ and $w \in G_Q$, by applying a sequence of commutations, from any reduced expression $w = t_{i_1}t_{i_2}...t_{i_k}$ we can obtain a reduced expression of the form:*

$$w = \left(\prod_{u=1}^{N} \overline{w}(\Gamma^i_{j_u})\right)\left(\prod_{v=1}^{N'} t_{j'_v}\overline{w}(\Gamma^i_{j'_v})\right)\left(\prod_{p=1}^{M} t_i\left[\prod_{q=1}^{N_p} t_{j_{pq}}\overline{w}(\Gamma^i_{j_{pq}})\right]\right), \tag{7.3}$$

where j_u, j'_v and j_{pq} are connected to i by an edge in Q, for all $1 \leq u \leq N$, $1 \leq v \leq N'$, $1 \leq p \leq M$ and $1 \leq q \leq N_p$.

Proof. Let Q be a quiver of mutation-Dynkin type $A_{n_1} \sqcup ... \sqcup A_{n_r}$, where $n_1, ..., n_r \in \mathbb{Z}^+$ and $n = \sum_{i=1}^{r} n_i$, with $r \geq 1$. Fix $1 \leq i \leq n$ and suppose $w \in G_Q$ has a reduced expression $w = t_{i_1}t_{i_2}...t_{i_k}$.

If $t_{i_j} \neq t_i$ for all $1 \leq j \leq k$ then, by Lemma 7.3.3, there is a sequence of commutations that can be applied to the reduced expression $t_{i_1}t_{i_2}...t_{i_k}$ to obtain a reduced expression of the form (7.2), which is of the form (7.3) (by taking $M = 0$). Assuming this is not the case means that there exist $1 \leq p_1 < ... < p_a \leq k$ such that $i_{p_b} = i$ for each $1 \leq b \leq a$ and $i_c \neq i$ for all $c \notin \{p_1, ..., p_a\}$.

Therefore, we can write this reduced expression as the factorisation

$$w = w_0t_{i_{p_1}}w_1t_{i_{p_2}}w_2...t_{i_{p_a}}w_a = w_0t_iw_1t_iw_2...t_iw_a$$

where $w_0 = t_{i_1}...t_{i_{p_1-1}}$, with $i_j \neq i$ for all $1 \leq j \leq p_1 - 1$, and, for $1 \leq b \leq a$, $w_b = t_{i_{p_b+1}}...t_{i_{p_{b+1}-1}}$, with $i_j \neq i$ for all $p_b + 1 \leq j \leq p_{b+1} - 1$ (taking $p_{a+1} - 1 = k$).

By Lemma 7.3.3, we can apply a sequence of commutations to $w_a = t_{i_{pa+1}}...t_{i_k}$ to obtain a reduced expression of w_a of the form:

$$w_a = \left(\prod_{u=1}^{N} \overline{w}(\Gamma^i_{j_u})\right)\left(\prod_{v=1}^{N'} t_{j'_v}\overline{w}(\Gamma^i_{j'_v})\right),$$

where j_u and j'_v are connected to i in Q, for all $1 \leq u \leq N$ and $1 \leq v \leq N'$.

As $t_i\overline{w}(\Gamma^i_{j_u}) = \overline{w}(\Gamma^i_{j_u})t_i$ for each $1 \leq u \leq N$, we have a reduced expression

$$t_i w_a = \left(\prod_{u=1}^{N} \overline{w}(\Gamma^i_{j_u})\right) t_i \left(\prod_{v=1}^{N'} t_{j'_v}\overline{w}(\Gamma^i_{j'_v})\right).$$

As $w_{a-1}\left(\prod_{u=1}^{N} \overline{w}(\Gamma^i_{j_u})\right)$ has no terms equal to t_i, we can then repeat this same process for $w_{a-1}\left(\prod_{u=1}^{N} \overline{w}(\Gamma^i_{j_u})\right)$ and so on, in descending order for each $1 \leq b \leq a-1$. The resulting expression, after relabelling, will be of the form

$$w_0\left(\prod_{u=1}^{N} \overline{w}(\Gamma^i_{j_u})\right)\left(\prod_{p=1}^{M} t_i\left[\prod_{q=1}^{N_p} t_{j_{pq}}\overline{w}(\Gamma^i_{j_{pq}})\right]\right).$$

Moreover, as only legal commutations have been applied, this expression is reduced. Finally, by Lemma 7.3.3, we can apply a sequence of commutations to obtain a reduced expression of $w_0\left(\prod_{u=1}^{N} \overline{w}(\Gamma^i_{j_u})\right)$ of the form (7.2). Substituting this into the expression above, the resulting expression of w will be reduced and of the form (7.3) (after relabelling). □

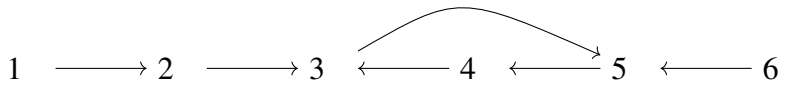
The next result follows immediately from Lemma 7.2.8 and Proposition 7.3.5.

Corollary 7.3.6. *Let Q be a quiver of mutation-Dynkin type on n vertices with connected components of type A and consider the associated cluster group, G_Q. For any $1 \leq i \leq n$, take $w \in {}^I G$ where $I = T \setminus \{t_i\}$ (i.e. all reduced expressions of w begin in t_i). By applying a sequence of commutations, from any reduced expression of w we can obtain a reduced expression of the form:*

$$w = \prod_{p=1}^{M} t_i\left[\prod_{q=1}^{N_p} t_{j_{pq}}\overline{w}(\Gamma^i_{j_{pq}})\right], \tag{7.4}$$

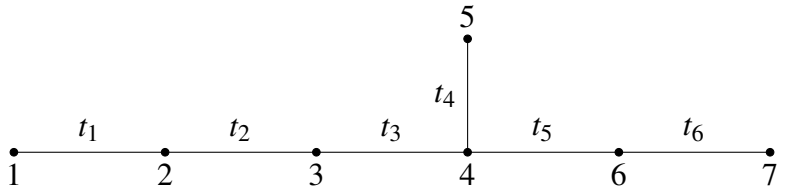
where j_{pq} is connected to i by an edge in Q for all $1 \leq p \leq M$ and $1 \leq q \leq N_p$, where $M, N_p \in \mathbb{Z}^+$.

Example 7.3.7. Let Q be the quiver



Take the following labelling of Γ_Q by the set $\{1,2,3,4,5,6,7\}$.

Γ_Q:



With respect to this labelling, π_Q is the isomorphism:

$$\pi_Q : G_Q \longrightarrow \Sigma_7,$$
$$\pi_Q : t_1 \longmapsto (1,2),$$
$$\pi_Q : t_2 \longmapsto (2,3),$$
$$\pi_Q : t_3 \longmapsto (3,4),$$
$$\pi_Q : t_4 \longmapsto (4,5),$$
$$\pi_Q : t_5 \longmapsto (4,6),$$
$$\pi_Q : t_6 \longmapsto (6,7).$$

Consider $w = t_3t_2t_4t_5t_6t_3t_5t_1t_2 \in G_Q$. To show this expression is reduced, by Proposition 4.2.11, we only need to show that $l(w) \notin \{1,3,5,7\}$. The computations required to verify this were performed using Maple^{TM} ([39]) and the exact code for this example is given in Appendix $A.4$, alongside a code that shows that any reduced expression of w begins in t_3.

By taking $i = 3$, this is a reduced expression of the form (7.4) where $M = 2$ and $N_1 = 3$ with $j_{11} = 2$, $j_{12} = 4$, $j_{13} = 5$ and $\overline{w}(\Gamma^3_{j_{11}}) = \overline{w}(\Gamma^3_{j_{12}}) = e$, $\overline{w}(\Gamma^3_{j_{13}}) = t_6$ and $N_2 = 3$ with $j_{21} = 5$, $j_{22} = 1$, $j_{23} = 2$ and $\overline{w}(\Gamma^3_{j_{21}}) = \overline{w}(\Gamma^3_{j_{22}}) = \overline{w}(\Gamma^3_{j_{23}}) = e$.

We note that the reduced expressions given in these results are not necessarily unique.

In the next section, we give a possible motivation for considering the forms of reduced expressions of elements lying in the set ${}^IG$ where $I = T \setminus \{t_i\}$, for some $1 \leq i \leq n$.

7.4 The Length Function on Parabolic Subgroups

Section 1.4 outlined many key properties for Coxeter group presentations and we have shown that several comparable results exist for cluster group presentations. For example,

Theorem 1.4.8 stated that a parabolic subgroup, W_I, of a finite Coxeter group, W, is a finite Coxeter group in its own right. Moreover, the length function on W_I, considered as a finite Coxeter group, agrees with the length function on the whole group, where we consider W_I as a subgroup of W. That is, $l = l_I$.

Analogously, we have shown that the parabolic subgroups of the cluster groups associated to quivers of mutation-Dynkin type $A_{n_1} \sqcup ... \sqcup A_{n_r}$, where $n_1, ..., n_r \in \mathbb{Z}^+$ and $n = \sum_{i=1}^{r} n_i$ with $r \geq 1$, are also cluster groups in their own right. Motivated by Theorem 1.4.8, a natural next step would be to study the length function on these parabolic subgroups.

Definition 7.4.1. *Let Q be a cluster quiver. For $w \in G_Q$ and $I \subseteq T$, we define $l_I(w)$ to be the smallest r such that $w = t_{i_1}...t_{i_r}$ where $t_{i_j} \in I$ for all $1 \leq j \leq r$.*

In this section we will discuss existing proofs for the theorem that the length function on a parabolic subgroup of a finite Coxeter group agrees with the length function on the parabolic subgroup considered as a Coxeter group in its own right. We also provide an alternative proof for this theorem in the type A_n case, not yet published in any existing literature. Finally, we will discuss a conjecture that, if proved, could be used to show that the same statement is true for cluster groups associated to quivers of mutation-Dynkin type $A_{n_1} \sqcup ... \sqcup A_{n_r}$, where $n_1, ..., n_r \in \mathbb{Z}^+$ and $n = \sum_{i=1}^{r} n_i$, with $r \geq 1$.

7.4.1 The Coxeter Case

In this section, we consider the following theorem.

Theorem. *[33, Theorem 5.5] Let (W, S) be a Coxeter system and take any subset $I \subseteq S$. For $w \in W_I$, if $w = s_{i_1}...s_{i_k}$ is a reduced expression then $s_{i_j} \in I$ for all $1 \leq j \leq k$. It follows that $l = l_I$.*

The proof of this theorem is achieved in [33, Theorem 5.5] by examining the coefficients of the simple roots in the linear expansion of the roots appearing in the set $\{w(\alpha_i) : w \in G_I, 1 \leq i \leq n\}$. In particular, recall that Theorem 1.4.4 stated the following.

> For any $w \in W$ and $s_i \in S$, if $l(ws_i) < l(w)$ then $w(\alpha_i) < 0$ and if $l(ws_i) > l(w)$ then $w(\alpha_i) > 0$.

This difference in sign of the coefficient of α_i in the roots $w(\alpha_i)$ and α_i, when $l(ws_i) < l(w)$, is key in proving that $l = l_I$ for the Coxeter case. The same result can be proved using the Tits representation [13, Corollary 4.1.5].

After establishing some groundwork, we will provide a different perspective, not yet published in any existing literature, proving [33, Theorem 5.5] for the type A_n case.

Let (W,S) be a finite Coxeter group system of type A_n, where $S = \{s_1,...,s_n\}$. By Lemma 3.2.1, we can consider each element of W as a permutation, $\sigma(w) \in \Sigma_{n+1}$, where σ is the group isomorphism:

$$\sigma : W \longrightarrow \Sigma_{n+1},$$
$$\sigma : s_{\alpha_i} \longmapsto (i, i+1).$$

Given an expression $x_1...x_k$, where $x_i \in S \cup \{e\}$ for each $1 \leq i \leq k$, we define a sequence $a(x_1...x_k)$, of length k, consisting of $(n+1)-$tuples in the following way.

$$a(x_1...x_k) = (a(x_1...x_k)_0, a(x_1...x_k)_1, ..., a(x_1...x_k)_k) = (a(x_1...x_k)_p)_{p=0}^{k},$$

where $a(x_1...x_k)_p = (a(x_1...x_k)_{p,1}, a(x_1...x_k)_{p,2}, ..., a(x_1...x_k)_{p,n+1}) = (a(x_1...x_k)_{p,q})_{q=1}^{n+1}$, for each $0 \leq p \leq k$ and $a(x_1...x_k)_{p,q} = \sigma(x_1...x_p)(q)$ for each $1 \leq q \leq n+1$, where $a(x_1...x_k)_{0,q} = \sigma(e)(q) = q$.

Remark 7.4.2. As $a(x_1...x_k)_{0,q} = \sigma(e)(q) = q$, it follows that $a(x_1...x_k)_0 = (1,2,...,n+1)$ for any expression $x_1...x_k$. Moreover, as $\sigma(s_i) = (i, i+1)$ for each $s_i \in S$, the difference between $a(x_1...x_k)_p$ and $a(x_1...x_k)_{p+1}$ when $x_{p+1} = s_m$ is a swap of the entries in the m and $m+1$ positions of $a(x_1...x_k)_p$, with no difference if $x_{p+1} = e$, for each $0 \leq p \leq k-1$.

Example 7.4.3. Let W be the Coxeter group with Coxeter graph

$$s_1 \text{ ——— } s_2 \text{ ——— } s_3$$

and take $w = s_1s_2s_3s_2 \in W$. Then

$$a(s_1s_2s_3s_2) = ((1,2,3,4),(2,1,3,4),(2,3,1,4),(2,3,4,1),(2,4,3,1)).$$

It is clear to see that if $w, w' \in W$ have reduced expressions $w = s_{i_1}...s_{i_k}$ and $w' = s_{i'_1}...s_{i'_{k'}}$, then $w = w'$ if and only if $a(s_{i_1}...s_{i_k})_k = a(s_{i'_1}...s_{i'_{k'}})_{k'}$.

Given any $1 \leq j \leq n$, we define the sets

$$X_j = \{1,...,j\} \text{ and } Y_j = \{j+1,...,n+1\}.$$

Using these sets, for an expression $x_1...x_k$, we obtain a new sequence, $a_j(x_1...x_k)$, from $a(x_1...x_k)$ by defining

$$a_j(x_1...x_k) = (a_j(x_1...x_k)_0, a_j(x_1...x_k)_1, ..., a_j(x_1...x_k)_k) = (a_j(x_1...x_k)_p)_{p=0}^{k},$$

where $a_j(x_1...x_k)_p = (a(x_1...x_k)_{p,1}, a(x_1...x_k)_{p,2}, ..., a(x_1...x_k)_{p,n+1}) = (a_j(x_1...x_k)_{p,q})_{q=1}^{n+1}$ is obtained in the following way. Let

$$a_j(x_1...x_k)_p^{X_j} = (a(x_1...x_k)_{p,r_1}, ..., a(x_1...x_k)_{p,r_j}),$$

where $r_1 < ... < r_j$ are such that $a(x_1...x_k)_{p,r_l}$ is an entry of $a(x_1...x_k)_p$ lying in X_j for all $1 \leq l \leq j$. Clearly, there are j such entries. Similarly, let

$$a_j(x_1...x_k)_p^{Y_j} = (a(x_1...x_k)_{p,r'_1}, ..., a(x_1...x_k)_{p,r'_{n+1-j}}),$$

where $r'_1 < ... < r'_{n+1-j}$ are such that $a(x_1...x_k)_{p,r'_l}$ is an entry of $a(x_1...x_k)_p$ lying in Y_j for all $1 \leq l \leq n+1-j$. Clearly, there are $n+1-j$ such entries. We take

$$a_j(x_1...x_k)_p = (a(x_1...x_k)_{p,r_1}, ..., a(x_1...x_k)_{p,r_j}, a(x_1...x_k)_{p,r'_1}, ..., a(x_1...x_k)_{p,r'_{n+1-j}}).$$

Thus the order that elements of the set X_j appear in each of the tuples $a(x_1...x_k)_p$ are preserved in the tuple $a_j(x_1...x_k)_p$. Similarly for Y_j.

Example 7.4.4. Let (W,S) be the finite Coxeter group given in Example 7.4.3 and recall that

$$a(s_1s_2s_3s_2) = ((1,2,3,4),(2,1,3,4),(2,3,1,4),(2,3,4,1),(2,4,3,1)).$$

Let $j = 2$, so $X_j = \{1,2\}$ and $Y_j = \{3,4\}$. Then

$$a_2(s_1s_2s_3s_2) = ((1,2,3,4),(2,1,3,4),(2,1,3,4),(2,1,3,4),(2,1,4,3)).$$

Lemma 7.4.5. *For every expression $x_1...x_k$, where $x_i \in S \cup \{e\}$, $a_j(x_1...x_k) = a(x'_1...x'_{k'})$, where $x'_l \in S \cup \{e\}$, for each $1 \leq j \leq k$.*

Proof. To show this, we show that the difference between the two tuples $a_j(x_1...x_k)_p$ and $a_j(x_1...x_k)_{p+1}$ is a swap between two neighbouring entries or no difference at all, so x'_{p+1} is equal to s_l, for some $1 \leq l \leq n$, or the identity element.

Consider $a_j(x_1...x_k)_p$ and $a_j(x_1...x_k)_{p+1}$ for some $0 \leq p \leq k-1$. For ease, we write $a(x_1...x_k)_p = (a_{p,1}, ..., a_{p,n+1})$ and $a(x_1...x_k)_{p+1} = (a_{p+1,1}, ..., a_{p+1,n+1})$. If $x_{p+1} = e$, by Remark 7.4.2, $a(x_1...x_k)_p = a(x_1...x_k)_{p+1}$ and so $a_j(x_1...x_k)_p = a_j(x_1...x_k)_{p+1}$. Thus x'_{p+1} is the identity element. So we suppose $x_{p+1} = s_m$ for some $1 \leq m \leq n$. So

$$a(x_1...x_k)_{p+1} = (a_{p,1}, ..., a_{p,m-1}, a_{p,m+1}, a_{p,m}, a_{p,m+2}, ..., a_{p,n+1}).$$

We consider each case when

(1) $a_{p,m}, a_{p,m+1} \in X_j$.

(2) $a_{p,m}, a_{p,m+1} \in Y_j$.

(3) $a_{p,m} \in X_j, a_{p,m+1} \in Y_j$.

(4) $a_{p,m} \in Y_j, a_{p,m+1} \in X_j$.

For cases (1) and (2) we consider $a_j(x_1...x_k)_p = (a_{p,r_1},...,a_{p,r_j},a_{p,r'_1},...,a_{p,r'_{n+1-j}})$, where $a_{p,r_1},...,a_{p,r_j} \in X_j$ and $a_{p,r'_1},...,a_{p,r'_{n+1-j}} \in Y_j$.

(1) Then $m = r_l, m+1 = r_{l+1}$ for some $1 \leq l \leq j-1$. Moreover,

$$a_j(x_1...x_k)_{p+1} = (a_{p,r_1},...,a_{p,r_{l-1}},a_{p,r_{l+1}},a_{p,r_l},a_{p,r_{l+2}}...,a_{p,r_j},a_{p,r'_1},...,a_{p,r'_{n+1-j}}).$$

So $x'_{p+1} = s_{r_l}$.

(2) Then $m = r'_l, m+1 = r'_{l+1}$ for some $1 \leq l \leq n-j$. Moreover,

$$a_j(x_1...x_k)_{p+1} = (a_{p,r_1},...,a_{p,r_j},a_{p,r'_1},,...a_{p,r'_{l-1}},a_{p,r'_{l+1}},a_{p,r'_l},a_{p,r'_{l+2}}...,a_{p,r'_{n+1-j}}).$$

So $x'_{p+1} = s_{j+r'_l}$.

For cases (3) and (4) we consider $a(x_1...x_k)_p = (a_{p,1},...,a_{p,m-1},a_{p,m},a_{p,m+1},a_{p,m+2},...,a_{p,n+1})$.

(3) Then $a(x_1...x_k)_{p+1} = (a_{p,1},...,a_{p,m-1},a_{p,m+1},a_{p,m},a_{p,m+2},...,a_{p,n+1})$. Clearly all entries which lie in X_j lying to the left (respectively, right) of $a_{p,m}$ in $a(x_1...x_k)_p$ still lie to the left (respectively, right) of $a_{p,m}$ in $a(x_1...x_k)_{p+1}$. Similarly for entries which lie in Y_j lying to the left (respectively, right) of $a_{p,m+1}$. Thus $a_j(x_1...x_k)_p = a_j(x_1...x_k)_{p+1}$ so $x'_{p+1} = e$.

(4) Similar to (3)

□

We are now able to give an alternative proof that the support of a reduced expression of a parabolic subgroup, W_I, of a finite Coxeter group of type A_n is contained in I, for any $I \subseteq S$.

Theorem. *[33, Theorem 5.5] Let (W,S) be a Coxeter system of type A_n and take $I \subseteq S$. For $w \in W_I$, if $w = s_{i_1}...s_{i_k}$ is a reduced expression then $s_{i_j} \in I$ for all $1 \leq j \leq k$.*

Proof. For a Coxeter system (W,S), let $I \subseteq S$ and $w \in W_I$. Take a reduced expression $w = s_{i_1}...s_{i_k}$. Suppose there exists some $1 \leq l \leq k$ such that $s_{i_l} \notin I$. Note that $i_l = j$, for some $1 \leq j \leq n$. We will show that $a_j(s_{i_1}...s_{i_k})_k = a(s_{i_1}...s_{i_k})_k$.

As $s_j \notin I$, $a(s_{i_1}...s_{i_k})_k = (a_{k,1},...a_{k,n+1})$ where $a_{k,1},...,a_{k,j} \in X_j$ and $a_{k,j+1},...,a_{k,n+1} \in Y_j$. From the definition, it follows that $a_j(s_{i_1}...s_{i_k})_k^{X_j} = (a_{k,1},...a_{k,j})$ and $a_j(s_{i_1}...s_{i_k})_k^{Y_j} = (a_{k,j+1},...a_{k,n+1})$ and so $a_j(s_{i_1}...s_{i_k})_k = (a_{k,1},...,a_{k,j},a_{k,j+1},...,a_{k,n+1}) = a(s_{i_1}...s_{i_k})_k$.

By Lemma 7.4.5, $a_j(s_{i_1}...s_{i_k}) = a(x'_1...x'_k)$ for some expression $x'_1...x'_k$ where $x'_i \in S \cup \{e\}$. In particular, we can see from the proof of Lemma 7.4.5 that $x'_j = e$. Letting $w' = x'_1...x'_k \in W$, it follows that $l(w') < k$. However, as $a_j(s_{i_1}...s_{i_k})_k = a(x'_1...x'_k)_k$, we must have that $w' = w$. So $l(w') = l(w) = k$, contradicting that $l(w') < k$. □

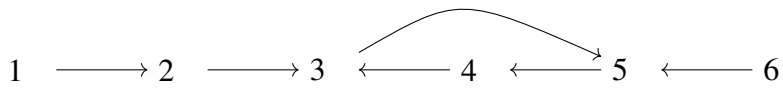
7.4.2 The Cluster Group Case

As highlighted previously, it is the difference in sign of the coefficient of α_i in the roots $w(\alpha_i)$ and α_i, when $l(ws_i) < l(w)$, that is key in proving that $l = l_I$ in the Coxeter case. We hoped to find a similar property for the cluster group, G_Q, associated to a quiver, Q, which appears in a seed of a cluster algebra of finite type A_n. In particular, we wanted to show the following holds for a positive companion basis $C = \{\gamma_i : 1 \leq i \leq n\}$ of Q.

> If $l(wt_i) < l(w)$ for $1 \leq i \leq n$ and $w \in G_Q$, then the coefficient of γ_i in the root $w(\gamma_i)$ is less than or equal to 0.

However, we can provide a counterexample to show that this is not the case. First, it is useful to introduce some notation. Let Q be a quiver of mutation-Dynkin type $A_{n_1} \sqcup ... \sqcup A_{n_r}$, where $n_1,...,n_r \in \mathbb{Z}^+$ and $n = \sum_{i=1}^r n_i$, with $r \geq 1$, and let $C = \{\gamma_i : 1 \leq i \leq n\}$ be a positive companion basis of Q, shown to exist by Lemma 5.3.4. For each $\gamma \in \Phi_C$ and $1 \leq i \leq n$, we write C_i^γ to denote the coefficient of γ_i in γ. Note that, by Lemma 5.3.8, $C_{\gamma_i}^\gamma \in \{-1,0,1\}$ for all $1 \leq i \leq n$ and $\gamma \in \Phi_C$.

Example 7.4.6. Let Q be the quiver



and consider the element $w = t_3t_2t_4t_5t_6t_3t_5t_1t_2t_3t_4t_3 \in G_Q$. To show this expression is reduced, by Proposition 4.2.11, we only need to show that $l(w) \notin \{0,2,4,6,8,10\}$. The computations required to verify this were performed using Maple^{TM} ([39]) and the exact code for this example is given in Appendix A.5.

We can see that $l(wt_3) < l(w)$, however

$$w(\gamma_3) = \gamma_3 - \gamma_5 + \gamma_6.$$

That is, $C_3^{w(\gamma_3)} > 0$.

However, the following conjecture, which has yet to be disproved, would be equally useful, if proved, for adapting the proof of [33, Theorem 5.5] to show that the length on a parabolic subgroup agrees with the length on the cluster group, for cluster groups associated to quivers appearing in the seeds of a cluster algebra of finite type A_n.

Conjecture 7.4.7. Let Q be a quiver of mutation-Dynkin type $A_{n_1} \sqcup ... \sqcup A_{n_r}$, where $n_1,...,n_r \in \mathbb{Z}^+$ and $n = \sum_{i=1}^{r} n_i$, with $r \geq 1$, and let $C = \{\gamma_i : 1 \leq i \leq n\}$ be a positive companion basis of Q. For any $w \in G_Q$ and $1 \leq i \leq n$, if

$$C_{\gamma_i}^{w(\gamma_j)} = \begin{cases} -1 & \text{if } i = j \text{ or } i \text{ and } j \text{ are connected by an arrow in } Q \\ 0 & \text{if } i \text{ and } j \text{ are not connected by an arrow in } Q \end{cases}$$

then $l(wt_i) < l(w)$.

A possible proof for this conjecture is by induction. In the induction step, for $w \in G_Q$, the result follows easily so long as there is a reduced expression $w = t_{i_1}...t_{i_k}$ such that $t_{i_1} \neq t_i$. The difficulty occurs in completing the induction step for elements whose reduced expressions all begin in t_i. That is, $w \in {}^I G$, where $I = T \setminus \{t_i\}$. By studying the forms of the reduced expressions of the elements in this set, we hope to find some insight into proving Conjecture 7.4.7.

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Appendices

Appendix A

The computations in the following sections were performed using Maple^{TM} ([39]). Maple is a trademark of Waterloo Maple Inc.

A.1 Example 7.2.2

To show the expression $t_2t_3t_1t_2$ is reduced, we perform the following computations using Maple^{TM} ([39]). To begin, we define the permutations `t[i]` to be $\pi_Q(t_i)$, for each $1 \leq i \leq 3$, and $w = \pi_Q(t_2t_3t_1t_2)$.

```
> with(GroupTheory);
> t[1] := Perm([[1, 2]]);
> t[2] := Perm([[2, 3]]);
> t[3] := Perm([[2, 4]]);
> w := PermProduct(t[2], t[1], t[3], t[2]);
```

The following code will check that no product of any two of these three permutations equals $\pi_Q(w)$. That is, we check that w has no expression of length 2 in G_Q.

```
> for i from 1 to 3 do
    for j from 1 to 3 do
     if not(i = j) then
      if PermProduct(t[i], t[j]) = w then
       print(i, j)
      fi
     fi
    od
```

```
od;
```

We remark that we include the instruction "`if not(i = j) then`" to reduce the number of computations, as for each $1 \leq i \leq 3$ we know that $t_i^2 = e$.

A.2 Proposition 7.2.4

To show the expression $t_1t_2t_3t_2$ is reduced, we perform the following computations using Maple^{TM} ([39]). To begin, we define the permutations `t[i]` to be $\pi_Q(t_i)$, for each $1 \leq i \leq 3$, and $w = \pi_Q(t_1t_2t_3t_2)$.

```
> with(GroupTheory);
> t[1] := Perm([[1, 2]]);
> t[2] := Perm([[2, 3]]);
> t[3] := Perm([[2, 4]]);
> w := PermProduct(t[1], t[2], t[3], t[2]);
```

The following code will check that no product of any two of these three permutations equals w. That is, we check that w has no expression of length 2 in G_Q.

```
> for i from 1 to 3 do
    for j from 1 to 3 do
     if not(i = j) then
      if PermProduct(t[i], t[j]) = w then
       print(i, j)
      fi
     fi
    od
   od;
```

A.3 Example 7.3.4

To show the expression $t_3t_2t_5t_3t_4$ is reduced, we perform the following computations using Maple^{TM} ([39]). To begin, we define the permutation `t[i]` to be $\pi_Q(t_i)$, for each $1 \leq i \leq 5$, and $w = \pi_Q(t_3t_2t_5t_3t_4)$.

```
> with(GroupTheory);
> t[1] := Perm([[1, 3]]);
> t[2] := Perm([[2, 3]]);
> t[3] := Perm([[3, 4]]);
> t[4] := Perm([[4, 5]]);
> t[5] := Perm([[4, 6]]);
> w := PermProduct(t[3], t[2], t[5], t[3], t[4]);
```

The following code will check that $w \neq t_i$ for all $1 \leq i \leq 5$.

```
> for i from 1 to 5 do
    if PermProduct(t[i]) = w then
    print(i)
    fi
  od:
```

The following code will check that no product of any three of these five permutations equals $\pi_Q(w)$. That is, we check that w has no expression of length 3 in G_Q.

```
> for i from 1 to 5 do
    for j from 1 to 5 do
     if not(i = j) then
      for k from 1 to 5 do
       if not(k = j) then
        if PermProduct(t[i], t[j], t[k]) = w then
         print(i, j, k)
        fi
       fi
```

```
      od
     fi
    od
   od:
```

A.4 Example 7.3.7

To show the expression $t_3t_2t_4t_5t_6t_3t_5t_1t_2$ is reduced, we perform the following computations using Maple^{TM} ([39]). To begin, we define the permutation `t[i]` to be $\pi_Q(t_i)$, for each $1 \leq i \leq 6$, and $w = \pi_Q(t_3t_2t_4t_5t_6t_3t_5t_1t_2)$.

```
> with(GroupTheory);
> t[1] := Perm([[1, 2]]);
> t[2] := Perm([[2, 3]]);
> t[3] := Perm([[3, 4]]);
> t[4] := Perm([[4, 5]]);
> t[5] := Perm([[4, 6]]);
> t[6] := Perm([[6, 7]]);
> w := PermProduct(t[3], t[2], t[4], t[5], t[6], t[3], t[5], t[1],
t[2]);
```

The following code will check that $w \neq t_i$ for all $1 \leq i \leq 6$.

```
> for i from 1 to 6 do
    if t[i] = w then
     print(i)
    fi
   od:
```

The following code will check that no product of any three of these six permutations equals $\pi_Q(w)$. That is, we check that w has no expression of length 3 in G_Q.

```
> for i from 1 to 6 do
    for j from 1 to 6 do
```

```
     if not(i = j) then
      for k from 1 to 6 do
       if not(k = j) then
        if PermProduct(t[i], t[j], t[k]) = w then
         print(i, j, k)
        fi
       fi
      od
     fi
    od
   od:
```

The following code will check that no product of any five of these six permutations equals $\pi_Q(w)$. That is, we check that w has no expression of length 5 in G_Q.

```
> for i from 1 to 6 do
    for j from 1 to 6 do
     if not(i = j) then
      for k from 1 to 6 do
       if not(k = j) then
        for l from 1 to 6 do
         if not(l = k) then
          for m from 1 to 6 do
           if not(m = l) then
            if PermProduct(t[i], t[j], t[k], t[l], t[m]) = w then
             print(i, j, k, l, m)
            fi
           fi
          od
         fi
        od
       fi
      od
     fi
```

```
     od
   od:
```

The following code will check that no product of any seven of these six permutations equals $\pi_Q(w)$. That is, we check that w has no expression of length 7 in G_Q.

```
> for i from 1 to 6 do
    for j from 1 to 6 do
    if not(i = j) then
    for k from 1 to 6 do
    if not(k = j) then
    for l from 1 to 6 do
    if not(l = k) then
    for m from 1 to 6 do
    if not(m = l) then
    for n from 1 to 6 do
    if not(n = m) then
    for 0 from 1 to 6 do
    if not(n = o) then
    if PermProduct(t[i], t[j], t[k], t[l], t[m], t[n], t[o]) = w
then
    print(i, j, k, l, m, n, o)
    fi
    fi
    od
    fi
    od
    fi
    od
    fi
    od
    fi
```

```
    od
   od:
```

The following code will show that no products of any nine of these six permutations, for which the first term in the product is not equal to `t[3]`, is equal to $\pi_Q(w)$. That is, all reduced expressions of w begin in t_3.

```
>for i from 1 to 6 do
    if not(i = 3) then
    for j from 1 to 6 do
    if not(i = j) then
    for k from 1 to 6 do
    if not(k = j) then
    for l from 1 to 6 do
    if not(k = l) then
    for m from 1 to 6 do
    if not(l = m) then
    for n from 1 to 6 do
    if not(n = m) then
    for o from 1 to 6 do
    if not(o = n) then
    for p from 1 to 6 do
    if not(p = o) then
    for q from 1 to 6 do
    if not(q = p) then
    if PermProduct(t[i], t[j], t[k], t[l], t[m], t[n], t[o], t[p],
t[q]) = w then
    print(i, j, k, l, m, n, o, p, q)
    fi
    fi
    od
    fi
    od
    fi
```

```
      od
      fi
     od
    fi
   od
  fi
 od
fi
od
fi
od;
```

A.5 Example 7.4.6

To show the expression $t_3t_2t_4t_5t_6t_3t_5t_1t_2t_3t_4t_3$ is reduced, we perform the following computations using Maple^{TM} ([39]). To begin, we define the permutation `t[i]` to be $\pi_Q(t_i)$, for each $1 \leq i \leq 6$, and $w = \pi_Q(t_3t_2t_4t_5t_6t_3t_5t_1t_2t_3t_4t_3)$.

```
> with(GroupTheory);
> t[1] := Perm([[1, 2]]);
> t[2] := Perm([[2, 3]]);
> t[3] := Perm([[3, 4]]);
> t[4] := Perm([[4, 5]]);
> t[5] := Perm([[4, 6]]);
> t[6] := Perm([[6, 7]]);
> w := PermProduct(t[3], t[2], t[4], t[5], t[6], t[3], t[5], t[1],
t[2], t[3], t[4], t[3]);
```

```
w:= (1, 5, 7, 4)(2, 6)
```

The following code will check that no product of any two of these six permutations equals $\pi_Q(w)$. That is, we check that w has no expression of length 2 in G_Q.

```
> for i from 1 to 6 do
    for j from 1 to 6 do
     if not(i = j) then
      if PermProduct(t[i], t[j]) = w then
       print(i, j)
      fi
     fi
    od
   od;
```

The following code will check that no product of any four of these six permutations equals $\pi_Q(w)$. That is, we check that w has no expression of length 4 in G_Q.

```
>for i from 1 to 6 do
    for j from 1 to 6 do
     if not(i = j) then
     for k from 1 to 6 do
      if not(k = j) then
       for l from 1 to 6 do
       if not(k = l) then
        if PermProduct(t[i], t[j], t[k], t[l]) = w then
         print(i, j, k, l)
        fi
       fi
      od
     fi
    od
   fi
  od
 od;
```

The following code will check that no product of any six of these six permutations equals $\pi_Q(w)$. That is, we check that w has no expression of length 6 in G_Q.

```
>for i from 1 to 6 do
```

```
    for j from 1 to 6 do
    if not(i = j) then
    for k from 1 to 6 do
    if not(k = j) then
    for l from 1 to 6 do
    if not(k = l) then
    for m from 1 to 6 do
    if not(l = m) then
    for n from 1 to 6 do
    if not(n = m) then
    if PermProduct(t[i], t[j], t[k], t[l], t[m], t[n]) = w then
    print(i, j, k, l, m, n)
    fi
    fi
    od
    fi
    od
    fi
    od
    fi
   od;
```

The following code will check that no product of any eight of these six permutations equals $\pi_Q(w)$. That is, we check that w has no expression of length 8 in G_Q.

```
>for i from 1 to 6 do
    for j from 1 to 6 do
    if not(i = j) then
    for k from 1 to 6 do
    if not(k = j) then
    for l from 1 to 6 do
    if not(k = l) then
```

```
      for m from 1 to 6 do
      if not(l = m) then
      for n from 1 to 6 do
      if not(n = m) then
      for o from 1 to 6 do
      if not(o = n) then
      for p from 1 to 6 do
      if not(p = o) then
      if PermProduct(t[i], t[j], t[k], t[l], t[m], t[n], t[o], t[p])
= w then
       print(i, j, k, l, m, n, o, p)
      fi
      fi
      od
      fi
      od
      fi
      od
      fi
      od
      fi
     od
     fi
    od
   od;
```

The following code will check that no product of any ten of these six permutations equals $\pi_Q(w)$. That is, we check that w has no expression of length 10 in G_Q.

```
>for i from 1 to 6 do
    for j from 1 to 6 do
    if not(i = j) then
```

```
     for k from 1 to 6 do
     if not(k = j) then
     for l from 1 to 6 do
     if not(k = l) then
     for m from 1 to 6 do
     if not(l = m) then
     for n from 1 to 6 do
     if not(n = m) then
     for o from 1 to 6 do
     if not(o = n) then
     for p from 1 to 6 do
     if not(p = o) then
     for q from 1 to 6 do
     if not(q = p) then
     for r from 1 to 6 do
     if not(q = r) then
     if PermProduct(t[i], t[j], t[k], t[l], t[m], t[n], t[o], t[p],
t[q], t[r]) = w then
     print(i, j, k, l, m, n, o, p, q, r)
     fi
     fi
     od
     fi
     od
     fi
     od
     fi
     od
     fi
     od
     fi
     od
```

```
      fi
     od
    fi
   od
od;
```