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Massey Products in Moment-Angle Complexes

by

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ABSTRACT

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This thesis presents systematic constructions of new non-trivial higher Massey products in the cohomology of moment-angle complexes. This is achieved by using combinatorial operations, such as stellar subdivision and edge contraction, on the underlying simplicial complex of a moment-angle complex. These techniques construct non-trivial higher Massey products of cohomology classes in degree three or higher and can be used on any simplicial complex. Consequently, we find new examples of non-trivial Massey products in moment-angle manifolds $\mathcal{Z}_P$ for simple polytopes P that are neither truncated n-cubes nor Pogorelov polytopes, both of which have already been studied in the literature.

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Declaration of Authorship

I, Abigail Linton, declare that this thesis entitled

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and the work presented in it are my own and have been generated by me as the result of my own original research.

I confirm that:

1. This work was done wholly or mainly while in candidature for a research degree at this University;
2. Where any part of this thesis has previously been submitted for a degree or any other qualification at this University or any other institution, this has been clearly stated;
3. Where I have consulted the published work of others, this is always clearly attributed;
4. Where I have quoted from the work of others, the source is always given. With the exception of such quotations, this thesis is entirely my own work;
5. I have acknowledged all main sources of help;
6. Where the thesis is based on work done by myself jointly with others, I have made clear exactly what was done by others and what I have contributed myself;
7. None of this work has been published before submission.

Signed:

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Chapter 1

Introduction

In this thesis, I study Massey products in toric topology. In particular, I present systematic methods to construct moment-angle complexes with non-trivial higher Massey products, using homotopy theory and the combinatorial structure in moment-angle complexes.

Massey products are higher cohomology operations that refine cup products and they are studied extensively throughout algebra, topology and geometry. In symplectic geometry, it is well known that all Kähler manifolds are formal [16], that is, Kähler manifolds have no non-trivial Massey products. Since the 1970s, there has been a well-known problem in symplectic geometry to find examples of non-Kähler manifolds. The first example was found by Thurston [37] in 1975, but in general the non-Kähler structure is difficult to detect and there are still relatively few examples known [2]. Massey products are obstructions to the Kähler structure, but being higher cohomology operations, Massey products are also very difficult to calculate. This is one of the reasons that we study Massey products in moment-angle complexes.

For a simplicial complex $\mathcal{K}$, the moment-angle complex $\mathcal{Z}_{\mathcal{K}}$ is a topological space formed by identifying products of discs and circles according to the intersection of simplices in $\mathcal{K}$. These spaces naturally arise out of the study of complements of coordinate subspace arrangements in combinatorics, the intersection of quadrics in complex geometry, and level sets for moment maps in symplectic geometry. Moment-angle complexes are fundamental objects in toric topology, which inherently relates to other fields such as combinatorics, commutative algebra, geometric group theory, and complex geometry. The homotopy type of moment-angle complexes is not yet fully understood, so Massey products are a valuable topological invariant for these spaces. Furthermore, the underlying combinatorial structure of moment-angle complexes is a useful platform from which to study Massey products.

The first examples of non-trivial higher Massey products in moment-angle complexes were found in 2003 by Baskakov [7], who gave an infinite family of moment-angle complexes with non-trivial triple Massey products. In 2007, Denham and Suciu [17] gave a classification of triple Massey products of classes in the lowest degree (degree three). Since then, there have been other families of examples such as Limonchenko's family of n-Massey products in moment-angle complexes over truncated cubes [26], and Zhuravleva found non-trivial triple Massey products over Pogorelov polytopes [39]. All of these results use a combinatorial interpretation of Massey products in moment-angle complexes, which alludes to the ability to understand the structure of Massey products better from a combinatorial perspective. However other than these few families of examples, not much is currently known about the existence or abundance of non-trivial Massey products in the cohomology of moment-angle complexes.

The main approach of this thesis is to use combinatorial operations on the underlying simplicial complex of a moment-angle complex. We construct non-trivial n-Massey products $\langle\alpha_1, \ldots, \alpha_n\rangle$ in $H^*(\mathcal{Z}_\mathcal{K})$ where $\mathcal{K}$ is obtained by performing stellar subdivisions on the join of n simplicial complexes $\mathcal{K}_1 * \cdots * \mathcal{K}_n$ and $\alpha_i \in H^*(\mathcal{K}_i)$. The degree of the Massey product is then determined by the degree of $\alpha_i \in H^*(\mathcal{K}_i)$. Additionally, we construct a non-trivial n-Massey product in $H^*(\mathcal{Z}_\mathcal{K})$ given the existence of a different non-trivial n-Massey product in $H^*(\mathcal{Z}_{\hat{\mathcal{K}}})$, where $\mathcal{K}$ maps to $\hat{\mathcal{K}}$ by edge contractions. These edge contractions are simplicial maps that maintain the homotopy type of $\mathcal{K}$, but change the combinatorics. Then we show that the Massey product in $H^*(\mathcal{Z}_\mathcal{K})$ corresponds to the Massey product in $H^*(\mathcal{Z}_{\hat{\mathcal{K}}})$ and is non-trivial. Its degree is determined by the original Massey product and the number of edge contractions performed. In particular, these higher Massey products are in arbitrary degree, not just in the lowest degree. The constructions introduced in this thesis also generalise all of the aforementioned families of examples, thereby proving that there are many more non-trivial Massey products in the cohomology of moment-angle complexes than previously shown.

When a simplicial complex $\mathcal{K}$ is a simplicial sphere, such as the boundary of the dual of a simple polytope P, the moment-angle complex $\mathcal{Z}_\mathcal{K}$ (or $\mathcal{Z}_P$) is a smooth manifold: a (polytopal) moment-angle manifold. In both of the constructions in this thesis, there is no constraint on the simplicial complexes used, and indeed we find new non-trivial Massey products in moment-angle manifolds. Hence these manifolds will be non-Kähler.

The first chapter of this thesis is dedicated to relevant background material and review of existing results. The subsequent chapters present the original results of the thesis. In the final chapter, we also show that there are non-trivial Massey products in moment-angle complexes that correspond to simplicial posets.

Chapter 2

Background

2.1 Moment-Angle Complexes

This section summarises the relevant definitions and existing results for moment-angle complexes and some of their wider context. Our aim is to be able to calculate the cohomology of moment-angle complexes, and importantly to understand the cup product.

Let $\mathbf{k}$ be a field or the integers. A *simplicial complex* $\mathcal{K}$ on the vertex set $V(\mathcal{K}) = [m] = \{1, \ldots, m\}$ is a collection of subsets $\sigma \subset [m]$ such that if $\sigma \in \mathcal{K}$ and $\tau \subset \sigma$, then $\tau \in \mathcal{K}$. These subsets $\sigma \in \mathcal{K}$ are called *simplices*. For a set of vertices $J \subset [m]$, the *full subcomplex* $\mathcal{K}_J$ is

$$\mathcal{K}_J = \{\sigma \in \mathcal{K} \colon V(\sigma) \subset J\}.$$

2.1.1 Introduction

Moment-angle complexes first appeared in work by Buchstaber and Panov [10], and are fundamental objects in toric topology. In particular, the study of moment-angle complexes is often very intradisciplinary since they also arise naturally as complements of coordinate subspace arrangements, intersections of quadrics, and level sets of moment-maps in symplectic geometry (see [4] for a summary). In this thesis, we focus on the relationship between the topology of moment-angle complexes and the combinatorics.

Definition 2.1.1. For a simplicial complex $\mathcal{K}$ on $[m]$ vertices, the *moment-angle complex* $\mathcal{Z}_\mathcal{K}$ for $\mathcal{K}$ is

$$\mathcal{Z}_\mathcal{K} = \bigcup_{I \in \mathcal{K}} \left(D^2, S^1\right)^I \subset (D^2)^m$$

where $(D^2, S^1)^I = \prod_{i=1}^m Y_i$ for $Y_i = D^2$ if $i \in I$, and $Y_i = S^1$ if $i \notin I$.

Example 2.1.2. Let $\mathcal{K}$ be two disjoint points. Then the moment-angle complex is a union of two solid tori taken over their boundary torus, which corresponds to the empty simplex $\varnothing \in \mathcal{K}$.

$$\begin{aligned}\mathcal{Z}_{\mathcal{K}} &= D^2 \times S^1 \bigcup_{S^1 \times S^1} S^1 \times D^2 \\ &= \partial(D^2 \times D^2) = S^3.\end{aligned}$$

In general, when $\mathcal{K}$ is a triangulation of a sphere, the moment-angle complex $\mathcal{Z}_{\mathcal{K}}$ is a *moment-angle manifold.*

Theorem 2.1.3 ([10]). *If $\mathcal{K}$ is a triangulation of an n-dimensional sphere on m vertices, then $\mathcal{Z}_{\mathcal{K}}$ is an $(m+n+1)$-dimensional (closed) topological manifold.* □

A proof of this theorem can be found in [10]. For a simple polytope P, let the *nerve complex* $\mathcal{K}_P = \partial(P^*)$ of P be the boundary complex of the dual polytope. Then $\mathcal{Z}_P = \mathcal{Z}_{\mathcal{K}_P}$ be the *(polytopal) moment-angle manifold* corresponding to the polytope P. Moment-angle manifolds $\mathcal{Z}_P$ are particularly studied since they are smooth and have rich geometrical properties (see [11, Sections 5, 6] for more information).

Example 2.1.4. Let $\mathcal{K}$ be the boundary of a pentagon. There are no n-simplices in $\mathcal{K}$ for $n \geqslant 2$, and as such the maximal simplices are one-dimensional. Therefore,

$$\begin{aligned}\mathcal{Z}_{\mathcal{K}} =&(D^2 \times D^2 \times S^1 \times S^1 \times S^1) \cup (S^1 \times D^2 \times S^1 \times D^2 \times S^1) \\ &\cup (S^1 \times S^1 \times D^2 \times D^2 \times S^1) \cup (S^1 \times S^1 \times D^2 \times S^1 \times D^2) \\ &\cup (D^2 \times S^1 \times S^1 \times S^1 \times D^2).\end{aligned}$$

In fact $\mathcal{Z}_{\mathcal{K}}$ is diffeomorphic to the connected sum $(S^3 \times S^4)^{\#5}$, by results of Bosio and Meersseman [9, Theorem 6.3].

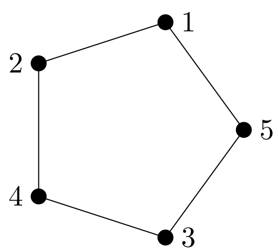


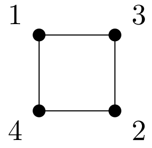
Figure 2.1: The boundary of a pentagon

For two simplicial complexes $\mathcal{K}_1$ and $\mathcal{K}_2$ on vertex sets V_1 and V_2 respectively, the join is $\mathcal{K}_1 * \mathcal{K}_2 = \{I_1 \cup I_2 \colon I_1 \in \mathcal{K}_1, I_2 \in \mathcal{K}_2\}$ on the set $V_1 \sqcup V_2$. This join also corresponds to the product of simple polytopes, as discussed in [11, Example 2.2.9].

Lemma 2.1.5. *For two simple polytopes P_1 and P_2, $\mathcal{K}_{P_1 \times P_2} = \mathcal{K}_{P_1} * \mathcal{K}_{P_2}$* □

We also have the following property of moment-angle complexes [11, Proposition 4.1.3].

Proposition 2.1.6. *Let $\mathcal{K}_1$ and $\mathcal{K}_2$ be simplicial complexes. Then $\mathcal{Z}_{\mathcal{K}_1 * \mathcal{K}_2} = \mathcal{Z}_{\mathcal{K}_1} \times \mathcal{Z}_{\mathcal{K}_2}$.* □



Example 2.1.7. Let $\mathcal{K}$ be the boundary of a square, the join of two pairs of disjoint points $\mathcal{K}_1 = \{\varnothing, \{1\}, \{2\}\}$, $\mathcal{K}_2 = \{\varnothing, \{3\}, \{4\}\}$. By Proposition 2.1.6 and Example 2.1.2, $\mathcal{Z}_\mathcal{K} = S^3 \times S^3$. Indeed, the moment-angle complex is

$$\begin{aligned}
\mathcal{Z}_\mathcal{K} = (D^2 \times S^1 \times D^2 \times S^1) \cup (S^1 \times D^2 \times D^2 \times S^1) \cup (S^1 \times D^2 \times S^1 \times D^2) \\
\cup (D^2 \times S^1 \times S^1 \times D^2) \\
= \partial(D^2 \times D^2) \times D^2 \times S^1 \cup \partial(D^2 \times D^2) \times S^1 \times D^2 \\
= \partial(D^2 \times D^2) \times \partial(D^2 \times D^2) = S^3 \times S^3.
\end{aligned}$$

For a topological pair (X, A) there is a construction $(X, A)^\mathcal{K}$ that generalises moment-angle complexes.

Definition 2.1.8. Let $\mathcal{K}$ be a simplicial complex on $[m]$ vertices, and let $\mathbf{X} = (X_1, \dots, X_m)$, $\mathbf{A} = (A_1, \dots, A_m)$ be families of pointed CW-spaces such that A_i is a pointed subset of X_i for all i. Then the *polyhedral product* $(\mathbf{X}, \mathbf{A})^\mathcal{K}$ is

$$(\mathbf{X}, \mathbf{A})^\mathcal{K} = \bigcup_{I \in \mathcal{K}} (\mathbf{X}, \mathbf{A})^I$$

where $(\mathbf{X}, \mathbf{A})^I = \prod_{i=1}^m Y_i$ for $Y_i = X_i$ if $i \in I$, and $Y_i = A_i$ if $i \notin I$. If $X_1 = \ldots = X_m$ and $A_1 = \ldots = A_m$, then we write $(X, A)^\mathcal{K}$ instead of $(\mathbf{X}, \mathbf{A})^\mathcal{K}$. If A is a point, then we write $(X)^\mathcal{K}$ instead of $(X, A)^\mathcal{K}$.

Polyhedral products are functorial with respect to continuous maps of topological pairs and inclusions of simplicial complexes. In particular, the following result is from [36, Proposition 3.1].

Proposition 2.1.9 ([36])**.** *Let $(\mathbf{X}, \mathbf{A})$ be a sequence of pointed, path-connected CW-pairs. Let $\mathcal{K}$ be a simplicial complex on $[m]$ and let $J \subset [m]$.*

- *The inclusion $\mathcal{K}_J \hookrightarrow \mathcal{K}$ induces an inclusion $(\mathbf{X}, \mathbf{A})^{\mathcal{K}_J} \hookrightarrow (\mathbf{X}, \mathbf{A})^\mathcal{K}$.*
- *The projection $\prod_{i=1}^m X_i \to \prod_{i \in J} X_i$ induces a map $(\mathbf{X}, \mathbf{A})^\mathcal{K} \to (\mathbf{X}, \mathbf{A})^{\mathcal{K}_J}$.*

Moreover, the composite $(\mathbf{X}, \mathbf{A})^{\mathcal{K}_J} \hookrightarrow (\mathbf{X}, \mathbf{A})^\mathcal{K} \to (\mathbf{X}, \mathbf{A})^{\mathcal{K}_J}$ is the identity map. □

As a result of the retraction $(\mathbf{X}, \mathbf{A})^\mathcal{K} \to (\mathbf{X}, \mathbf{A})^{\mathcal{K}_J}$, we have that $H^*(\mathcal{Z}_{\mathcal{K}_J}) \subset H^*(\mathcal{Z}_\mathcal{K})$ for any $J \subset [m]$. This is a fact we will use extensively when computing Massey products in the cohomology of moment-angle complexes.

Besides moment-angle complexes, other important examples of polyhedral products include real moment-angle complexes, $\mathcal{R}_{\mathcal{K}} = (D^1, S^0)^{\mathcal{K}}$ or right-angled Coxeter groups, $\pi_1((\mathbb{R}P^\infty, *)^{\mathcal{K}})$. A summary of the variety of polyhedral products can be found in [4].

Before the term "polyhedral product" was used, one of the first examples studied was Davis-Januszkiewicz space $DJ(\mathcal{K}) \simeq (\mathbb{C}P^\infty)^{\mathcal{K}}$. These were introduced by Davis and Januszkiewicz in [15] and are discussed in [11, Section 4.3]. Davis-Januszkiewicz spaces are important for the calculation of the cohomology of moment-angle complexes since there is a homotopy fibration $\mathcal{Z}_{\mathcal{K}} \to (\mathbb{C}P^\infty)^{\mathcal{K}} \hookrightarrow (\mathbb{C}P^\infty)^m$ [11, Theorem 4.3.2]. In other words, a moment-angle complex is the homotopy pullback making the following diagram commute

$$\begin{array}{ccc} \mathcal{Z}_{\mathcal{K}} & \longrightarrow & (\mathbb{C}P^\infty)^{\mathcal{K}} \\ \downarrow & & \downarrow i \\ \{pt\} & \longrightarrow & (\mathbb{C}P^\infty)^m. \end{array}$$

Given this diagram, there is an Eilenberg-Moore spectral sequence [19] and it was shown in [13] that the spectral sequence converges at the second page. Then there is a theorem by Eilenberg-Moore [33] that gives an isomorphism of algebras

$$H^*(\mathcal{Z}_{\mathcal{K}}) \cong \mathrm{Tor}_{H^*((\mathbb{C}P^\infty)^m)}(H^*((\mathbb{C}P^\infty)^{\mathcal{K}}), H^*(\{pt\})) \tag{2.1}$$

where Tor algebras will be discussed further in Section 2.1.2.1.

2.1.2 Face rings and cohomology of moment-angle complexes

An important property of Davis-Januszkiewicz spaces is that its cohomology ring is the *face ring* $\mathbf{k}[\mathcal{K}]$ of a simplicial complex. The face ring of a simplicial complex originates from the study of algebraic combinatorics and combinatorial commutative algebra, but it is also a fundamental tool for understanding the cohomology of moment-angle complexes.

Definition 2.1.10. Let $\mathcal{K}$ be a simplicial complex on m vertices. Let $\mathbf{k}[m] = \mathbf{k}[v_1, \ldots, v_m]$. The *face ring* (or *Stanley-Reisner ring*) $\mathbf{k}[\mathcal{K}]$ for $\mathcal{K}$ is given by

$$\mathbf{k}[\mathcal{K}] = \mathbf{k}[m]/\mathcal{I}_{\mathcal{K}}$$

where $\mathcal{I}_{\mathcal{K}} = (v_I : I \notin \mathcal{K})$ is the *Stanley-Reisner ideal* generated by square-free monomials $v_I = v_{i_1} \ldots v_{i_l} \in \mathbf{k}[m]$ corresponding to minimal missing faces $I = \{i_1, \ldots, i_l\} \subset [m]$.

Example 2.1.11. Let $\mathcal{K}$ be the boundary of a pentagon as in Figure 2.1. Then

$$\mathbf{k}[\mathcal{K}] = \mathbf{k}[v_1, v_2, v_3, v_4, v_5]/\mathcal{I}_{\mathcal{K}}$$

where $\mathcal{I}_{\mathcal{K}} = (v_{13}, v_{14}, v_{23}, v_{25}, v_{45})$ is the ideal generated by monomials whose simplices are not present in the pentagon.

The following proposition relates Davis-Januszkiewicz spaces to face rings of simplicial complexes, thus highlighting a relation between combinatorial algebra and toric topology.

Proposition 2.1.12 ([32])**.** *Let $\mathcal{K}$ be a simplicial complex on $[m]$. The cohomology ring of $(\mathbb{C}P^\infty)^\mathcal{K}$ is isomorphic to the face ring $\mathbf{k}[\mathcal{K}]$.*

Proof. The cellular structure of $\mathbb{C}P^\infty$ is given by one cell in every even dimension ([23]). Therefore $(\mathbb{C}P^\infty)^m$ is given by cells of the form $D^{2k_1}_{j_1}\times\cdots\times D^{2k_p}_{j_p}$ for $j_1,\ldots,j_p\in[m]$, where $D^{2k_i}_{j_i}$ is a $2k_i$-dimensional cell in the j_ith factor of $(\mathbb{C}P^\infty)^m$. Accordingly, the cochain group $C^*((\mathbb{C}P^\infty)^m)$ has a basis of cochains $(D^{2k_1}_{j_1}\cdots D^{2k_p}_{j_p})^*$ dual to $D^{2k_1}_{j_1}\times\cdots\times D^{2k_p}_{j_p}$.

Comparably, $(\mathbb{C}P^\infty)^\mathcal{K}$ has cells $D^{2k_1}_{j_1}\times\cdots\times D^{2k_p}_{j_p}$ where $\{j_1,\ldots,j_p\}\in\mathcal{K}$. So the cochain map $i_*\colon C^*((\mathbb{C}P^\infty)^m)\to C^*((\mathbb{C}P^\infty)^\mathcal{K})$ induced by the inclusion $i\colon(\mathbb{C}P^\infty)^\mathcal{K}\hookrightarrow(\mathbb{C}P^\infty)^m$ has a kernel generated by cochains $(D^{2k_1}_{j_1}\ldots D^{2k_p}_{j_p})^*$ for $\{j_1,\ldots,j_p\}\notin\mathcal{K}$.

We can identify $C^*((\mathbb{C}P^\infty)^m)$ with $\mathbf{k}[m]$ by identifying the cochains $(D^{2k_1}_{j_1}\ldots D^{2k_p}_{j_p})^*$ with monomials $v^{k_1}_{j_1}\ldots v^{k_p}_{j_p}$. Similarly we can identify $\ker(i_*)$ with the Stanley-Reisner ideal $\mathcal{I}_\mathcal{K}$. Therefore $C^*((\mathbb{C}P^\infty)^\mathcal{K})\cong\mathbf{k}[\mathcal{K}]$. Since there are only cells in even dimensions, all boundary maps in the cochain complex are trivial and so $H^*((\mathbb{C}P^\infty)^\mathcal{K})=C^*((\mathbb{C}P^\infty)^\mathcal{K})\cong\mathbf{k}[\mathcal{K}]$. □

Therefore, the identity in (2.1) becomes

$$H^*(\mathcal{Z}_\mathcal{K})\cong\mathrm{Tor}_{\mathbf{k}[m]}(\mathbf{k}[\mathcal{K}],\mathbf{k}).\tag{2.2}$$

Our aim is to understand the cup product in $H^*(\mathcal{Z}_\mathcal{K})$ in terms of $\mathcal{K}$, in order to calculate Massey products. However, it is difficult to see this using the isomorphism of algebras given by the Eilenberg-Moore theorem. Consequently, Panov [32, Theorem 4.7] gave an alternative proof of (2.2). We summarise this proof in Section 2.1.2.2 after a brief survey of Tor-algebras in Section 2.1.2.1.

2.1.2.1 Tor-algebras and the Koszul algebra

To understand the cohomology of moment-angle complexes, we need to be able to determine Tor (2.2), which can be difficult to calculate. The definition of Tor given here differs slightly from the standard definition in algebra since we use non-positive degrees rather than non-negative.

Definition 2.1.13. Let A be a commutative finitely generated $\mathbf{k}$-algebra with unit. For an A-module M, a *free* (or respectively, *projective*) *resolution* of M is an exact sequence of A-modules

$$\cdots\xrightarrow{d^{-(i+1)}}P^{-i}\xrightarrow{d^{-i}}\cdots\xrightarrow{d^{-2}}P^{-1}\xrightarrow{d^{-1}}P^0\xrightarrow{d^0}M\longrightarrow 0$$

where P^{-i} are free (projective) A-modules.

Definition 2.1.14. For graded A-modules M and N, the module $\mathrm{Tor}_A^{-i}(M,N)$ is defined as the cohomology module of the cochain complex

$$\ldots \longrightarrow P^{-i} \otimes_A N \longrightarrow \cdots \xrightarrow{d^{-2}\otimes id} P^{-1} \otimes_A N \xrightarrow{d^{-1}\otimes id} P^0 \otimes_A N \longrightarrow 0$$

obtained by omitting the term M in its free resolution and applying $\otimes_A N$ to each term. Therefore $\mathrm{Tor}_A^{-i}(M,N) = H^{-i}(P \otimes_A N)$. The Tor-algebra $\mathrm{Tor}_A(M,N)$ is then defined as

$$\mathrm{Tor}_A(M,N) = \bigoplus_{i,j\geqslant 0} \mathrm{Tor}_A^{-i,j}(M,N)$$

with a multiplication

$$\mathrm{Tor}_A^{-i_1,j_1}(M,N) \otimes \mathrm{Tor}_A^{-i_2,j_2}(M,N) \to \mathrm{Tor}_A^{-i_1-i_2,j_1+j_2}(M,N)$$

where $\mathrm{Tor}_A^{-i,j}(M,N)$ is the jth graded component of $\mathrm{Tor}_A^{-i}(M,N)$.

The choice of resolution in Definition 2.1.14 does not matter, since for any choice of free resolution of M, the cohomology modules are canonically isomorphic [23, Lemma 3A.2]. Therefore, regarding $\mathbf{k}$ as a $\mathbf{k}[m]$-module, we consider the *Koszul resolution* of $\mathbf{k}$

$$0 \xrightarrow{d} \Lambda^m[m] \otimes_{\mathbf{k}} \mathbf{k}[m] \xrightarrow{d} \cdots \xrightarrow{d} \Lambda^1[m] \otimes_{\mathbf{k}} \mathbf{k}[m] \xrightarrow{d} \mathbf{k}[m] \xrightarrow{\epsilon} \mathbf{k} \longrightarrow 0 \tag{2.3}$$

where $\Lambda[m] = \Lambda[u_1,\ldots,u_m]$ is the exterior algebra ($u_i^2 = 0$, $u_iu_j = -u_ju_i$), $\Lambda^i[m]$ is the subspace of $\Lambda[m]$ containing monomials $u_1\cdots u_i$ of length i and $\mathbf{k}[m] = \mathbf{k}[v_1,\ldots,v_m]$ is the polynomial algebra over $\mathbf{k}$ on m variables. For convenience we refer to $u_i \otimes v_j \in \Lambda[m] \otimes \mathbf{k}[m]$ as the monomial u_iv_j, and let $u_Iv_J = u_{i_1}\ldots u_{i_k}v_{j_1}\ldots v_{j_l}$ for $I = \{i_1,\ldots,i_k\}$ and $J = \{j_1,\ldots,j_l\}$. The augmentation map ϵ sends $v_i \mapsto 0$, and the differential d is such that

$$d(u_i) = v_i, \quad d(v_i) = 0 \quad \text{and} \quad d(a\cdot b) = (da)\cdot b + (-1)^i a \cdot db \tag{2.4}$$

for any $a \in \Lambda^i[m] \otimes \mathbf{k}[m]$ and $b \in \Lambda[m] \otimes \mathbf{k}[m]$. The sequence (2.3) can be checked to be exact, as in [11, Section A.2]. We let $\Lambda[m] \otimes \mathbf{k}[m]$ be a bigraded differential algebra with

$$\mathrm{bideg}(u_i) = (-1,2) \quad \text{and} \quad \mathrm{bideg}(v_i) = (0,2). \tag{2.5}$$

By tensoring the Koszul resolution by $\otimes_{\mathbf{k}[m]}\mathbf{k}[\mathcal{K}]$, and since $\mathrm{Tor}_A(M,N) \cong \mathrm{Tor}_A(N,M)$ [24, Propostion 3A.5], we define the *Tor-algebra of* $\mathcal{K}$ to be $\mathrm{Tor}_{\mathbf{k}[m]}(\mathbf{k}[\mathcal{K}],\mathbf{k})$. Also $(\Lambda[m] \otimes \mathbf{k}[\mathcal{K}])$ is called the *Koszul complex* of $\mathbf{k}[\mathcal{K}]$. There is an isomorphism of bigraded algebras,

$$\begin{aligned}\mathrm{Tor}_{\mathbf{k}[m]}(\mathbf{k}[\mathcal{K}],\mathbf{k}) \cong \mathrm{Tor}_{\mathbf{k}[m]}(\mathbf{k},\mathbf{k}[\mathcal{K}]) &= H(\Lambda[m] \otimes \mathbf{k}[m] \otimes_{\mathbf{k}[m]} \mathbf{k}[\mathcal{K}], d)\\ &\cong H(\Lambda[m] \otimes \mathbf{k}[\mathcal{K}], d).\end{aligned}$$

Therefore we give the Tor-algebra of $\mathcal{K}$ a bigrading that is inherited from the cohomology of $\Lambda[m] \otimes \mathbf{k}[\mathcal{K}]$ and

$$\mathrm{Tor}_{\mathbf{k}[m]}(\mathbf{k}[\mathcal{K}], \mathbf{k}) = \bigoplus_{i,j \geqslant 0} \mathrm{Tor}^{-i,2j}_{\mathbf{k}[m]}(\mathbf{k}[\mathcal{K}], \mathbf{k}).$$

Example 2.1.15. Let $\mathcal{K}$ be the boundary of a square as in Example 2.1.7. We calculate the Tor-algebra of $\mathcal{K}$ with the following cochain complex

$$0 \longrightarrow \Lambda^4[4] \otimes \mathbf{k}[\mathcal{K}] \xrightarrow{d_4} \Lambda^3[4] \otimes \mathbf{k}[\mathcal{K}] \xrightarrow{d_3} \Lambda^2[4] \otimes \mathbf{k}[\mathcal{K}] \xrightarrow{d_2} \Lambda^1[4] \otimes \mathbf{k}[\mathcal{K}] \xrightarrow{d_1} \mathbf{k}[\mathcal{K}] \longrightarrow 0.$$

Generators of $\Lambda^1[4] \otimes \mathbf{k}[\mathcal{K}]$ are of the form $u_i v_{j_1}^{p_1} \cdots v_{j_l}^{p_l}$ for $J = \{j_1, \ldots, j_l\} \in \mathcal{K}$ and $p_1, \ldots, p_l \in \mathbb{Z}_{\geqslant 0}$. For ease of notation, let $v_J^p = v_{j_1}^{p_1} \cdots v_{j_l}^{p_l}$. The map d_1 is given by $d_1(u_i v_J^p) = v_i v_J^p + (-1)^1 u_i \cdot 0 = v_i v_J^p$. Therefore $\mathrm{im}(d_1)$ is generated by all v_i corresponding to a vertex of $\mathcal{K}$, so

$$\mathrm{Tor}^0_{\mathbf{k}[m]}(\mathbf{k}[\mathcal{K}], \mathbf{k}) = \frac{\mathbf{k}[\mathcal{K}]}{\mathrm{im}(d_1)} \cong \mathbf{k}.$$

The kernel of d_1 is generated by elements of the form $u_i v_j v_J^p - u_j v_i v_J^p$ for $i \cup J, j \cup J \in \mathcal{K}$ and by monomials $u_i v_J^p$ such that $i \cup J \notin \mathcal{K}$, that is, $u_1 v_2^{p_2}, u_2 v_1^{p_1}, u_3 v_4^{p_4}, u_4 v_3^{p_3}$. The map d_2 is given by $d_2(u_{ij} v_J^p) = u_j v_i v_J^p - u_i v_j v_J^p$. For example, $d_2(u_{12}) = u_1 v_2 - u_2 v_1$, $d_2(u_{34}) = u_3 v_4 - u_4 v_3$, $d_2(u_{12} v_1) = u_2 v_1^2$, etc. Hence $\mathrm{im}\, d_2$ contains elements $u_1 v_2 - u_2 v_1$, $u_3 v_4 - u_4 v_3$ and $u_1 v_2^{p_2}, u_2 v_1^{p_1}, u_3 v_4^{p_4}, u_4 v_3^{p_3}$ for integers $p_1, \ldots, p_4 > 1$. Therefore

$$\mathrm{Tor}^{-1}_{\mathbf{k}[m]}(\mathbf{k}[\mathcal{K}], \mathbf{k}) \cong \langle [u_1 v_2], [u_3 v_4] \rangle.$$

The kernel of d_2 is generated by $u_{ij} v_J^p$ where $i \neq j$, $i \cup J, j \cup J \notin \mathcal{K}$, and elements of the form $u_{jk} v_i v_J^p - u_{ik} v_j v_J^p + u_{ij} v_k v_J^p$, $i \neq j \neq k \neq i$. The map d_3 is given by $d_3(u_{ijk} v_J^p) = u_{jk} v_i v_J^p - u_{ik} v_j v_J^p + u_{ij} v_k v_J^p$. So $d_3(u_{124} v_3) = u_{24} v_{13} - u_{14} v_{23}$, $d_3(u_{234} v_1) = -u_{24} v_{13} + u_{23} v_{14}$, and also $u_{23} v_{14} = d(u_{123} v_4) + u_{13} v_{24}$, $u_{14} v_{23} = -d(u_{134} v_2) + u_{13} v_{24}$. Additionally, for any edge $\{j_3, j_4\} \in \mathcal{K}$, $\{j_1, j_2\} = [4] \setminus \{j_3, j_4\}$, and any $p_3 - 1, p_4 > 0$, $d(u_{j_1 j_2 j_3} v_{j_3}^{p_3 - 1} v_{j_4}^{p_4}) = u_{j_1 j_2} v_{j_3}^{p_3} v_{j_4}^{p_4}$. Therefore

$$\mathrm{Tor}^{-2}_{\mathbf{k}[m]}(\mathbf{k}[\mathcal{K}], \mathbf{k}) \cong \langle [u_{13} v_{24}] \rangle.$$

In particular, $[u_{13} v_{24}]$ is the product of the generators $[u_1 v_2], [u_3 v_4]$ in $\mathrm{Tor}^{-1}_{\mathbf{k}[m]}(\mathbf{k}[\mathcal{K}], \mathbf{k})$.

The final non-trivial map, d_4, is given by $d_4(u_{1234}) = u_{234} v_1 - u_{134} v_2 + u_{124} v_3 - u_{123} v_4$, and $\ker d_3 = \mathrm{im}\, d_4$. So $\mathrm{Tor}^{-q}_{\mathbf{k}[m]}(\mathbf{k}[\mathcal{K}], \mathbf{k}) = 0$ for all $q \geqslant 3$.

The generators of $\mathrm{Tor}^{-1}_{\mathbf{k}[m]}(\mathbf{k}[\mathcal{K}], \mathbf{k})$ have bidegree $(-1, 4)$, and similarly the generator of $\mathrm{Tor}^{-2}_{\mathbf{k}[m]}(\mathbf{k}[\mathcal{K}], \mathbf{k})$ has bidegree $(-2, 8)$. Hence

$$\mathrm{Tor}^0_{\mathbf{k}[m]}(\mathbf{k}[\mathcal{K}], \mathbf{k}) = \mathrm{Tor}^{0,0}_{\mathbf{k}[m]}(\mathbf{k}[\mathcal{K}], \mathbf{k}) \cong \mathbf{k},$$
$$\mathrm{Tor}^{-1}_{\mathbf{k}[m]}(\mathbf{k}[\mathcal{K}], \mathbf{k}) = \mathrm{Tor}^{-1,4}_{\mathbf{k}[m]}(\mathbf{k}[\mathcal{K}], \mathbf{k}) \cong \langle [u_1 v_2], [u_3 v_4] \rangle,$$
$$\mathrm{Tor}^{-2}_{\mathbf{k}[m]}(\mathbf{k}[\mathcal{K}], \mathbf{k}) = \mathrm{Tor}^{-2,8}_{\mathbf{k}[m]}(\mathbf{k}[\mathcal{K}], \mathbf{k}) \cong \langle [u_{13} v_{24}] \rangle$$

and all other dimensions are trivial.

The aim of this example is to demonstrate that calculating $\mathrm{Tor}_{\mathbf{k}[m]}(\mathbf{k}[\mathcal{K}], \mathbf{k})$ is cumbersome. Nonetheless, the Tor-algebra of $\mathcal{K}$ is crucial for understanding $H^*(\mathcal{Z}_{\mathcal{K}})$. So, to make Tor more approachable, it is helpful to use the following "auxiliary" multigraded algebra $R^*(\mathcal{K})$. In Example 2.1.25, we will see a much more simplified version of Examples 2.1.15.

Definition 2.1.16. For a simplicial complex $\mathcal{K}$ on $[m]$, define the quotient algebra

$$R^*(\mathcal{K}) = \Lambda[u_1, \dots, u_m] \otimes \mathbf{k}[\mathcal{K}]/(v_i^2, u_i v_i, 1 \leqslant i \leqslant m)$$

with $\mathrm{bideg}(u_i) = (-1, 2), \mathrm{bideg}(v_i) = (0, 2)$ and

$$d(u_i) = v_i, \quad d(v_i) = 0, \quad \text{and} \quad d(a \cdot b) = (da) \cdot b + (-1)^i a \cdot db. \tag{2.6}$$

This is an algebra with a basis of monomials $u_I v_J$ for $I \subset [m]$, $J \in \mathcal{K}$ and $I \cap J = \varnothing$.

By the definition of $R^*(\mathcal{K})$, it is possible to define a quotient projection

$$\varrho \colon \Lambda[u_1, \dots, u_m] \otimes k[\mathcal{K}] \to R^*(\mathcal{K})$$

as well as a k-linear map

$$\iota \colon R^*(\mathcal{K}) \to \Lambda[m] \otimes k[\mathcal{K}]$$

that maps $u_I v_J \in R^*(\mathcal{K})$ to $u_I v_J \in \Lambda[m] \otimes k[\mathcal{K}]$. Since both the domain and range have the same differential, ι commutes with the differentials. Therefore ι is a homomorphism of bigraded differential k-vector spaces, with $\varrho \circ \iota = id$. However it is not a homomorphism of algebras since $\iota(u_i v_i) = 0$ but $\iota(u_i)\iota(v_i) \neq 0$.

Proposition 2.1.17. *The projection homomorphism* $\varrho \colon \Lambda[u_1, \dots, u_m] \otimes k[\mathcal{K}] \to R^*(\mathcal{K})$ *induces an isomorphism on cohomology.* □

One proof of Proposition 2.1.17 is by finding an explicit cochain homotopy between the identity map and $\iota \circ \varrho$. This proof can be found in [11, Lemma 3.2.6]. Importantly, Proposition 2.1.17 says that there is an isomorphism of algebras

$$\mathrm{Tor}_{\mathbf{k}[m]}(\mathbf{k}[\mathcal{K}], \mathbf{k}) \cong H(\Lambda[m] \otimes \mathbf{k}[\mathcal{K}], d) \cong H(R^*(\mathcal{K}), d).$$

Then calculations of $H(R^*(\mathcal{K}), d)$ are simplified compared to calculations of $\mathrm{Tor}_{\mathbf{k}[m]}(\mathbf{k}[\mathcal{K}], \mathbf{k})$ since $u_i v_i = 0$ and $v_i^2 = 0$, as in the following example.

Example 2.1.18. Let $\mathcal{K}$ be the boundary of a pentagon, as in Figure 2.1. Let $R^l(\mathcal{K})$ be generated by elements in $R^*(\mathcal{K})$ of total degree l, so that $R^0(\mathcal{K})$ is generated by the element 1, $R^1(\mathcal{K})$ generated by u_i, $R^2(\mathcal{K})$ generated by v_i and $u_i u_j$ for $i < j$, etc. The generators $u_{ij} v_{klm}$ and v_{ijkl} of $R^8(\mathcal{K})$ are zero because there are no 2 or 3-simplices in $\mathcal{K}$. Also the generators $u_{ijklmnop}$, $u_{ijklmn} v_o$, $u_{ijkl} v_{mn}$ are zero because

$i, j, k, l, m, n, o, p \in \{1, 2, 3, 4, 5\}$ and $u_i^2 = 0$, $u_i v_i = 0$, $v_i^2 = 0$. Therefore $R^8(\mathcal{K}) = 0$ and similarly $R^l(\mathcal{K}) = 0$ for $l > 7$. Then we have the cochain complex

$$0 \longrightarrow R^0(\mathcal{K}) \xrightarrow{d^0} R^1(\mathcal{K}) \xrightarrow{d^1} R^2(\mathcal{K}) \xrightarrow{d^2} \cdots \xrightarrow{d^6} R^7(\mathcal{K}) \longrightarrow 0$$

where d^0 is trivial and d^l the differential in (2.6) for $l > 0$. If we take the cohomology of this sequence, $H^0(R^*(\mathcal{K})) = \mathbf{k}$. Also, $d^1(u_i) = v_i$ for all i and $\ker(d^1) = 0$, so $H^1(R^*(\mathcal{K})) = 0$. Similarly $H^2(R^*(\mathcal{K})) = 0$ since $\ker(d^2) = \mathrm{im}(d^1)$, where $d^2(u_i u_j) = u_j v_i - u_i v_j$ for all i, j and $d^2(v_i) = 0$ but $v_i \in \mathrm{im}(d^1)$. The map d^3 takes the generators $u_i v_j$, u_{ijk} of $R^3(\mathcal{K})$ to $d^3(u_i v_j) = v_{ij}$ and $d^3(u_{ijk}) = u_{jk} v_i - u_{ik} v_j + u_{ij} v_k$. So $\ker d^3$ is generated by $u_i v_j$ for $\{i, j\} \notin \mathcal{K}$ and by $u_i v_j - u_j v_i = d^2(u_{ij})$ for $\{i, j\} \in \mathcal{K}$. Hence

$$H^3(R^*(\mathcal{K})) = \langle [u_1 v_3], [u_1 v_4], [u_2 v_3], [u_2, v_5], [u_4 v_5] \rangle.$$

Also, $R^4(\mathcal{K})$ is generated by monomials u_{ijkl}, $u_{ij} v_k$, and v_{ij}. So $\ker d^4$ contains $v_{ij} = d^3(u_i v_j)$, $u_{ij} v_k$ if $\{i, k\}, \{j, k\} \notin \mathcal{K}$ and $d^3(u_{ijk})$. Thus

$$H^4(R^*(\mathcal{K})) = \langle [u_{34} v_1], [u_{35} v_2], [u_{12} v_3], [u_{15} v_4], [u_{24} v_5] \rangle.$$

The group $R^5(\mathcal{K})$ is generated by monomials u_{ijklm}, $u_{ijk} v_l$, $u_i v_{jk}$. Since $\mathcal{K}$ has no 2-simplices, $d^5(u_i v_{jk}) = v_{ijk}$ is zero. Also for any $u_i v_{jk} \neq 0$, either $\{i, j\} \notin \mathcal{K}$ or $\{i, k\} \notin \mathcal{K}$ because $\mathcal{K}$ does not contain a cycle of length three. Thus either $u_i v_{jk} = d^4(-u_{ik} v_j)$ if $\{i, j\} \notin \mathcal{K}$ or $u_i v_{jk} = d^4(-u_{ij} v_k)$ if $\{i, k\} \notin \mathcal{K}$. Hence $\ker d^5 = \mathrm{im}\, d^4$.

The kernel of d^6 contains monomials of the form $u_{ij} v_{kl}$ since there are no 2-simplices in $\mathcal{K}$. Also for any $u_{ij} v_{kl} \neq 0$, either $\{i, k\}, \{j, k\} \notin \mathcal{K}$ or $\{i, l\}, \{j, l\} \notin \mathcal{K}$. Thus either $u_{ij} v_{kl} = u_{jl} v_{ik} - u_{il} v_{jk} + u_{ij} v_{kl} = d^5(u_{ijl} v_k)$ in the first case, or $u_{ij} v_{kl} = d^5(u_{ijk} v_l)$ in the latter. Hence $\ker d^6 = \mathrm{im}\, d^5$.

The generators of $R^7(\mathcal{K})$ are $u_{ijklmno}$, $u_{ijklm} v_n$ and $u_{ijk} v_{lm}$. We have that $u_{ijk} v_{lm} \in \ker d^7$ because there are no 2-simplices. For any $u_{ijkl} v_m \neq 0$, $d^6(u_{ijkl} v_m) = u_{jkl} v_{im} - u_{ikl} v_{jm} + u_{ijl} v_{km} - u_{ijk} v_{lm}$ and exactly two of these terms are non-zero. So there is no linear combination a of monomials $u_{ijkl} v_m \in R^6(\mathcal{K})$ such that $d(a) = u_{ijk} v_{lm}$. Thus $u_{ijk} v_{lm} \notin \mathrm{im}\, d^6$ and

$$H^7(R^*(\mathcal{K})) = \langle [u_{124} v_{35}] \rangle.$$

In particular, $[u_{124} v_{35}] = [u_1 v_3] \cdot [u_{24} v_5]$. In terms of the bigrading, the non-zero cohomology groups are

$$\begin{gathered} H^{0,0}\left(R^*(\mathcal{K})\right) = H^0\left(R^*(\mathcal{K})\right) = \mathbf{k}, \\ H^{-1,4}\left(R^*(\mathcal{K})\right) = H^3(R^*(\mathcal{K})) = \langle [u_1 v_3], [u_1 v_4], [u_2 v_3], [u_2, v_5], [u_4 v_5] \rangle, \\ H^{-2,6}\left(R^*(\mathcal{K})\right) = H^4(R^*(\mathcal{K})) = \langle [u_{34} v_1], [u_{35} v_2], [u_{12} v_3], [u_{15} v_4], [u_{24} v_5] \rangle, \\ H^{-3,10}\left(R^*(\mathcal{K})\right) = H^7(R^*(\mathcal{K})) = \langle [u_{124} v_{35}] \rangle. \end{gathered}$$

2.1.2.2 The relationship between $H^*(\mathcal{Z}_\mathcal{K})$ and Tor

We will first see that $H^*(\mathcal{Z}_\mathcal{K}) \cong \mathrm{Tor}_{\mathbf{k}[m]}(\mathbf{k}[\mathcal{K}], \mathbf{k})$ is an isomorphism of algebras via $H(R^*(\mathcal{K}), d)$, before giving a combinatorial interpretation of $H(R^*(\mathcal{K}), d)$ in Section 2.1.2.3. These results were first proved in [8] and [14]. We first summarise the arguments that show $H^*(\mathcal{Z}_\mathcal{K}) \cong H(R^*(\mathcal{K}))$ as an isomorphism of graded modules.

Let $C^*(\mathcal{Z}_\mathcal{K})$ be the cellular cochains of $\mathcal{Z}_\mathcal{K}$. We equip the product $\mathbb{D}^m$ of m polydiscs with a cell structure from [10], where we use the polydisc $\mathbb{D}$ instead of D^2 to emphasise the cellular decomposition. Each component $\mathbb{D}$ has the point $1 \in \mathbb{D}$ as a 0-cell, the complement of 1 in the boundary circle as a 1-cell T and the interior D as the 2-cell. It is convenient to label each cell of $\mathbb{D}^m$ by $\varkappa(I, J)$, where I parameterises the T-cells and J the D-cells so that each $\varkappa(I, J)$ is a product of $|I|$ T-cells and $|J|$ D-cells, with 0-cells in $m - |I| - |J|$ coordinates.

Since $\mathcal{Z}_\mathcal{K} \subset \mathbb{D}^m$, it has a cell decomposition consisting of $\varkappa(I, J)$ for each $J \in \mathcal{K}$ and $I \cap J = \varnothing$, by definition of $\mathcal{Z}_\mathcal{K}$. Then $C^*(\mathcal{Z}_\mathcal{K})$ has a basis of $\varkappa(I, J)^*$, dual to $\varkappa(I, J)$. Also $C^*(\mathcal{Z}_\mathcal{K}) = \bigoplus_{q=0}^{m} C^{*,2q}(\mathcal{Z}_\mathcal{K})$ has a bigrading given by

$$\mathrm{bideg}\, \varkappa(I, J)^* = (-|I|, 2|I| + 2|J|)$$

since the cellular differential preserves the second degree. Using this cell structure, we have the following Lemma.

Lemma 2.1.19 ([10])**.** *For a simplicial complex $\mathcal{K}$, there is an isomorphism of graded modules*

$$H(R^*(\mathcal{K})) \cong H^*(\mathcal{Z}_\mathcal{K}).$$

Proof. We define a map $g\colon R^*(\mathcal{K}) \to C^*(\mathcal{Z}_\mathcal{K})$, $u_I v_J \mapsto \varkappa(I, J)^*$. This map g is a bijection of basis elements, and as such is also an isomorphism of bigraded modules. To show that it is an isomorphism of cochains, we must show g commutes with the differentials δ (the coboundary map for $C^*(\mathcal{Z}_\mathcal{K})$) and d (the differential for $R^*(\mathcal{K})$). By construction of the cellular decomposition $\varkappa(\{i\}, \varnothing)$ is a T_i-cell, the boundary of the D_i-cell, $\varkappa(\varnothing, \{i\})$. Therefore,

$$\delta(g(u_i)) = \delta(\varkappa(\{i\}, \varnothing)^*) = \delta T_i^* = D_i^* = \varkappa(\varnothing, \{i\})^* = g(v_i) = g(d(u_i))$$

and

$$\delta(g(v_j)) = \delta(\varkappa(\varnothing, \{j\})^*) = \delta D_j^* = 0 = g(0) = g(d(v_j)).$$

Hence we have the isomorphism as required. □

This lemma above helps to prove the stronger case. The proof of Proposition 2.1.20 shows that the map $g\colon R^*(\mathcal{K}) \to C^*(\mathcal{Z}_\mathcal{K})$ in the proof of Lemma 2.1.19 is an isomorphism of algebras by explicitly defining a product in $C^*(\mathcal{Z}_\mathcal{K})$. This proof can be found in [10, Theorem 4.2.2].

Proposition 2.1.20 ([10]). *For a simplicial complex $\mathcal{K}$ on $[m]$ vertices, there is an isomorphism of graded algebras,*

$$H^*(\mathcal{Z}_\mathcal{K}) \cong H(R^*(\mathcal{K})) \cong \mathrm{Tor}_{\mathbf{k}[m]}(\mathbf{k}[\mathcal{K}], \mathbf{k}).$$

□

2.1.2.3 Hochster's Theorem

It is evident that the cohomology of moment-angle complexes is given by Tor, but Tor is difficult to calculate. Introduced by Hochster [24] in the 1970s, Hochster's formula has been an important tool for transitioning between algebraic and combinatorial problems as it relates the Tor-algebra of a simplicial complex $\mathcal{K}$ to the reduced cohomology $\widetilde{H}^*$ of full subcomplexes of $\mathcal{K}$.

Theorem 2.1.21 (Hochster's theorem [24]). *For a simplicial complex $\mathcal{K}$ on $[m]$, there is an isomorphism*

$$\mathrm{Tor}^{-i,2j}_{\mathbf{k}[m]}(\mathbf{k}[\mathcal{K}], \mathbf{k}) \cong \bigoplus_{J\subset[m]\colon |J|=j} \widetilde{H}^{j-i-1}(\mathcal{K}_J),$$

where we define $\widetilde{H}^{-1}(\mathcal{K}_\varnothing) = \mathbf{k}$.

The following proof, which comes from Panov [32], is not Hochster's original proof.

Proof. We define a multigrading on elements $v_1^{a_1}\dots v_m^{a_m} \in \mathbf{k}[m]$ so that $\mathrm{mdeg}\, v_1^{a_1}\dots v_m^{a_m} = (2a_1,\dots,2a_m)$. This induces a $\mathbb{Z}\oplus\mathbb{N}^m$-multigrading on $\mathrm{Tor}_{\mathbf{k}[m]}(\mathbf{k}[\mathcal{K}], \mathbf{k})$ through the Koszul complex $\Lambda[m]\otimes\mathbf{k}[\mathcal{K}]$, where $\mathrm{mdeg}\, u_1^{a_1}\dots u_m^{a_m} = (-(a_1+\dots+a_m), 2a_1,\dots,2a_m)$ for a_i either 0 or 1, and $\mathrm{mdeg}\, v_1^{a_1}\dots v_m^{a_m} = (0, 2a_1,\dots,2a_m)$ for $a_i\in\mathbb{N}$. We will denote $(a_1,\dots,a_m)\in\mathbb{N}^m$ by $\mathbf{a}$. Then

$$\mathrm{Tor}^{-i,2j}_{\mathbf{k}[m]}(\mathbf{k}[\mathcal{K}], \mathbf{k}) = \bigoplus_{\mathbf{a}\in\mathbb{N}^m\colon a_1+\dots+a_m=j} \mathrm{Tor}^{-i,2\mathbf{a}}_{\mathbf{k}[m]}(\mathbf{k}[\mathcal{K}], \mathbf{k}).$$

We also define a multigrading on $R^*(\mathcal{K})$. For $J\subset[m]$ whose elements have an order induced by the order on $[m]$, there is a corresponding $\mathbb{N}^m$-vector. We let the jth element be 1 if $j\in J$, and 0 otherwise. We also denote this vector J. Then let every non-zero monomial $u_Iv_L\in R^*(\mathcal{K})$, $I, L\subset[m]$, $I\cap L\neq\varnothing$ have multidegree $(-|I|, 2J)$, where we regard $J = I\cup L$ as a $\mathbb{N}^m$-vector. Since $u_iv_i = 0$ and $v_i^2 = 0$ in $R^*(\mathcal{K})$, $R^{-i,2\mathbf{a}}(\mathcal{K}) = 0$ for any vector $\mathbf{a}\in\mathbb{N}^m$ that is not a $(0,1)$-vector. By Proposition 2.1.17, we have an isomorphism of multigraded algebras $\mathrm{Tor}_{\mathbf{k}[m]}(\mathbf{k}[\mathcal{K}], \mathbf{k})\cong H(R^*(\mathcal{K}))$. Hence $\mathrm{Tor}^{-i,2\mathbf{a}}_{\mathbf{k}[m]}(\mathbf{k}[\mathcal{K}], \mathbf{k})\cong H^{-i,2\mathbf{a}}(R^*(\mathcal{K}))$ also vanishes. Therefore it only remains to show that $H^{-i,2J}(R^*(\mathcal{K}))\cong\widetilde{H}^{|J|-i-1}(\mathcal{K}_J)$. To do this, we will construct an isomorphism of cochain complexes $\bigoplus_{J\subset[m]} C^*(\mathcal{K}_J)$ to $R^*(\mathcal{K})$. To keep track of signs, let $\varepsilon(j, J) = (-1)^{r-1}$ for j

the rth element of J, and for $L \subset J$, let

$$\varepsilon(L,J) = \prod_{j \in L} \varepsilon(j,J).$$

Each simplicial cochain group $C^p(\mathcal{K}_J) = \mathrm{Hom}(C_p(\mathcal{K}_J), \mathbf{k})$ has a basis of χ_L for a p-simplex $L \in \mathcal{K}_J$, where χ_L takes the value 1 on L and 0 on any other simplex of $\mathcal{K}_J$. There is the augmented simplicial chain complex

$$0 \longrightarrow \mathbf{k} \xrightarrow{\epsilon} C^0(\mathcal{K}_J) \xrightarrow{d} \cdots \xrightarrow{d} C^{p-1}(\mathcal{K}_J) \xrightarrow{d} C^p(\mathcal{K}_J) \xrightarrow{d} \dots$$

where ϵ is the (co)augmentation map and the simplicial coboundary map d is given by

$$d(\chi_L) = \sum_{j \in J \setminus L,\, j \cup L \in \mathcal{K}_J} \varepsilon(j, j \cup L)\chi_{j \cup L}. \tag{2.7}$$

We define a $\mathbf{k}$-linear map

$$\begin{aligned} f\colon C^{p-1}(\mathcal{K}_J) &\longrightarrow R^{p-|J|,2J}(\mathcal{K}) \\ \chi_L &\longmapsto \varepsilon(L,J)u_{J\setminus L}v_L. \end{aligned} \tag{2.8}$$

This is an isomorphism of $\mathbf{k}$-vector spaces, but we will show that with the map $\mathbf{k} \to R^{-|J|,2J}(\mathcal{K})$ given by $1 \mapsto u_J$, we also have an isomorphism of cochain complexes

$$\begin{array}{ccccccccccc} 0 & \longrightarrow & \mathbf{k} & \xrightarrow{\epsilon} & C^0(\mathcal{K}_J) & \xrightarrow{d} & \dots & \xrightarrow{d} & C^{p-1}(\mathcal{K}_J) & \xrightarrow{d} & \dots \\ & & \downarrow\cong & & \downarrow\cong & & & & \downarrow\cong & & \\ 0 & \longrightarrow & R^{-|J|,2J}(\mathcal{K}) & \xrightarrow{d} & R^{1-|J|,2J}(\mathcal{K}) & \xrightarrow{d} & \dots & \xrightarrow{d} & R^{p-|J|,2J}(\mathcal{K}) & \xrightarrow{d} & \dots \end{array} \tag{2.9}$$

where $d\colon R^{p-1-|J|,2J}(\mathcal{K}) \to R^{p-|J|,2J}(\mathcal{K})$ the differential of $R^*(\mathcal{K})$ in (2.6).

First we show that f commutes with the differentials,

$$\begin{aligned} f(d(\chi_L)) &= f\left(\sum_{j \in J \setminus L,\, j \cup L \in \mathcal{K}_J} \varepsilon(j, j \cup L)\chi_{j \cup L}\right) \\ &= \sum_{j \in J \setminus L} \varepsilon(j, j \cup L)\varepsilon(j \cup L, J)u_{J \setminus (j \cup L)}v_{j \cup L}. \end{aligned}$$

We no longer need to include the condition $j \cup L \in \mathcal{K}_J$ in the sum because $v_{j \cup L} = 0$ if $j \cup L \notin \mathcal{K}_J$. Also,

$$\begin{aligned} d(f(\chi_L)) &= d(\varepsilon(L,J)u_{J\setminus L}v_L) \\ &= \varepsilon(L,J) \sum_{j \in J \setminus L} \varepsilon(j, J \setminus L)u_{J \setminus (j \cup L)}v_{j \cup L}. \end{aligned}$$

By definition of ε, $\varepsilon(j \cup L, J) = \varepsilon(j,J) \cdot \varepsilon(L,J)$ and $\varepsilon(j, j \cup L) \cdot \varepsilon(j,J) = \varepsilon(j, J \setminus L)$. Therefore the above calculations show that $fd(\chi_L) = df(\chi_L)$. Hence we have an isomorphism of cochain complexes. Taking cohomology, we have the relation $\widetilde{H}^{p-1}(\mathcal{K}_J) \cong H^{p-|J|,2J}(R^*(\mathcal{K}))$. Let $-i = p - |J|$, then this proves the result. □

Theorem 2.1.21 gives us an isomorphism of modules $\mathrm{Tor}_{\mathbf{k}[m]}(\mathbf{k}[\mathcal{K}], \mathbf{k}) \cong \bigoplus_{J \subset [m]} \widetilde{H}^*(\mathcal{K}_J)$. In Examples 2.1.15 and 2.1.18, the calculations of $\mathrm{Tor}_{\mathbf{k}[m]}(\mathbf{k}[\mathcal{K}], \mathbf{k})$ and $R^*(\mathcal{K})$ were complicated. The advantage of using Hochster's theorem is that the calculations of $\bigoplus_{J\subset[m]\colon |J|=j} \widetilde{H}^{j-i-1}(\mathcal{K}_J)$ are significantly easier.

Example 2.1.22. Let $\mathcal{K}$ be the boundary of a pentagon as in Figure 2.1. First consider the first non-trivial reduced cohomology groups, $\widetilde{H}^0(\mathcal{K}_J)$. By Theorem 2.1.21,

$$\mathrm{Tor}^{-1,4}_{\mathbf{k}[m]}(\mathbf{k}[\mathcal{K}], \mathbf{k}) \cong \bigoplus_{J\subset[m]\colon |J|=2} \widetilde{H}^{2-1-1}(\mathcal{K}_J) = \bigoplus_{J\subset[m]\colon |J|=2} \widetilde{H}^{0}(\mathcal{K}_J).$$

Some subcomplexes $\mathcal{K}_J$ on two vertices are contractible; for example the subcomplex $\mathcal{K}_{\{1,2\}}$ has $\widetilde{H}^0(\mathcal{K}_{\{1,2\}}) = 0$. Therefore these full subcomplexes do not contribute to $\mathrm{Tor}^{-1,4}_{\mathbf{k}[m]}(\mathbf{k}[\mathcal{K}], \mathbf{k})$. This means that we only need to consider the non-contractible full subcomplexes on two vertices. Hence

$$\begin{aligned}\mathrm{Tor}^{-1,4}_{\mathbf{k}[m]}(\mathbf{k}[\mathcal{K}], \mathbf{k}) &\cong \widetilde{H}^0(\mathcal{K}_{\{1,3\}}) \oplus \widetilde{H}^0(\mathcal{K}_{\{1,4\}}) \oplus \widetilde{H}^0(\mathcal{K}_{\{2,3\}}) \oplus \widetilde{H}^0(\mathcal{K}_{\{2,5\}}) \oplus \widetilde{H}^0(\mathcal{K}_{\{4,5\}})\\ &\cong \mathbf{k}^5.\end{aligned}$$

A similar situation occurs with the other cohomology groups. For example,

$$\begin{aligned}\mathrm{Tor}^{-2,6}_{\mathbf{k}[m]}(\mathbf{k}[\mathcal{K}], \mathbf{k}) &\cong \widetilde{H}^0(\mathcal{K}_{\{1,3,4\}})\\ &\oplus \widetilde{H}^0(\mathcal{K}_{\{2,3,5\}}) \oplus \widetilde{H}^0(\mathcal{K}_{\{1,2,3\}}) \oplus \widetilde{H}^0(\mathcal{K}_{\{1,4,5\}}) \oplus \widetilde{H}^0(\mathcal{K}_{\{2,4,5\}}) \cong \mathbf{k}^5\end{aligned}$$

since $\mathcal{K}_J$ is contractible for all other $J \subset [m]$, $|J| = 3$. The only other non-zero reduced cohomology groups are

$$\mathrm{Tor}^{0,0}_{\mathbf{k}[m]}(\mathbf{k}[\mathcal{K}], \mathbf{k}) \cong \widetilde{H}^{-1}(\mathcal{K}_{\varnothing}) \cong \mathbf{k} \quad \text{and} \quad \mathrm{Tor}^{-3,10}_{\mathbf{k}[m]}(\mathbf{k}[\mathcal{K}], \mathbf{k}) \cong \widetilde{H}^{1}(\mathcal{K}_{\{1,2,3,4,5\}}) \cong \mathbf{k}.$$

Therefore by using Theorem 2.1.21, we have the same result as in Example 2.1.18 but with much easier computation.

Putting together Proposition 2.1.20 and Theorem 2.1.21, we have an isomorphism of modules

$$H^*(\mathcal{Z}_{\mathcal{K}}) \cong \mathrm{Tor}_{\mathbf{k}[m]}(\mathbf{k}[\mathcal{K}], \mathbf{k}) \cong \bigoplus_{J\subset[m]} \widetilde{H}^*(\mathcal{K}_J). \tag{2.10}$$

However we want an isomorphism of algebras and to understand the cup product. Therefore we need a product on $\bigoplus_{J\subset[m]} \widetilde{H}^*(\mathcal{K}_J)$.

For two simplices $L = \{l_0, \dots, l_p\}, M = \{m_0, \dots, m_q\} \in \mathcal{K}$, $L \cap M = \varnothing$, let $L \cup M$ denote the simplex on the vertices $\{l_0, \dots, l_p, m_0, \dots, m_q\}$. Then there is a product on

$\bigoplus_{J\subset[m]} \widetilde{H}^*(\mathcal{K}_J)$ given by Baskakov [6],

$$\begin{aligned} C^{p-1}(\mathcal{K}_I)\otimes C^{q-1}(\mathcal{K}_J) &\to C^{p+q-1}(\mathcal{K}_{I\cup J}),\\ \chi_L\otimes\chi_M &\mapsto \begin{cases} c_{L\cup M}\,\chi_{L\cup M} & \text{if } I\cap J=\varnothing,\\ 0 & \text{otherwise}\end{cases}\end{aligned} \tag{2.11}$$

where

$$c_{L\cup M}=\varepsilon(L,I)\,\varepsilon(M,J)\,\zeta\,\varepsilon(L\cup M,I\cup J) \tag{2.12}$$

and

$$\zeta=\prod_{k\in I\setminus L}\varepsilon(k,k\cup J\setminus M). \tag{2.13}$$

For any simplicial complex $\mathcal{K}$, if $I,J\subset[m]$ and $I\cap J=\varnothing$, then $\mathcal{K}_{I\cup J}$ is a subcomplex of $\mathcal{K}_I*\mathcal{K}_J$. Then the product in (2.11) is the restriction to $\mathcal{K}_{I\cup J}$ of the standard product

$$C^{p-1}(\mathcal{K}_I)\otimes C^{q-1}(\mathcal{K}_J)\to C^{p+q-1}(\mathcal{K}_I*\mathcal{K}_J)$$

where the increase in degree from $(p-1)+(q-1)=p+q-2$ to the degree $p+q-1$ comes from the fact that $|\mathcal{K}_I|*|\mathcal{K}_J|\simeq\Sigma(|\mathcal{K}_I|\wedge|\mathcal{K}_J|)$, where Σ denotes suspension, $\wedge$ denotes the smash product and $|\mathcal{K}|$ is the geometric realisation of $\mathcal{K}$.

Proposition 2.1.23 ([8])**.** *There is an isomorphism of algebras* $\mathrm{Tor}_{\mathbf{k}[m]}(\mathbf{k}[\mathcal{K}],\mathbf{k})\cong\bigoplus_{J\subset[m]}\widetilde{H}^*(\mathcal{K}_J)$.

Proof. Theorem 2.1.21 says we already have an isomorphism of modules. In that proof, the map $f\colon C^{p-1}(\mathcal{K}_I)\to R^{p-|I|,2I}(\mathcal{K})$ in (2.8) takes $\chi_L\mapsto\varepsilon(L,I)\,u_{I\setminus L}v_L$. It only remains to show that for any $\chi_L\in C^{p-1}(\mathcal{K}_I)$ and $\chi_M\in C^{q-1}(\mathcal{K}_J)$, $f(\chi_L\chi_M)=f(\chi_L)f(\chi_M)$. If $I\cap J=\varnothing$, then

$$\begin{aligned} f(\chi_L\chi_M) &= f(c_{L\cup M}\,\chi_{L\cup M})\\ &=\varepsilon(L\cup M,I\cup J)c_{L\cup M}\,u_{I\cup J\setminus L\cup M}v_{L\cup M}\\ &=\varepsilon(L\cup M,I\cup J)^2\varepsilon(L,I)\,\varepsilon(M,J)\,\zeta\,u_{I\cup J\setminus L\cup M}v_{L\cup M}\\ &=\varepsilon(L,I)\,\varepsilon(M,J)\,u_{I\setminus L}u_{J\setminus M}v_{L\cup M}\\ &=(\varepsilon(L,I)\,u_{I\setminus L}v_L)\cdot(\varepsilon(M,J)\,u_{J\setminus M}v_M)\\ &=f(\chi_L)f(\chi_M).\end{aligned}$$

If $I\cap J\neq\varnothing$, then $\varepsilon(L,I)\,\varepsilon(M,J)\,\zeta\,u_{I\cup J\setminus L\cup M}v_{L\cup M}=0$ in $R^*(\mathcal{K})$ since $u_i^2=u_iv_i=v_i^2=0$ for any $i\in[m]$. Hence $f(\chi_L\chi_M)=0=f(\chi_L)f(\chi_M)$. □

Therefore by Proposition 2.1.20, the cup product in $H^*(\mathcal{Z}_\mathcal{K})$ corresponds to the product in (2.11). Assembling Propositions 2.1.20, 2.1.23 and Theorem 2.1.21, there is the following theorem.

Theorem 2.1.24 ([8])**.** *There is an isomorphism of cochains* $\widetilde{C}^{*-1}(\mathcal{K}_J)\to C^{*-|J|,2J}(\mathcal{Z}_\mathcal{K})\subset$

$C^{*+|J|}(\mathcal{Z}_{\mathcal{K}})$, *inducing an isomorphism of algebras*

$$H^*(\mathcal{Z}_{\mathcal{K}}) \cong \bigoplus_{J \subset [m]} \widetilde{H}^*(\mathcal{K}_J)$$

where $\widetilde{H}^{-1}(\mathcal{K}_{\varnothing}) = \mathbf{k}$ *and*

$$H^p(\mathcal{Z}_{\mathcal{K}}) \cong \bigoplus_{J \subset [m]} \widetilde{H}^{p-|J|-1}(\mathcal{K}_J).$$

Example 2.1.25. Let $\mathcal{L}_1$ and $\mathcal{L}_2$ be the disjoint pair of points $\{1,2\}$ and $\{3,4\}$ respectively. Let $\mathcal{K} = \mathcal{L}_1 * \mathcal{L}_2$ be the boundary of a square. By Theorem 2.1.24, the non-zero cohomology groups are

$$\begin{aligned} H^0(\mathcal{Z}_{\mathcal{K}}) &\cong \widetilde{H}^{-1}(\mathcal{K}_{\varnothing}) \\ H^3(\mathcal{Z}_{\mathcal{K}}) &\cong \widetilde{H}^0(\mathcal{K}_{\{1,2\}}) \oplus \widetilde{H}^0(\mathcal{K}_{\{3,4\}}) \\ H^6(\mathcal{Z}_{\mathcal{K}}) &\cong \widetilde{H}^1(\mathcal{K}_{\{1,2,3,4\}}). \end{aligned}$$

More specifically, let $\alpha = [\chi_1] \in \widetilde{H}^0(\mathcal{K}_{\{1,2\}})$ and let $\beta = [\chi_3] \in \widetilde{H}^0(\mathcal{K}_{\{3,4\}})$. Then by Proposition 2.1.23, the cup product $\alpha\beta \in H^*(\mathcal{Z}_{\mathcal{K}})$ is represented by the cochain

$$\varepsilon(1,\{1,2\})\,\varepsilon(3,\{3,4\})\,\zeta\,\varepsilon(\{1,3\},\{1,2,3,4\})\chi_{\{1,3\}} = \chi_{\{1,3\}} \in C^1(\mathcal{K}).$$

The cohomology class $\alpha\beta$ is non-zero since it is a generator of $\widetilde{H}^1(\mathcal{K})$. Equivalently by Theorem 2.1.24, this product $\alpha\beta$ could be represented by the product of $[u_2v_1]$ and $[u_4v_3]$ in $H^{-1,4}(R^*(\mathcal{K}))$, given by $[u_{24}v_{13}] \in H^{-2,8}(R^*(\mathcal{K}))$. This is the only non-zero cup product, therefore thus we see that $H^*(\mathcal{Z}_{\mathcal{K}})$ has the same cohomology ring structure as $H^*(S^3 \times S^3)$.

For a cochain $a \in C^p(\mathcal{K}_J)$, let the *support of a* be the set S_a of p-simplices $\sigma \in \mathcal{K}_J$ such that

$$a = \sum_{\sigma \in S_a} a_\sigma \chi_\sigma$$

for a nontrivial coefficient a_σ. For a cohomology class $\alpha \in \widetilde{H}^p(\mathcal{K}_J)$, we say that α is *supported on $\mathcal{K}_J$.*

Lemma 2.1.26. *For a simplicial complex $\mathcal{K}$, let $a \in C^p(\mathcal{K}_I)$ and $b \in C^q(\mathcal{K}_J)$. Let the order of vertices in $\mathcal{K}$ be such that $i < j$ for every $i \in I$ and $j \in J$. Suppose*

$$a = \sum_{\sigma \in S_a} a_\sigma \chi_\sigma \quad \text{and} \quad b = \sum_{\tau \in S_b} b_\tau \chi_\tau$$

for p-simplices $\sigma \in S_a \subset \mathcal{K}_I$, q-simplices $\tau \in S_b \subset \mathcal{K}_J$ and coefficients a_σ, b_τ. Then the *product* $ab \in C^{p+q+1}(\mathcal{K}_{I\cup J})$ *is given by*

$$ab = (-1)^{|I|(q+1)} \sum_{\sigma\in S_a} \sum_{\tau\in S_b} a_\sigma b_\tau \chi_{\sigma\cup\tau}.$$

Proof. By (2.12), the product ab is given by

$$\begin{aligned} ab &= \left(\sum_{\sigma\in S_a} a_\sigma \chi_\sigma\right)\left(\sum_{\tau\in S_b} b_\tau \chi_\tau\right) \\ &= \sum_{\sigma\in S_a} \sum_{\tau\in S_b} a_\sigma\, b_\tau\, \varepsilon(\sigma, I)\, \varepsilon(\tau, J)\, \zeta\, \varepsilon(\sigma\cup\tau, I\cup J)\, \chi_{\sigma\cup\tau} \end{aligned}$$

where $\zeta = 1$ since all vertices of I are ordered before vertices of J in $\mathcal{K}$.

By the definition of ε, and since all elements I are ordered before J, $\varepsilon(\sigma\cup\tau, I\cup J) = \varepsilon(\sigma, I)\varepsilon(\tau, I\cup J)$. Furthermore, for each q-simplex $\tau = \{i_1, \ldots, i_{q+1}\} \subset J$,

$$\begin{aligned} \varepsilon(\tau, I\cup J) = \prod_{j\in\{1,\ldots,q+1\}} \varepsilon(i_j, I\cup J) &= \prod_{j\in\{1,\ldots,q+1\}} (-1)^{|I|}\varepsilon(i_j, J) \\ &= (-1)^{|I|(q+1)} \prod_{j\in\{1,\ldots,q+1\}} \varepsilon(i_j, J) \\ &= (-1)^{|I|(q+1)}\varepsilon(\tau, J). \end{aligned}$$

Therefore, since $\varepsilon(I, J)^2 = 1$ for any sets I, J,

$$ab = \sum_{\sigma\in S_a} \sum_{\tau\in S_b} a_\sigma b_\tau (-1)^{|I|(q+1)} \chi_{\sigma\cup\tau}.$$

□

Example 2.1.27. Let $\mathcal{K}$ be the boundary of a pentagon as in Figure 2.1. Consider the non-zero cohomology classes $\alpha \in \widetilde{H}^0(\mathcal{K}_{1,2,3})$ and $\beta \in \widetilde{H}^0(\mathcal{K}_{4,5})$, which are represented by the cocycles

$$a = \chi_1 + \chi_2 \in C^0(\mathcal{K}_{1,2,3}) \quad \text{and} \quad b = \chi_5 \in C^0(\mathcal{K}_{4,5})$$

respectively. Then the product $ab \in C^1(\mathcal{K}_{1,2,3,4,5})$ is given by

$$ab = (-1)^{|\{1,2,3\}|(0+1)} (\chi_{1,5} + \chi_{2,5}) = -\chi_{1,5}$$

since the cochain $\chi_{2,5}$ is zero because there is no edge $\{2, 5\}$ in $\mathcal{K}$.

It is important for Lemma 2.1.26 that the vertices I come before the vertices J. For example, suppose the vertices 3 and 4 were labelled the other way around. Then $a = \chi_1 + \chi_2 \in C^0(\mathcal{K}_{1,2,4})$ and $b = \chi_5 \in C^0(\mathcal{K}_{3,5})$, and for each $i \in \{1, 2\}$,

$$\zeta = \varepsilon(i, \{i, 3\})\, \varepsilon(4, \{3, 4\}) = -1.$$

So by (2.12), the product $ab \in C^1(\mathcal{K}_{1,2,3,4,5})$ is given by

$$\begin{aligned}ab =&\varepsilon(1,\{1,2,4\})\,\varepsilon(5,\{3,5\})\,\zeta\,\varepsilon(\{1,5\},\{1,2,3,4,5\})\,\chi_{1,5}\\ &\qquad\qquad + \varepsilon(2,\{1,2,4\})\,\varepsilon(5,\{3,5\})\,\zeta\,\varepsilon(\{2,5\},\{1,2,3,4,5\})\,\chi_{2,5}\\ =&1\cdot(-1)\cdot(-1)\cdot 1\,\chi_{1,5} = \chi_{1,5}.\end{aligned}$$

2.2 Massey Products

In knot theory, the linking number of a knot or link is known to correspond to the cup product in cohomology. However, the linking number cannot distinguish all knots/links. For example, the complement of Borromean rings (in S^3) and the complement of three disjoint rings both have the same cohomology ring, so the cup product cannot distinguish them. When Massey [38] first introduced his triple products in 1957, he demonstrated that any triple Massey product of three disjoint rings was trivial but that there was a non-trivial triple product for the Borromean rings. Hence triple Massey products are a helpful invariant.

Triple Massey products are secondary operations. In 1958 they were generalised to higher Massey products by Kraines [29] in such a way that cup products are two-Massey products (up to sign). In this sense, Massey products can be thought of as generalisations of standard cup products. In general, higher Massey products are very difficult to compute, being higher cohomology operations.

Historically, Massey products have long been studied in algebra. For a graded $\mathbf{k}$-vector space $V = \bigoplus_i V^i$ with finite-dimensional graded components, the Poincaré series is $P_V(t) = \sum_{i\geqslant 0} \dim_k V^i t^i$. In 1962, Golod studied the Poincaré series of $\mathrm{Tor}_R(\mathbf{k},\mathbf{k})$ where R is a Noetherian local ring. He gave an expression [21] for the Poincaré series for any R where all Massey products in the Koszul complex are trivial, and such rings were termed *Golod*. In Sections 2.1.2, we saw that the cohomology of the Koszul complex $(\Lambda[m]\otimes\mathbf{k}[\mathcal{K}])$ is isomorphic to $\bigoplus_{J\subset[m]} \widetilde{H}^*(\mathcal{K}_J)$. This lead to the study of *Golod* simplicial complexes (such as in [22]), whose face ring $\mathbf{k}[\mathcal{K}]$ is Golod, that is, all Massey products in $\mathrm{Tor}_{\mathbf{k}[m]}(\mathbf{k}[\mathcal{K}],\mathbf{k}) = H(\Lambda[m]\otimes\mathbf{k}[\mathcal{K}])$ are trivial.

Kähler manifolds are complex manifolds with a Hermitian metric whose associated differential two-form is closed. There are many expository texts on Kähler manifolds, such as [5]. It turned out that many symplectic manifolds have a Kähler structure and it was not until Thurston in 1976 [37] that an example of a non-Kähler compact symplectic manifold was found. Now there are more known non-Kähler manifolds, but in general it is still hard to detect them; see [3] for an introduction. Deligne, Griffiths, Morgan, and Sullivan [16] showed that all Kähler manifolds are formal. They also showed that Massey products are also an obstruction to formality. That is, if there is a non-trivial Massey product, then the space is not formal [16]. Therefore, the existence of non-trivial Massey products in a manifold gives an explicit example of a non-Kähler manifold.

2.2.1 Introduction

We first define Massey products before calculating them via combinatorics in Section 2.2.2. Since Massey introduced triple Massey products, there have been a few alternative definitions of Massey products that differ at most by sign. The definition given here follows [29] and [31] which, in particular, differs by a sign to the definition of the triple Massey product given in [11].

Definition 2.2.1. Let (A, d) be a differential graded algebra with classes α_i in $H^{p_i}(A, d)$ for $1 \leqslant i \leqslant n$. Let $a_{i,i} \in A^{p_i}$ be a representative for α_i. A *defining system* associated to $\langle \alpha_1, \cdots, \alpha_n \rangle$ is a set of elements $(a_{i,k})$ for $1 \leqslant i \leqslant k \leqslant n$ and $(i, k) \neq (1, n)$ such that $a_{i,k} \in A^{p_i+\cdots+p_k-k+i}$ and

$$d(a_{i,k}) = \sum_{r=i}^{k-1} \overline{a_{i,r}} a_{r+1,k}$$

where $\overline{a_{i,r}} = (-1)^{1+\deg a_{i,r}} a_{i,r}$.

To each defining system of $\langle \alpha_1, \cdots, \alpha_n \rangle$, the *associated cocycle* is

$$\sum_{r=1}^{n-1} \overline{a_{1,r}} a_{r+1,n} \in A^{p_1+\cdots+p_n-n+2}$$

and indeed, this can easily be directly shown to be a cocycle.

The *n-Massey product* $\langle \alpha_1, \cdots, \alpha_n \rangle$ is the set of cohomology classes of associated cocycles for all possible defining systems.

For example for classes $\alpha_i = [a_i] \in H^{p_i}(A, d)$, the triple Massey product $\langle \alpha_1, \alpha_2, \alpha_3 \rangle \in H^{p_1+p_2+p_3-1}[A]$ is defined when $\alpha_1\alpha_2 = 0$ and $\alpha_2\alpha_3 = 0$. Since these cup products are zero, there exist choices of cochains $a_{12} \in A^{p_1+p_2-1}$ and $a_{23} \in A^{p_2+p_3-1}$ such that $d(a_{12}) = (-1)^{1+p_1} a_1 a_2$ and $d(a_{23}) = (-1)^{1+p_2} a_2 a_3$. Then the *triple Massey product* $\langle \alpha_1, \alpha_2, \alpha_3 \rangle \in H^{p_1+p_2+p_3-1}[A]$ is the set of classes represented by cochains

$$(-1)^{p_1+1} a_1 a_{23} + (-1)^{p_1+p_2} a_{12} a_3 \in A^{p_1+p_2+p_3-1}.$$

This cochain is a cocycle since

$$\begin{aligned} &d((-1)^{p_1+1} a_1 a_{23} + (-1)^{p_1+p_2} a_{12} a_3) \\ &= (-1)^{p_1+1} (da_1 \cdot a_{23} + (-1)^{p_1} a_1 \cdot da_{23}) + (-1)^{p_1+p_2} \left(da_{12} \cdot a_3 + (-1)^{p_1+p_2-1} a_{12} \cdot da_3\right) \\ &= (-1) a_1((-1)^{1+p_2} a_2 a_3) + (-1)^{p_1+p_2} (-1)^{1+p_1} a_1 a_2 a_3 \\ &= (-1)^{p_2} a_1 a_2 a_3 + (-1)^{p_2+1} a_1 a_2 a_3 = 0. \end{aligned}$$

Kraines [25, Theorem 3] showed that for each α_i, the choice of representing cocycle a_i does not change the Massey product.

Theorem 2.2.2 ([25])**.** *A Massey product $\langle \alpha_1, \ldots, \alpha_n \rangle$ depends only on $\alpha_1, \ldots, \alpha_n$.*

That is, for any $i \in \{1, \dots, n\}$ and any $b \in A^{p_i - 1}$,

$$\langle [a_1], \dots, [a_i + d(b)], \dots, [a_n] \rangle = \langle [a_1], \dots, [a_i], \dots, [a_n] \rangle.$$

However the choice of the cochains $a_{i,k}$ is very important for Massey products.

Definition 2.2.3. An n-Massey product is called *trivial* if it contains 0.

Therefore a Massey product is trivial if there exists a defining system $(a_{i,k})$ such that $[\omega] = 0$, where ω is the associated cocycle.

Definition 2.2.4. The *indeterminacy* of a n-Massey product is the set of differences between elements in $\langle \alpha_1, \cdots, \alpha_n \rangle$.

Lemma 2.2.5. *In a triple Massey product $\langle \alpha_1, \alpha_2, \alpha_3 \rangle$, any two elements differ by an element in*

$$\alpha_1 \cdot H^{p_2+p_3-1}[A] + \alpha_3 \cdot H^{p_1+p_2-1}[A] \subset H^{p_1+p_2+p_3-1}[A].$$

Proof. Let $[\omega], [\omega'] \in \langle \alpha_1, \alpha_2, \alpha_3 \rangle$, $[\omega] \neq [\omega']$ and suppose they are represented by cocycles $(-1)^{p_1+1} a_1 a_{23} + (-1)^{p_1+p_2} a_{12} a_3$ and $(-1)^{p_1+1} a_1 a'_{23} + (-1)^{p_1+p_2} a'_{12} a_3$ respectively. Define cocycles $x = a_{12} - a'_{12} \in A^{p_1+p_2-1}$ and $y = a_{23} - a'_{23} \in A^{p_2+p_3-1}$. The cohomology class $[\omega] - [\omega']$ is represented by the cocycle $(-1)^{p_1+1} a_1 y + (-1)^{p_1+p_2} x a_3$. Then cup products on the level of cochains induce cup products in cohomology, and hence $[\omega] - [\omega'] \in \alpha_1 \cdot H^{p_2+p_3-1}[A] + \alpha_3 \cdot H^{p_1+p_2-1}[A]$. □

Therefore for the triple Massey product, the *indeterminacy* is the set in Lemma 2.2.5. If a cohomology class $[\omega] \in \langle \alpha_1, \alpha_2, \alpha_3 \rangle$ is contained in the indeterminacy, then $[\omega]$ differs from 0 only by an element in the indeterminacy. Thus there is another choice of cochains a_{12}, a_{23} such that the associated cocycle is zero. Hence $0 \in \langle \alpha_1, \alpha_2, \alpha_3 \rangle$. In general, we can say a triple Massey product is trivial if and only if there is a choice of cochains a_{12}, a_{23} such that $[\omega] = [(-1)^{p_1+1} a_1 a_{23} + (-1)^{p_1+p_2} a_{12} a_3]$ and $[\omega] \in \alpha \cdot H^{p_2+p_3-1}[A] + \gamma \cdot H^{p_1+p_2-1}[A]$. There is no equivalent "nice" expression for the indeterminacy of higher Massey products. For example, *matric* Massey products are a generalisation of Massey products; they are studied thoroughly by May [30]. In [30, Proposition 2.3], May shows that elements in the indeterminacy of an n-fold matric Massey product are elements in a certain $(n-1)$-fold matric Massey product. In practice, this is a difficult expression to work with for calculations. This makes detecting non-trivial higher Massey products harder than non-trivial triple Massey products.

Importantly, Massey products are homotopy invariants [25, Property 2.1].

Theorem 2.2.6 ([25])**.** *Let A, B be differential graded algebras with a map $f \colon A \to B$. Let f^* be the induced homomorphism $f^* \colon H^*(A) \to H^*(B)$. Then for $\langle \alpha_1, \dots, \alpha_n \rangle \subset H^*(A)$,*

$$f^* \langle \alpha_1, \dots, \alpha_n \rangle \subset \langle f^* \alpha_1, \dots, f^* \alpha_n \rangle.$$

If f^* is an isomorphism, then $f^*\langle\alpha_1,\dots,\alpha_n\rangle = \langle f^*\alpha_1,\dots,f^*\alpha_n\rangle$. □

2.2.2 Massey products via combinatorics

Calculating Massey products in the cohomology of moment-angle complexes can be done through detecting Massey products in the corresponding Tor-algebra. By Theorem 2.1.24, we know there are isomorphisms of algebras

$$H^*(\mathcal{Z}_\mathcal{K}) \cong H^*(R^*(\mathcal{K})) \cong \bigoplus_{J\subset[m]} \widetilde{H}^*(\mathcal{K}_J).$$

Therefore by Theorem 2.2.6, a Massey product in $H^*(\mathcal{Z}_\mathcal{K})$ can also be expressed in terms of $\bigoplus_{J\subset[m]} \widetilde{H}^*(\mathcal{K}_J)$. Recall that for $a \in C^*(\mathcal{Z}_\mathcal{K})$, $\overline{a} = (-1)^{1+\deg a}a$.

Definition 2.2.7. For any element $\alpha \in \widetilde{H}^p(\mathcal{K}_J) \subset \bigoplus_{J\subset[m]} \widetilde{H}^*(\mathcal{K}_J)$, let $\overline{\deg}(\alpha) = p+|J|+1$. Also for $a \in C^p(\mathcal{K}_J)$, let $\overline{\overline{a}} = (-1)^{1+\overline{\deg}\, a}a = (-1)^{2+p+|J|}a$.

The map $f\colon C^p(\mathcal{K}_J) \to R^{p+1-|J|,2|J|}(\mathcal{K})$ given in (2.8) is a $\mathbf{k}$-linear isomorphism. Therefore, for $a \in C^p(\mathcal{K}_J)$, the degree of $f(a) \in R^{p+1-|J|,2|J|}(\mathcal{K})$ is $p+|J|+1$ and

$$\begin{aligned}\overline{f(a)} &= (-1)^{1+\deg f(a)} f(a) = f((-1)^{1+\deg f(a)}a)\\ &= f((-1)^{2+p+|J|)}a) = f(\overline{\overline{a}}).\end{aligned}$$

Let $\langle\alpha_1,\dots,\alpha_n\rangle \subset H^*(\mathcal{Z}_\mathcal{K})$ where each class $\alpha_i \in H^{p_i+|J_i|+1}(\mathcal{Z}_\mathcal{K})$ corresponds to $\alpha_i \in H^{p_i}(\mathcal{K}_{J_i})$. Let $a_i \in C^{p_i}(\mathcal{K}_{J_i})$ be a cocycle representative for α_i. Then using the product given in (2.11), the cochains $a_{i,k}$ such that $d(a_{i,k}) = \sum_{r=i}^{k-1} \overline{\overline{a_{i,r}}}a_{r+1,k}$ are elements $a_{i,k} \in C^{p_i+\cdots+p_k}(\mathcal{K}_{J_i\cup\cdots\cup J_k})$. Using the degree from Definition 2.2.7,

$$\begin{aligned}\overline{\deg}(a_{i,k}) &= p_i + \cdots + p_k + |J_i \cup \cdots \cup J_k| + 1\\ &= (p_i + |J_i| + 1) + \cdots + (p_k + |J_k| + 1) - k + i\\ &= \overline{\deg}(a_i) + \cdots + \overline{\deg}(a_k) - k + i\end{aligned}$$

which matches the definition of a defining system in Definition 2.2.1. Hence for a defining system $(a_{i,k}) \subset C^*(\mathcal{Z}_\mathcal{K})$, we have a corresponding defining system $(a_{i,k}) \subset \bigoplus_{J\subset[m]} C^*(\mathcal{K}_J)$. Furthermore, the associated cocycle $\omega \in C^{p_1+\cdots+p_n+|J_1\cup\cdots\cup J_n|+2}(\mathcal{Z}_\mathcal{K})$ corresponds to the associated cocycle $\omega \in C^{p_1+\cdots+p_n+1}(\mathcal{K}_{J_1\cup\cdots\cup J_n})$.

Let $\langle\alpha_1,\alpha_2,\alpha_3\rangle$ be a triple Massey product on $\alpha_i \in H^{p_i}(\mathcal{Z}_\mathcal{K})$ for $i = 1,2,3$. By Lemma 2.2.5, the indeterminacy of a triple Massey product is given by $\alpha_1 \cdot H^{p_2+p_3}(\mathcal{Z}_\mathcal{K}) + \alpha_3 \cdot H^{p_1+p_2}(\mathcal{Z}_\mathcal{K})$. Then Theorem 2.1.24 implies that each class α_i corresponds to a class $\alpha_i \in \widetilde{H}^{p_i-|J_i|-1}(\mathcal{K}_{J_i})$ and the *indeterminacy* of $\langle\alpha_1,\alpha_2,\alpha_3\rangle$ from Lemma 2.2.5 becomes

$$\alpha_1 \cdot \widetilde{H}^{p_2+p_3-|J_2|-|J_3|-2}(\mathcal{K}_{J_2\cup J_3}) + \alpha_3 \cdot \widetilde{H}^{p_1+p_2-|J_1|-|J_2|-2}(\mathcal{K}_{J_1\cup J_2}). \tag{2.14}$$

Example 2.2.8. Consider a simplicial complex $\mathcal{K}$ as shown in Figure 2.2. Let $\alpha_1, \alpha_2, \alpha_3 \in H^{-1,4}(\mathcal{Z}_\mathcal{K})$ be represented by cocycles $a_1 = -u_1v_2$, $a_2 = -u_3v_4$ and $a_3 = -u_5v_6$ respectively. Then $\alpha_1\alpha_2, \alpha_2\alpha_3 \in H^{-2,8}(\mathcal{Z}_\mathcal{K})$ but the cochain products a_1a_2 and a_2a_3 are not zero and it is potentially hard to show that $\alpha_1\alpha_2 = 0 = \alpha_2\alpha_3$ in $R^*(\mathcal{K})$. Alternatively, by the map in (2.8), a_1 corresponds to $\chi_2 \in C^0(\mathcal{K}_{12})$. Similarly, we let $a_2 = \chi_4 \in C^0(\mathcal{K}_{34})$ and $a_3 = \chi_6 \in C^0(\mathcal{K}_{56})$. So $\alpha_1 \in \widetilde{H}^0(\mathcal{K}_{12})$, $\alpha_2 \in \widetilde{H}^0(\mathcal{K}_{34})$, $\alpha_3 \in \widetilde{H}^0(\mathcal{K}_{56})$. Then $\alpha_1\alpha_2 \in \widetilde{H}^1(\mathcal{K}_{1234})$ and $\alpha_2\alpha_3 \in \widetilde{H}^1(\mathcal{K}_{3456})$ by the product induced by (2.11). Since the full subcomplexes $\mathcal{K}_{1234}$ and $\mathcal{K}_{3456}$ are paths in this simplicial complex,

$$\widetilde{H}^p(\mathcal{K}_{1234}) = \widetilde{H}^p(\mathcal{K}_{3456}) = 0 \text{ for all } p \in \mathbb{Z}. \tag{2.15}$$

Therefore the cup products $\alpha_1\alpha_2$ and $\alpha_2\alpha_3$ are zero and the triple Massey product $\langle\alpha_1, \alpha_2, \alpha_3\rangle$ is defined for this simplicial complex.

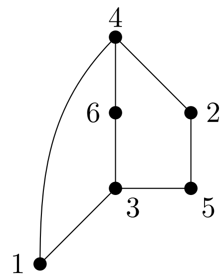


Figure 2.2: A simplicial complex for which $\mathcal{Z}_\mathcal{K}$ has a non-trivial 3-Massey product.

Let $a_{1,2} \in C^0(\mathcal{K}_{1234})$ be the cochain $-\chi_2$. Then $d(a_{1,2}) = \chi_{24} = \overline{\overline{a_1}}a_2$. Similarly let $a_{2,3} \in C^0(\mathcal{K}_{3456})$ be the cochain $-\chi_4$. So $d(a_{2,3}) = \chi_{46} = \overline{\overline{a_2}}a_3$. Then we have a defining system $(a_{i,k})$, $1 \leqslant i \leqslant k \leqslant 3$, $(i,k) \neq (1,3)$. Since $1 + \overline{\deg}\, a_{i,k}$ is even for all (i,k), the associated cocycle ω is given by $a_1a_{2,3} + a_{1,2}a_3 = -\chi_2\chi_4 - \chi_2\chi_6 = -\chi_2\chi_4 = -\chi_{24}$. The cohomology class $[\omega]$ is a generator of $H^1(\mathcal{K})$, and so $[\omega] \neq 0$.

By (2.14), the indeterminacy of this triple Massey product is given by $\alpha_1 \cdot \widetilde{H}^0(\mathcal{K}_{3456}) + \alpha_3 \cdot \widetilde{H}^0(\mathcal{K}_{1234})$. By (2.15), this is zero, and therefore $\langle\alpha_1, \alpha_2, \alpha_3\rangle$ contains only a single element. In this case, that element is $[\omega]$, and hence $\langle\alpha_1, \alpha_2, \alpha_3\rangle$ is non-trivial.

2.3 Massey Products in Toric Topology

The first family of non-trivial Massey products in the cohomology of moment-angle complexes were found by Baskakov [7] in 2003. Like these first examples, most of the existing results about non-trivial Massey products in moment-angle complexes are for the case of triple Massey products only. Other results are about n-Massey products for $n > 3$ but where the classes are in the lowest possible degree. This section summarises these results.

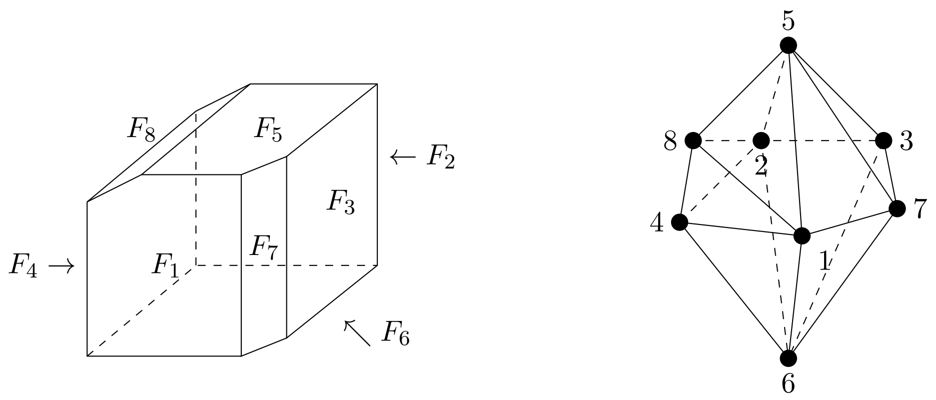
2.3.1 The first Massey products in moment-angle complexes

Before summarising some examples of Massey products in moment-angle complexes, we define truncation in polytopes.

Definition 2.3.1. Let $P \subset \mathbb{R}^n$ be a polytope and let F be a face of P. Let $H \subset \mathbb{R}^n$ be a hyperplane that does not include any vertex of P and suppose H separates all vertices of F from the rest of the vertices in P. Let H_1, H_2 be the half spaces defined by H and suppose H_2 is the half space such that $F \subset P \cap H_2$. We say that the polytope $\widetilde{P} = P \cap H_1$ is obtained from P by a *hyperplane cut* or *face truncation*.

Example 2.3.2. The 2-truncated cube in Figure 2.3a is obtained from the 3-dimensional cube by two edge truncations.

Baskakov's [7] infinite family of non-trivial triple Massey products were motivated by the discovery of a non-trivial Massey product in the cohomology of a moment-angle manifold $\mathcal{Z}_P$ where P is a 2-truncated cube.



(a) A cube truncated at two edges

(b) The one-skeleton of $\mathcal{K}_P$ when P is a 2-truncated cube.

Figure 2.3

Example 2.3.3. Let P be a cube, a simple 3-polytope, truncated at two edges as in Figure 2.3a. Then $\mathcal{K}_P = \partial P^*$ is a 2-dimensional simplicial complex with vertices $1, \ldots, 8$. We label the facets of P and the vertices of $\mathcal{K}_P$ so that pairwise adjacent facets $F_{i_1}, \ldots, F_{i_k}$ in P correspond to simplices $\{i_1, \ldots, i_k\}$ in $\mathcal{K}_P$. Then $\mathcal{K}_P$ is as in Figure 2.3b and the Stanley-Reisner ideal of $\mathcal{K}_P$ is

$$\mathcal{I}_{\mathcal{K}_P} = (v_1v_2, v_3v_4, v_5v_6, v_1v_3, v_2v_7, v_4v_7, v_4v_5, v_3v_8, v_6v_8, v_7v_8).$$

Let $\mathcal{K}$ be the full subcomplex of $\mathcal{K}_P$ on the vertices $\{1, \ldots, 6\}$, corresponding to the original six facets of the 3-cube. By Proposition 2.1.9, $H^{-1,4}(\mathcal{Z}_{\mathcal{K}}) \subset H^{-1,4}(\mathcal{Z}_{\mathcal{K}_P})$. Let $\alpha_1, \alpha_2, \alpha_3 \in H^{-1,4}(\mathcal{Z}_{\mathcal{K}})$ be represented by cocycles $a_1 = \chi_1 \in C^0(\mathcal{K}_{12})$, $a_2 = \chi_4 \in C^0(\mathcal{K}_{34})$ and $a_3 = \chi_6 \in C^0(\mathcal{K}_{56})$. Then $a_1a_2 = 0$ since $\{1, 4\} \notin \mathcal{K}$. Also $a_2a_3 = \chi_{46}$. So one set of choices for a_{12} and a_{23} is 0 and χ_6, respectively. In this way, the triple Massey product $\langle \alpha, \beta, \gamma \rangle$ contains the cohomology class $[\omega]$ represented by the cocycle

$$\omega = (-1)^{3+1} a_1 a_{23} + (-1)^{4+2} a_{12} a_3 = a_1 a_{23} = \chi_1 \chi_6 = \chi_{16}$$

which is not a coboundary in $C^1(\mathcal{K})$. Therefore ω is non-zero.

Furthermore, by (2.14) the indeterminacy is given by $\alpha_1 \cdot \widetilde{H}^0(\mathcal{K}_{3456}) + \alpha_3 \cdot \widetilde{H}^0(\mathcal{K}_{1234})$. Since both $\mathcal{K}_{1234}$ and $\mathcal{K}_{3456}$ are contractible, the indeterminacy is zero. Hence $\langle \alpha, \beta, \gamma \rangle \subset H^*(\mathcal{Z}_{\mathcal{K}_P})$ is non-trivial.

In this example, a non-trivial triple Massey product was constructed by truncating two edges in a cube, which is the product of three unit intervals. We wish to generalise this construction in order to create non-trivial higher Massey products in moment-angle complexes. By Lemma 2.1.5, the product of simple polytopes corresponds to the join of the relevant simplicial complexes. Additionally, truncating faces in a polytope P corresponds to performing *stellar subdivision* on a simplicial complex $\mathcal{K}_P$.

For a simplicial complex $\mathcal{K}$, the *star* and *link* of a simplex $I \in \mathcal{K}$ are

$$\operatorname{st}_{\mathcal{K}} I = \{J \in \mathcal{K} \colon I \cup J \in \mathcal{K}\}$$

$$\operatorname{lk}_{\mathcal{K}} I = \{J \in \mathcal{K} \colon I \cup J \in \mathcal{K}, I \cap J = \varnothing\}.$$

Furthermore, the *boundary of the star* of $I \in \mathcal{K}$ is

$$\partial \operatorname{st}_{\mathcal{K}} I = \{J \in \mathcal{K} \colon I \cup J \in \mathcal{K}, I \not\subset J\}.$$

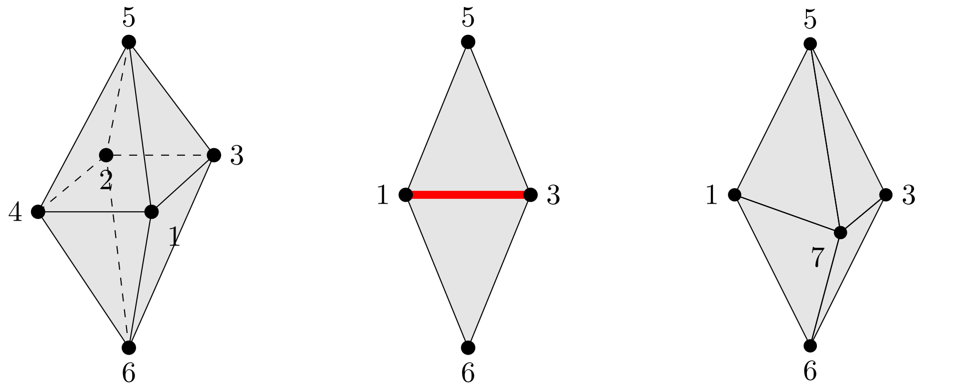
Definition 2.3.4. The *stellar subdivision* of $\mathcal{K}$ at I is

$$\operatorname{ss}_I \mathcal{K} = (\mathcal{K} \setminus \operatorname{st}_{\mathcal{K}} I) \cup (\operatorname{cone} \partial \operatorname{st}_{\mathcal{K}} I).$$

Stellar subdivision does not change homotopy type. If $\mathcal{K}$ is a triangulation of an n-sphere on m vertices, then $\operatorname{ss}_I \mathcal{K}$ is also a triangulation of S^n but on $m+1$ vertices. So both $\mathcal{Z}_{\mathcal{K}}$ and $\mathcal{Z}_{\operatorname{ss}_I \mathcal{K}}$ are manifolds, $(m+n+1)$ and $(m+n+2)$-dimensional manifolds, respectively. Therefore, if $\mathcal{K}$ is a triangulation of a sphere and we find a non-trivial Massey product in $H^*(\mathcal{Z}_{\operatorname{ss}_I \mathcal{K}})$, then $\mathcal{Z}_{\operatorname{ss}_I \mathcal{K}}$ is an example of a non-formal/non-Kähler manifold.

Example 2.3.5. Let $\mathcal{K}$ be the boundary of an octahedron as in Figure 2.4a. Consider the simplex $I = \{1,3\} \in \mathcal{K}$. The star $\operatorname{st}_{\mathcal{K}} I$ has maximal faces $\{1,3,5\}$ and $\{1,3,6\}$, see Figure 2.4b. Then $\partial \operatorname{st}_{\mathcal{K}} I = \{\{1,5\},\{1,6\},\{3,5\},\{3,6\}\}$. Thus $\operatorname{cone} \partial \operatorname{st}_{\mathcal{K}} I$ has four maximal 2-simplices on the vertex set $\{1,3,5,6,7\}$, where $\{7\}$ is the added cone-vertex (see Figure 2.4c). Then $\operatorname{ss}_I \mathcal{K}$ is a simplicial complex on seven vertices.

The simplicial complex $\mathcal{K}_P$ in Figure 2.3b is obtained from the octahedron in Figure 2.4a by stellar subdividing at the edges $\{1,3\}$ and $\{4,5\}$. These stellar subdivisions correspond to truncating the intersection of the facets F_1 and F_3 and the intersection of F_4 and F_5



(a) $\mathcal{K}$ is on 6 vertices **(b)** Star of the edge $\{1,3\}$ **(c)** Cone on $\partial\, \mathrm{st}_{\mathcal{K}}\{1,3\}$

Figure 2.4: Components of stellar subdivision on the boundary of an octahedron

in a 3-cube to obtain Figure 2.3a. Following Example 2.3.3, Baskakov generalised the construction of this cube for other moment-angle manifolds using stellar subdivision.

Construction 2.3.6 (Baskakov)**.** Let $\mathcal{K}_i$ be a triangulation of an (n_i-1)-dimensional sphere S^{n_i-1} on a set of m_i vertices, V_i, for $i=1,2,3$. Let $\mathcal{K}=\mathcal{K}_1 * \mathcal{K}_2 * \mathcal{K}_3$ be the join of these three spheres, so that $\mathcal{K}$ is a triangulation of an $(n-1)$-dimensional sphere on m vertices, for $n=n_1+n_2+n_3$, $m=m_1+m_2+m_3$. Choose maximal simplices $I_1 \in \mathcal{K}_1$, $I_2, I_2' \in \mathcal{K}_2$ such that $I_2 \cap I_2' = \varnothing$, and $I_3 \in \mathcal{K}_3$. Let $\widetilde{\mathcal{K}}$ be obtained from $\mathcal{K}$ by stellar subdivisions,

$$\widetilde{\mathcal{K}} = \mathrm{ss}_{I_1\cup I_2}\left(\mathrm{ss}_{I_2'\cup I_3}\mathcal{K}\right).$$

Therefore $\widetilde{\mathcal{K}}$ is a triangulation of a $(n-1)$-dimensional sphere on $m+2$ vertices.

Theorem 2.3.7 ([7])**.** *Let $\alpha_i \in H^{n_1-m_1,2m_1}(\mathcal{Z}_{\widetilde{\mathcal{K}}})$ correspond to $\alpha_i \in \widetilde{H}^{n_i-1}(\widetilde{\mathcal{K}}_{V_i})$ for $i=1,2,3$. Then $\langle \alpha_1,\alpha_2,\alpha_3\rangle \subset H^{n_1+m_1+n_2+m_2+n_3+m_3-1}(\mathcal{Z}_{\widetilde{\mathcal{K}}})$ is defined and non-trivial.*

Proofs of Theorem 2.3.7 can be found in [7] or [11], or later in Example 3.1.16. In particular, by the expression in (2.14), the indeterminacy in this case is given by

$$\alpha_1 \cdot \widetilde{H}^{n_2+n_3-m_2-m_3-4}(\widetilde{\mathcal{K}}_{V_2\cup V_3}) + \alpha_3 \cdot \widetilde{H}^{n_1+n_2-m_1-m_2-4}(\widetilde{\mathcal{K}}_{V_1\cup V_2}).$$

In the construction of $\widetilde{\mathcal{K}}$, we have that $\left(\mathrm{ss}_{I_2'\cup I_3}\mathcal{K}_2 * \mathcal{K}_3\right) \cap \left(\overset{\circ}{\mathrm{st}}_{\mathcal{K}} I_1 \cup I_2\right) = \varnothing$. Therefore $\mathrm{ss}_{I_2'\cup I_3}\mathcal{K}_2 * \mathcal{K}_3 \subset \widetilde{\mathcal{K}}$ is a full subcomplex of $\widetilde{\mathcal{K}}$. Hence $\widetilde{\mathcal{K}}_{V_2\cup V_3} = \left(\mathrm{ss}_{I_2'\cup I_3}\mathcal{K}_2 * \mathcal{K}_3\right)_{V_2\cup V_3}$. Furthermore, since the star of a simplex is connected and $\mathcal{K}_2 * \mathcal{K}_3$ is homotopy equivalent to a (n_2+n_3-1)-sphere, $\left(\mathrm{ss}_{I_2'\cup I_3}\mathcal{K}_2 * \mathcal{K}_3\right)_{V_2\cup V_3}$ has the homotopy type of a (n_2+n_3-1)-ball. Therefore $\widetilde{H}^*(\widetilde{\mathcal{K}}_{V_2\cup V_3}) = 0$. The same arguments are true for $\widetilde{\mathcal{K}}_{V_1\cup V_2}$, so similarly $\widetilde{H}^*(\widetilde{\mathcal{K}}_{V_1\cup V_2}) = 0$. Hence the indeterminacy for $\langle \alpha_1,\alpha_2,\alpha_3\rangle$ is always trivial. Thus to prove that $\langle \alpha_1,\alpha_2,\alpha_3\rangle$ is non-trivial, it is sufficient to find only one non-zero class in $\langle \alpha_1,\alpha_2,\alpha_3\rangle$.

In general, showing that a Massey product is non-trivial is significantly easier when the indeterminacy is trivial. Trivial indeterminacy implies that there is only one element in the Massey product, thus it is sufficient to only check if that element is zero or non-zero. In many cases, we do not know if the indeterminacy is trivial or not, and thus calculating non-trivial Massey products is more difficult than in this example.

2.3.2 A classification of triple Massey products in lowest degree

Baskakov's family of examples proved that moment-angle complexes can have non-trivial Massey products in their cohomology. Therefore moment-angle complexes have more "complexity" than was previously known. The examples also set out a new interpretation of Massey products in combinatorial terms. We give a combinatorial classification of triple Massey products of three dimensional classes in $H^*(\mathcal{Z}_\mathcal{K})$. This classification is largely based on a result by Denham and Suciu, but improves on [17, Theorem 6.1.1] by considering Massey products with non-trivial indeterminacy.

Before classifying lowest-degree triple Massey products in moment-angle complexes, we first consider examples of non-trivial triple Massey products with non-trivial indeterminacy.

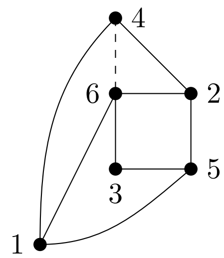


Figure 2.5: A simplicial complex that corresponds to a Massey product with non-trivial indeterminacy.

Example 2.3.8. Let $\mathcal{K}$ be the graph in Figure 2.5, where the dashed edge is optional. Let $\alpha_1, \alpha_2, \alpha_3 \in H^3(\mathcal{Z}_\mathcal{K})$ correspond to $\alpha_1 = [\chi_1] \in \widetilde{H}^0(\mathcal{K}_{12})$, $\alpha_2 = [\chi_3] \in \widetilde{H}^0(\mathcal{K}_{34})$, $\alpha_3 = [\chi_5] \in \widetilde{H}^0(\mathcal{K}_{56})$. Since $\widetilde{H}^1(\mathcal{K}_{1234}) = 0$ and $\widetilde{H}^1(\mathcal{K}_{3456}) = 0$, the products $\alpha_1\alpha_2 \in \widetilde{H}^1(\mathcal{K}_{1234})$ and $\alpha_2\alpha_3 \in \widetilde{H}^1(\mathcal{K}_{3456})$ are zero.

A cochain $a_{12} \in C^0(\mathcal{K}_{1234})$ such that $d(a_{12}) = \overline{\overline{\chi_1}}\chi_3 = 0$ is of the form $a_{12} = c_1\chi_3 + c_2(\chi_1 + \chi_4 + \chi_2)$, $c_1, c_2 \in \mathbf{k}$. A cochain $a_{23} \in C^0(\mathcal{K}_{3456})$ such that $d(a_{23}) = \overline{\overline{\chi_3}} \cdot \chi_5 = \chi_{35}$ is of the form $a_{23} = c_3\chi_4 + c_4(\chi_6 + \chi_3 + \chi_5) + \chi_5$, $c_3, c_4 \in \mathbf{k}$, where $c_3 = c_4$ if $\{4, 6\} \in \mathcal{K}$. The associated cocycle $\omega \in C^1(\mathcal{K})$ is

$$\omega = \overline{\overline{a}}_1 a_{23} + \overline{\overline{a}}_{12} a_3 = c_3\chi_{14} + c_4(\chi_{16} + \chi_{15}) + \chi_{15} + c_1\chi_{35} + c_2(\chi_{15} + \chi_{25}).$$

Since $d(\chi_5) = \chi_{15} + \chi_{35} + \chi_{25}$ and $d(\chi_1) = -\chi_{16} - \chi_{14} - \chi_{15}$ for $\chi_1, \chi_5 \in C^1(\mathcal{K})$,

$$\omega = (c_3 - c_4)\chi_{14} - c_4 d(\chi_1) + \chi_{15} + (c_1 - c_2)\chi_{35} + c_2 d(\chi_5).$$

Therefore $[\omega] = [(c_3 - c_4)\chi_{14} + \chi_{15} + (c_1 - c_2)\chi_{35}] \neq 0$ for any $c_1, c_2, c_3, c_4 \in \mathbf{k}$. By Hochster's formula, $a_{12} \in C^0(\mathcal{K}_{1234})$, $a_{23} \in C^0(\mathcal{K}_{3456})$ and $[\omega] \in H^1(\mathcal{K})$ correspond to $a_{12}, a_{23} \in C^5(\mathcal{Z}_\mathcal{K})$ and $[\omega] \in H^8(\mathcal{Z}_\mathcal{K})$, respectively. Hence $\langle \alpha_1, \alpha_2, \alpha_3 \rangle \subset H^8(\mathcal{Z}_\mathcal{K})$ is non-trivial and has non-trivial indeterminacy, given by $\alpha_1 \cdot \widetilde{H}^0(\mathcal{K}_{3456}) + \alpha_3 \cdot \widetilde{H}^0(\mathcal{K}_{1234})$. These calculations are similar to Example 2.2.8. In that case, the triple Massey product only contains one (non-zero) class, so the indeterminacy is trivial.

The simplicial complexes in Example 2.3.8 are also the last two graphs in Figure 2.6. We will show that all non-trivial triple Massey products of the form $\langle \alpha_1, \alpha_2, \alpha_3 \rangle$ for $\alpha_i \in H^3(\mathcal{Z}_\mathcal{K})$ are classified by these eight graphs.

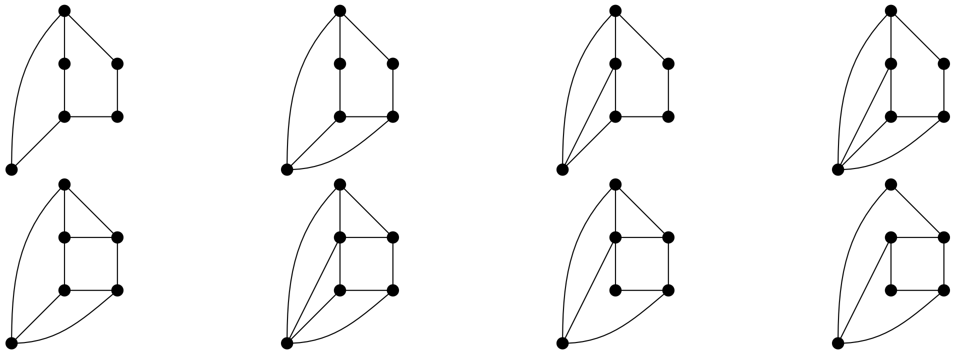


Figure 2.6: The obstruction graphs

Lemma 2.3.9. *None of the graphs in Figure 2.6 is isomorphic to another.*

Proof. Label the eight graphs a, b, c, d along the top row and e, f, g, h along the bottom. Two graphs are not isomorphic if their vertices have different valencies. For each graph, we list the valency of the vertices.

$a : 3, 3, 2, 2, 2, 2$	$b : 3, 3, 3, 3, 2, 2$	$c : 3, 3, 3, 3, 2, 2$	$d : 4, 3, 3, 3, 3, 2$
$e : 3, 3, 3, 3, 3, 3$	$f : 4, 4, 3, 3, 3, 3$	$g : 4, 3, 3, 3, 3, 2$	$h : 3, 3, 3, 3, 2, 2$

Thus graphs a, e, f are not isomorphic to any of the other graphs. Also the graphs d and g are not isomorphic because the vertices of valency 2 and 4 are adjacent in g but not adjacent in d. The graph c is different to b, h because the two vertices of valency 2 are adjacent in graph c but not adjacent in b or h. The graph b is different to h because the two vertices of valency 2 are at a minimal distance of 2 from each other, that is, there is one vertex in between them in b. In h, these two vertices are at a minimal distance of 3 from each other. Therefore each of these graphs is not isomorphic to another. □

Denham and Suciu [17, Theorem 6.1.1] showed that the first 6 graphs in Figure 2.6 classify non-trivial Massey products $\langle \alpha_1, \alpha_2, \alpha_3 \rangle$ for $\alpha_i \in H^3(\mathcal{Z}_\mathcal{K})$ but the proof works only when the indeterminacy is trivial. The following result is based on that Theorem, but also considers the case when the triple Massey product has non-trivial indeterminacy.

Theorem 2.3.10. *A Massey product $\langle \alpha_1, \alpha_2, \alpha_3 \rangle \subset H^8(\mathcal{Z}_\mathcal{K})$ for $\alpha_1, \alpha_2, \alpha_3 \in H^3(\mathcal{Z}_\mathcal{K})$ is* *defined and non-trivial if and only if the one-skeleton $\mathcal{K}^{(1)}$ of $\mathcal{K}$ contains a full subcomplex isomorphic to a graph in Figure 2.6.*

Proof. As $\mathcal{Z}_{\mathcal{K}_J}$ retracts off $\mathcal{Z}_{\mathcal{K}}$, it is sufficient to prove Theorem 2.3.10 when $\mathcal{K}$ has six vertices. Let $\mathcal{K}^{(1)}$ be a graph in Figure 2.6. Then the arguments to find non-trivial Massey products in Examples 2.2.8 and 2.3.8 apply to $\mathcal{K}^{(1)}$. Also, these calculations are not affected by 2-simplices in $\mathcal{K}$, and $\dim(\mathcal{K}) \leqslant 2$. Thus $\langle \alpha_1, \alpha_2, \alpha_3 \rangle$ is non-trivial.

Conversely, suppose $\langle \alpha_1, \alpha_2, \alpha_3 \rangle$ is non-trivial for $\alpha_1, \alpha_2, \alpha_3 \in H^3(\mathcal{Z}_{\mathcal{K}})$. Let G on $[6]$ be the edge complement of $\mathcal{K}^{(1)}$, so $\{i,j\} \in G$ if and only if $\{i,j\} \notin \mathcal{K}$. We will show that G is a graph (a) or (b) in Figure 2.7, where dashed edges are optional.

By Theorem 2.1.24,

$$\begin{aligned} H^3(\mathcal{Z}_{\mathcal{K}}) &= H^{-1,4}(\mathcal{Z}_{\mathcal{K}}) \oplus H^{-3,6}(\mathcal{Z}_{\mathcal{K}}) \oplus \dots \\ &\cong \bigoplus_{J \subset [m], |J|=2} \widetilde{H}^0(\mathcal{K}_J) \oplus \bigoplus_{J \subset [m], |J|=3} \widetilde{H}^{-1}(\mathcal{K}_J) \oplus \dots \\ &\cong \bigoplus_{J \subset [m], |J|=2} \widetilde{H}^0(\mathcal{K}_J). \end{aligned}$$

Therefore there are full subcomplexes $\mathcal{K}_{S_i}$ for $S_i \subset [m]$, $|S_i| = 2$ such that α_i corresponds to $\alpha_i \in \widetilde{H}^0(\mathcal{K}_{S_i})$. Since $\langle \alpha_1, \alpha_2, \alpha_3 \rangle$ is non-trivial, $S_i \cap S_j = \varnothing$ for $i \neq j$. Let $S_1 = \{1,2\}$, $S_2 = \{3,4\}$, $S_3 = \{5,6\}$. Since $\{v_1, v_2\} \notin \mathcal{K}$ for any $v_1, v_2 \in S_i$, G contains the edges $\{1,2\}, \{3,4\}, \{5,6\}$. Since $\alpha_i \alpha_{i+1} = 0$, $\mathcal{K}_{S_i \cup S_{i+1}}$ does not contain a cycle for $i = 1,2$. Thus there exist edges $\{v_1, v_2\}, \{v_2', v_3\} \in G$ for $v_i, v_i' \in S_i$.

Figure 2.7: Graph complements G, dashed edges optional.

Suppose $\{v_1, v_2\}, \{v_2, v_3\} \in G$ for $v_i \in S_i$. Let $a_1 = \chi_{v_1} \in C^0(\mathcal{K}_{12})$, $a_2 = \chi_{v_2} \in C^0(\mathcal{K}_{34})$, $a_3 = \chi_{v_3} \in C^0(\mathcal{K}_{56})$ be representing cocycles for $\alpha_1, \alpha_2, \alpha_3$. So $a_1 a_2 = 0$ and $a_2 a_3 = 0$. For $a_{12} = 0 = a_{23}$, $\omega = 0$, which contradicts the non-triviality of $\langle \alpha_1, \alpha_2, \alpha_3 \rangle$. Thus $\{v_1, v_2\}, \{v_2, v_3\} \notin G$ for $v_i \in S_i$.

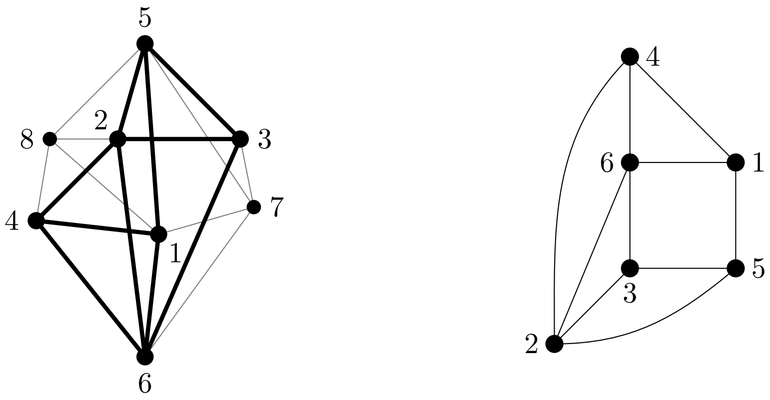
Label the vertices of G so that there is a path $1, \dots, 6$. Consider the case when $\{1,3\}, \{4,6\} \notin G$. Since $\{v_1, v_2\}, \{v_2, v_3\} \notin G$ for $v_i \in S_i$, vertices 3 and 4 have valency two. Suppose $\{2,5\} \in G$. Let $\chi_2 \in C^0(\mathcal{K}_{12}), \chi_3 \in C^0(\mathcal{K}_{34}), \chi_5 \in C^0(\mathcal{K}_{56})$ represent $\alpha_1, \alpha_2, \alpha_3$, respectively. Since $a_1 a_2 = 0$, let $a_{12} = 0$ and let $a_{23} = \chi_5$. Then $\omega = \chi_{25}$ is zero, contradicting the non-triviality of $\langle \alpha_1, \alpha_2, \alpha_3 \rangle$. Hence $\{2,5\} \notin G$ and G is graph (a), where $\{1,5\}$, $\{1,6\}$ and $\{2,6\}$ are optional.

There are three more cases. When $\{1,3\} \in G$ and $\{4,6\} \notin G$, it is necessary that $\{1,5\} \notin G$ otherwise in Example 2.3.8, $\omega = \chi_{15}$ is zero. Also $\{2,5\} \notin G$ as in the previous case. The edges $\{1,6\},\{2,6\}$ in G are optional as they do not change the calculations in the Example. If $\{1,6\},\{2,6\} \notin G$, G is graph (b) with $\{4,6\} \notin G$. For other selections of $\{1,6\},\{2,6\}$, G is isomorphic to a graph in (a). When $\{4,6\} \in G$ and $\{1,3\} \notin G$, G is symmetric to when $\{1,3\} \in G$ and $\{4,6\} \notin G$, so up to isomorphism we obtain the same graphs.

Finally if $\{1,3\},\{4,6\} \in G$, then $\{1,5\},\{2,5\},\{2,6\} \notin G$ for the same reasons as in the last two cases. Let $\alpha_1,\alpha_2,\alpha_3$ be represented by $\chi_1 \in C^0(\mathcal{K}_{12})$, $\chi_3 \in C^0(\mathcal{K}_{34})$, $\chi_6 \in C^0(\mathcal{K}_{56})$, respectively. Let $a_{12} = 0$, $a_{23} = \chi_6$. Then $\omega = \chi_{16}$ and therefore $\{1,6\} \notin G$. So G is graph (b) with $\{4,6\} \in G$.

Up to graph isomorphism, the graphs (a) and (b) are edge complements to exactly the one skeletons in Figure 2.6. □

Example 2.3.11. In Example 2.3.3, we found a non-trivial triple Massey product in $\mathcal{Z}_P$, where P is a 2-truncated 3-cube. In particular, the full subcomplex on the vertices $1,\dots,6$ of the one-skeleton of $\mathcal{K}_P$ is shown in Figure 2.8a. This full subcomplex can also be drawn as in Figure 2.8b, which is one of the obstruction graphs in Figure 2.6. Therefore the triple Massey product in Example 2.3.3 can also be regarded as an example of Theorem 2.3.10.



(a) **(b)**

Figure 2.8: An obstruction graph in $\mathcal{K}_P$ when P is a 2-truncated cube.

Example 2.3.12. For $k \geqslant 3$, a k-belt in a polytope P is a sequential collection of k facets $(F_{i_1},\dots,F_{i_k},F_{i_{k+1}})$, $F_{i_{k+1}} = F_{i_1}$, such that $F_{i_p} \cap F_{i_q} \neq \varnothing$ if and only if $q \equiv p+1 \mod k$, and $F_{i_1} \cap F_{i_2} \cap F_{i_3} = \varnothing$ if $k = 3$. Also a k-belt in a simple polytope P corresponds to a cycle of length k in $\mathcal{K}_P$. Buchstaber, Erokhovets, Masuda, Panov and Park [12] showed that for any polytope P with no 4-belts, all triple Massey products $\langle \alpha_1,\alpha_2,\alpha_3\rangle \subset H^8(\mathcal{Z}_P)$ for $\alpha_i \in H^3(\mathcal{Z}_P)$ are trivial. This is proved by considering the one-skeleton of $\mathcal{K}_P$ and noticing that all of the obstruction graphs in Figure 2.6 have a chordless cycle of length four. If P contains no 4-belts, then $K_P^{(1)}$ contains no cycles of length 4, thus $K_P^{(1)}$ does not contain an obstruction graph. Hence Denham and Suciu's result means that any triple Massey product on classes of degree three will be trivial.

Theorem 2.3.10 is a useful result for easily detecting Massey products, but it is limited to only triple Massey products in the lowest degree.

2.3.3 A family of higher Massey products

The examples of non-trivial Massey products have so far been examples only of triple products. In 2016, Limonchenko [26] also used Baskakov's original truncated cube example as inspiration for constructing non-trivial higher Massey products in the cohomology of moment-angle complexes.

Construction 2.3.13. [26] Let I^n be the n-dimensional cube with opposite facets labelled F_i, F_{n+i} for $i = 1, \dots, n$. Let P_n be a polytope obtained from I^n by making consecutive cuts of faces so that the Stanley-Reisner ideal of $\mathcal{K}_{P_n}$ is

$$\mathcal{I} = \Big(\{v_i v_{n+i} \colon 1 \leqslant i \leqslant n\}, \{v_i v_{n+i+1} \colon 1 \leqslant i \leqslant n-1\}, \dots, \{v_i v_{n+i+(n-2)} \colon 1 \leqslant i \leqslant 2\}, R\Big)$$

where v_i correspond to facets F_i of P_n, and R is the set of all relations coming from the new facets. For this construction, we write $\mathcal{Z}_{P_n}$ for the moment angle complex of $\mathcal{K}_{P_n}$.

Example 2.3.14. When $n = 2$, I^2 is a square. The Stanley-Reisner ideal corresponding to $\mathcal{K}_{P_2}$ is $\mathcal{I} = (v_1v_3, v_2v_4)$, and so no cuts are made to I^2. That is, $P_2 = I^2$. When $n = 3$, I^3 is the 3-cube and the Stanley-Reisner ideal of $\mathcal{K}_{I_3}$ is (v_1v_4, v_2v_5, v_3v_6). By Construction 2.3.13, the Stanley-Reisner ideal of $\mathcal{K}_{P_3}$ is

$$\mathcal{I} = (\{v_1v_4, v_2v_5, v_3v_6\}, \{v_1v_5, v_2v_6\}, v_7v_2, v_7v_4, v_8v_3, v_8v_5)\,.$$

This corresponds to obtaining P_3 from I_3 by truncating the intersection of the facets F_1 and F_5, and the intersection of F_2 and F_6, thus introducing new facets F_7 and F_8. Up to a relabelling of facets, P_3 is the same 2-truncated cube as in Figure 2.3a.

Theorem 2.3.15 ([26])**.** *Let $n \geqslant 2$ and let $\mathcal{K}$ denote $\mathcal{K}_{P_n}$. Also let $\alpha_i \in \widetilde{H}^0(\mathcal{K}_{i,n+i})$ for $1 \leqslant i \leqslant n$. Then all l-Massey products of l consecutive elements from $\alpha_1, \dots, \alpha_n$ are defined and the n-Massey product $\langle \alpha_1, \dots, \alpha_n \rangle$ is non-trivial.*

The idea of the proof of Theorem 2.3.15 is to ensure that the n-Massey product $\langle \alpha_1, \dots, \alpha_n \rangle$ is defined by cutting/truncating faces so that the two $(n-1)$-Massey products $\langle \alpha_1, \dots, \alpha_{n-1} \rangle$ and $\langle \alpha_2, \dots, \alpha_n \rangle$ are trivial. The proof uses induction on n, the dimension of the cube. For example, P_2 is the square and the 2-Massey product is the same as the cup product. Since K_{P_2} is the boundary of a square, the cup product is non-trivial (as seen in Example 2.1.25). In I^3, the cup products $\alpha_1\alpha_2$ and $\alpha_2\alpha_3$ are each supported on a full subcomplex that is the boundary of a square. Then we make two truncations to cut each of the squares and thus make both cup products $\alpha_1\alpha_2$ and $\alpha_2\alpha_3$ trivial. We have already seen that there is a non-trivial triple Massey product in $\mathcal{Z}_{P_3}$

in Example 2.3.3. When $n = 4$, the Stanley-Reisner ideal of I_4 is $(v_1v_5, v_2v_6, v_3v_7, v_4v_8)$. We then perform cuts to I_4 so that the two Massey products $\langle \alpha_1, \alpha_2, \alpha_3\rangle$ and $\langle \alpha_2, \alpha_3, \alpha_4\rangle$ are defined. That is, add monomials v_1v_6, v_2v_7, v_3v_8 to the Stanley-Reisner ideal and there are two copies of P_3 contained in this stage of the construction of P_4. Then we again cut to ensure that $\langle \alpha_1, \alpha_2, \alpha_3\rangle$ and $\langle \alpha_2, \alpha_3, \alpha_4\rangle$ are trivial. That is, add monomials v_1v_7, v_2v_8 to the Stanley-Reisner ring in order to obtain P_4. So

$$\mathcal{I} = (\{v_1v_5, v_2v_6, v_3v_7, v_4v_8\}, \{v_1v_6, v_2v_7, v_3v_8\}, \{v_1v_7, v_2v_8\}, \ldots)$$

where R is the set of relations coming from the new facets $F_9, \ldots, F_{13}$. A more detailed proof can be found in [26] and [27], or alternatively in Example 3.1.17.

Similar to Baskakov's family of examples and Denham and Suciu's classification, the indeterminacy in Limonchenko's family always turns out to be trivial [27, Theorem 3.3]. Additionally, all of the classes α_i are elements of $\widetilde{H}^0(\mathcal{K}_{i,n+i})$, where $\mathcal{K}_{i,n+i}$ is a simplicial complex on two disjoint vertices, so α_i corresponds to a class in $H^3(\mathcal{Z}_\mathcal{K})$. Both of these properties simplify the calculations to check the non-triviality of the Massey products.

2.3.4 Other families of examples of non-trivial Massey products

The Pogorelov class of polytopes is the class of combinatorial 3-polytopes that admit a right-angled realisation in Lobachevsky space $\mathbb{L}^3$ that is unique up to isometry [12]. It includes polytopes such as fullerenes, whose facets are either pentagons or hexagons. The moment-angle complexes corresponding to Pogorelov polytopes are key for the study of hyperbolic manifolds of Löbell type, as well as for the study of cohomological rigidity [12].

In 2017, Zhuravleva [39] showed that the simplicial complex $\mathcal{K} = \mathcal{K}_P$ corresponding to any Pogorelov polytope P has a full subcomplex as shown in Figure 2.9. Let $\alpha \in \widetilde{H}^0(\mathcal{K}_{567})$, $\beta \in \widetilde{H}^0(\mathcal{K}_{2,b_0,\ldots,b_n})$ and $\gamma \in \widetilde{H}^0(\mathcal{K}_{3,4})$. Then by Theorem 2.1.24, these classes correspond to $\alpha \in H^4(\mathcal{Z}_\mathcal{K})$, $\beta \in H^{n+3}(\mathcal{Z}_\mathcal{K})$ and $\gamma \in H^3(\mathcal{Z}_\mathcal{K})$. As in (2.14), the indeterminacy is given by

$$\alpha \cdot \widetilde{H}^0(\mathcal{K}_{2,b_0,\ldots,b_n,3,4}) + \gamma \cdot \widetilde{H}^0(\mathcal{K}_{2,b_0,\ldots,b_n,5,6,7}) = 0.$$

Zhuravleva showed that this Massey product $\langle \alpha, \beta, \gamma\rangle \subset H^{n+9}(\mathcal{Z}_\mathcal{K})$ is non-trivial.

Theorem 2.3.16 ([39])**.** *For any Pogorelov polytope P, there is a non-trivial triple Massey product $\langle \alpha, \beta, \gamma\rangle$ in $H^*(\mathcal{Z}_\mathcal{K})$.*

The proof of this theorem is by explicit calculation in similar style to Examples 2.2.8 and 2.3.3.

So far, the families of examples of non-trivial Massey products in moment-angle complexes have all had trivial indeterminacy. Additionally, the Massey products have all been on spherical classes. For example, both Limonchenko's family in Section 2.3.3 and

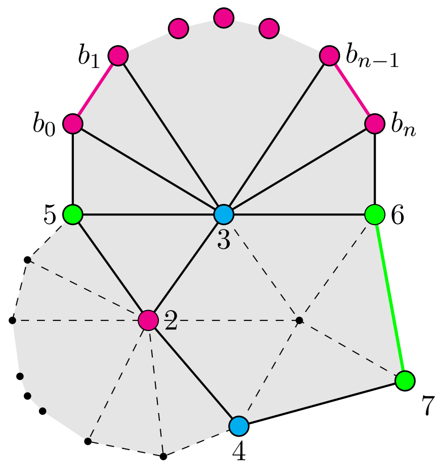


Figure 2.9: A full subcomplex of the simplicial complex corresponding to any Pogorelov polytope [39]

Denham and Suciu's classification in Section 2.3.2 are on classes $\alpha_i \in \widetilde{H}^0(S^0)$. Similarly, Baskakov's family in Section 2.3.1 is on classes $\alpha_i \in \widetilde{H}^n(S^n)$. In the rest of this thesis, we will see constructions of non-trivial higher Massey products in moment-angle complexes on arbitrary classes, including torsion classes.

Chapter 3

Combinatorial Operations and Massey Products

3.1 Join and stellar subdivision

The aim of this section is to develop a new systematic construction of non-trivial higher Massey products in the cohomology of moment-angle complexes.

We have already seen in Proposition 2.1.6 that the join $\mathcal{K}_1 * \mathcal{K}_2$ of simplicial complexes $\mathcal{K}_1$, $\mathcal{K}_2$ induces a product $\mathcal{Z}_{\mathcal{K}_1 * \mathcal{K}_2} = \mathcal{Z}_{\mathcal{K}_1} \times \mathcal{Z}_{\mathcal{K}_2}$ of corresponding moment-angle complexes. This product can be thought of as a 2-Massey product. For example, let $\bar{\mathcal{K}}$ be the boundary of a square, which is the join of $\mathcal{K}^1 = \{\{1\},\{2\}\}$ and $\mathcal{K}^2 = \{\{3\},\{4\}\}$, as in Figure 3.1a. In Example 2.1.25 we saw that the 2-Massey product of $\alpha \in \widetilde{H}^0(\bar{\mathcal{K}}_{1,2})$ and $\beta \in \widetilde{H}^0(\bar{\mathcal{K}}_{3,4})$ is non-trivial in the cohomology group $\widetilde{H}^1(\bar{\mathcal{K}}_{1,2,3,4})$, that is, in $H^*(\mathcal{Z}_{\bar{\mathcal{K}}})$.

(a) Non-trivial cup product of α and β

(b) Trivial cup product of α and β, after stellar subdivision

Figure 3.1: 2-Massey products (cup products) on a square

We use this motivating example to also create higher non-trivial Massey products. However, if all cup products are non-trivial, then the triple Massey product is not defined. Therefore we need to break the cycle that supports $\alpha\beta \in \widetilde{H}^1(\bar{\mathcal{K}}_{1,2,3,4})$ in order to make the cup product trivial. Let $\mathcal{K} = \mathrm{ss}_{\{1,4\}}\,\bar{\mathcal{K}}$, as in Figure 3.1b. Then for $\alpha \in \widetilde{H}^0(\mathcal{K}_{1,2})$ and $\beta \in \widetilde{H}^0(\mathcal{K}_{3,4})$, the product $\alpha\beta$ is trivial. This is how the families given by Baskakov and Limonchenko were constructed in Constructions 2.3.6 and 2.3.13.

In this section, we systematically construct simplicial complexes $\mathcal{K}$ such that $H^*(\mathcal{Z}_{\mathcal{K}})$ contains a non-trivial n-Massey product $\langle \alpha_1, \dots, \alpha_n \rangle$ of classes α_i in arbitrary degree.

The idea is to create $\mathcal{K}$ by first starting with the join $\bar{\mathcal{K}}$ of n simplicial complexes $\mathcal{K}^i$, each of which support α_i. In the join $\bar{\mathcal{K}} = \mathcal{K}^1 * \ldots * \mathcal{K}^n$, the cup product $\alpha_1 \cup \ldots \cup \alpha_n$ is non-trivial. To obtain $\mathcal{K}$, we further stellar subdivide the join $\bar{\mathcal{K}}$ in such a way that for every $l < n$, the relevant l-Massey products vanish simultaneously ensuring that the n-Massey product is defined.

3.1.1 Massey products and stellar subdivision

To construct a non-trivial n-Massey product, we will need to perform several stellar subdivisions to ensure that the l-Massey products for $l < n$ are trivial. We first show that the order of stellar subdivisions on a simplicial complex does not affect a full subcomplex on the original vertices.

Lemma 3.1.1. *Let $\mathcal{L}$ be a simplicial complex. For simplices $I_1, I_2 \in \mathcal{L}$, such that $I_1 \cap I_2 \neq I_1, I_2$, let $\mathcal{M} = \mathrm{ss}_{I_2} \mathrm{ss}_{I_1} \mathcal{L}$ and $\mathcal{N} = \mathrm{ss}_{I_I} \mathrm{ss}_{I_2} \mathcal{L}$. Then the full subcomplexes $\mathcal{M}_{V(\mathcal{L})}$ and $\mathcal{N}_{V(\mathcal{L})}$ are equal.*

Proof. Since $(\mathrm{ss}_{I_1} \mathcal{L})_{V(\mathcal{L})} = (\mathcal{L} \setminus \mathrm{st}_{\mathcal{L}} I_1) \cup \partial \, \mathrm{st}_{\mathcal{L}} I_1 = \{J \in \mathcal{L} \mid I_1 \not\subset J\}$, we can alternatively express $\mathrm{ss}_{I_1} \mathcal{L}$ as $\{J \in \mathcal{L} \mid I_1 \not\subset J\} \cup \{J \sqcup \{*_1\} \mid J, I_1 \cup J \in \mathcal{L} \text{ and } I_1 \not\subset J\}$. Since $I_1 \cap I_2 \neq I_1, I_2$, neither $I_1 \subset I_2$ nor $I_2 \subset I_1$ and so $I_1 \in \mathrm{ss}_{I_2} \mathcal{L}$ and $I_2 \in \mathrm{ss}_{I_1} \mathcal{L}$. Therefore after stellar subdividing $\mathrm{ss}_{I_1} \mathcal{L}$ at I_2,

$$\mathcal{M} = \{J \in \mathcal{L} \mid I_1, I_2 \not\subset J\} \cup \{J \sqcup \{*_1\} \mid J, I_1 \cup J \in \mathcal{L} \text{ and } I_1, I_2 \not\subset J\} \cup \\ \cup \{J \sqcup \{*_2\} \mid J, I_2 \cup J \in \mathrm{ss}_{I_1} \mathcal{L} \text{ and } I_2 \not\subset J\}.$$

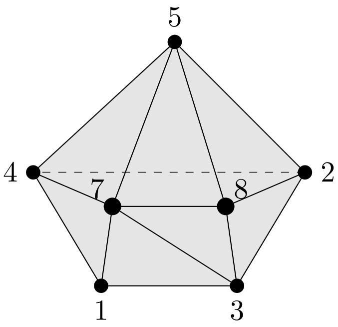
A similar expression can be made for $\mathcal{N}$. Then $\mathcal{M}_{V(\mathcal{L})} = \{J \in \mathcal{L} \mid I_1, I_2 \not\subset J\} = \mathcal{N}_{V(\mathcal{L})}$. □

Example 3.1.2. Let $\mathcal{L}$ be the simplicial complex in Figure 2.4a. Let $I_1 = \{1, 5\}$ and $I_2 = \{3, 5\}$. The star $\mathrm{st}_{\mathcal{L}} I_1$ contains maximal simplices $\{1, 4, 5\}$ and $\{1, 3, 5\}$, and $\mathrm{st}_{\mathcal{L}} I_2$ contains $\{1, 3, 5\}$ and $\{2, 3, 5\}$. If I_1 is stellar subdivided first, then $\mathrm{st}_{\mathrm{ss}_{I_1} \mathcal{L}} I_2$ contains maximal simplices $\{2, 3, 5\}$ and $\{3, 5, 7\}$, where $\{7\}$ is the new vertex introduced by the first stellar subdivision. Then $\mathcal{M} = \mathrm{ss}_{I_2} \mathrm{ss}_{I_1} \mathcal{L}$ and $\mathcal{N} = \mathrm{ss}_{I_I} \mathrm{ss}_{I_2} \mathcal{L}$ are different, as shown in Figures 3.2a and 3.2b, respectively. Restricting both $\mathcal{M}$ and $\mathcal{N}$ to the original vertices $V(\mathcal{L}) = \{1, \ldots, 6\}$, we have that $\mathcal{M}_{V(\mathcal{L})} = \mathcal{N}_{V(\mathcal{L})}$.

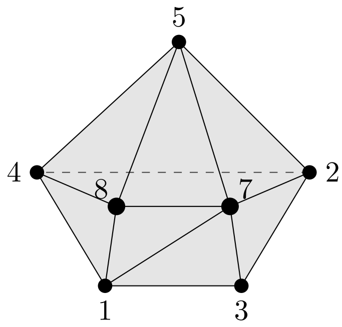
Construction 3.1.3. Let $\mathcal{K}$ be a simplicial complex on $[m]$ with a non-trivial $\alpha \in \widetilde{H}^p(\mathcal{K})$ for $p \geqslant 0$. Let α be represented by a cocycle a that is supported on the p-simplices in $S_a \subset \mathcal{K}$ so that $a = \sum_{\sigma \in S_a} c_\sigma \chi_\sigma \in C^p(\mathcal{K})$ for a non-zero coefficient $c_\sigma \in \mathbf{k}$ for each $\sigma \in S_a$. For every simplex $\sigma \in S_a$, let v_σ denote one particular choice of vertex in σ. Let P_σ be the set

$$P_\sigma = \{p\text{-simplices } \sigma' \in \mathcal{K} \mid \sigma \cap \sigma' = \sigma \setminus v_\sigma\}.$$

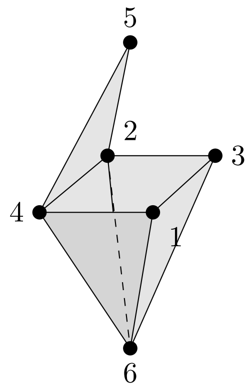
We fix an order on the simplices in S_a. Let $\sigma^{(1)}$ be the first element of S_a. Then let



(a) $\mathcal{M}_{\{1,2,3,4,5,7,8\}}$



(b) $\mathcal{N}_{\{1,2,3,4,5,7,8\}}$



(c) $\mathcal{M}_{V(\mathcal{L})} = \mathcal{N}_{V(\mathcal{L})}$

Figure 3.2: The order of stellar subdivisions does not affect the restriction to the original vertices

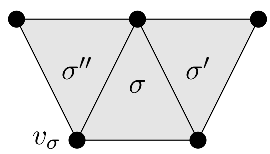


Figure 3.3: For this choice of a vertex $v_\sigma \in \sigma$, $\sigma' \in P_\sigma$ but $\sigma'' \notin P_\sigma$

$S_a^{(1)} = S_a \backslash P_{\sigma^{(1)}}$. Let $\sigma^{(2)}$ be the next element after $\sigma^{(1)}$ in $S_a^{(1)}$. Then let $S_a^{(2)} = S_a^{(1)} \backslash P_{\sigma^{(2)}}$. We continue inductively until $\sigma^{(l)}$ is the last element of $S_a^{(l-1)}$, and let

$$\widetilde{S}_a = S_a^{(l-1)} \setminus P_{\sigma^{(l)}}. \tag{3.1}$$

At each stage, $\sigma \notin P_\sigma$ so $\widetilde{S}_a$ contains at least the last element $\sigma^{(l)}$. Let

$$P_a = P_{\sigma^{(1)}} \cup \cdots \cup P_{\sigma^{(l)}}. \tag{3.2}$$

This set is non-empty as follows. If $p = 0$ and $\widetilde{H}^0(\mathcal{K}) \neq 0$, then $\mathcal{K}$ is a disjoint union of at least two vertices. For any $v, w \in \mathcal{K}$, $v \cap w = \varnothing = v \setminus v$. Hence $w \in P_v$. Alternatively let $p > 0$. Since $\alpha \in \widetilde{H}^p(\mathcal{K})$ is non-zero, there is a non-zero cycle $x \in C_p(\mathcal{K})$ such that $a(x) \neq 0$. Let $x = \sum_{\tau \in T_x} c_\tau \Delta_\tau$ for non-zero coefficients c_τ and a set of p-simplices $T_x \subset \mathcal{K}$. Let $\sigma \in S_a \cap T_x$. If ∂ is the boundary map and v_j is the jth vertex in σ, then $\partial(\sigma) = \sum_{j=1}^{p+1} (-1)^{j+1} \Delta_{\sigma \backslash v_j} \neq 0$. Since x is a cycle, there is a simplex $\tau \in T_x$, $\tau \neq \sigma$, and a vertex $v \in \tau$ such that $\tau \setminus v = \tau \cap \sigma = \sigma \setminus v_j$ for any vertex $v_j \in \sigma$. Hence for any $\sigma \in S_a \cap T_x$, P_σ is non-empty.

For $i \in \{2, \ldots, n-1\}$, let $\mathcal{K}^i$ be a simplicial complex on $[m_i]$ vertices that is not an $(m_i - 1)$-simplex. Since $\mathcal{K}^i$ is not a simplex, there is a non-zero cohomology class $\alpha_i \in \widetilde{H}^{p_i}(\mathcal{K}^i_{J_i})$ for $p_i \in \mathbb{N}$, $J_i \subseteq [m_i]$. For $i \in \{1, n\}$, let $\mathcal{K}^i$ be a simplicial complex on $[m_i]$ such that there exist $J_i \subset [m_i]$, $p_i \in \mathbb{N}$ and a non-zero $\alpha_i = [a_i] \in \widetilde{H}^{p_i}(\mathcal{K}^i_{J_i})$ for which there is a p_i-simplex $\sigma_i \in S_{a_i}$ (if $i = 1$) or $\sigma_i \in P_{a_i}$ (if $i = n$) that is maximal in $\mathcal{K}^i_{J_i}$.

Let $\bar{\mathcal{K}} = \mathcal{K}^1 * \cdots * \mathcal{K}^n$, so $\bar{\mathcal{K}}_{V(\mathcal{K}^i)} = \mathcal{K}^i$ for every $i \in \{1, \ldots, n\}$. The vertices in each vertex set $V(\mathcal{K}^i)$ have an induced order from $[m_i]$. Suppose the vertex set $V(\bar{\mathcal{K}}) = \bigsqcup_{i \in \{1,\ldots,n\}} V(\mathcal{K}^i)$ is ordered so that $u < v$ for all $u \in V(\mathcal{K}^i)$ and $v \in V(\mathcal{K}^j)$ for $i < j$. We construct a simplicial complex $\mathcal{K}$ by stellar subdividing $\bar{\mathcal{K}}$ as follows. For any $i < k$, $(i, k) \neq (1, n)$, let $\sigma_i \in S_{a_i}$ and $\sigma_k \in P_{a_k}$. If $\sigma_i \cup \sigma_k \in \bar{\mathcal{K}}$, we stellar subdivide $\bar{\mathcal{K}}$ at the simplex $\sigma_i \cup \sigma_k$. For ease of notation, let $\bar{\mathcal{K}}$ denote the resulting simplicial complex $\mathrm{ss}_{\sigma_i \cup \sigma_k} \bar{\mathcal{K}}$. We iteratively stellar subdivide $\bar{\mathcal{K}}$ at every $\sigma_i \cup \sigma_k$ for every $i < k$ and every $\sigma_i \in S_{a_i}$, $\sigma_k \in P_{a_k}$. Let $\mathcal{K}$ denote the resulting simplicial complex restricted to the original vertices of $\mathcal{K}^1 * \cdots * \mathcal{K}^n$.

Lemma 3.1.4. *The simplicial complex $\mathcal{K}$ is independent of the order of simplices in P_{a_k}.*

Proof. For any $\sigma_k, \sigma'_k \in P_{a_k}$, we have that $\sigma_i \cup \sigma_k \cap \sigma_i \cup \sigma'_k \neq \sigma_i \cup \sigma_k, \sigma_i \cup \sigma'_k$. So by Lemma 3.1.1, the order of P_{a_k} does not affect $\mathcal{K}$. □

Lemma 3.1.5. *The simplicial complex $\mathcal{K}$ is independent of the order in which the pairs $\{i, k\}$, $1 \leqslant i < k \leqslant n$, are chosen.*

Proof. For any a_i, the set S_{a_i} of simplices lies in $J_i \subset V(\mathcal{K}^i)$. Let $\{i_1, k_1\}$ and $\{i_2, k_2\}$ be two pairs of indices. For any $\sigma_{i_j} \in S_{a_{i_j}}$ and any $\sigma_{k_j} \in P_{a_{k_j}}$ such that $1 \leqslant i_j < k_j \leqslant n$, $j = 1, 2$, the intersection of $\sigma_{i_1} \cup \sigma_{k_1} \subset J_{i_1} \cup J_{k_1}$ and $\sigma_{i_2} \cup \sigma_{k_2} \subset J_{i_2} \cup J_{k_2}$ is empty. Therefore by Lemma 3.1.1, we can stellar subdivide at simplices $\sigma_{i_1} \cup \sigma_{k_1}$ and simplices $\sigma_{i_2} \cup \sigma_{k_2}$ in either order. □

Example 3.1.6. Let $\mathcal{K}^1$ be the disjoint union of two vertices $\{1\}, \{2\}$ and $\mathcal{K}^2$ the simplicial complex in Figure 3.4a. The join $\mathcal{K}^1 * \mathcal{K}^2$ is homotopy equivalent to $S^2 \vee S^1$. Let $\alpha_1 \in \widetilde{H}^0(\mathcal{K}^1)$, $\alpha_2 \in \widetilde{H}^0(\mathcal{K}^2)$ be represented by the cochains $a_1 = \chi_1$ and $a_2 = \chi_3 + \chi_4 + \chi_5$, respectively. Then $S_{a_1} = \{1\}$, and $S_{a_2} = \{\{3\}, \{4\}, \{5\}\}$. Following the construction above, for $\sigma_2 = \{3\}$ there is only one choice of a vertex $v = 3$. Then $P_{\{3\}} = \{\{4\}, \{5\}, \{6\}\}$ so $\widetilde{S}_{a_2} = S^{(1)}_{a_2} = \{\{3\}\}$ and $P_{a_2} = P_{\{3\}}$. Let

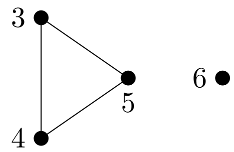
$$\bar{\mathcal{K}} = \mathrm{ss}_{\{1,6\}}\, \mathrm{ss}_{\{1,5\}}\, \mathrm{ss}_{\{1,4\}}\, \mathcal{K}^1 * \mathcal{K}^2.$$

Therefore $\mathcal{K} = \bar{\mathcal{K}}_{1,2,3,4,5,6}$, as in Figure 3.4b. Since $\mathcal{K}$ is contractible, the cup product $\alpha_1 \alpha_2$ is trivial.

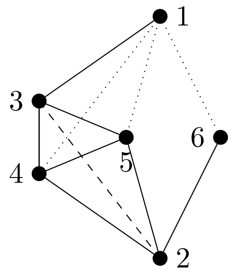
Example 3.1.7. In addition to $\mathcal{K}^1$ and $\mathcal{K}^2$ in Example 3.1.6, let $\mathcal{K}^3$ be the disjoint union of two vertices $\{7\}, \{8\}$. Let $\alpha_3 \in \widetilde{H}^0(\mathcal{K}^3)$ be represented by $a_3 = \chi_7$. Then $\widetilde{S}_{a_3} = S_{a_3} = \{7\}$ and $P_{a_3} = P_{\{7\}} = \{\{8\}\}$. By Construction 3.1.3, we stellar subdivide $\mathcal{K}^1 * \mathcal{K}^2 * \mathcal{K}^3$ at $\sigma_i \cup \sigma_k$ for every $\sigma_i \in S_{a_i}$ and $\sigma_k \in P_{a_k}$ for $i = 1, 2$ and $k = i + 1$. Since $S_{a_2} = \{\{3\}, \{4\}, \{5\}\}$, we obtain the simplicial complex

$$\bar{\mathcal{K}} = \mathrm{ss}_{\{5,8\}}\, \mathrm{ss}_{\{4,8\}}\, \mathrm{ss}_{\{3,8\}}\, \mathrm{ss}_{\{1,6\}}\, \mathrm{ss}_{\{1,5\}}\, \mathrm{ss}_{\{1,4\}}\, \mathcal{K}^1 * \mathcal{K}^2 * \mathcal{K}^3.$$

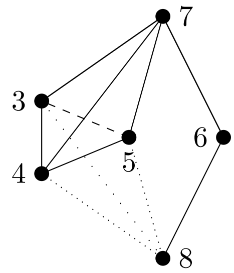
The resultant simplicial complex $\mathcal{K} = \bar{\mathcal{K}}_{1,2,3,4,5,6,7,8}$ has a 1-cycle on the edges $\{1,3\}$, $\{2,3\}$, $\{2,8\}$, $\{1,8\}$, which is not a boundary (proved by Lemma 3.1.11). Therefore $\widetilde{H}^1(\mathcal{K})$ is non-trivial.



(a) The simplicial complex $\mathcal{K}^2$



(b) $\mathcal{K}_{1,2,3,4,5,6}$ after stellar subdivisions at $\{1,4\}, \{1,5\}, \{1,6\}$



(c) $\mathcal{K}_{3,4,5,6,7,8}$ after stellar subdivisions at $\{3,8\}, \{4,8\}, \{5,8\}$

Figure 3.4: Example of Construction 3.1.3

Lemma 3.1.8. *In Construction 3.1.3, the simplicial complex $\mathcal{K}$ depends on the order of simplices in S_{a_k}.*

Proof. Suppose $\sigma_k \in S_{a_k}$, $\sigma'_k \in P_{\sigma_k}$ and let $\sigma_i \in S_{a_i}$ for an $i \in \{1, \dots, k-1\}$. If $\sigma'_k \in S_{a_k} \cap P_{\sigma_k}$, then either $\sigma'_k > \sigma_k$ or $\sigma'_k < \sigma_k$ in the order of simplices in S_{a_k}. In the first case, the simplex $\sigma'_k \in P_{a_k}$ and hence $\sigma_i \cup \sigma_k \in \mathcal{K}$ and $\sigma_i \cup \sigma'_k \notin \mathcal{K}$. Conversely in the second case, if the chosen vertex $v_{k'} \in \sigma'_k$ is such that $\sigma'_k \setminus v_{k'} = \sigma_k \setminus v_k$, then $\sigma_k \in P_{\sigma'_k}$. So $\sigma_k \in P_{a_k}$ and therefore $\sigma_i \cup \sigma'_k \in \mathcal{K}$ and $\sigma_i \cup \sigma_k \notin \mathcal{K}$. □

Lemma 3.1.9. *The choice of vertex $v_k \in \sigma_k$ affects the number of stellar subdivisions performed in Construction 3.1.3.*

Proof. Suppose the simplicial complex in Figure 3.3 is a full subcomplex of $\mathcal{K}^k_{J_k}$, for some $k \in \{2, \dots, n\}$. Further suppose that $S_{a_k} = \{\sigma_k, \sigma''_k\}$. In Figure 3.3, $\sigma''_k \notin P_{\sigma_k}$ so $P_{a_k} = P_{\sigma_k} \cup P_{\sigma''_k}$. Alternatively, if $v_k \in \sigma_k$ was chosen so that $v_k \notin \sigma_k \cap \sigma''_k$, then $\sigma''_k \in P_{\sigma_k}$. Therefore if $\sigma_k < \sigma''_k$ in S_{a_k}, then $P_{\sigma''_k} \not\subset P_{a_k}$ and thus fewer stellar subdivisions are made. □

We aim to show that there is a non-trivial n-Massey product in $H^*(\mathcal{Z}_{\mathcal{K}})$ where $\mathcal{K}$ is the simplicial complex created by Construction 3.1.3. We do this in stages, first showing that the n-Massey product is defined.

Proposition 3.1.10. *Let $\mathcal{K}$ be a simplicial complex constructed in Construction 3.1.3. Then $\langle \alpha_1, \dots, \alpha_n \rangle \subset H^*(\mathcal{Z}_{\mathcal{K}})$ is defined.*

Proof. Let $a_i = \sum_{\sigma_i \in S_{a_i}} c_{\sigma_i} \chi_{\sigma_i}$ be a representative cocycle for $\alpha_i \in \widetilde{H}^{p_i}(\mathcal{K}_{J_i})$ for each $i \in \{1, \dots, n-1\}$, as in Construction 3.1.3. As in Section 2.2.2, we construct a defining system $(a_{i,k})$ for the Massey product $\langle \alpha_1, \dots, \alpha_n \rangle \subset H^{p_1+\dots+p_k+|J_1 \cup \dots \cup J_k|+2}(\mathcal{Z}_{\mathcal{K}})$.

For $1 \leqslant i < k \leqslant n$, $(i,k) \neq (1,n)$, let $a_{i,k} \in C^{p_i+\dots+p_k}(\mathcal{K}_{J_i \cup \dots \cup J_k})$ be the cochain given by

$$a_{i,k} = \sum_{\sigma_i \in S_{a_i}} \sum_{\sigma_{i+1} \in \widetilde{S}_{a_{i+1}}} \cdots \sum_{\sigma_k \in \widetilde{S}_{a_k}} c_{\sigma_i} \dots c_{\sigma_k}\, \theta_{i,k}\, \chi_{\sigma_i \cup \dots \cup \sigma_k \setminus (v_{i+1} \cup \dots \cup v_k)} \tag{3.3}$$

where $\widetilde{S}_{a_i}$ is the set in (3.1), each vertex $v_i \in \sigma_i$ is the vertex chosen in Construction 3.1.3, and $\theta_{i,k}$ is the sign $\theta_{i,k} = 1$ when $i = k$ and

$$\theta_{i,k} = (-1)^{k-i}(-1)^{|J_i|(p_{i+1}+\dots+p_k)}(-1)^{|J_{i+1}|(p_{i+2}+\dots+p_k)} \dots (-1)^{|J_{k-1}|p_k} \cdot \\ \cdot \varepsilon(v_{i+1}, \sigma_{i+1}) \dots \varepsilon(v_k, \sigma_k). \tag{3.4}$$

Following the stellar subdivisions in Construction 3.1.3, for any simplex $\sigma_i \cup \sigma'_k$, the simplex $\sigma_i \cup \sigma_k \setminus v_k$ is contained in $\partial\, \mathrm{st}_{\mathcal{L}}(\sigma_i \cup \sigma'_k)$. Hence $\sigma_i \cup \dots \cup \sigma_k \setminus (v_{i+1} \cup \dots \cup v_k) \in \mathcal{K}$. Then since every coefficient c_{σ_i} is non-zero, the cochain $a_{i,k}$ is not trivial.

We show that $d(a_{i,k}) = \sum_{r=i}^{k-1} \overline{\overline{a_{i,r}}} \cdot a_{r+1,k}$, as in Section 2.2.2. By the definition of the coboundary map,

$$d(a_{i,k}) = \sum_{\sigma_i \in S_{a_i}} \sum_{\sigma_{i+1} \in \widetilde{S}_{a_{i+1}}} \cdots \sum_{\sigma_k \in \widetilde{S}_{a_k}} c_{\sigma_i} \dots c_{\sigma_k}\, \theta_{i,k} \cdot \\ \left(\sum_{\substack{j \in J_i \cup \dots \cup J_k \setminus (\sigma_i \cup \dots \cup \sigma_k \setminus (v_{i+1} \cup \dots \cup v_k)): \\ j \cup \sigma_i \cup \dots \cup \sigma_k \setminus (v_{i+1} \cup \dots \cup v_k) \in \mathcal{K}}} \varepsilon(j, j \cup \sigma_i \cup \dots \cup \sigma_k \setminus (v_{i+1} \cup \dots \cup v_k))\, \chi_{j \cup \sigma_i \cup \dots \cup \sigma_k \setminus (v_{i+1} \cup \dots \cup v_k)} \right). \tag{3.5}$$

We will show that the only non-zero summands in this summation are when $j \in v_{i+1} \cup \dots \cup v_k$. Suppose there is a vertex $j \in J_i \cup \dots \cup J_k \setminus (\sigma_i \cup \dots \cup \sigma_k)$ such that $j \cup \sigma_i \cup \dots \cup \sigma_k \setminus (v_{i+1} \cup \dots \cup v_k) \in \mathcal{K}$. Then there are two cases, that is, either $j \in J_i$ or $j \in J_l$ for $l \in \{i+1, \dots, k\}$.

If $j \in J_i$, then $j \cup \sigma_i \in \mathcal{K}^i$. Since a_i is a cocycle, $d(a_i) = 0$, so there are other simplices $\tau_1, \dots, \tau_s \in S_{a_i}$ such that there is a vertex $w_n \in J_i \setminus \tau_n$ with $w_n \cup \tau_n = j \cup \sigma_i$ for $n \in \{1, \dots, s\}$ and

$$0 = c_{\sigma_i}\, \varepsilon(j, j \cup \sigma_i)\, \chi_{j \cup \sigma_i} + \sum_{n=1}^{s} c_{\tau_n}\, \varepsilon(w_n, w_n \cup \tau_n)\, \chi_{w_n \cup \tau_n}.$$

So, similarly, there are summands in (3.5) corresponding to $w_n \cup \tau_n \cup \sigma_{i+1} \cup \dots \cup \sigma_k \setminus$

$(v_{i+1} \cup \cdots \cup v_k)$ since $w_n \cup \tau_n = j \cup \sigma_i$ and no simplex $j \cup \sigma_i \cup \cdots \cup \sigma_k \setminus (v_{i+1} \cup \cdots \cup v_k)$ was stellar subdivided for any $j \in J_i$ and any $\sigma_i \in S_{a_i}$. Therefore,

$$0 = c_{\sigma_i}\, \varepsilon(j, j \cup \sigma_i \cup \sigma_{i+1} \cup \cdots \cup \sigma_k \setminus (v_{i+1} \cup \cdots \cup v_k))\, \chi_{j\cup\sigma_i\cup\sigma_{i+1}\cup\cdots\cup\sigma_k\setminus(v_{i+1}\cup\cdots\cup v_k)} +$$
$$+ \sum_{n=1}^{s} c_{\tau_n}\, \varepsilon(w_n, w_n \cup \tau_n \cup \sigma_{i+1} \cup \cdots \cup \sigma_k \setminus (v_{i+1} \cup \cdots \cup v_k))\, \chi_{w_n\cup\tau_n\cup\sigma_{i+1}\cup\cdots\cup\sigma_k\setminus(v_{i+1}\cup\cdots\cup v_k)}.$$

If $j \in J_l$ for some $l \in \{i+1, \ldots, k\}$, then $j \cup \sigma_l \setminus v_l \in \mathcal{K}^l$ and in particular, $j \cup \sigma_l \setminus v_l \in P_{\sigma_l} \subset P_{a_l}$. So by Construction 3.1.3, $\sigma_i \cup j \cup \sigma_l \setminus v_l \notin \mathcal{K}$. Hence $j \cup \sigma_i \cup \cdots \cup \sigma_k \setminus (v_{i+1} \cup \cdots \cup v_k) \notin \mathcal{K}$ for any $j \in J_i \cup \cdots \cup J_k \setminus (v_{i+1} \cup \cdots \cup v_k)$.

Therefore, we only consider the case when $j \in v_{i+1} \cup \cdots \cup v_k$. Then,

$$d(a_{i,k}) = \sum_{\sigma_i \in S_{a_i}} \sum_{\sigma_{i+1} \in \widetilde{S}_{a_{i+1}}} \cdots \sum_{\sigma_k \in \widetilde{S}_{a_k}} c_{\sigma_i} \ldots c_{\sigma_k} \theta_{i,k} \cdot$$
$$\cdot \sum_{\substack{j \in v_{i+1}\cup\cdots\cup v_k : \\ j\cup\sigma_i\cup\cdots\cup\sigma_k\setminus(v_{i+1}\cup\cdots\cup v_k)\in\mathcal{K}}} \varepsilon(j, j \cup \sigma_i \cup \cdots \cup \sigma_k \setminus (v_{i+1} \cup \cdots \cup v_k))\, \chi_{j\cup\sigma_i\cup\cdots\cup\sigma_k\setminus(v_{i+1}\cup\cdots\cup v_k)}.$$

Let $j \in v_{i+1} \cup \cdots \cup v_k$ be denoted as v_{r+1} for $r \in \{i, \ldots, k-1\}$, then this may be rewritten as

$$d(a_{i,k}) = \sum_{r=i}^{k-1} \theta_{i,k} \sum_{\sigma_i \in S_{a_i}} \sum_{\sigma_{i+1} \in \widetilde{S}_{a_{i+1}}} \cdots \sum_{\sigma_k \in \widetilde{S}_{a_k}} c_{\sigma_i} \ldots c_{\sigma_k} \cdot$$
$$\cdot\, \varepsilon(v_{r+1}, \sigma_i \cup \cdots \cup \sigma_k \setminus (v_{i+1} \cup \cdots \cup \hat{v}_{r+1} \cup \cdots \cup v_k))\, \chi_{\sigma_i\cup\cdots\cup\sigma_k\setminus(v_{i+1}\cup\cdots\cup\hat{v}_{r+1}\cup\cdots\cup v_k)} \tag{3.6}$$

where $\hat{v}_{r+1}$ denotes that the vertex v_{r+1} is deleted from the sequence $v_{i+1}, \ldots, v_k$.

To show that $d(a_{i,k}) = \sum_{r=i}^{k-1} \overline{\overline{a_{i,r}}} \cdot a_{(r+1),k}$, consider the expression for $\sum_{r=i}^{k-1} \overline{\overline{a_{i,r}}} \cdot a_{(r+1),k}$,

$$\sum_{r=i}^{k-1} (-1)^{1+\overline{\deg}(a_{i,r})} \left(\sum_{\sigma_i \in S_{a_i}} \sum_{\sigma_{i+1} \in \widetilde{S}_{a_{i+1}}} \cdots \sum_{\sigma_r \in \widetilde{S}_{a_r}} c_{\sigma_i} \ldots c_{\sigma_r}\, \theta_{i,r}\, \chi_{\sigma_i\cup\cdots\cup\sigma_r\setminus(v_{i+1}\cup\cdots\cup v_r)} \right) \cdot$$
$$\cdot \left(\sum_{\sigma_{r+1} \in S_{a_{r+1}}} \sum_{\sigma_{r+2} \in \widetilde{S}_{a_{r+2}}} \cdots \sum_{\sigma_k \in \widetilde{S}_{a_k}} c_{\sigma_{r+1}} \ldots c_{\sigma_k}\, \theta_{r+1,k}\, \chi_{\sigma_{r+1}\cup\cdots\cup\sigma_k\setminus(v_{r+2}\cup\cdots\cup v_k)} \right).$$

For any $\sigma_{r+1} \in S_{a_{r+1}} \setminus \widetilde{S}_{a_{r+1}}$, we must have that $\sigma_{r+1} \in P_{a_{r+1}}$. Therefore $\sigma_i \cup \sigma_{r+1} \notin \mathcal{K}$. Hence by applying Lemma 2.1.26,

$$\sum_{r=i}^{k-1} \overline{\overline{a_{i,r}}} \cdot a_{(r+1),k} = \sum_{r=i}^{k-1} \sum_{\sigma_i \in S_{a_i}} \sum_{\sigma_{i+1} \in \widetilde{S}_{a_{i+1}}} \cdots \sum_{\sigma_k \in \widetilde{S}_{a_k}} (-1)^{1+\overline{\deg}(a_{i,r})} \cdot$$
$$\cdot\, (-1)^{|J_i\cup\cdots\cup J_r|(p_{r+1}+\cdots+p_k+1)} c_{\sigma_i} \ldots c_{\sigma_k}\, \theta_{i,r}\, \theta_{r+1,k}\, \chi_{\sigma_i\cup\cdots\cup\sigma_k\setminus(v_{i+1}\cup\cdots\cup\hat{v}_{r+1}\cup\cdots\cup v_k)}. \tag{3.7}$$

Since $\overline{\deg}(a_{i,r}) = |J_i \cup \cdots \cup J_r| + p_i + \cdots + p_r + 1$,

$$(-1)^{1+\overline{\deg}(a_{i,r})}(-1)^{|J_i\cup\cdots\cup J_r|(p_{r+1}+\cdots+p_k+1)} = (-1)^{(p_i+\cdots+p_r)+|J_i\cup\cdots\cup J_r|(p_{r+1}+\cdots+p_k)}.$$

To show that $d(a_{i,k}) = \sum_{r=i}^{k-1} \overline{\overline{a_{i,r}}} \cdot a_{(r+1),k}$, we need to prove that (3.5) is equal to (3.7). Thus we want to show that

$$\theta_{i,k}\ \varepsilon(v_{r+1}, \sigma_i \cup \cdots \cup \sigma_k \setminus (v_{i+1} \cup \cdots \cup \hat{v}_{r+1} \cup \cdots \cup v_k)) \tag{3.8}$$
$$= (-1)^{(p_i+\cdots+p_r)+|J_i\cup\cdots\cup J_r|(p_{r+1}+\cdots+p_k)}\theta_{i,r}\ \theta_{r+1,k}. \tag{3.9}$$

Since

$$\theta_{i,r} = (-1)^{r-i}(-1)^{|J_i|(p_{i+1}+\cdots+p_r)}\cdots(-1)^{|J_{r-1}|p_r}\ \varepsilon(v_{i+1},\sigma_{i+1})\cdots\varepsilon(v_r,\sigma_r)$$

and

$$\theta_{r+1,k} = (-1)^{k-r-1}(-1)^{|J_{r+1}|(p_{r+2}+\cdots+p_k)}\cdots(-1)^{|J_{k-1}|p_k}\ \varepsilon(v_{r+2},\sigma_{r+2})\cdots\varepsilon(v_k,\sigma_k),$$

the expression (3.9) becomes

$$(-1)^{k-i-1}(-1)^{(p_i+\cdots+p_r)}(-1)^{|J_i|(p_{i+1}+\cdots+p_k)}(-1)^{|J_{i+1}|(p_{i+2}+\cdots+p_k)}\cdots(-1)^{|J_{k-1}|p_k}$$
$$\varepsilon(v_{i+1},\sigma_{i+1})\ldots\varepsilon(v_r,\sigma_r)\varepsilon(v_{r+2},\sigma_{r+2})\ldots\varepsilon(v_k,\sigma_k).$$

This can be rewritten as

$$(-1)^{p_i+\cdots+p_r-1}\ \varepsilon(v_{r+1},\sigma_{r+1})\ \theta_{i,k}. \tag{3.10}$$

Next consider (3.8). For any $r \in \{i,\ldots,k-1\}$, suppose $v_{r+1} \in \sigma_{r+1}$ is the lth vertex in the vertex set of $\sigma_i \cup \cdots \cup \sigma_k \setminus (v_{i+1} \cup \cdots \cup \hat{v}_{r+1} \cup \cdots \cup v_k)$. Then

$$\varepsilon(v_{r+1}, \sigma_i \cup \cdots \cup \sigma_k \setminus (v_{i+1} \cup \cdots \cup \hat{v}_{r+1} \cup \cdots \cup v_k)) = (-1)^{l-1}.$$

Since $v_{r+1} \in \sigma_{r+1}$, l is given by

$$l = |\sigma_i| + (|\sigma_{i+1}| - 1) + \cdots + (|\sigma_r| - 1) + l_{r+1}$$

where l_{r+1} is the position of v_{r+1} in σ_{r+1} and $|\sigma_i| = p_i + 1$ for every i. So $l = (p_i + 1) + p_{i+1} + \cdots + p_r + l_{r+1}$, and hence

$$\varepsilon(v_{r+1}, \sigma_i\cup\cdots\cup\sigma_k\setminus(v_{i+1}\cup\cdots\cup\hat{v}_{r+1}\cup\cdots\cup v_k)) = (-1)^{l-1} = (-1)^{p_i+\cdots+p_r+1}\ \varepsilon(v_{r+1},\sigma_{r+1}). \tag{3.11}$$

Thus (3.8) may be rewritten as $(-1)^{p_i+\cdots+p_r+1}\ \theta_{i,k}\ \varepsilon(v_{r+1},\sigma_{r+1})$, which is equal to (3.10).

Hence (3.5) is equal to (3.7), and so $d(a_{i,k}) = \sum_{r=i}^{k-1} \overline{\overline{a_{i,r}}} \cdot a_{(r+1),k}$. Therefore $(a_{i,k})$ corresponds to a defining system for $\langle \alpha_1, \ldots, \alpha_n \rangle$. □

Since the n-Massey product $\langle \alpha_1, \ldots, \alpha_n \rangle$ is defined, it remains to show that this Massey product is non-trivial. To do this, we first verify that there is at least one non-zero cohomology class in $\langle \alpha_1, \ldots, \alpha_n \rangle$. Then we show that any other element of $\langle \alpha_1, \ldots, \alpha_n \rangle$ is also non-zero.

Let $\omega \in C^{p_1+\cdots+p_n+1}(\mathcal{K}_{J_1\cup\cdots\cup J_n})$ be the associated cocycle for the above defining system. Then,

$$\omega = \sum_{r=1}^{n-1} \overline{\overline{a_{1,r}}} \cdot a_{(r+1),n}.$$

In particular,

$$\omega = \sum_{r=1}^{n-1} \sum_{\sigma_1 \in S_{a_1}} \sum_{\sigma_2 \in \widetilde{S}_{a_2}} \cdots \sum_{\sigma_n \in \widetilde{S}_{a_n}} \theta\, \chi_{\sigma_1 \cup \cdots \cup \sigma_n \setminus (v_2 \cup \cdots \cup v_r \cup v_{r+2} \cup \cdots \cup v_n)} \tag{3.12}$$

as in (3.7) when $i = 1$ and $k = n$, where θ is the coefficient

$$\theta = c_{\sigma_1} \ldots c_{\sigma_n}\, (-1)^{p_1+\cdots+p_r-1}\, \varepsilon(v_{r+1}, \sigma_{r+1})\, \theta_{1,n}$$

as in (3.10).

Lemma 3.1.11. *The class $[\omega] \in \widetilde{H}^{p_1+\ldots+p_n+1}(\mathcal{K}_{J_1\cup\cdots\cup J_n})$ is non-zero.*

Proof. We construct a cycle $x \in C_{p_1+\ldots+p_n+1}(\mathcal{K}_{J_1\cup\cdots\cup J_n})$ such that $\omega(x) \neq 0$. If $[x]$ is a non-zero homology class, then this concludes that $[\omega] \neq 0$.

Let $\sigma_i \in \widetilde{S}_{a_i}$ for $2 \leqslant i < n$ be fixed. By assumption in Construction 3.1.3, J_1 and J_n were chosen such that there exist $\sigma_1 \in S_{a_1}$ and $\sigma_n \in P_{a_n}$ that are maximal in $\mathcal{K}_{J_1}$, $\mathcal{K}_{J_n}$, respectively. Since $\alpha_1 \in \widetilde{H}^{p_1}(\mathcal{K}_{J_1})$ is non-zero, there is a homology class $[x_1] \in \widetilde{H}_{p_1}(\mathcal{K}_{J_1})$ with a representative cycle $x_1 \in C_{p_1}(\mathcal{K}_{J_1})$ such that $a_1(x_1) \neq 0$. As for any general chain, we can write x_1 as

$$x_1 = \sum_{\tilde{\sigma}_1 \in S_{x_1}} c_{\tilde{\sigma}_1} \Delta_{\tilde{\sigma}_1}$$

for a collection of p_1-simplices $S_{x_1} \subset \mathcal{K}_{J_1}$ and non-zero coefficients $c_{\tilde{\sigma}_1}$.

Let $\partial(\sigma_2 \cup \sigma_n)$ be the boundary of the simplex $\sigma_2 \cup \sigma_n$. Then let $x_2 \in C_{p_2+p_n}(\partial(\sigma_2 \cup \sigma_n))$ be the cycle

$$x_2 = \sum_{w_2 \in \sigma_2 \cup \sigma_n} c_{w_2} \Delta_{\sigma_2 \cup \sigma_n \setminus w_2}$$

for vertices $w_2 \in \sigma_2 \cup \sigma_n$ and non-zero coefficients c_{w_2}, so that $[x_2] \in \widetilde{H}_{p_2+p_n}(\partial(\sigma_2 \cup \sigma_n))$ is the spherical class. Similarly for $3 \leqslant i \leqslant n-1$, let $x_i \in C_{p_i-1}(\partial(\sigma_i))$ be the cycle given by

$$x_i = \sum_{w_i \in \sigma_i} c_{w_i} \Delta_{\sigma_i \setminus w_i}$$

for vertices $w_i \in \sigma_i$ and non-zero coefficients c_{w_i}.

Let $x \in C_{p_1+\ldots+p_n+1}(\mathcal{K}_{J_1\cup\cdots\cup J_n})$ be the chain

$$x = \sum_{\tilde{\sigma}_1 \in S_{x_1}} \sum_{w_2 \in \sigma_2 \cup \sigma_n} \sum_{w_3 \in \sigma_3} \cdots \sum_{w_{n-1} \in \sigma_{n-1}} c_{\tilde{\sigma}_1} c_{w_2} \cdots c_{w_{n-1}} \Delta_{\tilde{\sigma}_1 \cup \sigma_2 \cup \cdots \cup \sigma_{n-1} \cup \sigma_n \setminus (w_2 \cup \cdots \cup w_{n-1})}.$$

For ease of notation, let T_x be the set of simplices t that supports x, where t has the form

$$t = \tilde{\sigma}_1 \cup \sigma_2 \cup \cdots \cup \sigma_{n-1} \cup \sigma_n \setminus (w_2 \cup \cdots \cup w_{n-1}) \tag{3.13}$$

for a p_i-simplex $\tilde{\sigma}_1 \in S_{x_1}$, and a choice of vertices $w_2 \in \sigma_2 \cup \sigma_n$, $w_i \in \sigma_i$ for $3 \leqslant i \leqslant n-1$. We will first show that x is a cycle, before showing that it is also not a boundary. The boundary $\partial(x)$ is given by

$$\partial(x) = \sum_{\tilde{\sigma}_1 \in S_{x_1}} \sum_{w_2 \in \sigma_2 \cup \sigma_n} \sum_{w_3 \in \sigma_3} \cdots \sum_{w_{n-1} \in \sigma_{n-1}} \sum_{v \in t} \varepsilon(v,t)\; c_{\tilde{\sigma}_1} c_{w_2} \cdots c_{w_{n-1}} \Delta_{t \setminus v}.$$

Since $\tilde{\sigma}_1 \subset J_1$, $\sigma_i \subset J_i$ for $2 \leqslant i \leqslant n$, and $J_i \cap J_j = \varnothing$ for $i \neq j$, every choice of vertex $v \in t$ is contained in a simplex $\tilde{\sigma}_1$ or σ_i for $2 \leqslant i \leqslant n$. If $v \in \tilde{\sigma}_1$, then $\varepsilon(v,t) = \varepsilon(v, \tilde{\sigma}_1)$. Also like in (3.11), if $v \in \sigma_i$ for $i > 1$, then

$$\varepsilon(v,t) = \begin{cases} (-1)^{p_1+1}\, \varepsilon(v, \sigma_2) & \text{if } w_2 \in \sigma_n \text{ and } i = 2, \\ (-1)^{p_1+\cdots+p_{i-1}+2}\, \varepsilon(v, \sigma_i \setminus \tilde{w}_i) & \text{if } w_2 \in \sigma_n \text{ and } i > 2, \\ (-1)^{p_1+\cdots+p_{n-1}+1}\, \varepsilon(v, \sigma_n) & \text{if } w_2 \in \sigma_2 \text{ and } i = n, \\ (-1)^{p_1+\cdots+p_{i-1}+1}\, \varepsilon(v, \sigma_i \setminus w_i) & \text{if } w_2 \in \sigma_2 \text{ and } i < n \end{cases}$$

where $\tilde{w}_i = w_i$ for $1 < i < n$, and $\tilde{w}_n = w_2$. We rewrite $\partial(x)$ as

$$\partial(x) = \sum_{\tilde{\sigma}_1 \in S_{x_1}} \sum_{w_2 \in \sigma_2 \cup \sigma_n} \sum_{w_3 \in \sigma_3} \cdots \sum_{w_{n-1} \in \sigma_{n-1}} \sum_{i=1}^{n} \sum_{v \in \tilde{\sigma}_i \setminus \tilde{w}_i} \varepsilon(v,t)\; c_{\tilde{\sigma}_1} c_{w_2} \cdots c_{w_{n-1}} \Delta_{t \setminus v}$$

where $\tilde{\sigma}_1 \setminus \tilde{w}_1 = \tilde{\sigma}_1$ and $\tilde{\sigma}_i = \sigma_i$ for $i > 1$. Let $\Delta_{t\setminus v}|_J$ denote the restriction of $\Delta_{t \setminus v}$ to its vertices in $J \subset [m]$. Then

$$\partial(x) = \sum_{\tilde{\sigma}_1 \in S_{x_1}} \sum_{w_2 \in \sigma_2 \cup \sigma_n} \sum_{w_3 \in \sigma_3} \cdots \sum_{w_{n-1} \in \sigma_{n-1}} \left(\sum_{i=1}^{n} \sum_{v \in \tilde{\sigma}_i \setminus \tilde{w}_i} \varepsilon(v,t)\; c_{\tilde{\sigma}_1} c_{w_2} \cdots c_{w_{n-1}} (\Delta_{t\setminus v}|_{J_i})(\Delta_{t\setminus v}|_{V(\mathcal{K})\setminus J_i}) \right).$$

We rearrange $\partial(x)$ into four collections of summands, one in which $v \in \tilde{\sigma}_1$, another for $v \in \sigma_2 \cup \sigma_n \setminus w_2$, and two more when $v \in \sigma_i \setminus w_i$ for $3 \leqslant i \leqslant n-1$ where either $w_2 \in \sigma_2$ or $w_2 \in \sigma_2$. Then writing $\varepsilon(v,t)$ more explicitly,

$$\partial(x) = \sum_{w_2 \in \sigma_2 \cup \sigma_n} \sum_{w_3 \in \sigma_3} \cdots \sum_{w_{n-1} \in \sigma_{n-1}} c_{w_2} \cdots c_{w_{n-1}} (\Delta_{t\setminus v}|_{V(\mathcal{K})\setminus J_1}) \left(\sum_{\tilde{\sigma}_1 \in S_{x_1}} \sum_{v \in \tilde{\sigma}_1} \varepsilon(v, \tilde{\sigma}_1)\; c_{\tilde{\sigma}_1} (\Delta_{t\setminus v}|_{J_1}) \right) +$$

$$
\begin{aligned}
&+ \sum_{\tilde{\sigma}_1 \in S_{x_1}} \sum_{w_3 \in \sigma_3} \cdots \sum_{w_{n-1} \in \sigma_{n-1}} c_{\tilde{\sigma}_1} c_{w_3} \cdots c_{w_{n-1}} (-1)^{p_1+p_3 \cdots + p_{n-1}+1} (\Delta_{t \setminus v}|_{V(\mathcal{K}) \setminus J_2 \cup J_n}) \\
&\qquad \left(\sum_{w_2 \in \sigma_2 \cup \sigma_n} \sum_{v \in \sigma_2 \cup \sigma_n \setminus w_2} \varepsilon(v, \sigma_2 \cup \sigma_n \setminus w_2)\, c_{w_2} (\Delta_{t \setminus v}|_{J_2 \cup J_n}) \right) + \\
&+ \sum_{\tilde{\sigma}_1 \in S_{x_1}} \sum_{w_2 \in \sigma_2} \sum_{w_3 \in \sigma_3} \cdots \sum_{w_{n-1} \in \sigma_{n-1}} c_{\tilde{\sigma}_1} c_{w_2} \cdots c_{w_{n-1}} \\
&\qquad \left(\sum_{i=3}^{n-1} (-1)^{p_1+\cdots+p_{i-1}+1} (\Delta_{t \setminus v}|_{V(\mathcal{K}) \setminus J_i}) \left(\sum_{v \in \sigma_i \setminus w_i} \varepsilon(v, \sigma_i \setminus w_i)(\Delta_{t \setminus v}|_{J_i}) \right) \right) + \\
&+ \sum_{\tilde{\sigma}_1 \in S_{x_1}} \sum_{w_2 \in \sigma_n} \sum_{w_3 \in \sigma_3} \cdots \sum_{w_{n-1} \in \sigma_{n-1}} c_{\tilde{\sigma}_1} c_{w_2} \cdots c_{w_{n-1}} \\
&\qquad \left(\sum_{i=3}^{n-1} (-1)^{p_1+\cdots+p_{i-1}+2} (\Delta_{t \setminus v}|_{V(\mathcal{K}) \setminus J_i}) \left(\sum_{v \in \sigma_i \setminus w_i} \varepsilon(v, \sigma_i \setminus w_i)(\Delta_{t \setminus v}|_{J_i}) \right) \right).
\end{aligned}
$$

Each collection of summands can be written in terms of $\partial(x_i)$, that is

$$
\begin{aligned}
\partial(x) = &\sum_{w_2 \in \sigma_2 \cup \sigma_n} \sum_{w_3 \in \sigma_3} \cdots \sum_{w_{n-1} \in \sigma_{n-1}} c_{w_2} \cdots c_{w_{n-1}} (\Delta_{t \setminus v}|_{V(\mathcal{K}) \setminus J_1})\, \partial(x_1) + \\
&+ \sum_{\tilde{\sigma}_1 \in S_{x_1}} \sum_{w_3 \in \sigma_3} \cdots \sum_{w_{n-1} \in \sigma_{n-1}} c_{\tilde{\sigma}_1} c_{w_3} \cdots c_{w_{n-1}} (-1)^{p_1+p_3 \cdots + p_{n-1}+1} (\Delta_{t \setminus v}|_{V(\mathcal{K}) \setminus J_2 \cup J_n})\, \partial(x_2) + \\
&+ \sum_{\tilde{\sigma}_1 \in S_{x_1}} \sum_{w_2 \in \sigma_2} \sum_{i=3}^{n-1} \sum_{w_3 \in \sigma_3} \cdots \widehat{\sum_{w_i \in \sigma_i}} \cdots \sum_{w_{n-1} \in \sigma_{n-1}} c_{\tilde{\sigma}_1} c_{w_2} \cdots \widehat{c_{w_i}} \cdots c_{w_{n-1}} \\
&\qquad \left((-1)^{p_1+\cdots+p_{i-1}+1} (\Delta_{t \setminus v}|_{V(\mathcal{K}) \setminus J_i})\, \partial(x_i) \right) + \\
&+ \sum_{\tilde{\sigma}_1 \in S_{x_1}} \sum_{w_2 \in \sigma_n} \sum_{i=3}^{n-1} \sum_{w_3 \in \sigma_3} \cdots \widehat{\sum_{w_i \in \sigma_i}} \cdots \sum_{w_{n-1} \in \sigma_{n-1}} c_{\tilde{\sigma}_1} c_{w_2} \cdots \widehat{c_{w_i}} \cdots c_{w_{n-1}} \\
&\qquad \left((-1)^{p_1+\cdots+p_{i-1}+2} (\Delta_{t \setminus v}|_{V(\mathcal{K}) \setminus J_i})\, \partial(x_i) \right)
\end{aligned}
$$

where $\widehat{\ }$ denotes omission. Since every x_i is a cycle, $\partial(x_i) = 0$. Therefore also $\partial(x) = 0$ and x is a cycle.

We will show that x is not a boundary. In particular, we will show that the link of a simplex $s \in T_x$ is empty, so that x cannot be a boundary of a collection of higher dimensional simplices. Let us consider t when $\tilde{\sigma}_1 = \sigma_1 \in S_{a_1}$, and $w_i = v_i$ for $i = 2, \ldots, n-1$. Let

$$
s = \sigma_1 \cup \cdots \cup \sigma_{n-1} \cup \sigma_n \setminus (v_2 \cup \cdots \cup v_{n-1}). \tag{3.14}
$$

Recall that in Construction 3.1.3, $a_1 \in C^{p_1}(\mathcal{K}_1)$ and $a_n \in C^{p_n}(\mathcal{K}_n)$ were chosen so that there are simplices $\sigma_1 \in S_{a_1}$ and $\sigma_n \in P_{a_n}$ that are maximal in $\mathcal{K}_{J_1}$ and $\mathcal{K}_{J_n}$ respectively. Hence we have that $\mathrm{lk}_{\mathcal{K}_{J_1}}(\sigma_1) = \varnothing$ and $\mathrm{lk}_{\mathcal{K}_{J_n}}(\sigma_n) = \varnothing$. Suppose there is a vertex $j \in \mathrm{lk}_{\mathcal{K}_{J_1 \cup \cdots \cup J_n}}(s)$. Then there is an $m \in \{2, \ldots, n-1\}$ such that $j \in J_m$. Therefore $\sigma_m \cup j \setminus v_m \in K_{J_m}$, and in particular $\sigma_m \cup j \setminus v_m \in P_{\sigma_m}$. Thus there would have been a stellar subdivision made at the simplex $\sigma_1 \cup \sigma_m \cup j \setminus v_m$ during the construction of $\mathcal{K}$. So $\sigma_1 \cup \sigma_m \cup j \setminus v_m \notin \mathcal{K}$, and subsequently also $\sigma_1 \cup \sigma_m \cup j \notin \mathcal{K}$. This contradicts the assumption that $j \in \mathrm{lk}_{\mathcal{K}_{J_1 \cup \cdots \cup J_n}}(s)$. Hence $\mathrm{lk}_{\mathcal{K}_{J_1 \cup \cdots \cup J_n}}(s) = \varnothing$ and the cycle x cannot be a boundary. Therefore, the homology class $[x] \in \widetilde{H}_{p_1+\cdots+p_n+1}(\mathcal{K}_{J_1\cup\cdots\cup J_n})$ is non-zero.

Furthermore by (3.12), the summands of ω are of the form

$$\chi_{\sigma_1\cup\cdots\cup\sigma_n\setminus(v_2\cup\cdots\cup v_r\cup v_{r+2}\cup\cdots\cup v_n)}$$

for $\sigma_1 \in S_{a_1}$, $\sigma_i \in \widetilde{S}_{a_i}$ for $2 \leqslant i \leqslant n$. Thus the only non-zero terms in the evaluation of ω on x are when $r = 1$, $w_i = v_i$ for $3 \leqslant i \leqslant n-1$, and $w_2 = v_n$. Therefore,

$$\begin{aligned}
\omega(x) &= \sum_{\sigma_1\in S_{a_1}} \sum_{\tilde{\sigma}_1\in S_{x_1}} \theta c_{\tilde{\sigma}_1} c_{v_2}\cdots c_{v_{n-1}} \chi_{\sigma_1}(\Delta_{\tilde{\sigma}_1}) \\
&= c_{\sigma_2}\ldots c_{\sigma_n}\ (-1)^{p_1-1}\ \varepsilon(v_2,\sigma_2)\ \theta_{1,n} c_{v_2}\cdots c_{v_{n-1}} \sum_{\sigma_1\in S_{a_1}} \sum_{\tilde{\sigma}_1\in S_{x_1}} c_{\sigma_1} c_{\tilde{\sigma}_1} \chi_{\sigma_1}(\Delta_{\tilde{\sigma}_1}) \\
&= c_{\sigma_2}\ldots c_{\sigma_n}\ (-1)^{p_1-1}\ \varepsilon(v_2,\sigma_2)\ \theta_{1,n} c_{v_2}\cdots c_{v_{n-1}} a_1(x_1).
\end{aligned}$$

Since $a_1(x_1) \neq 0$ and the coefficient $c_{\sigma_2}\ldots c_{\sigma_n}\ (-1)^{p_1-1}\ \varepsilon(v_2,\sigma_2)\ \theta_{1,n} c_{v_2}\cdots c_{v_{n-1}}$ is non-zero, $\omega(x)$ is also non-zero. Here, x a cycle representative of a non-zero homology class $[x] \in \widetilde{H}_{p_1+\cdots+p_n+1}(\mathcal{K}_{J_1\cup\cdots\cup J_n})$, so $[\omega]$ must also be non-zero. □

Example 3.1.12. Let $\mathcal{K}$ be the simplicial complex as in Figure 3.5a, where stellar subdivisions were performed at the simplices $\sigma_1 \cup \sigma_2'$, $\sigma_2 \cup \sigma_3'$. That is, $S_{a_1} = \{\sigma_1\}$, $S_{a_2} = \{\sigma_2\}$, $S_{a_3} = \{\sigma_3\}$, $P_{a_2} = \{\sigma_2'\}$, $P_{a_3} = \{\sigma_3'\}$. As in (3.13), the cycle x is supported on a collection of simplices of the form

$$t = \tilde{\sigma}_1 \cup \sigma_2 \cup \sigma_3' \setminus (w_2)$$

for simplices $\tilde{\sigma}_1 \in \{\sigma_1, \sigma_1'\}$, and a choice of vertex $w_2 \in \sigma_2 \cup \sigma_3'$. Therefore the set of simplices T_x that supports x contains $\sigma_1 \cup \sigma_2$, $\sigma_1' \cup \sigma_2$, $\sigma_1' \cup \sigma_3'$ and $\sigma_1 \cup \sigma_3'$, as shown in Figure 3.5b. The simplex $s \in T_x$ is $\sigma_1 \cup \sigma_3'$, and the stellar subdivisions performed to construct $\mathcal{K}$ secure that $\mathrm{lk}_{\mathcal{K}}(s) = \varnothing$.

As in (3.12), summands of ω are of the form $\chi_{\sigma_1\cup\sigma_2\cup\sigma_3\setminus(v_2)} = \chi_{\sigma_1\cup\sigma_3}$ and $\chi_{\sigma_1\cup\sigma_2\cup\sigma_3\setminus(v_3)} = \chi_{\sigma_1\cup\sigma_2}$. Therefore ω evaluates on exactly one simplex of T_x, $\sigma_1\cup\sigma_2$. So $\omega(x) \neq 0$.

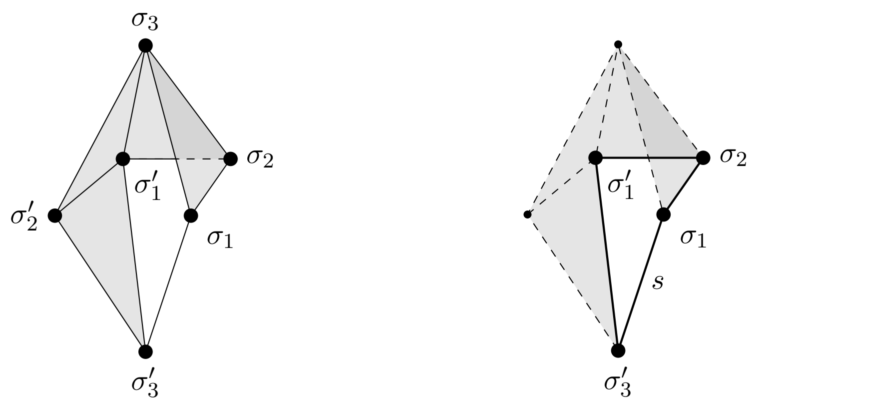
As in Section 2.2.2, the above lemma shows that the cohomology class

$$[\omega] \in H^{p_1+\cdots+p_n+|J_n\cup\cdots\cup J_n|+2}(\mathcal{Z}_{\mathcal{K}})$$

is non-zero. It remains to show that every other element of the Massey product $\langle\alpha_1,\ldots,\alpha_n\rangle$ is also non-zero.

Proposition 3.1.13. *The n-Massey product $\langle\alpha_1,\ldots,\alpha_n\rangle \subset H^*(\mathcal{Z}_{\mathcal{K}})$ is non-trivial.*

Proof. We will show that for the cycle x constructed in Lemma 3.1.11 and any element $[\omega'] \in \langle\alpha_1,\ldots,\alpha_n\rangle$, $[\omega']([x])$ is non-zero. Since $[x] \in \widetilde{H}_{p_1+\cdots+p_n+1}(\mathcal{K}_{J_1\cup\cdots\cup J_n})$ was shown to be non-zero, then this implies that $[\omega']$ is non-zero.



(a) A simplicial complex $\mathcal{K}$ constructed by stellar subdivisions at $\sigma_1 \cup \sigma_2'$ and $\sigma_2 \cup \sigma_3'$.

(b) The cycle x is supported by simplices that include s, whose link is empty.

Figure 3.5

Let $(a_{i,k})$ be any defining system of $\langle \alpha_1, \ldots, \alpha_n \rangle$, where $a_{i,k} \in C^{p_i+\cdots+p_k}(\mathcal{K}_{J_i \cup \cdots \cup J_k})$ as in Section 2.2.2. Also let $S_{a_{i,k}}$ be the set of $(p_i + \cdots + p_k)$-simplices such that

$$a_{i,k} = \sum_{\sigma \in S_{a_{i,k}}} c_\sigma \chi_\sigma$$

for non-zero coefficients $c_\sigma \in \mathbf{k}$. The differential increases the degree of cochains by one, which in particular corresponds to adding a vertex to the simplices in $S_{a_{i,k}}$. Since the base of the defining system is given by $a_i = \sum_{\sigma_i \in S_{a_i}} c_{\sigma_i} \chi_{\sigma_i}$ for every i, for any $\sigma_i \in S_{a_i}$ there is at least one simplex in $S_{a_{2,n}}$ of the form $\sigma_2 \cup \cdots \cup \sigma_n \setminus (u_2 \cup \cdots \cup u_{n-1})$ for $\sigma_i \in \widetilde{S}_{a_i}$ and vertices $u_i \in \sigma_2 \cup \cdots \cup \sigma_n$ for $2 \leqslant i \leqslant n$, $u_i \neq u_j$. So $a_1 a_{2,n}$ contains a summand supported on a simplex of the form

$$\sigma = \sigma_1 \cup \sigma_2 \cup \cdots \cup \sigma_{n-1} \cup \sigma_n \setminus (u_2 \cup \cdots \cup u_{n-1}) \tag{3.15}$$

for $\sigma_1 \in S_{a_1}$, $\sigma_i \in \widetilde{S}_{a_i}$ and vertices $u_i \in \sigma_2 \cup \cdots \cup \sigma_n$ for $2 \leqslant i \leqslant n$, $u_i \neq u_j$.

Let ω' be the associated cocycle for this defining system $(a_{i,k})$,

$$\omega' = \sum_{\tau \in S_{\omega'}} c_\tau \chi_\tau$$

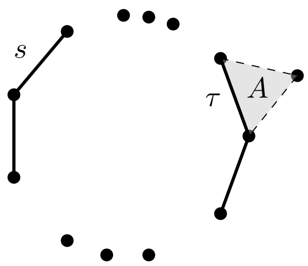
for non-zero coefficients $c_\tau \in \mathbf{k}$. By the definition of an associated cocycle, ω' has a summand $a_1 a_{2,n}$. Hence $S_{\omega'}$ contains σ in (3.15). We would like to compare the simplices σ in (3.15) and t in (3.13). Specifically, we want to show that $S_{\omega'} \cap T_x \neq \varnothing$. Since $\sigma_1 \in S_{a_1}$, $\sigma_i \in \widetilde{S}_{a_i}$ for $2 \leqslant i \leqslant n$, we have that $\sigma_i \notin P_{a_i}$ for every $i = 1, \ldots, n$. Therefore $\sigma_1 \cup \cdots \cup \sigma_n \in \mathcal{K}$ since it was not removed by stellar subdivision in Construction 3.1.3. Then both σ and t are $(p_1 + \cdots + p_n + 1)$-dimensional faces of $\sigma_1 \cup \cdots \cup \sigma_n$. If there is no appropriate set of choices of u_i and w_i such that $\sigma = t$, then there is a cochain $b \in C^{p_1+\cdots+p_n}(\mathcal{K})$ such that $\omega' + d(b)$ contains a summand χ_t and no summand χ_σ.

Rename this cocycle as ω'. Then $t \in S_{\omega'} \cap T_x$. Therefore, the evaluation $\omega'(x)$ has at least one non-zero term. However, there could be other simplices in $S_{\omega'} \cap T_x$ and so we cannot conclude that $\omega'(x)$ is non-zero.

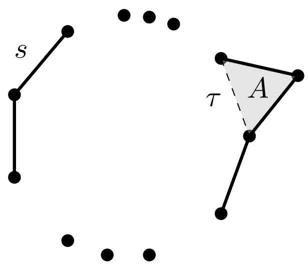
We construct a cocycle ω'' and a cycle x' such that $[\omega''] = [\omega']$, $[x'] = [x]$, and $\omega''(x') \neq 0$. Set an order on the simplices T_x such that the simplex s in (3.14) is the last in this order. We work inductively to remove simplices in $S_{\omega'} \cap T_x$ so that $S_{\omega''} \cap T_{x'}$ contains only the simplex s. Let us start at the first simplex $\tau \in T_x$.

Suppose that both T_x and $S_{\omega'}$ contain $\tau \neq s$. Then there is a non-zero term in the evaluation of ω' on x. The link of τ is non-empty, since $\tau \neq s$ and because $\mathcal{K}$ was constructed from the join of simplicial complexes. So there is a $(p_1 + \ldots + p_k + 2)$-dimensional simplex $A \in \mathcal{K}_{J_1 \cup \cdots \cup J_n}$ containing τ in its boundary.

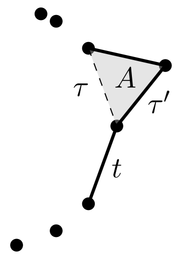
Suppose $S_{\omega'}$ does not contain any other face of A. Then replace x by x', where the simplex $\tau \in T_x$ is replaced by the $(p_1 + \ldots + p_k + 1)$-simplices in $\partial(A) \setminus \tau$ to form $T_{x'}$. Therefore x' is the cycle $x - c_\tau\, \epsilon(v, A)\, \partial(A)$, where c_τ is the coefficient of the summand Δ_τ in x, v is the vertex such that $v \cup \tau = A$, and $\epsilon(v, A)$ is the coefficient of Δ_τ in $\partial(A)$. Thus $[x] = [x']$. Moreover, $S_{\omega'}$ and $T_{x'}$ do not both contain the simplex τ.



(a) The cycle x



(b) The cycle x'



(c) There may be a simplex τ' that shares a boundary with τ

Figure 3.6: If the link of τ is non-empty, then the cycle x can be changed to x'

Alternatively, suppose $S_{\omega'}$ contains another face τ' of A. Since x is a cycle, there is another simplex $t \in T_x$ such that $\tau \cap \tau' \subset t$ (as shown in Figure 3.6c). If $t < \tau$ in the order on T_x, then again we create the cycle x' from x by replacing τ in T_x by the $(p_1 + \ldots + p_k + 1)$-simplices in $\partial(A) \setminus \tau$. Then τ is not contained in x'. By induction, $\tau' \notin T_x$, otherwise we would have considered it before τ. Similarly since $t < \tau$, the induction process means that $t \notin S_{\omega'}$.

On the other hand, suppose $S_{\omega'}$ contains another face τ' of A and $t > \tau$ in the order on T_x. Then $t \in \mathrm{st}_{\mathcal{K}_{J_1 \cup \cdots \cup J_n}}(\tau \cap \tau')$. Let $\omega'' = \omega' - c_\tau\, \varepsilon(\tau \setminus \tau \cap \tau', \tau)\, d(\chi_{\tau \cap \tau'})$ with a coefficient $-c_\tau$ where c_τ is the coefficient of the summand χ_τ in ω' and $\varepsilon(\tau \setminus \tau \cap \tau', \tau)$ is its coefficient in $d(\chi_{\tau \cap \tau'})$. So ω'' does not contain a summand χ_τ, but does have a summand χ_t, and $[\omega''] = [\omega']$. We continue the induction process on the next simplex in T_x.

Finally we come to a cocycle ω'' and a cycle x' such that $S_{\omega''} \cap T_{x'} = s$. Since the cycle x

only depended on particular choices of σ_i for all $i \in \{1, \ldots, n\}$, and since the link of the simplex s in $\mathcal{K}_{J_1 \cup \cdots \cup J_n}$ is empty, the induction process terminates. The evaluation of ω'' on x' only has one non-zero term, which is supported by the simplex s. Thus $\omega''(x') \neq 0$, and so $[\omega''] = [\omega']$ is non-zero. □

Example 3.1.14. For $i = 1, 2, 3$, let $\mathcal{K}^i$ be the simplicial complexes as in Examples 3.1.6 and 3.1.7. For $a_1 = \chi_1$, $a_2 = \chi_3 + \chi_4 + \chi_5$, $a_3 = \chi_7$, we previously saw that $S_{a_1} = \{1\}$ and $\widetilde{S}_{a_2} = S_{a_2}^{(1)} = \{\{3\}\}$. Therefore by (3.3),

$$a_{1,2} = \theta_{1,2}\chi_1 = -\chi_1.$$

Similarly $S_{a_2} = \{\{3\}, \{4\}, \{5\}\}$ and $\widetilde{S}_{a_3} = S_{a_3} = \{7\}$ so

$$a_{2,3} = \theta_{2,3}(\chi_3 + \chi_4 + \chi_5) = -(\chi_3 + \chi_4 + \chi_5).$$

Let ω be the associated cocycle for this defining system. Then

$$\omega = -\chi_1(\chi_3 + \chi_4 + \chi_5) - \chi_1\chi_7.$$

Therefore $[\omega]$ is supported on the 1-cycle given by the edges $\{1, 3\}$, $\{2, 3\}$, $\{2, 8\}$, $\{1, 8\}$. Alternatively, another defining system could have $a'_{2,3} = \chi_8 + \chi_6 + \chi_7$. Then, the associated cocycle ω' for this defining system is given by

$$\omega' = \chi_1(\chi_6 + \chi_7 + \chi_8) + -\chi_1\chi_7 = \chi_{17} + \chi_{18} - \chi_{17} = \chi_{18}.$$

Thus $[\omega']$ is also supported on the 1-cycle given by the edges $\{1, 3\}$, $\{2, 3\}$, $\{2, 8\}$, $\{1, 8\}$. By Proposition 3.1.13, this is true for all other defining systems and $\langle \alpha_1, \alpha_2, \alpha_3 \rangle$ is a non-trivial Massey product.

In summary, Proposition 3.1.10, Lemma 3.1.11 and Proposition 3.1.13 prove the following theorem.

Theorem 3.1.15. *For $i \in \{2, \ldots, n-1\}$, let $\mathcal{K}^i$ be a simplicial complex on $[m_i]$ that is not an $(m_i - 1)$-simplex. For $i \in \{1, n\}$, let $\mathcal{K}^i$ be a simplicial complex on $[m_i]$ such that there exist $J_i \subset [m_i]$, $p_i \in \mathbb{N}$ and a non-zero $[a_i] \in \widetilde{H}^{p_i}(\mathcal{K}^i_{J_i})$ for which there is a p_i-simplex $\sigma_i \in S_{a_i}$ (if $i = 1$) or $\sigma_i \in P_{a_i}$ (if $i = n$) that is maximal in $\mathcal{K}^i_{J_i}$. Then there exists a simplicial complex $\mathcal{K}$, obtained by performing stellar subdivisions on $\mathcal{K}^1 * \cdots * \mathcal{K}^n$, with a non-trivial n-Massey product in $H^*(\mathcal{Z}_\mathcal{K})$.* □

Two key examples of Theorem 3.1.15 are the families of Baskakov and Limonchenko.

Example 3.1.16 (Baskakov's family)**.** For $i = 1, 2, 3$, let $\mathcal{K}^i$ be a triangulation of a $(n_i - 1)$-sphere on $[m_i]$. Let $I_1 \in \mathcal{K}^1$, $I_2, I'_2 \in \mathcal{K}^2$, $I_3 \in \mathcal{K}^3$ be maximal simplices such that I_2 and I'_2 are adjacent. That is, there is a vertex $v_{2'} \in \mathcal{K}^2$ such that $I_2 \cap I'_2 \cup v_{2'} = I'_2$. Similarly let $I'_3 \in \mathcal{K}^3$ be a maximal simplex adjacent to I_3 so that there exists a vertex $v_{3'} \in \mathcal{K}^3$ such that $I_3 \cap I'_3 \cup v_{3'} = I'_3$. Let $a_1 = \chi_{I_1}$, $a_2 = \chi_{I_{2'}}$, and $a_3 = \chi_{I_{3'}}$ be cocycle representatives of $\alpha_i \in \widetilde{H}^{n_i-1}(\mathcal{K}^i)$ for $i = 1, 2, 3$. Then Construction 3.1.3 produces the same simplicial complex as Construction 2.3.6. Therefore Theorem 3.1.15 and Construction 3.1.3 recovers the family of examples of non-trivial triple Massey products in $H^*(\mathcal{Z}_\mathcal{K})$ given by Baskakov in [7].

Example 3.1.17 (Limonchenko's family)**.** For $i = 1, \ldots, n$, let $\mathcal{K}^i$ be a copy of two disjoint points labelled $\{\sigma_i\}, \{\sigma_i'\}$. Then the stellar subdivisions in Construction 3.1.3 correspond to the truncations in Construction 2.3.13. Therefore Theorem 3.1.15 recovers the infinite family of examples of non-trivial n-Massey products given by Limonchenko [26].

Theorem 3.1.15 does not just recover these existing results about non-trivial Massey products in the cohomology of moment-angle complexes. Theorem 3.1.15 creates non-trivial n-Massey products from any non-zero cohomology classes supported on a full subcomplex of any simplicial complex $\mathcal{K}^i$. Therefore there is no limit on the dimension of the classes α_i, nor on the size of n, that is, how many classes α_i there are. In particular, using this construction it is possible to have Massey products on torsion elements, as shown in Example 3.1.18.

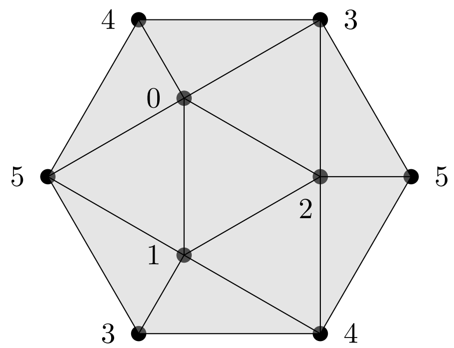


Figure 3.7: A 6-vertex triangulation of $\mathbb{R}P^2$.

Example 3.1.18. Let $\mathcal{K}^1$ be a triangulation of $\mathbb{R}P^2$ on 6 vertices as in Figure 3.7. Let $\mathcal{K}^2$, $\mathcal{K}^3$ be copies of two disjoint vertices labelled $6, 7$ and $8, 9$, respectively. Let $\alpha_1 \in \widetilde{H}^2(\mathcal{K}^1)$ be represented by χ_{012}. For $i = 2, 3$, let $\alpha_i \in \widetilde{H}^0(\mathcal{K}^i)$ be represented by $a_2 = \chi_6$ and $a_3 = \chi_8$, respectively. By Construction 3.1.3, $P_{a_2} = \{\{7\}\}$ and $P_{a_3} = \{\{9\}\}$. Then let

$$\bar{\mathcal{K}} = \mathrm{ss}_{\{0127\}}\, \mathrm{ss}_{\{69\}}\, \mathcal{K}^1 * \mathcal{K}^2 * \mathcal{K}^3$$

and let $\mathcal{K} = \bar{\mathcal{K}}_{0123456789}$. By Theorem 3.1.15, there is a non-trivial triple Massey product $\langle \alpha_1, \alpha_2, \alpha_3 \rangle \subset H^{14}(\mathcal{Z}_\mathcal{K})$. This is the smallest example of a non-trivial triple Massey product on a torsion class since $\mathcal{K}^1$ is the triangulation of $\mathbb{R}P^2$ on the least number of vertices.

Since α_1 is the generator of $\widetilde{H}^2(\mathcal{K}^1) \cong \widetilde{H}^2(\mathbb{R}P^2)$, α_1 is a torsion element. The cocycle constructed in (3.12) is $\omega = -\chi_{0126} - \chi_{0128} \in C^3(\mathcal{K})$, and it can be checked that the corresponding class $[\omega] \in \langle \alpha_1, \alpha_2, \alpha_3 \rangle$ is not a torsion element in $H^{14}(\mathcal{Z}_\mathcal{K})$.

Also, there is a cochain $a'_{1,2} = \chi_{126} + \chi_{124} - \chi_{147} - \chi_{347} + \chi_{037} + \chi_{027}$ such that $d(a'_{1,2}) = \chi_{0126} \in C^3(\mathcal{K}_{01234567})$, which is different to $a_{1,2}$ constructed in (3.3). The associated cocycle to this defining system is $\omega' = -\chi_{0126} + \chi_{1268} + \chi_{1248} - \chi_{1478} - \chi_{3478} + \chi_{0378} + \chi_{0278}$. It can be checked that $[\omega'] \neq 0$ and that $[\omega] \neq [\omega']$. Therefore $\langle \alpha_1, \alpha_2, \alpha_3 \rangle$ has non-trivial indeterminacy. In particular, the indeterminacy is given by $\alpha_1 \cdot \widetilde{H}^0(\mathcal{K}_{6789}) + \alpha_3 \cdot \widetilde{H}^2(\mathcal{K}_{01234567}) = \alpha_3 \cdot \widetilde{H}^2(\mathcal{K}_{01234567})$, where $\widetilde{H}^2(\mathcal{K}_{01234567}) \cong \mathbb{Z}$.

3.2 Edge contraction

The next section demonstrates a systematic method to create non-trivial higher Massey products given existing non-trivial higher Massey products. The idea is to contract edges of a simplicial complex $\mathcal{K}$ in a way that preserves the homotopy type of $\mathcal{K}$. We first define such an edge contraction.

3.2.1 Introduction

Definition 3.2.1. Let $\mathcal{K}$, $\hat{\mathcal{K}}$ be simplicial complexes with an edge $\{u, w\} \in \mathcal{K}$, and a vertex $z \in V(\hat{\mathcal{K}})$ such that $V(\hat{\mathcal{K}}) \setminus \{z\} = V(\mathcal{K}) \setminus \{\{u\}, \{w\}\}$. The simplicial complex $\hat{\mathcal{K}}$ is obtained from $\mathcal{K}$ by *an edge contraction* of $\{u, w\}$ if there is a map $\varphi_V \colon V(\mathcal{K}) \to V(\hat{\mathcal{K}})$

$$\varphi_V(v) = \begin{cases} z & \text{for } v \in \{u, w\} \\ v & \text{for } v \notin \{u, w\} \end{cases}$$

that extends to a surjective map $\varphi \colon \mathcal{K} \to \hat{\mathcal{K}}$, where $\varphi(I) = \{\varphi_V(v_1), \ldots, \varphi_V(v_n)\}$ for $I = \{v_1, \ldots, v_n\} \in \mathcal{K}$. The map $\varphi \colon \mathcal{K} \to \hat{\mathcal{K}}$ is called the *edge contraction of* $\{u, w\} \in \mathcal{K}$.

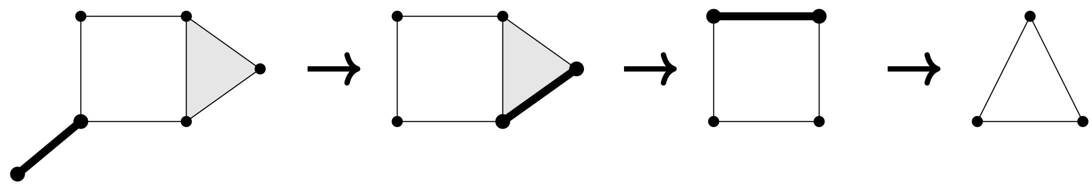
Edge contractions are simplicial maps, but they do not preserve the topology of $\mathcal{K}$ in general. Attali, Lieutier and Salinas [1] showed that the homotopy type of a simplicial complex is preserved under edge contractions that satisfy the *link condition*.

Theorem 3.2.2 ([1])**.** *For any simplicial complex $\mathcal{K}$, if an edge $\{u, w\} \in \mathcal{K}$ satisfies the link condition,*

$$\mathrm{lk}_{\mathcal{K}}(\{u\}) \cap \mathrm{lk}_{\mathcal{K}}(\{w\}) = \mathrm{lk}_{\mathcal{K}}(\{u, w\}), \tag{3.16}$$

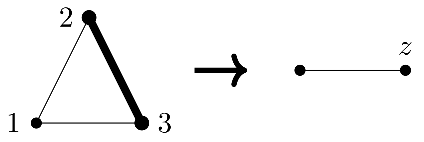
then the edge contraction of $\{u, w\}$ preserves the homotopy type of $\mathcal{K}$.

Example 3.2.3. The following is a series of edge contractions that satisfy the link condition.



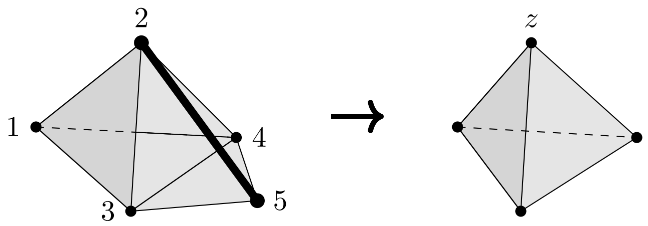
Here, despite the fact that the dimension of the simplicial complex has reduced, the homotopy type has remained the same as the link condition holds.

Example 3.2.4. Without the link condition, the homotopy type of a simplicial complex under edge contractions can change, such as in the following example.



The links of the vertices $\{2\}$ and $\{3\}$ both contain the vertex $\{1\}$, but $\mathrm{lk}_{\mathcal{K}}(\{2,3\})$ is empty, so the link condition is not satisfied.

Example 3.2.5. An edge contraction that does not satisfy the link condition may also create a cycle. For example, suppose $\hat{\mathcal{K}}$ is a triangulation of S^2 on four vertices, and let $\mathcal{K}$ be a 2-dimensional simplicial complex with facets $\{1,2,3\},\{1,2,4\},\{1,3,4\},\{3,4,5\},\{2,5\}$. So $H^2(\mathcal{K})=0$. The link of the edge $\{2,5\}$ is empty, while $\mathrm{lk}_{\mathcal{K}}\{2\}\cap\mathrm{lk}_{\mathcal{K}}\{5\}=\{\{3\},\{4\}\}$. The edge contraction of $\{2,5\}$ results in a 2-cycle in $\hat{\mathcal{K}}$, as shown below.



The following properties of edge contractions will help calculations of Massey products later. Let $\hat{\mathcal{K}}$ be a simplicial complex and let $\hat{x}\in C_q(\hat{\mathcal{K}})$. Then $\hat{x}$ is supported on a collection $T_{\hat{x}}$ of simplices so that $\hat{x}$ can be written as $\sum_{\hat{\sigma}\in T_{\hat{x}}} c_{\hat{\sigma}}\Delta_{\hat{\sigma}}$ where $c_{\hat{\sigma}}\in\mathbf{k}$ and $\Delta_{\hat{\sigma}}$ is a generator of $C_q(\hat{\mathcal{K}})$.

Corollary 3.2.6. *Let $[\hat{x}]\in\widetilde{H}_q(\hat{\mathcal{K}})$ be non-zero. Suppose a simplicial complex $\mathcal{K}$ maps to $\hat{\mathcal{K}}$ by one edge contraction $\varphi\colon\mathcal{K}\to\hat{\mathcal{K}}$ that satisfies the link condition. Then there is a non-zero class $[x]\in\widetilde{H}_q(\mathcal{K})$ and a representative $x\in C_q(\mathcal{K})$ such that for every $\hat{\sigma}\in T_{\hat{x}}$, there is exactly one lift of $\hat{\sigma}$ in the collection T_x of simplices that supports x.*

Proof. Let $\hat{x}\in C_q(\hat{\mathcal{K}})$ be a representative of $[\hat{x}]$. By Theorem 3.2.2 and since $\varphi\colon\mathcal{K}\to\hat{\mathcal{K}}$ satisfies the link condition, there exists a cycle $x\in C_q(\mathcal{K})$ such that $\varphi_\#(x)=\hat{x}$, where $\varphi_\#\colon C_q(\mathcal{K})\to C_q(\hat{\mathcal{K}})$ is the map induced by φ and

$$\varphi_\#(\Delta_\sigma)=\begin{cases}\mathrm{sgn}(\varphi(\sigma))\Delta_{\varphi(\sigma)} & \text{if } \sigma \text{ is not contracted,}\\ 0 & \text{otherwise,}\end{cases}$$

where sgn is the sign of the permutation. Let $x=\sum_{\sigma\in T_x} c_\sigma\Delta_\sigma$, where Δ_σ is a generator of $C_q(\mathcal{K})$ and $c_\sigma\in\mathbf{k}$ is non-zero. Suppose there are two different q-simplices $\sigma,\tau\in T_x$

such that $\varphi(\sigma) = \varphi(\tau) = \hat{\sigma} \in \hat{\mathcal{K}}$. Let $\{u, v\} \in \mathcal{K}$ be the edge that is contracted by φ. Since $\varphi(\sigma) = \varphi(\tau)$, let $u \in \sigma$ and $v \in \tau$. Then $\sigma \setminus u = \sigma \cap \tau = \tau \setminus v$, that is, $\sigma \cap \tau \in \mathrm{lk}_{\mathcal{K}}\{u\} \cap \mathrm{lk}_{\mathcal{K}}\{v\}$. By the link condition, $\sigma \cap \tau \in \mathrm{lk}_{\mathcal{K}}\{u, v\}$, so there is a $(q+1)$-simplex A on the vertices $V(\sigma) \cup V(\tau)$. Since φ contracts only one edge, there are no other q-simplices $\eta \in \mathcal{K}$ such that $\varphi(\eta) = \hat{\sigma}$.

Let $x' = x - c_\tau\, \varepsilon(u, A)\, \partial(A)$, where c_τ is the coefficient of the summand Δ_τ in x and $\varepsilon(u, A)$ is the coefficient of Δ_τ in $\partial(A)$. So $[x] = [x']$, but the support $T_{x'}$ of x' does not contain τ. Since σ, τ are the only q-simplices in $\partial(A)$ that do not get contracted, there are no new pairs $\sigma_1, \tau_1 \in T_{x'}$ such that $\varphi(\sigma_1) = \varphi(\tau_1)$ is a q-simplex. Also, $\varphi_\#(\partial(A)) = \varepsilon(u, A)\varphi_\#(\Delta_\tau) + \varepsilon(v, A)\varphi_\#(\Delta_\sigma)$. Thus

$$\varphi_\#(x') = \hat{x} - c_\tau\, \varepsilon(u, A)\, (\varepsilon(u, A)\, \mathrm{sgn}(\varphi(\tau)) + \varepsilon(v, A)\, \mathrm{sgn}(\varphi(\sigma)))\, \Delta_{\hat{\sigma}}.$$

Since x' is a cycle, $\varphi_\#(x')$ is also a cycle, which implies that

$$\varepsilon(u, A)\, \mathrm{sgn}(\varphi(\tau)) + \varepsilon(v, A)\, \mathrm{sgn}(\varphi(\sigma)) = 0.$$

Hence $\varphi_\#(x') = \hat{x}$. If $\tau, \sigma \notin T_{x'}$, then $\hat{\sigma} \notin T_{\hat{x}}$. Therefore whether $\sigma \in T_{x'}$ or $\sigma \notin T_{x'}$, there is exactly one lift $\sigma \in T_{x'}$ for every $\hat{\sigma} \in T_{\hat{x}}$. □

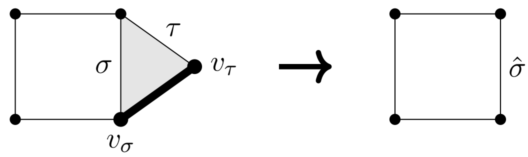


Figure 3.8: There are cycles x in $C_q(\mathcal{K})$ whose support contains only one of σ or τ

3.2.2 Massey products and edge contraction

The aim of this section is to create a non-trivial n-Massey product in the cohomology of a moment-angle complex $\mathcal{Z}_{\mathcal{K}}$, given an existing non-trivial n-Massey product in the cohomology of another moment-angle complex $\mathcal{Z}_{\hat{\mathcal{K}}}$, where $\mathcal{K}$ is mapped onto $\hat{\mathcal{K}}$ by a series of edge contractions.

Construction 3.2.7. Let $\hat{\mathcal{K}}$ be a simplicial complex with a non-trivial n-Massey product $\langle \hat{\alpha}_1, \ldots, \hat{\alpha}_n \rangle \subset H^*(\mathcal{Z}_{\hat{\mathcal{K}}})$. As in Section 2.2.2, every class $\hat{\alpha}_i \in H^*(\mathcal{Z}_{\hat{\mathcal{K}}})$ has a corresponding class

$$\hat{\alpha}_i \in \widetilde{H}^{p_i}(\hat{\mathcal{K}}_{\hat{J}_i})$$

for a set of vertices $\hat{J}_i \subset V(\hat{\mathcal{K}})$. Furthermore $\hat{J}_i \cap \hat{J}_j = \varnothing$ for any $i \neq j$ since $\langle \hat{\alpha}_1, \ldots, \hat{\alpha}_n \rangle$ is non-trivial; otherwise for any $a \in C^p(\hat{\mathcal{K}}_{\hat{J}_i})$, $b \in C^q(\hat{\mathcal{K}}_{\hat{J}_i})$, we have that $ab = 0$ if $\hat{J}_i \cap \hat{J}_j \neq \varnothing$.

Suppose there is a simplicial complex $\mathcal{K}$ and a series of edge contractions that satisfy the link condition, $\varphi\colon \mathcal{K} \to \hat{\mathcal{K}}$. Let the vertices in $V(\hat{\mathcal{K}})$ be ordered and suppose that all of the vertices in $\hat{J}_i$ come before those of $\hat{J}_{i+1}$. For a set of p-simplices $P \subset \hat{\mathcal{K}}$, let

$$\varphi_p^{-1}(P) = \{p\text{-simplex } \sigma \in \mathcal{K} \colon \varphi(\sigma) = \hat{\sigma} \text{ for } \hat{\sigma} \in P\}.$$

Then let the vertices $V(\mathcal{K})$ be ordered and suppose the order is such that for any vertex $\hat{v}$ that comes before $\hat{w}$ in $\hat{\mathcal{K}}$, we also have v before w for every $v \in \varphi_0^{-1}(\hat{v})$ and $w \in \varphi_0^{-1}(\hat{w})$. Let $J_i = \varphi_0^{-1}(\hat{J}_i) \subset V(\mathcal{K})$. Then by the order on $V(\mathcal{K})$, all vertices in J_i come before those in J_{i+1}. Also $J_i \cap J_j = \varnothing$ for any $i \neq j$ since $\hat{J}_i \cap \hat{J}_j = \varnothing$ and $\varphi_0^{-1}(\hat{v}) \cap \varphi_0^{-1}(\hat{w}) = \varnothing$ for any vertices $\hat{v}, \hat{w} \in \hat{\mathcal{K}}$, $\hat{v} \neq \hat{w}$.

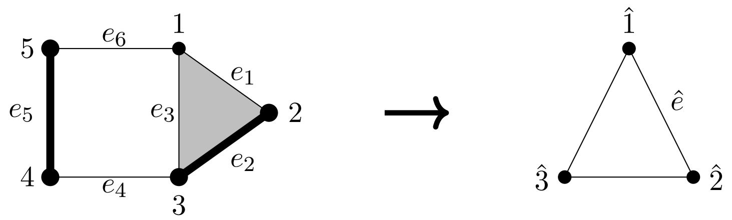
Let $\hat{a}_i$ be a cocycle representing $\hat{\alpha}_i \in \widetilde{H}^{p_i}(\hat{\mathcal{K}}_{\hat{J}_i})$. Let $S_{\hat{a}_i}$ be the set of p_i-simplices $\hat{\sigma}_i \in \hat{\mathcal{K}}_{\hat{J}_i}$ such that

$$\hat{a}_i = \sum_{\hat{\sigma} \in S_{\hat{a}_i}} c_{\hat{\sigma}} \chi_{\hat{\sigma}} \in C^{p_i}(\hat{\mathcal{K}}_{\hat{J}_i})$$

for non-zero coefficients $c_{\hat{\sigma}_i} \in \mathbf{k}$. Then, let $a_i \in C^{p_i}(\mathcal{K}_{J_i})$ be the cochain

$$a_i = \sum_{\hat{\sigma} \in S_{\hat{a}_i}} c_{\hat{\sigma}} \sum_{\sigma \in \varphi_{p_i}^{-1}(\hat{\sigma})} \chi_\sigma. \tag{3.17}$$

Example 3.2.8. Let $\mathcal{K}_{J_i}$, $\hat{\mathcal{K}}_{\hat{J}_i}$ be the simplicial complexes as shown below, where $\hat{\mathcal{K}}_{\hat{J}_i}$ is obtained from $\mathcal{K}_{J_i}$ by contracting the edges $e_2 = \{2,3\} \mapsto \{\hat{2}\}$ and $e_5 = \{4,5\} \mapsto \{\hat{3}\}$. The cohomology class $\hat{\alpha}_i \in \widetilde{H}^1(\hat{\mathcal{K}}_{\hat{J}_i})$ may be represented by the cocycle $\chi_{\hat{e}}$, so $S_{\hat{a}_i} = \{\hat{e}\}$.



The edge contraction of e_2 satisfies the link condition, since $\mathrm{lk}_{\mathcal{K}}(e_2) = \mathrm{lk}_{\mathcal{K}}\{2\} \cap \mathrm{lk}_{\mathcal{K}}\{3\} = \{1\}$. Under the map $\varphi \colon \mathcal{K} \to \hat{\mathcal{K}}$, $\varphi_1^{-1}(\hat{e}) = \{e_1, e_3\}$. So by (3.22), a_i is the cochain

$$a_i = \chi_{e_1} + \chi_{e_3} \in C^1(\mathcal{K}_{J_i}).$$

Lemma 3.2.9. *The cochain a_i is a cocycle.*

Proof. For ease of notation, we omit the index i in the following proof, that is, let $a = a_i$, $p = p_i$, $J = J_i$, etc. Let $V(\sigma)$ denote the vertices of a simplex σ. Applying the coboundary map to a,

$$d(a) = \sum_{\hat{\sigma} \in S_{\hat{a}}} c_{\hat{\sigma}} \sum_{\sigma \in \varphi_p^{-1}(\hat{\sigma})} \sum_{j \in J \setminus V(\sigma)} \varepsilon(j, j \cup \sigma) \chi_{j \cup \sigma}.$$

Since $V(\sigma) \subset \varphi_0^{-1}(V(\hat\sigma)) \subset J$ for any $\hat\sigma \in S_{\hat a}$, this may be written as

$$d(a) = \sum_{\hat\sigma \in S_{\hat a}} c_{\hat\sigma} \sum_{\sigma \in \varphi_p^{-1}(\hat\sigma)} \sum_{j \in J \setminus \varphi_0^{-1}(V(\hat\sigma))} \varepsilon(j, j \cup \sigma)\chi_{j\cup\sigma} \tag{3.18}$$

$$+ \sum_{\hat\sigma \in S_{\hat a}} c_{\hat\sigma} \sum_{\sigma \in \varphi_p^{-1}(\hat\sigma)} \sum_{j \in \varphi_0^{-1}(V(\hat\sigma)) \setminus V(\sigma)} \varepsilon(j, j \cup \sigma)\chi_{j\cup\sigma}. \tag{3.19}$$

For any p-simplex $\hat\sigma \in S_{\hat a}$ and any $\sigma \in \varphi_p^{-1}(\hat\sigma)$, if there is a vertex $j \in \varphi_0^{-1}(V(\hat\sigma)) \setminus V(\sigma)$ such that $j \cup \sigma \in \mathcal{K}$, then $\bar\sigma = j \cup \sigma \in \varphi_{p+1}^{-1}(\hat\sigma)$ and there is a vertex $i \in V(\sigma)$ such that $\varphi(i) = \varphi(j)$. Hence $j \cup \sigma \setminus i \in \varphi_p^{-1}(\hat\sigma)$. Moreover, i, j are consecutive vertices in $V(\bar\sigma)$ by the order of the vertices in $\mathcal{K}$ defined in Construction 3.2.7, so $\varepsilon(j, \bar\sigma) = -\varepsilon(i, \bar\sigma)$. Therefore, for any $\hat\sigma \in S_{\hat a}$,

$$\sum_{\sigma \in \varphi_p^{-1}(\hat\sigma)} \sum_{j \in \varphi_0^{-1}(V(\hat\sigma)) \setminus V(\sigma)} \varepsilon(j, j \cup \sigma)\chi_{j\cup\sigma} = \sum_{\substack{\bar\sigma \in \varphi_{p+1}^{-1}(\hat\sigma),\\ i,j \in \bar\sigma :\, \varphi(i) = \varphi(j)}} \varepsilon(j, \bar\sigma)\chi_{\bar\sigma} + \varepsilon(i, \bar\sigma)\chi_{\bar\sigma} = 0$$

so (3.19) is zero.

Next consider (3.18). For $j \in J \setminus \varphi_0^{-1}(V(\hat\sigma))$ such that $j \cup \sigma \in \mathcal{K}$, there is a corresponding simplex $\varphi(j \cup \sigma) = \varphi(j) \cup \varphi(\sigma) \in \hat{\mathcal{K}}$. Hence for any summand $\varepsilon(j, j \cup \sigma)\chi_{j\cup\sigma}$ in (3.18), there is a corresponding summand $\varepsilon(\varphi(j), \varphi(j \cup \sigma))\chi_{\varphi(j\cup\sigma)}$ in the expression for $d(\hat a)$.

Since $d(\hat a) = 0$, there are other simplices $\hat\tau_1, \dots, \hat\tau_s \in S_{\hat a}$ with a vertex $\hat w_n = \varphi(j \cup \sigma) \setminus \hat\tau_n$ for $n \in \{1, \dots, s\}$ such that

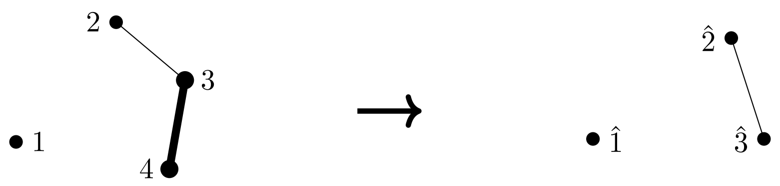
$$c_{\varphi(\sigma)}\, \varepsilon(\varphi(j), \varphi(j \cup \sigma))\, \chi_{\varphi(j\cup\sigma)} + \sum_{n=1}^{s} c_{\hat\tau_n}\, \varepsilon(\hat w_n, \hat w_n \cup \hat\tau_n)\, \chi_{\hat w_n \cup \hat\tau_n} = 0.$$

Therefore there is a p-simplex $\tau_n \in \mathcal{K}$ such that τ_n is a maximal face of $j \cup \sigma$, and so $\varphi(w_n \cup \tau_n) = \hat w_n \cup \hat\tau_n$ for the vertex $w_n = j \cup \sigma \setminus \tau_n$. Furthermore, $\varepsilon(w_n, w_n \cup \tau_n) = \varepsilon(\hat w_n, \hat w_n \cup \hat\tau_n)$ by the ordering of vertices in $\mathcal{K}$. Thus, (3.18) has summands $\varepsilon(w_n, w_n \cup \tau_n)\chi_{w_n\cup\tau_n}$ such that

$$c_\sigma\, \varepsilon(j, j \cup \sigma)\, \chi_{j\cup\sigma} + \sum_{n=1}^{s} c_{\tau_n}\, \varepsilon(w_n, w_n \cup \tau_n)\, \chi_{w_n\cup\tau_n} = 0.$$

This means that (3.18) is zero, and hence a is a cocycle. □

Example 3.2.10. Let $J_1 = \{1, 2, 3, 4\}$ and $\hat J_1 = \{\hat 1, \hat 2, \hat 3\}$. Suppose $\mathcal{K}_{J_1}$ and $\hat{\mathcal{K}}_{\hat J_1}$ are the simplicial complexes shown below, where $\mathcal{K}_{J_1}$ maps onto $\hat{\mathcal{K}}_{\hat J_1}$ by the edge contraction $\{3, 4\} \mapsto \{\hat 3\}$.



Suppose $\hat{a}_1 = \chi_{\hat{2}} + \chi_{\hat{3}} \in C^0(\hat{\mathcal{K}}_{\hat{J}_1})$. We have that $d(\hat{a}_1) = -\chi_{\hat{2},\hat{3}} + \chi_{\hat{2},\hat{3}} = 0$.

By (3.17), $a_1 = \chi_2 + \chi_3 + \chi_4 \in C^0(\mathcal{K}_{J_1})$. Then $d(a_1) = -\chi_{2,3} + (\chi_{2,3} - \chi_{3,4}) + \chi_{3,4} = 0$. The summands of the form $\chi_{2,3}$ correspond to the summands $\chi_{\hat{2},\hat{3}}$ in $d(\hat{a}_1)$ as summands in (3.18). The summands of the form $\chi_{3,4}$ cancel in pairs as summands in (3.19). So a_1 is a cocycle.

Furthermore, we check that a_i is not a coboundary.

Lemma 3.2.11. *The class $\alpha_i = [a_i] \in \widetilde{H}^{p_i}(\mathcal{K}_{J_i})$ is non-zero.*

Proof. Since the Massey product $\langle \hat{\alpha}_1, \ldots, \hat{\alpha}_n \rangle$ is non-trivial, the class $\hat{\alpha}_i \in \widetilde{H}^{p_i}(\hat{\mathcal{K}}_{\hat{J}_i})$ is non-zero. Therefore there is a homology class $[\hat{x}] \in \widetilde{H}_{p_i}(\hat{\mathcal{K}}_{\hat{J}_i})$ such that the evaluation $\hat{\alpha}_i(\hat{x})$ is non-zero. Let $\hat{x}$ be supported on a collection $T_{\hat{x}} \subset \hat{\mathcal{K}}_{\hat{J}_i}$ of p_i-simplices so that $\hat{x} = \sum_{\hat{\tau} \in T_{\hat{x}}} c_{\hat{\tau}} \Delta_{\hat{\tau}}$ for $c_{\hat{\tau}} \in \mathbf{k}$.

By Lemma 3.2.6, there is a non-zero class $[x] \in \widetilde{H}_{p_i}(\mathcal{K}_{J_i})$ and a representative $x \in C_{p_i}(\mathcal{K}_{J_i})$ such that for every $\hat{\tau} \in T_{\hat{x}}$, there is exactly one lift of $\hat{\tau}$ in the collection T_x of simplices that supports x. For any $\hat{\tau} \in T_{\hat{x}}$, let $c_\tau = c_{\hat{\tau}}$ for $\tau \in T_x$ such that $\varphi(\tau) = \hat{\tau}$. Let $x = \sum_{\tau \in T_x} c_\tau \Delta_\tau$ for $c_\tau \in \mathbf{k}$.

By definition, any $\sigma \in \varphi_{p_i}^{-1}(\hat{\sigma})$ for $\hat{\sigma} \in S_{\hat{a}_i}$ does not contract since both $\hat{\sigma}$ and σ are p_i-simplices. Therefore, evaluating the cocycle a_i on the cycle x,

$$\begin{aligned} a_i(x) &= \sum_{\hat{\sigma} \in S_{\hat{a}_i}} c_{\hat{\sigma}} \left(\sum_{\sigma \in \varphi_{p_i}^{-1}(\hat{\sigma})} \chi_\sigma \left(\sum_{\tau \in T_x} c_\tau \Delta_\tau \right) \right) \\ &= \sum_{\hat{\sigma} \in S_{\hat{a}_i}} \sum_{\hat{\tau} \in T_{\hat{x}}} c_{\hat{\sigma}} \, c_\tau \, \chi_{\hat{\sigma}}(\Delta_{\hat{\tau}}). \end{aligned}$$

Then since $c_\tau = c_{\hat{\tau}}$, this is equal to $\hat{\alpha}_i(\hat{x})$. So since the evaluation $\hat{\alpha}_i(\hat{x})$ is non-zero, then also $a_i(x)$ is non-zero. Therefore, $\alpha_i = [a_i] \in \widetilde{H}^{p_i}(\mathcal{K}_{J_i})$ is a non-zero cohomology class. □

By Section 2.2.2, for the Massey product $\langle \hat{\alpha}_1, \ldots, \hat{\alpha}_n \rangle \subset H^{(p_1+\cdots+p_n)+|\hat{J}_1 \cup \cdots \cup \hat{J}_n|+2}(\mathcal{Z}_{\hat{\mathcal{K}}})$, there is a defining system $(\hat{a}_{i,k})$ for cochains $\hat{a}_{i,k} \in C^{p_i+\cdots+p_k}(\hat{\mathcal{K}}_{\hat{J}_i \cup \ldots \cup \hat{J}_k})$, $1 \leqslant i \leqslant k \leqslant n$ and $(i,k) \neq (1,n)$. Suppose

$$\hat{a}_{i,k} = \sum_{\hat{\tau} \in S_{\hat{a}_{i,k}}} c_{\hat{\tau}} \chi_{\hat{\tau}} \tag{3.20}$$

for simplices $\hat{\tau} \in S_{\hat{a}_{i,k}} \subset \hat{\mathcal{K}}_{\hat{J}_i \cup \ldots \cup \hat{J}_k}$, non-zero coefficients $c_{\hat{\tau}} \in \mathbf{k}$. Then

$$d(\hat{a}_{i,k}) = \sum_{\hat{\tau} \in S_{\hat{a}_{i,k}}} c_{\hat{\tau}} \left(\sum_{\hat{j} \in \hat{J}_i \cup \cdots \cup \hat{J}_k \setminus V(\hat{\tau})} \varepsilon(\hat{j}, \hat{j} \cup \hat{\tau}) \chi_{\hat{j} \cup \hat{\tau}} \right)$$

is equal to

$$\sum_{r=i}^{k-1} (-1)^{1+\overline{\deg}(\hat{a}_{i,r})} \hat{a}_{i,r} \hat{a}_{r,k} = \sum_{r=i}^{k-1} (-1)^{1+\overline{\deg}(\hat{a}_{i,r})} \, c \left(\sum_{\hat{\nu} \in S_{\hat{a}_{i,r}}} \sum_{\hat{\eta} \in S_{\hat{a}_{r+1,k}}} c_{\hat{\nu}} \, c_{\hat{\eta}} \, \chi_{\hat{\nu} \cup \hat{\eta}} \right) \tag{3.21}$$

where $(-1)^{1+\overline{\deg}(\hat{a}_{i,r})} = (-1)^{(p_i+\ldots+p_r)+|\hat{J}_i \cup \cdots \cup \hat{J}_r|}$ and $c = (-1)^{|\hat{J}_i \cup \cdots \cup \hat{J}_r|(p_{r+1}+\cdots+p_k+1)}$ comes from the product of $\hat{a}_{i,r}$ and $\hat{a}_{r,k}$, as in Lemma 2.1.26.

Proposition 3.2.12. *Let $\mathcal{K}$ be a simplicial complex that maps to $\hat{\mathcal{K}}$ by edge contractions satisfying the link condition. Then there is a n-Massey product $\langle \alpha_1, \ldots, \alpha_n \rangle$ defined on $H^*(\mathcal{Z}_{\mathcal{K}})$.*

Proof. For every $i \in \{1, \ldots, n\}$, let $\alpha_i = [a_i]$ for a_i as in (3.17). We construct a defining system $(a_{i,k})$ for $\langle \alpha_1, \ldots, \alpha_n \rangle \subset H^*(\mathcal{Z}_{\mathcal{K}})$, where $a_{i,k} \in C^{p_i+\cdots+p_k}(\mathcal{K}_{J_i \cup \cdots \cup J_k})$ as in Section 2.2.2. For $i \neq k$, let

$$a_{i,k} = \theta_{i,k} \, \hat{\theta}_{i,k} \sum_{\hat{\tau} \in S_{\hat{a}_{i,k}}} c_{\hat{\tau}} \left(\sum_{\tau \in \varphi^{-1}_{p_i+\cdots+p_k}(\hat{\tau})} \chi_\tau \right) \tag{3.22}$$

for $S_{\hat{a}_{i,k}}$, $c_{\hat{\tau}} \in \mathbf{k}$ from (3.20), and

$$\begin{aligned} \theta_{i,k} &= (-1)^{|J_i|(p_{i+1}+\cdots+p_k)}(-1)^{|J_{i+1}|(p_{i+2}+\cdots+p_k)} \cdots (-1)^{|J_{k-1}|p_k} \\ \hat{\theta}_{i,k} &= (-1)^{|\hat{J}_i|(p_{i+1}+\cdots+p_k)}(-1)^{|\hat{J}_{i+1}|(p_{i+2}+\cdots+p_k)} \cdots (-1)^{|\hat{J}_{k-1}|p_k}. \end{aligned} \tag{3.23}$$

When $i = k$, let $\theta_{i,i} = 1 = \hat{\theta}_{i,i}$ so that $a_{i,i} = a_i$ as in (3.17). We will show that $d(a_{i,k}) = \sum_{r=i}^{k-1} \overline{a_{i,r}} a_{r,k}$, where $\overline{a_{i,r}} = (-1)^{1+\overline{\deg}\, a_{i,r}} a_{i,r}$ as in Definition 2.2.7.

Applying the coboundary map to $a_{i,k}$,

$$d(a_{i,k}) = \theta_{i,k} \, \hat{\theta}_{i,k} \sum_{\hat{\tau} \in S_{\hat{a}_{i,k}}} c_{\hat{\tau}} \left(\sum_{\tau \in \varphi^{-1}_{p_i+\cdots+p_k}(\hat{\tau})} \sum_{j \in J_i \cup \cdots \cup J_k \setminus V(\tau)} \varepsilon(j, j \cup \tau) \chi_{j \cup \tau} \right).$$

This may be rewritten as

$$d(a_{i,k}) = \theta_{i,k} \, \hat{\theta}_{i,k} \sum_{\hat{\tau} \in S_{\hat{a}_{i,k}}} c_{\hat{\tau}} \left(\sum_{\tau \in \varphi^{-1}_{p_i+\cdots+p_k}(\hat{\tau})} \sum_{j \in J_i \cup \cdots \cup J_k \setminus \varphi_0^{-1}(V(\hat{\tau}))} \varepsilon(j, j \cup \tau) \chi_{j \cup \tau} \right) + \tag{3.24}$$

$$+\,\theta_{i,k}\,\hat{\theta}_{i,k}\sum_{\hat{\tau}\in S_{\hat{a}_{i,k}}} c_{\hat{\tau}}\left(\sum_{\tau\in\varphi^{-1}_{p_i+\cdots+p_k}(\hat{\tau})}\ \sum_{j\in\varphi_0^{-1}(V(\hat{\tau}))\setminus V(\tau)} \varepsilon(j,j\cup\tau)\chi_{j\cup\tau}\right). \tag{3.25}$$

For any $\tau\in S_{\hat{a}_{i,k}}$ and any $\tau\in\varphi^{-1}_{p_i+\cdots+p_k}(\hat{\tau})$, first suppose there is a vertex $j\in\varphi_0^{-1}(V(\hat{\tau}))\setminus V(\tau)$ such that $j\cup\tau\in\mathcal{K}$. Then $j\cup\tau=\bar{\tau}\in\varphi^{-1}_{p_i+\cdots+p_k+1}(\hat{\tau})$ and there is a vertex $i\in V(\tau)$ such that $\varphi(i)=\varphi(j)$. Thus $j\cup\tau\setminus i\in\varphi^{-1}_{p_i+\cdots+p_k}(\hat{\tau})$. Moreover, i,j are consecutive vertices in $V(\bar{\tau})$ by the order of vertices in $\mathcal{K}$ defined in Construction 3.2.7, so $\varepsilon(j,\bar{\tau})=-\varepsilon(i,\bar{\tau})$. Therefore (3.25) is zero since all summands cancel out in pairs, that is, for any $\hat{\tau}\in S_{\hat{a}_{i,k}}$,

$$\sum_{\tau\in\varphi^{-1}_{p_i+\cdots+p_k}(\hat{\tau})}\ \sum_{j\in\varphi_0^{-1}(V(\hat{\tau}))\setminus V(\tau)}\varepsilon(j,j\cup\tau)\chi_{j\cup\tau}=\sum_{\substack{\bar{\tau}\in\varphi^{-1}_{p_i+\cdots+p_k+1}(\hat{\tau}),\\ i,j\in\bar{\tau}:\,\varphi(i)=\varphi(j)}}\varepsilon(j,\bar{\tau})\chi_{\bar{\tau}}+\varepsilon(i,\bar{\tau})\chi_{\bar{\tau}}=0.$$

Consider (3.24). For any $j\in J_i\cup\cdots\cup J_k\setminus\varphi_0^{-1}(V(\hat{\tau}))$, $\varphi(j)\notin V(\hat{j})$. So for any simplex $j\cup\tau\in\mathcal{K}$, for $j\in J_i\cup\cdots\cup J_k\setminus\varphi_0^{-1}(V(\hat{\tau}))$, there is a simplex $\varphi(j)\cup\hat{\tau}\in\hat{\mathcal{K}}$. Therefore any summand in (3.24) has a corresponding summand in the expression for $d(\hat{a}_{i,k})$. Hence (3.24) may be rewritten as

$$d(a_{i,k})=\theta_{i,k}\,\hat{\theta}_{i,k}\sum_{\hat{\tau}\in S_{\hat{a}_{i,k}}}c_{\hat{\tau}}\left(\sum_{\hat{j}\in\hat{J}_i\cup\cdots\cup\hat{J}_k\setminus V(\hat{\tau})}\ \sum_{j\cup\tau\in\varphi^{-1}_{p_i+\cdots+p_k+1}(\hat{j}\cup\hat{\tau})}\varepsilon(j,j\cup\tau)\chi_{j\cup\tau}\right) \tag{3.26}$$

where, by the order of vertices in $\mathcal{K}$, $\varepsilon(j,j\cup\tau)=\varepsilon(\hat{j},\hat{j}\cup\hat{\tau})$. Since $d(\hat{a}_{i,k})=\sum_{r=i}^{k-1}\overline{\overline{\hat{a}_{i,r}}}\hat{a}_{r,k}$, the expression in (3.26) can be written in terms of the expression in (3.21). That is, $d(a_{i,k})$ is equal to

$$\theta_{i,k}\,\hat{\theta}_{i,k}\sum_{r=i}^{k-1}(-1)^{1+\overline{\deg}(\hat{a}_{i,r})}\,c\left(\sum_{\hat{\nu}\in S_{\hat{a}_{i,r}}}\ \sum_{\hat{\eta}\in S_{\hat{a}_{r+1,k}}}c_{\hat{\nu}}\,c_{\hat{\eta}}\left(\sum_{\zeta\in\varphi^{-1}_{p_i+\cdots+p_k+1}(\hat{\nu}\cup\hat{\eta})}\chi_\zeta\right)\right) \tag{3.27}$$

where $(-1)^{1+\overline{\deg}(\hat{a}_{i,r})}=(-1)^{(p_i+\cdots+p_r)+|\hat{J}_i\cup\cdots\cup\hat{J}_r|}$ and $c=(-1)^{|\hat{J}_i\cup\cdots\cup\hat{J}_r|(p_{r+1}+\cdots+p_k+1)}$ comes from the product of $\hat{a}_{i,r}$ and $\hat{a}_{r,k}$, as in Lemma 2.1.26.

Any simplex $\zeta\in\varphi^{-1}_{p_i+\cdots+p_k+1}(\hat{\nu}\cup\hat{\eta})$ is on $p_i+\cdots+p_k+2$ vertices and so can be written as $\nu\cup\eta$ for ν the restriction of ζ to its first $p_i+\cdots+p_r+1$ vertices, and η the restriction of ζ to its last $p_{r+1}+\cdots+p_k+1$ vertices. Then $\nu\in\varphi^{-1}_{p_i+\cdots+p_r}(\hat{\nu})$ and $\eta\in\varphi^{-1}_{p_{r+1}+\cdots+p_k}(\hat{\eta})$. Furthermore, $\hat{\theta}_{i,k}\,(-1)^{1+\overline{\deg}(\hat{a}_{i,r})}\,c=(-1)^{(p_i+\cdots+p_r)}\,\hat{\theta}_{i,r}\,\hat{\theta}_{r+1,k}$. So (3.27) may be rewritten as

$$d(a_{i,k})=\sum_{r=i}^{k-1}(-1)^{(p_i+\cdots+p_r)}\,\theta_{i,k}\,\hat{\theta}_{i,r}\,\hat{\theta}_{r+1,k}.$$

$$\cdot\left(\sum_{\hat\nu\in S_{\hat a_{i,r}}}\ \sum_{\hat\eta\in S_{\hat a_{r+1,k}}} c_{\hat\nu}\, c_{\hat\eta}\left(\sum_{\nu\in\varphi^{-1}_{p_i+\cdots+p_r}(\hat\nu)}\ \sum_{\eta\in\varphi^{-1}_{p_{r+1}+\cdots+p_k}(\hat\eta)} \chi_{\nu\cup\eta}\right)\right). \tag{3.28}$$

Comparatively, the product $\sum_{r=i}^{k-1}(-1)^{1+\overline{\deg}(a_{i,r})}a_{i,r}a_{r,k}$ is

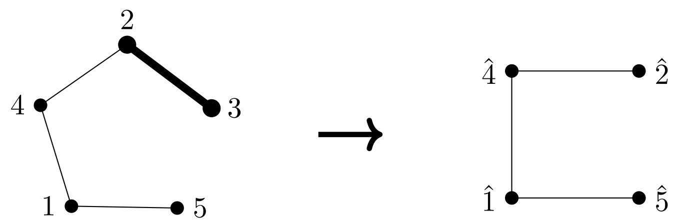
$$\sum_{r=i}^{k-1}(-1)^{1+\overline{\deg}(a_{i,r})}(-1)^{|J_i\cup\cdots\cup J_r|(p_{r+1}+\cdots+p_k+1)}\theta_{i,r}\ \theta_{r+1,k}\ \hat\theta_{i,r}\ \hat\theta_{r+1,k}\cdot$$

$$\cdot\left(\sum_{\hat\nu\in S_{\hat a_{i,r}}}\ \sum_{\hat\eta\in S_{\hat a_{r+1,k}}} c_{\hat\nu}\, c_{\hat\eta}\left(\sum_{\nu\in\varphi^{-1}_{p_i+\cdots+p_r}(\hat\nu)}\ \sum_{\eta\in\varphi^{-1}_{p_{r+1}+\cdots+p_k}(\hat\eta)} \chi_{\nu\cup\eta}\right)\right) \tag{3.29}$$

where $(-1)^{1+\overline{\deg}(a_{i,r})}=(-1)^{(p_i+\cdots+p_r)+|J_i\cup\cdots\cup J_r|}$ and the sign $(-1)^{|J_i\cup\cdots\cup J_r|(p_{r+1}+\cdots+p_k+1)}$ comes from the product of $a_{i,r}$ and $a_{r+1,k}$ as in Lemma 2.1.26. Using the expression for $\theta_{i,k}$ in (3.23), $(-1)^{1+\overline{\deg}(a_{i,r})}(-1)^{|J_i\cup\cdots\cup J_r|(p_{r+1}+\cdots+p_k+1)}\ \theta_{i,r}\ \theta_{r+1,k}\ =(-1)^{(p_i+\cdots+p_r)}\ \theta_{i,k}$. Therefore the expressions in (3.28) and (3.29) are equal.

Hence $d(a_{i,k})=\sum_{r=i}^{k-1}\overline{\overline{a_{i,r}}}a_{r,k}$, and so $(a_{i,k})$ is a defining system for the Massey product $\langle\alpha_1,\ldots,\alpha_n\rangle$. □

Example 3.2.13. Let $J_1=\{1,2,3\}$, $\hat J_1=\{\hat1,\hat2\}$, $J_2=\{4,5\}$ and $\hat J_2=\{\hat4,\hat5\}$. Suppose $\mathcal{K}_{J_1\cup J_2}$ and $\hat{\mathcal{K}}_{\hat J_1\cup\hat J_2}$ are the simplicial complexes shown below, where $\mathcal{K}_{J_1\cup J_2}$ maps onto $\hat{\mathcal{K}}_{\hat J_1\cup\hat J_2}$ by the edge contraction $\{2,3\}\mapsto\{\hat2\}$.



Suppose $\hat a_1=\chi_{\hat2}\in C^0(\hat{\mathcal{K}}_{\hat J_1})$, $\hat a_2=\chi_{\hat4}\in C^0(\hat{\mathcal{K}}_{\hat J_2})$, and $\hat a_{1,2}=-\chi_{\hat2}\in C^0(\hat{\mathcal{K}}_{\hat J_1\cup\hat J_2})$. Then $d(\hat a_{1,2})=\chi_{\hat2,\hat4}=(-1)^{1+\overline{\deg}\,\hat a_1}\hat a_1\hat a_2$. By (3.17), $a_1=\chi_2+\chi_3\in C^0(\mathcal{K}_{J_1})$ and $a_2=\chi_4\in C^0(\mathcal{K}_{J_2})$. By (3.22), $a_{1,2}=-\chi_2-\chi_3\in C^0(\mathcal{K}_{J_1\cup J_2})$, since $\theta_{1,2}=1$. We have that $d(a_{1,2})=(\chi_{2,4}+\chi_{2,3})-\chi_{2,3}=\chi_{2,4}=(-1)^{1+\overline{\deg}\,a_1}a_1a_2=\overline{\overline{a_1}}a_2$.

Since there is a n-Massey product $\langle\alpha_1,\ldots,\alpha_n\rangle$ defined on $H^*(\mathcal{Z}_\mathcal{K})$, it remains to show that this Massey product is non-trivial. To do this, we first check that there is at least one non-zero cohomology class in $\langle\alpha_1,\ldots,\alpha_n\rangle$.

Let $\omega\in C^{(p_1+\cdots+p_n)+|J_1\cup\cdots\cup J_n|+2}(\mathcal{Z}_\mathcal{K})$ be the associated cocycle for the defining system $(a_{i,k})$ for $\langle\alpha_1,\ldots,\alpha_n\rangle$. Then

$$\omega=\sum_{r=1}^{n-1}\overline{\overline{a_{1,r}}}a_{r,n}\in C^{p_1+\cdots+p_n+1}(\mathcal{K}_{J_1\cup\cdots\cup J_n}).$$

As in (3.29) when $i = 1$ and $k = n$, we can express ω as the cocycle

$$\omega = \sum_{r=1}^{n-1} (-1)^{1+\overline{\deg}(a_{1,r})}(-1)^{|J_1\cup\cdots\cup J_r|(p_{r+1}+\cdots+p_n+1)}\theta_{1,r}\ \theta_{r+1,n}\ \hat{\theta}_{1,r}\ \hat{\theta}_{r+1,n}\cdot$$
$$\cdot\left(\sum_{\hat{\nu}\in S_{\hat{a}_{1,r}}}\ \sum_{\hat{\eta}\in S_{\hat{a}_{r+1,n}}} c_{\hat{\nu}}\ c_{\hat{\eta}}\left(\sum_{\nu\in\varphi^{-1}_{p_1+\cdots+p_r}(\hat{\nu})}\ \sum_{\eta\in\varphi^{-1}_{p_{r+1}+\cdots+p_n}(\hat{\eta})} \chi_{\nu\cup\eta}\right)\right). \tag{3.30}$$

Lemma 3.2.14. *The class* $[\omega] \in H^{p_1+\cdots+p_n+1}(\mathcal{K}_{J_1\cup\cdots\cup J_n})$ *is non-zero.*

Proof. We will show that there is a cycle $x \in C_{p_1+\cdots+p_n+1}(\mathcal{K}_{J_1\cup\cdots\cup J_n})$ such that $[x] \neq 0$ and $\omega(x) \neq 0$. By assumption, there is a non-zero element $[\hat{\omega}] \in \langle\hat{\alpha}_1,\dots,\hat{\alpha}_n\rangle \subset H^*(\mathcal{Z}_{\hat{\mathcal{K}}})$. As in Section 2.2.2, this may be represented by the cocycle $\hat{\omega} = \sum_{r=1}^{n-1}\overline{\overline{\hat{a}_{1,r}}}\hat{a}_{r,n} \in C^{p_1+\cdots+p_n+1}(\hat{\mathcal{K}}_{\hat{J}_1\cup\cdots\cup\hat{J}_n})$ for $\hat{a}_{i,k}$ as in (3.20). Like (3.21) when $i = 1$ and $k = n$,

$$\hat{\omega} = \sum_{r=1}^{n-1} (-1)^{1+\overline{\deg}(\hat{a}_{1,r})}\ (-1)^{|\hat{J}_1\cup\cdots\cup\hat{J}_r|(p_{r+1}+\cdots+p_n+1)}\left(\sum_{\hat{\nu}\in S_{\hat{a}_{1,r}}}\ \sum_{\hat{\eta}\in S_{\hat{a}_{r+1,n}}} c_{\hat{\nu}}\ c_{\hat{\eta}}\ \chi_{\hat{\nu}\cup\hat{\eta}}\right)$$

where $(-1)^{1+\overline{\deg}(\hat{a}_{1,r})} = (-1)^{(p_1+\ldots+p_r)+|\hat{J}_1\cup\cdots\cup\hat{J}_r|}$ and the sign $(-1)^{|\hat{J}_1\cup\cdots\cup\hat{J}_r|(p_{r+1}+\cdots+p_n+1)}$ comes from the product of $\hat{a}_{1,r}$ and $\hat{a}_{r,n}$, as in Lemma 2.1.26 .

Since $[\hat{\omega}]$ is non-zero, there is a non-zero homology class $[\hat{x}] \in \widetilde{H}_{p_1+\cdots+p_n+1}(\hat{\mathcal{K}}_{\hat{J}_1\cup\cdots\cup\hat{J}_n})$ such that $\hat{\omega}(\hat{x}) \neq 0$. Let the representing cycle $\hat{x}$ be supported on a collection $T_{\hat{x}} \subset \hat{\mathcal{K}}_{\hat{J}_1\cup\cdots\cup\hat{J}_n}$ of $(p_1+\cdots+p_n+1)$-simplices,

$$\hat{x} = \sum_{\hat{\tau}\in T_{\hat{x}}} c_{\hat{\tau}}\Delta_{\hat{\tau}}$$

for $c_{\hat{\tau}} \in \mathbf{k}$ and $\Delta_{\hat{\tau}}$ a generator of $C_{p_1+\cdots+p_n+1}(\hat{\mathcal{K}}_{\hat{J}_1\cup\cdots\cup\hat{J}_n})$.

By Lemma 3.2.6, there is a non-zero class $[x] \in \widetilde{H}_{p_1+\cdots+p_n+1}(\mathcal{K}_{J_1\cup\cdots\cup J_n})$ and a representative $x \in C_{p_1+\cdots+p_n+1}(\mathcal{K}_{J_1\cup\cdots\cup J_n})$ such that for every $\hat{\sigma} \in T_{\hat{x}}$, there is exactly one lift of $\hat{\sigma}$ in the collection T_x of simplices that supports x. For any $\hat{\tau} \in T_{\hat{x}}$, let $c_\tau = c_{\hat{\tau}}$ for $\tau \in T_x$ such that $\varphi(\tau) = \hat{\tau}$. Then for $c_\tau \in \mathbf{k}$, let

$$x = \sum_{\tau\in T_x} c_\tau\Delta_\tau.$$

Since $(-1)^{1+\overline{\deg}(a_{1,r})}\ (-1)^{|J_1\cup\cdots\cup J_r|(p_{r+1}+\cdots+p_n+1)}\ \theta_{1,r}\ \theta_{r+1,n} = (-1)^{(p_1+\cdots+p_r)}\ \theta_{1,n}$, we evaluate the cocycle ω on the cycle x using the expression (3.30) for ω,

$$\omega(x) = \sum_{r=1}^{n-1} (-1)^{(p_1+\cdots+p_r)}\ \theta_{1,n}\ \hat{\theta}_{1,r}\ \hat{\theta}_{r+1,n}\cdot$$

$$\cdot \left(\sum_{\hat\nu \in S_{\hat a_{1,r}}} \sum_{\hat\eta \in S_{\hat a_{r+1,n}}} c_{\hat\nu}\, c_{\hat\eta} \left(\sum_{\nu \in \varphi^{-1}_{p_1+\cdots+p_r}(\hat\nu)} \sum_{\eta \in \varphi^{-1}_{p_{r+1}+\cdots+p_n}(\hat\eta)} \chi_{\nu\cup\eta} \left(\sum_{\tau\in T_x} c_\tau \Delta_\tau \right)\right)\right).$$

Since every $\hat\tau \in T_{\hat x}$ has exactly one lift $\tau \in T_x$, and $\varphi(\nu\cup\eta) = \hat\nu \cup \hat\eta$, we have that $\chi_{\nu\cup\eta}(\tau) \neq 0$ if and only if $\chi_{\hat\nu\cup\hat\eta}(\hat\tau) \neq 0$. Also $c_\tau = c_{\hat\tau}$ for $\tau \in T_x$ such that $\varphi(\tau) = \hat\tau$. Therefore,

$$\omega(x) = \sum_{r=1}^{n-1} (-1)^{(p_1+\cdots+p_r)}\, \theta_{1,n}\, \hat\theta_{1,r}\, \hat\theta_{r+1,n} \left(\sum_{\hat\nu \in S_{\hat a_{1,r}}} \sum_{\hat\eta \in S_{\hat a_{r+1,n}}} c_{\hat\nu}\, c_{\hat\eta}\, \chi_{\hat\nu\cup\hat\eta} \left(\sum_{\hat\tau \in T_{\hat x}} c_{\hat\tau} \Delta_{\hat\tau} \right)\right).$$

We also have $(-1)^{(p_1+\cdots+p_r)}\, \hat\theta_{1,r}\, \hat\theta_{r+1,n} = (-1)^{1+\overline{\deg}(\hat a_{1,r})}\, (-1)^{|\hat J_1\cup\cdots\cup \hat J_r|(p_{r+1}+\cdots+p_n+1)}\, \hat\theta_{1,n}$. Hence

$$\omega(x) = \theta_{1,n}\, \hat\theta_{1,n}\, \hat\omega(\hat x).$$

So since $\hat\omega(\hat x) \neq 0$, also $\omega(x) \neq 0$. Thus $[\omega] \in H^{p_1+\cdots+p_n+1}(\mathcal{K}_{J_1\cup\cdots\cup J_n})$ is non-zero. □

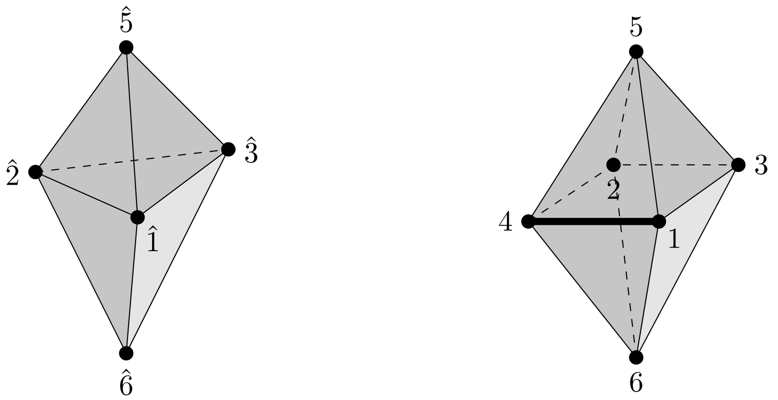
Therefore by Section 2.2.2, we have shown that $\langle \alpha_1, \ldots, \alpha_n \rangle \subset H^{(p_1+\cdots+p_n)+|J_1\cup\cdots\cup J_n|+2}(\mathcal{Z}_{\mathcal{K}})$ contains a non-zero element. It remains to show that every other element $[\omega'] \in \langle \alpha_1, \ldots, \alpha_n \rangle$ is also non-zero. In Lemma 3.2.14, ω was the associated cocycle of a defining system for $\langle \alpha_1, \ldots, \alpha_n \rangle$ that was constructed from a defining system for $\langle \hat\alpha_1, \ldots, \hat\alpha_n \rangle$. Therefore in a sense ω was derived from an associated cocycle $\hat\omega \in C^*(\mathcal{Z}_{\hat{\mathcal{K}}})$. However, not every defining system for $\langle \alpha_1, \ldots, \alpha_n \rangle$ can be directly constructed from a defining system for $\langle \hat\alpha_1, \ldots, \hat\alpha_n \rangle$ in this way.

Example 3.2.15. Let $\hat{\mathcal{K}}_1$ be a triangulation of S^1 on three vertices, $\{\hat 1, \hat 2, \hat 3\}$. Let $\hat{\mathcal{K}}_2 = \{\{\hat 5\}, \{\hat 6\}\}$, and let $\hat{\mathcal{K}}_3 = \{\{\hat 7\}, \{\hat 8\}\}$. Let $\hat\alpha_1 = [\chi_{\hat 1\hat 3}] \in \widetilde{H}^1(\hat{\mathcal{K}}_1)$, $\hat\alpha_2 = [\chi_{\hat 5}] \in \widetilde{H}^0(\hat{\mathcal{K}}_2)$ and $\hat\alpha_3 = [\chi_{\hat 7}] \in \widetilde{H}^0(\hat{\mathcal{K}}_3)$. Let $\hat{\mathcal{K}}$ be a simplicial complex on the vertices $\{\hat 1, \hat 2, \hat 3, \hat 5, \hat 6, \hat 7, \hat 8\}$, obtained from $\mathrm{ss}_{\{\hat 5, \hat 8\}}\, \mathrm{ss}_{\{\hat 1, \hat 3, \hat 6\}}\, \hat{\mathcal{K}}_1 * \hat{\mathcal{K}}_2 * \hat{\mathcal{K}}_3$ by restricting to the original vertices. The simplicial complex $\hat{\mathcal{K}}_{\hat 1,\hat 2,\hat 3,\hat 5,\hat 6}$ is shown in Figure 3.9a. Then by Theorem 3.1.15, there is a non-trivial triple Massey product $\langle \hat\alpha_1, \hat\alpha_2, \hat\alpha_3 \rangle \subset H^*(\mathcal{Z}_{\hat{\mathcal{K}}})$. There are a number of options for $\hat a_{1,2}$ such that $d(\hat a_{1,2}) = (-1)^{1+\deg(\hat a_1)}\hat a_1 \hat a_2 = (-1)^{3+1}\chi_{\hat 1\hat 3\hat 5} = \chi_{\hat 1\hat 3\hat 5}$. For example $\chi_{\hat 1\hat 3}$ or $-\chi_{\hat 1\hat 6} - \chi_{\hat 1\hat 2} - \chi_{\hat 1\hat 5}$.

Let $\mathcal{K}$ be the simplicial complex on vertices $\{1, \ldots, 8\}$ that edge contracts to $\hat{\mathcal{K}}$ by contracting the edge $\{1, 4\} \mapsto \{\hat 1\}$ as in Figure 3.9b. This edge contraction satisfies the link condition. By Construction 3.2.7, we have cocycles $a_1 = \chi_{13}$, $a_2 = \chi_5$, $a_3 = \chi_7$. Then $a_1 a_2 = \chi_{13}\chi_5 = (-1)^4\chi_{135} = \chi_{135}$. Using (3.22), we can construct options for $a_{1,2}$. For example, $\chi_{\hat 1\hat 3}$ becomes $\theta_{1,2}\hat\theta_{1,2}\chi_{13} = -\chi_{13}$. For the cochain $\hat a_{1,2} = -\chi_{\hat 1\hat 6} - \chi_{\hat 1\hat 2} - \chi_{\hat 1\hat 5}$, the support is $S_{\hat a_{1,2}} = \{\{\hat 1, \hat 6\}, \{\hat 1, \hat 2\}, \{\hat 1, \hat 5\}\}$. Then

$$\varphi_1^{-1}(\{\hat 1, \hat 6\}) = \{\{1,6\},\{4,6\}\}, \quad \varphi_1^{-1}(\{\hat 1, \hat 2\}) = \{\{2,4\}\}, \quad \varphi_1^{-1}(\{\hat 1, \hat 5\}) = \{\{1,5\},\{4,5\}\}.$$

Therefore by (3.22), $a_{1,2} = -\theta_{1,2}\hat\theta_{1,2}(\chi_{16}+\chi_{46}+\chi_{24}+\chi_{45}+\chi_{15}) = \chi_{16}+\chi_{46}+\chi_{24}+\chi_{45}+\chi_{15}$. Nevertheless, there are other options for $a_{1,2}$ that cannot be constructed from any $\hat a_{1,2}$.



(a) The simplicial complex $\hat{\mathcal{K}}_{\hat{1},\hat{2},\hat{3},\hat{5},\hat{6}}$, which is missing the simplex $\{\hat{1},\hat{3},\hat{6}\}$.

(b) The simplicial complex $\mathcal{K}_{1,2,3,4,5,6}$, which is missing the simplex $\{1,3,6\}$

Figure 3.9: The simplicial complex $\mathcal{K}_{1,2,3,4,5,6}$ maps to $\hat{\mathcal{K}}_{\hat{1},\hat{2},\hat{3},\hat{5},\hat{6}}$ by contracting the edge $\{1,4\} \mapsto \{\hat{1}\}$.

For example, let $a_{1,2} = -\chi_{16} - \chi_{14} - \chi_{15}$. For the edge $\{1,4\} \in \mathcal{K}$, $\{1,4\} \notin \varphi_1^{-1}(\hat{e})$ for any edge $\hat{e} \in \hat{\mathcal{K}}$, so $a_{1,2}$ does not directly correspond to any $\hat{a}_{1,2}$ as in (3.22). However,

$$\begin{aligned} a_{1,2} - d(\chi_1) &= -\chi_{16} - \chi_{14} - \chi_{15} - (\chi_{16} + \chi_{14} + \chi_{15} + \chi_{13}) \\ &= -\chi_{13} = \theta_{1,2}\hat{\theta}_{1,2} \sum_{\tau \in \varphi^{-1}_{p_i+\dots+p_k}(\hat{1}\hat{3})} \chi_\tau. \end{aligned}$$

Therefore $a_{1,2}$ does correspond to a choice of $\hat{a}_{1,2}$ after adding a coboundary. In a similar way, in the proof of Proposition 3.2.16 we will show that any defining system $(a_{i,k})$ for $\langle \alpha_1, \dots, \alpha_n \rangle$ corresponds to a defining system $(\hat{a}_{i,k})$ for $\langle \hat{\alpha}_1, \dots, \hat{\alpha}_n \rangle$.

Proposition 3.2.16. *The n-Massey product $\langle \alpha_1, \dots, \alpha_n \rangle$ is non-trivial.*

Proof. For any series of edge contractions $\varphi \colon \mathcal{K} \to \hat{\mathcal{K}}$, we can repeat the arguments in this proof for each edge contraction in turn. Suppose that $\varphi \colon \mathcal{K} \to \hat{\mathcal{K}}$ is the contraction of just one edge $\{u,v\} \in \mathcal{K}$. By Construction 3.2.7, $\{u,v\} \subset J_i$ for $i \in \{1,\dots,n\}$.

For $a_{i,i} = a_i$ as defined in (3.17), let $(a_{i,k})$ be a defining system for $\langle \alpha_1, \dots, \alpha_n \rangle$,

$$a_{i,k} = \sum_{\sigma \in S_{a_{i,k}}} c_\sigma \chi_\sigma \in C^{p_i+\dots+p_k}(\mathcal{K}_{J_i \cup \dots \cup J_k}).$$

We will show that any defining system $(a_{i,k})$ corresponds to a defining system $(\hat{a}_{i,k})$ for $\langle \hat{\alpha}_1, \dots, \hat{\alpha}_n \rangle$ in $H^*(\mathcal{Z}_{\hat{\mathcal{K}}})$. There are two main stages to this proof. Firstly, for a defining system $(a_{i,k})$ such that $\{u,v\} \notin \sigma$ for any $\sigma \in S_{a_{i,k}}$ and any pair $\{i,k\}$, we construct a corresponding defining system $(\varphi^*(a_{i,k}))$ for $\langle \hat{\alpha}_1, \dots, \hat{\alpha}_n \rangle$. Secondly, for any other defining system $(a_{i,k})$, we create a different defining system $(\widetilde{a}_{i,k})$ for $\langle \alpha_1, \dots, \alpha_n \rangle$ such that $\{u,v\} \notin \sigma$ for any $\sigma \in S_{\widetilde{a}_{i,k}}$ and any pair $\{i,k\}$. Then applying the first step to $(\widetilde{a}_{i,k})$, we have a defining system $(\varphi^*(\widetilde{a}_{i,k}))$ that corresponds to $(a_{i,k})$.

For this first step, suppose that $\{u,v\} \notin \sigma$ for any $\sigma \in S_{a_{i,k}}$ and any pair $\{i,k\}$. That is, no simplex $\sigma \in S_{a_{i,k}}$ is contracted. Let $a \in C^p(\mathcal{K}_{J_i\cup\cdots\cup J_k})$ be a general cochain such that $\{u,v\} \notin \sigma$ for any $\sigma \in S_a$, where either $p = p_i + \cdots + p_k$ or $p = p_i + \cdots + p_k + 1$. We will define $\varphi^*(a) \in C^p(\hat{\mathcal{K}}_{\varphi(J_i\cup\cdots\cup J_k)})$ when either a is a cocycle, or $a = a_{i,k}$ and $p = p_i + \cdots + p_k$, or $\varphi(\sigma) \neq \varphi(\sigma')$ for any $\sigma, \sigma' \in S_a$.

Let $a = \sum_{\sigma\in S_a} c_\sigma \chi_\sigma$. Suppose there are simplices $\sigma, \sigma' \in S_a$ such that $\varphi(\sigma) = \varphi(\sigma')$. We will show that $c_\sigma = c_{\sigma'}$. Without loss of generality, let $u \in \sigma$ and $v \in \sigma'$. By the link condition, $\sigma \cup v = \sigma' \cup u$ is a simplex in $\mathcal{K}$. So, $d(a)$ contains summands such as

$$d(c_\sigma \chi_\sigma + c_{\sigma'}\chi_{\sigma'}) = c_\sigma\ \varepsilon(v, \sigma\cup v)\chi_{\sigma\cup v} + c_{\sigma'}\ \varepsilon(u, \sigma'\cup u)\chi_{\sigma'\cup u} + \text{ other terms.}$$

Due to the labelling of vertices in Construction 3.2.7, the labels u and v are always consecutive in $\sigma \cup v = \sigma' \cup u$. That is, $\varepsilon(v, \sigma \cup v) = \varepsilon(u, \sigma \cup v)$. So,

$$d(c_\sigma \chi_\sigma + c_{\sigma'}\chi_{\sigma'}) = \pm(c_\sigma - c_{\sigma'})\chi_{\sigma\cup v} + \text{ other terms.}$$

If a is a cocycle, then $c_\sigma = c_{\sigma'}$ because $d(a) = 0$. If $a = a_{i,k}$, then $d(a_{i,k}) = \sum_{r=i}^{k-1} \overline{\overline{a_{i,r}}} a_{r,k}$. Hence if $c_\sigma \neq c_{\sigma'}$, there is an index $r \in \{i, \dots, k-1\}$ such that $\sigma \cup v = \tau \cup \eta$ for $\tau \in S_{i,r}$ and $\eta \in S_{r+1,k}$. Since $\{u,v\} \subset J_i$ for some $i \in \{1,\dots,n\}$ and $S_{i,k} \subset J_i \cup\cdots\cup J_k$ for any $\{i,k\}$, either $\{u,v\} \in \tau$ or $\{u,v\} \in \eta$. This contradicts the assumption that $\{u,v\} \notin \sigma$ for any $\sigma \in S_{a_{i,k}}$ and any $\{i,k\}$. So $c_\sigma = c_{\sigma'}$.

For $J \subset [m]$, let $\hat{J} = \varphi(J)$. Then when either a is a cocycle, or $a = a_{i,k}$ and $p = p_i + \cdots + p_k$, or $\varphi(\sigma) \neq \varphi(\sigma')$ for any $\sigma, \sigma' \in S_a$, let

$$\varphi^*(a) = c_{i,k} \sum_{\hat{\sigma}\in\varphi(S_a)} c_{\hat{\sigma}}\chi_{\hat{\sigma}} \ \in\ C^p(\hat{\mathcal{K}}_{\hat{J}_i\cup\cdots\cup\hat{J}_k}) \tag{3.31}$$

where $c_{\hat{\sigma}} = c_\sigma$ for any $\sigma \in S_a$ such that $\varphi(\sigma) = \hat{\sigma}$, $c_{i,i} = 1$ and

$$c_{i,k} = (-1)^{(|J_i|-|\hat{J}_i|)p_{i+1}}(-1)^{(|J_i\cup J_{i+1}|-|\hat{J}_i\cup\hat{J}_{i+1}|)p_{i+2}}\cdots(-1)^{(|J_i\cup\cdots\cup J_{k-1}|-|\hat{J}_i\cup\cdots\cup\hat{J}_{k-1}|)p_k}.$$

To show that $(\varphi^*(a_{i,k}))$ is a defining system for $\langle\hat{\alpha}_1, \dots, \hat{\alpha}_n\rangle$, we will check three properties of $\varphi^*(a)$. Firstly, for any constant $c' \in \mathbf{k}$ and for $a = c_\sigma\chi_\sigma$, $b = c_\tau\chi_\tau$ in $C^p(\mathcal{K}_{J_i\cup\cdots\cup J_k})$ where p is either $p_i + \cdots + p_k$ or $p_i + \cdots + p_k + 1$ and $\{u,v\} \notin \sigma, \tau$,

$$\begin{gathered}\varphi^*(c'a) = c_{i,k}\ c'c_\sigma\chi_{\varphi(\sigma)} = c'\varphi^*(a) \text{ and}\\ \varphi^*(a+b) = c_{i,k}\ (c_\sigma\chi_{\varphi(\sigma)} + c_\tau\chi_{\varphi(\tau)}) = \varphi^*(a) + \varphi^*(b).\end{gathered} \tag{3.32}$$

Secondly, let $a = \sum_{\sigma\in S_a} c_\sigma\chi_\sigma \in C^{p_i+\cdots+p_k}(\mathcal{K}_{J_i\cup\cdots\cup J_k})$. Then

$$d(a) = \sum_{\sigma\in S_a} \sum_{\substack{j\in J_i\cup\cdots\cup J_k\setminus\sigma,\\ j\cup\sigma\in\mathcal{K}_{J_i\cup\cdots\cup J_k}}} c_\sigma\ \varepsilon(j, j\cup\sigma)\ \chi_{j\cup\sigma}.$$

Suppose that for every summand $\chi_{j\cup\sigma}$ that is not cancelled by other summands in $d(a)$,

$j \cup \sigma$ does not contract. That is, $|j \cup \sigma| = p_i + \cdots + p_k + 2 = |\hat{j} \cup \hat{\sigma}|$ where $\hat{j} = \varphi(j)$ and $\hat{\sigma} = \varphi(\sigma)$. We want to show that $\varphi^*(d(a))$ is a coboundary. We have

$$\varphi^*(d(a)) = c_{i,k} \sum_{\hat{\sigma} \in \varphi(S_a)} c_{\hat{\sigma}} \Bigg(\sum_{\substack{\hat{j} \in \hat{J}_i \cup \cdots \cup \hat{J}_k \setminus \hat{\sigma}, \\ \hat{j} \cup \hat{\sigma} \in \hat{\mathcal{K}}_{\hat{J}_i \cup \cdots \cup \hat{J}_k}, \\ |\hat{j} \cup \hat{\sigma}| = p_i + \cdots + p_k + 2}} \varepsilon(\hat{j}, \hat{j} \cup \hat{\sigma})\, \chi_{\hat{j} \cup \hat{\sigma}} \Bigg)$$

where $\varepsilon(j, j \cup \sigma) = \varepsilon(\hat{j}, \hat{j} \cup \hat{\sigma})$ due to the order on vertices in $\mathcal{K}$ and since $j \cup \sigma$ does not contract. Let $\hat{S} = \{\varphi(\sigma) \mid \sigma \in S_a, |\varphi(\sigma)| = p_i + \cdots + p_k + 1\}$ and let $b = \sum_{\hat{\sigma} \in \hat{S}} c_{\hat{\sigma}} \chi_{\hat{\sigma}} \in C^{p_i + \cdots + p_k}(\hat{\mathcal{K}}_{\hat{J}_i \cup \cdots \cup \hat{J}_k})$. Then

$$\varphi^*(d(a)) = d(b). \tag{3.33}$$

In particular, let $a = a_{i,k}$ for some i, k, so $\{u, v\} \notin \sigma$ for any $\sigma \in S_{a_{i,k}}$. Suppose that for a simplex $\sigma \in S_{a_{i,k}}$, there is a simplex $j \cup \sigma \in \mathcal{K}_{J_i \cup \cdots \cup J_k}$ for $j \in J_i \cup \cdots \cup J_k \setminus \sigma$ that is contracted. That is, $\{u, v\} \in j \cup \sigma$. By the definition of a defining system, $d(a_{i,k}) = \sum_{r=i}^{k-1} \overline{\overline{a_{i,r}}} a_{r,k}$. Therefore either $c_\sigma\ \varepsilon(j, j \cup \sigma)\ \chi_{j \cup \sigma}$ is cancelled by other terms in $d(a_{i,k})$, or there exists $i \leqslant r < k$ and simplices $\tau \in S_{a_{i,r}}$, $\eta \in S_{a_{r+1,k}}$ such that $\tau \cup \eta = j \cup \sigma$. In the latter case, if $\{u, v\} \in j \cup \sigma$, then $\{u, v\} \in \tau \cup \eta$. This implies that either $\{u, v\} \in \tau$ or $\{u, v\} \in \eta$, since by construction $\{u, v\} \subset J_i$ for an $1 \leqslant i \leqslant n$ and $\tau \in S_{a_{i,r}} \subset J_i \cup \cdots \cup J_r$, $\eta \in S_{a_{r+1,k}} \subset J_{r+1} \cup \cdots \cup J_k$. This then contradicts the assumption that $\{u, v\} \notin \sigma$ for any $\sigma \in S_{a_{i,k}}$ and any $\{i, k\}$. Hence a summand of the form $c_\sigma\ \varepsilon(j, j \cup \sigma)\ \chi_{j \cup \sigma}$, where $\{u, v\} \in j \cup \sigma$, is cancelled out by other summands. Therefore $\varphi^*(d(a_{i,k})) = d(\varphi^*(a_{i,k}))$.

Thirdly, let $a_{i,r} = \sum_{\tau \in S_{a_{i,r}}} c_\tau \chi_\tau$, $a_{i,r} \in C^{p_i + \cdots + p_r}(\mathcal{K}_{J_i \cup \cdots \cup J_r})$ and $a_{r+1,k} = \sum_{\eta \in S_{a_{r+1,k}}} c_\eta \chi_\eta$, $a_{r+1,k} \in C^{p_{r+1} + \cdots + p_k}(\mathcal{K}_{J_{r+1} \cup \cdots \cup J_k})$. Then

$$\begin{aligned}
&\sum_{r=i}^{k-1} \overline{\overline{\varphi^*(a_{i,r})}} \varphi^*(a_{r+1,k}) \\
&\quad = \sum_{r=i}^{k-1} (-1)^{1 + \overline{\deg}\, \varphi^*(a_{i,r})} \left(c_{i,r} \sum_{\hat{\tau} \in \varphi(S_{a_{i,r}})} c_\tau \chi_{\hat{\tau}} \right) \cdot \left(c_{r+1,k} \sum_{\hat{\eta} \in \varphi(S_{a_{r+1,k}})} c_\eta \chi_{\hat{\eta}} \right) \\
&\quad = \sum_{r=i}^{k-1} C \left(\sum_{\hat{\tau} \in \varphi(S_{a_{i,r}})} \sum_{\hat{\eta} \in \varphi(S_{a_{r+1,k}})} c_\tau c_\eta \chi_{\hat{\tau} \cup \hat{\eta}} \right)
\end{aligned}$$

where

$$C = (-1)^{1 + \overline{\deg}\, \varphi^*(a_{i,r})} (-1)^{|\hat{J}_i \cup \cdots \cup \hat{J}_r|(p_{r+1} + \cdots + p_k + 1)} c_{i,r} c_{r+1,k}.$$

Since $(-1)^2 = 1$, $(-1)^{|\hat{J}_i \cup \cdots \cup \hat{J}_r|(p_{r+1} + \cdots + p_k + 1)} = (-1)^{-|\hat{J}_i \cup \cdots \cup \hat{J}_r|(p_{r+1} + \cdots + p_k + 1)}$. So using the expressions for $c_{i,r}$ and $c_{r+1,k}$, and using $\overline{\deg}\, \varphi^*(a_{i,r}) = 1 + p_i + \cdots + p_r + |\hat{J}_i \cup \cdots \hat{J}_r|$,

$$C = (-1)^{2 + p_i + \cdots + p_r + |\hat{J}_i \cup \cdots \hat{J}_r|} (-1)^{-|\hat{J}_i \cup \cdots \cup \hat{J}_r|(p_{r+1} + \cdots + p_k + 1)}$$

$$
\begin{aligned}
&\cdot (-1)^{(|J_i|-|\hat{J}_i|)p_{i+1}} \cdots (-1)^{(|J_i\cup\cdots\cup J_{r-1}|-|\hat{J}_i\cup\cdots\cup\hat{J}_{r-1}|)p_r} \\
&\cdot (-1)^{(|J_{r+1}|-|\hat{J}_{r+1}|)p_{r+2}} \cdots (-1)^{(|J_{r+1}\cup\cdots\cup J_{k-1}|-|\hat{J}_{r+1}\cup\cdots\cup\hat{J}_{k-1}|)p_k} \\
=&(-1)^{p_i+\cdots+p_r}(-1)^{(|J_i|-|\hat{J}_i|)p_{i+1}} \cdots (-1)^{(|J_i\cup\cdots\cup J_{r-1}|-|\hat{J}_i\cup\cdots\cup\hat{J}_{r-1}|)p_r} \cdot (-1)^{|\hat{J}_i\cup\cdots\cup\hat{J}_r|p_{r+1}} \\
&\cdot (-1)^{(|J_{r+1}|-|\hat{J}_{r+1}|-|\hat{J}_i\cup\cdots\cup\hat{J}_r|)p_{r+2}} \cdots (-1)^{(|J_{r+1}\cup\cdots\cup J_{k-1}|-|\hat{J}_{r+1}\cup\cdots\cup\hat{J}_{k-1}|-|\hat{J}_i\cup\cdots\cup\hat{J}_r|)p_k} \\
=&(-1)^{p_i+\cdots+p_r}(-1)^{|J_i\cup\cdots\cup J_r|(p_{r+1}+\cdots+p_k)}c_{i,k} \\
=&(-1)^{1+\overline{\deg}\, a_{i,r}}(-1)^{|J_i\cup\cdots\cup J_r|(p_{r+1}+\cdots+p_k+1)}c_{i,k}.
\end{aligned}
$$

By assumption, $\{u,v\} \notin \sigma$ for any $\sigma \in S_{a_{i,k}}$ and any $\{i,k\}$. Thus $\{u,v\} \notin \tau$ and $\{u,v\} \notin \eta$ for any $i \leqslant r < k$ and any simplices $\tau \in S_{a_{i,r}}$, $\eta \in S_{a_{r+1,k}}$. Also by construction, $\{u,v\} \subset J_i$ for an index $1 \leqslant i \leqslant n$, so $\{u,v\} \notin \tau \cup \eta$. Hence $\varphi(\tau\cup\eta) = \varphi(\tau)\cup\varphi(\eta)$ is a $(p_i+\cdots+p_k+1)$-simplex. Therefore using the definition of $\varphi^*(a)$, the properties in (3.32), and the fact that $\varphi(\tau\cup\eta) = \varphi(\tau)\cup\varphi(\eta) = \hat{\tau}\cup\hat{\eta}$,

$$
\begin{aligned}
&\sum_{r=i}^{k-1} \overline{\overline{\varphi^*(a_{i,r})}}\varphi^*(a_{r+1,k}) \\
&\quad = \sum_{r=i}^{k-1}(-1)^{1+\overline{\deg}\, a_{i,r}}(-1)^{|J_i\cup\cdots\cup J_r|(p_{r+1}+\cdots+p_k+1)}c_{i,k}\left(\sum_{\hat{\tau}\in\varphi(S_{a_{i,r}})}\ \sum_{\hat{\eta}\in\varphi(S_{a_{r+1,k}})} c_\tau c_\eta \chi_{\hat{\tau}\cup\hat{\eta}}\right) \\
&\quad = \varphi^*\left(\sum_{r=i}^{k-1} \overline{\overline{a_{i,r}}}a_{r+1,k}\right).
\end{aligned}
$$

Pairing this with the fact that $\varphi^*(d(a_{i,k})) = d(\varphi^*(a_{i,k}))$ for any i,k, we have that

$$
d(\varphi^*(a_{i,k})) = \varphi^*(d(a_{i,k})) = \varphi^*\left(\sum_{r=i}^{k-1}\overline{\overline{a_{i,r}}}a_{r+1,k}\right) = \sum_{r=i}^{k-1}\overline{\overline{\varphi^*(a_{i,r})}}\varphi^*(a_{r+1,k}).
$$

Lastly, by the definition of $a_i = a_{i,i}$ in (3.17), $\varphi^*(a_{i,i}) = \hat{a}_{i,i} = \hat{a}_i$. Hence $(\varphi^*(a_{i,k}))$ is a defining system for $\langle\hat{\alpha}_1,\ldots,\hat{\alpha}_n\rangle$ if $(a_{i,k})$ is a defining system such that $\{u,v\}\notin\sigma$ for any $\sigma\in S_{a_{i,k}}$ and any pair $\{i,k\}$. Also, if ω is the associated cocycle for $(a_{i,k})$, then

$$
\varphi^*(\omega) = \varphi^*(\sum_{r=i}^{k-1}\overline{\overline{a_{i,r}}}a_{r+1,k}) = \sum_{r=i}^{k-1}\overline{\overline{\varphi^*(a_{i,r})}}\varphi^*(a_{r+1,k})
$$

so $\varphi^*(\omega)$ is the associated cocycle for $(\varphi^*(a_{i,k}))$. Moreover if $[\omega]=0$, then there is a cochain $a \in C^{p_1+\cdots+p_n}(\mathcal{K}_{J_1\cup\cdots\cup J_n})$ such that $\omega + d(a) = 0$. No simplices in S_ω contract, and $d(a) = -\omega$. Then by (3.32) and (3.33), $\varphi^*(\omega+d(a)) = \varphi^*(\omega)+d(b) = 0$ for a cochain $b \in C^{p_i+\cdots+p_k}(\hat{\mathcal{K}}_{\hat{J}_i\cup\cdots\cup\hat{J}_k})$. So $[\varphi^*(\omega)]=0$, which contradicts the non-triviality of $\langle\hat{\alpha}_1,\ldots,\hat{\alpha}_n\rangle$. Hence $[\omega]\neq 0$.

For the second stage of this proof, suppose that $(a_{i,k})$ is a defining system for $\langle\alpha_1,\ldots,\alpha_n\rangle$ such that there is a pair of indices $\{i,k\}$ with $\{u,v\}\in\sigma$ for some $\sigma\in S_{a_{i,k}}$. We will create another defining system $(\widetilde{a}_{i,k})$ for $\langle\alpha_1,\ldots,\alpha_n\rangle$ such that $\{u,v\}\notin\sigma$ for any $\sigma\in S_{\widetilde{a}_{i,k}}$ and such that $[\omega]=[\widetilde{\omega}]$ where ω, $\widetilde{\omega}$ are the associated cocycles for $(a_{i,k})$, $(\widetilde{a}_{i,k})$, respectively.

The cocycle $a_i = a_{i,i}$ as defined in (3.17) is such that $\{u,v\} \notin \sigma$ for every $\sigma \in S_{a_i}$. Therefore, let $\{i,k\}$ be a pair of indices such that there is a simplex $\sigma \in S_{a_{i,k}}$ with $\{u,v\} \in \sigma$, and for every $i < i' < k' < k$, $\{u,v\} \notin \sigma$ for any $\sigma \in S_{a_{i',k'}}$. Let $\sigma \in S_{a_{i,k}}$ be a simplex such that $\{u,v\} \in \sigma$, and let c_σ be the (non-zero) coefficient of χ_σ in $a_{i,k}$. Then for every pair $\{i',k'\} \subset [n]$, let $c = -(-1)^{1+\overline{\deg}\, a_{i,k}} c_\sigma\, \varepsilon(u,\sigma)$ and let

$$\widetilde{a}_{i',k'} = \begin{cases} a_{i',k'} - c_\sigma\, \varepsilon(u,\sigma)\, d(\chi_{\sigma\setminus u}) & \text{if } i' = i < k = k', \\ a_{i',k'} + c_\sigma\, \varepsilon(u,\sigma)\, a_{i',i-1}\chi_{\sigma\setminus u} & \text{if } i' < i < k = k', \\ a_{i',k'} + c\, \chi_{\sigma\setminus u} a_{k+1,k'} & \text{if } i' = i < k < k', \\ a_{i',k'} & \text{if } i' < i < k < k' \text{ or } i < i' < k' < k \end{cases} \tag{3.34}$$

where $\chi_{\sigma\setminus u} \in C^{p_i+\cdots p_k-1}(\mathcal{K}_{J_i\cup\cdots\cup J_k})$. We will show that $(\widetilde{a}_{i',k'})$ is a defining system for $\langle \alpha_1, \ldots, \alpha_n \rangle$. Firstly since $k - i > 1$, $\widetilde{a}_{i',i'} = a_{i',i'}$ for every $i' \in [n]$. We also need to show that $d(\widetilde{a}_{i',k'}) = \sum_{r=i'}^{k'-1} \overline{\widetilde{a}_{i',r}}\widetilde{a}_{r+1,k'}$ for every $\{i',k'\}$.

For $i < i' < k' < k$, we have that $\widetilde{a}_{i',k'} = a_{i',k'}$ so

$$d(\widetilde{a}_{i',k'}) = d(a_{i',k'}) = \sum_{r=i'}^{k'-1} \overline{a_{i',r}} a_{r+1,k'} = \sum_{r=i'}^{k'-1} \overline{\widetilde{a}_{i',r}} \widetilde{a}_{r+1,k'}.$$

For $i' = i < k = k'$,

$$d(\widetilde{a}_{i,k}) = d(a_{i,k} - c_\sigma\, \varepsilon(u,\sigma)\, d(\chi_{\sigma\setminus u})) = d(a_{i,k})$$

since $d(d(\chi_{\sigma\setminus u})) = 0$. Since $\chi_{\sigma\setminus u} \in C^{p_i+\cdots p_k-1}(\mathcal{K}_{J_i\cup\cdots\cup J_k})$, $d(\chi_{\sigma\setminus u}) \in C^{p_i+\cdots p_k}(\mathcal{K}_{J_i\cup\cdots\cup J_k})$. Hence $\widetilde{a}_{i,k} \in C^{p_i+\cdots p_k}(\mathcal{K}_{J_i\cup\cdots\cup J_k})$ and $\overline{\deg}\, \widetilde{a}_{i,k} = \overline{\deg}\, a_{i,k}$. Additionally,

$$d(\chi_{\sigma\setminus u}) = \sum_{\substack{j \in J_i\cup\cdots\cup J_k \setminus (\sigma\setminus u), \\ j\cup\sigma\setminus u \in \mathcal{K}_J}} \varepsilon(j, j\cup\sigma\setminus u)\chi_{j\cup\sigma\setminus u}.$$

So χ_σ is the only summand of $d(\chi_{\sigma\setminus u})$ such that $\{u,v\} \in \sigma$. Thus $a_{i,k} - c_\sigma\, \varepsilon(u,\sigma)\, d(\chi_{\sigma\setminus u})$ no longer contains the summand χ_σ and also

$$|\{\tau \in S_{\widetilde{a}_{i,k}} : \{u,v\} \in \tau\}| < |\{\tau \in S_{a_{i,k}} : \{u,v\} \in \tau\}|.$$

Next, for $i' < i < k = k'$, we have $a_{i',i-1} \in C^{p_{i'}+\cdots+p_{i-1}}(\mathcal{K}_{J_{i'}\cup\cdots\cup J_{i-1}})$ and so $a_{i',i-1}\chi_{\sigma\setminus u} \in C^{p_{i'}+\cdots+p_k}(\mathcal{K}_{J_{i'}\cup\cdots\cup J_k})$. Hence $\widetilde{a}_{i',k} \in C^{p_{i'}+\cdots+p_k}(\mathcal{K}_{J_{i'}\cup\cdots\cup J_k})$. Also,

$$\begin{aligned} d(\widetilde{a}_{i',k}) &= d(a_{i',k} + c_\sigma\, \varepsilon(u,\sigma)\, a_{i',i-1}\chi_{\sigma\setminus u}) \\ &= d(a_{i',k}) + c_\sigma\, \varepsilon(u,\sigma) \left(d(a_{i',i-1})\chi_{\sigma\setminus u} + (-1)^{\overline{\deg}\, a_{i',i-1}} a_{i',i-1} d(\chi_{\sigma\setminus u}) \right) \\ &= \sum_{r=i'}^{k-1} \overline{a_{i',r}} a_{r+1,k} + c_\sigma\, \varepsilon(u,\sigma) \left(\sum_{r=i'}^{i-2} \overline{a_{i',r}} a_{r+1,i-1} \right) \chi_{\sigma\setminus u} - c_\sigma\, \varepsilon(u,\sigma)\, \overline{a_{i',i-1}} d(\chi_{\sigma\setminus u}) \\ &= \sum_{r=i'}^{i-2} \overline{a_{i',r}} (a_{r+1,k} + c_\sigma\, \varepsilon(u,\sigma)\, a_{r+1,i-1}\chi_{\sigma\setminus u}) + \overline{a_{i',i-1}}(a_{i,k} - c_\sigma\, \varepsilon(u,\sigma)\, d(\chi_{\sigma\setminus u})) \end{aligned}$$

$$+ \sum_{r=i}^{k-1} \overline{\overline{a_{i',r}}} a_{r+1,k}$$

$$= \sum_{r=i'}^{k-1} \overline{\overline{\widetilde{a}_{i',r}}} \widetilde{a}_{r+1,k}.$$

For $i' = i < k < k'$, we have that $\widetilde{a}_{i,k'} \in C^{p_i+\cdots+p_{k'}}(\mathcal{K}_{J_i \cup \cdots \cup J_{k'}})$ since $\chi_{\sigma \setminus u} a_{k+1,k'} \in C^{p_i+\cdots+p_{k'}}(\mathcal{K}_{J_i \cup \cdots \cup J_{k'}})$. Furthermore,

$$\begin{aligned} d(\widetilde{a}_{i,k'}) &= d(a_{i,k'} - (-1)^{1+\overline{\deg}\, a_{i,k}} c_\sigma\ \varepsilon(u,\sigma)\ \chi_{\sigma \setminus u} a_{k+1,k'}) \\ &= \sum_{r=i}^{k'-1} \overline{\overline{a_{i,r}}} a_{r+1,k'} - (-1)^{1+\overline{\deg}\, a_{i,k}} c_\sigma\ \varepsilon(u,\sigma)\ d(\chi_{\sigma \setminus u}) a_{k+1,k'} \\ &\qquad - (-1)^{1+\overline{\deg}\, a_{i,k}} c_\sigma\ \varepsilon(u,\sigma)\ (-1)^{\overline{\deg}\, \chi_{\sigma \setminus u}} \chi_{\sigma \setminus u} \left(\sum_{r=k+1}^{k'-1} \overline{\overline{a_{k+1,r}}} a_{r+1,k'} \right) \\ &= \sum_{r=i}^{k-1} \overline{\overline{a_{i,r}}} a_{r+1,k} + (-1)^{1+\overline{\deg}\, a_{i,k}} (a_{i,k} - c_\sigma\ \varepsilon(u,\sigma)\ d(\chi_{\sigma \setminus u})) a_{k+1,k'} \\ &\qquad + \sum_{r=k+1}^{k'-1} \left(-(-1)^{1+\overline{\deg}\, a_{i,k}} c_\sigma\ \varepsilon(u,\sigma)\ (-1)^{\overline{\deg}\, \chi_{\sigma \setminus u}} \chi_{\sigma \setminus u} \overline{\overline{a_{k+1,r}}} + \overline{\overline{a_{i,r}}} \right) a_{r+1,k'}. \end{aligned}$$

More specifically, let $c = -(-1)^{1+\overline{\deg}\, a_{i,k}} c_\sigma\ \varepsilon(u,\sigma)$. Then in the last summand,

$$\begin{aligned} c\ (-1)^{\overline{\deg}\, \chi_{\sigma \setminus u}} \chi_{\sigma \setminus u} \overline{\overline{a_{k+1,r}}} \\ &= (-1)^{p_i+\cdots+p_k-1+|J_i \cup \cdots \cup J_k|+1} (-1)^{2+p_{k+1}+\cdots+p_r+|J_{k+1} \cup \cdots \cup J_r|}\ c\ \chi_{\sigma \setminus u} a_{k+1,r} \\ &= (-1)^{2+p_i+\cdots+p_k+|J_i \cup \cdots \cup J_k|}\ c\ \chi_{\sigma \setminus u} a_{k+1,r} \\ &= (-1)^{1+\overline{\deg}\, a_{i,r}}\ c\ \chi_{\sigma \setminus u} a_{k+1,r}. \end{aligned}$$

Therefore

$$\begin{aligned} d(\widetilde{a}_{i,k'}) &= \sum_{r=i}^{k-1} \overline{\overline{a_{i,r}}} a_{r+1,k} + (-1)^{1+\overline{\deg}\, a_{i,k}} (a_{i,k} - c_\sigma\ \varepsilon(u,\sigma)\ d(\chi_{\sigma \setminus u})) a_{k+1,k'} \\ &\qquad + \sum_{r=k+1}^{k'-1} (-1)^{1+\overline{\deg}\, a_{i,r}} (\ c\ \chi_{\sigma \setminus u} a_{k+1,r} + a_{i,r}) a_{r+1,k'} \\ &= \sum_{r=i'}^{k-1} \overline{\overline{\widetilde{a}_{i',r}}} \widetilde{a}_{r+1,k}. \end{aligned}$$

Lastly when $i' < i < k < k'$, consider

$$\sum_{r=i'}^{k'-1} \overline{\overline{\widetilde{a}_{i',r}}} \widetilde{a}_{r+1,k'} = \overline{\overline{a_{i',i-1}}} \widetilde{a}_{i,k'} + \overline{\overline{\widetilde{a}_{i',k}}} a_{k+1,k'} + \sum_{r \in \{i',\ldots,\widehat{i-1},\ldots,\widehat{k},\ldots k'-1\}} \overline{\overline{\widetilde{a}_{i',r}}} \widetilde{a}_{r+1,k'}$$

where $\widehat{l}$ denotes omission. Thus

$$\begin{aligned}\sum_{r=i'}^{k'-1}\overline{\overline{\widetilde{a}_{i',r}}}\widetilde{a}_{r+1,k'} &= \overline{\overline{a_{i',i-1}}}\Big(a_{i,k'}-(-1)^{1+\overline{\deg}\,a_{i,k}}c_\sigma\ \varepsilon(u,\sigma)\ \chi_{\sigma\backslash u}a_{k+1,k'}\Big)\\ &+(-1)^{\overline{\deg}\,a_{i',k}}\Big(a_{i',k'}+c_\sigma\ \varepsilon(u,\sigma)\ a_{i',i-1}\chi_{\sigma\backslash u}\Big)\,a_{k+1,k'}+\sum_{r\in\{i',\ldots,\widehat{i-1},\ldots,\widehat{k},\ldots k'-1\}}\overline{\overline{\widetilde{a}_{i',r}}}\widetilde{a}_{r+1,k'}\\ &=-(-1)^{1+\overline{\deg}\,a_{i',i-1}}(-1)^{1+\overline{\deg}\,a_{i,k}}c_\sigma\ \varepsilon(u,\sigma)\ \chi_{\sigma\backslash u}a_{i',i-1}a_{k+1,k'}\\ &\qquad+(-1)^{\overline{\deg}\,a_{i',k}}c_\sigma\ \varepsilon(u,\sigma)\ a_{i',i-1}\chi_{\sigma\backslash u}a_{k+1,k'}+d(a_{i',k'})\\ &=\Big((-1)^{1+\overline{\deg}\,a_{i',k}}-(-1)^{1+\overline{\deg}\,a_{i',i-1}}(-1)^{1+\overline{\deg}\,a_{i,k}}\Big)\,c_\sigma\ \varepsilon(u,\sigma)\ a_{i',i-1}\chi_{\sigma\backslash u}a_{k+1,k'}+d(a_{i',k'})\end{aligned}$$

where

$$\begin{aligned}(-1)^{1+\overline{\deg}\,a_{i',i-1}}(-1)^{1+\overline{\deg}\,a_{i,k}} &= (-1)^{2+p_{i'}+\cdots+p_{i-1}+|J_{i'}\cup\cdots\cup J_{i-1}|}(-1)^{2+p_i+\cdots+p_k+|J_i\cup\cdots\cup J_k|}\\ &=(-1)^{2+p_{i'}+\cdots+p_k+|J_{i'}\cup\cdots\cup J_k|}=(-1)^{1+\overline{\deg}\,a_{i',k}}.\end{aligned}$$

Hence

$$\begin{aligned}\sum_{r=i'}^{k'-1}\overline{\overline{\widetilde{a}_{i',r}}}\widetilde{a}_{r+1,k'} &= \Big((-1)^{1+\overline{\deg}\,a_{i',k}}-(-1)^{1+\overline{\deg}\,a_{i',k}}\Big)\,c_\sigma\ \varepsilon(u,\sigma)\ a_{i',i-1}\chi_{\sigma\backslash u}a_{k+1,k'}+d(a_{i',k'})\\ &=0+d(a_{i',k'})=d(\widetilde{a}_{i',k'}).\end{aligned}$$

Therefore for all $\{i',k'\}$, $\widetilde{a}_{i',k'}\in C^{p_{i'}+\cdots+p_{k'}}(\mathcal{K}_{J_{i'}\cup\cdots\cup J_{k'}})$ and $d(\widetilde{a}_{i',k'})=\sum_{r=i'}^{k'-1}\overline{\overline{\widetilde{a}_{i',r}}}\widetilde{a}_{r+1,k'}$. So $(\widetilde{a}_{i',k'})$ is a defining system for $\langle\alpha_1,\ldots,\alpha_n\rangle$. Also $\sigma\notin\tau$ for any $\tau\in S_{\widetilde{a}_{i',k'}}$ and any $\{i',k'\}$. The associated cocycle $\widetilde{\omega}$ for this defining system is given by $\sum_{r=1}^{n-1}\overline{\overline{\widetilde{a}_{1,r}}}\widetilde{a}_{r+1,n}$. Thus by calculating $\sum_{r=1}^{n-1}\overline{\overline{\widetilde{a}_{1,r}}}\widetilde{a}_{r+1,n}$ in a similar manner as in the above calculations,

$$\widetilde{\omega}=\begin{cases}\omega & \text{if } i\neq 1, k\neq n,\\ \omega+c_\sigma\ \varepsilon(u,\sigma)\ d(a_{i',i-1}\chi_{\sigma\backslash u}) & \text{if } 1=i<k=n,\\ \omega-(-1)^{1+\overline{\deg}\,a_{i,k}}c_\sigma\ \varepsilon(u,\sigma)d(\chi_{\sigma\backslash u}a_{k+1,k'}) & \text{if } 1=i<k<n\end{cases}\tag{3.35}$$

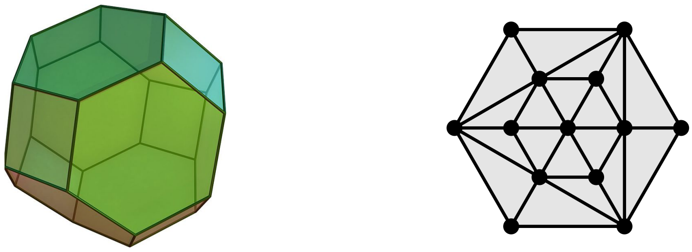
where ω is the associated cocycle for $(a_{i',k'})$. (There is no $a_{1,n}$, so it is not possible for $i=1,k=n$.) So in terms of the cohomology classes, $[\widetilde{\omega}]=[\omega]$. Therefore $[\widetilde{\omega}]=0$ if and only if $[\omega]=0$.

If there is a cochain $\widetilde{a}_{i',k'}$ such that $\{u,v\}\in\sigma$ for some $\sigma\in S_{\widetilde{a}_{i',k'}}$, then we can repeat the above procedure to construct $(\widetilde{\widetilde{a}}_{i',k'})$, etc. In each iteration, $\sigma\notin\tau$ for any $\tau\in S_{\widetilde{a}_{i',k'}}$ and any $\{i',k'\}$. Thus after a finite number of iterations, we obtain a defining system $(\widetilde{a}_{i',k'})$ such that $\{u,v\}\notin\sigma$ for any $\sigma\in S_{\widetilde{a}_{i',k'}}$ and any pair $\{i',k'\}$. Then we can construct a defining system $(\varphi^*(\widetilde{a}_{i',k'}))$ for $\langle\hat{\alpha}_1,\ldots,\hat{\alpha}_n\rangle$. So if $[\omega]=[\widetilde{\omega}]=0$, then $[\varphi^*(\widetilde{\omega})]=0$, which contradicts the assumption that $\langle\hat{\alpha}_1,\ldots,\hat{\alpha}_n\rangle$ is non-trivial. Hence if $\langle\hat{\alpha}_1,\ldots,\hat{\alpha}_n\rangle$ is non-trivial, then $\langle\alpha_1,\ldots,\alpha_n\rangle$ is non-trivial. □

Proposition 3.2.12, Lemma 3.2.14 and Proposition 3.2.16 prove the following statement.

Theorem 3.2.17. *Let $\hat{\mathcal{K}}$ be a simplicial complex with a non-trivial n-Massey product in $H^*(\mathcal{Z}_{\hat{\mathcal{K}}})$. Let $\mathcal{K}$ be a simplicial complex that maps onto $\hat{\mathcal{K}}$ by a series edge contractions $\varphi\colon \mathcal{K} \to \hat{\mathcal{K}}$ that satisfy the link condition. Then there is a non-trivial n-Massey product in $H^*(\mathcal{Z}_{\mathcal{K}})$.*

Remark 3.2.18. The degree of the classes in the new Massey product are different to the degree of classes in the original Massey product. The original Massey product $\langle \hat{\alpha}_1, \dots, \hat{\alpha}_n \rangle \subset H^{|\hat{J}_1 \cup \dots \cup \hat{J}_n| + (p_1 + \dots + p_n + 1) + 1}(\mathcal{Z}_{\hat{\mathcal{K}}})$ had classes $\hat{\alpha}_i \in H^{|\hat{J}_i| + p_i + 1}(\mathcal{Z}_{\hat{\mathcal{K}}})$. Theorem 3.2.17 gives an n-Massey product on classes whose degree is determined by $|J_i| \geqslant |\hat{J}_i|$, so $\alpha_i \in H^{|J_i| + p_i + 1}(\mathcal{Z}_{\mathcal{K}})$. Therefore $\langle \alpha_1, \dots, \alpha_n \rangle \subset H^{(p_1 + \dots + p_n) + |J_1 \cup \dots \cup J_n| + 2}(\mathcal{Z}_{\mathcal{K}})$.



(a) A truncated octahedron P

(b) A full subcomplex of $\mathcal{K}$, missing just one vertex

Figure 3.10

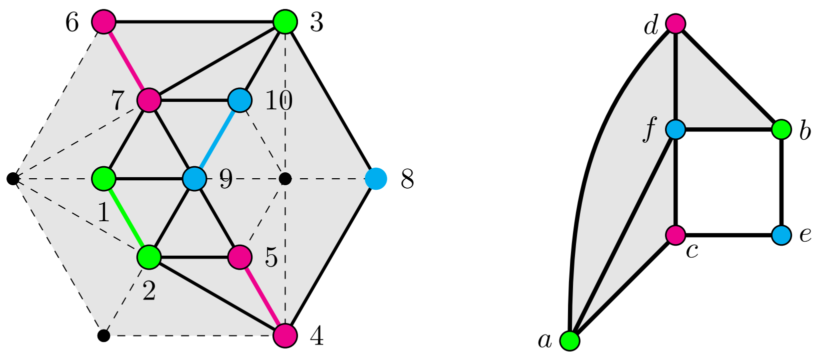
Example 3.2.19. Let P be the truncated octahedron as shown in Figure 3.10a. Its facets are squares and hexagons. Let $\mathcal{K} = \mathcal{K}_P = \partial P^*$ be the simplicial complex that is the nerve complex of the simple polytope P. Then $\mathcal{K}$ is the simplicial complex in Figure 3.10b, with one more vertex and six 2-simplices joining this vertex to the boundary of the disc in Figure 3.10b.

Consider a full subcomplex $\mathcal{K}_V$ of $\mathcal{K}$, such as in Figure 3.11a. It is possible to edge contract $\mathcal{K}_V$ to a simplicial complex on 6 vertices by contracting the coloured edges, as shown in Figure 3.11b. These edge contractions satisfy the link condition.

Up to graph isomorphism, the one skeleton of the simplicial complex $\hat{\mathcal{K}}$ in Figure 3.11b is one of the obstruction graphs from Figure 2.6. Therefore by Theorem **??**, there is a non-trivial triple Massey product $\langle \hat{\alpha}_1, \hat{\alpha}_2, \hat{\alpha}_3 \rangle \subset H^8(\mathcal{Z}_{\hat{\mathcal{K}}})$ for $\hat{\alpha}_i \in H^3(\mathcal{Z}_{\hat{\mathcal{K}}})$ and where $\hat{\alpha}_1$, $\hat{\alpha}_2$, $\hat{\alpha}_3$ are supported on $\hat{J}_1 = \{a, b\}$, $\hat{J}_2 = \{c, d\}$, $\hat{J}_3 = \{e, f\}$, respectively.

Since the edge contractions taking $\mathcal{K}_V$ to $\hat{\mathcal{K}}$ satisfy the link condition, by Theorem 3.2.17 there is also a non-trivial triple Massey product $\langle \alpha_1, \alpha_2, \alpha_3 \rangle \subset H^*(\mathcal{Z}_{\mathcal{K}})$. Moreover, $\alpha_1, \alpha_2, \alpha_3$ are supported on the vertex sets $J_1 = \{1, 2, 3\}, J_2 = \{4, 5, 6, 7\}, J_3 = \{8, 9, 10\}$, respectively. Hence $\alpha_1 \in H^4(\mathcal{Z}_{\mathcal{K}})$, $\alpha_2 \in H^5(\mathcal{Z}_{\mathcal{K}})$, $\alpha_3 \in H^4(\mathcal{Z}_{\mathcal{K}})$, and $\langle \alpha_1, \alpha_2, \alpha_3 \rangle \subset H^{12}(\mathcal{Z}_{\mathcal{K}})$. Therefore, Theorem 3.2.17 gives a new triple Massey product. In particular, it shows that $\mathcal{Z}_{\mathcal{K}} = \mathcal{Z}_P$ is a non-formal manifold, when P is a truncated octahedron.

Example 3.2.20. In Section 2.3.4 we described how Zhuravleva [39] showed that for any Pogorelov polytope P, moment-angle complexes $\mathcal{Z}_P = \mathcal{Z}_{\mathcal{K}_P}$ have a non-trivial triple Massey product using the full subcomplex in Figure 3.12.



(a) A full subcomplex of $\mathcal{K}$

(b) A full subcomplex of $\mathcal{K}$ after edge contracting the coloured edges

Figure 3.11

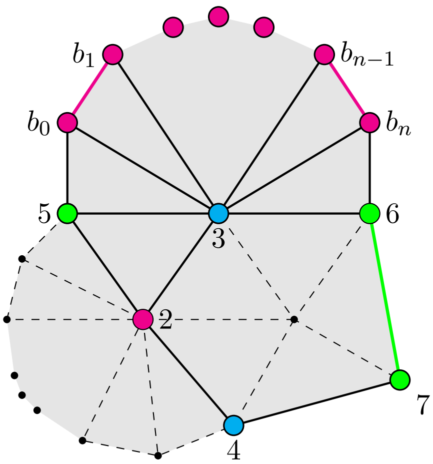


Figure 3.12: A full subcomplex of the simplicial complex corresponding to any Pogorelov polytope [39]

Applying edge contractions to the coloured edges of the full subcomplex in Figure 2.9, we obtain the simplicial complex in Figure 3.13. Up to graph isomorphism, the one-skeleton of this simplicial complex is one of the obstruction graphs in Figure 2.6, given by Denham and Suciu [17]. These edge contractions satisfy the link condition, and therefore Theorem 3.2.17 recovers the non-trivial triple Massey products in Zhuravleva's work from Denham and Suciu's classification of non-trivial triple Massey products of classes in degree three.

Furthermore, similar conclusions can be made about other simple polytopes, rather than just those in the Pogorelov class or the truncated octahedron. For example, when P is a polytope such as the 3-dimensional permutahedron, 3-dimensional stellahedron, truncated dodecahedron, etc, then $\mathcal{K}_P$ contains full subcomplexes that also edge contract to an obstruction graph. Therefore we can conclude the existence of non-trivial triple Massey products in many other moment-angle manifolds, other than those related to Pogorelov polytopes. This also provides more explicit examples moment-angle manifolds that are non-formal.

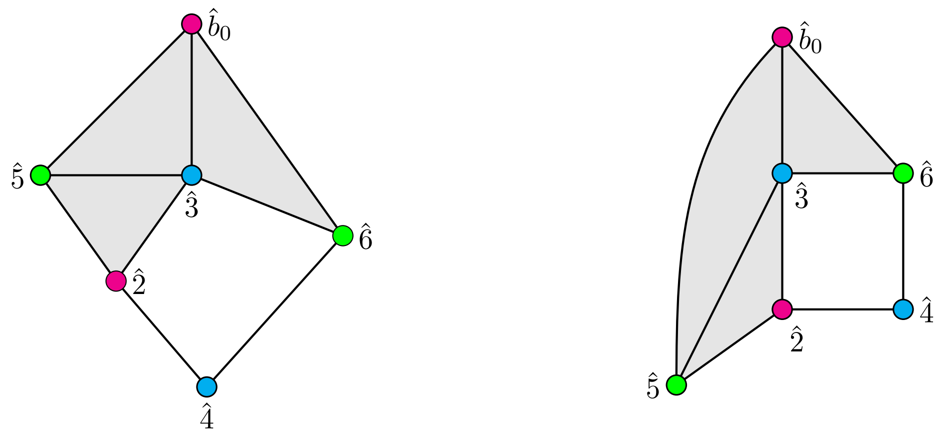


Figure 3.13: An edge-contracted full subcomplex of a simplicial complex corresponding to any Pogorelov polytope

Remark 3.2.21. Theorem 3.2.17 can be used for detecting non-trivial Massey products in moment-angle complexes. For example, let $\mathcal{K}$ be a simplicial complex with $\alpha_1, \dots, \alpha_n \in H^*(\mathcal{Z}_\mathcal{K})$ such that $\alpha_i \in \widetilde{H}^{p_i}(\mathcal{K}_{J_i})$ and $J_i \cap J_j = \varnothing$ for any $i \neq j$. Suppose $\mathcal{K}$ edge contracts to a simplicial complex $\hat{\mathcal{K}}$ that has non-trivial Massey product $\langle \hat{\alpha}_1, \dots, \hat{\alpha}_n \rangle \in H^*(\mathcal{Z}_{\hat{\mathcal{K}}})$ such that $\langle \hat{\alpha}_1, \dots, \hat{\alpha}_n \rangle$ can be lifted to $\langle \alpha_1, \dots, \alpha_n \rangle$ as in the proof of Theorem 3.2.17. Then $\langle \alpha_1, \dots, \alpha_n \rangle \subset H^*(\mathcal{Z}_\mathcal{K})$ is also a non-trivial higher Massey product.

We can also use Theorem 3.2.17 to reduce known non-trivial Massey products to other non-trivial Massey products of smaller degree.

Corollary 3.2.22. *Let $\mathcal{K}$ be a simplicial complex with a non-trivial n-Massey product $\langle \alpha_1, \dots, \alpha_n \rangle \subset H^*(\mathcal{Z}_\mathcal{K})$ for $\alpha_i \in \widetilde{H}^{p_i}(\mathcal{K}_{J_i})$ and $J_i \cap J_j = \varnothing$ for any $i \neq j$. Suppose $\varphi\colon \mathcal{K} \to \hat{\mathcal{K}}$ is a series of edge contractions such that there are non-trivial classes $\hat{\alpha}_i \in \widetilde{H}^{p_i}(\hat{\mathcal{K}}_{\varphi(J_i)})$ for $i = 1, \dots, n$. Then there is a non-trivial n-Massey product $\langle \hat{\alpha}_1, \dots, \hat{\alpha}_n \rangle$ in $H^*(\mathcal{Z}_{\hat{\mathcal{K}}})$.*

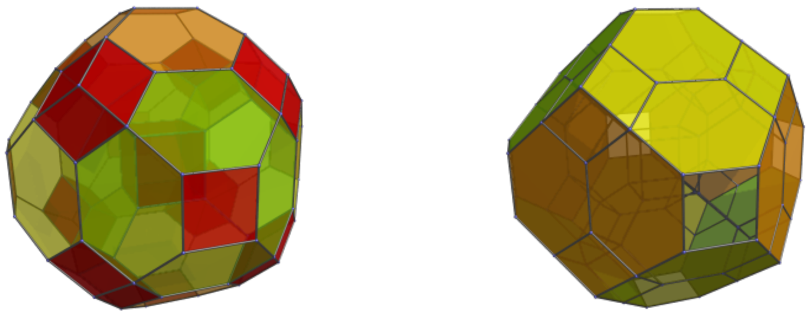
Proof. If the Massey product $\langle \hat{\alpha}_1, \dots, \hat{\alpha}_n \rangle$ was trivial, then lifting it as in Theorem 3.2.17 would also give a trivial Massey product $\langle \alpha_1, \dots, \alpha_n \rangle$. □

3.3 A moment-angle manifold with non-trivial higher Massey product

In this section, we use both Theorem 3.1.15 and Theorem 3.2.17 to show the existence of a non-trivial 4-Massey product in the cohomology of $\mathcal{Z}_P$ where P is a 4-polytope that is not a truncated 4-cube.

Example 3.3.1. Let P be the truncated 24-cell, which is a simple 4-polytope composed of 24 truncated octahedrons and 24 cubes. It is derived from the 24-cell by performing vertex cuts at every vertex, which exposes the 24 cubes.

In a cube-first projection of the truncated 24-cell, there is one cube closest in the four-dimensional viewpoint. Six truncated octahedra surround that central cube, and eight more cubes fill gaps between three adjacent truncated octahedra. There are 12 truncated octahedra and six cubes in the equator of the truncated 24-cell. Then on the far side of the truncated 24-cell, a further six truncated octahedra and $8+1$ cubes have the same arrangement as those on the near side. Therefore in total there are $6+12+6=24$ truncated octahedra and $1+8+6+8+1=24$ cubes.



(a) Near (far) side of the truncated 24-cell, plus six equatorial cubes shown as flat squares due to the $4D$ viewpoint.

(b) Twelve equatorial truncated octahedra shown as flat hexagons due to the $4D$ viewpoint.

Figure 3.14: Visualisation of the truncated 24-cell [35]

We find a non-trivial 4-Massey product in $H^*(\mathcal{Z}_P)$, as constructed by Theorems 3.1.15 and 3.2.17. Let $\mathcal{K}_P$ be the simplicial complex that is dual to P.

Label the nearest cube in P as C_1. Label the furthest cube $C_{1'}$. Let the six equatorial cubes be $C_2, C_{2'}, C_3, C_{3'}, C_4, C_{4'}$, such that opposite cubes along the same axis are C_i and $C_{i'}$. Every truncated octahedron is adjacent to exactly two of the cubes $C_1, C_{1'}, \dots, C_4, C_{4'}$, so let each truncated octahedron be labelled $O_{i,j}$ for C_i, C_j the adjacent cubes.

Let $\mathcal{K}$ be a full subcomplex of $\mathcal{K}_P$ after removing the vertices $O_{1,3'}, O_{1,2'}, O_{2,3'}, O_{2,4'}, O_{3,4'}$, and all of the vertices that correspond to cubes other than $C_i, C_{i'}$ for $i=1,\dots,4$. Let $\hat{\mathcal{K}}_1$ be the result of contracting the edges $\{C_{1'}, O_{1',j}\} \mapsto \hat{C}_{1'}$ for $j \in \{2, 2', \dots, 4, 4'\}$. Then the one-skeleton of $\hat{\mathcal{K}}_1$ is the simplicial complex in Figure 3.15, where $\hat{C}_{1'}$ is connected to every vertex except C_1, $O_{1,2}$, $O_{1,3}$, $O_{1,4}$ and $O_{1,4'}$.

Performing edge contractions at each of the bold edges, we obtain a simplicial complex $\hat{\mathcal{K}}$. These edge contractions satisfy the link condition. The one-skeleton of $\hat{\mathcal{K}}$ is shown in Figure 3.16, where $\hat{C}_{1'}$ is connected to every vertex except $\hat{C}_1$.

Rearranging the vertices of $\hat{\mathcal{K}}$, we have the simplicial complex whose one-skeleton is shown in Figure 3.16b. This simplicial complex is exactly the dual to the 4-cube, after stellar subdivisions have been performed at the edges $\{\hat{C}_1, \hat{C}_{2'}\}$, $\{\hat{C}_1, \hat{C}_{3'}\}$, $\{\hat{C}_2, \hat{C}_{3'}\}$, $\{\hat{C}_2, \hat{C}_{4'}\}$, $\{\hat{C}_3, \hat{C}_{4'}\}$. That is,

$$\hat{\mathcal{K}} = \mathrm{ss}_{\{\hat{C}_1,\hat{C}_{2'}\}}\, \mathrm{ss}_{\{\hat{C}_1,\hat{C}_{3'}\}}\, \mathrm{ss}_{\{\hat{C}_2,\hat{C}_{3'}\}}\, \mathrm{ss}_{\{\hat{C}_2,\hat{C}_{4'}\}}\, \mathrm{ss}_{\{\hat{C}_3,\hat{C}_{4'}\}}\, \mathcal{K}_1 * \mathcal{K}_2 * \mathcal{K}_3 * \mathcal{K}_4$$

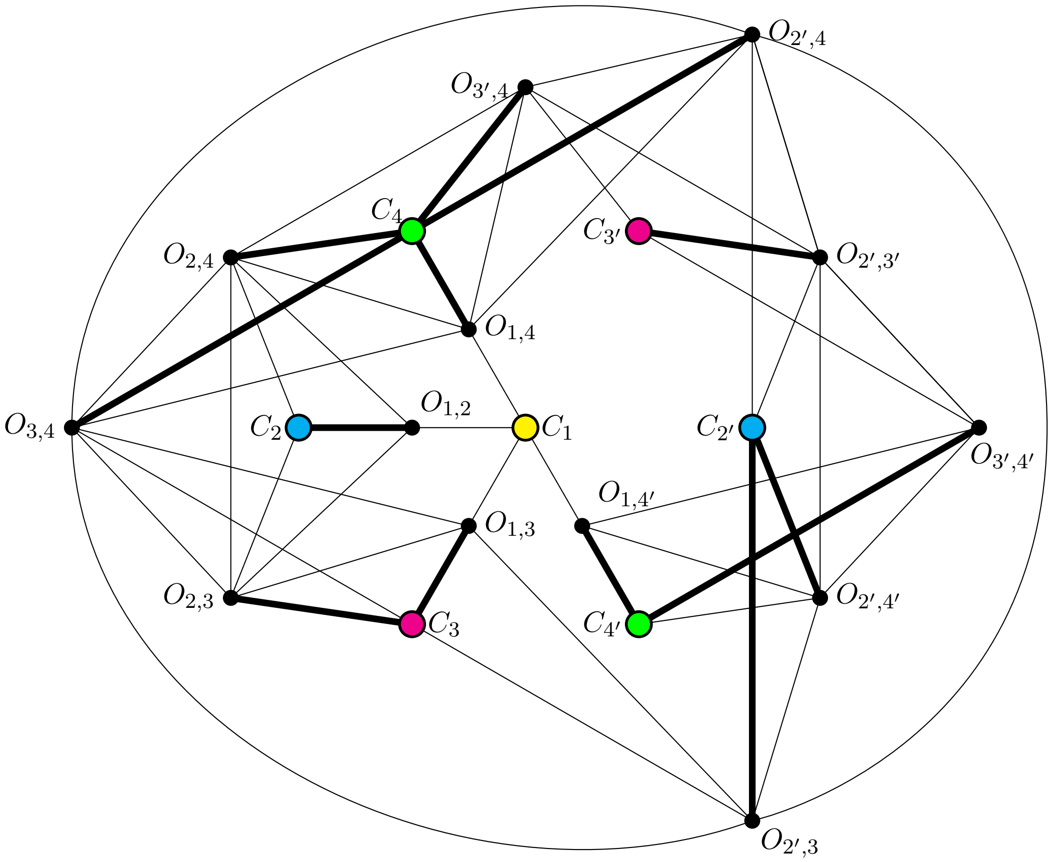


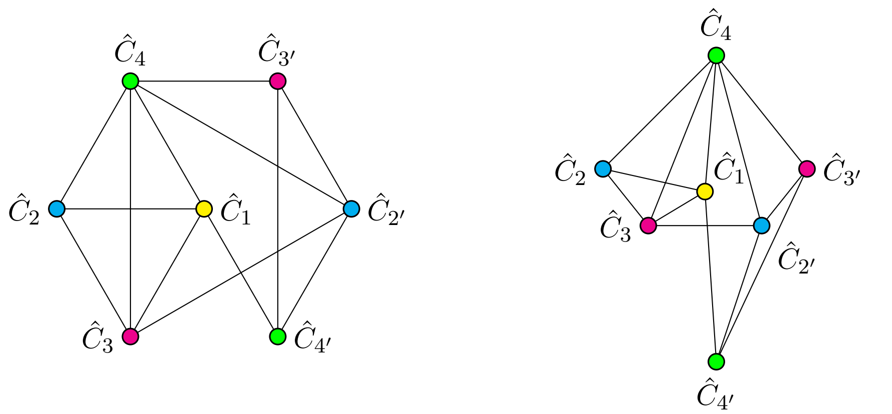
Figure 3.15: A full subcomplex of $\hat{\mathcal{K}}_1$

where $\mathcal{K}_i = \{\{\hat{C}_i\}, \{\hat{C}_{i'}\}\}$, for $i = 1, 2, 3, 4$.

Therefore by Theorem 3.1.15, there is a non-trivial 4-Massey product $\langle \hat{\alpha}_1, \hat{\alpha}_2, \hat{\alpha}_3, \hat{\alpha}_4 \rangle \subset H^{10}(\mathcal{Z}_{\hat{\mathcal{K}}})$. Each cohomology class $\hat{\alpha}_i$ can be thought of as a class in $\widetilde{H}^0(\hat{\mathcal{K}}_i)$. By Theorem 3.2.17, there is a non-trivial 4-Massey product $\langle \alpha_1, \alpha_2, \alpha_3, \alpha_4 \rangle \subset H^{29}(\mathcal{Z}_{\mathcal{K}}) \subset H^{29}(\mathcal{Z}_P)$, where $\alpha_1 \in H^9(\mathcal{Z}_P)$, $\alpha_2 \in H^6(\mathcal{Z}_P)$, $\alpha_3 \in H^6(\mathcal{Z}_P)$, $\alpha_4 \in H^{10}(\mathcal{Z}_P)$. Alternatively, $\alpha_i \in \widetilde{H}^0(\mathcal{K}_{J_i})$ for

$$J_1 = \{C_1, C_{1'}, O_{1',2}, O_{1',2'}, O_{1',3}, O_{1',3'}, O_{1',4}, O_{1',4'}\},$$
$$J_2 = \{C_2, C_{2'}, O_{1,2}, O_{2',4'}, O_{2',3}\}, \qquad J_3 = \{C_3, C_{3'}, O_{1,3}, O_{2,3}, O_{2',3'}\},$$
$$J_4 = \{C_4, C_{4'}, O_{1,4}, O_{1,4'}, O_{2,4}, O_{2',4}, O_{3,4}, O_{3',4}, O_{3',4'}\}.$$

Therefore $\mathcal{Z}_P$ is a non-formal manifold, when P is a truncated 24-cell.



(a) The one-skeleton of a full subcomplex of $\hat{\mathcal{K}}$

(b) A rearranged one-skeleton of a full subcomplex of $\hat{\mathcal{K}}$

Figure 3.16

Chapter 4

Simplicial Posets

4.1 Introduction

Simplicial posets are a generalisation of simplicial complexes. They first arose as quotients of simplicial complexes under group actions [20], but also correspond to *ideal triangulations* in low-dimensional topology. It was shown by Lü and Panov [28] that we can generalise moment-angle complexes to correspond with simplicial posets instead of just simplicial complexes. One advantage of this is that we can obtain moment-angle complexes such as spheres in even dimension.

The faces of a simplex form a partially ordered set (poset) with respect to inclusion, with the empty set as the initial element.

Definition 4.1.1. A simplicial poset $\mathcal{S}$ is a finite poset with order relation $\leqslant$, an initial element $\hat{0}$ and the property that for any $\sigma \in \mathcal{S}$, the lower segment $[\hat{0}, \sigma] = \{\tau \in \mathcal{S} \colon \hat{0} \leqslant \tau \leqslant \sigma\}$ is the face poset of a standard simplex.

All simplicial complexes $\mathcal{K}$ are also simplicial posets by considering $\mathcal{K}$ as its face poset. We use the term *simplicial poset* to also refer to the cell complex obtained by assigning a geometric simplex Δ^σ to every $\sigma \in \mathcal{S}$ and gluing these simplices along the poset category. Therefore we refer to an element $\sigma \in \mathcal{S}$ as a *simplex*. The *rank* of a simplex $|\sigma|$ is k if the face Δ^σ is a $(k-1)$-dimensional standard simplex. A *vertex* is a 0-dimensional simplex. The *dimension* of a simplicial poset $\mathcal{S}$ is the maximum of ranks of its simplices minus one.

Example 4.1.2. The simplest example of a simplicial complex that is not a simplicial poset is the "doubling" of a standard one-simplex on two vertices, as shown in Figure 4.1a. Figure 4.1b is the simplicial poset obtained by gluing two standard 2-simplices (triangles) along their boundary, and Figure 4.1c is the result of adding two copies of the standard 1-simplex to one edge of a triangle.

Stanley [34] defined a face ring for simplicial posets that coincides with the definition of

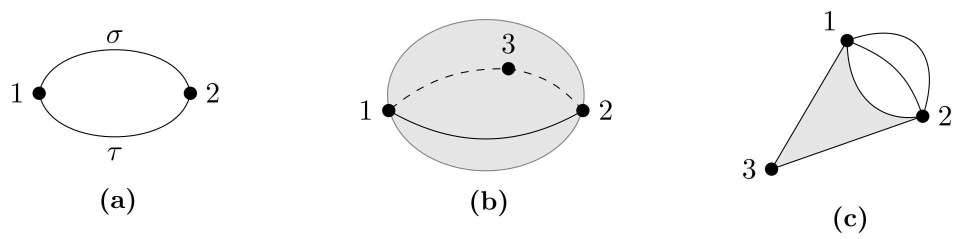


Figure 4.1: Examples of simplicial posets

a face ring for simplicial complexes.

Definition 4.1.3. For any two simplices $\sigma, \tau \in \mathcal{S}$, the *join* $\sigma \vee \tau$ is the set of their least common upper bounds, and the *meet* $\sigma \wedge \tau$ is the set of their greatest common lower bounds.

Example 4.1.4. In Figure 4.1a, for simplices $\{1\}, \sigma \in \mathcal{S}$, $\{1\} \wedge \sigma = \{1\}$ and $\{1\} \vee \sigma = \{\sigma\}$. Additionally, $\{1\} \wedge \{2\} = \varnothing$ and $\{1\} \vee \{2\} = \{\sigma, \tau\}$. For the maximal simplices $\sigma, \tau \in \mathcal{S}$, $\sigma \wedge \tau = \{1, 2\}$ while $\sigma \vee \tau = \varnothing$.

Definition 4.1.5 ([34])**.** For a simplicial poset $\mathcal{S}$, the *face ring* $\mathbf{k}[\mathcal{S}]$ is the quotient

$$\mathbf{k}[\mathcal{S}] = \mathbf{k}[v_\sigma : \sigma \in \mathcal{S}]/\mathcal{I}_\mathcal{S}$$

where the generator v_σ has degree $\deg v_\sigma = 2|\sigma|$, and $\mathcal{I}_\mathcal{S}$ is the Stanley-Reisner ideal generated by

$$v_{\hat{0}} - 1 \quad \text{and} \quad v_\sigma v_\tau - v_{\sigma \wedge \tau} \cdot \sum_{\eta \in \sigma \vee \tau} v_\eta.$$

The sum over the empty set is taken to be zero, so if $\sigma \vee \tau = \varnothing$, there is the relation $v_\sigma v_\tau = 0$.

Remark 4.1.6. For a simplicial complex $\mathcal{K}$, the monomial $v_\sigma v_\tau$ corresponds to a missing face of $\mathcal{K}$ if and only if $\sigma \vee \tau = \varnothing$, in which case $v_\sigma v_\tau = 0$. If, however, $\sigma \cup \tau$ is not a missing face, then $\sigma \vee \tau$ contains one element so $v_\sigma v_\tau = v_{\sigma \wedge \tau} v_{\sigma \vee \tau}$. Therefore for simplicial complexes, Definition 4.1.5 of the face ring is the same as Definition 2.1.10.

Example 4.1.7. For $\mathcal{S}$ given by Figure 4.1a, the join and meet of simplices is calculated in Example 4.1.4. Then the Stanley-Reisner ideal is generated by the relators

$$v_\varnothing - 1, \quad v_1 v_2 - v_\varnothing \cdot (v_\sigma + v_\tau) = v_1 v_2 - (v_\sigma + v_\tau) \quad \text{and} \quad v_\sigma v_\tau.$$

Additionally, Lü and Panov [28] gave a definition of moment-angle complexes for simplicial posets that generalises the definition of moment-angle complexes for simplicial complexes. For a simplicial poset $\mathcal{S}$ on $[m]$, and any simplex $\sigma \in \mathcal{S}$, let

$$(D^2, S^1)^\sigma = \{(z_1, \ldots, z_m) \in D^{2m} \colon |z_j| = 1 \text{ if } j \not\leq \sigma\}.$$

Therefore $(D^2, S^1)^\sigma$ is a subspace of D^{2m} homeomorphic to a product of $|\sigma|$ discs and

$m - |\sigma|$ circles. There is a natural inclusion $(D^2, S^1)^\sigma \subset (D^2, S^1)^\tau$ for $\sigma \leqslant \tau$, so in a categorical sense, there is a diagram $(D^2, S^1)^{\mathcal{S}} \colon \textsc{cat}(\mathcal{S}) \to \textsc{top}$, $\sigma \mapsto (D^2, S^1)^\sigma$, from the face poset of $\mathcal{S}$ to the category of topological spaces.

Definition 4.1.8. For a simplicial poset $\mathcal{S}$, the *moment-angle complex* $\mathcal{Z}_{\mathcal{S}}$ is

$$\mathcal{Z}_{\mathcal{S}} = \operatorname*{colim}_{\sigma \in \textsc{cat}(\mathcal{S})} (D^2, S^1)^\sigma.$$

Example 4.1.9. Consider again Figure 4.1a. For the maximal simplices σ, τ, both $(D^2, S^1)^\sigma$, $(D^2, S^1)^\tau$ are copies of $D^4 = D^2 \times D^2$. The boundary of both σ and τ is the set $\{1, 2\}$ of disjoint points. This corresponds to $S^3 = D^2 \times S^1 \cup S^1 \times D^2$ since the union of $(D^2, S^1)^{\{1\}}$ and $(D^2, S^1)^{\{2\}}$ is taken over $(D^2, S^1)^\varnothing = S^1 \times S^1$. Hence in the colimit, these copies of S^3 in both $(D^2, S^1)^\sigma$ and $(D^2, S^1)^\tau$ are identified, and so $\mathcal{Z}_{\mathcal{S}} = S^4$. This example is a moment-angle manifold that we cannot obtain from simplicial complexes.

As for simplicial complexes, Duval [18] showed that there is a simplicial poset equivalent of Hochster's theorem. Using this, Lü and Panov [28] showed that there is a simplicial poset version of Theorem 2.1.24.

Definition 4.1.10. For a simplicial poset $\mathcal{S}$ on $[m]$ and an index set $J \subset [m]$, let a *full subposet* $\mathcal{S}_J$ be the set of simplices $\sigma \in \mathcal{S}$ such that $V(\sigma) \subset J$.

Theorem 4.1.11 ([18, 28])**.** *For a simplicial poset $\mathcal{S}$ on $[m]$, there is an isomorphism of cochains $\widetilde{C}^{*-1}(\mathcal{S}_J) \to C^{*-|J|,2J}(\mathcal{Z}_{\mathcal{S}}) \subset C^{*+|J|}(\mathcal{Z}_{\mathcal{S}})$, inducing an isomorphism of bigraded algebras*

$$H^*(\mathcal{Z}_{\mathcal{S}}; \mathbf{k}) \cong \operatorname{Tor}^*_{\mathbf{k}[m]}(\mathbf{k}[\mathcal{S}], \mathbf{k}) \cong \bigoplus_{J \subset [m]} \widetilde{H}^*(\mathcal{S}_J)$$

where in particular, $\widetilde{H}^{-1}(\mathcal{S}_\varnothing) = \mathbf{k}$ and

$$H^p(\mathcal{Z}_{\mathcal{S}}) \cong \bigoplus_{J \subset [m]} \widetilde{H}^{p-|J|-1}(\mathcal{S}_J).$$

Additionally, the differentials in $C^*(\mathcal{Z}_{\mathcal{S}}; \mathbf{k})$ and $\widetilde{C}^*(\mathcal{S}_J)$ are the same as those for simplicial complexes in Section 2.1.2.3. The main ideas for this proof are similar to the ideas in Section 2.1.2.1. Some of the technical algebraic arguments do not work for simplicial posets and instead need to be proved by topological and categorical arguments. Specifically, we still have an auxiliary algebra $R^*(\mathcal{S})$ similar to Definition 2.1.16, but proving the map $\varrho \colon \Lambda[u_1, \ldots, u_m] \otimes \mathbf{k}[\mathcal{S}] \to R^*(\mathcal{S})$ is a quasi-isomorphism is much harder to achieve than in the proof of Proposition 2.1.17. Instead, this proof uses a deformation retraction $D^2 \hookrightarrow S^\infty \to D^2$ and formality of polyhedral products to obtain a deformation retraction

$$\mathcal{Z}_{\mathcal{S}} = (D^2, S^1)^{\mathcal{S}} \hookrightarrow (S^\infty, S^1)^{\mathcal{S}} \to (D^2, S^1)^{\mathcal{S}}$$

where $(X, A)^{\mathcal{S}} = \operatorname*{colim}_{\sigma \in \textsc{cat}(\mathcal{S})} (X, A)^\sigma$. Then the proof constructs isomorphisms of cochain complexes $\Lambda[u_1, \ldots, u_m] \otimes \mathbf{k}[\mathcal{S}] \to C^*((S^\infty, S^1)^{\mathcal{S}})$ and $R^*(\mathcal{S}) \to C^*(\mathcal{Z}_{\mathcal{S}})$. From these maps it is possible to form a commutative square and show ϱ is a quasi-isomorphism. Specific constructions and more detail may be found in [28, Theorem 3.6]. Additionally, Lü and Panov [28] also concluded that $\mathcal{S} \mapsto \mathcal{Z}_{\mathcal{S}}$ is a covariant functor with respect to maps of simplicial posets.

Example 4.1.12. We may use Theorem 4.1.11 to calculate the cohomology of the moment-angle complex for $\mathcal{S}$ in Figure 4.1a.

$$H^0(\mathcal{Z}_{\mathcal{S}}) \cong \widetilde{H}^{-1}(\mathcal{S}_{\varnothing}) = \mathbf{k}$$
$$H^4(\mathcal{Z}_{\mathcal{S}}) \cong \widetilde{H}^{(4-2-1)}(S_{12}) \cong \widetilde{H}^1(S^1) \cong \mathbf{k}$$

and all other cohomology groups are zero. This agrees with Example 4.1.9.

Theorem 4.1.11 means that we can also study Massey products in moment-angle complexes corresponding to simplicial posets in the same way as in Section 2.2.2.

4.2 Massey products and simplicial posets

In this section we first show that non-trivial Massey products in moment-angle complexes that correspond to simplicial complexes lift to non-trivial Massey products in moment-angle complexes that correspond to simplicial posets. Subsequently we also show that there are non-trivial Massey products in moment-angle complexes from simplicial posets that cannot be obtained from such a lift.

For every simplicial poset, there is a corresponding simplicial complex.

Definition 4.2.1. The *associated simplicial complex* $\mathcal{K}_{\mathcal{S}}$ is the simplicial complex whose simplices are on the vertex set $V(\sigma)$ for $\sigma \in \mathcal{S}$. The *folding map* is $\mathcal{S} \to \mathcal{K}_{\mathcal{S}}$, $\sigma \mapsto V(\sigma)$ for $V(\sigma)$ the vertex set of $\sigma \in \mathcal{S}$.

Example 4.2.2. Let $\mathcal{S}$ be the simplicial poset in Figure 4.1a. Then $\mathcal{K}_{\mathcal{S}}$ is a 1-simplex. For the simplicial posets in Figures 4.1b and 4.1c, $\mathcal{K}_{\mathcal{S}}$ is a 2-simplex.

Since $\mathcal{Z}_{\mathcal{S}}$ is functorial, the folding map in Definition 4.2.1 induces a map $f^*\colon H^*(\mathcal{Z}_{\mathcal{K}_{\mathcal{S}}}) \to H^*(\mathcal{Z}_{\mathcal{S}})$. We use this map to lift Massey products from $H^*(\mathcal{Z}_{\mathcal{K}_{\mathcal{S}}})$ to $H^*(\mathcal{Z}_{\mathcal{S}})$.

Proposition 4.2.3. *Any non-trivial Massey product $\langle \alpha_1, \ldots, \alpha_n \rangle$ in $H^*(\mathcal{Z}_{\mathcal{K}_{\mathcal{S}}})$ lifts to a non-trivial Massey product in $H^*(\mathcal{Z}_{\mathcal{S}})$.*

Proof. The folding map in Definition 4.2.1 induces a map $f^*\colon H^*(\mathcal{Z}_{\mathcal{K}_{\mathcal{S}}}) \to H^*(\mathcal{Z}_{\mathcal{S}})$ in cohomology. By Theorem 4.1.11, there is also a corresponding map $\widetilde{H}^*((\mathcal{K}_{\mathcal{S}})_I) \to \widetilde{H}^*(\mathcal{S}_I)$ that we will also denote f^*.

Let $[\omega] \in \langle \alpha_1, \ldots, \alpha_n \rangle \subset H^*(\mathcal{Z}_{\mathcal{K}_{\mathcal{S}}})$. By Theorem 4.1.11, $[\omega] \in H^*(\mathcal{Z}_{\mathcal{K}_{\mathcal{S}}})$ corresponds to a class $[\omega] \in \widetilde{H}^p((\mathcal{K}_{\mathcal{S}})_I)$ for $I \subset V(\mathcal{S})$ and an integer p. Let ω be a representative of $[\omega]$.

We wish to show that $f^*[\omega] \in H^*(\mathcal{Z}_\mathcal{S})$ is non-zero, that is, that $f^*(\omega) \in C^p(\mathcal{S}_I)$ is not a coboundary.

Suppose $f^*(\omega) \in C^p(\mathcal{S}_I)$ is a coboundary and let $c \in C^{p-1}(\mathcal{S})$ be a cochain such that $d(c) = f^*(\omega)$. The folding map $f\colon \mathcal{S} \to \mathcal{K}_\mathcal{S}$ preserves the rank of simplices, so c has a preimage $b = (f^*)^{-1}(c) \in C^{p-1}((\mathcal{K}_\mathcal{S})_I)$. As for simplicial complexes in (2.7), the differentials in $C^*(\mathcal{S}_I)$ are given by

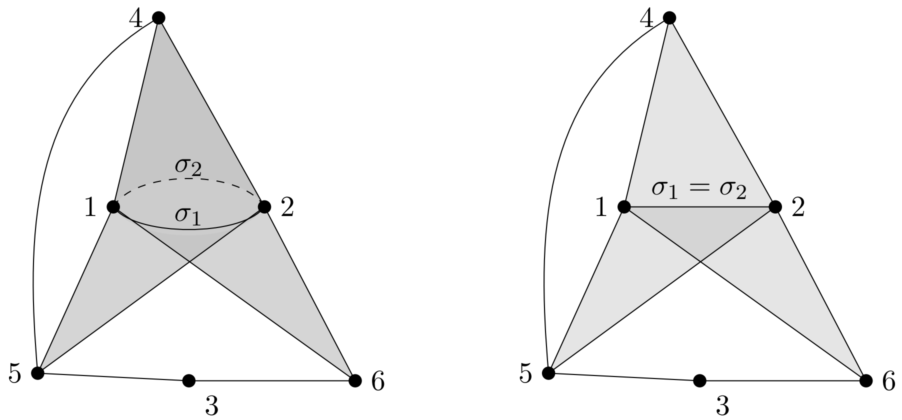
$$d(\chi_\sigma) = \sum_{j\in\sigma\setminus I, j\cup\sigma\in\mathcal{S}_I} \varepsilon(j, j\cup\sigma)\chi_{j\cup\sigma}.$$

Hence the differentials in $C^*((\mathcal{K}_\mathcal{S})_I)$ and $C^*(\mathcal{S}_I)$ are the same and $f^*\colon C^p((\mathcal{K}_\mathcal{S})_I) \to C^p(\mathcal{S}_I)$ commutes with the differential. Thus $d(b) = \omega \in C^{p-1}((\mathcal{K}_\mathcal{S})_I)$, and this contradicts the assumption that $\langle \alpha_1, \ldots, \alpha_n\rangle \subset H^*(\mathcal{Z}_{\mathcal{K}_\mathcal{S}})$ is non-trivial. □

On the other hand, there are also non-trivial Massey products in the moment-angle complexes of simplicial posets that cannot be lifted from an associated simplicial complex, such as in the following example.

Example 4.2.4. Let $\mathcal{S}_{12}$ be the simplicial poset with two edges σ_1, σ_2 on two vertices, $\{1\}, \{2\}$. We take the join of this subposet with the vertex $\{i\}$ for $i = 4, 5, 6$, thus creating a union of three cones on $\mathcal{S}_{12}$. We add the edge $\{4,5\}$ and for another vertex $\{3\}$, we add the edges $\{3,5\}$ and $\{3,6\}$. Let $\mathcal{S}$ be the resulting simplicial poset, as drawn in Figure 4.2a,

$$\mathcal{S} = \{\varnothing, \{1\}, \{2\}, \{3\}, \{4\}, \{5\}, \{6\}, \sigma_1, \sigma_2, \{2,4\}, \{1,4\}, \{3,5\}, \{3,6\}, \{4,5\},$$
$$\{4,\sigma_1\}, \{4,\sigma_2\}, \{5,\sigma_1\}, \{5,\sigma_2\}, \{6,\sigma_1\}, \{6,\sigma_2\}\}.$$



(a) A simplicial poset with non-trivial Massey product in $H^*(\mathcal{Z}_\mathcal{S})$

(b) The associated simplicial complex $\mathcal{K}_\mathcal{S}$ for $\mathcal{S}$.

Figure 4.2: A simplicial poset with non-trivial Massey product in $H^*(\mathcal{Z}_\mathcal{S})$, and its associated simplicial complex

Let $\alpha_1 \in \widetilde{H}^1(\mathcal{S}_{1,2})$, $\alpha_2 \in \widetilde{H}^0(\mathcal{S}_{3,4})$, $\alpha_3 \in \widetilde{H}^0(\mathcal{S}_{5,6})$. These correspond to classes $\alpha_1 \in$

$H^4(\mathcal{Z}_{\mathcal{S}})$, $\alpha_2, \alpha_3 \in H^3(\mathcal{Z}_{\mathcal{S}})$.

Next we show that $\langle \alpha_1, \alpha_2, \alpha_3 \rangle$ is defined and non-trivial. By Theorem 4.1.11, $\alpha_1\alpha_2 \in \widetilde{H}^2(\mathcal{S}_{1234})$. Since $\widetilde{H}^2(\mathcal{S}_{1234}) = 0$, the product $\alpha_1\alpha_2$ is therefore zero. Similarly, $\alpha_2\alpha_3 \in \widetilde{H}^1(\mathcal{S}_{3456}) = 0$ and so $\langle \alpha_1, \alpha_2, \alpha_3 \rangle$ is defined. Furthermore, $\widetilde{H}^0(\mathcal{S}_{3456}) = 0$ and $\widetilde{H}^1(\mathcal{S}_{1234}) = 0$. By (2.14), the indeterminacy of $\langle \alpha_1, \alpha_2, \alpha_3 \rangle$ is

$$\alpha_1 \cdot \widetilde{H}^0(\mathcal{S}_{3456}) + \alpha_3 \cdot \widetilde{H}^1(\mathcal{S}_{1234}).$$

Therefore the indeterminacy is trivial and it only remains to show that there is a non-zero element in $\langle \alpha_1, \alpha_2, \alpha_3 \rangle$.

Let α_1 be represented by the cocycle $a_1 = \chi_{\sigma_1}$, α_2 be represented by $a_2 = \chi_4$, and α_3 by $a_3 = \chi_6$. Then let $a_{1,2} = \chi_{\sigma_1} \in C^1(\mathcal{S}_{1234})$ so that $d(a_{1,2}) = \chi_{\sigma_1 \cup 4} = \overline{\overline{a_1}} a_2$, where $\overline{\overline{a_1}} = (-1)^{1+\overline{\deg a_1}} a_1$ as in Definition 2.2.7. Also let $a_{2,3} = 0$ since $\overline{\overline{a_2}} a_3 = \chi_{46}$. Then the associated cocycle to this defining system is given by $\omega = \overline{\overline{a_1}} a_{2,3} + \overline{\overline{a_{1,2}}} a_3 = -\chi_{\sigma_1 \cup 6}$. The class $[\omega]$ is a generator of $\widetilde{H}^2(\mathcal{S}_{123456})$, so $[\omega] \neq 0$. Since the indeterminacy is trivial, $[\omega]$ is the only element in this triple Massey product. Hence $\langle \alpha_1, \alpha_2, \alpha_3 \rangle$ in $H^*(\mathcal{Z}_{\mathcal{S}})$ is non-trivial.

In particular this triple Massey product cannot be obtained as a lift induced by the folding map $\mathcal{S} \to \mathcal{K}_{\mathcal{S}}$, where $\mathcal{K}_{\mathcal{S}}$ is shown in Figure 4.2b. Since $\widetilde{H}^p((\mathcal{K}_{\mathcal{S}})_{12}) = 0$ for all p, there is no non-zero class $\alpha_1 \in \widetilde{H}^p((\mathcal{K}_{\mathcal{S}})_{12})$. Therefore there is no corresponding non-trivial Massey product in $H^*(\mathcal{Z}_{\mathcal{K}_{\mathcal{S}}})$. Hence we have shown the following.

Proposition 4.2.5. *There exist simplicial posets $\mathcal{S}$ with non-trivial Massey products in $H^*(\mathcal{Z}_{\mathcal{S}})$ that do not exist in $H^*(\mathcal{Z}_{\mathcal{K}_{\mathcal{S}}})$, where $\mathcal{K}_{\mathcal{S}}$ is the associated simplicial complex for $\mathcal{S}$.* □

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