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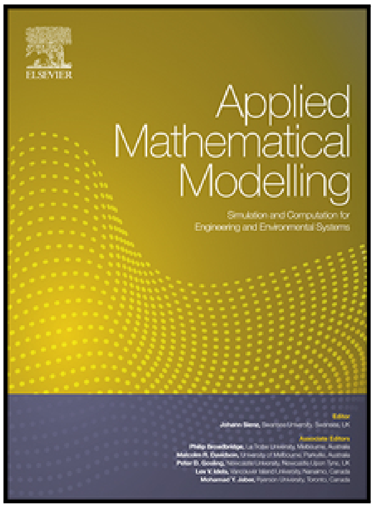

The continuous resupply policy for deteriorating items with stock-dependent observable demand in a two-warehouse and two-echelon supply chain

Yousef Ghiami, Patrick Beullens

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Highlights:

Retailer with stock-dependent demand, two warehouses, and deterioration

Continuous resupply from RW to OW proposed to keep stock in OW high

Net present value functions to maximise profits derived

Integrated distribution pattern achieved across retailer and wholesaler

Continuous resupply better than common OW/RW policy for some situations

The continuous resupply policy for deteriorating items with stock-dependent observable demand in a two-warehouse and two-echelon supply chain

Yousef Ghiami[1]
Department of Supply Chain Analytics,
Vrije Universiteit Amsterdam, Amsterdam, The Netherlands, Y.Ghiami@vu.nl

Patrick Beullens
Mathematical Sciences and Southampton Business School and CORMSIS,
University of Southampton, United Kingdom, P.Beullens@soton.ac.uk

Abstract

This paper expands previous work on stock-dependent demand for a retailer with a two-warehouse (OW/RW) situation to the case of deteriorating items and where the retailer seeks to obtain the integrated optimal distribution policy from collaboration with a supplier. Motivated by practical applications and recent literature, a policy is considered whereby products in good order from the retailer's back-room (RW) are frequently transferred to its capacitated main store OW. Because the demand depends on the stock of good products in the OW, the aim is to keep this stock at its full capacity with products in good condition, and this can be done for as long as the RW stock of good products is positive. A firm's objective function is the Net Present Value (NPV) of the firm's future cash-flows. The profit functions are developed for both this continuous resupply policy and the commonly used policy in the OW/RW literature. Numerical examples are included and have been solved with grid search methods. The examples illustrate the benefits of adopting the continuous resupply policy, and also collaboration between the retailer and the wholesaler. Moreover, it is shown how these benefits can be shared by small adjustments to the product's unit price between the firms.

Keywords: Net Present Value; deteriorating inventory; capacity constraint; shortages

[1]Corresponding Author

1 Introduction

In the grocery retail industry, a major part of the product portfolio consists of deteriorating items; products that have a limited shelf life and lose quality over time. The quality of these items highly depends on the keeping and handling conditions at different stages of the supply chain. These conditions include factors such as temperature, pressure, and humidity. All the actors in the chain would then need to utilise appropriate infrastructure that well respects the keeping condition requirements of such items. However, this is hard to achieve since in real-life cases, actors in the chain are usually quite diverse in terms of facility and equipment [1]. This highlights the importance of employing logistics models that not only incorporate the deterioration property, but also consider the keeping conditions of different stages of the chain.

A grocery retailer meets the demand for numerous deteriorating items on a daily basis. To this end, the retailer stores some quantity of those items at a temperature-controlled back-room and uses this inventory to replenish the refrigerators (shelves) in the front room. Customers frequently visit these fridges to pick the products of their choice which results in a loss of energy and exposes the remaining items to a fluctuating temperature. The quantities stored in the back-room, however, do not undergo such conditions. This negative impact of keeping items on shelves is also observed for another group of items that do not necessarily have strict temperature requirements, such as fruits. For such items, the deterioration rate increases when unpacked and displayed on shelves compared to the situation where these are packed (and inaccessible by customers) in the back-room.

Although it is relatively more expensive to display the items on shelves (e.g., due to a higher deterioration rate and energy losses), it is crucial for fulfilling the demand. Having full shelves becomes more important for items with stock-dependent demand. Dealing with such setting requires making decisions on quantity and frequency of the replenishment. These decisions are part of almost every retailer's day to day business and yet not sufficiently investigated by the research community. The subset of the deteriorating item inventory management literature that is devoted to such applications is known as "two-warehouse" models.

In a two-warehouse setting, a retailer has a limited capacity at the own warehouse (OW) and therefore it may be necessary to rent some extra warehouse capacity (RW) to be able to fulfill the demand. Based on the assumptions made in the literature, not all the OW/RW models are applicable to the setting introduced in this paper. In some cases the RW represents a warehouse that belongs to an independent firm, whereas, in others the RW could be interpreted as the backroom of the same retailer. The continuous resupply policy studied in this paper might improve its relevance for the latter case. Nevertheless, the logic mostly adopted in the OW/RW literature does have its own important areas of applications.

In practice, adopting the continuous resupply policy would result in a higher cost for the system, e.g., more frequent replenishment would require a larger number of personnel. The implication of this policy would be higher values for holding cost parameters compared to systems that adopt the conventional resupply policy; in a supermarket setting, a manager should assign more personnel to frequently resupply/replenish the shelves.

In the main stream of the two-warehouse models, the researchers assume that since the holding cost is higher at the RW, the retailer starts fulfilling the demand from the RW until its inventory level reaches zero. The retailer then uses the inventory stored at the OW to meet the demand. Since in such models both warehouses are directly used to fulfill the end customers' demand, the implicit assumption is that, from customers' point of view, the OW and RW are located in the same place or sufficiently close. Moreover, the researchers assume that items are not moved between the two warehouses while the demand is met using the inventory stored at the RW. This implies that during the time that inventory level at the RW is positive, the inventory level at the OW goes down only due to deterioration. This means that for the case of stock-dependent demand[2], the retailer loses some potential demand while it could have been avoided by constantly transferring items from the RW to the OW. This assumption is not unrealistic since the two warehouses are located in the same place.

The contribution of this paper is threefold; (1) we introduce a new resupply policy between the two warehouses, (2) we include the supplier (wholesaler) and evaluate the performance of the two-echelon model with the net present value (NPV) approach, and (3) we compare the continuous resupply policy with the

[2]In the literature when the demand is assumed to be stock-dependent, it is a function of the inventory level at the OW which is visible by the end customer, i.e. shelves.

conventional policy introduced in the literature to see which policy performs better for the two-echelon supply chain defined in this paper. The paper is further organised as follows. In Section 2, we present an overview of the two-warehouse inventory literature. Section 3 presents the notations, the modelling assumptions, the inventory activities, and the cash-flow structures of the retailer and the wholesaler. In Section 4, the NPV functions for the retailer and the wholesaler under the continuous resupply policy are derived. Section 5 presents these functions under the common resupply policy. Numerical examples are presented in Section 6 to demonstrate the value of collaboration under the continuous resupply policy. Finally, Section 7 summarises the findings of this study and suggests future research possibilities.

2 Literature review

The literature of deteriorating inventory expands across several decades now. For an overview of the literature, see Nahmias [2], Raafat [3], Goyal and Giri [4], and Li et al. [5]. First applications of deteriorating item inventory models were inspired by blood bank systems and later were expanded to embrace examples such as fruits, vegetables, and flowers. In terms of deterioration pattern, perishable items are quite diverse, however, we can generally identify two main categories; (1) instantaneous deteriorating items, the quality of this group stays unchanged for a specific period of time, i.e., expiry date, after which the items immediately lose the whole value, hence should be discarded, examples of such items include blood units and dairy products, (2) non-instantaneous deteriorating items, this group is highly sensitive to the keeping conditions and part of the on-hand inventory loses its quality and therefore seems less attractive to the end customer. Similar to the first category, this group of items may also have expiry date, however, since they are highly sensitive to keeping condition, they may start gradually losing their fresh appearance before the expiry date. Retailers have different policies towards these deteriorating items when quality degradation occurs, for instance they may sell the items with a markdown. Fresh salad, cut flowers, and loose fruits are in this category of deteriorating items. In this paper, we focus on the subset of the literature that studies the latter category.

In the literature of inventory management, most researchers assume an unlimited warehouse capacity, see, e.g., Ghiami and Williams [6] Tai et al. [7]. This assumption, however, is not practical for some cases. A retailer that is located on a high street, for instance, needs to be very precise about replenishment policies due to the warehouse capacity constraints. Sarma [8] was first to assume a limited capacity for an inventory model and suggested the use of an external warehouse. Thereafter, different streams of literature evolved around the OW/RW setting for various applications for deteriorating items.

A group of researchers study applications in which a retailer can have permissible delays in payment. This agreement would be helpful especially when the retailer has limited cash to instantaneously settle the account with the suppliers. Liang and Zhou [9] study an inventory model with a constant demand rate that does not allow for shortages. In their model, they assume that the deterioration rate at the OW is higher than the RW. Jaggi et al. [10] consider a two-warehouse inventory model with price-sensitive demand and study the effect of trade credit period on the retailer's performance . Tiwari et al. [11] extend the model introduced by Liang and Zhou [9] by allowing for shortages and study the effects of inflation on the inventory policies. Chakraborty et al. [12] analyse a system with permissible delay in payment in which there is a ramp type demand and items are subject to Weibull distribution deterioration.

A stream of research on two-warehouse models focuses on different inventory policies that the retailer may adopt. Pakkala and Achary [13] study a production-inventory model with Last-In-First-out (LIFO) policy in which the manufacturer first stores produced items at the OW and if there is any production surplus, they are stored at the RW. To fulfill the demand, however, the manufacturer uses the inventory stored at the RW first. Lee [14] builds upon the model developed by Pakkala and Achary [13] and compare LIFO with First-In-First-out (FIFO) policy over a finite horizon. Jaggi et al. [10] study a special case of these production-inventory models where production rate is infinite, i.e., inventory model, and compare different dispatching policies, including the effects of permissible delays in payments. Assuming a finite planning horizon, Xu et al. [15] extend the model developed by Jaggi et al. [10] modifying those policies for the problem under study. In another study, Alamri and Syntetos [16] develop a two-warehouse system in which a percentage of every replenishment quantity is defective and therefore there is a need for a screening operation. The authors investigate the effect of applying a policy that simultaneously uses the items stored at the OW and RW.

A subset of the literature on the OW/RW models investigates the effect of time-value of money on inventory and production-inventory settings. Yang [17] studies a model with two warehouses in which shortages are allowed. He compares two different scenarios, (1) having the stock-out period at the start of the inventory cycle, and (2) the common style in the literature where the shortage period occurs at the end of each cycle. Dey et al. [18] study an OW/RW setting with finite horizon in which shortages are allowed and the aggregated inventory level of the two warehouses is considered for the analysis. Yang and Chang [19] investigate a two-warehouse model in which the retailer has the option of permissible delay in settling the account with the supplier. Bhunia et al. [20] extend the model developed by Yang [17] assuming a time-dependent demand function. Tiwari et al. [21] investigate a similar system assuming that shortages are allowed and partially backlogged. Considering complete backlogging, Jonas [22] develops a two-echelon production-inventory model in which a manufacturer and a supplier jointly optimise their policies and the manufacturer has a credit period for paying the supplier.

In the literature of OW/RW models, stock-dependent demand has not received much attention. Zhou and Yang [23] may be the first to consider this demand pattern. The stock-dependent demand has then been used in several papers, see, e.g., Tiwari et al. [11] and [21]. It is interesting and insightful to see how the application of the continuous resupply policy may boost the sale since it keeps the OW (shelves) full. An interesting paper that further motivates our paper is Panda et al. [24], in which the value of having full shelf space in the store is discussed and in which the trade-off between a larger capacity in an OW/RW situation versus more extensive use of a RW is analysed (the capacity of the OW being the shelf space allocated to the product). The expansion to the case of item deterioration, as undertaken in this paper, implies that choosing a continuous resupply policy as well would be a natural choice.

While many articles in the deteriorating inventory literature develop NPV functions based on costs only, it is important to develop an objective function that also takes into account the revenues (Ghiami and Beullens [25]). There are two main reasons for this. First, demand increases with the inventory level of good products at the OW. Because, in general, it cannot be guaranteed that this level will equal the capacity of the OW at all times, one must account for demand and hence revenue fluctuations for the retailer over time. Second, a model only based on cost will also prove difficult for accurately deriving the

NPV function of the supplier since part of the costs of the retailer will translate into revenues for the supplier, and hence the need for considering the revenue stream for the supplier, too. See also Beullens [26] for a discussion on the impact that a lack of recognition of revenues shifting over time has had on inventory research.

To our knowledge, no study has analysed the effect of continuous resupply policy when the demand is stock-dependent. This would be interesting to study since the continuous resupply policy tends to keep the inventory level at the OW as high as possible, hence, introduces a totally different dynamics into the inventory system. Moreover, including the supplier in the model would provide the managers with insights into the buyer-supplier relation and the implications of such collaborations for each of the players in this supply chain. In the OW/RW literature, Ghiami et al. [27] and Jonas [22] have presented a multi-echelon model and item deterioration with respectively stock-dependent and constant demand rates. In a single-echelon setting, Dey et al. [18] study a two-warehouse model applying a continuous resupply policy with time-dependent demand.

In situations where the RW is not in the vicinity of the OW, a continuous resupply policy might be too expensive. Such models assume a transport costs from RW to OW. Zhou and Yang [23] study a two-warehouse model in which the demand is met using the inventory stored in the OW, and the RW is used to replenish the OW in bulks. This two-echelon system is to deliver a non-deteriorating items for which shortages are not allowed.

3 Modelling assumptions and parameters

3.1 Notation

Table 1 lists the notations that we use in this paper.

Table 1: Parameters used for modelling the inventory system

notation	description	notation	description
$D(t)$	demand rate which is $y + zI_o(t)$	d_W	unit disposal cost at the wholesaler
y	constant component of the demand rate	θ_o	deterioration rate at the OW
z	coefficient that relates the demand to the inventory level	θ_r	deterioration rate at the RW
		θ	deterioration rate at the wholesaler
s_R	fixed ordering cost at the retailer	f_o	unit holding cost per unit of time at the OW
s_W	fixed ordering cost at the wholesaler	f_r	unit holding cost per unit of time at the RW
p	sales price	f	unit holding cost per unit of time at the wholesaler
p_R	retailer purchasing price	g	deposit paid by customer in the case of shortage
p_W	wholesaler purchasing price	r	reduction in price for backordered items
α	discounting rate	b	unit shortage cost per unit of time
β	backlogging rate	π	unit lost sale cost
d_R	unit disposal cost at the retailer	W	capacity at the OW

3.2 The activity

In this study, we consider an integrated system that includes a retailer and a wholesaler. The retailer cooperates with the wholesaler to deliver a deteriorating product. Figure 1 illustrates how the stock position changes over time at the retailer. The demand at the retailer is defined as $D(t) = y + zI_o(t)$, where y and z are constants and $I_o(t)$ represents the inventory level at the OW at time t. Similar demand pattern has been used in the literature by, e.g., Min et al. [28], Chung and Cárdenas-Barrón [29], and Zanoni and Jaber [30]. The retailer places an order to the supplier and instantaneously receives the batch at $t = 0$. The retailer then stores W units of the received quantity in the OW and uses the RW for storing the excess inventory. During the time interval between $t = 0$ and $t = t_r$, the demand realises at the rate $D(t) = y + zW$ as the OW is full. As long as the inventory level at the RW is positive, the retailer continuously replenishes the OW using the items stored at the RW. We assume that the retailer follows a FIFO policy for replenishing the OW, e.g., putting the newly replenished quantity at the back of the shelf.

At time $t = t_r$, the RW inventory level reaches zero, therefore the retailer starts using the OW to meet the demand. During the time interval between $t = t_r$ and $t = t_o$, the inventory level at the OW decreases with the varying demand rate of $D(t) = y + zI_o(t)$. At $t = t_o$ the OW runs out of inventory, hence the demand rate drops to y and stays at this level until the next replenishment ($t = T_R$). During this shortage period, only a percentage of the demand (β) is backlogged while the rest is lost. On the arrival of the next batch, the backlogged demand is met immediately. The described inventory position between 0 and T_R at the retailer takes place over intervals of length T_R (decision variable) at infinitum. Figure 1 graphically illustrates the inventory level during one inventory cycle. In section 4, we show how T_R consists of two independent components, i.e., t_r and t_s, for which the model should find the optimal values.

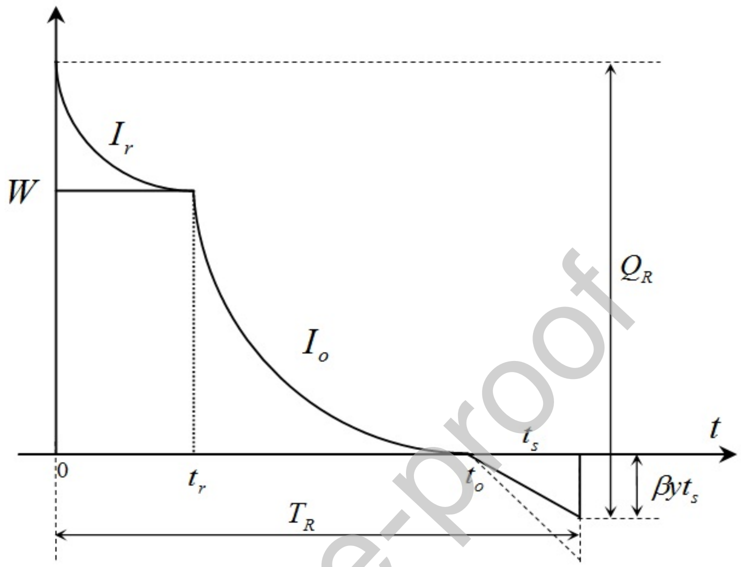

Figure 1: Inventory level at the Retailer

Each inventory cycle at the wholesaler is to cover k inventory cycles of the retailer ($T_W = kT_R$). This is another decision variable of the model. At the start of each inventory cycle of the wholesaler, a quantity of Q_W is received from the upstream manufacturer/supplier. The wholesaler then immediately dispatches a quantity of Q_R to the retailer. This results in drops of size Q_R in the wholesaler inventory level in T_R intervals. Between each two deliveries to the retailer, the inventory level at the wholesaler decreases due to a constant rate of deterioration (θ). This pattern continues until the inventory level reaches zero at $t = (k-1)T_R$. At time $t = kT_R$ the wholesaler replenishes the inventory and immediately sends the next batch (Q_R) to the retailer. At this point, the inventory position is exactly as it was at $t = 0$. This logistics pattern starts at $t = ikT_R$ ($i = 0, 1, 2, \dots$) and lasts for kT_R units of time, see Figure 2. It should be noted that since there is no backlogged demand at $t = 0$, the quantity sent to the retailer for the first period is $Q_R - \beta y t_s$.

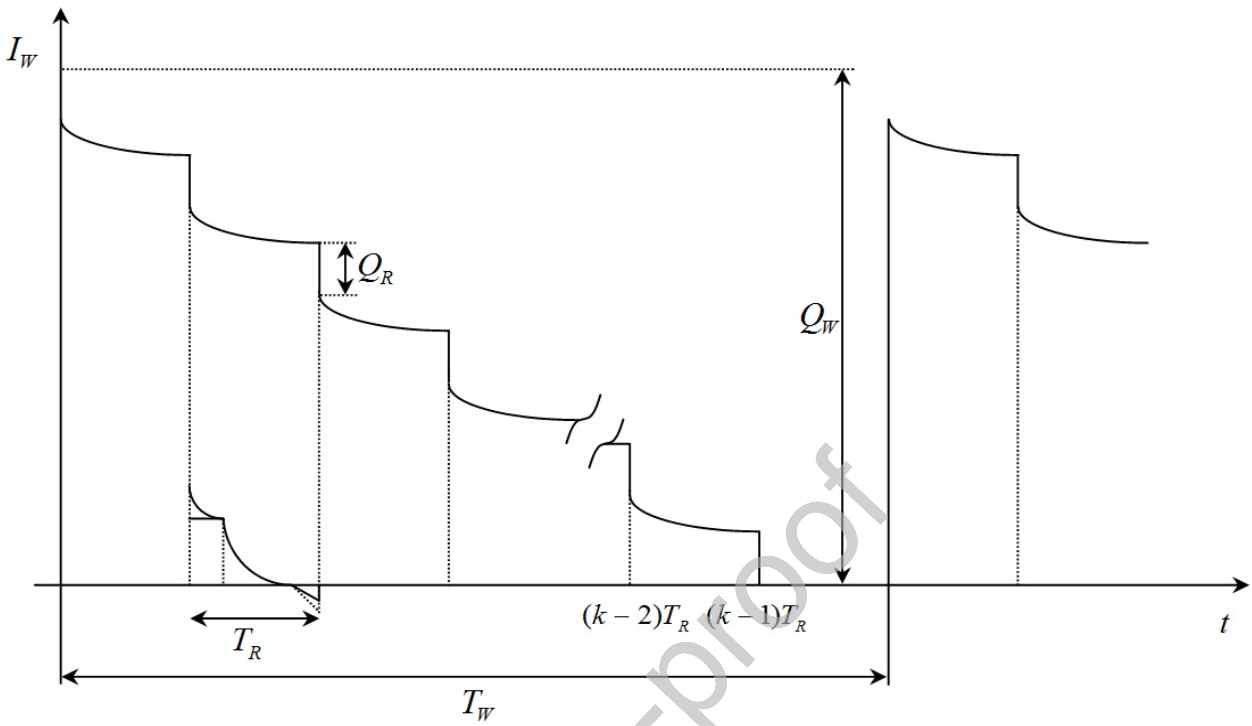

Figure 2: Inventory level at the Wholesaler

3.3 The cash-flows

The objective of this integrated supply chain is to maximise the NPV of all future cash-flows, see Ghiami and Beullens [25]. With this regard, the payment structure of the whole supply chain is considered. The transfer prices between the two members of this two-echelon model do not have any impact on the optimal solution of the integrated system as payment symmetry and equal opportunity cost are assumed, see Beullens and Janssens [31]. For comparison purposes, however, it is useful to explicitly include the transfer prices as this will allow the comparison of an integrated approach to the one in which the firms act independently. The payment structures of the retailer and the wholesaler are illustrated in Figures 3 and 4.

It is assumed that when the demand arises and the retailer's stock position is strictly positive, the relevant revenue is immediately received by the retailer. Since the inventory level at the OW is at maximum during the period associated with t_r, the annuity stream of the revenue over this period is $p(y + zW)$. During the time interval between t_r and t_o, the revenue at the retailer is a continuous function of the

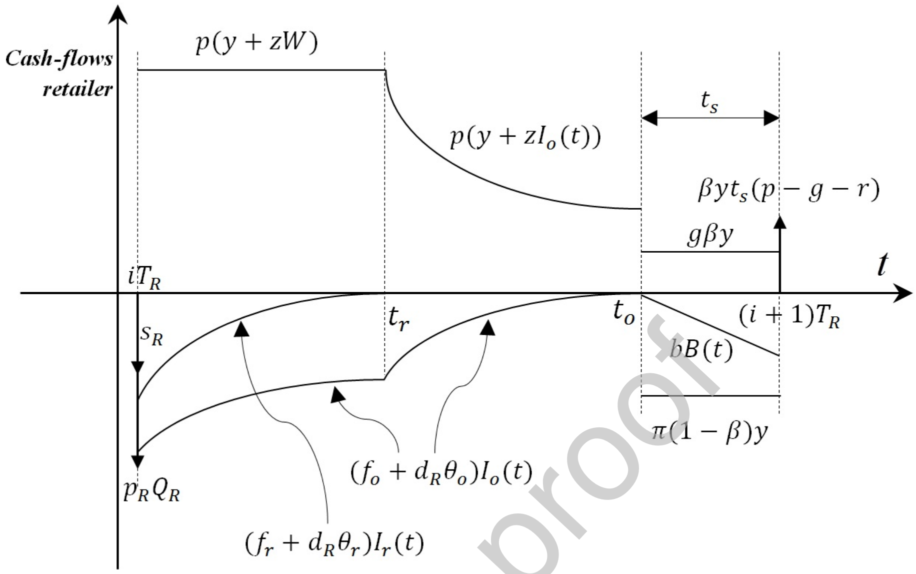

Figure 3: Cash-flow structure at the retailer

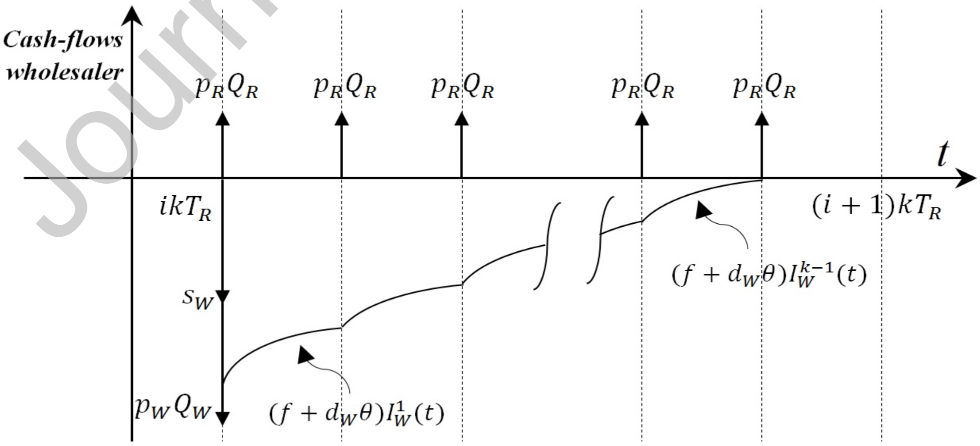

Figure 4: Cash-flow structure at the wholesaler

inventory level, $p(y + zI_o(t))$. Shortage period starts at $t = t_o$ after which the retailer receives a deposit $g \geq 0$ for each unit of the demand which is backlogged. This creates an annuity steam revenue of $g\beta y$ between t_o and T_R. During this shortage period, the retailer incurs a unit backorder cost of b per unit of time. The retailer also has to pay a penalty of π for each unit of lost sale which creates an annuity stream cost of $\pi(1-\beta)y$ during the stock-out period. At the end of an inventory period, when the retailer replenishes the inventory level, the backlogged demand is met immediately and the retailer gives a price reduction of $r \geq 0$ per unit of backordered item which creates a revenue of $\beta y t_s(p - g - r)$.

The retailer incurs a set-up cost of s_R at the beginning of each inventory period when placing an order to the wholesaler. At the same time, the retailer should pay the purchasing price of $p_R Q_R$ to the wholesaler, except for the first inventory period where it is $p_R(Q_R - \beta y t_s)$. The retailer pays an out-of-pocket holding cost of f_o and f_r per unit of time per unit of item stored at the OW and the RW, respectively. Each deteriorated item at the OW and the RW creates an instantaneous out-of-pocket cost of d_R which is paid to a recycling company to dispose of.

The wholesaler receives lump sum revenues of $p_R Q_R$ in T_R intervals, except for the first revenue at $t = 0$ where it is $p_R(Q_R - \beta y t_s)$. At the start of each inventory period, i.e., at $t = iT_W$ $(i = 0, 1, 2, \dots)$, the wholesaler incurs a fixed set-up cost of s_W to place an order to the upstream manufacturer/supplier. The wholesaler also pays the purchasing cost to the upstream manufacturer/supplier at $t = iT_W$ $(i = 0, 1, 2, \dots)$ proportional to the order quantity Q_W. There is an out-of-pocket holding cost of f for inventory at the wholesaler which should be paid per item per unit of time. During the in-stock period, each deteriorated item creates a disposal cost of d_W for the wholesaler that should be paid to a recycling company.

4 Model under the continuous resupply policy

4.1 Inventory and shortage levels at the retailer

In this research, it is assumed that the demand is a function of the on-hand inventory in the OW. The following shows the demand function

$$D(t) = y + zI_o(t). \tag{1}$$

Under the continuous resupply policy, the retailer aims at keeping the inventory level of the OW at maximum. To this end, the retailer replaces both deteriorated and sold items at the OW with fresh ones from the RW. At the same time, the inventory level at the RW further decreases due to deterioration. The inventory level finally reaches zero at $t = t_r$, therefore $I_r(t_r) = 0$, which is used as a boundary condition to find the inventory level at the RW. The following differential equation represents the change of inventory level at the RW:

$$\frac{dI_r(t)}{dt} = -(zW + y) - \theta_r I_r(t) - \theta_o W, \qquad 0 \leq t \leq t_r. \tag{2}$$

By solving the differential equation presented in (2), the inventory level of the RW during this interval is obtained:

$$I_r(t) = \frac{(z + \theta_o)W + y}{\theta_r}(e^{\theta_r(t_r - t)} - 1), \qquad 0 \leq t \leq t_r. \tag{3}$$

The inventory level at the OW remains unchanged between $t = 0$ and $t = t_r$ $(dI_o(t)/dt = 0)$ with the initial inventory level of W, therefore

$$I_o(t) = W, \qquad 0 \leq t \leq t_r. \tag{4}$$

The retailer starts using the items stored at the OW at $t = t_r$ when the inventory at the RW is totally depleted. The following differential equation shows how the inventory level at the RW changes due to demand and deterioration until it reaches zero at $t = t_o$:

$$\frac{dI_o(t)}{dt} = -zI_o(t) - y - \theta_o I_o(t), \qquad t_r \leq t \leq t_o, \tag{5}$$

which results in the following inventory level for the OW:

$$I_o(t) = \frac{y}{z + \theta_o}(e^{(z+\theta_o)(t_o - t)} - 1), \qquad t_r \leq t \leq t_o. \tag{6}$$

The shortage period starts at $t = t_o$ when the retailer runs out of inventory at the OW and lasts until the end of the cycle, i.e., $t = T_R$. During this shortage period only a percentage, β, of the demand is backordered. The shortage level is then presented by the following differential equation:

$$\frac{dB(t)}{dt} = \beta y, \qquad t_o \leq t \leq T_R. \tag{7}$$

Considering the relevant boundary condition, $B(t_o) = 0$, the shortage level is

$$B(t) = \beta y(t - t_o), \qquad t_o \leq t \leq T_R. \tag{8}$$

To formalise the interdependency between the decision variables, one should note that (4) and (6) give the same value for $I_o(t)$ at $t = t_r$, therefore

$$t_o = t_r + \frac{1}{z+\theta_o} ln\left(1 + \frac{z+\theta_o}{y} W\right). \tag{9}$$

The inventory period at the retailer is then obtained as follows

$$T_R = t_r + \frac{1}{z+\theta_o} ln(1 + \frac{z+\theta_o}{y} W) + t_s. \tag{10}$$

The retailer's batch size for the second period onward is the sum of the backordered items and the initial inventory level:

$$\begin{aligned} Q_R =& I_r(0) + I_o(0) + B(T_R) \\ =& \frac{(z+\theta_o)W + y}{\theta_r}(e^{\theta_r t_r} - 1) + W + \beta y t_s. \end{aligned} \tag{11}$$

4.2 Inventory level at the wholesaler

The wholesaler covers k inventory cycles of the retailer during each inventory period, T_W. This divides the inventory cycle at the wholesaler into k intervals of length T_R. The stock level at the wholesaler drops by Q_R at the beginning of each of these intervals when a batch is sent to the retailer. During each interval, the inventory level at the wholesaler goes down due to deterioration. Based on intuition, in any inventory cycle, it is optimal for the wholesaler to have exactly Q_R units in stock at the time of sending the k^{th} replenishment quantity to the retailer. This means that after dispatching this quantity, the wholesaler will be out of stock until the next order from the upstream manufacturer/supplier is received. The following differential equation illustrates the change in the stock level over the i^{th} time interval:

$$\frac{dI_W^i(t)}{dt} = -\theta I_W^i(t), \qquad i = 1, 2, \dots k-1. \tag{12}$$

The inventory level during this interval is then (see Appendix A)

$$I_W^i(t) = Q_R e^{\theta(iT_R - t)} \frac{e^{\theta(k-i)T_R} - 1}{e^{\theta T_R} - 1}, \qquad (i-1)T_R \leq t \leq iT_R, \qquad i = 1, 2, ..., k-1. \tag{13}$$

Using (13), we obtain the inventory level at the wholesaler at $t = 0$, just after sending the first batch to the retailer:

$$I_W^1(0) = Q_R \frac{e^{k\theta T_R} - e^{\theta T_R}}{e^{\theta T_R} - 1}. \tag{14}$$

The wholesaler order quantity is then given by $I_W^1(0) + Q_R$, therefore

$$Q_W = Q_R \frac{e^{k\theta T_R} - 1}{e^{\theta T_R} - 1}. \tag{15}$$

The first order quantity of the wholesaler ($t = 0$) is $Q_W - \beta y t_s$, since there is no backlogged demand at the retailer. In the following sections, the relevant revenue and cost functions of the parties involved in this supply chain are analysed.

4.3 Revenues and costs at the retailer

In this research work, in order to aggregate all the costs and revenues of the supply chain as a whole, we normalise all the cash-flows taking place between this two-echelon system and external parties. To this end, the equivalent annuity stream of all these cash-flows are obtained and added to the total profit function. For detailed explanation and mathematical analysis of annuity stream functions see Grubbström [32]. To see a detailed explanation on how to calculate the annuity stream of costs and revenues for deteriorating item inventory models see, e.g., Ghiami [33] and Ghiami and Beullens [25].

In each inventory cycle, the retailer earns a revenue of $p(y + zW)$ over the time interval between $t = 0$ and $t = t_r$. The equivalent annuity stream of such revenues over an infinite horizon is

$$ASR_{R1} = p(y + zW)\frac{1 - e^{-\alpha t_r}}{1 - e^{-\alpha T_R}}. \tag{16}$$

The revenue function of the retailer changes to $p(y + zI_o(t))$ between t_r and t_o. The equivalent annuity stream of these revenues over an infinite horizon is hence given by

$$\begin{aligned} ASR_{R2} =& \frac{\alpha}{1 - e^{-\alpha T_R}} \int_{t_r}^{t_o} p(y + zI_o(t))e^{-\alpha t} dt \\ =& pye^{-\alpha t_o} \frac{e^{\alpha(t_o - t_r)} - 1}{1 - e^{-\alpha T_R}} \\ &+ \frac{\alpha pyze^{-\alpha t_o}}{(z + \theta_o)(1 - e^{-\alpha T_R})} \left[\frac{1}{\alpha + z + \theta_o}(e^{(\alpha + z + \theta_o)(t_o - t_r)} - 1) - \frac{1}{\alpha}(e^{\alpha(t_o - t_r)} - 1) \right]. \end{aligned} \tag{17}$$

Over the shortage period (between $t = t_o$ and $t = T_R$) in each inventory cycle, the retailer receives a deposit of g for each backordered item which creates an annuity revenue of $g\beta y$ in that cycle. The equivalent annuity stream of revenues obtained from the deposits over an infinite horizon is

$$ASR_{R3} = g\beta y \frac{e^{-\alpha t_o} - e^{-\alpha T_R}}{1 - e^{-\alpha T_R}}. \tag{18}$$

Immediately after each replenishment at the retailer, the backordered demand is met, creating a lump sum revenue of $(p - g - r)\beta y t_s$. The equivalent annuity stream of all these revenues over an infinite horizon is as follows:

$$ASR_{R4} = (p - g - r)\beta y t_s \frac{\alpha e^{-\alpha T_R}}{1 - e^{-\alpha T_R}}. \tag{19}$$

The annuity stream of all revenues at the retailer over an infinite horizon is then given by

$$ASR_R = ASR_{R1} + ASR_{R2} + ASR_{R3} + ASR_{R4}. \tag{20}$$

At the start of each inventory cycle, the retailer incurs a set-up cost of s_R. The annuity stream of all set-up costs over an infinite horizon is given by

$$SC_R = s_R \frac{\alpha}{1 - e^{-\alpha T_R}}. \tag{21}$$

The retailer pays the purchasing price at the beginning of each inventory cycle when placing an order. It should be noted that the order quantity of the first inventory cycle is smaller as there is not backordered demand. The annuity stream of all purchasing costs over an infinite horizon is then

$$PC_R = p_R Q_R \frac{\alpha}{1 - e^{-\alpha T_R}} - \alpha p_R \beta y t_s. \tag{22}$$

The equivalent annuity stream of all out-of-pocket holding costs at the OW and RW over an infinite horizon are respectively given by

$$\begin{aligned} HC_{OW} =& \frac{\alpha}{1 - e^{-\alpha T_R}} \left[f_o \int_0^{t_o} I_o(t) e^{-\alpha t} dt \right] \\ =& f_o W \frac{1 - e^{-\alpha t_r}}{1 - e^{-\alpha T_R}} \\ &+ \frac{f_o y}{z + \theta_o} \left[\frac{\alpha}{z + \theta_o + \alpha} \left(\frac{e^{(z+\theta_o)(t_o - t_r) - \alpha t_r} - e^{-\alpha t_o}}{1 - e^{-\alpha T_R}} \right) - \frac{e^{-\alpha t_r} - e^{-\alpha t_o}}{1 - e^{-\alpha T_R}} \right], \end{aligned} \tag{23}$$

and

$$\begin{aligned} HC_{RW} =& \frac{\alpha}{1 - e^{-\alpha T_R}} \left[f_r \int_0^{t_r} I_r(t) e^{-\alpha t} dt \right] \\ =& \frac{f_r \left((z + \theta_o) W + y \right)}{\theta_r} \left[\frac{\alpha}{\alpha + \theta_r} \left(\frac{e^{\theta_r t_r} - 1}{1 - e^{-\alpha T_R}} \right) - \frac{\theta_r}{\alpha + \theta_r} \left(\frac{1 - e^{-\alpha t_r}}{1 - e^{-\alpha T_R}} \right) \right]. \end{aligned} \tag{24}$$

The annuity stream of the overall holding cost at the retailer is then

$$HC_R = HC_{OW} + HC_{RW}. \tag{25}$$

Since we need the terms $\int_0^{t_o} I_o(t) e^{-\alpha t} dt$ and $\int_0^{t_r} I_r(t) e^{-\alpha t} dt$ for also deterioration cost calculations, we have included the detailed mathematical workout of these two terms in Appendix B.

The retailer pays a cost of d_R per unit of deteriorated item to a recycling company to dump/recycle the item. This cost is incurred immediately after an item deteriorates. The annuity stream of the deterioration cost of all inventory cycles at the OW and RW over an infinite horizon are respectively given by

$$\begin{aligned} DC_{OW} =& d_R\theta_o W \frac{1-e^{-\alpha t_r}}{1-e^{-\alpha T_R}} \\ &+ \frac{d_R\theta_o y}{z+\theta_o}\left[\frac{\alpha}{z+\theta_o+\alpha}\left(\frac{e^{(z+\theta_o)(t_o-t_r)-\alpha t_r}-e^{-\alpha t_o}}{1-e^{-\alpha T_R}}\right)-\frac{e^{-\alpha t_r}-e^{-\alpha t_o}}{1-e^{-\alpha T_R}}\right], \end{aligned} \tag{26}$$

and

$$DC_{RW} = d_R\left((z+\theta_o)W+y\right)\left[\frac{\alpha}{\alpha+\theta_r}\left(\frac{e^{\theta_r t_r}-1}{1-e^{-\alpha T_R}}\right)-\frac{\theta_r}{\alpha+\theta_r}\left(\frac{1-e^{-\alpha t_r}}{1-e^{-\alpha T_R}}\right)\right]. \tag{27}$$

Using these two cost components, the overall deterioration cost at the retailer is obtained as follows

$$DC_R = DC_{OW} + DC_{RW}. \tag{28}$$

During the stock-out period, each backordered item creates a penalty cost of b per unit of time. The present value of this shortage cost in the first period is

$$\begin{aligned} BC =& \int_{t_o}^{T_R} b\beta y(t-t_o)e^{-\alpha t}dt \\ =& \frac{b\beta y}{\alpha}\left[\frac{e^{-\alpha t_o}}{\alpha}(1-e^{-\alpha t_s})-t_s e^{-\alpha T_R}\right]. \end{aligned} \tag{29}$$

The equivalent annuity stream of all shortage costs over an infinite horizon is then given by

$$\begin{aligned} BC_R =& \frac{\alpha}{1-e^{-\alpha T_R}}BC \\ =& b\beta y\left[\frac{e^{-\alpha t_o}}{\alpha}\left(\frac{1-e^{-\alpha t_s}}{1-e^{-\alpha T_R}}\right)-t_s\frac{e^{-\alpha T_R}}{1-e^{-\alpha T_R}}\right]. \end{aligned} \tag{30}$$

A penalty of π per unit of lost sale is paid by the retailer which creates a cost of $\pi y(1-\beta)$ over the time interval between $t=t_o$ and $t=T_R$. The equivalent annuity stream of all lost sale costs over an infinite horizon is

$$\begin{aligned} LC_R =& \frac{\alpha}{1-e^{-\alpha T_R}} \pi y(1-\beta) \int_{t_o}^{T_R} e^{-\alpha t} dt \\ =& \pi y(1-\beta) \left(\frac{e^{-\alpha t_o} - e^{-\alpha T_R}}{1-e^{-\alpha T_R}} \right). \end{aligned} \tag{31}$$

Considering the annuity streams of revenue in (16)–(19) and annuity streams of cost in (21)–(27), (30), and (31), the annuity stream profit function at the retailer is given by

$$ASP_R = ASR_R - (SC_R + PC_R + HC_R + BC_R + LC_R + DC_R). \tag{32}$$

4.4 Revenues and costs at the wholesaler

The wholesaler receives revenues of $p_R Q_R$ at $t = iT_R$ $(i = 1, 2, ...)$ associated with the batches sent to the retailer. The revenues at $t = 0$, however, is less as there is no backordered demand at the retailer yet. The equivalent annuity stream of all the revenues over an infinite horizon is given by

$$\begin{aligned} ASR_W =& \alpha p_R(Q_R - \beta y t_s) + \alpha p_R Q_R e^{-\alpha T_R}(1 + e^{-\alpha T_R} + e^{-2\alpha T_R} + ...) \\ =& p_R Q_R \frac{\alpha}{1-e^{-\alpha T_R}} - \alpha p_R \beta y t_s. \end{aligned} \tag{33}$$

The set-up cost of purchasing for the wholesaler, s_W, is incurred at the beginning of each inventory cycle. The equivalent annuity stream of all set-up costs paid over an infinite horizon is then

$$SC_W = s_W \frac{\alpha}{1-e^{-\alpha k T_R}}. \tag{34}$$

The wholesaler purchases the item in batches of size Q_W at the unit price of p_W. These costs are incurred at the beginning of the wholesaler's inventory cycles. The corresponding annuity stream of all purchasing costs at the wholesaler is then as follows

$$PC_W = p_W Q_W \frac{\alpha}{1-e^{-\alpha k T_R}} - \alpha p_W \beta y t_s. \tag{35}$$

In the next step, we obtain the annuity stream of all holding costs at the wholesaler over an infinite horizon as presented below, see Appendix C for the detailed mathematical analysis.

$$HC_W = \frac{\alpha f Q_R(e^{(\theta+\alpha)T_R} - 1)}{(\theta+\alpha)(e^{\theta T_R} - 1)(1 - e^{-\alpha k T_R})}\left[e^{\theta k T_R}\frac{e^{-(\theta+\alpha)T_R} - e^{-(\theta+\alpha)kT_R}}{1 - e^{-(\theta+\alpha)T_R}} - \frac{e^{-\alpha T_R} - e^{-\alpha k T_R}}{1 - e^{-\alpha T_R}}\right]. \tag{36}$$

The deterioration cost at the wholesaler is incurred exactly at the same time as the out-of-pocket holding cost, and occurs at a rate of $d_W\theta$. The annuity stream of this cost over an infinite horizon is given by

$$DC_W = \frac{\alpha d_W \theta Q_R(e^{(\theta+\alpha)T_R} - 1)}{(\theta+\alpha)(e^{\theta T_R} - 1)(1 - e^{-\alpha k T_R})}\left[e^{\theta k T_R}\frac{e^{-(\theta+\alpha)T_R} - e^{-(\theta+\alpha)kT_R}}{1 - e^{-(\theta+\alpha)T_R}} - \frac{e^{-\alpha T_R} - e^{-\alpha k T_R}}{1 - e^{-\alpha T_R}}\right]. \tag{37}$$

Using the revenues and costs presented in (33)–(37), the annuity stream of the profit at the wholesaler is then

$$ASP_W = ASR_W - (SC_W + PC_W + HC_W + DC_W). \tag{38}$$

Considering the same capital rate for both firms, the annuity stream profit function of the supply chain is

$$\begin{aligned} ASP_{SC} =& ASP_R + ASP_W \\ =& ASR_R \\ &- (SC_R + HC_R + BC_R + LC_R + DC_R + SC_W + PC_W + HC_W + DC_W). \end{aligned} \tag{39}$$

5 Model under the common OW/RW policy

As discussed in Section 1, in the literature, two-warehouse systems are typically modelled in a different way in terms of the stock movements between the OW and the RW. In this research work, we argue that although the conventional two-warehouse models from the literature can be useful in many applications, in a retailer context with demand dependent on observable stock at the OW and with item deterioration, it may not perform as well as the continuous resupply policy. In order to quantify how the two policies may deviate from each other in terms of optimal policies, in this paper, we model a two-warehouse setting using the common OW/RW logic introduced in the literature. In its logistical set-up, this model is very similar to Ghiami et al. [27] in which this common OW/RW method was also used. However the objective function in that paper was not based on the NPV. Figure 5 graphically illustrates how the inventory levels at the OW and the RW change under this policy in the case of item deterioration.

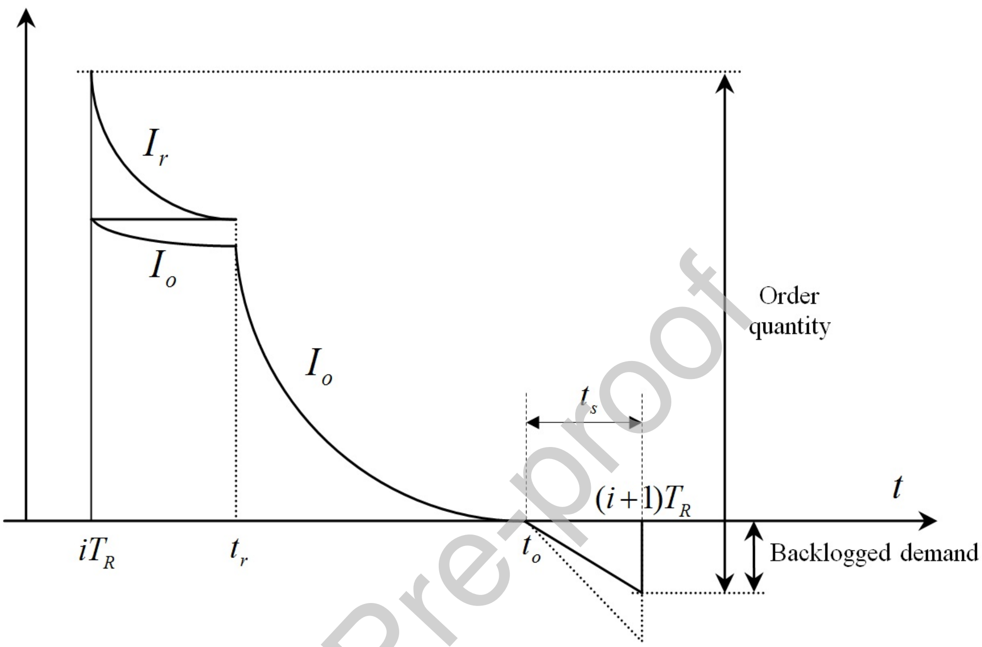

Figure 5: The inventory level at the RW and the OW based on the existing literature (Ghiami et al. [27])

The following differential equation represents the change in the inventory level at the RW

$$\frac{dI'_r(t)}{dt} = -\theta_r I'_r(t) - (y + zI'_o(t)), \qquad 0 \leq t \leq t'_r. \tag{40}$$

Considering the boundary condition, $I'_r(t_r) = 0$, for the differential equation presented in (40), the inventory level of this time interval is obtained as follows

$$I'_r(t) = \frac{y}{\theta_r}(e^{\theta_r(t'_r - t)} - 1) + \frac{zWe^{-\theta_o t}}{\theta_r - \theta_o}(e^{(\theta_r - \theta_o)(t'_r - t)} - 1), \qquad 0 \leq t \leq t'_r. \tag{41}$$

In the literature, it is assumed that during the time that the RW is in use, the inventory level at the OW goes down due to deterioration, therefore

$$\frac{dI'_o(t)}{dt} = -\theta_o I'_o(t), \qquad 0 \leq t \leq t'_r. \tag{42}$$

Using the boundary condition, $I'_o(0) = W$, the inventory level of the OW during this time period is obtained:

$$I'_o(t) = We^{-\theta_o t}, \qquad 0 \leq t \leq t'_r. \tag{43}$$

The change in the inventory level of the OW between $t = t'_r$ and $t = t'_o$ is as shown in the following differential equation

$$\frac{dI'_o(t)}{dt} = -zI'_o(t) - y - \theta_o I'_o(t), \qquad t'_r \leq t \leq t'_o, \tag{44}$$

and therefore

$$I'_o(t) = \frac{y}{z+\theta_o}(e^{(z+\theta_o)(t'_o-t)} - 1), \qquad t'_r \leq t \leq t'_o. \tag{45}$$

Considering the unique value for $I'_o(t)$ at $t = t'_r$ obtained from both (43) and (45), the value of t'_o is obtained as a function of t'_r:

$$t'_o = t'_r + \frac{1}{z+\theta_o} ln\left(1 + \frac{z+\theta_o}{y} We^{-\theta_o t'_r}\right), \tag{46}$$

and since $T_R = t_o + t_s$, therefore

$$T'_R = t'_r + \frac{1}{z+\theta_o} ln\left(1 + \frac{z+\theta_o}{y} We^{-\theta_o t'_r}\right) + t'_s. \tag{47}$$

The retailer's batch size for the second period onwards is then

$$Q'_R = \frac{y}{\theta_r}(e^{\theta_r t'_r} - 1) + \frac{zW}{\theta_r - \theta_o}(e^{(\theta_r-\theta_o)t'_r} - 1) + W + \beta y t'_s. \tag{48}$$

The revenue received by the retailer between $t = 0$ and $t = t'_r$ is $p(y + zI'_o(t))$. The annuity stream of all such revenues over an infinite horizon is hence given by

$$ASR'_{R1} = py\frac{1-e^{-\alpha t'_r}}{1-e^{-\alpha T'_R}} + \frac{\alpha pzW}{\theta_o+\alpha}\left(\frac{1-e^{-(\theta_o+\alpha)t'_r}}{1-e^{-\alpha T'_R}}\right). \tag{49}$$

The other revenue terms for the retailer are as presented in (17)–(19), however, since they are functions of t'_o, t'_r, and T'_R (and not t_o, t_r, and T_R), we respectively use ASP'_{R2}, ASP'_{R3}, and ASP'_{R4} to denote them. The annuity stream of revenues at the retailer is then

$$ASR'_R = ASR'_{R1} + ASR'_{R2} + ASR'_{R3} + ASR'_{R4}. \tag{50}$$

Under the common policy, the retailer's setup and purchasing costs are parametrically the same as (21) and (22). In this section, we respectively use SC'_R and PC'_R to denote these two cost functions. Taking the same approach as in Section 4.3, the present value of holding cost at the OW and the RW only for the first period are

$$HC'_o = f_o\int_0^{t'_o} I'_o(t)e^{-\alpha t}dt, \tag{51}$$

and

$$HC'_r = f_r\int_0^{t'_r} I'_r(t)e^{-\alpha t}dt, \tag{52}$$

respectively. The annuity stream of holding cost at the retailer over an infinite horizon is hence given by

$$HC'_R = HC'_{OW} + HC'_{RW}, \tag{53}$$

where

$$\begin{aligned} HC'_{OW} =& \frac{\alpha}{1-e^{-\alpha T'_R}}HC'_o \\ =& \frac{\alpha f_o W}{\alpha+\theta_o}\left(\frac{1-e^{-(\alpha+\theta_o)t'_r}}{1-e^{-\alpha T'_R}}\right) \\ &+ \frac{f_o y}{z+\theta_o}\left[\frac{\alpha}{\alpha+z+\theta_o}\left(\frac{e^{(z+\theta_o)(t'_o-t'_r)-\alpha t'_r}-e^{-\alpha t'_o}}{1-e^{-\alpha T'_R}}\right) - \frac{e^{-\alpha t'_r}-e^{-\alpha t'_o}}{1-e^{-\alpha T'_R}}\right], \end{aligned} \tag{54}$$

and

$$\begin{aligned} HC'_{RW} =& \frac{\alpha}{1-e^{-\alpha T'_R}} HC'_r \\ =& \frac{\alpha f_r (e^{\theta_r t'_r} - e^{-\alpha t'_r})}{(\alpha+\theta_r)(1-e^{-\alpha T'_R})} \left(\frac{y}{\theta_r} + \frac{zWe^{-\theta_o t'_r}}{\theta_r - \theta_o} \right) \\ &- \frac{f_r y}{\theta_r} \left(\frac{1-e^{-\alpha t'_r}}{1-e^{-\alpha T'_R}} \right) - \frac{\alpha f_r z W}{(\alpha+\theta_o)(\theta_r-\theta_o)} \left(\frac{1-e^{-(\alpha+\theta_o)t'_r}}{1-e^{-\alpha T'_R}} \right). \end{aligned} \tag{55}$$

The present value of deterioration cost at the OW and the RW for the first period are respectively

$$DC'_o = d_R \int_0^{t'_o} \theta_o I'_o(t) e^{-\alpha t} dt, \tag{56}$$

and

$$DC'_r = d_R \int_0^{t'_r} \theta_r I'_r(t) e^{-\alpha t} dt. \tag{57}$$

Therefore, the annuity stream of deteriorating cost at the retailer over an infinite horizon is

$$DC'_R = DC'_{OW} + DC'_{RW}, \tag{58}$$

where

$$\begin{aligned} DC'_{OW} =& \frac{\alpha}{1-e^{-\alpha T'_R}} DC'_o \\ =& \frac{\alpha d_R \theta_o W}{\alpha+\theta_o} \left(\frac{1-e^{-(\alpha+\theta_o)t'_r}}{1-e^{-\alpha T'_R}} \right) \\ &+ \frac{d_R \theta_o y}{z+\theta_o} \left[\frac{\alpha}{\alpha+z+\theta_o} \left(\frac{e^{(z+\theta_o)(t'_o-t'_r)-\alpha t'_r} - e^{-\alpha t'_o}}{1-e^{-\alpha T'_R}} \right) - \frac{e^{-\alpha t'_r} - e^{-\alpha t'_o}}{1-e^{-\alpha T'_R}} \right], \end{aligned} \tag{59}$$

and

$$\begin{aligned} DC'_{RW} =& \frac{\alpha}{1-e^{-\alpha T'_R}} DC'_r \\ =& \frac{\alpha d_R \theta_r (e^{\theta_r t'_r} - e^{-\alpha t'_r})}{(\alpha+\theta_r)(1-e^{-\alpha T'_R})} \left(\frac{y}{\theta_r} + \frac{zWe^{-\theta_o t'_r}}{\theta_r - \theta_o} \right) \\ &- \frac{d_R \theta_r y}{\theta_r} \left(\frac{1-e^{-\alpha t'_r}}{1-e^{-\alpha T'_R}} \right) - \frac{\alpha d_R \theta_r z W}{(\alpha+\theta_o)(\theta_r-\theta_o)} \left(\frac{1-e^{-(\alpha+\theta_o)t'_r}}{1-e^{-\alpha T'_R}} \right). \end{aligned} \tag{60}$$

The annuity stream of backorder cost and lost sale cost at the retailer are obtained as in (30) and (31):

$$BC'_R = b\beta y \left(\frac{e^{-\alpha t'_o}}{\alpha} \left(\frac{1 - e^{-\alpha t'_s}}{1 - e^{-\alpha T'_R}} \right) - t'_s \frac{e^{-\alpha T'_R}}{1 - e^{-\alpha T'_R}} \right), \tag{61}$$

and

$$LC'_R = \pi y (1 - \beta) \left(\frac{e^{-\alpha t'_o} - e^{-\alpha T'_R}}{1 - e^{-\alpha T'_R}} \right). \tag{62}$$

Therefore, the annuity stream of the profit function at the retailer is

$$ASP'_R = ASR'_R - (SC'_R + PC'_R + HC'_R + DC'_R + BC'_R + LC'_R). \tag{63}$$

At the wholesaler, the annuity stream of revenue and costs are parametrically the same as (33)–(37), however, they are functions of t'_o, t'_r, and T'_R. We, therefore, denote them by ASR'_W, SC'_W, PC'_W, HC'_W, and DC'_W, respectively. The annuity stream profit function of the supply chain is then given by

$$ASP'_{SC} = ASR'_R - (SC'_R + HC'_R + BC'_R + LC'_R + DC'_R + SC'_W + PC'_W + HC'_W + DC'_W). \tag{64}$$

In the next section, the difference between the two models presented in Sections 4 and 5 is analysed in more detail.

6 Numerical examples

In this section, we investigate the effect of integration and continuous resupply on four cases. These cases are different in terms of deterioration and holding cost rates at the OW and the RW, and each could represent a real-case situation, see Table 2. For instance, Case 1 could be a small retailer on a high street with limited warehouse capacity that rents a nearby warehouse at a higher cost and since the warehouse is not well-specialised in handling that specific deteriorating item, the resulting deterioration rate is higher. Case 4 could be a grocery retailer and what happens between the front-room and back-room; the items that are placed in the refrigerators and are accessible by the customers will have higher inventory cost since customers keep opening the refrigerators. This results in a higher cost, compared to the situation at the back-room. This situation would also result in a higher deterioration, due to fluctuations in the temperature and possible damages.

In order to solve the model, we perform an exhaustive search. To this end, we define three nested loops, enumerating t_r, t_s, and k. More specifically for t_r and t_s, we assign values between 0 and 30 with incremental steps of 0.01. The integer values assigned to k are between 1 and 15.

Table 2: Cases studied in this research

	$\theta_o < \theta_r$	$\theta_o \geq \theta_r$
$f_o < f_r$	Case 1	Case 2
$f_o \geq f_r$	Case 3	Case 4

6.1 Impact of integration

This section illustrates the differences between the integrated model and the non-integrated one where the retailer follows the continuous resupply policy. In the integrated model, the inventory policies at the supplier and the retailer are determined simultaneously to maximise the integrated supply chain profit function, ASP_{SC}. In the non-integrated supply chain, the retailer first sets the inventory policy to maximise the profit function. The wholesaler then sets the inventory policy aiming at profit maximisation. The sum of the two profit functions gives the total profit of the supply chain, ASP_{Seq}. We quantify the improvement obtained as a result of integration (δ_{imp}) by measuring the relative change observed in the profit of the supply chain:

$$\delta_{imp} = 100 \times \frac{ASP^*_{SC} - ASP^*_{Seq}}{ASP^*_{Seq}}. \tag{65}$$

To numerically study the effect of integration, the following data set is used: $W = 200$, $y = 200$, $z = 0.2$, $\theta = 0.03$, $\beta = 0.7$, $p = 13$, $p_R = 8$, $p_W = 3.5$, $\alpha = 0.05$, $g = 0$, $r = 0$, $s_R = 500$, $s_W = 2000$, $f = 0.3$, $b = 2$, $\pi = 0$, $d_R = 0$, and $d_W = 0$. The values of deterioration rates and holding cost parameters are as follows:

- Case 1: $\theta_o = 0.05$, $\theta_r = 0.09$, $f_o = 0.4$, and $f_r = 0.8$,

- Case 2: $\theta_o = 0.09$, $\theta_r = 0.05$, $f_o = 0.4$, and $f_r = 0.8$,
- Case 3: $\theta_o = 0.05$, $\theta_r = 0.09$, $f_o = 0.8$, and $f_r = 0.4$,
- Case 4: $\theta_o = 0.09$, $\theta_r = 0.05$, $f_o = 0.8$, and $f_r = 0.4$.

Table 3: Numerical results for integrated and sequential approach

Case	Approach	t_r	t_o	t_s	T_R	Q_R	ASP_R	k	T_W	Q_W	ASP_W	ASP_{SC}
1	Sequential	0.71	1.60	0.38	1.98	437	580.73	3	5.94	1392	355.89	936.62
	Integrated	1.59	2.48	0.00	2.48	628	487.14	2	4.96	1304	553.66	1040.79
											δ_{imp} (%)	11.12
2	Sequential	0.76	1.64	0.47	2.11	466	551.06	2	4.22	962	359.56	910.62
	Integrated	1.68	2.56	0.00	2.56	652	462.74	2	5.12	1356	569.83	1032.57
											δ_{imp} (%)	13.39
3	Sequential	0.78	1.67	0.48	2.15	470	548.59	2	4.30	970	354.95	903.54
	Integrated	3.33	4.22	0.78	5.00	1280	235.10	1	5.00	1280	824.86	1059.96
											δ_{imp} (%)	17.32
4	Sequential	0.85	1.73	0.57	2.30	504	521.52	2	4.60	1044	379.27	900.79
	Integrated	3.78	4.66	0.69	5.35	1371	225.49	1	5.35	1370	565.90	1091.39
											δ_{imp} (%)	21.16

As Table 3 illustrates, the integrated approach results in a better performance for the whole supply chain in all cases. For the data set introduced in this paper, the increase in the supply chain profit, as a result of integration, can be as significant as 20%. The results of this experiment show that this shift is experienced by the two players differently; the integration results in a loss of profit for the retailer, while the wholesaler gains some profit. The success of the integration, therefore, depends on how these two players distribute the value generated by adopting the integrated policy amongst themselves. This gaining can be divided between the retailer and the wholesaler by reaching an agreement on the transfer price (p_R). In the current setting, the purchasing cost paid by the retailer is immediately received by the wholesaler as a revenue. This indicates that in the integrated approach, these two values cancel out and therefore do not have any influence on the *supply chain's optimal solution. Both players, however, see their profit functions* sensitive to p_R. Table 4 shows how the profits gained by the retailer and the wholesaler in Case 1 change when different values are assigned to p_R.

In this numerical example, the integration results in some improvement in the profit function for all the cases. To see the impact of integration when the profit margin of the item changes, a range of values

Table 4: The retailer's and the wholesaler's profit when p_R changes, integrated approach

p_R	T_R	Q_R	ASP_R	k	T_W	Q_W	ASP_W	ASP_{SC}
4	2.48	628	1562.01	2	4.96	1304	-521.22	1040.79
6	2.48	628	1024.57	2	4.96	1304	16.22	1040.79
8	2.48	628	487.14	2	4.96	1304	553.66	1040.79
10	2.48	628	-50.30	2	4.96	1304	1091.09	1040.79

is assigned to p. In order to capture only the effect of margins, we proportionately change the values of p_R and p_W, e.g., when p changes from 13 to 9, then $p_R = 5.54$ and $p_W = 2.42$. Figure 6 illustrates that how integration results in an improvement in the profit function of the cases under study. As the figure shows, for tighter margins, e.g., the case of a more competitive market, the improvement gained by integration is greater, while with higher margins the increase in the total profit of the supply chain gained after integration tends to be smaller.

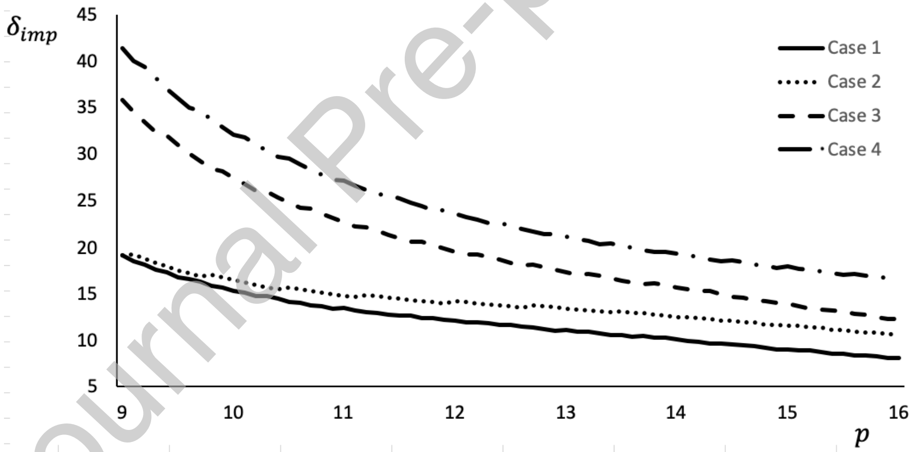

Figure 6: Improvement (%) in the ASP after integration for different margins

The results of the experiment also show that the performance of the two approaches could be significantly different in terms of fill rate ($f_l = 1 - \frac{t_s}{T_R}$). We numerically investigate the effect of integration on the fill rate for the cases. To this end, we change the sales prices as specified earlier in this numerical example and measure the fill rate offered by the two approaches. Figure 7 depicts how integration could increase the fill rate at the retailer level for all the cases. The jump that is seen in Case 2 is due to the integrality constraint on k. In this specific case, when increasing the sales price, the optimal value of k

changes from 1 to 2, which results in a jump in the fill rate.

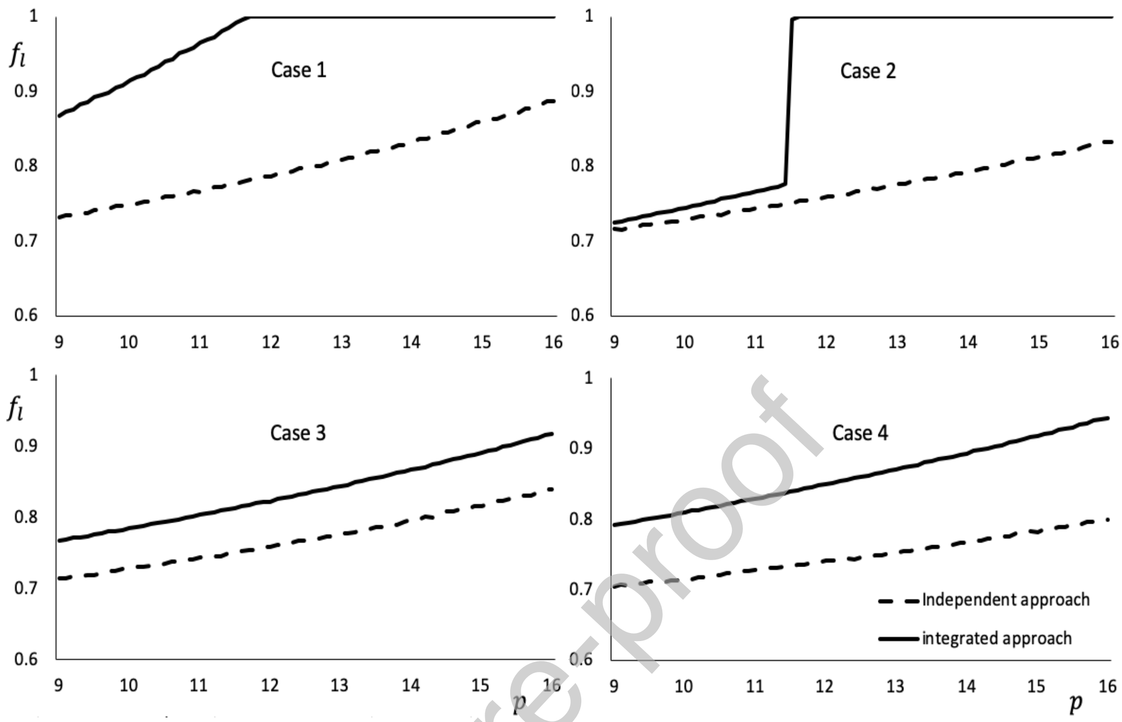

Figure 7: Resulting fill rate (%) for different margins

Experiments performed in this paper show that the effect of integration is more significant when the deterioration rate is higher. To study the combined effect of integration and deterioration, in each case, we measure the relative improvement in the profit function, obtained as a result of integration, for a range of values assigned to θ_o, i.e., $[0, 0.25]$. This range embraces a large group of products from non-deteriorating to highly perishable items. In this experiment, we proportionately change the value of θ and θ_r. As Figure 8 shows, in this numerical experiment, the effect of integration is more significant for cases 3 and 4 in which $f_o > f_r$. The figure also confirms that in all cases, the gaining obtained from integration increases when deterioration rate goes up.

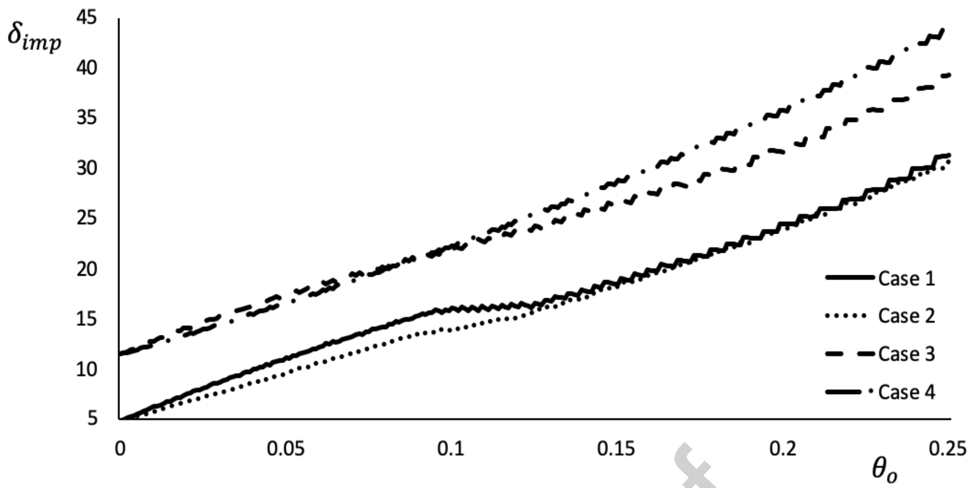

Figure 8: Improvement (%) in the ASP after integration for different deterioration rates

6.2 Impact of continuous resupply policy

6.2.1 Supply chain perspective

In this section, we investigate the differences between the continuous resupply policy (Model 1) and the common resupply policy (Model 2) in an integrated supply chain. For this purpose, the models developed in Sections 4 and 5 are solved using the following data set: $W = 200$, $y = 50$, $z = 0.7$, $\theta = 0.03$, $\beta = 0.7$, $p = 7.5$, $p_R = 5$, $p_W = 3$, $\alpha = 0.05$, $g = 3$, $r = 1$, $s_R = 300$, $s_W = 1200$, $f = 0.1$, $b = 2$, $\pi = 0$, $d_R = 0$, and $d_W = 0$. We define 4 cases that are different in terms of holding cost and deterioration rate parameters:

- Case 1: $\theta_o = 0.08$, $\theta_r = 0.09$, $f_o = 0.4$, and $f_r = 0.5$,
- Case 2: $\theta_o = 0.09$, $\theta_r = 0.08$, $f_o = 0.4$, and $f_r = 0.5$,
- Case 3: $\theta_o = 0.08$, $\theta_r = 0.09$, $f_o = 0.5$, and $f_r = 0.4$,
- Case 4: $\theta_o = 0.09$, $\theta_r = 0.08$, $f_o = 0.5$, and $f_r = 0.4$.

Table 5 presents the results of this numerical analysis including δ which captures the difference in the profit functions of the two models, i.e., the improvement in the profit if the retailer changes from the common policy to the continuous resupply policy:

$$\delta = 100 \times \frac{ASP^*_{SC}(Model\ 1) - ASP^*_{SC}(Model\ 2)}{ASP^*_{SC}(Model\ 2)}. \tag{66}$$

As discussed in Section 1, the implications of adopting the continuous resupply policy would be higher holding costs for the retailer. To incorporate this effect, we optimise the model with continuous resupply policy for a range of values for f_o. Table 5 shows that, for the data set presented in this study, the continuous resupply policy seems to be a superior option in Cases 1 and 2. This policy would also be the better option in Cases 3 and 4 unless it results in an increase of at least 20% in the holding cost parameter. One should note that f_o accounts for the keeping and handling costs that incurred by common resources, e.g., human resources and utilities. Therefore, adopting the continuous resupply policy would not greatly increase the holding cost parameter of one specific item.

Table 5: Numerical results of the comparison between Models 1 and 2

Case	Policy	f_o	t_r	t_o	t_s	T_R	Q_R	k	T_W	Q_W	ASP_{SC}	$\delta(\%)$
1	Common	0.4	1.59	3.28	0.00	3.28	510	2	6.57	1073	48.94	
	Continuous	0.4	1.54	3.36	0.00	3.36	541	2	6.72	1138	61.18	25.01
	resupply	0.4 +5%	1.54	3.36	0.00	3.36	541	2	6.72	1138	58.44	19.41
		0.4+10%	1.54	3.36	0.00	3.36	541	2	6.72	1138	55.70	13.81
		0.4+20%	1.53	3.35	0.00	3.35	538	2	6.70	1133	50.24	2.66
2	Common	0.4	1.62	3.29	0.00	3.29	512	2	6.58	1076	44.67	
	Continuous	0.4	1.57	3.37	0.00	3.37	548	2	6.74	1155	58.29	30.49
	resupply	0.4 +5%	1.56	3.36	0.00	3.36	546	2	6.72	1150	55.54	24.33
		0.4+10%	1.56	3.36	0.00	3.36	546	2	6.72	1150	52.79	18.18
		0.4+20%	1.55	3.35	0.00	3.35	544	2	6.70	1145	47.30	5.89
3	Common	0.5	1.66	3.35	0.00	3.35	524	2	6.70	1104	43.80	
	Continuous	0.5	1.61	3.42	0.00	3.42	557	2	6.84	1175	55.79	27.37
	resupply	0.5 +5%	1.6	3.41	0.00	3.41	555	2	6.82	1169	52.33	19.48
		0.5+10%	1.6	3.41	0.00	3.41	555	2	6.82	1169	48.88	11.60
		0.5+20%	1.59	3.40	0.00	3.40	553	2	6.80	1164	41.99	-4.13
4	Common	0.5	1.7	3.36	0.00	3.36	527	2	6.72	1110	39.79	
	Continuous	0.5	1.63	3.43	0.00	3.43	563	2	6.86	1186	53.12	33.50
	resupply	0.5 +5%	1.63	3.43	0.00	3.43	563	2	6.86	1186	49.65	24.78
		0.5+10%	2.92	4.72	1.08	5.80	922	1	5.80	922	46.26	16.26
		0.5+20%	2.90	4.70	1.17	5.87	920	1	5.87	920	39.76	-0.08

In order to see how the profit margin of the item makes an impact on δ, these two models are solved across a range of values assigned to p. As discussed in Section 6.1, the transfer price does not have any influences on the optimal solution, therefore here the profit margin of the supply chain $(p - p_W)$ and the relevant effects on the optimal solution are studied. For this purpose, p_W is first set equal to 3 while

p is given a range of values, the two models are then solved and the values for δ are obtained. In this experiment, we assume an increase of 10% in f_o as a result of adopting the continuous resupply policy. The result of this analysis is illustrated in Figure 9. The figure shows that, for all cases, the difference between the two policies is significant when the margins are low. Moreover, it shows that for high margin items the effect of adopting the continuous resupply policy is almost the same for all four cases discussed in this example. The experiment also points out that for lower margins the common policy may not be

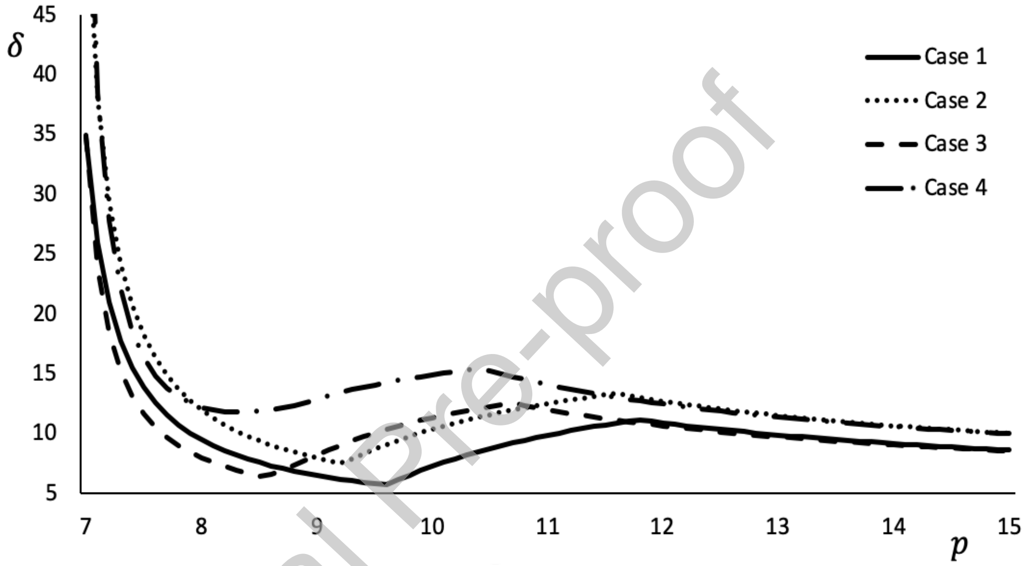

Figure 9: The difference between the two policies when p changes

able to find a feasible region while the continuous resupply policy obtains an optimal solution, for instance in Case 2 if $p = 6.7$, the profit obtained by the common and continuous policy are respectively -6.24 and 0.23. This indicates that the two replenishment policies discussed in this paper could give different perceptions of the same inventory system. It then depends on the planner of such settings which policy to adopt considering the features of the system.

6.2.2 Single-echelon setting

In Section 6.1, we investigate the effect of integration when the continuous resupply policy is adopted. Integration in a supply chain, however, is not an easy goal to achieve (Fawcett et al. [34]) and many retailers manage their operations in a non-integrated setting. Therefore, it is worthwhile to study the effects of the continuous resupply policy on the performance of such retailers. In this section, we exclude the supplier from the model developed in Section 4 and analyse the performance of this single-echelon model using the data presented in Section 6.2.1. The result of this analysis is presented in Table 6.

Our study shows that, in all four cases studied, if a retailer adopts the continuous resupply policy and this change results in an increase of less than 5% in the holding cost parameter, the retailer will experience an increase in the profit. Contrarily to the integrated setting discussed in Section 6.2.1, the relative difference between the two policies in the single-echelon setting increases for items with higher margins, see Figure 10.

Table 6: Numerical results of the comparison between Models 1 and 2, single-echelon setting

Case	Policy	f_o	t_r	t_o	t_s	T_R	Q_R	ASP_R	$\delta(\%)$
1	Common	0.4	0.79	2.54	0.04	2.58	353	49.48	
	Continuous resupply	0.4	0.77	2.59	0.00	2.59	364	52.26	5.62
		0.4 +5%	0.77	2.59	0.04	2.63	366	49.94	0.95
		0.4+10%	0.76	2.58	0.07	2.66	364	47.67	-3.66
		0.4+20%	0.75	2.57	0.14	2.71	365	43.20	-12.69
2	Common	0.4	0.80	2.54	0.12	2.66	357	44.11	
	Continuous resupply	0.4	0.78	2.58	0.08	2.66	371	47.02	6.60
		0.4 +5%	0.77	2.57	0.11	2.68	369	44.76	1.48
		0.4+10%	0.77	2.57	0.15	2.72	370	42.54	-3.57
		0.4+20%	0.75	2.55	0.21	2.76	368	38.17	-13.46
3	Common	0.5	0.81	2.56	0.17	2.73	362	40.95	
	Continuous resupply	0.5	0.79	2.61	0.13	2.74	373	43.41	5.99
		0.5 +5%	0.79	2.61	0.18	2.79	375	40.63	-0.78
		0.5+10%	0.78	2.60	0.22	2.82	374	37.91	-7.43
		0.5+20%	0.76	2.58	0.30	2.88	373	32.58	-20.43
4	Common	0.5	0.83	2.56	0.25	2.81	367	35.88	
	Continuous resupply	0.5	0.80	2.60	0.21	2.81	379	38.44	7.13
		0.5 +5%	0.80	2.60	0.25	2.85	381	35.73	-0.43
		0.5+10%	0.79	2.59	0.29	2.88	380	33.06	-7.87
		0.5+20%	0.76	2.56	0.37	2.93	376	27.84	-22.41

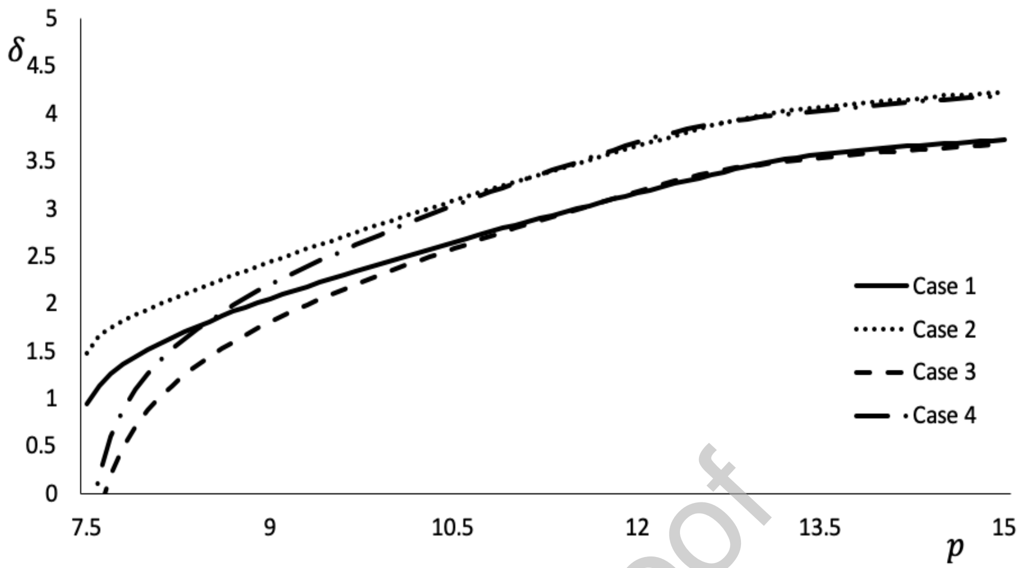

Figure 10: The difference between the two policies when p changes, single-echelon setting

6.3 Sensitivity analysis

In this section, we conduct a sensitivity analysis on the parameters of the model developed in Section 4. In order to avoid a lengthy report, we focus on Case 4 that represents the setting in a typical grocery retail store. To this end, we change the value of each parameter by steps of $\pm10\%$ and $\pm20\%$, when applicable, and study the effects of the changes on the decision variables and profit function. We use the same data sets introduced in Sections 6.1 and 6.2. We assign values of $10\%p$ and $20\%p$ to parameters g, r, and π when the initial value is zero. In order to perform sensitivity analysis on the deterioration cost parameters (d_R, d_W), we set them equal to 10% and 20% of the purchasing price of that echelon; $d_R = 0.1p_R$ & $0.2p_R$ and $d_W = 0.1p_W$ & $0.2p_W$. Tables 7 and 8 illustrate the results of the analyses. Since the transfer price (p_R) has no effect on the result, we do not present it in these tables (see Table 4).

In general, the model shows much higher sensitivity in the cases with lower margins; compare the columns under δ in Tables 7 and 8. The model shows the highest sensitivity to p and p_W. This is intuitive since these two parameters directly make an impact on the profit of the supply chain. Changes in the demand parameters, y and z, also make an impact on the solution. The significance of this impact, however, depends on their relative values, for instance, in the first example in which $y = 200$ and $z = 0.2$, the model is more sensitive to y, while in the second example where $y = 50$ and $z = 0.7$ the model shows

Table 7: The results of sensitivity analysis on the data set from Section 6.1

Par.		T_R	Q_R	T_W	Q_W	ASP_{SC}	δ	Par.		T_R	Q_R	T_W	Q_W	ASP_{SC}	δ
y	160	5.99	1264	5.99	1264	799.68	-26.73	z	0.16	5.50	1357	5.50	1357	1041.93	-4.53
	180	5.64	1318	5.64	1318	944.45	-13.47		0.18	5.43	1364	5.43	1364	1066.51	-2.28
	220	5.09	1419	5.09	1419	1240.25	13.64		0.22	5.27	1377	5.27	1377	1116.58	2.31
	240	4.86	1465	4.86	1465	1390.83	27.44		0.24	5.18	1383	5.18	1383	1142.11	4.65
s_R	400	5.22	1340	5.22	1340	1112.93	1.97	s_W	1600	4.79	1241	4.79	1241	1180.69	8.18
	450	5.29	1357	5.29	1357	1102.10	0.98		1800	5.08	1309	5.08	1309	1134.96	3.99
	550	5.42	1387	5.42	1387	1080.79	-0.97		2200	5.61	1431	5.61	1431	1049.64	-3.82
	600	5.48	1401	5.48	1401	1070.30	-1.93		2400	5.86	1489	5.86	1489	1009.48	-7.51
p	10.4	5.80	1396	5.80	1396	519.15	-52.43	p_W	2.80	5.30	1410	5.30	1410	1296.95	18.83
	11.7	5.60	1390	5.60	1390	802.01	-26.51		3.15	5.33	1392	5.33	1392	1193.02	9.31
	14.3	5.08	1345	5.08	1345	1387.56	27.14		3.85	5.37	1351	5.37	1351	991.95	-9.11
	15.6	4.78	1310	4.78	1310	1691.01	54.94		4.20	5.39	1332	5.39	1332	894.64	-18.03
α	0.040	5.42	1406	5.42	1406	1126.90	3.25	β	0.56	4.73	1296	4.73	1296	1076.03	-1.41
	0.045	5.39	1390	5.39	1390	1109.05	1.62		0.63	5.08	1336	5.08	1336	1080.12	-1.03
	0.055	5.32	1354	5.32	1354	1073.91	-1.60		0.77	5.52	1395	5.52	1395	1107.20	1.45
	0.060	5.28	1337	5.28	1337	1056.60	-3.19		0.84	5.64	1417	5.64	1417	1126.17	3.19
θ_o	0.072	5.34	1361	5.34	1361	1106.73	1.41	θ_r	0.040	5.46	1391	5.46	1391	1109.54	1.66
	0.081	5.35	1366	5.35	1366	1099.03	0.70		0.045	5.40	1380	5.40	1380	1100.35	0.82
	0.099	5.34	1375	5.34	1375	1083.78	-0.70		0.055	5.30	1362	5.30	1362	1082.64	-0.80
	0.108	5.35	1382	5.35	1382	1076.22	-1.39		0.060	5.26	1354	5.26	1354	1074.10	-1.58
θ	0.024	5.35	1370	5.35	1370	1091.39	0.00	f_o	0.64	5.30	1369	5.30	1369	1117.40	2.38
	0.027	5.35	1370	5.35	1370	1091.39	0.00		0.72	5.33	1371	5.33	1371	1104.35	1.19
	0.033	5.35	1370	5.35	1370	1091.39	0.00		0.88	5.38	1373	5.38	1373	1078.51	-1.18
	0.036	5.35	1370	5.35	1370	1091.39	0.00		0.96	5.39	1371	5.39	1371	1065.72	-2.35
f_r	0.32	5.50	1435	5.50	1435	1124.61	3.04	f	0.24	5.35	1370	5.35	1370	1091.39	0.00
	0.36	5.42	1401	5.42	1401	1107.58	1.48		0.27	5.35	1370	5.35	1370	1091.39	0.00
	0.44	5.28	1342	5.28	1342	1076.00	-1.41		0.33	5.35	1370	5.35	1370	1091.39	0.00
	0.48	5.22	1315	5.22	1315	1061.34	-2.75		0.36	5.35	1370	5.35	1370	1091.39	0.00
W	160	5.45	1347	5.45	1347	1062.37	-2.66	b	1.6	5.46	1384	5.46	1384	1093.94	0.23
	180	5.40	1359	5.40	1359	1076.90	-1.33		1.8	5.41	1379	5.41	1379	1092.57	0.11
	220	5.30	1384	5.30	1384	1105.83	1.32		2.2	5.31	1367	5.31	1367	1090.36	-0.09
	240	5.25	1392	5.25	1392	1120.23	2.64		2.4	5.27	1361	5.27	1361	1089.47	-0.18
d_R	0.8	5.31	1344	5.31	1344	1064.56	-2.46	d_W	0.35	5.35	1370	5.35	1370	1091.39	0.00
	1.6	5.26	1317	5.26	1317	1038.83	-4.82		0.70	5.35	1370	5.35	1370	1091.39	0.00
g	1.3	5.37	1373	5.37	1373	1091.75	0.03	r	1.3	4.93	1322	4.93	1322	1077.41	-1.28
	2.6	5.38	1375	5.38	1375	1092.13	0.07		2.6	4.73	1296	4.73	1296	1076.03	-1.41
π	1.3	5.19	1355	5.19	1355	1083.67	-0.71								
	2.6	5.00	1332	5.00	1332	1078.49	-1.18								

much higher sensitivity to z.

In both examples, the model shows a high sensitivity to changes in s_W. In the second example, for higher values of s_W, the model tends to choose longer replenishment cycles at the wholesaler, however, in order to avoid larger quantities at the retailer, the model sets $k = 2$ compared to smaller values of s_W, see

Table 8.

In the second example where margins are tight, the model is very sensitive to the interest rate parameter, α. We consider a conventional payment structure for this model; fixed ordering costs and purchasing costs are incurred at the time of replenishment and revenues are obtained while the demand is fulfilled. This means lower interest rates would be beneficial, since they lower the negative impact of such delayed revenues. In the second example, the model shows high sensitivity also to the warehouse capacity, W. This should be looked at together with the high value of z; a demand function with $y = 50$ and $z = 0.7$ makes W the main driver of the demand for the product.

7 Conclusions

In this paper, a two-warehouse supply chain for a deteriorating product is considered that consists of a retailer and a wholesaler. The retailer has a limited capacity for this product at the main store (OW). There is, however, an opportunity of keeping extra stock in the back-room (RW) which may have a different rate of deterioration and holding cost. Demand for the product is dependent on the stock level of good products at the OW. A linear function of the inventory level is used to specify the demand rate. We introduce and analyse the continuous resupply policy that aims at keeping the stock of good products at the OW at full capacity for as long as there is stock of good products available at the RW. When the stock at the RW runs out, the model considers the possibility of letting the stock at the OW drop to zero or even to negative values, and allows for partial backorders. During each inventory cycle, the wholesaler covers an integer number of the retailer's order size. The profits of the firms are based on the cash-flow functions that are associated with the logistical activities in the system and found as their Laplace transform. This produces the annuity stream profit functions for each of the firms, and their integrated supply chain.

In the inventory management literature, it is known that collaboration arguably results in financial improvement. In this research work, we illustrate how this financial benefit could be more significant for items with higher deterioration rates. Moreover, we numerically investigate the effect of margins on the gaining in the supply chain. Our study shows that the supply chain can significantly gain profit from

Table 8: The results of sensitivity analysis on the data set from Section 6.2

Par.		T_R	Q_R	T_W	Q_W	ASP_{SC}	δ	Par.		T_R	Q_R	T_W	Q_W	ASP_{SC}	δ
y	40	6.20	896	6.20	896	28.79	-45.79	z	0.56	6.59	527	6.76	1110	24.28	-96.75
	45	5.95	910	5.95	910	40.68	-23.42		0.63	6.16	553	6.64	1164	53.54	-50.09
	55	3.34	571	6.68	1202	68.24	28.47		0.77	3.37	607	6.42	1275	113.87	55.08
	60	3.26	580	6.52	1219	83.42	57.05		0.84	3.32	632	6.32	1325	144.85	111.25
s_R	240	3.39	551	6.76	1160	72.25	36.03	s_W	960	4.79	846	4.79	846	104.80	97.30
	270	3.41	558	6.82	1175	62.65	17.95		1080	5.27	886	5.27	886	77.84	46.54
	330	5.82	930	5.82	930	46.89	-11.73		1320	3.48	574	6.96	1212	32.62	38.59
	360	5.93	941	5.93	941	41.00	-22.82		1440	3.53	586	7.06	1238	12.37	76.72
p	6.00	7.23	868	7.23	868	-133.66	-351.64	p_W	2.4	3.69	625	7.38	1322	178.51	236.08
	6.75	6.56	904	6.56	904	-45.99	-186.59		2.7	3.56	594	7.12	1254	115.05	116.60
	8.25	4.87	924	4.87	924	166.99	214.39		3.3	6.14	896	6.14	896	0.44	-99.18
	9.00	4.97	951	4.97	951	287.92	442.04		3.6	6.49	866	6.49	866	-47.33	-189.10
α	0.04	3.54	589	7.08	1243	79.18	49.08	β	0.56	3.43	563	6.86	1186	53.12	0.00
	0.045	3.48	574	6.96	1212	66.09	24.43		0.63	3.43	563	6.86	1186	53.12	0.00
	0.055	5.77	910	5.77	910	44.41	-16.40		0.77	5.76	927	5.76	927	54.23	2.09
	0.06	5.84	901	5.84	901	36.12	-32.01		0.84	5.79	930	5.79	930	55.61	4.69
θ_o	0.072	3.48	563	6.96	1188	64.44	21.31	θ_r	0.064	5.72	945	5.72	945	64.78	21.96
	0.081	3.45	562	6.90	1184	58.77	10.64		0.072	5.72	934	5.72	934	58.68	10.47
	0.099	3.41	563	6.82	1187	47.48	-10.62		0.088	3.40	558	6.80	1175	50.23	-5.44
	0.108	5.81	928	5.81	928	42.07	-20.80		0.096	3.37	553	6.74	1164	47.41	-10.74
θ	0.024	3.47	572	6.94	1193	60.09	13.14	f_o	0.40	3.45	567	6.90	1196	67.02	26.17
	0.027	3.45	567	6.90	1190	56.61	6.57		0.45	3.44	565	6.88	1191	60.06	13.08
	0.033	5.71	921	5.71	921	52.87	-0.46		0.55	5.80	922	5.80	922	46.26	-12.90
	0.036	5.71	921	5.71	921	52.87	-0.46		0.60	5.87	920	5.87	920	39.76	-25.15
f_r	0.32	5.70	969	5.70	969	68.82	29.57	f	0.08	3.46	570	6.92	1201	59.61	12.22
	0.36	5.70	944	5.70	944	60.56	14.02		0.09	3.45	567	6.90	1196	56.35	6.09
	0.44	3.40	555	6.80	1170	49.60	-6.61		0.10	5.71	921	5.71	921	52.87	-0.46
	0.48	3.36	546	6.72	1150	46.21	-13.00		0.11	5.71	921	5.71	921	52.87	-0.46
W	160	3.41	506	6.82	1066	10.78	-79.71	b	1.6	5.93	929	5.93	929	54.12	1.88
	180	3.42	535	6.84	1127	32.37	-39.07		1.8	5.81	925	5.81	925	53.44	0.60
	220	5.39	957	5.39	957	76.40	43.83		2.2	3.43	563	6.86	1186	53.12	0.00
	240	5.05	991	5.05	991	100.88	89.93		2.4	3.43	563	6.86	1186	53.12	0.00
d_R	0.5	3.39	553	6.78	1165	43.39	-18.32	d_W	0.3	5.71	921	5.71	921	52.87	-0.47
	1.0	3.35	544	6.70	1144	33.86	-36.26		0.6	5.71	921	5.71	921	52.87	-0.47
g	2.4	3.43	563	6.86	1186	53.12	0.00	r	0.8	5.79	924	5.79	924	53.94	1.56
	2.7	3.43	563	6.86	1186	53.12	0.00		0.9	5.75	923	5.75	923	53.40	0.53
	3.3	3.43	563	6.86	1186	53.12	0.00		1.1	3.43	563	6.86	1186	53.12	0.00
	3.6	3.43	563	6.86	1186	53.12	0.00		1.2	3.43	563	6.86	1186	53.12	0.00
π	0.75	3.43	563	6.86	1186	53.12	0.00								
	1.50	3.43	563	6.86	1186	53.12	0.00								

integration when the margins are small. This increase in profit would be less for items with high margins.

Our experiments illustrate that the retailer's profit shrinks as a result of adopting the integrated optimal solution. Since the optimal profits of the wholesaler and the supply chain increase after integration, it then seems logical to seek for a solution that makes the integration feasible and interesting also for the retailer. In this paper, we show how small adjustments of the transfer price can provide a means to redistribute the profits between the firms and could therefore be an instrument by which the benefits of collaboration is shared in any relative degree.

In this paper, we conduct a comparison between the continuous resupply policy and the common policy from the literature. The numerical experiment shows that if the retailer shifts from the common policy to the continuous resupply, the integrated supply chain could experience a significant increase in the profit, for instance, in the settings studied in this paper, the gaining can be as high as 30%. Moreover, we show how this increase would change as a function of the product margin. The difference between the two policies also shows a high sensitivity to the deterioration rate. This shows that, when both policies are logical to adopt, the retailer should carefully choose which one to go for depending on the characteristics of the item at hand, e.g., margin or deterioration rate. We also study the effects of the continuous resupply policy for the case of a single-echelon supply chain, where there is no close collaboration between the retailer and the supplier. Our analyses on four different data sets show that as long as the adoption of the continuous resupply policy does not greatly increase the holding cost parameters, it could be a better option.

For the integrated supply chain, the continuous resupply policy presents a significantly better performance than the common policy for the cases investigated in this study. It is observed that the latter policy might not even produce a feasible result (i.e., the supply chain cannot generate a positive NPV), while under the continuous resupply policy an optimal solution with positive profits for the same data is found.

Further research could extend the model with, e.g., the costs of transferring stock from the RW to the OW, which would reduce the benefit of frequent resupply. Another extension for the case of deteriorating items could consider that out-of-pocket holding costs may depend on the time that products need to be stored in the RW, as in Alfares [35]. The model could also be extended to consider specific cases of deteriorating items where the situation at the RW and OW, in terms of holding cost and deterioration rate, are defined based on real data.

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A Inventory level at the wholesaler

The inventory level at the wholesaler at $t = (k-1)T_R$, just before sending the last batch to the retailer, is Q_R. Considering this boundary condition the inventory level at the wholesaler between $(k-2)T_R$ and $(k-1)T_R$ is given by

$$I_W^{k-1}(t) = Q_R e^{\theta[(k-1)T_R - t]}, \qquad (k-2)T_R \leq t \leq (k-1)T_R. \tag{67}$$

According to (67), the inventory level at the wholesaler at $t = (k-2)T_R$, just before sending a batch to the retailer is $Q_R(e^{\theta T_R} + 1)$. Using this inventory level as a boundary condition, the inventory level of $(k-2)^{th}$ interval is obtained as

$$I_W^{k-2}(t) = Q_R(e^{\theta T_R} + 1)e^{\theta[(k-2)T_R - t]}, \qquad (k-3)T_R \leq t \leq (k-2)T_R. \tag{68}$$

The inventory level at the wholesaler during i^{th} interval is hence given by

$$\begin{aligned} I_W^i(t) =& Q_R e^{\theta(iT_R - t)} \sum_{m=0}^{k-i-1} e^{m\theta T_R} \\ =& Q_R e^{\theta(iT_R - t)} \frac{e^{\theta(k-i)T_R} - 1}{e^{\theta T_R} - 1}, \qquad (i-1)T_R \leq t \leq iT_R, \qquad i = 1, 2, ..., k-1. \end{aligned} \tag{69}$$

B Calculation of $\int_0^{t_o} I_o(t)e^{-\alpha t}dt$ and $\int_0^{t_r} I_r(t)e^{-\alpha t}dt$

$$\begin{aligned} \int_0^{t_o} I_o(t)e^{-\alpha t}dt =& \int_0^{t_r} I_o(t)e^{-\alpha t}dt + \int_{t_r}^{t_o} I_o(t)e^{-\alpha t}dt \\ =& \int_0^{t_r} We^{-\alpha t}dt + \int_{t_r}^{t_o} \frac{y}{z+\theta_o}(e^{(z+\theta_o)(t_o-t)} - 1)e^{-\alpha t}dt \\ =& \frac{W}{-\alpha}\left[e^{-\alpha t}\right]_0^{t_r} + \frac{y}{z+\theta_o}\int_{t_r}^{t_o}\left(e^{(z+\theta_o)t_o}e^{-(\alpha+z+\theta_o)t} - e^{-\alpha t}\right)dt \\ =& \frac{W}{\alpha}[1 - e^{-\alpha t_r}] + \frac{y}{z+\theta_o}\left[\frac{e^{(z+\theta_o)t_o}}{-(\alpha+z+\theta_o)}\left[e^{-(\alpha+z+\theta_o)t}\right]_{t_r}^{to} + \frac{1}{\alpha}\left[e^{-\alpha t}\right]_{t_r}^{t_o}\right] \\ =& \frac{W}{\alpha}[1 - e^{-\alpha t_r}] + \frac{y}{z+\theta_o}\left[\frac{1}{\alpha+z+\theta_o}\left[e^{(z+\theta_o)(t_o-t_r)-\alpha t_r} - e^{-\alpha t_o}\right] + \frac{1}{\alpha}\left[e^{-\alpha t_o} - e^{-\alpha t_r}\right]\right] \end{aligned} \tag{70}$$

$$
\begin{aligned}
\int_0^{t_r} I_r(t)e^{-\alpha t}dt =& \int_0^{t_r} \frac{(z+\theta_o)W+y}{\theta_r}\Big(e^{\theta_r(t_r-t)}-1\Big)e^{-\alpha t}dt \\
=& \frac{(z+\theta_o)W+y}{\theta_r}\int_0^{t_r}\Big(e^{\theta_r t_r}e^{-(\alpha+\theta_r)t}-e^{-\alpha t}\Big) \\
=& \frac{(z+\theta_o)W+y}{\theta_r}\left[\frac{e^{\theta_r t_r}}{-(\alpha+\theta_r)}\Big[e^{-(\alpha+\theta_r)t}\Big]_0^{t_r}+\frac{1}{\alpha}\Big[e^{-\alpha t}\Big]_0^{t_r}\right] \\
=& \frac{(z+\theta_o)W+y}{\theta_r}\left[\frac{e^{\theta_r t_r}}{-(\alpha+\theta_r)}\Big[e^{-(\alpha+\theta_r)t_r}-1\Big]+\frac{1}{\alpha}\Big[e^{-\alpha t_r}-1\Big]\right] \\
=& \frac{(z+\theta_o)W+y}{\theta_r}\left[\frac{1}{\alpha+\theta_r}\Big[e^{\theta_r t_r}-e^{-\alpha t_r}\Big]-\frac{1}{\alpha}\Big[1-e^{-\alpha t_r}\Big]\right] \\
=& \frac{(z+\theta_o)W+y}{\theta_r}\left[\frac{1}{\alpha+\theta_r}\Big[e^{\theta_r t_r}-1\Big]+\frac{1}{\alpha+\theta_r}-\frac{e^{-\alpha t_r}}{\alpha+\theta_r}-\frac{1}{\alpha}\Big[1-e^{-\alpha t_r}\Big]\right] \\
=& \frac{(z+\theta_o)W+y}{\theta_r}\left[\frac{1}{\alpha+\theta_r}\Big[e^{\theta_r t_r}-1\Big]-\frac{\theta_r}{\alpha(\alpha+\theta_r)}\Big[1-e^{-\alpha t_r}\Big]\right]
\end{aligned}
\tag{71}
$$

C Annuity stream of holding and deterioration costs at the wholesaler

Considering the inventory level of the i^{th} time interval presented in (13), the present value of holding and deterioration costs at the wholesaler for the first inventory cycle are

$$
HC_{W1} = f\sum_{i=1}^{k-1}\int_{(i-1)T_R}^{iT_R} I_W^i(t)e^{-\alpha t}dt \tag{72}
$$

and

$$
DC_{W1} = d_W\theta\sum_{i=1}^{k-1}\int_{(i-1)T_R}^{iT_R} I_W^i(t)e^{-\alpha t}dt, \tag{73}
$$

respectively. To obtain the holding and deterioration costs, we need to first mathematically simplify the exponential sum as follows

$$\begin{aligned}
A_{ret} =& \sum_{i=1}^{k-1} \int_{(i-1)T_R}^{iT_R} I_W^i(t) e^{-\alpha t} dt \\
=& \sum_{i=1}^{k-1} \int_{(i-1)T_R}^{iT_R} Q_R e^{\theta(iT_R - t)} \frac{e^{\theta(k-i)T_R} - 1}{e^{\theta T_R} - 1} e^{-\alpha t} dt \\
=& \sum_{i=1}^{k-1} Q_R \frac{e^{\theta k T_R} - e^{\theta i T_R}}{e^{\theta T_R} - 1} \int_{(i-1)T_R}^{iT_R} e^{-(\theta+\alpha)t} dt \\
=& \sum_{i=1}^{k-1} Q_R \frac{e^{\theta k T_R} - e^{\theta i T_R}}{e^{\theta T_R} - 1} \left[\frac{1}{\theta+\alpha} \left(e^{-(\theta+\alpha)(i-1)T_R} - e^{-(\theta+\alpha)iT_R} \right) \right] \\
=& \sum_{i=1}^{k-1} Q_R \frac{e^{\theta k T_R} - e^{\theta i T_R}}{e^{\theta T_R} - 1} \left[\frac{e^{-(\theta+\alpha)iT_R}}{\theta+\alpha} \left(e^{(\theta+\alpha)T_R} - 1 \right) \right] \\
=& \frac{Q_R(e^{(\theta+\alpha)T_R} - 1)}{(\theta+\alpha)(e^{\theta T_R} - 1)} \sum_{i=1}^{k-1} \left(e^{\theta k T_R} e^{-(\theta+\alpha)iT_R} - e^{\alpha i T_R} \right) \\
=& \frac{Q_R(e^{(\theta+\alpha)T_R} - 1)}{(\theta+\alpha)(e^{\theta T_R} - 1)} \left[e^{\theta k T_R} \left(\frac{e^{-(\theta+\alpha)T_R} - e^{-(\theta+\alpha)kT_R}}{1 - e^{-(\theta+\alpha)T_R}} \right) - \left(\frac{e^{-\alpha T_R} - e^{-\alpha k T_R}}{1 - e^{\alpha T_R}} \right) \right],
\end{aligned} \tag{74}$$

therefore

$$HC_{W1} = \frac{fQ_R(e^{(\theta+\alpha)T_R} - 1)}{(\theta+\alpha)(e^{\theta T_R} - 1)} \left[e^{\theta k T_R} \left(\frac{e^{-(\theta+\alpha)T_R} - e^{-(\theta+\alpha)kT_R}}{1 - e^{-(\theta+\alpha)T_R}} \right) - \left(\frac{e^{-\alpha T_R} - e^{-\alpha k T_R}}{1 - e^{\alpha T_R}} \right) \right]. \tag{75}$$

The annuity stream of all holding costs at the wholesaler over an infinite horizon is then

$$\begin{aligned}
HC_W =& HC_{W1} \frac{\alpha}{1 - e^{-\alpha k T_R}} \\
=& \frac{\alpha f Q_R(e^{(\theta+\alpha)T_R} - 1)}{(\theta+\alpha)(e^{\theta T_R} - 1)(1 - e^{-\alpha k T_R})} \left[e^{\theta k T_R} \frac{e^{-(\theta+\alpha)T_R} - e^{-(\theta+\alpha)kT_R}}{1 - e^{-(\theta+\alpha)T_R}} - \frac{e^{-\alpha T_R} - e^{-\alpha k T_R}}{1 - e^{-\alpha T_R}} \right].
\end{aligned} \tag{76}$$

In a similar way, the annuity stream of deterioration cost at the wholesaler over an infinite horizon is

$$DC_W = \frac{\alpha d_W \theta Q_R(e^{(\theta+\alpha)T_R} - 1)}{(\theta+\alpha)(e^{\theta T_R} - 1)(1 - e^{-\alpha k T_R})} \left[e^{\theta k T_R} \frac{e^{-(\theta+\alpha)T_R} - e^{-(\theta+\alpha)kT_R}}{1 - e^{-(\theta+\alpha)T_R}} - \frac{e^{-\alpha T_R} - e^{-\alpha k T_R}}{1 - e^{-\alpha T_R}} \right]. \tag{77}$$